Programming Project – 2019-08-28

MATB22: Linear Algebra

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This assignment has 3 tasks.

The goal of this project is to demonstrate how with help of programming skills examples for a more theoretical course can be generated. A side-effect of this project is that your programming skills are kept alive.

The projects follows the line of a previous exam in the course MATB22 (Aug, 2017)

*For this project you should work in **groups of two or three**. Solve the project tasks during the course and upload your project answers as a single file having one of the file types* `*.py` *or* `*.ipynb` *in Canvas.*

Deadline: October 31, 2019. You will get an appointment for an oral presentation of your results to one of the teaching assistants. This presentation is a mandatory part of the project.

All questions and discussion with regards to the tasks should be done using Canvas.

Task 1

This is related to Task 1 in the above mentioned exam:

1. Find the vector $\boldsymbol{x} \in \mathbf{R}^3$ that minimises $\|A\boldsymbol{x} - \boldsymbol{y}\|$ where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

- Solve this task by setting up and solving the normal equations in Python.
- You learned in class that the solution minimizes $\min_x \|Ax-b\|$ or alternatively $\min_x (Ax-b)^2$. Use the function `scipy.optimize.fmin` and solve for x. Compare the results.
- Replace the given vector y now by $y = (1, a, 1, a)^{\mathrm{T}}$, where a is a free parameter. Solve in Python the least squares problem and compute the norm of the residual. Make a plot of this norm versus a. Are there values of a for which

the curve has a zero? Can you explain your observation by theoretical results from the course?

Task 2

This is related to Task 2 in the above mentioned exam:

$$\begin{cases} a_{n+1} = a_n + 3b_n + 2c_n \\ b_{n+1} = -3a_n + 4b_n + 3c_n \\ c_{n+1} = 2a_n + 3b_n + c_n \end{cases}, \qquad \begin{cases} a_0 = 8 \\ b_0 = 3 \\ c_0 = 12 \end{cases}.$$

- Write a function to show the iterates $z_n = [a_n, b_n, c_n]$ do not converge when $n \to \infty$.
- Do the normalized iterates $z_n/\|z_n\|$ converge[1]? If they converge, determine experimentally the limit z. Which property does this vector have?
- Compute for every normalized iterate the quantity $q_n = z_n^{\mathrm{T}} A z_n$. Do these quantities converge?
- How many iterates do you need to fullfil the requirement $\|z_n - z\| < \varepsilon$ for $\varepsilon = 10^{-8}$?
- Change ε from 0.1 to 10^{-16} and plot the number of iterates versus ε in a semilog plot.
- Do the same for q_n and its limit q.
- Which sequence converges faster?

Task 3

This is related to Task 3 in the above mentioned exam:

3. A surface has, with respect to an orthonormal coordinate system for 3-space, the equation

$$2x_1^2 - x_2^2 + 2x_3^2 - 10x_1x_2 - 4x_1x_3 + 10x_2x_3 = 1.$$

Identify its type and specify the points on the surface closest to the origin.

Make a 3D plot of this surface for $x_1, x_2 \in [-1, 1]$ without simplifying the above equation by hand. Hint: You might want to express one variable, e.g. x_3 in the equation as a function of the other two. This can be done by writing a function which itself calls a numerical method for solving nonlinear equations. You can use the build-in function `scipy.optimize.fsolve(f, x0)` for this end. Note

[1] Normalize in every step.

that the resulting plot consists of more than one part and you can find all parts by choosing suitable initial values `x0` in `fsolve`.

Alternative approach: Determine the solutions of x_3 in dependence of x_1 and x_2 using `sympy`.