PHYS 380 - Fall 2024

Assignment 7

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Problem 1

In order to derive approximation methods for derivatives, one must recall the Taylor series in the following form: for $x \in \mathbb{R}$, and any C^∞ function, $f : \mathbb{R} \to \mathbb{R}$, with finite, and small, quantity $h \in \mathbb{R}$, the function f translated by an amount h can be written precisely as

$$f(x+h) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f(x)}{dx^n} h^n \tag{1}$$

We also note that $f(x-h)$ is equivalent to the above with the stipulation that $h \to -h$, since h is an arbitrary finite real number. For an approximation to second order in h, consider the following:

$$f(x+h) - f(x-h) = \left(f(x) + \frac{df}{dx}h + \frac{d^2 f}{dx^2}h^2 \right) - \left(f(x) - \frac{df}{dx}h + \frac{d^2 f}{dx^2}h^2 \right), \tag{2}$$

where we have neglected terms of order h^3 and higher. This equation can be see to be equivalent to

$$\frac{df}{dx} = \frac{f(x+h) - f(x-h)}{2h}, \tag{3}$$

and the equality holds *only* in the second order approximation. For sufficiently small h, we conclude that the derivative of f is approximately,

$$\frac{df}{dx} \approx \frac{f(x+h) - f(x-h)}{2h} \tag{4}$$

In order to derive a third order approximation, we consider that $f(x + \Delta x)$, where $\Delta x = \alpha h$ for integer α, is, to third order in h,

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2} f''(x)\Delta x^2 + \frac{1}{6} f'''(x)\Delta x^3, \tag{5}$$

where “ $'$ ” is understood to denote a derivative with respect to x. It can be shown that the following combination of $f(x+h)$, $f(x-h)$, $f(x+2h)$, and $f(x-2h)$, where appropriate values of α are picked by comparison with equation 5, produces an approximation of df/dx accurate to third order in h:

$$f'(x) = \frac{8(f(x+h) - f(x-h)) - (f(x+2h) - f(x-2h))}{12h}. \tag{6}$$

Again, we note that the equality holds *only* in the third order approximation, otherwise, we conclude that for small h, df/dx may be approximated as

$$f'(x) \approx \frac{8(f(x+h) - f(x-h)) - (f(x+2h) - f(x-2h))}{12h}. \tag{7}$$

Problem 2

This problem asks for a generalization of the software written in a previous problem which calculates the scalar potential and the electric field of a single point charge. The problem essentially consists of recalling that the wave equation, $\Box\phi = J^0$, is linear, and so the potential of a distribution of point charges is a linear combination of the contribution of each charge:

$$\phi = \sum_{i=1}^{N} \phi_i = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0 r_i}, \tag{8}$$

where N is the number of point charges, and ϕ_i is the Coulomb potential of the ith charge. So, in principle, if the charges and their positions are known, the potential of the system is known. The electric field is given by $E^j = -\partial_i \phi \delta^{ij}$:

$$\mathbf{E} = -\nabla\phi \tag{9}$$

Employing the provided random point charge distribution software to modify the electric potential software previously written, figures 1 and 2 were obtained.

Problem 3

This problem asks for a solution to the damped harmonic oscillator via Euler's method. The statement of the problem is as follows:

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega^2 y = 0, \tag{10}$$

where γ is the damping parameter, and ω is the natural frequency of the system. Consider the following initial conditions: $y(0) = 2$ and $\dot{y}(0) = 0$; then, the analytical solution is given by:

$$y(t) = Ae^{kt} + Be^{-kt}. \tag{11}$$

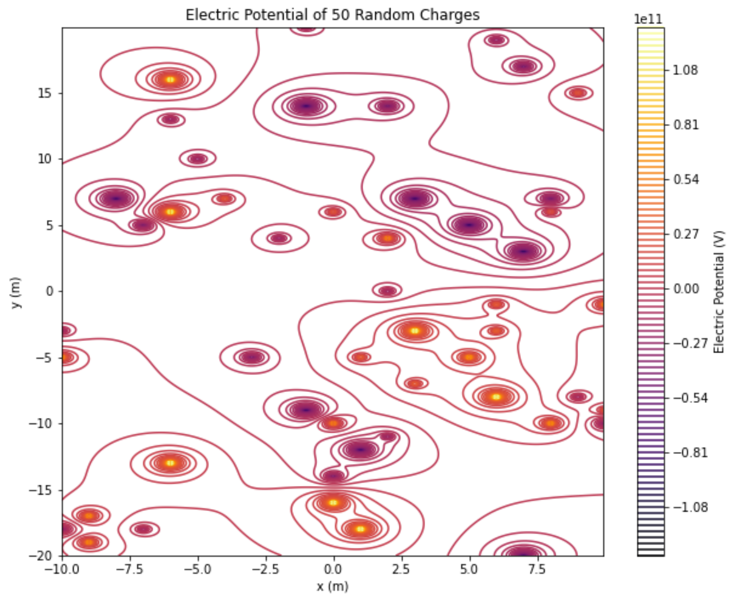

Figure 1: The scalar potential of a random distribution of 50 point charges with $|q| < 1$.

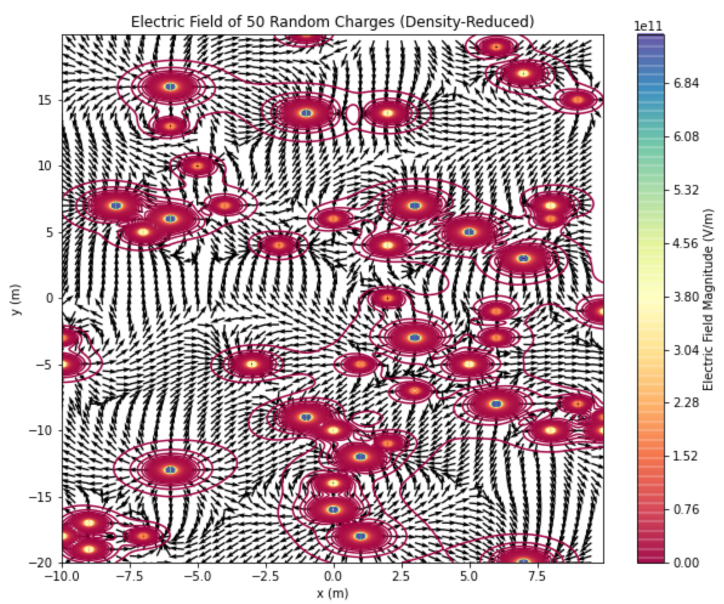

Figure 2: The electric field of a random distribution of 50 point charges with $|q| < 1$.

where k is given by the characteristic equation:

$$k^2 + \gamma k + \omega^2 = 0 \to k = -\omega_d \pm \omega \sqrt{\frac{\omega_d^2}{\omega^2} - 1}. \tag{12}$$

The initial conditions imply that $A = B$ and $A = 1$. Then, the solution is exactly: