PHYS 380 - Fall 2024

Assignment Final Project

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Problem 1

The first problem in this assignment asks for a solution to Laplace's equation. Specifically, the problem poses a two dimensional box problem, where three of the four sides of the box are grounded and the remaining side is held at 1 Volt. This problem is solved numerically by recalling that the problem obeys an averaging property, whereby the potential at point A is found by averaging the values of the potential around A. Numerically, this means that the potential at a point on the grid is the average of the potential at the four points in immediate contact with that point. The analytical solution is as follows:

The PDE we want to solve is $\nabla^2 V(x, y) = 0$, and the boundary conditions are:

BC1: $V(0, y) = 0$,

BC2: $V(a, y) = 0$,

BC3: $V(x, 0) = 1$,

BC4: $V(x, b) = 0$.

We now posit a separable ansatz: $V(x, y) = h(x)\phi(y)$, which turns our boundary conditions into:

BC1: $h(0) = 0$,

BC2: $h(a) = 0$,

BC3: $h(x)\phi(0) = 1$,

BC4: $\phi(b) = 0$.

From this, we see that ϕ does not have two homogeneous boundary conditions, but that h is a well-defined boundary value problem, and hence we solve h's eigenvalue problem first to calculate the separation constant:

$$\frac{d^2h}{dx^2} = \lambda h \rightarrow h(x) = \sin(\gamma x) \tag{1}$$

BC1 takes care of itself, and BC2 tells us that

$$0 = \sin(\gamma a) \rightarrow \gamma a = n\pi \quad \forall \quad n \in \mathbb{N} \tag{2}$$

Thus,

$$h(x) = \sin\left(\frac{n\pi x}{a}\right). \tag{3}$$

This implies that $\gamma^2 = \lambda$. We calculate the y dependence by assuming a solution of the form

$$\phi(y) = C_1 \cosh\left(\frac{n\pi(y-b)}{a}\right) + C_2 \sinh\left(\frac{n\pi(y-b)}{a}\right) \tag{4}$$

From this BC4 implies that $C_1 = 0$. From here, we form the general solution by superposing the product states $V_n = h_n\phi_n$ over all physical n:

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(y-b)}{a}\right) \tag{5}$$

All that is left to do is calculate the components of V, $C_n \sinh\left(-\frac{n\pi b}{a}\right)$. This potential V is a vector in the Hilbert space of Lebesgue integrable functions on $x \in [0,a]$, and so the orthogonality of the basis vectors $\sin\left(\frac{n\pi x}{a}\right)$ implies that

$$C_n = \frac{-2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a \sin\left(\frac{n\pi x}{a}\right) dx \tag{6}$$

This integrates to

$$C_n = \frac{-2}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \left((-1)^{n-1} + 1\right) \tag{7}$$

We can see that $C_n = 0$ for all even $n \in \mathbb{N}$, and equals, for all odd $n \in \mathbb{N}$,

$$C_n = \frac{-4}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \tag{8}$$

With this, we are equipped to write down the complete solution to the problem:

$$V(x,y) = \sum_{\text{odd } n}^{\infty} \frac{4}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right), \tag{9}$$

where the sum is explicitly taken over only odd natural numbers. The numerical solution is shown in figure 1.

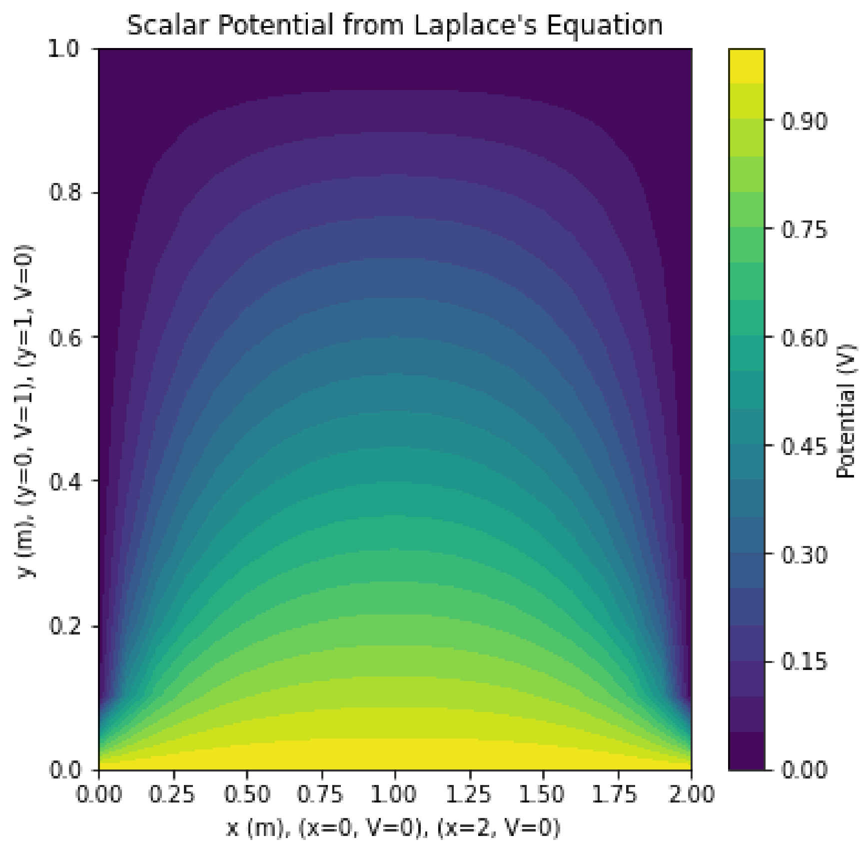

Figure 1: Scalar potential distribution in a 2D rectangular box with three sides grounded ($V = 0$) and one side held at $V = 1$ Volt.

Problem 2

This problem is much the same as the first, but with different boundary conditions. We are still solving $\nabla^2 V = 0$, but the new boundary conditions are

BC1: $V(0, y) = 0$,

BC2: $V(a, y) = 0$,

BC3: $V(x, 0) = \cos^2\left(\frac{\pi x}{a}\right)$,

BC4: $V(x, b) = -\sin^2\left(\frac{\pi x}{a}\right)$.

We now posit the same separable ansatz: $V(x, y) = h(x)\phi(y)$, which turns our boundary conditions into:

BC1: $h(0) = 0$,

BC2: $h(a) = 0$,

BC3: $h(x)\phi(0) = \cos^2\left(\frac{\pi x}{a}\right)$,

BC4: $h(x)\phi(b) = -\sin^2\left(\frac{\pi x}{a}\right)$.

Calculating the components A_n and B_n of V in this case is tricky, so I will just write down the general solution with undetermined A_n and B_n and leave the finnicky stuff up to the numerical solution, which is performed in essentially identical manner as the first problem.

$$V(x, y) = \sum_{n \in \mathbb{N}} \sin\left(\frac{n\pi x}{a}\right)\left[A_n \cosh\left(\frac{n\pi(y-b)}{a}\right) + B_n \sinh\left(\frac{n\pi(y-b)}{a}\right)\right], \tag{10}$$

where the precise natural numbers to sum over are determined by A_n and B_n as in the previous problem. The numerical solution is shown in figure 2.

Problem 3

The final problem of this assignment grounds all walls of the box, but places a parallel plate capacitor inside the box. I did not try to calculate the analytical solution for this situation; it may or may not be possible, I don't know. The numerical solution was much the same as before, however, with the only changes coming in the application of boundary conditions. I plotted the result twice, once as a contour plot, once as a surface plot. The results are shown below in figures 3 and 4. The results are those expected: the potential is a constant ± 1 on the plates and vanishes on the boundary.

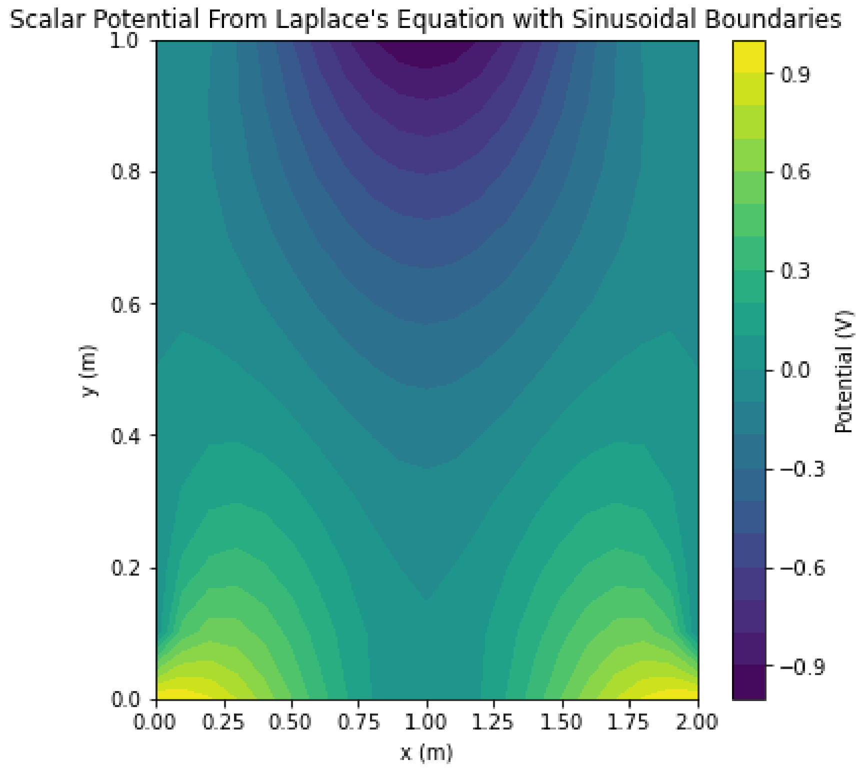

Figure 2: Scalar potential distribution in a 2D rectangular box with two sides grounded ($V = 0$) and the others held at sinusoidal voltages.

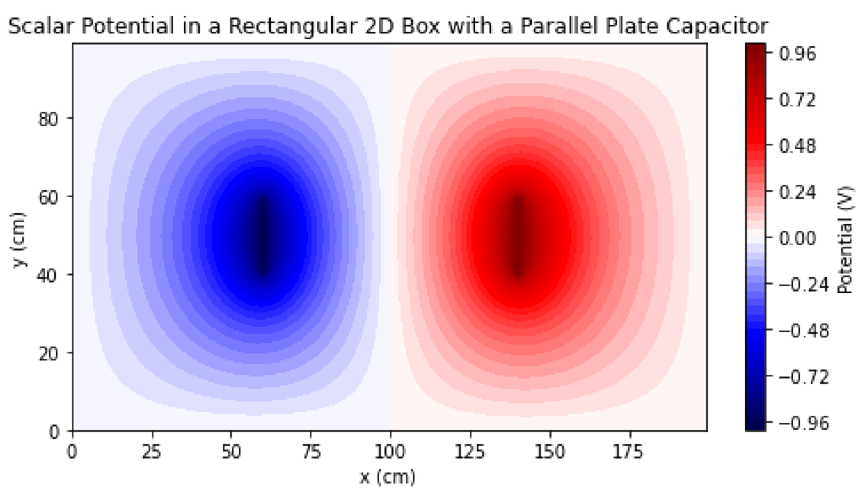

Figure 3: Contour plot of the scalar potential in a 2D rectangular box (with all sides grounded, $V = 0$) containing a parallel plate capacitor.

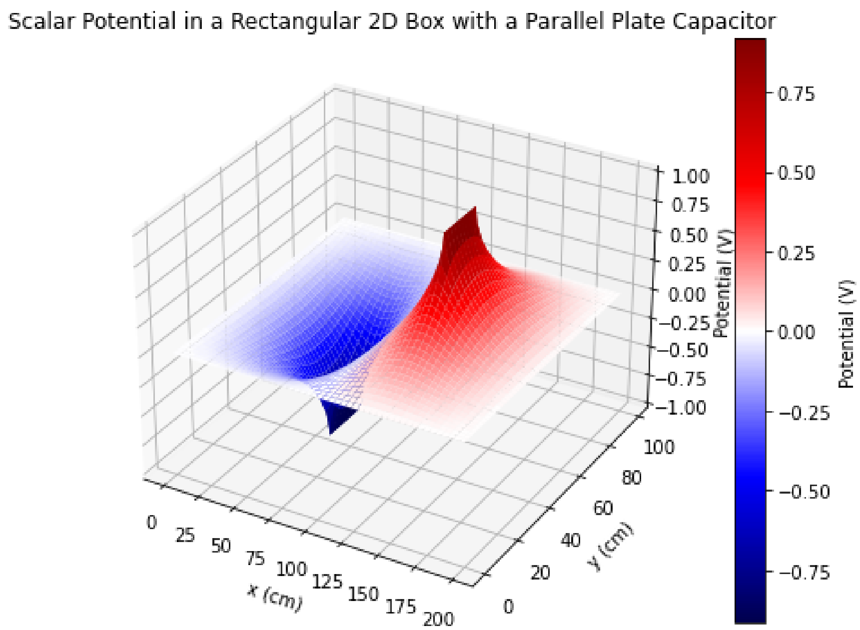

Figure 4: Surface plot scalar potential in a 2D rectangular box (with all sides grounded, $V = 0$) containing a parallel plate capacitor.