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Machine Learning for Degeneration Families of Calabi-Yau Manifolds in $\mathbb{CP}^n$

Author:

Frank She

Supervisor:

Dr. Daniel Platt

Submitted in partial fulfilment of the requirements for the degree at Imperial College London

CID: 02099231

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Statement of Originality

The work contained in this thesis is my own work unless otherwise stated.

Abstract

While Yau's solution [Yau, 1978] to the Calabi conjecture [Calabi, 1957] guarantees the existence and uniqueness of a Ricci-flat Kähler metric in a given Kähler class on a Calabi-Yau manifold, it is non-constructive: the explicit local formula for the Calabi-Yau metrics are generally unknown, which motivates a large literature on its numerically approximation.

In this thesis, we study some explicit examples of K3 surfaces in $\mathbb{CP}^3$ as Calabi-Yau degeneration family. We classify and numerically verify the corresponding types of complex structure limits, by computing the growth rate of the holomorphic volume of the degeneration family. We also use a neural-network based PDE solver [Douglas et al., 2021], which computes the approximated Calabi-Yau volume and the associated metric tensor. We finally propose two conjectures, supported by numerical evidence, relating dimension-collapsing of the degeneration family and eigenvalues of the Calabi-Yau metric tensors.

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I thank my family and friends for their constant support throughout my work on this thesis. I am also proud of the dedication and skills that I developed while undertaking this project.

Contents

1 Introduction

Calabi–Yau (CY) manifolds — compact Kähler manifolds with trivial canonical bundle — occupy a central place in modern geometry and mirror symmetry. The formulation and proof of the Calabi conjecture [Calabi, 1957; Yau, 1978] was pioneering in the field of geometric analysis. While Yau's solution to the Calabi conjecture guarantees the existence and uniqueness of a Ricci-flat Kähler metric in a given Kähler class, it is non-constructive: the explicit local formula for the Calabi-Yau metrics are generally unknown, which motivates a large literature on its numerically approximation.

A principal analytic approach to approximation is via algebraic embeddings and Bergman metrics Tian's theorem [Tian, 1990] show that Kähler metrics induced from projective embeddings by $H^0(X, L^k)$ can approximate any Kähler metric as $k \to \infty$, and Donaldson developed an iteration scheme which make these approximations numerically implementable [Donaldson, 2001, 2009]. Since Donaldson's work, a wide variety of refinements and attempts were made towards numerically approximating the Calabi-Yau metric, including but are not limited to the multi-grid PDE solver [Headrick and Wiseman, 2005] and machine learning solvers [Ashmore et al., 2020; Douglas et al., 2021; Larfors et al., 2022]. Degeneration families of Calabi-Yau manifolds, and K3 surfaces in particular, provide rich testbeds for both analytic theory and numerical experiments. Degenerations capture phenomena such as complex structure limits, dimension/volume collapsing, formation of necks and bubbles, and singular limits. The study of large complex structure limits (LCSL) is indispensable in the Strominger–Yau–Zaslow (SYZ) picture of mirror symmetry [Strominger et al., 1996; Gross and Wilson, 2000; Sun and Zhang, 2019; Li, 2022], and it is fascinating how these physical motivations both drove, and in turn benefited from, the mathematical advancements in Calabi-Yau geometry. [Candelas et al., 1991; Morrison, 1993] In this thesis, my main original contributions are as follows:

- A concise proof of the constancy of the Fubini-Study volume of the Calabi-Yau degeneration family.
- Verification of the transversality condition for concrete quartic K3 examples.
- Refactoring and extending the code by [Douglas et al., 2021], enabling computation for the holomorphic volume for a larger range of deformation parameter t.
- Numerically approximation of the Calabi-Yau volumes and metrics, for concrete examples of K3 surfaces and elliptic curves.
- Proposing two conjectures, supported by numerical evidences, relating dimension-collapsing of the degeneration family to the eigenvalues of the Calabi-Yau metric tensor.

The paper is structured as follows:

- In Section 2, we review the basic tools: we define Calabi-Yau manifolds, Calabi-Yau metrics, Gromov-Hausdorff convergence, degeneration family, complex structure limits, currents of integration and set up some other notations.
- In Section 3, we provide some well-studied examples of K3 surfaces in $\mathbb{CP}^3$ as Calabi-Yau degeneration family and classify the corresponding types of complex structure limits.
- In Section 4, these classifications of are numerically verified by computing the growth rate of holomorphic volume of the degeneration family. We also explain the neural-network PDE solver for the Monge–Ampère equation, which approximates the Calabi-Yau metric on a given K3 surface. Using this neural-network, we then compute the approximated Calabi-Yau volume and the associated metric tensor. We finally propose two conjectures, supported by numerical evidence, relating dimension-collapsing of the degeneration family and eigenvalues of the Calabi-Yau metric tensors.

2 Background

For understanding Calabi-Yau manifolds, we need to quickly recall the general theory about Kähler and Riemannian manifolds. In this section, we build from Kähler and Riemannian geometry, and introduce some basic properties of Calabi-Yau manifolds, including a brief discussion about Yau's theorem [Yau, 1978]. We then formalise the notion of *'a family of Calabi-Yau manifolds converging to some limiting space'* via the discussion regarding *polarised degeneration family*, *Gromov-Hausdorff convergence* and *complex structure limits*. We also provide a brief overview of the known convergence results about Calabi-Yau degeneration families. Finally, we review the basic theory of *currents of integration* and *neural network*, which paves the way for the numerical work in Section 4.

2.1 Kähler Manifolds

The material in this subsection is primarily adapted from Chapter 3.1 of [Huybrechts, 2005].

Definition 2.1. (Positive form (on $\mathbb{C}^n$)). Given $V \subset \mathbb{C}^n$, a (1,1)-form $\omega \in C^\infty(V, \Omega_V^{1,1})$ is called positive, if ω has representation:

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{jk} dz_j \wedge d\bar{z}_k \tag{1}$$

where the matrix-valued function $(h_{jk})_{j,k} : V \to \mathbb{C}^{n \times n}$ gives a positive definite matrix at every $x \in V$.

Definition 2.2. (Positive form (on complex manifold X)). Let X be a complex manifold (of dimension n), $\omega \in C^\infty\big(X, \Omega_X^{1,1}\big)$. We say ω is a positive form on X, if for every $x \in X$, there exists a holomorphic chart (U, ϕ) around x, with $\phi : U \to V \subset \mathbb{C}^n$ such that $(\phi^{-1})^*\omega$ is a positive form on $V \subset \mathbb{C}^n$.

Definition 2.3. (Kähler manifold). A complex manifold X is called a Kähler manifold if it admits a Kähler form. A (1, 1)-form $\omega \in C^\infty\big(X, \Omega_X^{1,1}\big)$ is called Kähler if it is real ($w = \bar{w}$), closed ($d\omega = 0$) and positive.

Definition 2.4. (Induced almost complex structure). Let X be a complex manifold, with complex charts $\{(U, \phi)\}$. The almost complex structure induced by the complex structure is a bundle morphism $J : TX \to TX$, fibre-wisely defined by:

$$\begin{aligned} \forall x \in X, J_x : T_xX &\longrightarrow T_xX, \\ [\gamma] &\mapsto [\phi^{-1}(i \cdot \phi\gamma)] \end{aligned} \tag{2}$$

and $J_x^2 = -\mathrm{Id}$..

2.2 Riemannian Manifolds

The material in this subsection is primarily adapted from [Lee, 2018, Chap. 2 - 7], which collects the minimal Riemannian geometry we require. We recall the notions of *Kähler metric*, *Levi–Civita* *connection* associated to a Riemannian metric, *Riemann curvature tensor* and *Ricci curvature*. These notions are the exact ingredients needed to state the Calabi–Yau condition (Ricci-flat Kähler metric) in the next subsection. We also define C^∞_{loc}-convergence of differential forms, which we will discuss in greater detail in Remark 2.67.

Definition 2.5. (Riemannian manifold). Let M be a (real) smooth manifold of dimension n. A Riemannian metric g is a smooth assignment of points to inner products, $p \mapsto g_p(\cdot,\cdot)$. Concretely:

1. $\forall p \in M, g_p : T_pM \times T_pM \to \mathbb{R}$, and g_p is bilinear, symmetric $\big(g_p(u,v) = g_p(v,u)\big)$ and positive-definite $\big(g_p(v,v) > 0$, for every non-zero tangent vectors $v \in T_pM\big)$.
2. Given any two smooth vector fields X, Y on M, the map $p \mapsto g_p(X(p), Y(p))$ is smooth.

The manifold with the Riemannian metric g is called a Riemannian manifold, denoted (M, g).

Definition 2.6. (Kähler metric). Let (X,ω) be a Kähler manifold, J be the almost complex structure induced by the complex structure. The Kähler form ω induces a Riemannian metric g, defined by:

$$\begin{aligned} g_x : T_xX \times T_xX &\longrightarrow \mathbb{R}, \\ (u,v) &\mapsto \omega_x(u, Jv) \end{aligned} \tag{3}$$

Moreover, $h_x(u,v) := g_x(u,v) + i\omega_x(u,v)$ is a Hermitian inner product.

To introduce the notion of *Ricci curvature*, we need to define *Levi-Civita connection* and *Riemann curvature tensor* first. Since all numerical computations are done in local charts, we will, for the most part of this paper, state our definitions in terms of local coordinates.

Definition 2.7. (Directional derivatives of a vector field v along u). Let u, v be two vector fields on a smooth manifold M. In local coordinates $(U, \phi = (x_1, ..., x_n))$, write $v = v_1(x)\frac{\partial}{\partial x_1} + \ldots + v_n(x)\frac{\partial}{\partial x_n}$, $u = u_1(x)\frac{\partial}{\partial x_1} + \ldots u_n(x)\frac{\partial}{\partial x_n}$, where $v_1, \ldots, v_n, u_1, \ldots, u_n \in C^\infty(U)$.
We define

$$u(v) = \sum_{j=1}^{n} \Big(u_1(x)\frac{\partial v_j}{\partial x_1} + \ldots + u_n(x)\frac{\partial v_j}{\partial x_n} \Big) \frac{\partial}{\partial x_j} \tag{4}$$

Equivalently, for every point $x \in M$, we have $u(v)(x) = \text{Jac}_v(x) \cdot u(x)$, where $\cdot$ denotes the matrix-vector multiplication.

Remark 2.8. Note that the above definition depends on the choice of local coordinates ϕ.

Definition 2.9. (Lie derivative of vector fields). Let u, v be two vector fields on a smooth manifold M, the Lie derivative takes the form $\mathcal{L}_u(v) = [u,v] = u(v) - v(u)$.

Definition 2.10. (Affine connection). Let $\pi : E \to M$ be a vector bundle, a connection on E is a map $\nabla : \mathfrak{X}(M) \times C^\infty(X,E) \longrightarrow C^\infty(X,E)$, where $\nabla(V,T)$ is denoted $\nabla_V T$, satisfies the following:

(1) $C^\infty(M)$-linearity over the 'vector field' entry:

$$\nabla_{fV+gW}(T) = f\nabla_V T + g\nabla_W T$$

(2) $\mathbb{C}$-linearity over the 'section' entry:

$$\nabla_V(aS + bT) = a\nabla_V S + b\nabla_V T$$

(3) Leibniz rule:

$$\nabla_V(fT) = V(f)\cdot T + f\cdot\nabla_V T \tag{5}$$

An affine connection is a connection on the tangent bundle TM.

Definition 2.11. (Covariant derivative). Let ∇ be an affine connection on TM, $u, v \in \mathfrak{X}(M)$, $\nabla_u v = \nabla(v)(u)$ is called the covariant derivative of v along direction u.

Definition 2.12. (Christoffel symbols). The Christoffel symbols are smooth functions $\Gamma^k_{ij} : U \to R$, for $i, j, k \in \{1, \ldots, n\}$, defined by:

$$\nabla_{e_i} e_j = \Gamma^k_{ij} e_k \tag{6}$$

where Einstein summation notation is used.

Proposition 2.13. *(Local formula for covariant derivative). Let $\{e_i\}_{i=1}^n$ be a local coordinate frame on M with affine connection ∇, u, v be vector fields on M. Then locally, we can express*

$$\nabla_u v = \big(u(v^k) + u^i v^j \Gamma^k_{ij}\big)e_k \tag{7}$$

where $u = u^i e_i, v = v^i e_i$ and Einstein summation notation is used.

Remark 2.14. Since every connection ∇ is of the form $\nabla = d + A$ in a local trivialisation $(\pi^{-1}(U), \Psi)$, for some matrix-valued 1-form A. Here take $A = (A^k_j)_{k,j}$, where each entry A^k_j is given by $A^k_j = \Gamma^k_{ij} dx^i$.

Since the above local formula for covariant derivative involves Christoffel symbols, it would be helpful if we also write down the local formula for Christoffel symbols.

Proposition 2.15. *(Local formula for Christoffel symbol). Let (U, ϕ) be a chart on a Riemannian manifold (M, g) with the Levi-Civita connection ∇ (2.17). In local coordinates $\{x^i\}$, we have the following expression for the Christoffel symbol Γ^k_{ij}, with respect to ∇ in the local coordinate frame $\{\frac{\partial}{\partial x_i}\}$, in terms of the components of g:*

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\Big(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{ij}}{\partial x_l}\Big) \tag{8}$$

where $(g_{ij})(x)$ denotes the matrix corresponding to the metric g at point x, and (g^{kl}) denotes the inverse matrix of (g_{ij}).

Remark 2.16. (An equivalent definition of a connection). Alternatively, given a vector bundle $\pi : E \to X$, we can define

$$\nabla : C^\infty(X, E) \longrightarrow C^\infty(X, \Omega^1_X \otimes E)$$

such that:

(1) ∇ is $\mathbb{C}$-linear:

$$\nabla(\alpha \cdot s + \beta \cdot t) = \alpha \nabla(s) + \beta \nabla(t)$$

(2) ∇ satisfies the Leibniz rule:

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s) \tag{9}$$

The two definitions are equivalent since we define $V(f) := df(V)$, for vector field V and smooth function f. In local coordinates $(x_1, ..., x_n)$, we have $V = v_1 \frac{\partial}{\partial x_1} + \ldots v_n \frac{\partial}{\partial x_n}$, for some smooth functions $v_1, ..., v_n$, and $df(V)(x) = \sum_{i=1}^{n} v_i(x) \dfrac{\partial f}{\partial x_i}(x), \forall x \in M$. Then by evaluating the TM-valued 1-form from equation (9) at a given vector field V, we recover the Leibniz rule from equation (5):

$$\begin{aligned}
\nabla(f \cdot T)(V) &= (df \otimes T)(V) + f \cdot \nabla(T)(V) \\
&= df(V) \otimes T + f \cdot \nabla_V T \\
&= V(f) \otimes T + f \cdot \nabla_V T
\end{aligned}$$

Theorem 2.17. *(Fundamental Theorem of Riemannian Geometry). Let (M, g) be a Riemannian manifold. There exists a unique affine connection ∇ on M which is torsion free, and compatible with the metric g, called the Levi-Civita connection.*
Here, torsion free means $\nabla_u v - \nabla_v u = [u, v]$, $\forall u, v \in \mathfrak{X}(M)$, and compatible with the metric g means $\nabla g = 0$, where $\nabla g : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M)$ is $C^\infty(M)$-multilinear, given by:

$$(\nabla g)(X, Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \tag{10}$$

Definition 2.18. (C^∞_{loc} convergence). [Sergeev, 2014, Example 4, p. 4] Let (M, g) be a smooth Riemannian manifold, ∇ be the Levi-Civita connection on M, $K_1 \subset K_2 \subset \ldots \subset M$ be an exhaustion of M by compact subset, $C^k(K_i)$ be the space of k-times differentiable functions on each K_i. We equip $C^k(K_i)$ with the following seminorm:

$$\|f\|_{C^k(K_i)} := \sum_{j=0}^{k} \sup_{x \in K_i} \left|\nabla^j f(x)\right|_g \quad , \quad \forall f \in C^k(K_i) \tag{11}$$

where $(\nabla^j f)_g$ denotes the j-th covariant derivative (with respect to the Levi-Civita connection). We say a sequence of functions $\{f_n\} \subset C^\infty(M)$ smoothly converges to some $f \in C^\infty(M)$ on compact subsets, denoted $\{f_n\} \xrightarrow{C^\infty_{\text{loc}}} f$, if:

$$\|f_n - f\|_{C^k(K_i)} \xrightarrow{n \to \infty} 0, \quad \forall\, i, k \in \mathbb{N}. \tag{12}$$

Definition 2.19. (Convergence of differential forms). Let (M, g) be a smooth Riemannian manifold of dimension n, ∇ be the Levi-Civita connection on M, $\{\omega_k\}, \omega$ be top-degree smooth differential forms on M. In local coordinates $\big(U, \phi := (x_1, \ldots, x_n)\big)$, we may express:

$$\omega_k = f_k \cdot dx_1 \wedge \ldots \wedge dx_n, \ \forall k \quad ; \quad \omega = f \cdot dx_1 \wedge \ldots \wedge dx_n \tag{13}$$

for some coefficient functions $f_k, f \in C^\infty(M)$.

We say $\{\omega_k\}$ converge to ω as differential forms if the coefficient functions $\{f_k\}$ converge to f. In particular, we say $\omega_k \xrightarrow{C^\infty_{\text{loc}}} \omega$ if:

$$f_k \xrightarrow{C^\infty_{\text{loc}}} f \tag{14}$$

Remark 2.20. Note that the above definition is independent of the choice of local coordinates. Let $\big(V, \psi := (y_1, \ldots, y_n)\big)$ be another chart. On the overlap $U \cap V$, the coefficient function in y-coordinate is given by:

$$\tilde{f}_k = \det\big(\text{Jac}(\psi \circ \phi^{-1})\big) \cdot f_k \tag{15}$$

Denote the transition map $F := \psi \circ \phi^{-1}$ and its Jacobian by DF. Since F is a diffeomorphism, by chain rule [Lee, 1997, Prop. C.4], the total derivative DF is invertible, with inverse $D(F^{-1})$, satisfying

$$(DF_p)^{-1} = D(F^{-1})_{F(p)} \quad , \forall p \in M \tag{16}$$

In particular, since DF is invertible, we have $\det(DF) \neq 0$. On the other hand, since determinant is a continuous map, on every compact subset of M, it attains a maximum, hence $0 < |\det(\text{Jac})| < M$, for some constant $M < \infty$. The above argument also holds for derivative of higher orders, by taking partial derivatives of equation (11). Hence $\tilde{f}_k \xrightarrow{C^\infty_{\text{loc}}} \tilde{f} \iff f_k \xrightarrow{C^\infty_{\text{loc}}} f$.

Definition 2.21. (Riemann curvature tensor). Let (M, g) be a Riemannian manifold, with the Levi-Civita connection ∇. The Riemann curvature tensor of (M, g) is a map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{17}$$

for vector fields X, Y, Z on M.

Proposition 2.22. *(Local formula for Riemann curvature tensor). In local coordinates, we have*

$$R = R^l_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l} \tag{18}$$

where $R^l_{ijk} = \partial_{x^i} \Gamma^l_{jk} - \partial_{x^j} \Gamma^l_{jk} + \Gamma^m_{jk} \Gamma^l_{im} - \Gamma^m_{ik} \Gamma^l_{jm}$.

Definition 2.23. (Ricci curvature). Let (M, g) be a Riemannian manifold, with Riemann curvature tensor R. The Ricci curvature is a (0, 2)-tensor, $Ric : TM \times TM \to \mathbb{R}$, pointiwsely defined by

$Ric_p : T_pM \times T_pM \to \mathbb{R}$,

$$Ric_p(X(p), Y(p)) := tr(R(\cdot, X(p))Y(p)) \tag{19}$$

where $X, Y \in \mathfrak{X}(M)$. Sometimes we write $Ric(g)(X, Y)$ to emphasise its dependence on the metric g.

Proposition 2.24. *(Local formula for Ricci curvature tensor). Pick $\{e_i\}$ orthonormal frame (with respect to g), we have*

$$Ric(X, Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i) \tag{20}$$

2.3 Calabi-Yau Manifolds

The above discussion naturally motivates the notion of *Calabi-Yau manifolds*, which has many equivalent definitions. In this subsection, we will formally define the notion of *Calabi-Yau metrics*, *Calabi-Yau manifolds*, *Calabi-Yau volume* and state *Yau's theorem*. The material in this subsection is primarily adapted from [Huybrechts, 2005, Chap. 4B, Chap. 6],

Definition 2.25. (Calabi-Yau metric). Let (M^{2n}, J) be a Kähler manifold (of complex dimension n), J be the almost complex structure induced by the complex structure, g be the Kähler metric (obtained by equation (3)). We say g is a Calabi-Yau metric if g is Ricci-flat:

$$Ric(g)(X, Y) = 0 \tag{21}$$

for all vector fields X, Y on M.

Definition 2.26. (Calabi-Yau manifold). Let (M^{2n}, g, J, ω) be a compact Kähler manifold (of complex dimension n). We say M is a Calabi-Yau manifold if it has vanishing first Chern class, $c_1(M) = 0 \in H^2(M, \mathbb{R})$.

Remark 2.27. (Convention). A Calabi-Yau manifold is often additionally required to be simply connected, but in this thesis, we do not make this assumption.

We will see in Theorem 2.33 [Yau, 1978] that a Calabi-Yau manifold admits a Calabi-Yau metric.

Definition 2.28. (Volume form). Let (M, g) be an oriented Riemannian manifold of dimension n. The volume form is the unique top-form dV_g such that

$$dV_g|_p(E_1, \ldots, E_n) = 1$$

whenever $\{E_1, \ldots, E_n\}$ is an oriented orthonormal basis of T_pM.

Proposition 2.29. *(Local formula for volume form). In local coordinates $\{x^i\}$, we have $dV_g = \sqrt{\det(g_{ij})}dx^1 \wedge \ldots \wedge dx^n$, and define the volume of M as $Vol(M) := \int_M dV_g$.*

Remark 2.30. Let M be a Calabi-Yau manifold, Yau's theorem (2.33) guarantees the existence of a Ricci-flat Kähler metric g_{CY} on M, but does not provide an explicit formula for the local components

$g_{CY,ij}$. By definition, M has vanishing first Chern class. As a consequence, it admits a nowhere vanishing holomorphic top-form $\Omega \in C^\infty(M, \Omega_M^{n,0})$ (hence a Riemannian volume form $(i)^n \Omega \wedge \overline{\Omega}$). We will firstly define the Calabi-Yau volume purely in terms of the holomorphic volume form induced by Ω.

Definition 2.31. (Holomorphic volume form). Let X be a Calabi-Yau manifold of complex dimension n, Ω be a holomorphic, nowhere vanishing $(n, 0)$-form on X, the holomorphic volume form dV_{hol} is given by:

$$dV_{hol} = \left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega} \tag{22}$$

We now state Yau's theorem and a corollary [Huybrechts, 2005, Thm. 4.B.19 & Thm. 4.B.22] to characterise Calabi-Yau manifolds.

Theorem 2.32. *(Calabi-Yau). [Huybrechts, 2005, Thm. 4.B.19] Let (X, g_0) be a compact Kähler manifold of complex dimension n, with Kähler form ω_0. For any real-differentiable function f on X with*

$$\int_X e^f \cdot \omega_0^n = \int_X \omega_0^n \tag{23}$$

There exists a unique Kähler metric g with associated Kähler form ω such that

$$[\omega] = [\omega_0] \quad \textit{and} \quad \omega^n = e^f \cdot \omega_0^n. \tag{24}$$

Corollary 2.33. *[Huybrechts, 2005, Cor. 4.B.22] Let (M, ω) be a compact Kähler manifold. The following are equivalent:*

1. *M has vanishing first Chern class, $c_1(M) = 0 \in H^2(M, \mathbb{R})$.*
2. *M has trivial canonical bundle (there exists a nowhere vanishing holomorphic top form Ω).*
3. *For every Kähler class $[\omega]$, there exists a unique Ricci-flat Kähler form $\omega_{CY} \in [\omega]$, with the associated Ricci-flat metric g_{CY} on M.*

Remark 2.34. (Holomorphic and Calabi-Yau volume form). [Huybrechts, 2005, Cor. 4.B.23] Note that the holomorphic volume form defined in definition 2.31 is independent (up to a multiplicative constant) of the choice of Ricci-flat metric or the Kähler class $[\omega_{CY}]$. Moreover, for a fixed nowhere-vanishing holomorphic volume form Ω on M, for every Kähler form ω on M, with the unique Ricci-flat Kähler form ω_{CY} (in the class $[\omega]$) provided by Yau's theorem, we have:

$$\omega_{CY}^n = A_{[\omega]} \cdot \Omega \wedge \overline{\Omega} \tag{25}$$

where the constant $A_{[\omega]}$ depends on the class $[\omega]$. In particular, since the integral of a top-degree form over a compact oriented manifold only depends on the de Rham cohomology class [Huybrechts, 2005, p. 284], we have:

$$\int_M \omega_{CY}^n = \int_M \omega^n \tag{26}$$

We therefore define $A_{[\omega]}$ to be as follows:

$$A_{[\omega]} = \frac{\displaystyle\int_M \omega^n}{\displaystyle\int_M \Omega \wedge \overline{\Omega}} \tag{27}$$

Consequently, after the normalization $\Omega \mapsto (A_{[\omega]})^{-1/2}\,\Omega$, we shall use the names 'holomorphic volume form' and 'Calabi-Yau volume form' interchangeably on Calabi-Yau manifolds.

2.4 Gromov-Hausdorff Convergence

To analyse the limiting behaviour of Calabi–Yau metric in Section 2.7, we must first make precise what it means for a family of Riemannian manifolds to converge. An appropriate notion of convergence in this context is the Gromov-Hausdorff convergence [Shioya, 2010, Def. 3.8]. In this subsection, we define the notion of *Hausdorff distance*, *Gromov-Hausdorff distance* and *Gromov-Hausdorff convergence*.

Definition 2.35. (Hausdorff distance). Let (X, d) be a metric space, $A, B \subset X$ be non-empty, closed subsets. We define the Hausdorff distance between A and B,

$$\mathrm{dist}_{\mathrm{H}}(A, B) := \max\left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\} \tag{28}$$

Definition 2.36. (Gromov-Hausdorff distance). Given two closed metric spaces A, B, we define the Gromov-Hausdorff distance between A and B as:

$$\mathrm{dist}_{\mathrm{GH}}(A, B) := \inf_{f,g} \left\{ \mathrm{dist}_{\mathrm{H}}\big(f(A), g(B)\big) \right\} \tag{29}$$

where $f : A \to X$ and $g : B \to X$ are isometric embeddings into some ambient metric space X.

Remark 2.37. The Hausdorff distance measures the largest distance that one must travel from a 'most-distant' point of a set to reach the other set. Informally, the Gromov-Hausdorff distance measures how isometric two metric spaces are. Concretely [Shioya, 2010, Lemma. 3.11], if $\mathrm{dist}_{\mathrm{GH}}(A, B) < \epsilon$, then there exists an almost isometry (ϵ-isometry) $f : A \to B$ whose distance-distortion satisfies:

$$\left| d_A(x, x') - d_B\big(f(x), f(x')\big) \right| < 2\epsilon \tag{30}$$

for every $x, x' \in A$.

Definition 2.38. (Gromov-Hausdorff convergence). Let $\{(X_n, d_n)\}_{n \geq 1}$ be a sequence of compact metric spaces. We say (X_n, d_n) converges to some metric space (Y, d) in the Gromov-Hausdorff sense, denoted $(X_n, d_n) \xrightarrow[n\to\infty]{GH} (Y, d)$, if $\mathrm{dist}_{\mathrm{GH}}(X_n, Y) \xrightarrow{n\to\infty} 0$.

2.5 Measured Gromov-Hausdorff Convergence

While the Gromov-Hausdorff convergence captures the collapsing of distances in a sequence of compact metric spaces, it conveys no information about the underlying space as a measure space (such as mass distribution, probability measures, volume and many more). In section 2.7, we will be analysing sequences of Calabi-Yau manifolds, viewed as compact metric measure spaces, hence the limiting behaviour of measures is equally important. In this subsection, we may remedy this by considering the measured Gromov-Hausdorff convergence [Shioya, 2010, Def. 4.3].

Definition 2.39. (Prokhorov distance). [Shioya, 2010, Def. 9.7] Let (X, d_X) be a metric-measure space, ν, ν' be two Borel probability measures on X. We define their Prokhorov distance:

$$\mathrm{dist}_P(\nu, \nu') := \inf\big\{\epsilon > 0 \text{ such that } \nu(A) \leq \nu'(A^\epsilon) + \epsilon, \forall \text{ Borel } A \subset X\big\} \tag{31}$$

where $A^\epsilon := \{x \in X : d_X(x, A) < \epsilon\}$.

Definition 2.40. (Measured Gromov-Hausdorff distance). Let (X, d_X, μ_X) and (Y, d_Y, μ_Y) be compact metric-measure spaces. We define their measured Gromov-Hausdorff distance:

$$\mathrm{dist}_{\mathrm{mGH}}\Big((X, d_X, \mu_X), (Y, d_Y, \mu_Y)\Big) := \inf_{f,g}\Big\{\max\{\mathrm{dist}_{\mathrm{H}}\big(f(X), g(Y)\big), \mathrm{dist}_P(f_*\mu_X, g_*\mu_Y)\}\Big\} \tag{32}$$

where $f : X \to Z$ and $g : Y \to Z$ are isometric embeddings into some ambient metric space Z, and $f_*\mu_X, g_*\mu_Y$ are the (respective) pushforward measures on Z.

Definition 2.41. (Measured Gromov-Hausdorff Convergence). We say a sequence of compact metric-measure spaces $\{(X_n, d_n, \mu_n)\}_{n\geq 1}$ converges to (X, d, μ) in the measured Gromov-Hausdorff sense, denoted $(X_n, d_n, \mu_n) \xrightarrow{mGH} (X, d, \mu)$, if:

$$\mathrm{dist}_{\mathrm{mGH}}\Big((X_n, d_n, \mu_n), (X, d, \mu)\Big) \xrightarrow{n\to\infty} 0 \tag{33}$$

The following theorem allows us to characterise measured Gromov-Hausdorff convergence in terms of almost isometries and Prokhorov convergence [Sturm, 2006].

Theorem 2.42. *[Sturm, 2006, p. 98] Let $(X, d, \mu), (X_n, d_n, \mu_n)$ be compact metric-measure spaces, for every n. The following are equivalent:*

1. $(X_n, d_n, \mu_n) \xrightarrow{mGH} (X, d, \mu)$
2. *There exists a sequence of real numbers $\epsilon_n \to 0$, Borel maps $f_n : X_n \to X$ which are ϵ_n-isometries, and the pushforward measure $(f_n)_*\mu_n$ converges to μ in the Prokhorov distance.*

Remark 2.43. (Motivation). Given measured Gromov-Hausdorff convergence $(X_n, d_n, \mu_n) \xrightarrow{mGH} (X, d, \mu)$, many geometric quantities on each X_n may be naturally passed onto the limiting space X under additional hypotheses. For instance, we will discuss the convergence of Calabi-Yau metrics in greater detail in Remark 2.67.

2.6 Singularities of Varieties

Viewing quartic K3 surfaces in $\mathbb{CP}^3$ as Calabi-Yau manifolds, varieties with singularities commonly arise as the Gromov-Hausdorff limit of a degeneration family of K3 surfaces. In this subsection, we define the notion of *singularity*, *ordinary double points* and provide some basic examples and non-examples of varieties with singularities. The material in this subsection is primarily adapted from [Fulton, 1969, Chap. 3].

Definition 2.44. (Affine variety/zero locus). Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial. The set

$$Z(f) := \{(x_1, \dots, x_n) \in \mathbb{C}^n : f(x_1, \dots, x_n) = 0\}$$

is called the zero locus of f. It is also called an affine variety.

Definition 2.45. (Singularity). We say a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is singular at a point $p := (x_1, \dots, x_n) \in Z(f)$ if $f(p) = 0$ and $\mathrm{grad}(f(p)) = 0$.

Definition 2.46. (Smooth affine variety). We say an affine variety is smooth if $Z(f)$ has no singularities.

Theorem 2.47. *[Liu, 2002, Thm. 2.19] Let $f \in \mathbb{C}[x_1, \dots, x_n]$. The following are equivalent:*
(1) $Z(f)$ admits a smooth atlas.
(2) $Z(f)$ has no singularity.

Definition 2.48. (Projective variety). Let $f \in \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous polynomial. The set $Z(f) := \{(x_0, \dots, x_n) \in \mathbb{CP}^n : f(x_0, \dots, x_n) = 0$

Definition 2.49. (Nodes/ordinary double points (for affine varieties)). Let $f \in \mathbb{C}[x_1, \dots, x_n]$. We say $p \in Z(f)$ in $\mathbb{C}^n$ is an ordinary double point (or node) if $f(p) = 0 = \mathrm{grad} f(p)$ and the Hessian matrix $H_f(p)$ is invertible.

Definition 2.50. (Ordinary double points (for projective varieties)). Let f be a homogeneous polynomial in $\mathbb{C}[x_0, \dots, x_n]$. We say $p \in \mathbb{CP}^n$ is an ordinary double point if $f(p) = 0 = \mathrm{grad} f(p)$ and the Hessian matrix $H_f(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{0 \le i,j \le n}(p)$ has rank n.

Remark 2.51. The defining homogeneous polynomial (of degree d) corresponds to a holomorphic section of the line bundle $\mathcal{O}_{\mathbb{CP}^n}(d)$ [Huybrechts, 2005, Prop. 2.4.1], it is not a globally defined function on $\mathbb{CP}^n$. After restricting to an affine chart $U_i := \{x_i \neq 0\}$ and normalising, $\tilde{f}_i := \frac{f}{x_i^d}$ is indeed a well-defined holomorphic function with an $n \times n$ Hessian matrix $H_{\tilde{f}}$. We note that requiring rank $H_{\tilde{f}_i} = n$ on every patch U_i is equivalent to the invertibility condition in definition 2.50.

The invertibility condition requires requires $H_f(p)$ to have the highest possible rank, for every singular point $p \in \mathbb{CP}^n$. In fact, we will show in the following claim, that $H_f(p)$ has rank at most n.

Claim 2.52. $\mathrm{rank}\big(H_f(p)\big) < n+1$ *at every singular point* $p \in \mathbb{CP}^n$*.*

Proof. Let $p \in \mathbb{CP}^n$ be a singular point of f and $p' \in \mathbb{C}^{n+1} \setminus \{0\}$ such that $[p'] = p$. The above claim means that $\mathrm{rank}\big(H_f(p')\big) < n+1$. By Euler's Homogeneous Function Theorem [Apostol, 1975, p. 287], we have:

$$p' \cdot \mathrm{grad}(f)(p') = \deg(f) \cdot f(p') \quad , \quad \forall p' \in \mathbb{C}^{n+1} \tag{34}$$

where the first $\cdot$ denotes the dot product between vectors, and the second $\cdot$ denotes scalar multiplication. Differentiating yields:

$$\mathrm{grad}(f)(p') + H_f(p') \cdot p' = \deg(f) \cdot \mathrm{grad}(f)(p') \tag{35}$$

where the first $\cdot$ denotes the matrix-vector multiplication, and the second $\cdot$ denotes the scalar-vector multiplication. At the singular point p', we have $\mathrm{grad}(f)(p') = 0$. Then:

$$H_f(p') \cdot p' = 0 \tag{36}$$

Since $p' \in \mathbb{C}^{n+1} \setminus \{0\}$, it is non-zero. Concretely, $H_f(p')$ has a non-zero vector p' in its kernel. By Rank-Nullity Theorem [Axler, 2015, Thm. 3.21], we have $\mathrm{rank}\big(H_f(p')\big) < n+1$. $\square$

Example 2.53. The affine variety $\{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}$ is called *Fermat quartic*. It is smooth in $\mathbb{CP}^3$.

Proof. Denote the Fermat quartic by F. Assume towards contradiction that $Z(F)$ admits a singularity at some point $p := [p_0 : p_1 : p_2 : p_3] \in \mathbb{CP}^3$. Then by definition 2.45, $F(p) = 0 = \mathrm{grad}(F)(p)$. Here,

$$\mathrm{grad}(F) = \begin{pmatrix} 4x_0^3 \\ 4x_1^3 \\ 4x_2^3 \\ 4x_3^3 \end{pmatrix} \tag{37}$$

and $\mathrm{grad}(F)(p) = 0$ forces $p = [0 : 0 : 0 : 0]$. However the origin is not a point in $\mathbb{CP}^3$, which yields a contradiction. $\square$

Example 2.54. Consider $f := (x_1^4+x_2^4+x_3^4)+x_0\cdot(x_1^3+x_2^3+x_3^3)+x_0^2\cdot(x_1^2+x_2^2+x_3^2) \in \mathbb{C}[x_0, x_1, x_2, x_3]$. We claim that $Z(f) \subset \mathbb{CP}^3$ is smooth, expect for $P := [1 : 0 : 0 : 0]$.

Proof. Consider the affine patch $U_i := \{x : x_i = 1\}$ on $Z(f)$.

1. On U_1, $f(1, x_1, x_2, x_3) = (x_1^4+x_2^4+x_3^4)+(x_1^3+x_2^3+x_3^3)+(x_1^2+x_2^2+x_3^2)$. We compute $\mathrm{grad} f(P) = (0, 0, 0)$, and $H_f(P) = \mathrm{diag}(2, 2, 2)$ of rank 3, hence P is indeed a node.

2. For showing that P is the unique singularity of $Z(f)$, we can verify the smoothness of f on the affine patches U_1, U_2, U_3 via the magma code [Bosma et al., 1997].

□

2.7 Complex Structure Limits

Complex structure limits describe how a family of Calabi–Yau manifolds $\{X_\epsilon\}_{\epsilon>0}$ degenerates as the parameter $\epsilon \to 0$, and they are central to understanding the corresponding behaviour of Ricci-flat metrics. In mirror symmetry, large complex structure limits (LCSL) are of special importance, Calabi-Yau manifolds admits an approximate special Lagrangian torus fibration near the LCSL, and the degeneration family collapses to a certain lower-dimensional space called the *essential skeleton* in the Gromov-Hausdorff limit [Gross and Wilson, 2000; Li, 2022]. In this subsection, we introduce the notion of *complex structure limits* via the discussion of *polarised one-parameter degeneration*. We then give two equivalent formulations, which will be used to interpret the numerical phenomena in Section 4. We finally present a brief summary of some known convergence results of Calabi-Yau degeneration families.

2.7.1 Polarised Degeneration

In this subsection, we will firstly give a general definition for *polarised degenerations*, and explain the special case of Calabi-Yau hypersurfaces in $\mathbb{CP}^{n+1}$.

Definition 2.55. (Polarised one-parameter degeneration). [Li, 2023a, p. 1] Denote the unit disc by $\mathbb{D}$. Let $\mathcal{X}$ be a complex manifold of dimension $n+1$, and $\pi : \mathcal{X} \to \mathbb{D}$ be a proper holomorphic map, satisfying the following conditions:

1. Each fibre $X_\epsilon := \pi^{-1}(\epsilon)$ is a smooth Calabi-Yau n-fold, for all $\epsilon \neq 0$.

2. There is a relatively ample line bundle [Hartshorne, 1977, p. 153] [Grothendieck and Dieudonné, 1961, Def. 4.6.1] $L \to \mathcal{X}$.

3. The central fibre $X_0 := \pi^{-1}(0)$ is allowed to be singular (and will be studied as the degeneration).

We will focus on the following two examples of polarised degeneration throughout thesis thesis.

Definition 2.56. (Calabi-Yau hypersurface one-parameter degeneration family). [Li, 2023a, Eqn. 1] [Sun and Zhang, 2019, Eqn. 1.1] Let $X_\epsilon := Z(F_0 \dots F_m + \epsilon F) \subset \mathbb{CP}^{n+1}$ be a family of Calabi-Yau n-varieties, $1 \leq m \leq n-1$ be an integer, where $F := x_0^{n+2} + \ldots + x_{n+1}^{n+2}$ denotes the $(n+2)$-th Fermat polynomial. Denote the punctured disk by $\mathbb{D}^\times := \mathbb{D} \setminus \{0\}$. Here, we assume the following conditions holds:

1. For each $i = 0, \ldots, m$, the hypersurface $Y_i := Z(F_i)$ is smooth.

2. For any $k \in \{1, \ldots, m+1\}$, any subset $I \subset \{0, \ldots, m\}$ with size $|I| = k$, the complete intersection,

$$\bigcap_{i \in I} Z(F_i) = \{F_i = 0, \forall i \in I\} \tag{38}$$

is smooth, of codimension k. This condition is often called *transversal intersection*.

3. For all $\epsilon \in \mathbb{D}^{\times}$, the hypersurface X_ϵ is smooth.

Definition 2.57. (Dwork family). Let $X_0 := Z(f_0) \subset \mathbb{CP}^3$, where $f_0(x) := x_0^4 + x_1^4 + x_2^4 + x_3^4$ is the Fermat quartic. Define

$$f_\epsilon(x) := f_0(x) - \epsilon \cdot x_0 x_1 x_2 x_3 \tag{39}$$

and define the Dwork deformation family $\{X_\epsilon\}$, where $X_\epsilon := Z(f_\epsilon)$.

Remark 2.58. (Rank condition for transversality of the quartic K3 family). Consider a family of quartic K3 surfaces of the form $X_\epsilon := Z(F_0 F_1 + \epsilon F) \subset \mathbb{CP}^3$. Let $d_0 := \deg(F_0)$ and $d_1 := \deg(F_1)$. Assume each hypersurface $Y_i := Z(F_i)$ is smooth, for $i \in \{0, 1\}$. Then:

1. When $I \subset \{0, 1\}$ has size $|I| = 1$, the transversal intersection condition (38) is simply the smoothness condition on each hypersurface (with codimension 1 by definition), which holds by assumption.

2. When $|I| = 2$, transversality (38) is equivalent to checking that $Y_0 \bigcap Y_1$ is a smooth subvariety of complex codimension 2 in $\mathbb{CP}^3$ (hence of complex dimension 1). To invoke the Regular Level Set Theorem [Lee, 1997, Cor. 5.14], we argue locally on affine patches. Without loss of generality, fix the affine patch $U_0 := \{x_0 \neq 0\}$ with affine coordinates $z_j := \dfrac{x_j}{x_0}$, $\forall j \in \{1, 2, 3\}$. Let $\tilde{F}_0 := \dfrac{F_0}{x_0^{d_0}}$ and $\tilde{F}_1 := \dfrac{F_1}{x_0^{d_1}}$ be the restriction of F_0 and F_1 to the affine patch U_0, as discussed in Remark 2.51. We need to construct a holomorphic map $\Phi_0 : U_0 \to \mathbb{C}^2$ such that $\Phi_0^{-1}\big((0,0)\big) = U_0 \bigcap (Y_0 \bigcap Y_1)$, and $(0,0) \in \mathbb{C}^2$ is a regular value of Φ_0. In other words, $d\Phi_0(p)$ is surjective on $U_0 \bigcap (Y_0 \bigcap Y_1)$. Here, a natural candidate is:

$$\begin{aligned} \Phi_0 : U_0 &\longrightarrow \mathbb{C}^2 \\ p &\longmapsto \begin{pmatrix} \tilde{F}_0(p) \\ \tilde{F}_1(p) \end{pmatrix} \end{aligned} \tag{40}$$

Clearly $\Phi_0^{-1}\big((0,0)\big) = U_0 \bigcap (Y_0 \bigcap Y_1)$. Now the regular value condition requires:

$$\operatorname{rank}_{\mathbb{C}}\big(\operatorname{Jac}(\Phi_0)\big)(p) = 2 \tag{41}$$

for every $p \in U_0 \bigcap (Y_0 \bigcap Y_1)$, where:

$$\mathrm{Jac}(\Phi_0)(p) := \begin{pmatrix} \mathrm{grad}(\tilde{F}_0) \\ \mathrm{grad}(\tilde{F}_1) \end{pmatrix} (p) = \begin{pmatrix} \partial_{z_1}\tilde{F}_0 & \partial_{z_2}\tilde{F}_0 & \partial_{z_3}\tilde{F}_0 \\ \partial_{z_1}\tilde{F}_1 & \partial_{z_2}\tilde{F}_1 & \partial_{z_3}\tilde{F}_1 \end{pmatrix} (p) \tag{42}$$

and complex rank two means the two rows of $\mathrm{Jac}(\Phi_0)(p)$ are linearly independent over $\mathbb{C}$.

Claim 2.59. *The rank condition in equation (41) is independent of the choice of affine patches.*

Proof. Let U_i be another affine patch, on the overlap $(U_0 \bigcap U_i) \bigcap (Y_0 \bigcap Y_1)$, we have:

$$\tilde{F}_0^i = (x_0/x_i)^{d_0} \tilde{F}_0 \quad ; \quad \tilde{F}_1^i = (x_0/x_i)^{d_1} \tilde{F}_1 \tag{43}$$

where $\tilde{F}_0^i$ and $\tilde{F}_1^i$ denote the restriction of F_0 and F_1 to the affine patch U_i. For every $p \in Y_0 \bigcap Y_1$, by Leibniz rule, differentiating equation (45) yields:

$$d\tilde{F}_0^i = d\big((x_0/x_i)^{d_0}\big)\tilde{F}_0(p) + (x_0/x_i)^{d_0}\ d\tilde{F}_0 \quad ; \quad d\tilde{F}_1^i = d\big((x_0/x_i)^{d_1}\big)\tilde{F}_1(p) + (x_0/x_i)^{d_1}\ d\tilde{F}_1 \tag{44}$$

Since $\tilde{F}_0(p) = 0 = \tilde{F}_1(p),\ \forall p \in Y_0 \bigcap Y_1$, we have:

$$d\tilde{F}_0^i = (x_0/x_i)^{d_0}\ d\tilde{F}_0 \quad ; \quad d\tilde{F}_1^i = (x_0/x_i)^{d_1}\ d\tilde{F}_1 \tag{45}$$

Since $d\phi_0$ and $d\phi_i$ differ by an invertible Jacobian, the rank is preserved under change of affine coordinates $z^{(0)} \to z^{(i)}$. ☐

2.7.2 Algebraic formulation of complex structure limits

We now present a formulation for the degeneration family from equation (2.56), from the perspective of algebraic geometry [Li, 2023b] [Sun and Zhang, 2019].

Definition 2.60. (Algebraic formulation for complex structure limits). If $X_\epsilon := Z(F_0 \dots F_m + \epsilon F) \subset \mathbb{CP}^{n+1}$ satisfies all conditions in example 2.56, then X_0 is called a complex structure limit. In particular, we say:

1. X_0 is non-collapsing if $m = 0$. In this thesis, we allow the central fibre X_0 to have at worst ordinary double points (which is an example of *KLT singularities* [Kollár and Mori, 1998, Def. 2.34]).
2. X_0 is a small complex structure limit if $m = 1$.
3. X_0 is an intermediate complex structure if $2 \leq m \leq n - 1$.
4. X_0 is a large intermediate complex structure if $m = n$.

Remark 2.61. Note that when $n = 2$ (e.g. K3 surfaces), the deformation family only admits non-collapsing ($m = 0$), small ($m = 1$), and large ($m = 2$) complex structure limits.

Remark 2.62. (Terminology). [Sun, 2019, Sec. 4.2] For K3 surfaces, the term *small complex structure limit (SCSL)* is often used to refer to *Type II Kulikov degeneration*.

2.7.3 Analytic formulation of complex structure limits

We now give an explicit construction for a nowhere vanishing holomorphic volume form on Calabi-Yau hypersurfaces, and present an analytic formulation of complex structure limits.

Proposition 2.63. *(Nowhere vanishing holomorphic volume form on Calabi-Yau hypersurfaces). [Douglas et al., 2021, Eqn. 10] Let $X \subset \mathbb{CP}^n$ be a smooth projective variety (of codimension 1), defined by a homogeneous polynomial of degree $n+1$. Let $U_0, \dots, U_n$ be affine patches on $\mathbb{CP}^n$. Then:*

1. *Existence: X admits a nowhere vanishing holomorphic volume form Ω.*
2. *Local formula: On $U_i \cap \{\frac{\partial f}{\partial x_k} \neq 0\}$, with multi-index notation, $\Omega_{U_i} := \Omega|_{U_i}$ is of the form:*

$$\Omega_{U_i} = \frac{1}{\frac{\partial f}{\partial x_k}} dz_I \tag{46}$$

where $I := \{0, \dots, n\} \setminus \{i, k\}$ and $i \neq k$.

Remark 2.64. (Gluing and independence). Note that by smoothness of each X_ϵ, at least one of $\frac{\partial f_\epsilon}{\partial x_i}$ and $\frac{\partial f_\epsilon}{\partial x_j}$ is non-zero on any overlap $U_i \cap U_j$. By gluing over all patches $U_0, \dots, U_n$, $\Omega_{U_i,\epsilon}$ extends to holomorphic volume form on X_ϵ, independent of the choice of i and k.

By [Li, 2023a, Sec. 1 and Sec. 2.3], we have the following characterisation:

Theorem 2.65. *(Analytic Characterisation of the Complex Structure Limits). Let $X_0 := Z(f_0) \subset \mathbb{CP}^n$ be a Calabi-Yau manifold defined by a homogeneous polynomial f_0 of degree $n+1$, $\{X_\epsilon\}$ be any one-parameter deformation family, with volume $\mathrm{Vol}(X_\epsilon) := \int_{X_\epsilon} \Omega_\epsilon \wedge \overline{\Omega}_\epsilon$, where Ω_ϵ is the holomorphic volume obtained by Proposition 2.63. The following holds:*

1. *If X_0 is a non-collapsing complex structure limit, then $\mathrm{Vol}(X_\epsilon) \xrightarrow{\epsilon \to 0} c$, for some constant c.*
2. *If X_0 is a small complex structure limit, then $\mathrm{Vol}(X_\epsilon) \cdot |\log(\epsilon)|^{-1} \xrightarrow{\epsilon \to 0} c$, for some constant c.*
3. *If X_0 is an intermediate complex structure limit, then $\mathrm{Vol}(X_\epsilon) \cdot |\log(\epsilon)|^{-m} \xrightarrow{\epsilon \to 0} 0$, where $m = \dim(X_\epsilon)$, but $\mathrm{Vol}(X_\epsilon) \xrightarrow{\epsilon \to 0} \infty$.*
4. *If X_0 is a large complex structure limit, then $\mathrm{Vol}(X_\epsilon) \cdot |\log(\epsilon)|^{-m} \xrightarrow{\epsilon \to 0} c$, for some constant $c > 0$.*

Remark 2.66. The proof of Theorem 2.65 is rather complicated. Here, we are facing two main obstacles:

1. The exponent m depends on the choice of the holomorphic volume form on X_ϵ. Consider a one-parameter degeneration family $\{X_\epsilon\}$, with some nowhere vanishing volume form Ω_ϵ on each X_ϵ. Let m_0 be the small integer such that equation (2.65) holds. For instance, suppose we rescale Ω_ϵ by $|\log \epsilon|$, the minimal exponent for equation (2.65) to hold changes accordingly to $m_0 + 2$. The statement of the theorem only holds when each X_ϵ is equipped with the holomorphic volume form Ω_ϵ from proposition 2.63.

2. It is a non-trivial fact [Cattani et al., 1986, Equation 4.9] that the volume growth is at most $\mathcal{O}(|\log \epsilon|^m)$, where $m := \dim(X_\epsilon)$. Instead, in Section 3, we will restrict our attention to some well-studied degeneration families, for which the equivalence of definition 2.60 and definition 2.65 has been proven explicitly in the literature. We will verify the algebraic condition (2.60) by hand and provide numerical evidence for the predicted exponents in Section 4.3.

2.7.4 Known Calabi-Yau Degeneration Results

Equipped with the above algebraic and analytic formulation of complex structure limits, in this subsection, we will summarise some known results about Calabi-Yau degeneration families.

Remark 2.67. (Heuristics). The heuristics is as follows:
As $\epsilon \to 0$:

1. The quartic polynomial f_ϵ converges to f_0 coefficient-wise.

2. The natural complex structure $J_\epsilon \to J_0$, where J_ϵ denotes the restriction of $J_{\mathbb{CP}^3}$ to TX_ϵ.

3. Denote the Fubini-Study form on $\mathbb{CP}^3$ by ω_{FS}. Let $\omega_{CY,\epsilon} \in [\omega_{\mathrm{FS}}|_{X_\epsilon}]$ be the unique Ricci-flat Kähler form on each X_ϵ (exists by Yau's theorem 2.33). After rescaling (to $\mathrm{diam}(X_\epsilon) = 1$), the family of Calabi-Yau manifolds $\left\{(X_\epsilon, \mathrm{dist}_{CY,\epsilon})\right\}_{\epsilon>0}$ converges to $(X_0, \mathrm{dist}_{CY,0})$ in the Gromov-Hausdorff sense (2.38), where $\mathrm{dist}_{CY,\epsilon}$ denotes the Riemannian distance with respect to the Calabi-Yau metric.

4. Moreover, if the degeneration family is non-collapsing (2.65), X_0 is a manifold with singularities, and admits a 'singular' Ricci-flat Kähler metric $\omega_{CY,0}$, unique in its Kähler class [Eyssidieux et al., 2009, p.4, Cor. E]. It suffices for us to note that $\omega_{CY,0}$ is a well-defined Calabi-Yau metric on the smooth locus of X_0 (denoted X_0^{reg}). In particular, we have $\omega_{CY,\epsilon} \to \omega_{CY,0}$ in the sense of C^∞_{loc} (2.19) on the smooth locus of X_0 [Tosatti, 2015, Thm. 1.1]. More precisely, it means that there exists a fibre-preserving embedding [Tosatti, 2020, Thm. 4.2]:

$$F : X_0^{\mathrm{reg}} \times \Delta \longrightarrow \mathcal{X} \tag{47}$$

such that:

$$F|_{X_0^{\mathrm{reg}} \times \{0\}} = \mathrm{Id}. \quad ; \quad (F|_{X_0^{\mathrm{reg}} \times \{\epsilon\}})^* \omega_{CY,\epsilon} \xrightarrow{C^\infty_{\mathrm{loc}}(X_0^{\mathrm{reg}})} \omega_{CY,0}. \tag{48}$$

Here, fibre-preserving means that given the fibrations $\pi : \mathcal{X} \to \Delta$, $\text{proj}_2 : X_0^{\text{reg}} \times \Delta \to \Delta$, we have:

$$F \circ \text{proj}_2 = \pi \tag{49}$$

Remark 2.68. While it is intuitive to think that coefficient-wise convergence of $f_\epsilon \to f_0$ leads to the Gromov-Hausdorff convergence of $X_\epsilon \to X_0$ (after rescaling to $\text{diam}(X_\epsilon) = 1$), rigorously establishing it in fact requires deep results. Namely:

1. Gromov's Pre-compactness Theorem [Gromov, 1999, Prop. 5.2], which guarantees the existence of a convergent subsequence (with respect to the Gromov-Hausdorff distance) in a family of closed Ricci-flat manifolds with uniformly bounded diameter.

2. [Rong and Zhang, 2013, Thm. 1.1], which proves the uniqueness of subsequential Gromov-Hausdorff limit of the one-parameter degeneration family of a Calabi-Yau n-variety ($n \geq 2$), and identifies the whole family's limit (in the metric completion of the central fibre).

We also include a structural theorem of Gromov-Hausdorff limits.

Theorem 2.69. *[Sun and Zhang, 2021, Thm. 1.1] Let $\{X_\epsilon\}_{\epsilon>0}$ be a sequence of K3 surfaces in $\mathbb{CP}^3$, satisfying diam$(X_\epsilon) = 1$. If $X_\epsilon \xrightarrow{\epsilon \to 0} X_0$ in the Gromov-Hausdorff sense, and X_0 is a collapsed limit, then X_0 is isometric to one of the following:*

- *(dimension 3) A flat orbifold $\mathbb{T}^3/\mathbb{Z}^2$.*
- *(dimension 2) A singular special Kähler metric on S^2 with local integral monodromy.*
- *(dimension 1) A one-dimensional unit interval.*

2.8 Currents of Integration

In Section 4.1, we will establish a technical lemma, which shows that the Fubini–Study volume of the Calabi–Yau degeneration family remains constant for every $\epsilon > 0$. The proof rests on an application of *convergence of currents*, therefore in this subsection, we pause for a brief review of *integration currents* and their basic properties. The material in this subsection is primarily adapted from [de Rham, 1984, Chapter 3] and [Griffiths and Harris, 1994, Chapter 3.1].

Definition 2.70. (Currents). Let M be a smooth manifold of dimension n and $\Omega_c^p(M)$ be the vector space of smooth, compactly-supported p-forms on M, where $p \in \{0, \dots, n\}$. A p-current is a continuous linear functional T on $\Omega_c^p(M)$. Here, continuity of T is understood in the sense of distributions: for any sequence of smooth forms (of degree p), $\{\omega_k\}_{k=1}^{\infty}$, all supported on the same compact set on M, if all derivatives of all their coefficients converge uniformly to 0 as $n \to \infty$, then $T(\omega_k) \xrightarrow{k\to\infty} 0$. We denote the space of p-currents on M as $\mathcal{D}^p(M)$.

A smooth, oriented submanifold naturally gives rise to a current via integration:

Definition 2.71. (Currents of integration). Let $X \subset M$ be an oriented p-dimensional submanifold, $\iota : X \hookrightarrow M$ be the natural inclusion, the p-current $[X]$ defined as follows is called the current of integration (associated with X):

$$[X](\omega) := \int_X \iota^* \omega, \quad \forall \omega \in \Omega_c^p(M) \tag{50}$$

Remark 2.72. (Indexing convention for currents in the complex setting). [Griffiths and Harris, 1994, p. 386] In the real setting, a p-current acts on p-forms. On a complex manifold M of complex dimension n, we use indices according to the Dolbeault decomposition: a current of type (p, q) acts on test forms of bidegree $(n-p,\ n-q)$. Thus an analytic variety $Z \subset M$ of complex codimension p defines a (p, p)-current, its action on a holomorphic $(n-p, n-p)$-form α on M is $[Z](\alpha) := \int_Z \alpha|_Z$.

Definition 2.73. (Convergence of currents). As continuous linear functionals, a natural notion of convergence on the space of p-currents is *weak-$*$ convergence*. We say a sequence $\{T_k\}_{k=1}^{\infty} \subset \mathcal{D}^p(M)$ converges to $T \in \mathcal{D}^p(M)$ in the sense of currents if:

$$T_k(\omega) \longrightarrow T(\omega), \quad \forall \omega \in \Omega_c^p(M). \tag{51}$$

We now state a corollary of the Poincaré-Lelong Equation [Griffiths and Harris, 1994, p. 388], which provides an explicit formula for the integration currents associated with a projective hypersurface $Z(f) \subset \mathbb{CP}^{n+1}$.

Corollary 2.74. *(Poincaré-Lelong Equation). Let f be a homogeneous polynomial of degree $n+2$ in $\mathbb{CP}^{n+1}$, and $Z \subset \mathbb{CP}^{n+1}$ be the corresponding hypersurface of complex dimension n. Then the integration current (associated to Z) is of the form:*

$$[Z] = \frac{i}{\pi} \partial \bar{\partial} \log |f| \tag{52}$$

Here, we interpret $\partial\bar{\partial} \log |f|$ as the second distributional derivative of $\log |f|$, and acts on (n, n)-forms on $\mathbb{CP}^{n+1}$ by:

$$\Big(\partial\bar{\partial} \log |f|\Big)(\alpha) := \int_Z \log |f| \cdot \partial\bar{\partial}\alpha, \quad \forall \alpha \in \Omega^{n,n}(\mathbb{CP}^{n+1}). \tag{53}$$

hence equation (52) becomes:

$$\frac{i}{\pi} \int_M \log |f| \cdot \partial\bar{\partial}\alpha = \int_Z \alpha|_Z, \quad \forall \alpha \in \Omega^{n,n}(\mathbb{CP}^{n+1}). \tag{54}$$

3 Example of Calabi-Yau Degeneration Families

In this section, we will provide some examples of non-collapsing, small and large complex structure limits, with a particular focus on on the degeneration family from equation (2.56) and Dwork (2.57) deformation of Calabi-Yau 2-folds (K3 surfaces) defined by homogeneous quartic polynomials in $\mathbb{CP}^3$. The classification of these examples have been treated with detail in literature [Sun, 2019; Li, 2022]. In Section 4, we will numerically verify these predicted classifications, by computing the holomorphic volumes ((2.60) and (2.65)).

The following is a well-known fact from algebraic geometry:

Theorem 3.1. *(Calabi-Yau Hypersurface Criterion). [Huybrechts, 2005, Cor.2.4.9] Let $X := Z(f) \subset \mathbb{CP}^n$ be a smooth projective variety, defined by a homogeneous polynomial f of degree d. Then X is a Calabi-Yau manifold exactly when $d = n + 1$.*

3.1 Non-Collapsing Complex Structure Limits

Example 3.2. We claim that the Dwork family of the Fermat quartic, $X_\epsilon := Z(x_0^4 + x_1^4 + x_2^4 + x_3^4 + \epsilon \cdot x_0x_1x_2x_3)$ is a non-collapsing complex structure degeneration, as $\epsilon \to 0$.

Proof. We have shown in example 2.53 that the central fibre X_0 is smooth. In the notation from definition 2.60, this gives $m = 0$, hence $\{X_\epsilon\}_{\epsilon\to 0}$ is a non-collapsing complex structure degeneration. □

Example 3.3. Let $Z_\epsilon := Z(\epsilon \cdot F + f)$, where F denotes the Fermat quartic, and $f := (x_1^4 + x_2^4 + x_3^4) + x_0 \cdot (x_1^3 + x_2^3 + x_3^3) + x_0^2 \cdot (x_1^2 + x_2^2 + x_3^2) \in \mathbb{C}[x_0, x_1, x_2, x_3]$. We claim that $\{Z_\epsilon\}_{\epsilon\to 0}$ is a non-collapsing complex structure degeneration.

Proof. We have shown in example 2.54 that the central fibre $Z(f) \subset \mathbb{CP}^3$ has a single node at $p := [1 : 0 : 0 : 0]$, satisfying the condition of having singularities no worse then nodes. In the notation from definition 2.60, this gives $m = 0$, hence $\{Z_\epsilon\}_{\epsilon\to 0}$ is a non-collapsing complex structure degeneration. □

3.2 Small Complex Structure Limits

As shown in [Sun, 2019, Sec. 4.2], the degeneration family in equation 2.56, with $m = 2$, is an example of small complex structure limit of Calabi-Yau hypersurfaces. Here, we consider an explicit example:

Example 3.4. We claim that the degeneration family $W_\epsilon := Z(\{\epsilon \cdot F + F_0F_1\})$ has a small complex structure limit, as $\epsilon \to 0$. Here, F denotes the Fermat quartic, $F_0 := x_0$, $F_1 := x_0^3 + x_1^3 + x_2^3 + x_3^3$.

Proof. We will use the rank condition (38), with $\Phi := (F_0, F_1)$, on affine patch U_1, with affine coordinates $z_0 := \frac{x_0}{x_1}, z_2 := \frac{x_2}{x_1}, z_3 := \frac{x_3}{x_1}$. The restriction of F_0 and F_1 on U_0 are given by $\tilde{F}_0 := \dfrac{x_0}{x_1} = z_0$ and

$\tilde{F}_1 := \frac{F_1}{x_1^3} = z_0^3 + 1 + z_2^3 + z_3^3$. Assume towards contradiction that

$$\mathrm{Jac}(\Phi_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3z_2^2 & 3z_3^2 \end{pmatrix} \tag{55}$$

has rank less than 2 at some $p \in Y_0 \bigcap Y_1$. This means at point p, row 1 and row 2 of the Jacobian are linearly dependent, so there exists some scalar λ with $(0, 3z_2^2, 3z_3^2) = \lambda(1, 0, 0)$, which yields a contradiction. Hence $\mathrm{Jac}(\Phi)$ has full rank everywhere on $Y_0 \bigcap Y_1$, and the intersection is transverse. Since $m = 1$, W_0 is indeed a small complex structure limit. □

3.3 Large Complex Structure Limits

Example 3.5. [Li, 2022, p. 2] The degeneration family $V_\epsilon := Z(\{\epsilon \cdot (x_0^4 + x_1^4 + x_2^4 + x_3^4) + x_0x_1x_2x_3\})$ has a large complex structure limit, as $\epsilon \to 0$.

4 Numerics

In this section, we revisit the examples of Calabi-Yau degeneration families that we discussed in Section 3. By computing the holomorphic volume, we can numerically verify the classifications of their corresponding complex structure limits, using the criterion from definition (2.60). Computation of the holomorphic volume relies on numerically integrating the holomorphic volume form. To make the numerical methodology rigorous, in Section 4.1, we start by recalling the general theory of numerical integration, especially *Monte-Carlo integration*, and underpin some technicalities (e.g. the constancy of the Fubini-Study volume of the degeneration family). In Section 4.2, we then discuss how the Calabi-Yau metric $g_{CY,\epsilon}$ on each X_ϵ is numerically approximated. After that, in Section 4.3, we present our numerical results of holomorphic volumes (using the procedure discussed in Section 4.1) for these degeneration families and comment on the numerical stability, followed by the machine learning approximation of the Calabi-Yau volumes and Calabi-Yau metrics in Section 4.4. Finally, in Section 4.5, we propose two conjectures, supported by numerical evidence, regarding the dimension-collapsing of the Calabi-Yau degeneration family and eigenvalues of Calabi-Yau metric tensors.

4.1 Numerical Integration

We begin by recalling the Monte-Carlo integration on a finite measure space:

Proposition 4.1. *(Monte-Carlo integration on a finite measure space). [Robert and Casella, 2004, Chap. 3.2] Let (X, μ) be a finite measure space, $f : X \to \mathbb{R}$ be a μ-integrable function. Let points $p_1, \dots, p_N$ be sampled i.i.d. with respect to the induced probability measure $\nu := \frac{\mu}{\mu(X)}$, then:*

$$\left(\frac{1}{N} \cdot \sum_{i=1}^{N} f(p_i) \right) \cdot \mu(X) \xrightarrow[N\to\infty]{a.s.} \int_X f d\mu \tag{56}$$

Definition 4.2. (Fubini-Study measure). In our case, on each X_ϵ, the restriction of the Fubini-Study form (denoted $\omega_{\mathrm{FS}}|_{X_\epsilon}$) defines the Fubini-Study volume form:

$$dV_{\mathrm{FS},\epsilon} := \frac{1}{n!}(\omega_{\mathrm{FS}}|_{X_\epsilon})^n \tag{57}$$

which induces the Fubini-Study measure, $\mu_{\mathrm{FS},\epsilon}$:

$$\mu_{\mathrm{FS},\epsilon}(A) := \int_A \frac{1}{n!}(\omega_{\mathrm{FS}}|_{X_\epsilon})^n \tag{58}$$

for any measurable $A \subseteq X_\epsilon$.

As submanifolds of $\mathbb{CP}^3$, X_ϵ has finite Fubini-Study volume, hence we may rescale $\mu_{\mathrm{FS},\epsilon}(X_\epsilon) = 1$ and compute the volume of X_ϵ with respect to other volume forms (e.g. we will discuss the computation of the holomorphic volume in Remark 4.6), by taking the average of their pointwise ratio (against $\mu_{\mathrm{FS},\epsilon}$) over X_ϵ. It is a worth noting that for every $\epsilon > 0$, X_ϵ and X_0 have the same Fubini-Study volume:

Lemma 4.3. *Denote the Fubini-Study volume of X_ϵ and X_0 by $\mathrm{Vol}_{FS}(X_\epsilon)$ and $\mathrm{Vol}_{FS}(X_0)$ respectively. For every given $\epsilon > 0$, we have $\mathrm{Vol}_{FS}(X_\epsilon) < \infty$. In particular,*

$$\mathrm{Vol}_{FS}(X_\epsilon) = \mathrm{Vol}_{FS}(X_0) \tag{59}$$

Proof. We appeal to the Poincaré-Lelong equation (2.74) and convergence of currents (2.73). Denote the Fubini-Study form on $\mathbb{CP}^3$ by ω_{FS}, we wish to prove the following claim:

Claim 4.4. *For every $\epsilon > 0$, we have $[X_\epsilon](\frac{1}{2}\omega_{FS}^2) = [X_0](\frac{1}{2}\omega_{FS}^2)$.*

If Claim 4.4 holds, then:

$$\begin{aligned} \mathrm{Vol}_{\mathrm{FS}}(X_\epsilon) &= [X_\epsilon](\frac{1}{2}\omega_{\mathrm{FS}}^2) \\ &= [X_0](\frac{1}{2}\omega_{\mathrm{FS}}^2) \\ &= \mathrm{Vol}_{\mathrm{FS}}(X_0) \end{aligned} \tag{60}$$

Proof. (of claim 4.4). Viewing X_ϵ and X_0 as currents of integration, we wish to invoke the Poincaré-Lelong equation (2.74). A concern is that Poincaré-Lelong equation only holds for holomorphic functions. However as discussed in Remark 2.51, the defining quartic polynomials f_ϵ and f are not globally defined functions on $\mathbb{CP}^3$. We remedy this by restricting each polynomial to an affine patch, and gluing using a partition of unity. Define:

$$\tilde{f}_{\epsilon,i} := \frac{f_\epsilon}{x_i^4},\ \forall\epsilon > 0 \quad ; \quad \tilde{f}_{0,i} := \frac{f_0}{x_i^4} \tag{61}$$

which are holomorphic functions on each affine patch U_i. Note that the zero set of $\tilde{f}_{\epsilon,i}$ is precisely $X_\epsilon \cap U_i$, for every $\epsilon > 0$, and the zero set of $\tilde{f}_{0,i}$ is $X_0 \cap U_i$. Now we can apply Poincaré-Lelong equation

(2.74) and remark 2.72, to $X_\epsilon \cap U_i$ and $X_0 \cap U_i$ as submanifolds in $\mathbb{CP}^3$ of codimension 1, for any ϵ. Let $\{\chi_i\}_{i=0}^3$ be a partition of unity subcoordinate to the affine patches $\{U_i\}_{i=0}^3$. For any $\alpha \in \Omega^{2,2}(\mathbb{CP}^3)$, we have:

$$\begin{aligned}
[X_\epsilon - X_0](\alpha) &= [X_\epsilon - X_0](\sum_{i=0}^{3} \chi_i \alpha) \\
&= \sum_{i=0}^{3} [X_\epsilon - X_0](\chi_i \alpha) \\
&= \sum_{i=0}^{3} [X_\epsilon \cap U_i - X_0 \cap U_i](\chi_i \alpha) \\
&= \frac{i}{\pi} \sum_{i=0}^{3} \int_{\mathbb{CP}^3} \big(\log|\tilde{f}_{\epsilon,i}| - \log|\tilde{f}_{0,i}|\big) \cdot \partial\bar{\partial}(\chi_i \alpha) \\
&= \frac{i}{\pi} \sum_{i=0}^{3} \int_{\mathbb{CP}^3} \log\left|\frac{f_\epsilon}{f_0}\right| \cdot \partial\bar{\partial}(\chi_i \alpha) \\
&= \frac{i}{\pi} \int_{\mathbb{CP}^3} \log\left|\frac{f_\epsilon}{f_0}\right| \cdot \partial\bar{\partial}(\alpha)
\end{aligned} \tag{62}$$

where in line 1, we used definition of partition of unity. In line 2, we used linearity of currents as linear functionals. In line 3, it follows from that fact that each $\mathrm{supp}(\chi_i \alpha) \subset U_i$. In line 4, we used the Poincaré-Lelong equation (2.74). In line 5, we made the simple observation that $\frac{\tilde{f}_{\epsilon,i}}{\tilde{f}_{0,i}} = \frac{f_\epsilon / x_i^4}{f_0 / x_i^4} = \frac{f_\epsilon}{f_0}$. In line 6, we used linearity of $\partial\bar{\partial}$ to exchange the finite sum. Now since ω_{FS} is d-closed, it is also $\bar{\partial}$-closed [Huybrechts, 2005, Lemma. 1.3.6]. By the Leibniz rule, we have:

$$\begin{aligned}
\partial\bar{\partial}\omega_{\mathrm{FS}}^2 &= \partial(\bar{\partial}\omega_{\mathrm{FS}} \wedge \omega_{\mathrm{FS}} + \omega_{\mathrm{FS}} \wedge \bar{\partial}\omega_{\mathrm{FS}}) \\
&= 0
\end{aligned} \tag{63}$$

With $\alpha = \frac{1}{2}\omega_{\mathrm{FS}}^2$, from line 6 of equation (62), we conclude that:

$$[X_\epsilon - X_0](\frac{1}{2}\omega_{\mathrm{FS}}^2) = 0. \tag{64}$$

□

4.2 Numerical Approximation of the Calabi-Yau Metric

As observed in remark 2.30, Yau's theorem, while powerful, remains non-constructive. In this subsection, we will explain how the Calabi-Yau metric is numerically approximated, via numerically solving the Monge-Ampére equation. We will then discuss some practical implementations, including Donaldson ν-balanced iteration [Donaldson, 2009] and the recent machine learning techniques [Douglas et al., 2021; Qi, 2025], which approximates the Calabi-Yau metric g_{CY} of projective Calabi-Yau 2-folds in $\mathbb{CP}^3$, defined by homogenous quartic polynomials.

Remark 4.5. (Heuristics: Solving for the Calabi–Yau Metric via Monge–Ampère). Consider $X_0 :=$

$Z(f_0) \subset \mathbb{CP}^3$, where f_0 is a homogeneous quartic polynomial, with polarised deformation family $\{X_\epsilon\}$. As discussed in remark 2.34, on each X_ϵ, for every Kähler class $[\omega_\epsilon]$ on X_ϵ, the holomorphic volume form coincide with the Calabi-Yau volume form, up to a constant (depending on the class $[\omega_\epsilon]$):

$$A_{[\omega_\epsilon]} \cdot \Omega_\epsilon \wedge \overline{\Omega}_\epsilon = \omega_{\epsilon,CY}^2 \tag{65}$$

Such $\omega_{\epsilon,CY}$ is unique by Yau's theorem (2.33). For obtaining the Calabi-Yau form $\omega_{\epsilon,CY}$, one might naively wish to 'take the square root' in the above equation. More formally, Yau's theorem (2.33) further shows that our intuition of 'taking the square root' is equivalent to solving the complex Monge-Ampère equation by finding ϕ:

$$\omega_{\epsilon,\phi} = \omega_0 + i\partial\bar{\partial}\phi \tag{66}$$

where $\phi : X_\epsilon \to \mathbb{R}$ is the unknown potential, $\omega_{\epsilon,\phi}^2 = A_{[\omega_\epsilon]} \cdot \Omega_\epsilon \wedge \overline{\Omega}_\epsilon$, and ω_0 is a reference Kähler form in $[\omega_\epsilon]$ (e.g. the restriction of the Fubini-Study form to X_ϵ). Once $\omega_{\epsilon,\phi}$ is found, we may recover the Calabi-Yau metric (on X_ϵ) by $g_{CY,\epsilon}(\cdot,\cdot) := \omega_{\epsilon,\phi}(\cdot, J\cdot)$.
As mentioned in Remark 2.67, when the degeneration family is non-collapsing, as $\epsilon \to 0$, $(X_\epsilon, \omega_{\epsilon,CY})$ converges to (X_0, ω_{CY}) in the Gromov-Hausdorff sense, with the Calabi-Yau metric on X_ϵ converges to the singular Calabi-Yau metric on X_0 in the sense of $C^\infty_{\text{loc}}(X_0^{\text{reg}})$.

Remark 4.6. (Numerical Solution to the Monge-Ampère Equation). For fixed ϵ, denote the approximated Calabi-Yau metric by $\omega_{\text{approx},\epsilon}$. Since $\Omega_\epsilon \wedge \overline{\Omega}_\epsilon$ and $\frac{1}{n!}\omega^n_{\text{approx},\epsilon}$ are both top-forms on X_ϵ, we may define their ratio as a smooth function:

$$r := \frac{\Omega_\epsilon \wedge \overline{\Omega}_\epsilon}{\frac{1}{n!}\omega^n_{\text{approx},\epsilon}} \tag{67}$$

Hence we may express the volume of X_ϵ as:

$$\int_{X_\epsilon} \Omega_\epsilon \wedge \overline{\Omega}_\epsilon = \int_{X_\epsilon} r \cdot \frac{1}{n!}\omega^n_{\text{approx},\epsilon} \tag{68}$$

The ratio $r(x)$ measures how well the Calabi-Yau volume form is approximated by $\frac{1}{n!}\omega^n_{\text{approx},\epsilon}$ (at point x), where $\frac{1}{n!}\omega^n_{\text{approx},\epsilon}$ is obtained by numerically solving the Monge-Ampère equation (66). In particular, if $r \approx 1$, then $\frac{1}{n!}\omega^n_{\text{approx},\epsilon} \approx \Omega_\epsilon \wedge \overline{\Omega}_\epsilon$.
In practice, with sample points $x_1, \ldots, x_N$ i.i.d. from the Fubini-Study measure (4.2) on X_ϵ, the ratio r at each point is given by:

$$r(x_i) := \frac{\Omega_\epsilon \wedge \overline{\Omega}_\epsilon(x_i)}{\omega_{\text{approx},\epsilon}(x_i)^n/n!} \tag{69}$$

Define the Monge-Ampére loss $\mathcal{L}$ [Larfors et al., 2022, p. 17, Eqn. 4.3]:

$$\mathcal{L} := \frac{1}{N}\sum_{i=1}^{N}\big(r(x_i)-1\big)^2 \tag{70}$$

Minimising $\mathcal{L}$ forces $r(x_i) \approx 1$, at every sample point x_i. To verify that the associated $\omega_{\text{approx},\epsilon}$ is indeed a good approximation to $\omega_{CY,\epsilon}$, we will check if the holomorphic volume matches with the Calabi-Yau volume, by performing Monte-Carlo integration over a fresh sample set, $x'_1, \ldots, x'_N$. The holomorphic volume is given by:

$$\int_{X_\epsilon} \Omega_\epsilon \wedge \overline{\Omega}_\epsilon = \frac{1}{N}\sum_{i=1}^{N} r(x'_i)\cdot \frac{\omega^n_{\text{approx},\epsilon}(x'_i)}{n!} \tag{71}$$

and the Calabi-Yau volume is given by:

$$\int_{X_\epsilon} \frac{\omega^n_{\text{approx},\epsilon}}{n!} = \frac{1}{N}\sum_{i=1}^{N} \frac{\omega_{\text{approx},\epsilon}(x'_i)^n}{n!} \tag{72}$$

If $\mathrm{Vol}_{\text{hol}}(X_\epsilon) \approx \mathrm{Vol}_{CY}(X_\epsilon)$, it is an evidence for the fact that $\omega_{\text{approx},\epsilon}$ is good numerical approximation to the true Ricci-flat form $\omega_{CY,\epsilon}$.

Remark 4.7. (Practical numerical approach). Instead of solving the Monge–Ampère PDE on the infinite-dimensional space of potentials, one usually restricts to a finite-dimensional family, within the space of Bergman metrics. The full definitions, proofs of existence/uniqueness of ν-balanced metric, and convergence statements [Donaldson, 2001, 2009] are rather technical and beyond the scope of this thesis, we therefore keep the discussion concise. Fix an very ample line bundle $L \to X$ and integer $k \gg 1$, let $\{s_1, \ldots, s_N\}$ be a basis of $H^0(X, L^k)$ (e.g. restriction of homogeneous polynomials of degree k on $\mathbb{CP}^n$). On $\mathbb{CP}^n$, the Fubini-Study form is given by:

$$\omega_{\text{FS}} = \partial\bar{\partial}\log\Big(\sum_{i=0}^{n}|x_i|^2\Big) \tag{73}$$

We define:

$$\omega_{\text{approx}} = \partial\bar{\partial}\log\Big(\sum_{i=0}^{N} h^{ij}\ s_i\overline{s_j}\Big) \tag{74}$$

where $P = (h^{ij}) = (h_{ij})^{-1}$ and $H := (h_{ij}) \in \mathbb{C}^{(N+1)\times(N+1)}$ is the unique (up to a scale) minimiser of the Monge-Ampére loss (70).

4.3 Results: Holomorphic Volumes

Remark 4.8. (Notation). In this paper, we denoted the polarised deformation parameter as ϵ, writing the deformation family as X_ϵ. In the accompanying Python code, the same parameter is called `t`.

Our Python implementation is built on the MLGeometry library [Qi, 2025], [Douglas et al., 2021], which provides a neural-network PDE solver, trained using the Adam optimiser [Kingma and Ba, 2017]. It computes an approximated Calabi-Yau volume form $\omega_{\text{approx},\epsilon}$ on a fixed hypersurface X_ϵ, which can be used to estimate the Calabi-Yau volume. Based on MLGeometryGuide's pipeline, we modified the code and introduced a continuous parameter $t \in (0, 1)$.

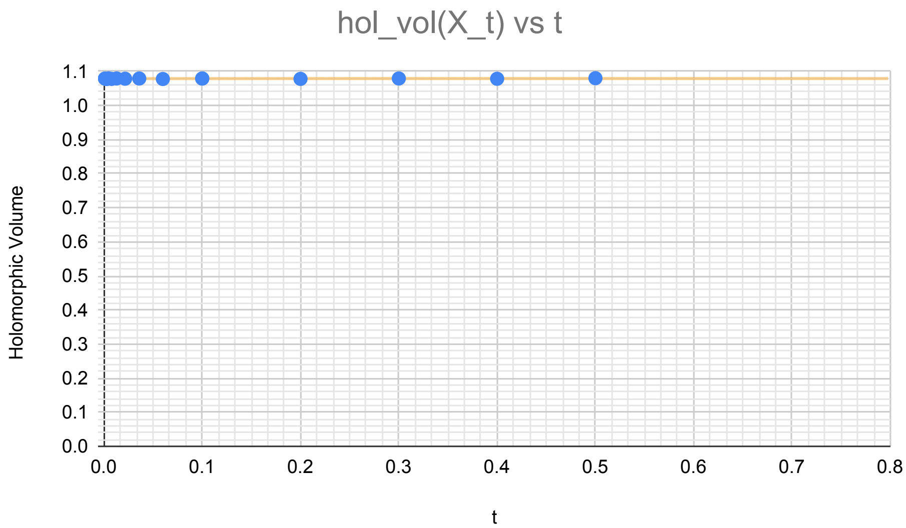
In the original MLGeometryGuide code, the holomorphic volume and Calabi-Yau volume are computed simultaneously, where the latter is obtained by training a neural network to approximate the Fubini–Study potential. Since the training step is rather computationally expensive, in practice $\text{Vol}(X_t)$ could be evaluated for five values of t.

Since the holomorphic volume is directly obtained from holomorphic volume form $\Omega_\epsilon \wedge \overline{\Omega}_\epsilon$, and there is no need to compute $\omega_{\text{approx},\epsilon}$. Hence in our refactored code, computation for the holomorphic volume is separated from the rest of the code and each evaluation reduces to a Monte-Carlo integration. This improves the runtime significantly, allowing us to compute $\text{Vol}_{\text{hol}}(X_t)$ for more than 30 different values of t, with 12800 sample points on each X_t. By Theorem 2.65, we can numerically classify non-collapsing, small and large complex structure limits at X_0, by examining the growth rate of $\text{Vol}_{\text{hol}}(X_t)$ as $t \to 0$.

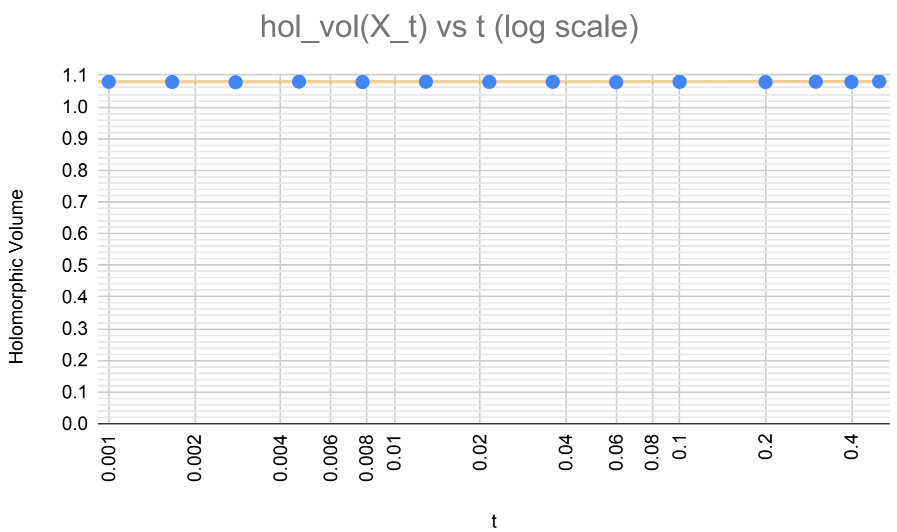
In this subsection, we will present two graphs for one representative member of each degeneration type. The first plot displays the holomorphic volume against the deformation parameter t, and the second plot displays the holomorphic volume normalised by $(\log |t|)^{-m}$ against t, where $m \in \{0, 1, 2\}$ is the integer characterising the type of complex structure limit of the deformation family (2.60). For the second graph, as $t \to 0$, we expect the curve to approach to a constant and appear as a horizontal line asymptotically.

Example 4.9. Let $X_\epsilon := Z(\{x_0^4 + x_1^4 + x_2^4 + x_3^4 + \epsilon \cdot x_0x_1x_2x_3\})$, as $\epsilon \to 0$

In figure 1, the holomorphic volume is constant (within Monte-Carlo error) as $t \to 0$, providing a strong numerical evidence that the degeneration family $\{X_\epsilon\}$ admits a non-collapsing complex structure limit 2.65 at X_0.



(a) holomorphic volume: $\{X_t\}$

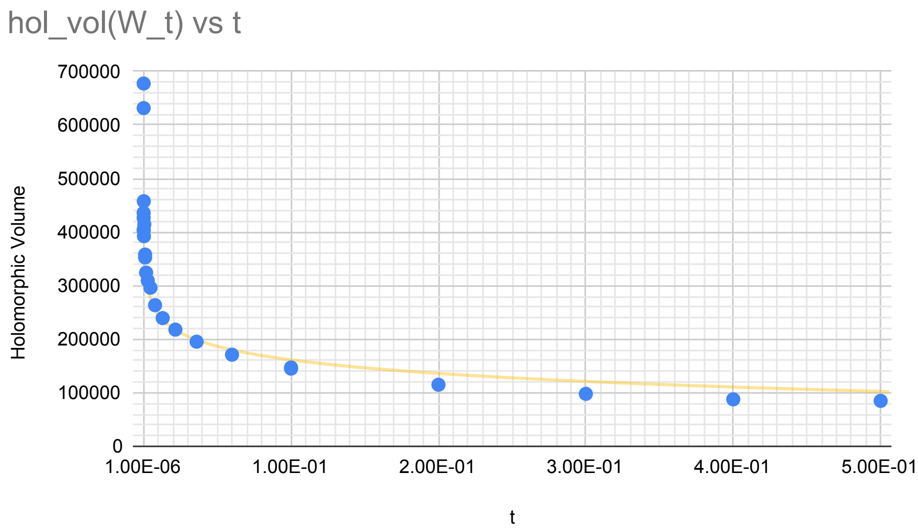


(b) holomorphic volume: $\{X_t\}$ (log scale)

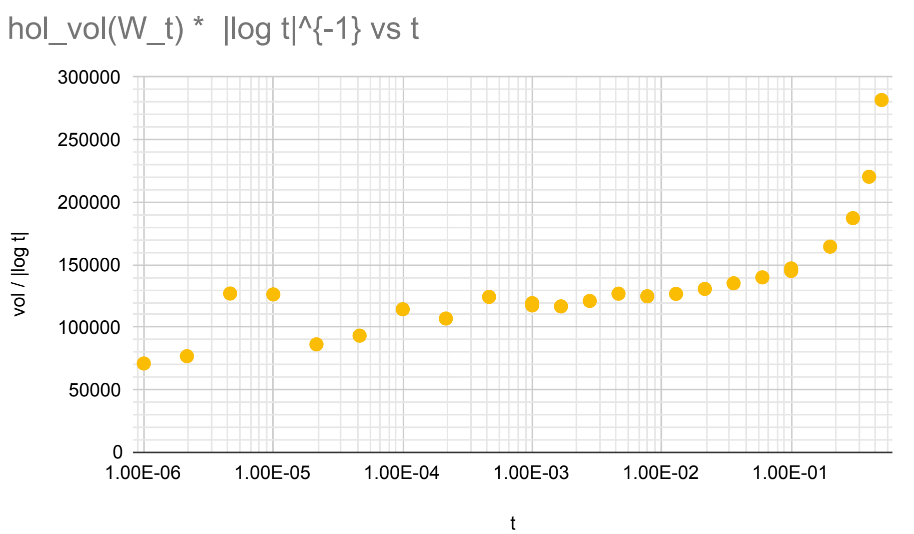
Figure 1: Holomorphic volume: $\{X_t\}$

In figure 2(b), we observe that the (normalised) holomorphic volume is approximately constant for $10^{-5} < t < 10^{-1}$. For very small values of t (e.g. $t \approx 10^{-6}$), the volume growth slightly disagrees with the prediction. The discrepancy is likely due to the following fact: since the volume form is sharply peaked near the singular fibre, some sample points might carry very large weights. If those points are not drawn, the Monte-Carlo estimate will differ from the actual integral. Despite the discrepancy for small t, these data provides a strong numerical evidence that the degeneration family $\{W_\epsilon\}$ admits a small complex structure limit 2.65 at W_0.

Example 4.10. Let $W_\epsilon := Z(\{\epsilon \cdot (x_0^4 + x_1^4 + x_2^4 + x_3^4) + (x_0^2 + x_1x_2 + x_3^2)(x_0^2 + x_2x_3 + x_1^2)\})$, as $\epsilon \to 0$



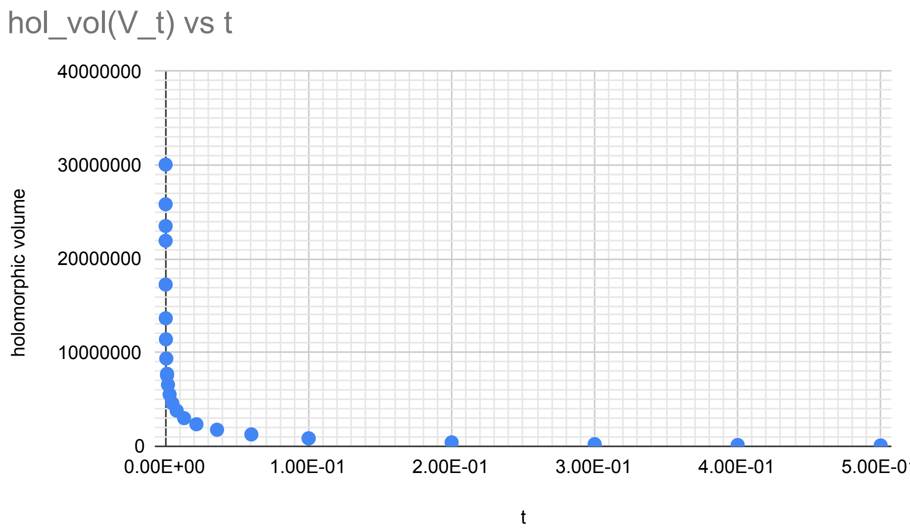
(a) holomorphic volume: $\{W_t\}$



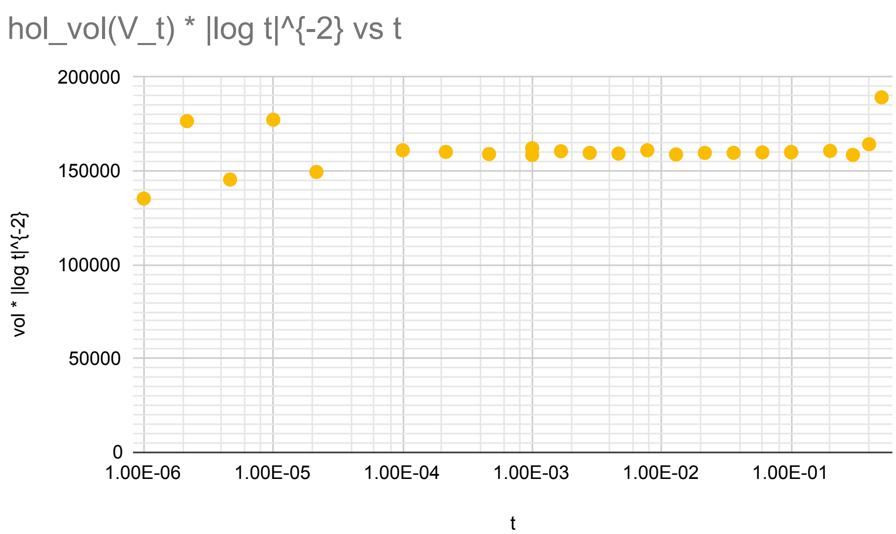
(b) holomorphic volume: $\{W_t\}$ (log scale)

Figure 2: Holomorphic volume: $\{W_t\}$

Example 4.11. Let $V_\epsilon := Z(\{\epsilon \cdot (x_0^4 + x_1^4 + x_2^4 + x_3^4) + x_0x_1x_2x_3\})$, as $\epsilon \to 0$



(a) holomorphic volume: $\{V_t\}$



(b) holomorphic volume: $\{V_t\}$ (log scale)

Figure 3: Holomorphic volume: $\{V_t\}$

We observe the same phenomenon in figure 3, that the (normalised) holomorphic volume is approximately constant for $10^{-4} < t < 10^{-1}$, and for very small values of t (e.g. $t \approx 10^{-6}$), the volume growth slightly disagrees with the prediction. Again, the discrepancy is likely due to the volume form being sharply peaked near the singular fibre. Despite the discrepancy for small t, these data provides a strong numerical evidence that the degeneration family $\{V_\epsilon\}$ admits a large complex structure limit 2.65 at V_0.

4.4 Results: Calabi-Yau Volumes

In this subsection, we present the approximated Calabi-Yau volume of one representative member of each degeneration type. We sampled 12800 points on each K3 surface and the neural network is trained for 200 epochs. The Monge-Ampére loss (70) per epoch is given by the following learning curves:

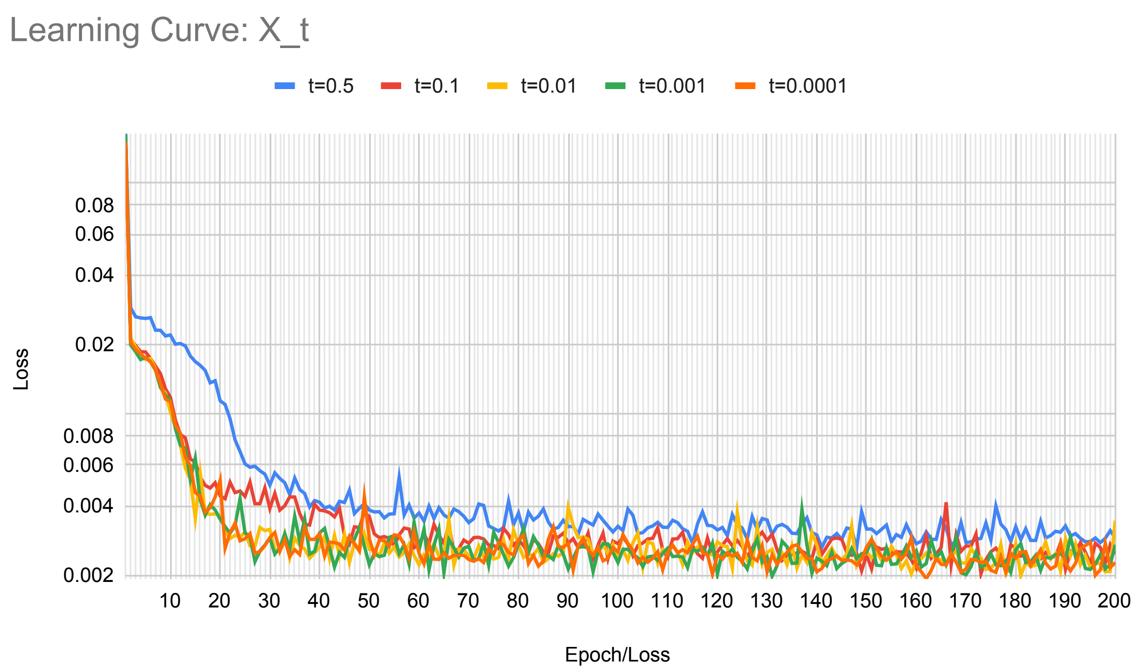


Figure 4: Learning Curve: $\{X_t\}$.

In figure 4, we observe that the Monge-Ampére loss decreases rapidly and plateaus near 0.003, for all values of t (in the test range), indicating an approximation to the Calabi-Yau volume for non-collapsing family $\{X_t\}$.

In contrast, for the collapsing families, in figure 5 and figure 6, the loss decays more slowly and plateaus at a higher value (around 0.02), for $t \in \{0.1, 0.5\}$. For smaller values of t (e.g. $t = 0.001$), the loss shows little or no improvement after training for 200 epochs. This behaviour is consistent with our prediction: near the singular fibre, the Calabi-Yau volume form becomes more sharply peaked, correspondingly the Monge-Ampére equation becomes harder to approximate (under the same model capacity and sample size), leading to slower convergence and larger final errors.

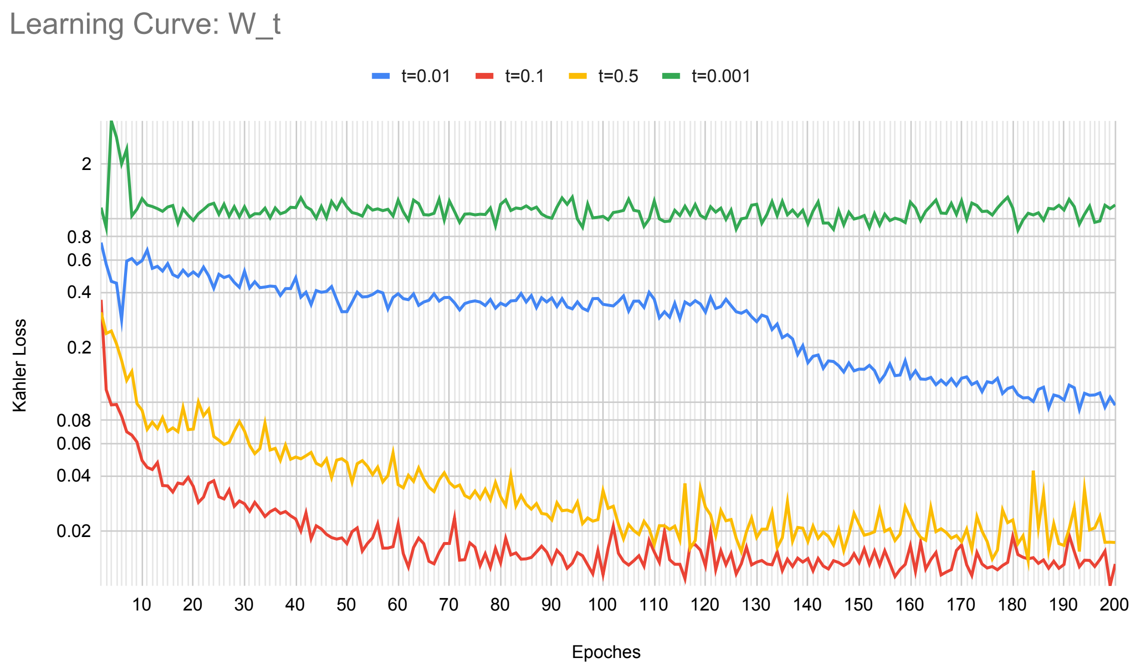


Figure 5: Learning Curve: $\{W_t\}$.

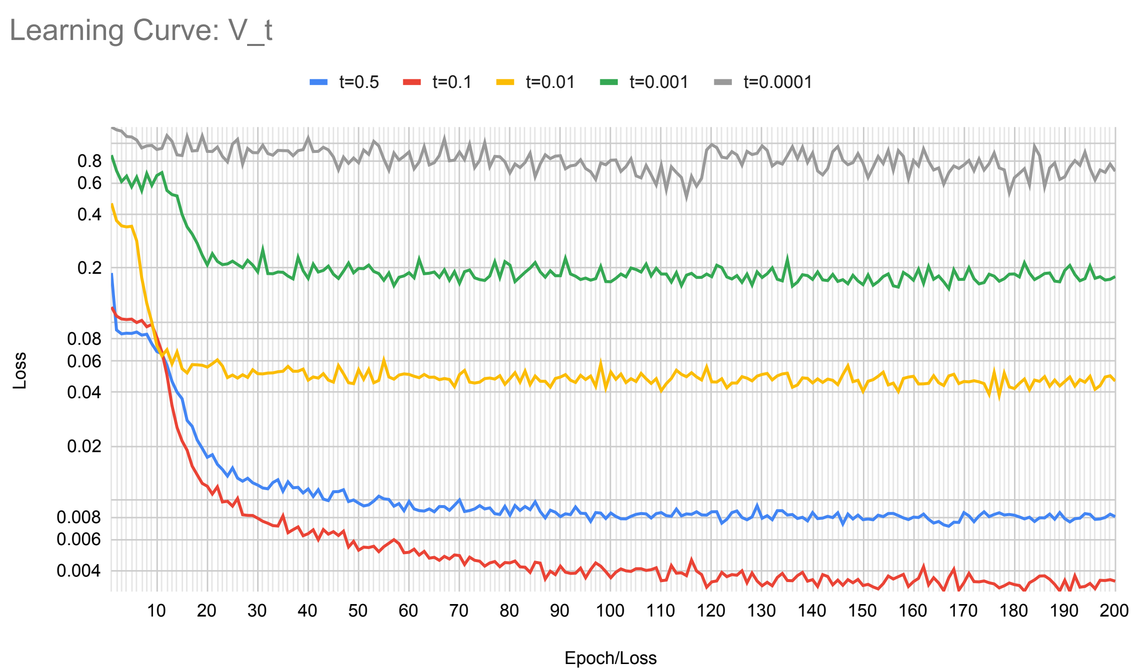


Figure 6: Learning Curve: $\{V_t\}$.

A comparison of the holomorphic volume and the approximated Calabi-Yau volume for each degeneration family is given as follows:

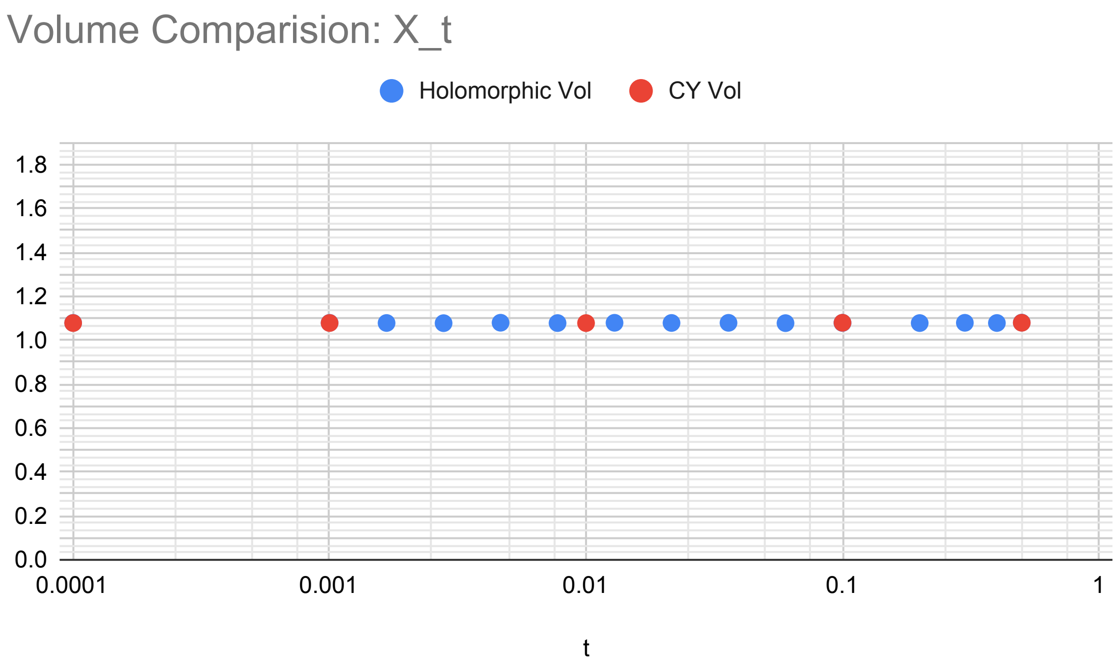


Figure 7: Learning Curve: $\{X_t\}$.

In figure 9, we observe that the approximated Calabi-Yau volume of the non-collapsing family $\{X_t\}$ approximately agrees with the holomorphic volume.

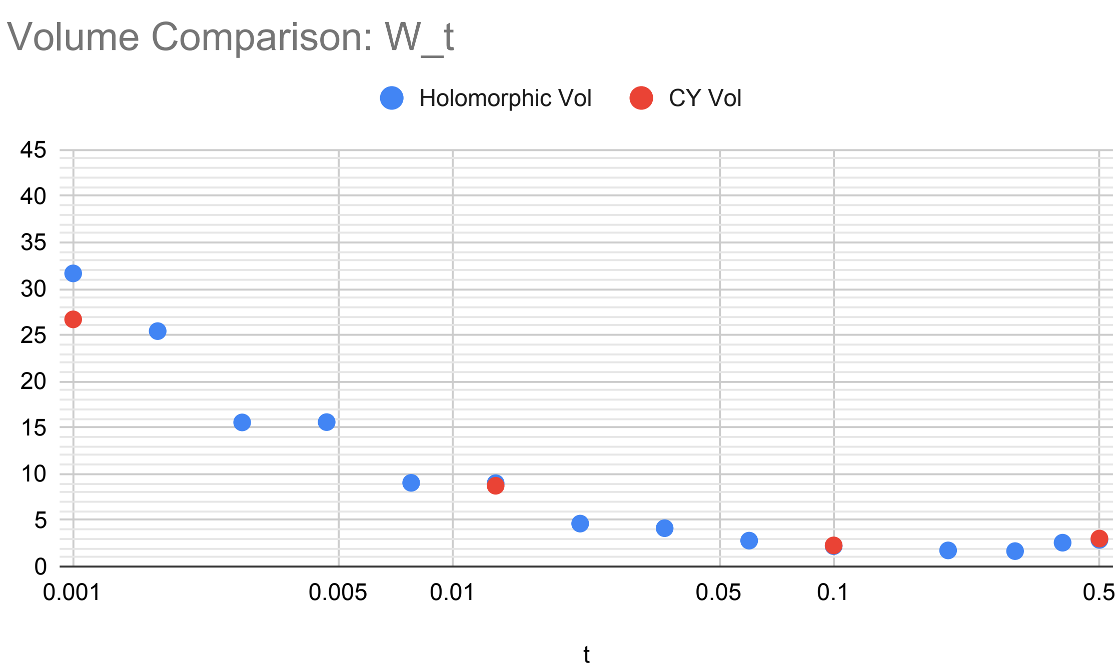


Figure 8: Learning Curve: $\{W_t\}$.

In figure 8, for the collapsing family $\{W_t\}$ approaching to a small complex structure limit , we observe that the approximated Calabi-Yau volume agrees with the holomorphic volume, for $0.1 < t < 0.5$. A small discrepancy appears as $t \to 0$ (e.g. at $t = 0.001$), consistent with the discrepancy of the Monge-Ampére loss observed in figure 5.

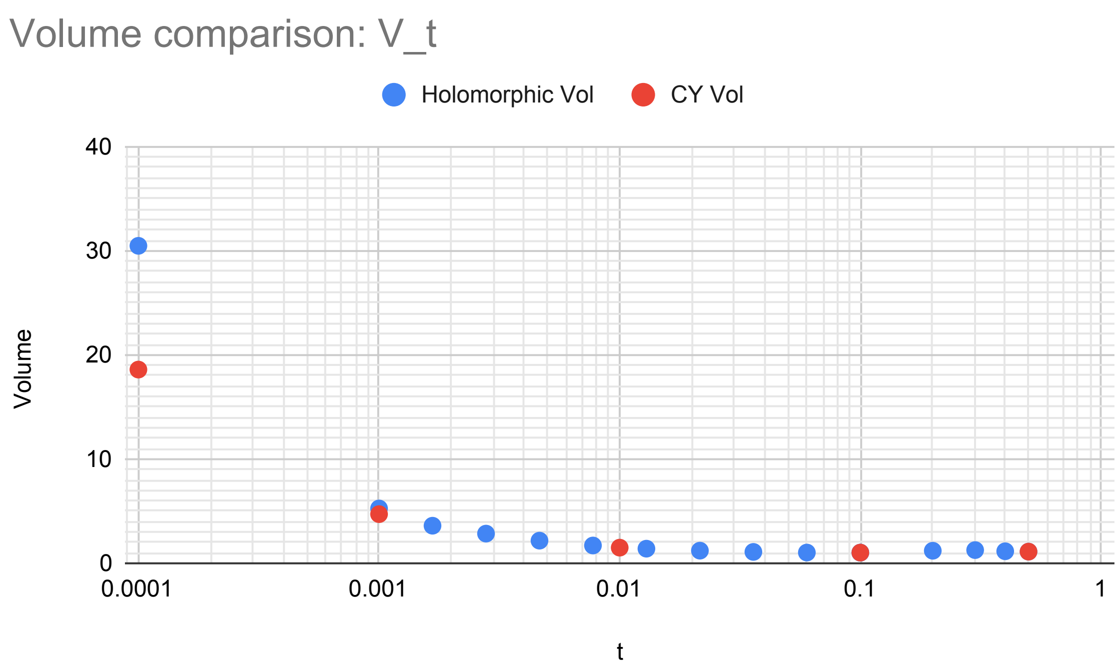


Figure 9: Learning Curve: $\{V_t\}$.

In figure 9, we note a similar phenomenon for the collapsing family $\{V_t\}$ approaching to a large complex structure limit: the approximated Calabi-Yau volume agrees with the holomorphic volume, for $0.1 < t < 0.5$, however the discrepancy as $t \to 0$ (e.g. at $t = 0.001$) is larger than the discrepancy in the case of $\{W_t\}$.

4.5 Dimension Collapsing of Elliptic Curves

From numerical experiments, we observe that for a family of elliptic curves $\{X_t\} \subset \mathbb{CP}^2$ as 1-dimensional Calabi-Yau manifolds, the eigenvalues of the Calabi-Yau metric tensors blow up as $t \to 0$. In this subsection, we will explain this observation in details and propose two conjectures, supported by numerical evidence.

In figure 10, we compared the distribution of eigenvalues of the Calabi-Yau metric tensor, for 4 different elliptic curves in $\mathbb{CP}^2$. We sampled 1280 points on each elliptic curve and the neural network is trained for 50 epochs. Here, $V_t := Z\big(t \cdot (x_0^3 + x_1^3 + x_2^3) + x_0x_1x_2\big)$ has a large complex structure limit, $X_t := Z\big(x_0^3 + x_1^3 + x_2^3 + t \cdot x_0x_1x_2\big)$ has a non-collapsing complex structure limit, and the Fermat cubic $Z(x_0^3 + x_1^3 + x_2^3)$ is smooth in $\mathbb{CP}^2$.

We observe that the eigenvalues of the Calabi-Yau metric tensors of the Fermat cubic (and $X_{0.0001}$ are mainly distributed between 3 and 5, whereas V_t show a much heavier right tail, with a much higher

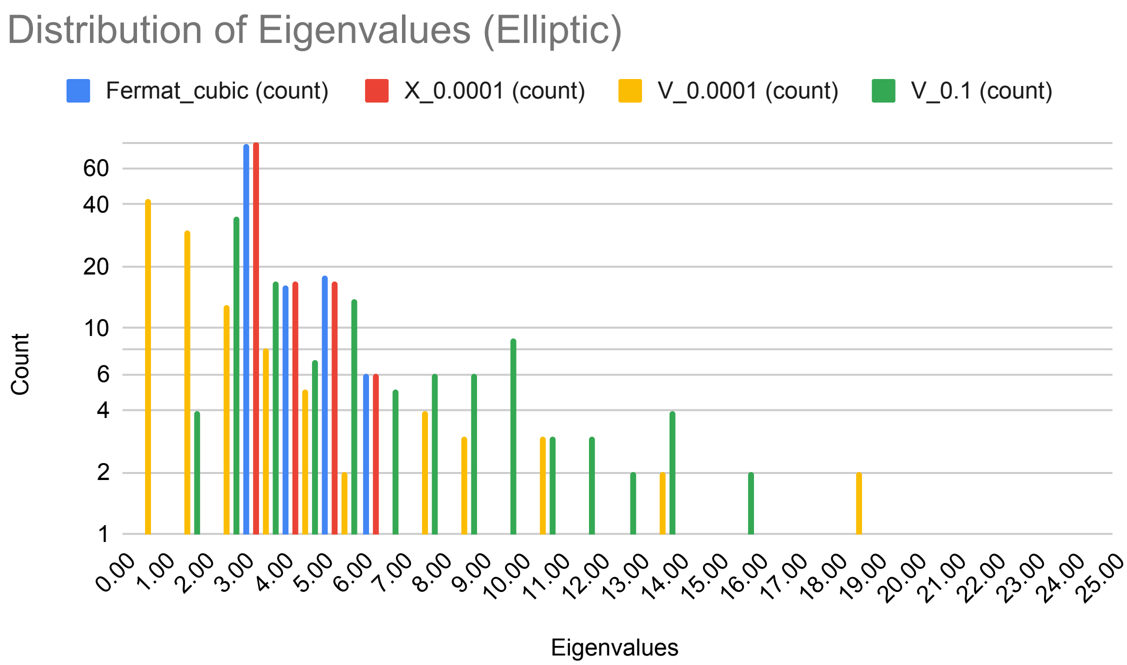


Figure 10: Distribution of Eigenvalues: Elliptic Curves in $\mathbb{CP}^2$

proportion of large eigenvalues. In particular, as $t \to 0$, we make two observations:

- The proportion of sample points whose eigenvalues exceed any fixed threshold increases.
- The 90th-percentile of the eigenvalue distribution also increases.

In other words, near the large complex structure limit, not only do large eigenvalues occur more frequently, they tend to be larger on average.

For proposing the conjecture, we first need to make precise the notion of *eigenvalues of the Calabi-Yau metric tensor*. Given two frames $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ on $T_x X_t$, let G_t and H_t be the matrix representation for the Calabi-Yau metric $g_{CY,t}$ in each frame. Since change of frame of a quadratic form acts by congruence:

$$H_t = P^T G_t P \tag{75}$$

where P is some transition matrix. Note that the eigenvalues of G_t are not preserved unless P is orthogonal. To obtain a coordinate-independent spectrum, we instead fix the Fubini-Study metric on $\mathbb{CP}^2$, $\hat{g} := g_{FS}$, and consider the $\hat{g}$-self-adjoint endomorphism:

$$A_t = \hat{g}^{-1} g_{CY,t} \tag{76}$$

Or in local coordinates:

$$g_{CY,t}(v,w) = \hat{g}(A_t v, w) \quad , \quad \forall v, w \in T_x X_t. \tag{77}$$

Claim 4.12. *The eigenvalues of A_t is independent of the choice of frame, for every $t > 0$.*

Proof. Given two frames $E := \{e_1, \dots, e_n\}$ and $E' := \{e'_1, \dots, e'_n\}$, with change of frame matrix P. Let G_t and $\hat{G}_t$ be the matrices associated to $g_{CY,t}$ and $\hat{g}$ in frame E, G'_t and $\hat{G}'_t$ be the matrices associated to $g_{CY,t}$ and $\hat{g}$ in frame E'. In frame E, A_t is represented by $\hat{G}_t^{-1} G_t$. In frame E', we have:

$$\begin{aligned} A'_t &= {\hat{G}'_t}{}^{-1} G'_t \\ &= (P^T \hat{G} P)^{-1} (P^T G_t P) \\ &= P^{-1} ({\hat{G}'_t}{}^{-1} G'_t) P \\ &= P^{-1} A_t P \end{aligned} \tag{78}$$

Since A_t and A'_t are similar matrices, they have the same spectrum. □

Remark 4.13. (Notation). We denote the eigenvalues of A_t at point $x \in X_t$ by $\kappa_1\big(A_t(x)\big), \dots, \kappa_n\big(A_t(x)\big)$. In the case of elliptic curves in $\mathbb{CP}^2$, for every $t > 0$, A_t is represented by a 1×1 matrix, hence we simply denote its eigenvalue by $\kappa\big(A_t(x)\big)$.

We can now propose the following conjecture:

Conjecture 4.14. *Let $\{X_t\}_{t>0}$ be a family of elliptic curves in $\mathbb{CP}^2$, approaching to the large complex structure limit as $t \to 0$. Let $g_{CY,t}$ be the Calabi-Yau metric on each X_t, and A_t be the $\hat{g}$-self-adjoint-endomorphism obtained from (76). Then for every $M \in (0, \infty)$, there exists a sufficiently small $t > 0$, and a point $x_t \in X_t$ such that $\kappa\big(A_t(x_t)\big) > M$.*

We further expect the following stronger conjecture to hold:

Conjecture 4.15. *Let $\{X_t\}_{t>0}$ be a family of Calabi-Yau hypersurfaces of dimension n in $\mathbb{CP}^{n+1}$, with Calabi-Yau metrics $\{g_{CY,t}\}$. Assume that $\{(X_t, g_{CY,t})\}$ collapses to an essential skeleton of dimension k in the Gromov-Hausdorff limit (after rescaling to diamter 1), where $k \leq n$. Let A_t be the $\hat{g}$-self-adjoint-endomorphism obtained from (76). Then as $t \to 0$, there exists a point $x_t \in X_t$ such that exactly $n - k$ eigenvalues of A_t goes to infinity.*

5 Appendix: code

The code and notebooks used to generate the numerical experiments in this thesis are publicly available at: https://drive.google.com/drive/folders/1qPgAeGcEkoxi1k_VD83-X3-BQRs8E1PT?usp=drive_link

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