**Introduction to the BPR-x**

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Link travel time performance functions (or volume-delay functions) have been widely used in static traffic assignment (STA) for transport planning. Among a variety of link performance functions, the BPR function, created by the US Bureau of Public Roads in 1964, is recognized as an analytical building block for system-wide performance evaluation. Focusing the relationship of traffic volume and delay, the BPR function in its polynomial form is computationally efficient and simple for implementation in the transport planning software. On the other hand, the planning community has long recognized that the static BPR function cannot capture traffic flow dynamics and queue evolution processes, particularly related to the queue formation, propagation, and dissipation. Besides, it has difficulties in using the average travel time measure to describe an oversaturated bottleneck with high density but low throughput. Compared to the STA model, the dynamic traffic assignment (DTA) model aims to embed a queueing model or other types of dynamic traffic flow models to capture the evolution processes of traffic congestion. With discretized time and space dimensions (such as the cell transmission model and link transmission model), DTA models have to address many computational challenges due to the introduced finer resolution.

Many practitioners have the following interesting questions: 1) where are those coefficients of alpha and beta in the BPR function coming from? 2) how to use observed traffic dynamics data to calibrate these coefficients? Aiming to help transportation planners, this project will develop a new generation of travel time performance model, namely BPR-x functions, for system-wide performance evaluation. Based on continuous and polynomial-based approximation, the proposed model explicitly establishes a coherent connection between the average performance relationship and the deterministic dynamic queuing model during a single oversaturated period. The derivative process for the BPR-x is very simple: 1) assume the inflow rate function follows a polynomial form and determine the order of the polynomial inflow rate function; 2) establish the equivalent factor form of the polynomial inflow rate function; 3) calculate the time-dependent queue length based on cumulative arrival and departure curves; 4) calculate the total delay and average delay with the integral of queue length function; 5) obtain the analytical form of the average travel time function during the oversaturated period. Table 1 summarizes the graphical illustration and analytical formulation of BPR-x, including the constant form, linear form, quadratic form, and cubic form for the inflow rate function.

The proposed BPR-x function has four advantages compared with existing models. Firstly, it has a similar structure with BPR function, but with more intuitional meanings for the coefficients (including the coefficient of the highest order term in the polynomial inflow rate function, the capacity or practical discharge rate, and the peak duration) compared to the original BPR function. With the assumption of constant capacity, the peak duration (i.e., degree of peakness) equals to the peak period demand over capacity, thus describing how inflow coverage and capacity parameters influence the average travel time. Secondly, it establishes the equivalence between the analytical BPR average performance functional form and the corresponding queue evolution form. Thirdly, a new type of calibration process from the point queue bottleneck perspective is proposed, and it is very simple to fully utilize the sensor data (e.g., flow, speed, density) to calibrate the coefficients based on the three-detector model. Lastly, with well-posed assumptions, the BPR-x function can be smoothly transferred for a simplified approximation of the full-scale DTA model. It also has broad applications in the transport system management (e.g., dynamic signal control, traffic system evaluation and optimization, bus and metro demand management), as long as we can obtain the system’s cumulative arrival and departure curves.

The BPR-x function opens a window for the performance evaluation of a single oversaturated freeway bottleneck. However, it still has many challenges in the future research. As the freeway has different facility types with merge and diverge ramps, so how to model the BPR-x with merging and diverging is still an open question. To extend the BPR-x from freeway to atrial streets, one of the inevitable problems is how to consider the travel time functions with oversaturated signals and how to optimize the signal control with BPR-x. Many theoretical issues also arise in the BPR-x. Which order should we choose to best represent the traffic system performance? The higher order may have a better fitness, but the structure may be tedious and result in a slow convergence for equilibrium traffic assignment; so how to make a tradeoff between the fitness and parsimonious structure is worthy of deep consideration. In the proposed BPR-x functions, the derivative process is based on point queue model. It is widely recognized that the point queue model cannot capture the spatial dynamics of traffic flows, so how to establish the analytical form of BPR-x from point queue model to spatial queue and kinematic wave models should be investigated. In addition, when multi-bottlenecks are considered, the impact of queue spillbacks should be studied seriously.

