Quadratic Unconstrained Binary Optimisation for Portfolio Optimisation Problems

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Abstract

Portfolio optimisation is incredibly important in the world of finance. Finding the optimal portfolio, however, proves to be very hard and could take extremely long periods of time. This literature review discusses optimisation problems in general, with a narrowing towards the quadratic unconstrained binary optimisation form of portfolio optimisation problems. This form can potentially be used to efficiently solve portfolio optimisation problems, which hasn't been possible before. This literature review will also discuss possible simulated annealing and how it can be used to solve these optimisation problems. Finally, this paper will discuss any gaps found in this area of research and propose a project which will cover some of these gaps.

Index Terms

Optimisation, QUBOs, Quadratic Unconstrained Binary Optimisation, Simulated Annealing, Portfolio Optimisation.

I certify that all material in this dissertation which is not my own work has been identified : Sam Shailer

I. INTRODUCTION

THE quadratic unconstrained binary optimisation (QUBO) model is currently an actively researched model within the field of optimisation[1][2]. This is due to its recently discovered potential to model large numbers of combinatorial optimisation (CO) problems. Previously these CO problems may have been inefficient to solve, however when formulated using the QUBO model it has been shown that a solution can be found efficiently. This is because the QUBO model underpins an area of quantum computing known as quantum annealing[1][2]. Quantum computing, and in turn quantum annealing, is a rapidly developing area of computing that is still in its infancy. This means any research carried out into effectively formulating problems for solving on a quantum computer, will not only speed up the development of functioning quantum computers but could also help form the foundations of how quantum computing works in the future.

This paper specifically describes the method of formulating portfolio optimisation problems in the QUBO model. Portfolio optimisation problems are a form of CO which find the best risk-return ratio possible for a portfolio. Being able to efficiently optimise large portfolios would produce better risk-return rates and could change how portfolios are managed altogether. To successfully model an accurate portfolio optimisation problem several constraints might need adding. This can lead to the optimisation problem being rendered as an NP-hard problem. This means there has yet to be found an algorithm that will efficiently solve these problems. Current attempts at solving portfolio optimisation problems mainly use meta-heuristic algorithms, however these algorithms can only find near-optimal solutions efficiently. This indicates that potentially better solutions to the problem exist. By formulating portfolio optimisation problems as QUBOs, meta-heuristic algorithms can still solve them, but there is also the possibility to find the best solution efficiently when the quantum technology becomes available.

The purpose of this literature review is to discuss the current state of research in portfolio optimisation and its integration with the QUBO model. This will start with a section on general combinatorial optimisation problems, what a combinatorial optimisation problem is, and how are they currently used. The next section will delineate portfolio optimisation, focusing on the most widely used methods and why this is an important research area. The following section will describe the QUBO model in depth and research further why it is currently such a popular model within the field of optimisation. Consequentially the next section will detail current research on methods of forming combinatorial optimisation problems as QUBOs. This will then narrow down to what the current research on formulating combinatorial portfolio optimisation problems as QUBOs is. The final section will identify current research on simulated annealing methods for solving QUBOs.

II. LITERATURE REVIEW

A. *Optimisation Problems*

Optimisation, in essence, is the process of finding the solution to a problem which is perceived to have the best outcome[3]. As described by J. Nocedal, this requires the problem to have quantitative values which can be changed in order to maximise or minimise the overall outcome[4]. This is incredibly common in the modern world, for example, finding the shortest path for a salesman to take between multiple cities [5],[6] or working out which stocks are the best to invest in to maximise returns while minimising risk[7]. This is only a tiny subset of the problems encompassed in the area of optimisation problems, with the research in this area growing ever larger and the number of possible problems to solve constantly increasing. As highlighted by P.R. Adby [8], due to the advent of the digital age, the ability to solve optimisation problems on computers has sparked extensive research into this area.

As discussed in the following books by A. Riccardi [3] and J. Nocedal [4], current optimisation really begins with the formulation of a problem, which requires optimisation, as a mathematical model. This model must be able to produce an optimal solution by changing the value of variables within it. The following set of steps describing the method to do this are explored further in the book by J. Nocedal [4]:

- The first step in this process is to identify the vector of variables/unknowns in the system (denoted by x).
- The next step is to find the objective function (a scalar function denoted by $f(x)$) of x that is required to be maximized or minimised.

- The last step in this process is to determine whether there are any constraints that the vector x must satisfy (these are denoted by c_i), these constraints must also be a scalar function.

When these steps are followed optimisation problems can be modelled in the following form:

$$\text{minimise: } f(x) \quad \text{subject to } c_i$$

While the form of this model may differ depending on where it's found, such as in the book by L.Wolsey[9], the basic principles remain the same. This model, and its other interpretations, are generally regarded as the standard optimisation model as it will allow all optimisation problems to be modelled mathematically in some form [3],[4],[8].

One of the largest areas researched in the field of optimisation is known as *quadratic optimisation* or *quadratic programming*. The definition of quadratic programming problems (QP) described in a paper by C. Floudas[10], is that they are optimisation problems where the objective function is of the quadratic form and where there is a linear set of equality or inequality constraints. A number of research articles mathematically define the basic QP as being of the following form[11],[12][13]. They state that the input data must include an $n \times n$ real symmetric matrix H, an $n \times 1$ real matrix c, an $n \times m$ real matrix A and an $m \times 1$ real matrix b. The mathematical definition is then given by:

$$\begin{aligned} \text{minimise:} \quad & f(x) = \frac{1}{2}x^T H x + c^T x \\ \text{subject to (st):} \quad & \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

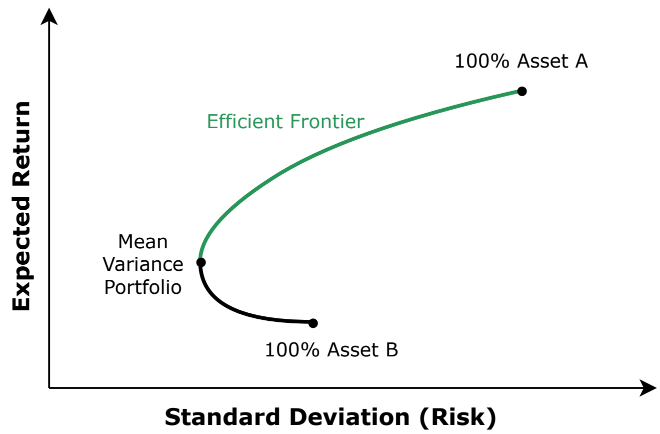
Where x is an $n \times 1$ real matrix whose values should minimise $f(x)$. While the exact layout of this formula may vary from paper to paper the definition remains the same. It was first shown by S. Sahni [14] that QP is NP-hard when the matrix H is negative definite. This result was later proven by S. Vavasis[12] who formulated the QP as a Turing Machine language-recognition problem and proved that this version of QP could be reduced to the satisfiability problem problem (SAT), which is a well-known NP-hard problem. NP-hard problems are problems where there is currently no efficiently known algorithm which can solve them. K. Murty[13] went a step further and showed that in special cases QP was an NP-complete problem. As also defined by K. Murty[13], the NP-complete class of problems are such that if a single problem within the class is found to be solved in polynomial time then every other problem within the class can also be efficiently solved. The areas in which quadratic programming can be used are extensive. For example a paper by O. Gupta[15], gives clear uses of QP within industries. These industries involve: Finance, Agriculture, Economics, Production and Operations, Marketing and Public Policy. These studies have begun to show the importance and usability of quadratic programming and why it is such a heavily researched area of optimisation.

B. Portfolio Optimisation Problems

The ability to optimise the risk return ratio of portfolios has been well researched. In the 1952 paper by H. Markowitz a method of portfolio optimisation was introduced which compared the risk return ratio between multiple assets[16]. What this optimisation method showed was that by having multiple assets in a portfolio you could increase the return of the portfolio while also lowering the risk. It was also shown that the more assets you had, which were less correlated with each other, the further you could maximise the risk return ratio. This concept is known as diversification and is the key point Markowitz made. Figure 1 represents this theory graphically by showing how the relationship of the portfolio between two assets is not linear but instead curves as you change the weight of the individual assets in the portfolio. The minimum variance portfolio is the point at which the weight of each asset minimises the overall risk of the portfolio. As shown in figure 1, any portfolio lying below the minimum variance portfolio is inefficient. This is because there is another point on the graph that has a higher return with the same risk. The section of this curve above the minimum variance portfolio is known as the *efficient frontier*.

In 1959 Markowitz formulated a solution to the problem of finding the efficient frontier for N number of assets using quadratic programming [18]. This was done by describing the risk between all assets as a matrix which is called

Fig. 1. Example of the possible portfolios between two assets[17]

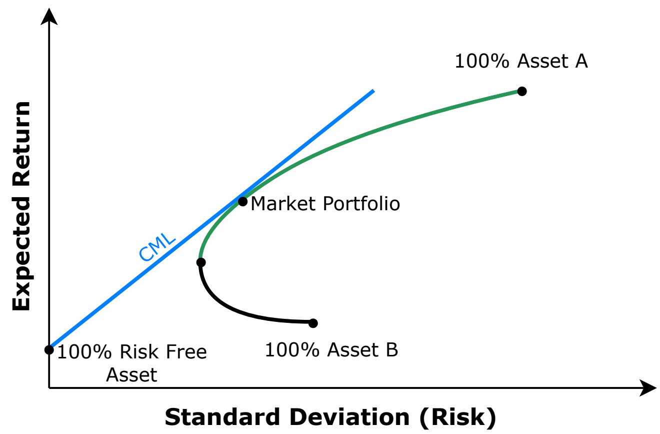


the co-variance matrix (C). An attribute of the covariance matrix is that it must be positive semi-definite. He then takes a non-empty vector (X) describing the weights of each asset in the portfolio. Using these variables he could then formulate a quadratic programming problem in the same form shown in the previous section. The complete mathematical derivation of this can be found in Markowitz's book [18]. Markowitz's theory is what pioneered further research into the field of portfolio optimisation, and this area of research is called modern portfolio theory (MPT).

A number of studies have also derived new portfolio optimisation methods from MPT, which attempt to model the real world stock market more accurately[19], [20]. For example, the capital asset pricing model (CAPM) developed by Sharpe [21] and Lintner [22] and based off MPT. CAPM describes the expected returns on an asset calculated using its relationship to the current market and comparing it with that of a risk-free asset. A risk-free asset is described by Sharpe and Lintner as an asset with 0 risk. Any asset which is guaranteed to only increases in value, no matter how long it takes, can be classed as a risk-free asset, however trying to find real world risk-free assets is proven to be difficult as you cannot predict what the price of an asset will do[23]. Sharpe describes the prices the market gives an investor as *the price of time* and *the price of risk* [21]. This concept is shown graphically in figure 2, this graph takes the risky asset as a portfolio, which was described earlier, and plots a new line showing how the risk varies as a larger portion of the investment is placed in the risky asset compared to the risk-free asset. This line is known as the Capital Market Line (CML), which describes how the investor trades off time for return by increasing the risk. In the paper by S. Ross, the correlation between the CML and efficient frontier is further discussed [24]. It explains that the point where the CML is tangent to the efficient frontier is known as the market portfolio as it will have the same Sharpe ratio as the market. The Sharpe ratio will be further discussed so in short it is used to compare the risk return ratio of a portfolio [25]. The other point discussed by Ross is the fact that the Sharpe ratio is the same at any point on the line. These models have begun to provide insight into how MPT can be expanded and the potential that it offers for modelling portfolio optimisation.

The use of the Sharpe Ratio has been shown in multiple papers to be incredibly important in assessing the optimisation of portfolios. While the original concept was introduced by Sharpe in his 1966 and 1975 papers [25], [26], his 1994 paper coins the term *Sharpe ratio* and fully describes exactly what the Sharpe ratio is[27]. While multiple versions of the Sharpe ratio are explained in this paper they all describe a similar overall concept, which is the use of the Sharpe ratio as a performance measure. For example, the ex-ante Sharpe ratio is a good measure of the estimated risk before a decision is actually made, and this can be described with the following

Fig. 2. Example of the CML with a two asset portfolio



mathematical formula.

$S =$ The ex-ante Sharpe ratio

$\overline{R_p} =$ Estimated return of the portfolio to measure

$\overline{R_b} =$ Estimated return of the benchmark

$\overline{d} = \overline{R_p} - \overline{R_b}$

$\sigma_d =$ The standard deviation of d

The ex-ante Sharpe ratio is:

$$S = \frac{\overline{d}}{\sigma_d}$$

The ex-post Sharpe ratio is similar to the ex-ante Sharpe ratio however instead of measuring the assets expected return per unit of risk it indicates the historical average return per unit of historic variability of the differential return. The formula is also similar but the average d is taken over a specified time period instead. Full details on this version of the Sharpe ratio can again be found in Sharpe's paper[27].

Issues with portfolio optimisation problems have also been studied in depth in an attempt to determine the practicality of portfolio optimisation. Although some studies suggest that portfolio optimisation is not practical due to a number of input errors resulting from a fundamental inability to predict the future[28], a large amount of studies still suggest that these errors can be reduced and optimised portfolio will nonetheless outperform inefficient portfolios. For example, a study into whether small input errors in portfolio optimisation will produce large output errors concluded that by using reasonable assumptions, small input errors have little effect on estimates of risk and return[29]. These findings were later demonstrated practically in an extensive study of how the $1/N$ portfolio (a portfolio where an equal fraction of wealth is distributed to each of the N assets) performs against different optimised portfolios[30]. This study found that in a variety of applications optimised portfolios produce a superior out-of-sample performance compared with equally weighted portfolios. The insights from these studies have shown why portfolio optimisation is still relevant and why the challenge to further optimise current portfolios is not purposeless.

C. Binary Portfolio Optimisation Problems

The occurrence of yes or no options in optimisation is found to be incredibly common, therefore research has been done to find out how to incorporate these options within the QP model. In the chapter 9 of the book "Applied Mathematical Programming" (AMP) [31], these types of options and ways to apply them in the QP are discussed. Initially AMP describes scenarios where these options may come up such as: whether a company should build a new plant; undertake in an advertisement campaign or develop a new product. In each of these cases it is clear that a yes or no decision must be made. The way to model this mathematically according to AMP is to use binary variables. Binary variables are variables which can hold the value of either 1 - Yes, or 0 - No. In the section of this paper on optimisation problems, the variables to optimise in the quadratic programming problem are referred in a matrix of real values x. According to AMP the first way to model the QP using binary variables is to make sure x is a matrix of binary values (0 or 1). The second method is to introduce a constraint to the problem where each member of the matrix, x_j, must be a real value less than or equal to 1. As AMP mentions, binary variables often occur in problems addressing long range, high-cost strategic decisions that are undertaken in capital investment planning.

Binary variables can also occur in portfolio optimisation. An example of this is when the assets for a portfolio are selected with equal weights, where this is discussed in the paper by D. Venturelli [32]. This article goes on to identify the choice of either selecting or not selecting an asset for the portfolio as the binary decision. In order to increase the allocation given to a particular asset in this model the asset should be cloned and then treated as a new asset. This will increase the weighting given to that asset while also decreasing the weights given to other assets. The model in this paper also contains coefficients which can model penalties on diversification as positive coefficients or model rewards as negative coefficients. A similar method to modelling portfolio optimisation using binary variables is to use assets which are binary in nature. As discussed in the paper by P. Mercurio[33], examples of assets that are binary by nature include: binary options, digital options, fixed-return options and sports bets. Binary options, which are particularly discussed in this paper, offer a fixed probability of success, a fixed rate of return and a fixed loss. Modeling a portfolio using these as the assets involves measuring the predicted rate of return assuming the assets were repeatedly bought after expiry. These papers show ways in which the portfolio can be changed to work with binary variables and how this can be modeled using optimisation methods.

The ways in which current portfolio models, like the mean variance model, can be adapted to work with binary variables has also been researched. The methods to convert integer variables to binary variables has been greatly researched in the following paper by K. Tamura[34]. In their paper the analyse the effectiveness of 3 schemes used to convert integer variables to binary variables when used on the quadratic knapsack problem. These schemes are one-hot encoding, binary encoding and unary encoding. These schemes are fully discussed within their paper but a short version of the descriptions in the paper are:

- one-hot encoding, assumes that each integer value, up to the desired integer variables max value, is mapped to a single binary variable. The coefficient of each variable is the integer value that variable corresponds to. If the bit corresponding to a value is 1 then the rest must be 0 and therefore that is the value the original integer variable should take. For example if x_1 has a max value of 3 then the binary encoding will be $x_1 = y_0 + 1y_1 + 2y_2 + 3y_3$ where y_i are the binary variables.
- binary encoding finds the max value of the integer variable to the nearest binary representation. Each binary variable is then assigned a coefficient equal to 2^i. As the nearest binary representation may allow for integer values greater than the max value, a constant value of C must be deducted. C is equal to 2 to the power of the number of binary variables required to represent the integer value, minus the max value of the integer variable minus 1. For example if x_1 has a max value of 10, the binary encoding will be $x_1 = y_0 + 2y_1 + 4y_2 + 8y_3 - 5$ where 5 has been found from $C = 2^4 - 10 - 1$.
- unary encoding, assumes that each integer value, up to the desired integer variables max value, is mapped to a single binary variable. The amount of variables equal to 1 before the first 0 indicates the value of the integer encoded. For example if x_1 has a max value of 2, the unary encoding will be $x_1 = y_0 + y_1 + y_2$. In this example a value of 0 for x_1 would be represented as $x_1 = 0 + 1 + 1$, a value of 1 would be $x_1 = 1 + 0 + 1$ and a value of 2 would be $x_1 = 1 + 1 + 0$.

This paper concluded with the findings that unary encoding appeared to have the highest performance when tested using simulated annealing, and binary encoding also performed well. One-hot encoding did not perform well and failed to find feasible solutions even when given optimal conditions. Binary encoding does have a lower number of variables compared to the other encoding schemes which is another factor taken into account in the paper. A paper by S. Karimi also used the binary encoding method as a way to formulate integer quadratic optimisation problems as binary quadratic optimisation problems. The use of these encoding methods has also shown to be work for single period discrete portfolio optimisation problem in a paper by G. Rosenberg[35]. This portfolio optimisation problem is further described within the paper however it is noted that the problem is NP-complete. This paper concluded with the statement that their solution had high success rates with efficiently solving the binary quadratic portfolio optimisation but current hardware would still need to advance in order to solve larger problems.

D. Quadratic Unconstrained Binary Optimisation

The Quadratic Unconstrained Binary Optimisation (QUBO) model for formulating optimisation problems has been researched greatly in recent years. The reason is that the QUBO model has been shown to cover a large number of combinatorial optimisation problems that are found in industry[36],[37]. The reason this is significant is because the QUBO model has been shown to be close to the Ising model in physics[1]. Quantum annealing is often used in an attempt to solve the Ising problem, which involves finding the lowest energy state of a physical system in order to minimise the objective function. The QUBO model therefore lends itself well to minimising optimisation problems via quantum annealing methods. The advantage of quantum annealing over classical methods is that it can minimise optimisation problems in polynomial time, compared to classical methods of solving the same algorithms. This is the reason why the QUBO model is currently so highly regarded and is instigating more research in this field of optimisation.

The paper by F. Glover[1] discusses the process of formulating an optimisation problem using the QUBO model. These steps involve firstly forming the problem as a quadratic optimisation problem, which was discussed in the first section of this paper. Next the variables within the problem will need to be encoded as binary variables and appropriate constraints will be added so that the variables can only be binary. Methods for this were also discussed in the previous section of this paper. Finally the constraints of the problem need to be identified and incorporated into the co-variance matrix. The co-variance matrix also needs to be translated to upper-triangular form which involves multiplying all values above the diagonal by 2 and setting all values below the diagonal to 0.

The method to incorporate constraints into the co-variance matrix is detailed in the paper by F. Glover, where they use quadratic penalties. Quadratic penalties apply when the constraint is not met and add a heavy penalty to the overall cost of the function. For this reason solvers will make sure to avoid situations in which the penalties are applied. An example of this used in the paper is the constraint $x_1 + x_3 + x_6 = 1$ being recast as the penalty $P(x_1 + x_3 + x_6 - 1)^2$, where P is the integer penalty value and x_i are binary variables. Just like the constraint, a solver will avoid all solutions where the sum of the variables, in this example, are not equal to 1 because there will be a large penalty applied if it uses those solutions. More information about methods to constrain variables can be found within the paper.

Another concept to note in the paper, is that if the original objective function has any linear part, for example $-5x_1 - 3x_2 - 8x_3 - 6x_4$ then after this has been binary encoded each variable can be set to x_i^2. This is because for binary variables $x_i = x_i^2$ as $0^2 = 0$ and $1^2 = 1$. This allows for the incorporation of the $c^T x$ term from the standard quadratic programming model into the quadratic term $x^T H x$. The overall mathematical form of the QUBO model is therefore given in the paper as:

$$\begin{aligned} \text{minimise:} \quad & f(x) = x^T Q x \\ \text{subject to (st):} \quad & \\ & x_i \in \{0, 1\} \end{aligned}$$

Where the matrix Q now represents the both the constraints and the coefficients of the original linear term. The only constraint on the model is that each variable x_i must be binary. This constraint is handled by the system used to solve the QUBO which is why it isn't included in the matrix Q.

The use of the QUBO model to solve portfolio optimisation problems has also been analysed. In the paper by F. Philipson[38], the ability to model portfolio optimisation problems using the QUBO model is tested. The portfolio optimisation model used for these tests made a binary decision on the inclusion/exclusion of assets. The assets used during the tests were 50 stocks taken from the S&P500 and the Nikkei225. The results of the tests showed the ability to successfully model and solve portfolio optimisation problems using QUBOs. A hybrid quantum solver was even able to find a solution within 3% of the optimal solution for the S&P500 with 400 stocks. The disadvantages discussed in the conclusion of this paper explained that quantum computing is still in its infancy. Quantum technology therefore still needs to progress before optimising portfolios using the QUBO model becomes commercially viable.

E. Simulated Annealing

While the ability to solve the QUBO model using quantum annealing does not appear viable yet, solving QUBOs using heuristic algorithms is being researched. The classical version of quantum annealing called simulated annealing (SA), attempts to simulate the effects of quantum annealing and is therefore well suited to being used to solve QUBO problems, this is documented in the paper by H. Kagawa[39]. in this paper, simulated annealing is described as being inspired by the physical phenomena present in the solidification of metals. This phenomenon occurs when metals are heated up to a point of near liquidation and are then steadily cooled until the atomic structure reaches a minimum energy level. The same process can be seen in the search algorithm presented for QUBOs in this paper. Instead of atoms in metals however, the algorithm changes the values of the binary variables, in a QUBO model, randomly while reducing the chance that the randomly selected variable is actually changed. This method makes sure that the algorithm checks many different minimum values so that it doesn't get stuck in a sub-optimal solution.

III. Summary and Conclusion

As shown in the literature review, optimisation is a massively developed field of research. Quadratic programming is a section of that field with countless papers describing how to formulate problems using quadratic programming and how QP can be modified to model different problems. Most of the literature describing these two areas are very well-reviewed and have been proven with lots of examples. This main area that is being researched a lot more recently is the applications that quadratic programming has to solve modern-day problems. This is what leads to the possibility of being able to solve portfolio optimisation using a version of the QP problem.

The topic of portfolio optimisation appears to be researched periodically. Looking at the dates of a lot of the literature reviewed in this section there appear points in time where the research of portfolio optimisation gains traction and then dies down again. These changes seem to coincide with the problems that occur within portfolio optimisation. As mentioned in the literature review, the inability to know the future means any estimates on the risk or return of an asset have some probability of being wrong. The more accurate of a model that's built to try and estimate future returns/risks the less computationally viable the problem becomes to solve. This is a probable reason why portfolio optimisation seems to have lost traction in research over the past 20 years. With the recent hype of quantum computing, however, and new potentially more time-efficient methods of solving portfolio optimisation problems there seems to have been an uptake in research in this area once more.

The development of the efficient frontier and the Sharpe ratio are great tools in the field of portfolio optimisation. They allow for a standard in which to measure how well a portfolio is predicted to perform against any other portfolio. I feel the use of these methods is vital in evaluating whether a portfolio that has been found is useful or not.

Binary portfolio optimisation problems until recently seem to have been formulated only using assets which in themselves are binary. This leads to a similar overall theme throughout the research, with many papers questioning whether just selecting (or not selecting) assets will produce an optimal portfolio. For assets that are themselves binary, this method of formulation will work a lot better but the final model won't follow the original mean-variance idea of optimisation. While these papers are still important as they develop the idea of using quadratic programming to solve portfolio optimisation problems, I don't think the models used are practical in developing optimal portfolios within a real-world setting.

The use of integer to binary encoding to formulate integer problems using the quadratic programming model seems to be a rather recent idea. In my opinion, integer to binary encoding is the way forward if an accurate model of mean-variance portfolio optimisation problems is to be formulated using the QUBO model. These papers offered multiple ways to approach integer to binary programming which allow for different directions that future research can take. The main disadvantage to using binary to integer encoding is the increase in variables within the final optimisation model. This causes a problem because current quantum hardware is limited in the number of binary variables it can handle. Until further developments are made in this area large amounts of assets being modelled still isn't viable on current quantum or classical. As the majority of the literature in this area is still relatively new, the practicality of the methods described is yet to be tested on a large scale. This could lead to a lot of potential adjustments in the future, but further development in this method is still very important in my opinion. If the use of integer to binary encoding to adapt portfolio optimisation problems is researched now, then when the hardware becomes available the process of implementing these problems will be a lot easier.

The quadratic unconstrained binary optimisation model offers a very useful way of formulating portfolio optimisation problems. This method could give portfolio optimisation another chance at being computationally viable. The literature clearly shows new paths being opened up for further research into this area and there seems to be keen interest from both individuals and industry to expand this field. One of the key areas that I couldn't find any literature on the use of the QUBO model to solve a portfolio optimisation problem where the classic integer weights were converted to binary variables. In my opinion, based on the current research being undertaken, there is a good chance to see large advances in this field soon.

Finally simulated annealing offers a bridge currently between quantum systems and classical systems. As discussed in the main literature review, quantum hardware is still not quite at a level where large scale optimisation problems can be solved using it. Another issue with quantum hardware is that it can be unreliable at times. This is due to potential interference from other quantum particles and is the main reason why scaling quantum hardware up is difficult. In the meantime, classical hardware offers a good way to show proof of concept. Using simulated annealing, the method in which QUBOs are solved can be shown and the potential speedup, if quantum hardware becomes available, is made evident.

IV. Specification

The main problem I can devise from the research undertaken in my literature review, is that there is a lack of portfolio optimisation problems which have been formulated as QUBOs using binary to integer encoding. This currently leaves a rather large gap in the analysis of how well the QUBO model works for formulating mean-variance portfolio optimisation problems. Without this problem being covered it would be hard to properly compare the optimal portfolios found using current known methods and the optimal portfolios found when formulated as a QUBO.

I therefore propose a project which attempts to formulate different mean-variance portfolio optimisation problems using integer to binary encoding and the QUBO model. I think this project would help to address the problem stated above. Below is a table describing the functional and non-functional requirements of such a project.

A. Functional Requirements

ID	Functional Requirement
1	The portfolio optimisation problems will need to use real market data such as the historic open and close prices for stocks on a particular market.
2	The quadratic optimisation model will need to model the variables associated with the integer weights of the portfolio.
3	The quadratic optimisation model will need to model the co-variance matrix of the portfolio.
4	The quadratic optimisation model will need to model the constraints of the portfolio optimisation problems.
5	The variables in the quadratic optimisation model will need to be converted to binary variables using the binary encoding method.
6	The constant needed to reduce the max value of the binary variables will need to be calculated.
7	The co-variance matrix will need to be converted to upper diagonal form.
8	The coefficients of the binary variables will need to be integrated into the co-variance matrix.
9	The constant for each binary variable will need to be integrated into the co-variance matrix.
10	The quadratic penalties for each constraint in the problem will need to be formulated.
11	The quadratic penalties will need to be incorporated into the co-variance matrix.
12	A simulated annealing algorithm will need to be coded in order to find optimal solutions to the QUBO problems.
13	The simulated annealing algorithm will need to accept an input of the number of variables in the QUBO model.
14	The simulated annealing algorithm will need to accept a co-variance matrix that has already been formed for the QUBO problem.
15	The simulated annealing algorithm must check that the input matrix has a width and height equal to the number of variables chosen.
16	Another algorithm could be written that takes the data of the optimal portfolio for a particular portfolio optimisation problem and plotted them on a graph.

B. Nonfunctional Requirements

ID	Nonfunctional Requirement
1	Multiple different portfolio optimisation problems need to be tested.
2	The simulated annealing algorithm must be tested with published data to make sure it works correctly
3	The time taken for the algorithm to find an optimal solution to the portfolio optimisation problem needs to be reasonable.
4	The number of assets in the portfolio optimisation problem must be kept low to reduce computation time.
5	The portfolios found need to be as optimal as possible .

V. Evaluation

ID	Evaluation Criteria
1	If known portfolio optimisation problems are used then the Sharpe ratio of the optimal portfolio can be compared to the Sharpe ratio of the optimal portfolio I find is. If the portfolio I find has a Sharpe ratio close to the known Sharpe ratio then this can be classed as a success.
2	Testing known quadratic portfolio optimisation problems and formulating them as quadratic portfolio optimisation problems myself. If the outcomes are the same the this will be a success.
3	Testing known QUBO portfolio optimisation problems and formulating them as QUBO portfolio optimisation problems myself. If the outcomes are the same the this will be a success.
4	If any optimal portfolio is found in a reasonable time then this criteria will be successful
5	If the algorithm drawing the graph actually draws a graph resembling the efficient frontier then that will be a success.
6	If the algorithm reading in the QUBO checks validates the size of the matrix correctly before running then that will be a success.

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