Table 1: Graphical illustration and analytical formulation of queue evolution

|  |  |  |
| --- | --- | --- |
| **Graphical illustration of queue evolution** | **Analytical formulation** | |
| (1) Constant form for inflow rates | Inflow rate |  |
|  | Queue length |  |
|  | Time-dependent delay |  |
|  | Average delay |  |
|  | Average travel time |  |
| (2) Linear form for inflow rate | Inflow rate |  |
|  | Queue length |  |
|  | Time-dependent delay |  |
|  | Average delay |  |
|  | Average travel time |  |
| (3) Quadratic form for inflow rate | Inflow rate |  |
|  | Queue length |  |
|  | Time-dependent delay |  |
|  | Average delay |  |
|  | Average travel time |  |
| (4) Cubic form for inflow rate | Inflow rate |  |
|  | Queue length |  |
|  | Time-dependent delay |  |
|  | Average delay |  |
|  | Average travel time |  |

|  |  |
| --- | --- |
| Symbols | Definitions |
|  | start time of congestion period |
|  | time index with maximum inflow rate |
|  | time index with maximum queue length |
|  | end time of congestion period |
|  | capacity (or discharge rate), assumed to be a constant value |
|  | total demand during the whole peak period |
|  | free flow travel time |
|  | parameters for the constant form of inflow rates |
|  | parameter for the linear form of inflow rates |
|  | parameter for the quadratic form of inflow rates |
|  | parameter for the cubic form of inflow rates |
|  | inflow rate function at time *t* |
|  | queue length at time *t* |
|  | traffic delay departing at time *t* |
|  | average delay during the whole peak period |
|  | average travel time during the whole peak period |

Assumption 1: The inflow rate follows a polynomial form.

Assumption 2: The capacity (or discharge rate) is a constant.



Figure 1: General graphical illustration of queue evolution for a single oversaturated bottleneck

**Newell’s method:** **Quadratic function for inflow rate**

Let  be the time with maximum inflow rate as shown in Figure 1(*b*). Newell assumes that the inflow rate near time  can be approximated by a quadratic function shown as follow (Newell, 1982):



It is obvious that , and further let , then Eq. can be transformed to



The discharge rate , where , can be estimated in terms of Eq. :



Then we can obtain two real roots of  and  as follows:





Now we can write  by a factored form:



It is obvious that  in Eq. passes two points  and , and the second derivative with respect to  is , which is consist with Eq. .

The queue length at time  equals , which can be obtained as follow:



After substituting Eq. into Eq. , we can obtain the queue length at time  in terms of ,  and :



Set , then Eq. can be calculated as follows:



The maximum queue length at time  is:



The queue will dissipate at time , i.e., , then we can obtain  as follow:



Therefore, Eq. can also be written as follow:



The total delay between time  and  can also be calculated by the area between  and  in Figure 1(*a*), which can be obtained by integration of Eq. as follow:



Above is the introduction of Newell’s method based on the assumption of quadratic form of inflow rate. Based on the total delay in Eq. , we can further obtain the average delay during the congestion period  to  as follow:



Denote the total demand during the peak period  to  as  (i.e. ), and denote the peak period as  (i.e. ), then the discharge rate (or capacity) can be represented by  and :



Substituting Eq. into Eq. , we can obtain the average delay between  and  as follow:



Discussion 1: Take a bit time before  into consideration, i.e., the analyzed period considers the off-peak time, and denote the total period and the peak period by  and , respectively. Denote the total demand by . Let , then . Therefore, we can obtain the following average delay and average travel time function:





In the next section, we will derive the total system delay based on the assumption of cubic form of inflow rate and show the superiority of the cubic model.

**Cubic form**

It is assumed that  is third-order differentiable at time , then  can be approximated in terms of the Taylor series expansion as follow:



Similarly, because , and further let  and , then Eq. can be transformed to



It is known that  passes through two control points, i.e.,  and , and we assume the third point passed through by  is , then we can rewrite Eq. with a factored form:



Now we can calculate the queue length as follow:



As for the inflow rate function , it should satisfy two strong constraints, i.e.,  when  and  when , and one weak constraint that the change of inflow rate at time  should not be very large, which means that the queue formation is not immediate. With the above constraints, it only has two cases of the type for the inflow rate function shown in Figure 2.



Figure 2: Two possible cases for the type of inflow rate function

Assumption 1: The derivative of inflow rate at time  is zero, which means that it has two equal roots for Eq. , i.e. .

Based on this assumption, we can further derive the queue length function as follow:



The maximum queue length at time  is:



The queue will dissipate at time , i.e., , then we can obtain  as follow:



Therefore, Eq. can also be written as follow:



The total delay between time  and  can be obtained as follow:



Based on the total delay in Eq. , we can further obtain the average delay and the travel time functions during the congestion period  to  as follow:





In order to have a concise form, we define  as follow:



Then Eq. has a concise form which is very similar to the BPR function:



Based on the queue length function defined in Eq. , we can obtain the time-dependent delay and time-dependent travel time departing at time  during time  and time  as follows:





Discussion 2: Take a bit time before  into consideration, i.e., the analyzed period considers the off-peak time, and denote the total period and the peak period by  and , respectively. Denote the total demand by . Let , then . Therefore, we can obtain the following average delay and travel time functions during the whole period:





Denote  as follow:



and replace the discharge rate  by the capacity , then Eq. has a concise form as follow:

