Fundamental Concepts

Introduction of Variables and Expressions

We know that there are infinitely many numbers. However, there are only 26 alphabets in the English language.

Can we represent numbers with the help of letters?
Yes, we can represent the numbers with the help of alphabets(letters).
Alphabets like *a*, *b*, *c*, *x*, *y*, *z*, *l*, *m*, *n*, etc. are used in Mathematics to denote variables.

Let us take an example.
In a class, the number of boys are 20 more than the number of girls. How many boys are there in the class?

The number of boys in the class varies with the number of girls.
If there are 10 girls in the class, then the number of boys = 10 + 20 = 30

If there are 55 girls in the class, then the number of boys = 55 + 20 = 75
Therefore, we can write a rule to find the number of boys as follows:

Number of boys = Number of girls + 20

Here, number of girls in the class can vary, and can take different values. On the other hand, number of boys in the class varies according to the number of girls in the class.

If we replace the number of boys and girls in the class with letters m and n respectively, then we get following expression:
$m = n + 20$

Here, n can take any value such as 0, 1, 2, 3... etc., and the value of m will change accordingly. Since the values of m and n can vary, these are known as variables.

A variable is something that does not have a fixed value. The value of a variable varies.

Also, a symbol with a fixed numeric value is known as a constant.
For example, 2, –4, $\sqrt{8}$, –3.4, $\frac{1}{2}$ etc., are constants as each of them have a fixed numeric value.

A combination of variables, numbers, and operators (+, –, ×, and ÷) is known as an algebraic expression.

For example,

1. $x + 7$
2. $2 – y$
3. $(5 \times y) + 9$
4. $11xyz + ab^2 – 2p^4q^3 + \frac{3}{4}$
5. $\frac{a^2b}{cd} + 2p^5 – 1.5z$

Let us try to form few simple mathematical expressions by applying the four operations on numbers.

1. 42 is added to 58 and then the result is divided by 25 = $\frac{58+42}{25}$
2. The product of 24 and 15 is subtracted from the product of 30 and 43 = $(30 \times 43) – (24 \times 15)$

In the same way, we can form algebraic expressions by applying the four operations on variables.

1. z divided by 5 and 5 added to the result = $\frac{z}{5}+5$
2. x multiplied by 6 and 4 added to the product = $6x + 4$
3. 39 subtracted from $5m$ = $5m – 39$
4. 5 added to $6m$ and the result is subtracted from $8n$ = $8n – (6m + 5)$
5. 16 subtracted from $7x$ and the result is subtracted from $–2y$ = $–2y – (7x – 16)$

In this way, we can represent a given real-life situation by using variables.
Let us now solve some more examples to understand the concept better.

Example 1:
Write down the following expressions in words.

1. $5x + 19$
2. $6y – 2$
3. $\frac{z}{2}+2y$
4. $(2y + 5) – 8$
5. $6 – (m + 7)$

6. $\frac{2x-5}{5}$

Solution:
(i) *x* is multiplied with 5 and then 19 is added to the product.
(ii) *y* is multiplied by 6 and then 2 is subtracted from the product.
(iii) *z* is divided by 2 and then 2*y* is added to the result.
(**iv)** 5 is added to 2*y* and 8 is subtracted from the result.
Or, 5 is added to the product of 2 and *y* and then 8 is subtracted from the result.
(v) 7 is added to *m* and the result is subtracted from 6.
(vi) 5 is subtracted from the product of 2 and *x* and then the result is divided by 5.

Example 2:
Form six expressions using two numbers 8 and 11 and variable *a.*

Solution:

1. $8a + 11$
2. $11a + 8$
3. $8a - 11$
4. $\frac{a}{8}+11$
5. $\frac{a}{11}+8$
6. $\frac{a}{11}-8$

Example 3:
Write down the expressions for the following situations.

1. **Diganta's age is 2 years more than 4 times Arjun's age.**
2. **What is the length of a rectangular field, if its length is 3 m less than twice its breadth?**
3. **The number of boys in a class is 8 less than 3 times the number of girls. Find the total number of students in the class.**
4. **Sonu is two times taller than Monu. Find the height of Sonu.**
5. **What will be the age of Aman after 14 years from now?**

Solution:

1. Let Arjun's age be *z* years. Therefore, four times Arjun's age is 4*z*.

   ∴ Diganta's age = $4z + 2$

2. Let the breadth of the rectangular field be *a* m.

   ∴ Length of the rectangular field = (2*a* – 3) m

3. Let the number of girls be *p.*

   ∴ Number of boys = 3*p* – 8

   Thus, total number of students in the class $= p + 3p - 8 = 4p - 8$

4. Let the height of Monu be *h* cm.

   ∴ The height of Sonu will be 2*h* cm.

4. Let the present age of Aman be *y* years.

   ∴ The age of Aman after 14 years will be *y* + 14 years.

Example 4:
Sonu is twice as old as Monu. Find the rule to find Sonu's age if Monu's age is taken as *x*.

Solution:
It is given that Sonu is twice as old as Monu.
The rule can be written as:
Sonu's age = 2 × Monu's age = 2*x*, where *x* = 1, 2, 3, 4, 5 …

Example 5:
The price of Mohit's book is Rs 3 less than 3 times the price of Rohit's book. Find the rule to find the price of Mohit's book.

Solution:

Here, the price of Mohit's book is given in terms of the price of Rohit's book.
Let the price of Rohit's book be Rs *x*.

∴ Price of Mohit's book = Rs (3 × price of Rohit's book – 3)
= Rs (3*x* – 3)

Example 6:

Sachin has 5 apples more than Suhaan. Find the rule to find the number of apples with Suhaan.

Solution:
Sachin has 5 apples more than Suhaan. This means that Suhaan has 5 apples less than Sachin.

Let the number of apples with Sachin be *n*.

∴ Number of apples with Suhaan = Number of apples with Sachin – 5
= $n - 5$

Example 7:
The speed of a train is 70 km/h. Find a rule for the total distance covered by the train in *x* hours.

Solution:

Speed of the train = 70 km/h
Total time taken by the train to cover the distance = x h

∴ Total distance covered by the train = speed × total time taken
= 70 × total time taken

= $70 \times x$
= $70x$ km

Concept of Polynomials

Polynomials

Consider this situation involving trains.

The speed of an express train is ten less than twice that of a passenger train. If each travels for as many hours as its speed, then what is the difference between the distances travelled by them?


Let the speed of the passenger train be x km/hr.

Then, travelling time of the train = x hours

Distance travelled by it = Speed × Time = $x \times x = x^2$ km

Now, speed of the express train = $(2x - 10)$ km/hr

Its travelling time = $(2x - 10)$ hours

Distance travelled by it = $(2x - 10)(2x - 10) = (4x^2 - 40x + 100)$ km

Thus, required difference = $4x^2 - 40x + 100 - x^2 = (3x^2 - 40x + 100)$ km

The expression $3x^2 - 40x + 100$ is an example of a polynomial. Different real-life problems such as the one given above can be expressed in the form of polynomials. Go through this lesson to familiarize yourself with these useful expressions.

Topics to be covered in this lesson:

- Identifying polynomials
- Constant polynomials
- Classification of polynomials according to the number of terms

Did You Know?

Ancient Babylonians developed a unique system to calculate things using formulae. These formulae consisted of letters, mathematical operators (+, −, ×, ÷) and numbers. It was this system that led to the development of algebra. The word ‘algebra’ is derived from the Arabic word ‘al-jabr’ meaning ‘the reunion of broken parts’.

Another Arabian connection with algebra is the Arab mathematician Muhammad ibn Musa al-Khwarizmi, whose theories greatly influenced this branch of mathematics.

Solved Examples

Easy

Example 1:

Which of the following expressions are polynomials? Justify your answers.

i) $2x^{1/2} + 3x + 4$

ii) $8x^3 + 7 + x$

iii) $2x^3 - \sqrt{2}y^5$

iv) $\sqrt{121}x - \frac{2}{3}y^2 + x^3$

v) $5\left(\sqrt{x}\right)^2 + 9x^2$

vi) $\frac{14}{x^2} - 9x^2$

Solution:

i) $2x^{1/2} + 3x + 4$

This expression is not a polynomial because the exponent of the first term is 1/2, which is not a whole number.

ii) $8x^3 + 7 + x$

This expression is a polynomial as all the coefficients are real numbers and the exponents of the variable x are whole numbers.

iii) $2x^3 - \sqrt{2}y^5$

This expression is a polynomial as all the coefficients are real numbers and the exponents of the variables x and y are whole numbers.

iv) $\sqrt{121}x - \frac{2}{3}y^2 + x^3 = 11x - \frac{2}{3}y^2 + x^3$

This expression is a polynomial as all the coefficients are real numbers and the exponents of the variables x and y are whole numbers.

v) $5\left(\sqrt{x}\right)^2 + 9x^2$

This expression is a polynomial because it can be written as $5x + 9x^2$, in which all the coefficients are real numbers and the exponents of the variable x are whole numbers.

vi) $\frac{14}{x^2} - 9x^2$

This expression is not a polynomial because it can be written as $14x^{-2} - 9x^2$, in which the variable in the first term has a negative exponent.

Example 2:

For each of the given polynomials, state whether it is a monomial, binomial or trinomial.

i) $4x^3$

ii) $13y^5 - y$

iii) $29t^3 + 14t - 9$

iv) $x - x^2$

Solution:

i) $4x^3$ is a monomial as it has only one term.

ii) $13y^5 - y$ is a binomial as it has two terms.

iii) $29t^3 + 14t - 9$ is a trinomial as it has three terms.

iv) $x - x^2$ is a binomial as it has two terms.

Medium

Example 1:

For each of the given polynomials, state whether it is a monomial, binomial or trinomial.

i) $(4x^3 + 3y) - (3x^3 + x^2) + (y - x^3)$

ii) $(t^2 + 1)^2$

iii) $[(m - 1)(m + 1)] + 2m^2 + 1$

Solution:

i) $(4x^3 + 3y) - (3x^3 + x^2) + (y - x^3)$

$= 4x^3 + 3y - 3x^3 - x^2 + y - x^3$

$= (4x^3 - 3x^3 - x^3) - x^2 + (3y + y)$

$= - x^2 + 4y$

The polynomial can be reduced to $-x^2 + 4y$, which has two terms; so, it is a binomial.

ii) $(t^2 + 1)^2$

= $(t^2)^2 + 2t^2 + 1^2$ [$\because (a + b)^2 = a^2 + 2ab + b^2$]

= $t^4 + 2t^2 + 1$ [$\because (a^m)^n = a^{m \times n}$]

The polynomial can be reduced to $t^4 + 2t^2 + 1$, which has three terms; so, it is a trinomial.

iii) $[(m - 1)(m + 1)] + 2m^2 + 1$

$= (m^2 - 1) + 2m^2 + 1$ [$\because (a - b)(a + b) = a^2 - b^2$]

$= m^2 + 2m^2 - 1 + 1$

$= 3m^2$

The polynomial can be reduced to $3m^2$, which has only one term; so, it is a monomial.

Hard

Example 1:

State whether or not the following expressions are polynomials. Justify your answers.

i) $\frac{x^3}{xy^{-2}} + \frac{xy}{10} - \frac{1}{7} - \frac{\sqrt{2}y^2}{y^{-1}}$

ii) $\frac{y^2}{3} + \frac{14x^{-4}}{x^2} - 5xy + \frac{x}{2}$

Solution:

i) $\frac{x^3}{xy^{-2}}+\frac{xy}{10}-\frac{1}{7}-\frac{\sqrt{2}y^2}{y^{-1}}$

$$=x^3(x^{-1}y^2)+\frac{1}{10}xy-\frac{1}{7}-\sqrt{2}y^2(y)$$
$$=x^{3-1}y^2+\frac{1}{10}xy-\frac{1}{7}-\sqrt{2}y^{2+1}$$
$$=x^2y^2+\frac{1}{10}xy-\frac{1}{7}-\sqrt{2}y^3$$

In the reduced form of the given expression, all the coefficients are real numbers and the exponents of the variables *x* and *y* are whole numbers. Hence, the given expression is a polynomial.

ii) $\frac{y^2}{3}+\frac{14x^{-4}}{x^2}-5xy+\frac{x}{2}$

$$=\frac{1}{3}y^2+14x^{-4}(x^{-2})-5xy+\frac{1}{2}x$$
$$=\frac{1}{3}y^2+14x^{-4-2}-5xy+\frac{1}{2}x$$
$$=\frac{1}{3}y^2+14x^{-6}-5xy+\frac{1}{2}x$$

In the reduced form of the given expression, the exponent of the second term (i.e., −6) is not a whole number. Hence, the given expression is not a polynomial.

Generalized Form of a Polynomial

Did You Know?

The word ‘polynomial’ is a combination of the Greek words ‘poly’ meaning ‘many’ and ‘nomos’ meaning ‘part or portion’. Thus, a polynomial is an algebraic expression having many parts.

Solved Examples

Easy

Example 1:

Find the coefficient of *x* in the following polynomials.

i) $\frac{\pi}{2}x^3 - 3x^2$

ii)(2x^2 − x) + 7 + 3x

Solution:

i) $\frac{\pi}{2}x^3 - 3x^2$

This expression can also be written as $\frac{\pi}{2}x^3 - 3x^2 + 0.x$.

Thus, in the given polynomial, the coefficient of x is 0.

ii)(2x^2 − x) + 7 + 3x

= 2x^2 − x + 7 + 3x

= 2x^2 + 2x + 7

Thus, in the given polynomial, the coefficient of x is 2.

Medium

Example 1:

For each of the following polynomials, write the constant term and the coefficient of the variable having the highest exponent.

i) $-\left(x^4\right)^{\frac{1}{2}}$ **− 3** $\left(\sqrt{x^3}\right)^2$ **+ (x + 2) − 4**

ii)3 (y + 2)^2 − 5y (y^2 + 2y2y**) + y^3**

Solution:

i) $-\left(x^4\right)^{\frac{1}{2}}$ − 3 $\left(\sqrt{x^3}\right)^2$ + (x + 2) − 4

= $-x^{4\times\frac{1}{2}}$ − 3x^3 + (x + 2) − 4 [∵ $(a^m)^n = a^{m \times n}$ and $\left(\sqrt{a}\right)^2 = a$]

$= -x^2 - 3x^3 + x - 2$

$= -3x^3 - x^2 + x - 2$

In this reduced form of the given polynomial, we have:

Constant term = -2

Term with the highest exponent = $-3x^3$

So, coefficient of the variable having the highest exponent = -3

ii) $3(y + 2)^2 - 5y(y^2 + 2/y) + y^3$

$= 3(y^2 + 4y + 4) - 5y^3 - 10 + y^3$

$= 3y^2 + 12y + 12 - 4y^3 - 10$

$= -4y^3 + 3y^2 + 12y + 2$

In this reduced form of the given polynomial, we have:

Constant term = 2

Term with the highest exponent = $-4y^3$

So, coefficient of the variable having the highest exponent = -4

Hard

Example 1:

Find the coefficients of x^3, x^2, x, y^3, y^2 and y in the following polynomials. Also, find the real constants.

i) $17x^4 - 3y - x(2x^3 + x) + 3(2y - 4x^2 + 1) - 4$

ii) $y^4\left(\frac{3}{2y} - 2\right) - \frac{1}{2}(6x^2 + y + 1) + y - 4x^2$

Solution:

i) $17x^4 - 3y - x(2x^3 + x) + 3(2y - 4x^2 + 1) - 4$

$= 17x^4 - 3y - 2x^4 - x^2 + 6y - 12x^2 + 3 - 4$

$= 15x^4 - 13x^2 + 3y - 1$

This reduced form of the given polynomial can be further written as:

$15x^4 + 0.x^3 - 13x^2 + 0.x + 0.y^3 + 0.y^2 + 3y - 1$

Therefore, we have:

Coefficient of $x^3 = 0$ Coefficient of $x^2 = -13$ Coefficient of $x = 0$

Coefficient of $y^3 = 0$ Coefficient of $y^2 = 0$ Coefficient of $y = 3$

Real constant = −1

ii) $y^4\left(\frac{3}{2y} - 2\right) - \frac{1}{2}(6x^2 + y + 1) + y - 4x^2$

$$= \frac{3}{2y} \times y^4 - 2y^4 - \frac{1}{2} \times 6x^2 - \frac{1}{2} \times y - \frac{1}{2} + y - 4x^2$$
$$= \frac{3}{2}y^3 - 2y^4 - 3x^2 - \frac{1}{2}y + y - 4x^2 - \frac{1}{2}$$
$$= \frac{3}{2}y^3 - 2y^4 - 7x^2 + \frac{1}{2}y - \frac{1}{2}$$
$$= -7x^2 - 2y^4 + \frac{3}{2}y^3 + \frac{1}{2}y - \frac{1}{2}$$

This reduced form of the given polynomial can be further written as:

$$0.x^3 - 7x^2 + 0.x - 2y^4 + \frac{3}{2}y^3 + 0.y^2 + \frac{1}{2}y - \frac{1}{2}$$

Therefore, we have:

Coefficient of $x^3 = 0$ Coefficient of $x^2 = -7$ Coefficient of $x = 0$

Coefficient of $y^3 = \frac{3}{2}$ Coefficient of $y^2 = 0$ Coefficient of $y = \frac{1}{2}$

Real constant = $-\frac{1}{2}$

Different Forms of a Polynomial

A polynomial can found and written in different forms. These forms are explained below.

Standard form: If the terms of a polynomial are written in descending order or ascending order of the powers of the variables then the polynomial is said to be in the standard form.

For example, the polynomial $3x + 15x^4 - 1 - 13x^2$ is not in the standard form. It can be written in the standard form as $15x^4 - 13x^2 + 3x - 1$ or $-1 + 3x - 13x^2 + 15x^4$.

Index form: Observe the polynomial $x^6 - 2x^4 - 10x^3 + 5$. In this polynomial, terms having x^5, x^2 and x are missing. These terms can be added to the polynomial with coefficient 0. Thus, the obtained polynomial will be $x^6 + 0x^5 - 2x^4 - 10x^3 + 0x^2 + 0x + 5$.

The polynomial obtained on adding the missing terms is said to be in the index form.

Coefficient form: When the coefficients of all the terms of a polynomial are written in a bracket by separating with comma then the polynomial is said to be written in the coefficient form.

It should be noted that if a term is missing then its coefficient is taken as 0. So, it is better to write the given polynomial in the index form before writing it in the coefficient form.

For example, to write the polynomial $x^6 - 2x^4 - 10x^3 + 5$ in the coefficient form, we will first write it in the index form as $x^6 + 0x^5 - 2x^4 - 10x^3 + 0x^2 + 0x + 5$.

Now, it can be written in the coefficient form as $(1, 0, -2, -10, 0, 0, 5)$.

Solved Examples

Easy

Example 1:

Express the given polynomials in the standard form.

(i) $-2y^3 + 5y^5 - 2 + y$

(ii) $11a - a^6 - 2a^3 + a^2 - 1$

Solution:

We know that if the terms of a polynomial are written in descending order or ascending order of the powers of the variables then the polynomial is said to be in the standard form.

Given polynomials can be written in the standard form as follows:

(i) **Given form:** $-2y^3 + 5y^5 - 2 + y$

Standard form: $5y^5 - 2y^3 + y - 2$ or $-2 + y - 2y^3 + 5y^5$

(ii) **Given form:** $11a - a^6 - 2a^3 + a^2 - 1$

Standard form: $-a^6 - 2a^3 + a^2 + 11a - 1$ or $-1 + 11a + a^2 - 2a^3 - a^6$

Example 2:

Express the given polynomials in the index form and coefficient form.

(i) $2m^7 + 12m^5 - 7m^2 - m$

(ii) $-4p^6 + 3p^3 - 2p + 7$

Solution:

We know that the polynomial obtained on adding the missing terms is said to be in the index form.

Also, when the coefficients of all the terms of a polynomial are written in a bracket by separating with comma then the polynomial is said to be written in the coefficient form.

Given polynomials can be written in the index form and coefficient form as follows:

(i) **Given form:** $2m^7 + 12m^5 - 7m^2 - m$

Index form: $2m^7 + 0m^6 + 12m^5 + 0m^4 + 0m^3 - 7m^2 - m$

Coefficient form: (2, 0, 12, 0, 0, −7, −1)

(ii) **Given form:** $-4p^6 + 3p^3 - 2p + 7$

Index form: $-4p^6 + 0p^5 + 0p^4 + 3p^3 + 0p^2 - 2p + 7$

Coefficient form: (−4, 0, 0, 3, 0, −2, 7)

Example 3:

Express the given coefficient forms in the index forms by taking x as the variable.

(i) (10, 0, −2, 1, 0, 0, 7)

(ii) (−1, 2, 0, 3, 0, 6)

Solution:

Given polynomials can be written in the index form as follows:

(i) **Given form:** (10, 0, −2, 1, 0, 0, 7)

Index form: $10x^6 + 0x^5 - 2x^4 + x^3 + 0x^2 + 0x + 7$

(ii) **Given form:** (−1, 2, 0, 3, 0, 6)

Index form: $-x^5 + 2x^4 + 0x^3 + 3x^2 + 0x + 6$

Degree of Polynomial

More about Polynomials

We know that a polynomial comprises a number of terms, which may have variables or numbers or both. Also, each term can be represented with a variable having some **exponent**. Exponents of the variables in a given polynomial can be the same or different.

Let us consider a polynomial $2x^5 + 4x^2 + 9$.

The terms of this polynomial and their exponents are as follows:

First term = $2x^5$; exponent in the first term = 5

Second term = $4x^2$; exponent in the second term = 2

Third term = 9 = $9x^0$; exponent in the third term = 0

Note that all the exponents in the above polynomial are different. These exponents help us to identify the degrees of polynomials. Polynomials are categorized based on their degrees.

In this lesson, we will learn about the degrees of polynomials and the classification of polynomials based on the same.

The Degree of a Polynomial

Whiz Kid

When a polynomial has an equals sign (=), then it becomes an equation. The maximum number of solutions of an equation is less than or equal to the degree of that equation.

Solved Examples

Easy

Example 1:

Find the degree of each term of the polynomial $3x^6 + 3x^4 - 6x + 3$. Also find the degree of the polynomial.

Solution:

The degree of the term $3x^6$ is 6.

The degree of the term $3x^4$ is 4.

The degree of the term $-6x$ is 1.

The degree of the term 3 is 0.

Here, the highest degree is 6. Hence, the degree of the polynomial is 6.

Medium

Example 1:

Write the degree of each of the following polynomials.

i) $\frac{x^2}{2x} - 9x^7 + \frac{1}{x^{-4}} + 7$

ii) $\frac{x^2}{3} + \frac{4x^{-1}}{x^{-2}} - 5x^2 + \frac{x}{2} - 9$

Solution:

i) $\frac{x^2}{2x} - 9x^7 + \frac{1}{x^{-4}} + 7$

$$= \frac{x^{2-1}}{2} - 9x^7 + x^4 + 7 \qquad \left(\because \frac{a^m}{a^n} = a^{m-n}, \text{where } m > n;\ \frac{1}{a^{-m}} = a^m \right)$$

$$= \frac{x}{2} - 9x^7 + x^4 + 7$$

In the given polynomial, the highest degree is 7. Hence, the degree of the polynomial is 7.

ii) $\frac{x^2}{3} + \frac{4x^{-1}}{x^{-2}} - 5x^2 + \frac{x}{2} - 9$

$$= \frac{x^2}{3} + 4x^{2-1} - 5x^2 + \frac{x}{2} - 9 \qquad \left(\because \frac{a^m}{a^n} = a^{n-m}, \text{where } m < n \right)$$

$$= \frac{x^2}{3} + 4x - 5x^2 + \frac{x}{2} - 9$$

$$= -\frac{14x^2}{3} + \frac{9x}{2} - 9$$

In the given polynomial, the highest degree is 2. Hence, the degree of the polynomial is 2.

The Degree of a Polynomial in more than one Variable

In case of the polynomials in one variable, the degree of a polynomial is the highest exponent of the variable in the polynomial, but what about the degree of the polynomial in more than one variable?

In this case, the sum of the powers of all variables in each term is obtained and the highest sum among all is the degree of the polynomial.

For example, find the degree of the polynomial $2xy + 3y^2z + 4x^2yz^2 – xyz – 2x^3$. Let us find the sum of the powers of all variables in each term of this polynomial.

Sum of the powers of all variables in the term $2xy$ = 1 + 1 = 2

Sum of the powers of all variables in the term $3y^2z$ = 2 + 1 = 3

Sum of the powers of all variables in the term $4x^2yz^2$ = 2 + 1 + 2 = 5

Sum of the powers of all variables in the term $–xyz$ = 1 + 1 + 1 = 3

Sum of the powers of all variables in the term $–2x^3$ = 3

Among all the sums, 5 is the highest and thus, the degree of the polynomial $2xy + 3y^2z + 4x^2yz^2 - xyz - 2x^3$ is 5.

Similarly, we can find the degree of any polynomial in more than one variable.

Solved Examples

Easy

Example 1:

Write the degree of each of the following polynomials.

(i) $24a^2b - abc + 11abc^2 + ab^2c^3 - 7a^2b^2$

(ii) $5p^5 + 10p^2qr - 9p^2qr^2 - p^2r^2 + 2pq^2 - 2p^3q^2r$

Solution:

(i) Let us find the sum of the powers of all variables in each term of $24a^2b - abc + 11abc^2 + ab^2c^3 - 7a^2b^2$.

Sum of the powers of all variables in the term $24a^2b = 2 + 1 = 3$

Sum of the powers of all variables in the term $-abc = 1 + 1 + 1 = 3$

Sum of the powers of all variables in the term $11abc^2 = 1 + 1 + 2 = 4$

Sum of the powers of all variables in the term $ab^2c^3 = 1 + 2 + 3 = 6$

Sum of the powers of all variables in the term $-7a^2b^2 = 2 + 2 = 4$

Among all the sums, 6 is the highest and thus, the degree of the polynomial $24a^2b - abc + 11abc^2 + ab^2c^3 - 7a^2b^2$ is 6.

(ii) Let us find the sum of the powers of all variables in each term of $5p^5 + 10p^2qr - 9p^2qr^2 - p^2r^2 + 2pq^2 - 2p^3q^2r$.

Sum of the powers of all variables in the term $5p^5 = 5$

Sum of the powers of all variables in the term $10p^2qr = 2 + 1 + 1 = 4$

Sum of the powers of all variables in the term $-9p^2qr^2 = 2 + 1 + 2 = 5$

Sum of the powers of all variables in the term $-p^2r^2 = 2 + 2 = 4$

Sum of the powers of all variables in the term $2pq^2 = 1 + 2 = 3$

Sum of the powers of all variables in the term $-2p^3q^2r = 3 + 2 + 1 = 6$

Among all the sums, 6 is the highest and thus, the degree of the polynomial $5p^5 + 10p^2qr - 9p^2qr^2 - p^2r^2 + 2pq^2 - 2p^3q^2r$ is 6.

Classification of Polynomials According to Their Degrees

Whiz Kid

If all the terms in a polynomial have the same exponent, then the expression is referred to as a **homogenous polynomial**.

Did You Know?

The graphs of linear polynomials are always straight lines. This is why these polynomials are called 'linear' polynomials.

Solved Examples

Easy

Example 1:

Classify each of the given polynomials according to its degree.

i) $\mathbf{11x^3 + 7x + 3}$

ii) $\mathbf{8x^2 + 3x}$

iii) $\mathbf{x + 5}$

iv) $\mathbf{9t^3}$

Solution:

i) $\mathbf{11x^3 + 7x + 3}$

The degree of this polynomial is 3. Hence, it is a cubic polynomial.

ii) $\mathbf{8x^2 + 3x}$

The degree of this polynomial is 2. Hence, it is a quadratic polynomial.

iii) $\boldsymbol{x + 5}$

The degree of this polynomial is 1. Hence, it is a linear polynomial.

iv) $\boldsymbol{9t^3}$

The degree of this polynomial is 3. Hence, it is a cubic polynomial.

Example 2:

Give an example of each of the following polynomials.

i) A monomial of degree 50

ii) A binomial of degree 17

iii) A trinomial of degree 99

Solution:

i) A monomial of degree 50 means a polynomial having one term and 50 as the highest exponent. An example of such a polynomial is $23y^{50}$.

ii) A binomial of degree 17 means a polynomial having two terms and 17 as the highest exponent. An example of such a polynomial is $41t^{17} + 53t$.

iii) A trinomial of degree 99 means a polynomial having three terms and 99 as the highest exponent. An example of such a polynomial is $p^{99} + 5p - 12$.

Medium

Example 1:

Classify each of the given polynomials according to its degree.

i) $\frac{x^2}{3} + 4x^3 - (5x^2 + 4x^3) + \frac{x}{2} - 9$

ii) $\boldsymbol{x + 3x^2 + (x + 2)(x^2 + 4 - 2x) + 54}$

Solution:

i) $\frac{x^2}{3}+4x^3-(5x^2+4x^3)+\frac{x}{2}-9$

$$=\frac{x^2}{3}+4x^3-5x^2-4x^3+\frac{x}{2}-9$$
$$=\frac{x^2}{3}-5x^2+\frac{x}{2}-9$$
$$=-\frac{14x^2}{3}+\frac{x}{2}-9$$

The degree of this polynomial is 2. Hence, it is a quadratic polynomial.

ii) $x + 3x^2 + (x + 2)(x^2 + 4 - 2x) + 54$

$= x + 3x^2 + (x^3 + 2^3) + 54$ $[\because a^3 + b^3 = (a + b)(a^2 + b^2 - ab)]$

$= x + 3x^2 + x^3 + 8 + 54$

$= x + 3x^2 + x^3 + 62$

The degree of this polynomial is 3. Hence, it is a cubic polynomial.

Factors, Coefficients and Terms of Algebraic Expressions

Shiva and Somesh are brothers. Shiva's age is 3 years less than Somesh's age.

Now, how can we represent this situation with an algebraic expression?

Let us assume Somesh's age as x years. Therefore, Shiva's age = $(x - 3)$ years

Here, $(x - 3)$ is an algebraic expression that represents Shiva's age.

Here, we can notice one thing. The ages of both Somesh and Shiva can vary, but the difference between the ages, i.e. 3 years, is always constant. In this algebraic expression $(x - 3)$, x can vary but the number 3 does not. Hence, x is known as a **variable (or algebraic number)** and 3 is called a **constant (absolute term**).

Let us consider some algebraic expressions given below.

(i) $3x + 5$

(ii) $4x^2 - 21$

Here, the first expression ($3x + 5$) is formed by adding $3x$ and 5. In this case, $3x$ and 5 are called **algebraic terms or simply terms** of the expression. The terms are always added to form an algebraic expression. They are never subtracted to form an algebraic expression. However, an expression may have positive or negative terms.

In the expression ($3x - 5$), the terms of the expression are $3x$ and (-5), and not $3x$ and 5. Thus, we added the terms $3x$ and (-5) to get the expression ($3x - 5$).

In expression (ii), $4x^2$ and (-21) are added to form $4x^2 - 21$. Therefore, $4x^2$ and (-21) are terms of the expression $4x^2 - 21$.

Let us again consider the expression $3x + 5$. Here, the term $3x$ is a product of 3 and x. We cannot factorise 3 and x further. Hence, 3 and x are called **factors** of the term $3x$.

The term 5 cannot be expressed as the product of variables and constant. Therefore, 5 is itself a factor of 5.

In expression (ii), the term $4x^2$ can be written as

$4x^2 = 4 \times x^2$

$4x^2 = 4x \times x$

But x^2 and $4x$ cannot be the factors of $4x^2$ as they can be factorised further.

$x^2 = x \times x$ and $4x = 4\times x$

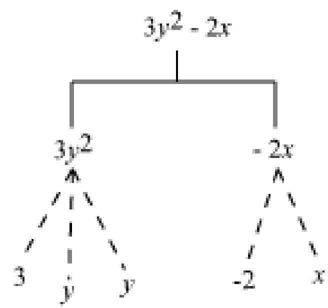
We can write $4x^2$ as the product of 4, x, and x as shown below.

$4x^2 = 4 \times x \times x$

Therefore, 4, x and x are called factors of $4x^2$.

We can also represent the factors and terms of an algebraic expression by a tree diagram.

Similarly, we can represent the tree diagram of an expression ($3y^2 - 2x$) as shown below.



In the expression $3y^2 - 2x$, we can see that the term $3y^2$ is the product of a numerical, i.e. 3, and other variables. This numerical, i.e. 3, is known as the **numerical coefficient** of the term, $3y^2$. Similarly, the coefficient of the term $-2x$ is -2. Generally, we define the numerical coefficient or efficient as

The numerical factor of a term is called the numerical coefficient (or constant coefficient) of the term.

Using this definition, we can say that the numerical coefficient of $-15xy$ is -15, since

$-15xy = -15 \times x \times y$

We can also write $-15xy$ as $-15xy = x \times (-15y) = y \times (-15x)$

Thus, we can say that the coefficient of x is $-15y$ and the coefficient of y is $-15x$.

Can we find the numerical coefficients of x in the expression $(x - 5)$ and that of xy in the expression $(7 - xy)$?

In case of $(x - 5)$, 1 is the numerical coefficient of x. In case of $(7 - xy)$, -1 is the numerical coefficient of xy.

Note: In the expression $-15xy$, xy is said to be the algebraic coefficient.

Thus, we can say that

If the coefficient of a term is 1, then it is not written before the term. If the coefficient of the term is −1, then only the '−' sign is put before the term.

Let us look at the factorization of the terms $5x^3yz^2$ and $-23x^3yz^2$.

$$5x^3yz^2 = 5\underbrace{x \times x \times x}_{3} \times \underbrace{y}_{1} \times \underbrace{z \times z}_{2}$$

$$-23x^3yz^2 = -23 \times \underbrace{x \times x \times x}_{3} \times \underbrace{y}_{1} \times \underbrace{z \times z}_{2}$$

Here, we can see that the two terms have different numerical factors 5 and −23, but same algebraic factors (each of these term contains the same variable, i.e. *x*, *y*, and *z*. Also, powers of these variables of each term are the same, i.e. power of *x*, *y*, and *z* are 3, 1, and 2 respectively). These terms are known as **like terms.** We can define them as

The terms having the same algebraic factors are called like terms. Like terms may have different numerical factors.

Let us consider the terms 6*xy* and 6*x*. Now, 6*xy* = 6 × *x* × *y* and 6*x* = 6 × *x*

Here, we can see that the two terms have the same numerical factor 6. Their algebraic factors *xy* and *x* are different. Such type of terms having different algebraic factors are said to be **unlike terms.** We define them as

The terms having different algebraic factors are called unlike terms.

Let us discuss some examples to understand these concepts better.

Example 1:

Find the terms in the algebraic expression $\left(-\frac{xy}{7}+14xy^2-3\right)$**.**

Solution:

The terms of the expression are $-\frac{xy}{7}, 14xy^2$, and −3.

Example 2:

Find the factors of $\left(-3x^2yz^3\right)$**.**

Solution:

$-3x^2yz^3 = -3 \times x \times x \times y \times z \times z \times z$

Therefore, the factors of $-3x^2yz^3$ are -3, x, x, y, z, z, and z.

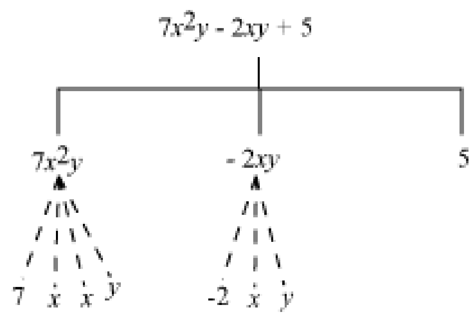
Example 3:

Represent the terms and factors of the algebraic expression $7x^2y - 2xy + 5$ through a

tree diagram.

Solution:

The tree diagram representation of the algebraic expression, $7x^2y - 2xy + 5$ is



Example 4:

Find the like terms in the algebraic expression $51x^2y - 21x^2y^2 + \frac{x^2y}{2} + 31 - 6xy - x^2y$.

Solution:

Here, the like terms are $51x^2y$, $\frac{x^2y}{2}$, and $-x^2y$.

Example 5:

Find the coefficients of *pq* in the following terms.

pq^2, $-3pq$, $15p^2q^2$, $-\frac{31}{5}p^2q$

Solution:

Terms	Coefficients of pq
pq^2	q
$-3pq$	-3
$15p^2q^2$	$15pq$
$-\frac{31}{5}p^2q$	$-\frac{31}{5}p$

Addition and Subtraction of Polynomials

The most important point to remember in this topic is as follows.

We can add and subtract like terms. In case of addition or subtraction of like terms, only their numerical coefficients are added or subtracted. The algebraic part of the terms remains as it is.

Adding and subtracting like monomials by arranging them vertically:

In this method, we have to write given like monomials one below other to perform the addition or subtraction. Also, write the coefficients for pure variable terms as 1. To add or subtract like monomials, we just need to add or subtract the coefficients and write the result with the variable.

For example, let us add monomials a^2, $-2a^2$ and $5a^2$. These monomials can be added by arranging vertically as follows:

$$\begin{array}{rr} & 1a^2 \\ + & -2a^2 \\ + & 5a^2 \\ \hline & 4a^2 \\ \hline \end{array}$$

Here, the sum of 1 and –2 is obtained as –1. Further, the sum of –1 and 5 is obtained as 4. So, we wrote 4 with a^2 and obtained the sum of given polynomials as $4a^2$.

Similarly, we can perform subtraction for the given monomials.

$$
\begin{array}{rr}
 & 1a^2 \\
- & -2a^2 \\
- & 5a^2 \\
\hline
 & -2a^2 \\
\hline
\end{array}
$$

When we subtracted –2 from 1, we got 3 [1 – (–2) = 3]. Further, on subtracting 5 from 3, we got –2 (3 – 5 = –2). So, we wrote –2 with a^2 and obtained the result as $–2a^2$.

Adding and subtracting polynomials by arranging them vertically:

To add the polynomials, we can arrange them vertically such that each term of lower polynomial is written below its like term in the upper polynomial. Also, write the coefficients for pure variable terms as 1.

Let us add the polynomials $–3x^2 + 4xy – z$ and $2xy + x^2 – 3z$ to learn the concept.
We can observe that $–3x^2$ and x^2 are like terms as they have same variable having same powers. Similarly, $4xy$ and $2xy$, and $– z$ and $–3z$ are other pairs of like terms.

These polynomials can be arranged vertically as follows:

$$
\begin{array}{rrrr}
 & -3x^2 & +4xy & -z \\
+ & 1x^2 & +2xy & -3z \\
\hline
 & & & \\
\hline
\end{array}
$$

Now, we just need to add the coefficients of like terms and write the variables as they are.

$$
\begin{array}{rrrr}
 & -3x^2 & +4xy & -z \\
+ & 1x^2 & +2xy & -3z \\
\hline
 & -2x^2 & +6xy & -4z \\
\hline
\end{array}
$$

Thus, the sum of the given polynomials is $–2x^2 + 6xy – 4z$.

Let us now study the concept of subtraction of polynomials by subtracting $2xy + x^2 – 3z$ from $–3x^2 + 4xy – z$.
Let us arrange them vertically first as we have done before.

$$
\begin{array}{rrrr}
 & -3x^2 & +4xy & -z \\
- & 1x^2 & +2xy & -3z \\
\hline
 & & & \\
\hline
\end{array}
$$

To subtract a polynomial from other, we add its opposite.
Now, we get

$$\begin{array}{r} -3x^2 + 4xy - z \\ + \quad -1x^2 - 2xy + 3z \\ \hline \\ \hline \end{array}$$

Now, we perform the addition as we have done before.

$$\begin{array}{r} -3x^2 + 4xy - z \\ + \quad -1x^2 - 2xy + 3z \\ \hline -4x^2 + 2xy + 2z \\ \hline \end{array}$$

Thus, the required difference is $-4x^2 + 2xy + 2z$.

Let us discuss some more examples to understand the concept better.

Example 1:

Add the following monomials by arranging them horizontally as well as vertically.

(a) $-3p^2$, $6p^2$ and $-11p^2$

(b) $8x^2y$, $-10x^2y$ and $-2x^2y$

Solution:

Addition by arranging horizontally:

(a) $-3p^2 + 6p^2 + (-11p^2) = (-3 + 6 - 11)p^2 = -8p^2$

(b) $8x^2y + (-10x^2y) + (-2x^2y) = (8 - 10 - 2)x^2y = -4x^2y$

Addition by arranging vertically:

(a)

$$\begin{array}{rr} & -3p^2 \\ + & 6p^2 \\ + & -11p^2 \\ \hline & -8p^2 \\ \hline \end{array}$$

(b)

$$\begin{array}{rr} & 8x^2y \\ + & -10x^2y \\ + & -2x^2y \\ \hline & -4x^2y \\ \hline \end{array}$$

Example 2:

Subtract the following monomials by arranging them horizontally as well as vertically.

(a) 25mn^2 from the sum of 14mn^2 and –mn^2

(b) (–x^2y^2 + 12x^2y^2) from 19x^2y^2

Solution:

Subtraction by arranging horizontally:

(a) $(14mn^2 - mn^2) - 25mn^2 = 13mn^2 - 25mn^2 = -12mn^2$

(b) $19x^2y^2 - (-x^2y^2 + 12x^2y^2) = 19x^2y^2 - 11x^2y^2 = 8x^2y^2$

Subtraction by arranging vertically:

(a)

$$\begin{array}{rr} & 14m^2n^2 \\ + & -1m^2n^2 \\ \hline & 13m^2n^2 \\ \hline \end{array} \qquad \begin{array}{rr} & 13m^2n^2 \\ - & 25m^2n^2 \\ \hline & -12m^2n^2 \\ \hline \end{array}$$

(b)

$$\begin{array}{rr} & -x^2y^2 \\ + & 12x^2y^2 \\ \hline & 11x^2y^2 \\ \hline \end{array} \qquad \begin{array}{rr} & 19x^2y^2 \\ - & 11x^2y^2 \\ \hline & 8x^2y^2 \\ \hline \end{array}$$

Example 3:

Add the expressions $5x^2 + 6xy - 11$, $7x^2y - 3y$, and $12x^2y - 3xy + 4$.

Solution:

$(5x^2 + 6xy - 11) + (7x^2y - 3y) + (12x^2y - 3xy + 4)$

$= 5x^2 + 6xy - 11 + 7x^2y - 3y + 12x^2y - 3xy + 4$

$= 5x^2 + 6xy - 3xy + 7x^2y + 12x^2y - 3y - 11 + 4$ (Rearranging the terms)

$=5x^2 + (6 - 3)xy + (7 + 12)x^2y - 3y + (-11 + 4)$

$= 5x^2 + 3xy + 19x^2y - 3y - 7$

Example 4:

Which expression when subtracted from the expression $(7x - 3y + 45xy + 7)$ gives

$(2x - 21y - 42xy)$?

Solution:

To get the required expression, we have to subtract $(2x - 21y - 42xy)$ from $(7x - 3y + 45xy + 7)$.

$(7x - 3y + 45xy + 7) - (2x - 21y - 42xy)$

$= 7x - 3y + 45xy + 7 - 2x + 21y + 42xy$

$= 7x - 2x - 3y + 21y + 45xy + 42xy + 7$

$= (7 - 2)x + (-3 + 21)y + (45 + 42)xy + 7$

$= 5x + 18y + 87xy + 7$

Example 5:

Subtract the sum of ($4y^2 - 6y$) and ($-2y^2 + 3y - 3$) from the sum of ($5y + 7$) and

($3y^2 - 9y + 2$).

Solution:

$(5y + 7) + (3y^2 - 9y + 2)$

$= 5y + 7 + 3y^2 - 9y + 2$

$= 3y^2 + 5y - 9y + 7 + 2$ [Rearranging the terms]

$= 3y^2 + (5y - 9y) + (7 + 2)$

$= 3y^2 + (5 - 9)\ y + 9$

$= 3y^2 + (-4)\ y + 9$

$= 3y^2 - 4y + 9$

$(4y^2 - 6y) + (-2y^2 + 3y - 3)$

$= 4y^2 - 6y - 2y^2 + 3y - 3$

$= 4y^2 - 2y^2 - 6y + 3y - 3$ [Rearranging the terms]

$= (4y^2 - 2y^2) + (-6y + 3y) - 3$

$= (4 - 2)\ y^2 + (-6 + 3)\ y - 3$

$= 2y^2 - 3y - 3$

Now, subtracting the sum of ($4y^2 - 6y$) and ($-2y^2 + 3y - 3$) from the sum of ($5y + 7$)and ($3y^2 - 9y + 2$) is the same as subtracting ($2y^2 - 3y - 3$) from ($3y^2 - 4y + 9$).

This can be done as

$(3y^2 - 4y + 9) - (2y^2 - 3y - 3)$

$= 3y^2 - 4y + 9 - 2y^2 + 3y + 3$

$= 3y^2 - 2y^2 - 4y + 3y + 9 + 3$ [Rearranging the terms]

$= (3 - 2)\, y^2 + (-4 + 3)\, y + (9 + 3)$

$= y^2 - y + 12$

Addition and Subtraction of Algebraic Expressions

Addition and subtraction are very important concepts in mathematics. In case of algebraic expressions, we add and subtract like terms. Like terms have same variables and same powers.

Let us add the expressions $-2x^2 + 5xy - z$ and $3xy + x^2 - 2z$ to learn the concept.

It can be observed that $-2x^2$ and x^2 are like terms as they have same variables that have same powers. Similarly, $5xy$ and $3xy$ and $-z$ and $-2z$ are other pairs of like terms.

For adding, these expressions can be arranged vertically such that each term of lower expression is written below its like term in the upper expression. The coefficients for pure variable terms can be written as 1.

This can be done in the following manner:

$$\begin{array}{r} -2x^2 + 5xy - 1z \\ +\quad 1x^2 + 3xy - 2z \\ \hline \\ \hline \end{array}$$

Now, the coefficients of like terms are added and the variables are written without making any changes.

$$\begin{array}{r} -2x^2 + 5xy - 1z \\ +\quad 1x^2 + 3xy - 2z \\ \hline -1x^2 + 8xy - 3z \\ \hline \end{array}$$

Thus, the sum of the given expressions is $-1x^2 + 8xy - 3z$ or $-x^2 + 8xy - 3z$.

This method of addition is called **vertical method**.

Now, let us add the same algebraic expressions by arranging them horizontally.

To obtain the sum, the like terms will have to be arranged together and added.

This can be done in the following manner:

$$\left(-2x^2+5xy-z\right)+\left(3xy+x^2-2z\right)$$

$$=\left(-2x^2+x^2\right)+\left(5xy+3xy\right)+\left(-z-2z\right)$$

$$=-x^2+8xy-3z$$

This method of addition is called **horizontal method**.

So, one can add algebraic expressions using any of the two methods.

Let us now study the concept of subtraction of algebraic expressions by subtracting $-2x^2+5xy-z$ from $3xy+x^2-2z$.

Let us first arrange them horizontally.

$$\begin{array}{rl} & 3xy+1x^2-2z \\ - & 5xy-2x^2-1z \\ \hline & \\ \hline \end{array}$$

Now, we write the signs opposite to the original signs, below each term of lower expression.

$$\begin{array}{rl} & 3xy+1x^2-2z \\ - & 5xy-2x^2-1z \\ & (-)\quad(+)\quad(+) \\ \hline & \\ \hline \end{array}$$

Now, we perform the operations according to the second signs of the lower expression.

$$\begin{array}{l} 3xy+1x^2-2z \\ 5xy-2x^2-1z \\ (-)\quad(+)\quad(+) \\ \hline -2xy+3x^2-z \\ \hline \end{array}$$

Thus, the required difference is $-2xy+3x^2-z$.

This method of subtraction is known as **vertical method**.

Now, let us find the required difference for the same algebraic expressions by arranging them horizontally.

In this method, we change the sign of each term of subtrahend and then proceed further by arranging the like terms together.

This can be done in the following manner:

$$\left(3xy + x^2 - 2z\right) - \left(-2x^2 + 5xy - z\right)$$

$$= 3xy + x^2 - 2z + 2x^2 - 5xy + z$$

$$= \left(3xy - 5xy\right) + \left(x^2 + 2x^2\right) + \left(-2z + z\right)$$
$$= -2xy + 3x^2 - z$$

This method of subtraction is known as **horizontal method**

It should be noted that after solving an algebraic expression, we always arrange the terms in the descending order of exponents.

Let us solve some examples to understand the concept better.

Example 1:

Add the following algebraic expressions.

(i) $ab + \frac{a^2}{3} - 5b + 2$ and $\frac{2ab}{5} - 3a^2 + b$

(ii) $x^2y - \sqrt{x} + 2\sqrt{xy} - 4y^2$ and $x^3 - 3\sqrt{xy} + 4y^2 - 7$

Solution:

(i) $\left(ab+\frac{a^2}{3}-5b+2\right)+\left(\frac{2ab}{5}-3a^2+b\right)$

$=\left(ab+\frac{2ab}{5}\right)+\left(\frac{a^2}{3}-3a^2\right)+(-5b+b)+2$

$=\left(1+\frac{2}{5}\right)ab+\left(\frac{1}{3}-3\right)a^2+(-5+1)b+2$

$=\left(\frac{5+2}{5}\right)ab+\left(\frac{1-9}{3}\right)a^2+(-4)b+2$

$=\frac{7ab}{5}-\frac{8}{3}a^2-4b+2$

(ii) $\left(x^2y-\sqrt{x}+2\sqrt{xy}-4y^2\right)+\left(x^3-3\sqrt{xy}+4y^2-7\right)$

$=x^2y-\sqrt{x}+\left(2\sqrt{xy}-3\sqrt{xy}\right)+\left(-4y^2+4y^2\right)+x^3-7$

$=x^2y-\sqrt{x}+(2-3)\sqrt{xy}+0+x^3-7$

$=x^2y-\sqrt{x}+(-1\sqrt{xy})+x^3-7$

$=x^2y-\sqrt{x}-\sqrt{xy}+x^3-7$

$=x^3+x^2y-\sqrt{xy}-\sqrt{x}-7$

Example 2:

Subtract the following algebraic expressions.

(i) $3x^6-\frac{3}{2}xy+9y-7$ from $-x^6+5yx+2-\frac{7}{2}y$

(ii) $6pq-2pqr+\frac{2}{3}\frac{p^2}{r}+3\frac{r}{q}$ from $\frac{3}{2}pq+2\frac{r}{q}+19-5\frac{p^2}{r}-2pqr$

Solution:

$$\text{(i)}\left(-x^6+5yx+2-\frac{7}{2}y\right)-\left(3x^6-\frac{3}{2}xy+9y-7\right)$$

$$=-x^6+5yx+2-\frac{7}{2}y-3x^6+\frac{3}{2}xy-9y+7$$

$$=(-x^6-3x^6)+\left(5yx+\frac{3}{2}xy\right)+(2+7)+\left(-\frac{7}{2}y-9y\right)$$

$$=(-1-3)x^6+\left(5+\frac{3}{2}\right)xy+9+\left(-\frac{7}{2}-9\right)y$$

$$=-4x^6+\frac{13}{2}xy+9-\frac{25}{2}y$$

$$=-4x^6+\frac{13}{2}xy-\frac{25}{2}y+9$$

$$\text{(ii)}\left(\frac{3}{2}pq+2\frac{r}{q}+19-5\frac{p^2}{r}-2pqr\right)-\left(6pq-2pqr+\frac{2}{3}\frac{p^2}{r}+3\frac{r}{q}\right)$$

$$=\frac{3}{2}pq+2\frac{r}{q}+19-5\frac{p^2}{r}-2pqr-6pq+2pqr-\frac{2}{3}\frac{p^2}{r}-3\frac{r}{q}$$

$$=\left(\frac{3}{2}pq-6pq\right)+\left(2\frac{r}{q}-3\frac{r}{q}\right)+19+\left(-5\frac{p^2}{r}-\frac{2}{3}\frac{p^2}{r}\right)+(-2pqr+2pqr)$$

$$=\left(\frac{3}{2}-6\right)pq+(2-3)\frac{r}{q}+19+\left(-5-\frac{2}{3}\right)\frac{p^2}{r}+0$$

$$=\left(-\frac{9}{2}\right)pq+(-1)\frac{r}{q}+19+\left(-\frac{17}{3}\right)\frac{p^2}{r}$$

$$=-\frac{9}{2}pq-\frac{r}{q}+19-\frac{17}{3}\frac{p^2}{r}$$

$$=-\frac{9}{2}pq-\frac{17}{3}\frac{p^2}{r}-\frac{r}{q}+19$$

Example 3:

(i) What should be subtracted from $ab+\frac{2}{5}a^2-6b+3$ to get $3a^2+9b-a+6ab$?

(ii) What should be added to $u^3+\frac{2}{7}w-uv+3$ to get $\frac{3}{4}u^3+8uv+w-7$?

Solution:

(i) Let the required algebraic expression be *X*.

$$\text{So, } ab+\frac{2}{5}a^2-6b+3-X=3a^2+9b-a+6ab$$

$$\Rightarrow X=\left(ab+\frac{2}{5}a^2-6b+3\right)-\left(3a^2+9b-a+6ab\right)$$

$$=ab+\frac{2}{5}a^2-6b+3-3a^2-9b+a-6ab$$

$$=(ab-6ab)+\left(\frac{2}{5}a^2-3a^2\right)+\left(-6b-9b\right)+3+a$$

$$=(1-6)ab+\left(\frac{2}{5}-3\right)a^2+(-6-9)b+3+a$$

$$=-5ab-\frac{13}{5}a^2-15b+3+a$$

$$=-\frac{13}{5}a^2-5ab-15b+a+3$$

Thus, $-\frac{13}{5}a^2-5ab-15b+a+3$ must be subtracted from $ab+\frac{2}{5}a^2-6b+3$ to get $3a^2+9b-a+6ab$.

(ii) Let the required algebraic expression be *X*.

$$\text{So, } u^3+\frac{2}{7}w-uv+3+X=\frac{3}{4}u^3+8uv+w-7$$

$$\Rightarrow X=\left(\frac{3}{4}u^3+8uv+w-7\right)-\left(u^3+\frac{2}{7}w-uv+3\right)$$

$$=\frac{3}{4}u^3+8uv+w-7-u^3-\frac{2}{7}w+uv-3$$

$$=\left(\frac{3}{4}u^3-u^3\right)+\left(8uv+uv\right)+\left(w-\frac{2}{7}w\right)+\left(-7-3\right)$$

$$=\left(\frac{3}{4}-1\right)u^3+(8+1)uv+\left(1-\frac{2}{7}\right)w-10$$

$$=-\frac{1}{4}u^3+9uv+\frac{5}{7}w-10$$

Thus, $-\frac{1}{4}u^3+9uv+\frac{5}{7}w-10$ must be added to $u^3+\frac{2}{7}w-uv+3$ to get $\frac{3}{4}u^3+8uv+w-7$.

Example 4:

Subtract $\frac{5}{9}xy+9x^2-6y+2x$ from the sum of $11x-6x^2-xy$ and $\frac{2}{9}xy+4x+6x^2-3$.

Solution:

Sum of $11x-6x^2-xy$ and $\frac{2}{9}xy+4x+6x^2-3$

$$=11x-6x^2-xy+\frac{2}{9}xy+4x+6x^2-3$$

$$=(11x+4x)+(-6x^2+6x^2)+\left(-xy+\frac{2}{9}xy\right)-3$$

$$=(11+4)x+0+\left(-1+\frac{2}{9}\right)xy-3$$

$$=15x-\frac{7}{9}xy-3$$

$$\begin{aligned}\text{Required result } &=\left(15x-\frac{7}{9}xy-3\right)-\left(\frac{5}{9}xy+9x^2-6y+2x\right)\\ &=15x-\frac{7}{9}xy-3-\frac{5}{9}xy-9x^2+6y-2x\\ &=(15x-2x)+\left(-\frac{7}{9}xy-\frac{5}{9}xy\right)-3-9x^2+6y\\ &=(15-2)x+\left(-\frac{7}{9}-\frac{5}{9}\right)xy-3-9x^2+6y\\ &=13x+\left(-\frac{12}{9}\right)xy-3-9x^2+6y\\ &=-9x^2-\frac{4}{3}xy+13x+6y-3\end{aligned}$$

Multiplication of Monomials with Polynomials

Let us discuss another example based on the above concept.

Suppose we have two monomials 4*x* and 5*y*. By multiplying them, we get

$4x \times 5y = (4 \times x) \times (5 \times y)$

$= (4 \times 5) \times (x \times y)$

$= 20xy$

Hence, we can say that $4x \times 5y = 20xy$

Now, $10x \times 5x^2z = (10 \times x) \times (5 \times x^2 \times z)$

This becomes,

$(10 \times 5) \times (x \times x^2 \times z) = 50x^3z$

Now, what if you have three monomials and you want to multiply them? How will you do so?

Suppose you want to multiply $10x$, $2xy$, and $5z$.

$10x \times 2xy \times 5z = (10 \times x) \times (2 \times x \times y) \times (5 \times z)$

First, we multiply the first two monomials.

$= \{(10 \times 2) \times (x \times x \times y)\} \times (5 \times z)$

$= (20 \times x^2 \times y) \times (5 \times z)$ $[\because x \times x = x^2]$

$= (20 \times 5) \times (x^2 \times y \times z)$

$= 100x^2yz$

This method of multiplication of three monomials can be extended to find out the product of any number of monomials.

Let us discuss some more examples based on multiplication of monomials.

Example 1:

Find the product of the following:

(a) $-2x^2y$ **and** $15\ xy^2z^3$

(b) $7ap$, $2qa^2x^3$ **and** $-5rx$

$(\mathbf{c})$ $\boldsymbol{ab, -2bc, -3cd}$ **and** $\boldsymbol{4ad}$

Solution:

$(\mathbf{a})$ $-2x^2y$ and $15\ xy^2z^3$

$= \left(-2x^2y\right) \times \left(15\ xy^2z^3\right)$

$= (-2 \times 15) \times \left(x^2y \times xy^2z^3\right)$

$= -30x^3y^3z^3 \qquad \left\{\text{as } x^2 \times x = x^3 \text{ and } y \times y^2 = y^3\right\}$

$(\mathbf{b})$ $7ap$, $2qa^2x^3$, and $-5rx$

$= (7ap) \times \left(2qa^2x^3\right) \times (-5rx)$

$= \left\{(7 \times 2) \times \left(ap \times qa^2x^3\right)\right\} \times (-5rx)$

$= \left(14a^3pqx^3\right) \times (-5rx)$

$= (14 \times -5) \times \left(a^3pqx^3 \times rx\right)$

$= -70a^3pqrx^4 \qquad \left[\text{as } x^3 \times x = x^4\right]$

$(\mathbf{c})$ ab, $-2bc$, $-3cd$, and $4ad$

$= (ab) \times (-2bc) \times (-3cd) \times (4ad)$

$= \left[(1 \times -2) \times (ab \times bc)\right] \times \left[(-3 \times 4) \times (cd \times ad)\right]$
$\{$Multiplying the 1st to 2nd and 3rd to 4th term$\}$

$= \left(-2ab^2c\right) \times \left(-12cd^2a\right)$

$= (-2 \times -12) \times \left(ab^2c \times cd^2a\right)$

$= 24a^2b^2c^2d^2$

Example 2.

If the side of a square is 4*x* cm, what is its area?

Answer:

Side of square = $4x$ cm

∴ Area of square = side × side = $4x \times 4x = 16x^2$ cm^2

Example 3:

The length, breadth, and height of three cuboids are given below in the table. Find the volume and area of the base of these cuboids.

	Length	**Breadth**	**Height**
(i)	$3ab$	$2bx$	$5xy$
(ii)	a^2b	b^2c	c^2a
(iii)	$3x$	$9x^2$	$27x^3$

Solution:

We know that, area of the base = Length × Breadth

Volume of cuboid = Length × Breadth × Height

(i) Area of the base = $3ab \times 2bx = (3 \times 2) \times (ab \times bx) = 6ab^2x$

Volume of the cuboid = $3ab \times 2bx \times 5xy$

$$= \{(3 \times 2) \times (ab \times bx)\} \times (5xy)$$
$$= (6ab^2x) \times (5xy)$$
$$= (6 \times 5) \times (ab^2x \times xy)$$
$$= 30\ ab^2x^2y$$

(ii) Area of the base = $a^2b \times b^2c = a^2b^3c$

Volume of the cuboid = $(a^2b \times b^2c) \times c^2a = a^2b^3c \times c^2a = a^3\,b^3\,c^3$

(iii) Area of the base = $3x \times 9x^2 = (3 \times 9) \times (x \times x^2) = 27x^3$

Volume of the cuboid = $3x \times 9x^2 \times 27x^3$

$= \left[(3 \times 9) \times \left(x \times x^2\right)\right] \times 27x^3$

$= 27x^3 \times 27x^3$

$= (27 \times 27)\left(x^3 \times x^3\right)$

$= 729\ x^6$

So far, we know how two multiply any number of monomials. But, what if we need to multiply a monomial with a binomial or a trinomial, etc.?

Can we multiply them?

We can multiply them easily.

To understand the method, look at the following video.

The method discussed in the above video shows the **horizontal arrangement** of multiplying monomials with polynomials.

Let us now learn about the **vertical arrangement** for the same by performing the multiplication of ($4x^2$ + $2x$) and $3x$*.*

This is similar to vertical method of multiplication of whole numbers.

Here, we will first multiply $3x$ with $2x$ and write the product with sign at the bottom. After doing this, we will multiply $3x$ with $4x^2$ and write the product with sign at the bottom. The expression obtained at the bottom will be the required product.

This can be done as follows:

$$\begin{array}{r} 4x^2 + 2x \\ \underline{\times \qquad 3x} \\ \underline{12x^3 + 6x^2} \end{array}$$

Similarly, we can multiply a trinomial with monomial as follows:

$$\begin{array}{r} 2y^3 - 5y + 1 \\ \underline{\times \qquad\qquad 2y} \\ \underline{4y^4 - 10y^2 + 2y} \end{array}$$

Let us discuss some more examples based on the multiplication of a monomial with polynomials.

Example 4:

Multiply the following in horizontal and vertical arrangements:

(a) $\frac{3}{5}p$ **and** $p-6q$

(b) $(1.5x+y+3z)$ **and** $7z$

Also find the values of the above expressions if $p=-5$, $q=-1$, $x=2$, $y=-3$, **and** $z=-1$.

Solution:

(a) Horizontal arrangement:

$$\frac{3}{5}p\times(p-6q)=\frac{3}{5}p\times p-\frac{3}{5}p\times 6q=\frac{3}{5}p^2-\frac{18}{5}pq$$

Vertical arrangement:

$$\begin{array}{r} p-6q \\ \times\qquad \frac{3}{5}p \\ \hline \frac{3}{5}p^2-\frac{18}{5}pq \\ \hline \end{array}$$

Substituting the values of p and q, we get

$$=\frac{3}{5}\times(-5)^2-\frac{18}{5}\times(-5)(-1)$$

$$=\frac{3}{5}\times 25-18$$
$$=15-18$$
$$=-3$$

(b) Horizontal arrangement:

$(1.5x + y + 3z) \times 7z = 1.5x \times 7z + y \times 7z + 3z \times 7z = 10.5xz + 7yz + 21z^2$

Vertical arrangement:

$$\begin{array}{r} 1.5x + y + 3z \\ \times \qquad\qquad 7z \\ \hline 10.5xz + 7yz + 21z^2 \\ \hline \end{array}$$

Substituting the values of *x*, *y*, and *z*, we get

$$\begin{aligned} &10.5xz + 7yz + 21z^2 \\ &= 10.5 \times 2 \times (-1) + 7 \times (-3) \times (-1) + 21 \times (-1)^2 \end{aligned}$$

$= -21 + 21 + 21$

= 21

Example 5:

(a) Add $\boldsymbol{a(b-c), b(c-a), \text{and } c(a-b)}$

(b) $\textbf{Subtract } \boldsymbol{5x(x - y + z) - 2z(-3x + 4y + 5z)} \textbf{ from } \boldsymbol{3y(4x + 3y - 2z)}$

Solution:

(a) Addition of *a* (*b* − *c*), *b* (*c* − *a*), and *c* (*a* − *b*)

$$\begin{aligned} &= a(b-c) + b(c-a) + c(a-b) \\ &= ab - ac + bc - ab + ac - bc \\ &= 0 \end{aligned}$$

(b) Subtraction of $5x(x - y + z) - 2z(-3x + 4y + 5z)$ from $3y(4x + 3y - 2z)$

$$\begin{aligned} &= 3y(4x + 3y - 2z) - \{5x(x - y + z) - 2z(-3x + 4y + 5z)\} \\ &= (3y)(4x) + (3y)(3y) + (3y)(-2z) - \begin{Bmatrix} (5x)(x) + (5x)(-y) + (5x)(z) \\ -2z(-3x) - 2z(4y) - 2z(5z) \end{Bmatrix} \end{aligned}$$

$$= 12xy + 9y^2 - 6yz - \left\{5x^2 - 5xy + 5zx + 6zx - 8yz - 10z^2\right\}$$

$$= 12xy + 9y^2 - 6yz - \left\{5x^2 - 5xy + 11zx - 8yz - 10z^2\right\}$$

$$= 12xy + 9y^2 - 6yz - 5x^2 + 5xy - 11zx + 8yz + 10z^2$$

$$= -5x^2 + 9y^2 + 10z^2 + 17xy + 2yz - 11zx$$

Multiplication of Two Polynomials

Suppose you want to buy ($2x + y$) metres of rope at the rate of Rs ($a - 3b$) per metre.

Can you calculate the amount of money you require?

The amount you require is $(2x + y) \times (a - 3b)$.

Now, how will you carry out this type of multiplication?

Here, first multiply $3x – y$ with $3y$ and then multiply $3x – y$ with x. After doing so, add the like terms as shown below:

$$\begin{array}{rr}
 & 3x - y \\
\times & x + 3y \\
\hline
 & 9xy - 3y^2 \\
+3x^2 - xy & \\
\hline
3x^2 + 8xy - 3y^2 & \\
\hline
\end{array}$$

The process of multiplying a binomial with a trinomial is not too different from that of multiplying two binomials. Let us learn more about it.

In the video, we have multiplied binomial with trinomial in **horizontal arrangement**. Let us now multiply the binomial ($x + y$) with trinomial ($2x + 3y + 1$) in **vertical arrangement**.

$$\begin{array}{r} 2x+3y+1 \\ \times \qquad\qquad x+y \\ \hline 2xy+3y^2+y \\ +2x^2+3xy+x \qquad\quad \\ \hline 2x^2+5xy+3y^2+x+y \\ \hline \end{array}$$

Thus, we can perform multiplication of binomials with binomials and trinomials using any of the horizontal or vertical arrangement method.

Let us now solve examples based on the above concepts.

Example 1:

Multiply the following using horizontal and vertical arrangement:

(a) 2 (x + y) and x − 3y

(b) (l + 3m) and (l + 6m + 7n)

Solution:

(a) Horizontal arrangement:

$2(x + y) = 2x + 2y$

Now, we have to multiply $(2x + 2y)$ and $x - 3y$.

$(2x + 2y) \times (x - 3y) = 2x \times (x - 3y) + 2y \times (x - 3y)$

(Using distributive property)

$= 2 \times x \times x - 2 \times 3 \times x \times y + 2 \times x \times y - 2 \times 3 \times y \times y$

$= 2x^2 - 6xy + 2xy - 6y^2$

$= 2x^2 - 4xy - 6y^2$ (Combining the like terms)

Vertical arrangement:

$$\begin{array}{r} x+y \\ \times \quad 2 \\ \hline 2x+2y \\ \hline \end{array} \qquad \begin{array}{r} x-3y \\ \times \qquad 2x+2y \\ \hline 2xy-6y^2 \\ +2x^2-6xy \qquad \\ \hline 2x^2-4xy-6y^2 \\ \hline \end{array}$$

(b) H**orizontal arrangement:**

$(l + 3m) \times (l + 6m + 7n)$

$= l \times (l + 6m + 7n) + 3m \times (l + 6m + 7n)$

(Using distributive property)

$= l \times l + l \times 6m + l \times 7n + 3m \times l + 3m \times 6m + 3m \times 7n$

$= l^2 + 6lm + 7ln + 3ml + 18m^2 + 21mn$

$= l^2 + 9ml + 7ln + 21mn + 18m^2$

[Combining the like terms $6lm$ and $3ml$]

Vertical arrangement:

$$\begin{array}{r} l+6m+7n \qquad \\ \times \qquad\qquad l+3m \qquad \\ \hline 3lm+18m^2+21mn \\ +l^2+6lm+7nl \qquad\qquad\qquad \\ \hline l^2+9lm+18m^2+21mn+7nl \\ \hline \end{array}$$

Example 2:

Simplify the following:

(a) $(x^2 + y^2) \times (x^3 + y + z^2) + 2(z^2 + 5z)$

(b) $(l - m)(l + m) + (m - n)(m + n) - (l - n)(n + l)$

(c) $(x - 4)(y - 4) - 16$

(d) $(a + b + c)(a - b + c)$

Solution:

(a) $(x^2 + y^2) \times (x^3 + y + z^2) + 2(z^2 + 5z)$

$= x^2(x^3 + y + z^2) + y^2(x^3 + y + z^2) + 2(z^2 + 5z)$

(Using distributive property)

$= x^2 \times x^3 + x^2 \times y + x^2 \times z^2 + y^2 \times x^3 + y^2 \times y + y^2 \times z^2 + 2 \times z^2 + 2 \times 5z$

$= x^5 + x^2y + x^2z^2 + x^3y^2 + y^3 + y^2z^2 + 2z^2 + 10z$

(b) $(l - m)(l + m) + (m - n)(m + n) - (l - n)(n + l)$

$= l(l + m) - m(l + m) + m(m + n) - n(m + n) - l(n + l) + n(n + l)$

(Using distributive property)

$= l^2 + lm - ml - m^2 + m^2 + mn - nm - n^2 - ln - l^2 + n^2 + nl$

$=(l^2 - l^2) + (lm - lm) + (-m^2 + m^2) + (mn - mn) + (-n^2 + n^2) + (-ln + ln)$

{lm and ml, ln and nl, mn and nm are like terms}

$= 0$

(c) $(x - 4) \times (y - 4) - 16$

$= x \times (y - 4) - 4(y - 4) - 16$ (Using distributive property)

$= xy - 4x - 4y + 16 - 16$

$= xy - 4x - 4y$

(d) $(a + b + c)(a - b + c)$

$= a(a - b + c) + b(a - b + c) + c(a - b + c)$ (Using distributive property)

$= a^2 - ab + ac + ba - b^2 + bc + ca - cb + c^2$ (Combining the like terms)

$= a^2 + c^2 - b^2 + 2ac$

Division of Polynomials by Polynomials

Let us discuss some more examples, in which we will not only divide binomials, but also divide other types of polynomials.

Example 1:

Factorise the following expressions and divide as directed.

(i) $6ab\,(9a^2 - 16b^2) \div 2ab\,(3a + 4b)$

(ii) $(x^2 - 14x - 32) \div (x + 2)$

(iii) $36abc\,(5a - 25)\,(2b - 14) \div 24(a - 5)\,(b - 7)$

Solution:

(i) We can factorise the given expressions as

$6ab\,(9a^2 - 16b^2) = 2 \times 3 \times a \times b\,[(3a)^2 - (4b)^2]$

$= 2 \times 3 \times a \times b\,[(3a + 4b)\,(3a - 4b)]\ [a^2 - b^2 = (a + b)(a - b)]$

And, $2ab\,(3a + 4b) = 2 \times a \times b \times (3a + 4b)$

$$\Rightarrow \frac{6ab\left(9a^2 - 16b^2\right)}{2ab\left(3a + 4b\right)} = \frac{2 \times 3 \times a \times b \times \left(3a + 4b\right) \times \left(3a - 4b\right)}{2 \times a \times b \times \left(3a + 4b\right)}$$

$= 3 \times (3a - 4b)$

$= 9a - 12b$

$\therefore 6ab\,(9a^2 - 16b^2) \div 2ab\,(3a + 4b) = 9a - 12b$

(ii) We can factorise the given expression as

$x^2 - 14x - 32 = x^2 - (16 - 2)\,x - 32$

$= x^2 - 16x + 2x - 32$

$= x\,(x - 16) + 2\,(x - 16)$

$= (x - 16)\,(x + 2)$

= (x + 2) = (x + 2)

$$\Rightarrow \frac{x^2-14x-32}{x+2} = \frac{(x-16)(x+2)}{(x+2)}$$

= $x - 16$

∴ $(x^2 - 14x - 32) \div (x + 2) = (x - 16)$

(iii) We can factorise the given expression as

$$\begin{aligned} 36abc(5a-25)\ (2b-14) &= 2\times2\times3\times3\times a\times b\times c(5\times a-5\times5)\ (2\times b-2\times7) \\ &= 2\times2\times3\times3\times5\times a\times b\times c\ (a-5)\times2(b-7) \\ &= 2\times2\times2\times3\times3\times5\times a\times b\times c\ (a-5)\ (b-7) \end{aligned}$$

$$24(a-5)(b-7) = 24\ (a-5)\ (b-7) = 2\times2\times2\times3(a-5)\ (b-7)$$

$$\therefore \frac{36abc(5a-25)(2a-14)}{24(a-5)(b-7)}$$

$$= \frac{2\times2\times2\times3\times3\times5\times a\times b\times c(a-5)(b-7)}{2\times2\times2\times3(a-5)(b-7)}$$

$$= 3\times5\times a\times b\times c$$

$$= 15abc$$

Hence, $36abc\ (5a - 25)\ (2b - 14) \div 24(a - 5)\ (b - 7) = 15abc$

Division of Polynomials by Polynomials (Degree 1) Using Long Division Method

Long Division Method for Division of Polynomials

Dividing one number by another is something that we know well. For example, let us divide 434 by 9.

$$\begin{array}{r} 48 \\ 9\overline{)\ 434} \\ -\underline{36} \\ 74 \\ -\underline{72} \\ \underline{2} \end{array}$$

In the above division, 434 is the **dividend**, 9 is the **divisor**, 48 is the **quotient** and 2 is the **remainder**.

We also know how to represent any division using the division algorithm, which states that:

Dividend = Divisor × Quotient + Remainder

Thus, we can write 434 as:

$434 = 9 \times 48 + 2$

We can divide one polynomial by another in the same way as we divide one number by another. In this lesson, we will learn to carry out the division of polynomials and verify the same using the division algorithm.

Division Algorithm

Polynomials also satisfy the division algorithm.

Consider the division of $2x^2 - 9x + 4$ by $x - 2$.

In this division, we have

Dividend = $2x^2 - 9x + 4$

Divisor = $x - 2$

Quotient = $2x - 5$

Remainder = -6

Now,

Divisor × Quotient + Remainder = $[(x - 2)(2x - 5)] + (-6)$

$= [x(2x - 5) - 2(2x - 5)] - 6$

$= 2x^2 - 5x - 4x + 10 - 6$

$= 2x^2 - 9x + 4$

= Dividend

Thus, the given division satisfies the division algorithm, i.e,

Dividend = Divisor × Quotient + Remainder

Solved Examples

Easy

Example 1:

Divide the polynomial by monomial and write the quotient and remainder.

(i) $9y^3 – 6y^2 + 3y \div 3y$

(ii) $8x^4 + 4x^2 – 5x + 1 \div 2x$

(iii) $4z^6 – 18z^4 + 24z^2 \div 12z$

Solution:

(i) $9y^3 – 6y^2 + 3y$ can be divided by $3y$ using long division method as follows:

$$\begin{array}{r|l}
 & 3y^2 - 2y + 1 \\ \hline
3y & 9y^3 - 6y^2 + 3y \\
 & \underset{-}{9y^3} \\ \hline
 & \quad -6y^2 + 3y \\
 & \quad \underset{+}{-6y^2} \\ \hline
 & \qquad\quad 3y \\
 & \qquad\quad \underset{-}{3y} \\ \hline
 & \qquad\quad 0 \\ \hline
\end{array}$$

Therefore,

Quotient = $3y^2 – 2y + 1$

Remainder = 0

(ii) $8x^4 + 4x^2 – 5x + 1$ can be divided by $2x$ using long division method as follows:

$$
\begin{array}{r l}
 & 4x^3 + 2x - \frac{5}{2} \\
\cline{2-2}
2x \big) & 8x^4 + 4x^2 - 5x + 1 \\
 & 8x^4 \\
 & - \\
\hline
 & \quad + 4x^2 - 5x + 1 \\
 & \quad + 4x^2 \\
 & \quad - \\
\hline
 & \qquad - 5x + 1 \\
 & \qquad - 5x \\
 & \qquad + \\
\hline
 & \qquad\qquad 1 \\
\hline
\end{array}
$$

Therefore,

Quotient = 4x3 + 2x − 524x3 + 2x - 52

Remainder = 1

(iii) $4z^6 - 18z^4 + 24z^2$ can be divided by $12z$ using long division method as follows:

$$
\begin{array}{r l}
 & \frac{1}{3}z^5 - \frac{3}{2}z^3 + 2z \\
\cline{2-2}
12z \big) & 4z^6 - 18z^4 + 24z^2 \\
 & \underset{-}{4z^6} \\
\hline
 & \quad -18z^4 + 24z^2 \\
 & \quad \underset{+}{-18z^4} \\
\hline
 & \qquad 24z^2 \\
 & \qquad \underset{-}{24z^2} \\
\hline
 & \qquad\quad 0 \\
\hline
\end{array}
$$

Therefore,

Quotient = $\frac{1}{3}z^5 - \frac{3}{2}z^3 + 2z$

Remainder = 0

Example 2:

Divide $5x^2 + 7x - 4$ by $x + 1$.

Solution:

Let $p(x) = 5x^2 + 7x - 4$ and $q(x) = x + 1$

The terms of $p(x)$ and $q(x)$ are arranged in decreasing order of their powers. Let us divide $p(x)$ by $q(x)$.

$$
\begin{array}{r|l}
 & 5x + 2 \\
\hline
x+1 & 5x^2 + 7x - 4 \\
 & 5x^2 + 5x \\
 & - \quad - \\
\hline
 & \quad\quad 2x - 4 \\
 & \quad\quad 2x + 2 \\
 & \quad\quad - \quad - \\
\hline
 & \quad\quad\quad -6 \\
\hline
\end{array}
$$

Example 3:

What is the quotient when $-2x^2 + x^3 - 9x + 18$ is divided by $x - 2$?

Solution:

Let $p(x) = -2x^2 + x^3 - 9x + 18$

$= x^3 - 2x^2 - 9x + 18$ (arranging the terms in decreasing order of their powers)

Let $q(x) = x - 2$

The terms of $p(x)$ and $q(x)$ are arranged in decreasing order of their powers. Let us divide $p(x)$ by $q(x)$.

$$
\begin{array}{r|l}
 & x^2 - 9 \\
\hline
x-2 & x^3 - 2x^2 - 9x + 18 \\
 & x^3 - 2x^2 \\
 & \underline{-\quad +} \\
 & \qquad -9x + 18 \\
 & \qquad -9x + 18 \\
 & \qquad \underline{+\quad -} \\
 & \qquad\qquad \underline{0}
\end{array}
$$

Hence, when $-2x^2 + x^3 - 9x + 18$ is divided by $x - 2$, the quotient is $x^2 - 9$.

Medium

Example 1:

Find the quotient and remainder on dividing $f(x)$ by $g(x)$, where

$f(x) = 6x^3 + 13x^2 + x - 2$ and $g(x) = 2x + 1$.

Solution:

We have $f(x) = 6x^3 + 13x^2 + x - 2$ and $g(x) = 2x + 1$.

The terms of $f(x)$ and $g(x)$ are arranged in decreasing order of their powers. Let us divide $f(x)$ by $g(x)$.

$$
\begin{array}{r|l}
 & 3x^2 + 5x - 2 \\
\hline
2x+1 & 6x^3 + 13x^2 + x - 2 \\
 & 6x^3 + 3x^2 \\
 & \underline{-\quad -} \\
 & \qquad 10x^2 + x \\
 & \qquad 10x^2 + 5x \\
 & \qquad \underline{-\quad -} \\
 & \qquad\qquad -4x - 2 \\
 & \qquad\qquad -4x - 2 \\
 & \qquad\qquad \underline{+\quad +} \\
 & \qquad\qquad\qquad \underline{0}
\end{array}
$$

Thus, when $6x^3 + 13x^2 + x - 2$ is divided by $2x + 1$, we get $3x^2 + 5x - 2$ as the quotient and zero as the remainder.

Example 2:

What are the quotient and remainder when $x^4 + 2x^3 - 33x^2 + 18x + 28$ is divided by $x + 7$?

Solution:

Let $p(x) = x^4 + 2x^3 - 33x^2 + 18x + 28$ and $q(x)$
$= x$ +https://www.google.co.in/?gws_rd=ssl 7

The terms of $p(x)$ and $q(x)$ are arranged in decreasing order of their powers. Let us divide $p(x)$ by $q(x)$.

$$
\begin{array}{r|l}
 & x^3 - 5x^2 + 2x + 4 \\
x+7 & x^4 + 2x^3 - 33x^2 + 18x + 28 \\
 & x^4 + 7x^3 \\
 & - \quad - \\
\hline
 & -5x^3 - 33x^2 + 18x + 28 \\
 & -5x^3 - 35x^2 \\
 & + \quad + \\
\hline
 & 2x^2 + 18x + 28 \\
 & 2x^2 + 14x \\
 & - \quad - \\
\hline
 & 4x + 28 \\
 & 4x + 28 \\
 & - \quad - \\
\hline
 & 0 \\
\hline
\end{array}
$$

$$\left(\frac{x^4}{x} = x^3\right)$$

$$\left(-\frac{5x^3}{x} = -5x^2\right)$$

$$\left(\frac{2x^2}{x} = 2x\right)$$

$$\left(\frac{4x}{x} = 4\right)$$

Thus, when $x^4 + 2x^3 - 33x^2 + 18x + 28$ is divided by $x + 7$, we get $x^3 - 5x^2 + 2x + 4$ as the quotient and zero as the remainder.

Hard

Example 1:

If the division of $x^3 + 2x^2 + kx + 3$ by $x - 3$ yields 21 as the remainder, then find the quotient and the value of *k.*

Solution:

Let $p(x) = x^3 + 2x^2 + kx + 3$ and $q(x) = x - 3$. It is given that the division of $p(x)$ by $q(x)$ gives 21 as the remainder.

According to the division algorithm:

Dividend = Divisor × Quotient + Remainder

$\Rightarrow x^3 + 2x^2 + kx + 3 = (x - 3) \times \text{Quotient} + 21$

$\Rightarrow x^3 + 2x^2 + kx + 3 - 21 = (x - 3) \times \text{Quotient}$

$$\therefore \text{Quotient} = \frac{x^3 + 2x^2 + kx - 18}{x - 3}$$

$$\begin{array}{r|l}
 & x^2 + 5x + (k+15) \\
\hline
x - 3 & x^3 + 2x^2 + kx - 18 \\
 & x^3 - 3x^2 \\
 & \underline{-\quad +\qquad\qquad} \\
 & \qquad 5x^2 + kx \\
 & \qquad 5x^2 - 15x \\
 & \qquad \underline{-\quad +\qquad\qquad} \\
 & \qquad\qquad (k+15)x - 18 \\
 & \qquad\qquad (k+15)x - 3k - 45 \\
 & \qquad\qquad \underline{-\qquad\quad +\quad +} \\
 & \qquad\qquad\qquad\quad \underline{3k + 27}
\end{array}$$

Now, $3k + 27 = 0$

$\Rightarrow 3k = -27$

$\Rightarrow k = -9$

So, quotient $= x^2 + 5x + (k + 15)$

$= x^2 + 5x + (-9 + 15)$

$= x^2 + 5x + 6$

Example 2:

Divide $3x^3 + 5x^2 + 4x + 7$ by $3x - 1$. Find the quotient and the remainder, and verify using the division algorithm.

Solution:

Let $p(x) = 3x^3 + 5x^2 + 4x + 7$ and $q(x) = 3x - 1$

The terms of $p(x)$ and $q(x)$ are arranged in decreasing order of their powers. Let us divide $p(x)$ by $q(x)$.

$$
\begin{array}{r|l}
 & \quad x^2 + 2x + 2 \\
\cline{2-2}
3x - 1 & 3x^3 + 5x^2 + 4x + 7 \\
 & 3x^3 - x^2 \\
 & \underline{-\quad +\qquad\qquad\qquad} \\
 & \qquad 6x^2 + 4x \\
 & \qquad 6x^2 - 2x \\
 & \underline{\qquad -\quad\ +\qquad\quad} \\
 & \qquad\qquad\quad 6x + 7 \\
 & \qquad\qquad\quad 6x - 2 \\
 & \underline{\qquad\qquad\quad -\quad +} \\
 & \underline{\qquad\qquad\qquad\qquad 9}
\end{array}
$$

We have obtained the quotient as $x^2 + 2x + 2$ and the remainder as 9.

Let us verify our result using the division algorithm, which states that:

Dividend = Divisor × Quotient + Remainder

Now,

Divisor × Quotient + Remainder = $[(3x - 1)\ (x^2 + 2x + 2)] + 9$

$= [3x\ (x^2 + 2x + 2) - 1\ (x^2 + 2x + 2)] + 9$

$= 3x^3 + 6x^2 + 6x - x^2 - 2x - 2 + 9$

$= 3x^3 + 5x^2 + 4x + 7$

= Dividend

Thus, the division satisfies the division algorithm.

Factorisation of Algebraic Expressions Using the Identity $(x + a)(x + b) = x^2 + (a + b)x + ab$

How can we factorise the algebraic expression $x^2 + 8x + 15$?

Note that we cannot express this expression as $(a + b)^2$, since 15 is not the square of any natural number. What do we do in such a case?

In this case, we can use the identity $x^2 + (a + b)x + ab = (x + a)(x + b)$.

If we compare $x^2 + 8x + 15$ with $x^2 + (a + b)x + ab$, then we obtain $a + b = 8$ and $ab = 15$.

Hence, we need to find two numbers, a and b, such that their sum is 8 and their product is 15.

The only numbers that fulfil these two conditions are 3 and 5.

Hence, we can write $x^2 + 8x + 15$ as

$x^2 + (5 + 3)x + 5 \times 3$

$= (x^2 + 5x) + (3 \times x + 5 \times 3)$

$= x(x + 5) + 3(x + 5)$

$= (x + 5)(x + 3)$ {Taking a common factor from each group}

Let us practise some more questions based on this concept.

Example 1:

Factorise the expression $x^2 - x - 42$.

Solution:

On comparing $x^2 - x - 42$ with $x^2 + (a + b)x + ab$, we obtain $a + b = -1$ and $ab = -42$.

Here, we have to find two numbers, a and b, such that their sum is -1 and their product is -42.

Since the product (−1) of the numbers a and b is negative and their sum (−42) is also negative, we have to choose two numbers such that the bigger number is negative and the smaller number is positive.

The numbers that fulfil these conditions are −7 and 6.

Thus, obtain $a + b = -1$ and $ab = -42$.
Hence,

$$
\begin{aligned}
x^2 - x - 42 &= x^2 - 7x + 6x - 42 \\
&= x(x - 7) + 6(x - 7) \\
&= (x - 7)(x + 6)
\end{aligned}
$$

Example 2:

Factorise the expression $y^2 - 13y + 36$.

Solution:

On comparing the expression $y^2 - 13y + 36$ with $y^2 + (a + b)y + ab$, we obtain

$a + b = -13$ and $ab = 36$.

Since the product of the numbers a and b is positive and their sum is negative, we have to choose two negative numbers.

The numbers that fulfil these conditions are −9 and −4.

$$
\begin{aligned}
\therefore y^2 - 13y + 36 &= y^2 - 4y - 9y + 36 \\
&= y(y - 4) - 9(y - 4) = (y - 4)(y - 9)
\end{aligned}
$$

Example 3:
Factorise the expression $7(x^2 - x)(2x^2 - 2x - 1) - 42$.

Solution:
We have

$7(x^2 - x)(2x^2 - 2x - 1) - 42$

$= 7(x^2 - x)\{2(x^2 - x) - 1)\} - 42$

Let $x^2 - x = m$

Thus, $7(x^2 - x)\{2(x^2 - x) - 1)\} - 42 = 7m(2m - 1) - 42$

$= 14m^2 - 7m - 42$
$= 7(2m^2 - m - 6)$
$= 7(2m^2 - 4m + 3m - 6)$
$= 7\{2m\,(m - 2) + 3(m - 2)\}$
$= 7(2m + 3)(m - 2)$

On re-substituting the value of m, we obtain

$$7(x^2 - x)\{2(x^2 - x) - 1)\} - 42 = 7\{2(x^2 - x) + 3)\}(\, x^2 - x - 2)$$
$$= 7(2x^2 - 2x + 3)(\, x^2 - x - 2)$$

Using Identities for "Square of Sum or Difference of Two Terms"

Let us try to find the square of the number 102. The square of a number, as we know, is the product of the number with itself. One way to do this is by writing the numbers one below the other, and then multiplying them as we normally do. The other way is to break the numbers and then apply distributive property. This will make our work much easier.

Let us see how.

$102^2 = 102 \times 102$

$= (100 + 2)\ (100 + 2)$

$= 100\ (100 + 2) + 2\ (100 + 2)$

$= 100 \times 100 + 100 \times 2 + 2 \times 100 + 2 \times 2$

$= 10000 + 200 + 200 + 4$

$= 10404$

Observing the similar expressions as above, we obtain the following identities.

$\boldsymbol{(a + b)^2 = a^2 + 2ab + b^2}$

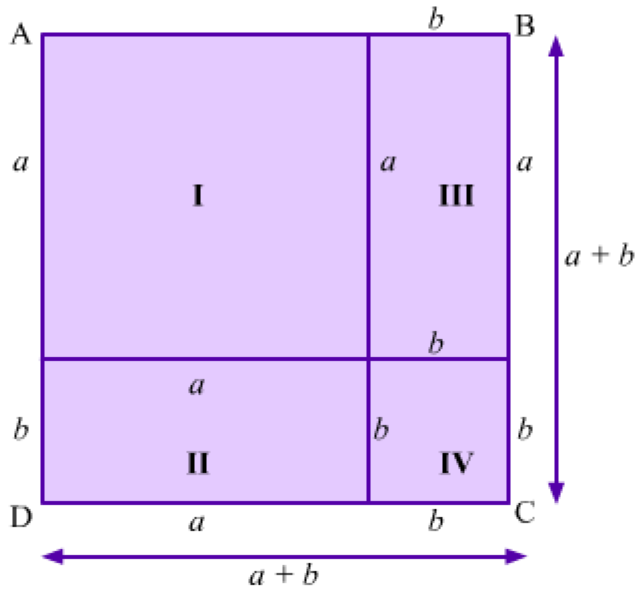
$\boldsymbol{(a - b)^2 = a^2 - 2ab + b^2}$

Deriving the identities geometrically:

These identities can be derived by geometrical construction as well. Let us learn the same.

(1) $(a + b)^2 = a^2 + 2ab + b^2$:

Let us consider a square ABCD whose each side measures $(a + b)$ unit.



It can be seen that, we have drawn two line segments at a distance of *a* unit from A such that one is parallel to AB and other is parallel to AD.

Also, the figure is divided into four regions named as I, II, III and IV.

Now,

∴ Area of square ABCD = $(a + b)^2$ sq. unit ...(i)

Region I is a square of side measuring *a* unit.

∴ Area of region I = a^2 sq. unit ...(ii)

Each of regions II and III is a rectangle having length and breadth as *a* unit and *b* unit respectively.

∴ Area of region II = *ab* sq. unit ...(iii)

And,

Area of region III = *ab* sq. unit ...(iv)

Region IV is a square of side measuring *b* unit.

∴ Area of region IV = b^2 sq. unit ...(v)

From the figure, we have

Area of square ABCD = Area of region I + Area of region II + Area of region III + Area of region IV

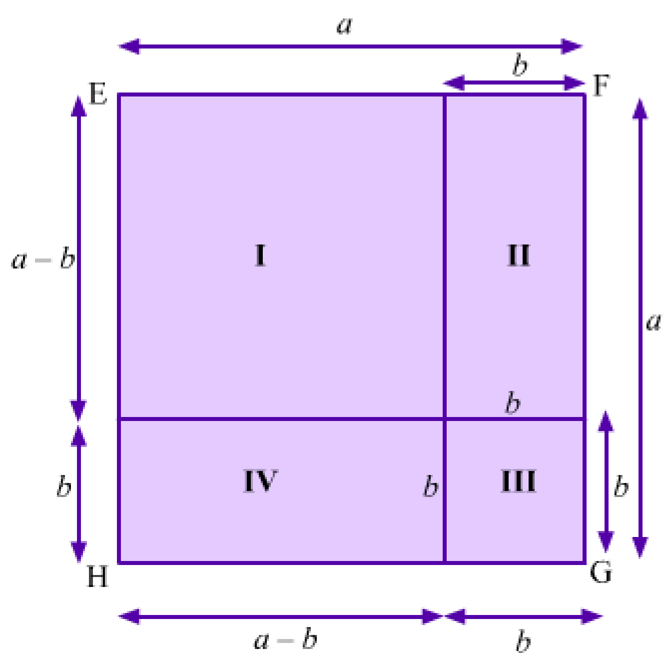
On substituting the values from (i), (ii), (iii), (iv) and (v), we get

$(a + b)^2 = a^2 + ab + ab + b^2$

$\Rightarrow (a + b)^2 = a^2 + 2ab + b^2$

(2) $(a - b)^2 = a^2 - 2ab + b^2$:

Let us consider a square EFGH whose each side measures *a* unit.



It can be seen that, we have drawn two line segments at a distance of *b* unit from G such that one is parallel to GH and other is parallel to FG.

Also, the figure is divided into four regions named as I, II, III and IV.

Now,

$\therefore$ Area of square EFGH = a^2 sq. unit ...(i)

Region I is a square of side measuring $(a - b)$ unit.

$\therefore$ Area of region I = $(a - b)^2$ sq. unit ...(ii)

Each of regions II and IV is a rectangle having length and breadth as $(a - b)$ unit and b unit respectively.

$\therefore$ Area of region II = $b(a - b)$ sq. unit ...(iii)

And,

Area of region IV = $b(a - b)$ sq. unit ...(iv)

Region III is a square of side measuring b unit.

$\therefore$ Area of region III = b^2 sq. unit ...(v)

From the figure, we have

Area of region I = Area of square ABCD − (Area of region II + Area of region III + Area of region IV)

On substituting the values from (i), (ii), (iii), (iv) and (v), we get

$(a - b)^2 = a^2 - [b(a - b) + b^2 + b(a - b)]$

$(a - b)^2 = a^2 - [ab - b^2 + b^2 + ab - b^2]$

$(a - b)^2 = a^2 - [2ab - b^2]$

$$\boxed{\Rightarrow (a - b)^2 = a^2 - 2ab + b^2}$$

The identities we have proved above are known as identity because for any value of a and b, the LHS is always equal to the RHS. The difference between an identity and an equation is that for an equation, its LHS and RHS are equal only for some values of the variable.
On the other hand, as we discussed, for an identity, the LHS equals the RHS for any value of the variable.

Many a times, these identities help in shortening our calculations. Let us discuss some examples using the above identities to understand this better.

Example 1:
Simplify the following expressions using suitable identities:
(a) $(2m + 3n)^2$
(b) $(4p - 7q)^2$

Solution:

(a)
On comparing the given expression $(2m + 3n)^2$ with $(a + b)^2$, we get $a = 2m$ and $b = 3n$.

Now,
$(a + b)^2 = a^2 + 2ab + b^2$

Thus,

$$\begin{aligned}(2m+3n)^2 &= (2m)^2 + 2(2m)(3n) + (3n)^2 \\ &= 4m^2 + 12mn + 9n^2\end{aligned}$$

(b)

On comparing the given expression $(4p - 7q)^2$ with $(a - b)^2$, we get $a = 4p$ and $b = 7q$.

Now,
$(a - b)^2 = a^2 - 2ab + b^2$

Thus,
$(4p - 7q)^2 = (4p)^2 - 2(4p)(7q) + (7q)^2$

$= 16p^2 - 56pq + 49q^2$

Example 2:

Simplify the following expressions using suitable identities:

(a) $(3ax + 5by)^2$

(b) $(0.6a^2 - 0.04b^3)^2$

(c) $\left(\frac{3}{7}l + \frac{4}{5}m\right)^2$

Solution:

(a) The given expression is $(3ax + 5by)^2$, which is of the form $(a+ b)^2$.

Thus, we can use the identity $(a + b)^2 = a^2 + 2ab + b^2$.

$\therefore (3ax + 5by)^2 = (3ax)^2 + 2\,(3ax)\,(5by) + (5by)^2$

$= 9a^2x^2 + 30abxy + 25b^2y^2$

(b) The given expression is $(0.6a^2 - 0.04b^3)^2$, which is of the form $(a - b)^2$.

Thus, we can use the identity $(a - b)^2 = a^2 - 2ab + b^2$.

$\therefore (0.6a^2 - 0.04b^3)^2 = (0.6a^2)^2 - 2\,(0.6a^2)\,(0.04b^3) + (0.04b^3)^2$

$= 0.36a^4 - 0.048a^2b^3 + 0.0016b^6$

(c) The given expression is $\left(\frac{3}{7}l + \frac{4}{5}m\right)^2$, which is of the form $(a + b)^2$.

Thus, we can use the identity $(a + b)^2 = a^2 + 2ab + b^2$.

$$\therefore \left(\frac{3}{7}l + \frac{4}{5}m\right)^2 = \left(\frac{3}{7}l\right)^2 + 2\left(\frac{3}{7}l \times \frac{4}{5}m\right) + \left(\frac{4}{5}m\right)^2$$
$$= \frac{9}{49}l^2 + \frac{24}{35}lm + \frac{16}{25}m^2$$

Example 3:

Find the value of $(208)^2$ using a suitable identity.

Solution:

$208 = 200 + 8$

$\therefore (208)^2 = (200 + 8)^2$

Thus, we can use the identity $(a + b)^2 = a^2 + 2ab + b^2$.

$\therefore (208)^2 = (200 + 8)^2$

$= (200)^2 + 2\,(200)\,(8) + (8)^2$

$= 40000 + 3200 + 64$

= 43264

Example 4:

Find the value of $(99)^2$ using a suitable identity.

Solution:

99 = 100 − 1

∴ $(99)^2 = (100 - 1)^2$

Thus, we can use the identity $(a - b)^2 = a^2 - 2ab + b^2$.

∴ $(99)^2 = (100 - 1)^2 = (100)^2 - 2\,(100)\,(1) + (1)^2$

= 10000 − 200 + 1

= 9800 + 1

= 9801

Example 5:

(a) If $x - \frac{1}{x} = 3$, then find the value of the expressions $x^2 + \frac{1}{x^2}$ and $x^4 + \frac{1}{x^4}$.

(b) If $2y + \frac{3}{y} = 5$, then find the value of the expression $4y^2 + \frac{9}{y^2}$.

(c) If $3x - 5y = -1$ and $xy = 6$, then find the value of the expression $9x^2 + 25y^2$.

Solution:

(a) It is given that $x - \frac{1}{x} = 3$.

On squaring both sides, we get

$$\left(x-\frac{1}{x}\right)^2=3^2$$

$$\Rightarrow (x)^2-2(x)\left(\frac{1}{x}\right)+\left(\frac{1}{x}\right)^2=9 \qquad \left[\text{Using the identity } (a-b)^2=a^2-2ab+b^2\right]$$

$$\Rightarrow x^2-2+\frac{1}{x^2}=9$$

$$\Rightarrow x^2+\frac{1}{x^2}=9+2$$

$$\Rightarrow x^2+\frac{1}{x^2}=11$$

Now, on squaring both sides again, we get

$$\left(x^2+\frac{1}{x^2}\right)^2=11^2$$

$$\Rightarrow \left(x^2\right)^2+2\left(x^2\right)\left(\frac{1}{x^2}\right)+\left(\frac{1}{x^2}\right)^2=121 \qquad \left[\text{Using the identity } (a+b)^2=a^2+2ab+b^2\right]$$

$$\Rightarrow x^4+2+\frac{1}{x^4}=121$$

$$\Rightarrow x^4+\frac{1}{x^4}=119$$

Thus, the value of the expression $\left(x^2+\frac{1}{x^2}\right)$ is 11 and the value of the expression $\left(x^4+\frac{1}{x^4}\right)$ is 119.

(b) It is given that $2y+\frac{3}{y}=5$.

On squaring both sides, we get

$$\left(2y+\frac{3}{y}\right)^2=5^2$$

$$\Rightarrow (2y)^2 + 2(2y)\left(\frac{3}{y}\right) + \left(\frac{3}{y}\right)^2 = 25$$
$$\Rightarrow 4y^2 + 12 + \frac{9}{y^2} = 25$$
$$\Rightarrow 4y^2 + \frac{9}{y^2} = 13$$

Thus, the value of the expression $\left(4y^2 + \frac{9}{y^2}\right)$ is 13.

(c) It is given that $3x - 5y = -1$.

On squaring both sides, we get

$$(3x-5y)^2 = (-1)^2$$
$$\Rightarrow (3x)^2 - 2(3x)(5y) + (5y)^2 = 1$$
$$\Rightarrow 9x^2 - 30xy + 25y^2 = 1$$
$$\Rightarrow 9x^2 - 30\times 6 + 25y^2 = 1 \qquad (xy = 6)$$
$$\Rightarrow 9x^2 + 25y^2 = 1 + 180$$
$$\Rightarrow 9x^2 + 25y^2 = 181$$

Thus, the value of the expression ($9x^2 + 25y^2$) is 181.

Example 6:

Prove that

(a) $(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$

(b) $\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab$

Solution:

(a) We know that $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$. Thus,

LHS = $(a + b)^2 + (a - b)^2$

$= a^2 + 2ab + b^2 + a^2 - 2ab + b^2$

$= (a^2 + a^2) + (b^2 + b^2) + (2ab - 2ab)$

$= 2a^2 + 2b^2$

$= 2(a^2 + b^2) =$ RHS

Hence, proved.

(b) We know that $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$. Thus,

$$\text{LHS} = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

$$= \left(\frac{a^2+2ab+b^2}{4}\right) - \left(\frac{a^2-2ab+b^2}{4}\right)$$

$$= \frac{a^2+2ab+b^2-a^2+2ab-b^2}{4}$$

$$= \frac{4ab}{4}$$

$$= ab = \text{RHS}$$

Hence, proved.

Using Identity for "Difference of Two Squares"

Suppose we need to find the product of the numbers 79 and 81. Instead of multiplying these two numbers, we can use the identity $(a + b)(a - b)$. This identity is very important and is applicable in various situations.

Let us first understand this identity.

$(a + b)(a - b) = a(a - b) + b(a - b)$ (By distributive property)

$= a^2 - ab + ba - b^2$

$= a^2 - ab + ab - b^2$ $(ab = ba)$

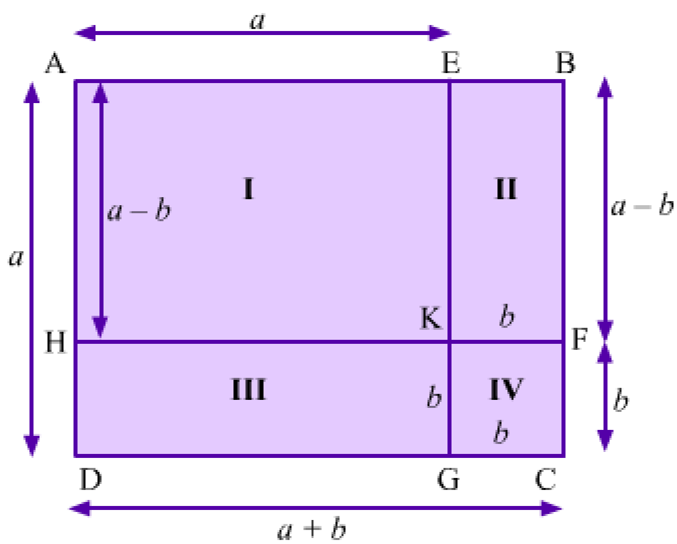
$= a^2 - b^2$

$\therefore$ **(a + b) (a − b) = a^2 − b^2**

Deriving the identity geometrically:

This identity can be derived from geometric construction as well.

For this, let us consider a square AEGD whose each side measures *a* units.



It can be seen that, a segment BC is drawn outside the square AEGD such that BC is parallel to EG and at a distance of *b* unit from G.

Another segment HF is drawn inside the square AEGD such that HF is parallel to GD and at a distance of *b* unit from G.

From the figure, it can be observed that

Area of rectangle ABFH = Area of rectangle ABCD – (Area of rectangle HKGD + Area of square KFCG)

$\Rightarrow (a + b)(a - b) = a(a + b) - [(a \times b) + (b \times b)]$

$\Rightarrow (a + b)(a - b) = a^2 + ab - ab - b^2$

$\Rightarrow$ **(a + b) (a − b) = a^2 − b^2**

Now, let us solve some examples in which the above identity can be applied.

Example 1:

Simplify the following expressions.

(a) (x + 3) (x − 3)

(b) (11 + y) (11 − y)

Solution:

(a) (x + 3) (x − 3)

This expression is of the form (a + b) (a − b).

Hence, we can use the identity (a + b) (a − b) = $a^2 - b^2$.

$(x + 3)(x - 3) = x^2 - 3^2 = x^2 - 9$

(b) (11 + y) (11 − y)

This expression is of the form (a + b) (a − b).

Hence, we can use the identity (a + b) (a − b) = $a^2 - b^2$.

$(11 + y)(11 - y) = 11^2 - y^2 = 121 - y^2$

Example 2:

Simplify the following expressions.

(a) $\left(\frac{3}{7}l+\frac{4}{5}m\right)\left(\frac{3}{7}l-\frac{4}{5}m\right)$

(b) $\left(x^2-y^3\right)\left(x^2+y^3\right)+\left(y^3-z^4\right)\left(y^3+z^4\right)+\left(z^4-x^2\right)\left(z^4+x^2\right)$

Solution:

(a) The given expression is $\left(\frac{3}{7}l+\frac{4}{5}m\right)\left(\frac{3}{7}l-\frac{4}{5}m\right)$.

Using the identity (a + b) (a − b) **=** $a^2 - b^2$, we get

$$\therefore \left(\frac{3}{7}l+\frac{4}{5}m\right)\left(\frac{3}{7}l-\frac{4}{5}m\right)=\left(\frac{3}{7}l\right)^2-\left(\frac{4}{5}m\right)^2$$
$$=\frac{9}{49}l^2-\frac{16}{25}m^2$$

(b) $\left(x^2-y^3\right)\left(x^2+y^3\right)+\left(y^3-z^4\right)\left(y^3+z^4\right)+\left(z^4-x^2\right)\left(z^4+x^2\right)$

Using the identity $(a + b)(a - b) = a^2 - b^2$, we get

$$\left\{\left(x^2\right)^2-\left(y^3\right)^2\right\}+\left\{\left(y^3\right)^2-\left(z^4\right)^2\right\}+\left\{\left(z^4\right)^2-\left(x^2\right)^2\right\}$$
$$=x^4-y^6+y^6-z^8+z^8-x^4$$
$$=\left(x^4-x^4\right)+\left(-y^6+y^6\right)+\left(-z^8+z^8\right)$$
$$=0$$

Example 3:

Find the values of the following expressions using suitable identities.

(a) 195 × 205

(b) $(993)^2 - (7)^2$

(c) 24.5 × 25.5

Solution:

(a) 195 = 200 − 5
and, 205 = 200 + 5

∴ 195 × 205 = (200 − 5) × (200 + 5)

$= (200)^2 - (5)^2$ [$\because (a + b)(a - b) = a^2 - b^2$]

= 40000 − 25

= 39975

(b) $(993)^2 - (7)^2$
= (993 + 7) (993 − 7) [$\because (a + b)(a - b) = a^2 - b^2$]

= (1000) (986)

= 986000

(c) 24.5 × 25.5

= (25 − 0.5) (25 + 0.5)

= $(25)^2 - (0.5)^2$ [$\because (a + b)(a - b) = a^2 - b^2$]

= 625 − 0.25

= 624.75

Factorisation of Algebraic Expressions Using Identities (a + b)2, (a - b)2 and a2 - b2

We know the identities

(i). $a^2 + 2ab + b^2 = (a + b)^2$

(ii). $a^2 - 2ab + b^2 = (a - b)^2$

(iii). $a^2 - b^2 = (a + b)(a - b)$

We can use these identities to factorise algebraic expressions as well. Let us discuss each identity one by one.

Application of the identity $a^2 + 2ab + b^2 = (a + b)^2$ to factorise an algebraic expression

Let us factorise the expression $x^2 + 6x + 9$.

In this expression, the first term is the square of x, the last term is the square of 3, and the middle term is positive and is twice the product of x and 3.

Thus, $x^2 + 6x + 9$ can be written as

$x^2 + 6x + 9 = (x)^2 + 2 \times x \times 3 + (3)^2$

The right hand side of this expression is in the form of $a^2 + 2ab + b^2$, where $a = x$ and $b = 3$.

We know the identity $a^2 + 2ab + b^2 = (a + b)^2$.

$\therefore (x)^2 + 2 \times x \times 3 + (3)^2 = (x + 3)^2$

Thus, $x^2 + 6x + 9 = (x + 3)^2$.

Application of the identity $a^2 - 2ab + b^2 = (a - b)^2$ to factorise an algebraic expression

Let us factorise the expression $9y^2 - 12y + 4$.

In this expression, the first term is the square of $3y$, the last term is the square of 2, and the middle term is negative and is twice of the product of $3y$ and 2.

Thus, $9y^2 - 12y + 4$ can be written as

$9y^2 - 12y + 4 = (3y)^2 - 2 \times 3y \times 2 + (2)^2$

The right hand side of this expression is in the form of $a^2 - 2ab + b^2$, where $a = 3y$ and

$b = 2$.

We know the identity $a^2 - 2ab + b^2 = (a - b)^2$.

$\therefore (3y)^2 - 2 \times 3y \times 2 + (2)^2 = (3y - 2)^2$

Thus, $9y^2 - 12y + 4 = (3y - 2)^2$.

Application of the identity $a^2 - b^2 = (a + b)(a - b)$ to factorise an algebraic expression

We use this identity when an expression is given as the difference of two squares.

Let us factorise the expression $x^2 - 25$.

We can write it as

$x^2 - 25 = (x)^2 - (5)^2$

The right hand side of this expression is in the form of $a^2 - b^2$, where $a = x$ and $b = 5$

On using the identity $a^2 - b^2 = (a + b)(a - b)$, we obtain

$x^2 - 25 = (x + 5)(x - 5)$

Thus, $(x + 5)$ and $(x - 5)$ are the factors of $x^2 - 25$.

To factorise an algebraic expression, we have to observe the given expression. If it has a form that fits the left hand side of one of the identities mentioned in the beginning, then the expression corresponding to the right hand side of the identity gives the desired factorisation.

Let us discuss some more examples based on what we have discussed so far.

Example 1:

Factorise the given expressions.

1. $\mathbf{25x^2 + 40xy + 16y^2}$
2. $\mathbf{81x^3 + x - 18x^2}$
3. $\mathbf{(p + 1)^2 - (p - 1)^2}$
4. $\mathbf{16a^2 - 25b^2 + 60bc - 36c^2}$
5. $\mathbf{al^2 - bm^2 - am^2 + bl^2}$
6. $\mathbf{81x^4 - 256\ y^4}$
7. $\mathbf{16x^4 - (3a + 5c)^4}$

Solution:

(1) The given expression is $25x^2 + 40xy + 16y^2$.

$25x^2 + 40xy + 16y^2 = (5x)^2 + 2 \times 5x \times 4y + (4y)^2$

$= (5x + 4y)^2$ [$a^2 + 2ab + b^2 = (a + b)^2$]

$\therefore 25x^2 + 40xy + 16y^2 = (5x + 4y)^2$

(2) The given expression is $81x^3 + x - 18x^2$.

Here, x is a factor common to all terms in the expression.

$\therefore 81x^3 + x - 18x^2 = x\ (81x^2 + 1 - 18x)$

$= x\ [(9x)^2 + (1)^2 - 2 \times 9x \times 1]$

$= x\ [9x - 1]^2$ [$a^2 - 2ab + b^2 = (a - b)^2$)

$\therefore 81x^3 + x - 18x^2 = x\ (9x - 1)^2$

(3) The given expression is $(p + 1)^2 - (p - 1)^2$.

On using the identity $a^2 - b^2 = (a + b)(a - b)$, we obtain

$(p + 1)^2 - (p - 1)^2 = \{(p + 1) + (p - 1)\}\{(p + 1) - (p - 1)\}$

$= (p + 1 + p - 1)(p + 1 - p + 1)$

$= (2p)(2) = 4p$

$\therefore (p + 1)^2 - (p - 1)^2 = 4p$

(4) The given expression is $16a^2 - 25b^2 + 60bc - 36c^2$.

$16a^2 - 25b^2 + 60bc - 36c^2$

$= 16a^2 - (25b^2 - 60bc + 36c^2)$

$= (4a)^2 - \{(5b)^2 - 2(5b)(6c) + (6c)^2\}$

= $(4a)^2 - (5b - 6c)^2$ [Using the identity $a^2 - 2ab + b^2 = (a - b)^2$]

= $\{(4a) + (5b - 6c)\}\{(4a) - (5b - 6c)\}$[Using the identity $a^2 - b^2 = (a + b)(a - b)$]

= $(4a + 5b - 6c)(4a - 5b + 6c)$

(5) The given expression is $al^2 - bm^2 - am^2 + bl^2$.

$al^2 - bm^2 - am^2 + bl^2$

$= al^2 - am^2 + bl^2 - bm^2$ {Regrouping the terms}

$= a(l^2 - m^2) + b(l^2 - m^2)$

$= (l^2 - m^2)(a + b)$

$= (l + m)(l - m)(a + b)$ [Using the identity $a^2 - b^2 = (a + b)(a - b)$]

(6) The given expression is $81x^4 - 256y^4$.

$81x^4 - 256y^4$

$$= \left(9x^2\right)^2 - \left(16y^2\right)^2$$

$$= \left(9x^2 + 16y^2\right)\left(9x^2 - 16y^2\right) \qquad \left[\text{Using the identity } a^2 - b^2 = (a+b)(a-b)\right]$$

$$= \left(9x^2 + 16y^2\right)\left\{(3x)^2 - (4y)^2\right\}$$

$$= \left(9x^2 + 16y^2\right)(3x + 4y)(3x - 4y) \qquad \left[\text{Using the identity } a^2 - b^2 = (a+b)(a-b)\right]$$

(7) The given expression is $16x^4 - (3a + 5c)^4$.

$16x^4 - (3a + 5c)^4$

$$= \left(4x^2\right)^2 - \left[(3a+5c)^2\right]^2$$

$$= \left\{4x^2 + (3a+5c)^2\right\}\left\{4x^2 - (3a+5c)^2\right\}$$

$$\left[a^2 - b^2 = (a+b)(a-b)\right]$$

$$= \left\{4x^2 + (3a)^2 + 2(3a)(5c) + (5c)^2\right\}\left\{(2x)^2 - (3a+5c)^2\right\}$$

$$\left[(a+b)^2 = a^2 + 2ab + b^2\right]$$

$$= \left(4x^2 + 9a^2 + 30ac + 25c^2\right)\left\{(2x) + (3a+5c)\right\}\left\{(2x) - (3a+5c)\right\}$$

$$\left[a^2 - b^2 = (a+b)(a-b)\right]$$

$$= \left(4x^2 + 9a^2 + 30ac + 25c^2\right)(2x + 3a + 5c)(2x - 3a - 5c)$$

Simplification of Algebraic Expressions Having Integral Denominators

Consider the fractions $\frac{12x^2y}{3y}, \frac{8x^2+14xy}{2x}, \frac{12x}{9}, \frac{7p^2+pq}{4q}, \frac{2p^2+5p+2}{p+2}, \frac{p+1}{2} + \frac{p}{3}$.

Observe that in these fractions, the numerator as well as the denominator both are polynomials. Such types of fractions are called **algebraic expressions**.

Fractions involving polynomial either in numerator or denominator (or both) are called algebraic fractions.

Now, look at the fraction $\frac{12x}{9}$.

$$\frac{12x}{9} = \frac{4x}{3}$$

Now, $\frac{4x}{3}$ cannot be simplified further as the numerator and denominator have no common factor other than 1. Therefore, $\frac{4x}{3}$ is the lowest form of the fraction $\frac{12x}{9}$.

An algebraic fraction is said to be in its simplest form or lowest form, if the numerator and denominator have no common factor, except 1.

Now, observe that amongst the above mentioned fractions, the fractions $\frac{12x}{9}$ and $\frac{p+1}{2}+\frac{p}{3}$ have integral denominators.

Can we simplify such type of fractions?

Now, we have seen that $\frac{12x}{9}$ can be simplified as $\frac{4x}{3}$.

In this way, we can simplify a given algebraic fraction with integral denominator.

Let us now look at some more examples to understand the concept better.

Example:

Simplify the following algebraic expressions:

(a) $\frac{4x}{3}-\frac{x}{4}+2$

(b) $6-\left(\frac{2p+1}{5}+\frac{p}{2}\right)$

(c) $\left(\frac{y}{2}+\frac{8y}{3}\right)$ of $\frac{12y}{18}$

(d) $\left(\frac{4p}{3}-\frac{4p}{5}\right)\div\left(\frac{1}{3p}+\frac{1}{p}\right)$

Solution:

(a) $\frac{4x}{3}-\frac{x}{4}+2$

$=\frac{4\times 4x-3\times x+12\times 2}{12}$ (Since LCM of 3 and 4 is 12)

$=\frac{16x-3x+24}{12}$

$=\frac{13x+24}{12}$

(b) $6-\left(\frac{2p+1}{5}+\frac{p}{2}\right)$

$=6-\left[\frac{2(2p+1)+5\times p}{10}\right]$ (LCM of 5 and 2 is 10)

$=6-\left(\frac{4p+2+5p}{10}\right)$

$=6-\left(\frac{9p+2}{10}\right)$

$=\frac{6\times 10-(9p+2)}{10}$

$=\frac{60-9p-2}{10}$

$=\frac{58-9p}{10}$

(c) $\left(\frac{y}{2}+\frac{8y}{3}\right)$ of $\frac{12y}{18}$

$=\left(\frac{3\times y+2\times 8y}{6}\right)$ of $\frac{12y}{18}$ [LCM of 2 and 3 is 6]

$=\left(\frac{3y+16y}{6}\right)$ of $\frac{2y}{3}$

$$= \frac{19y}{6} \times \frac{2y}{3}$$
$$= \frac{19y^2}{9}$$

(d) $\left(\frac{4p}{3} - \frac{4p}{5}\right) \div \left(\frac{1}{3p} + \frac{1}{p}\right)$

We have $\left(\frac{4p}{3} - \frac{4p}{5}\right) = \left(\frac{5 \times 4p - 3 \times 4p}{15}\right)$ [LCM of 3 and 5 is 15]

$$= \left(\frac{20p - 12p}{15}\right) = \frac{8p}{15}$$

$\left(\frac{1}{3p} + \frac{1}{p}\right) = \left(\frac{1 + 3 \times 1}{3p}\right)$ [LCM of 3*p* and *p* is 3*p*]

$$= \frac{4}{3p}$$
$$\therefore \left(\frac{4p}{3} - \frac{4p}{5}\right) \div \left(\frac{1}{3p} + \frac{1}{p}\right) = \frac{8p}{15} \div \frac{4}{3p}$$
$$= \frac{8p}{15} \times \frac{3p}{4}$$
$$= \frac{2p^2}{5}$$

Simplification of Expressions Involving Brackets

An algebraic expression may contain some brackets, namely line bracket, common bracket, curly bracket, or rectangular brackets, and some mathematical operations.

An expression enclosed within a bracket is considered as a single quantity even though it may consist of many terms.

Therefore, for simplifying an expression, we remove the bracket by the following rules.

(i) If '+' sign occurs before a bracket, then the signs of all the terms inside the bracket do not change.

(ii) If '−' sign occurs before a bracket, then the signs of all the terms inside the bracket change.

Brackets are removed in order of

(a) line brackets (b) common brackets

(c) curly brackets and lastly (d) rectangular brackets

It can be noted that the above rules apply when we insert a bracket.

Let us see how we simplify an algebraic expression by taking an example.

$$x-\left[2y+2\left\{y-\left(z-\overline{x+y}\right)\right\}\right]$$

Change the signs of the terms inside the line bracket as '−' sign occurs before the line bracket.

$$=x-\left[2y+2\{y-(z-x-y)\}\right]$$

Similarly, change the signs of terms inside the common bracket as '−' sign occurs before the common bracket.

$$=x-\left[2y+2\{y-z+x+y\}\right]=x-\left[2y+2\{2y-z+x\}\right]$$

Signs of terms inside the curly bracket remain unchanged as '+' sign occurs before it.

$$=x-[2y+4y-2z+2x]=x-[6y-2z+2x]$$

Change the signs of terms inside rectangular bracket as '−' sign occurs before it.

$$=x-6y+2z-2x$$
$$=-x-6y+2z$$

Example 1:

Simplify the following:

(a) $2y-\left\{y-\left(x-\overline{y+z}\right)\right\}$

(b) $4(2p-q)-3(p-\overline{q+2p})$

(c) $3e^2-\left[d^2-4\left\{f^2-\left(2e^2-\overline{f^2+d^2}\right)\right\}\right]$

Solution:

(a) $2y-\left\{y-\left(x-\overline{y+z}\right)\right\}$

$= 2y - \{y - (x - y - z)\}$[Line bracket is removed]

$= 2y - \{y - x + y + z\}$ [Line bracket is removed]

$= 2y - \{2y - x + z\}$

$= 2y - 2y + x - z = x - z$

(b) $4(2p-q)-3(p-\overline{q+2p})$

$= 8p - 4q - 3\ (p - q - 2p)$[One common bracket is removed and line bracket is removed in the other common bracket]

$= 8p - 4q - 3\ (- q - p)$

$= 8p - 4q + 3q + 3p$[Common bracket is removed]

$= 11p - q$

(c) $3e^2-\left[d^2-4\left\{f^2-\left(2e^2-\overline{f^2+d^2}\right)\right\}\right]$

$= 3e^2 - [d^2 - 4\ \{f^2 - (2e^2 - f^2 - d^2)\}]$[Line bracket is removed]

$= 3e^2 - [d^2 - 4\ \{f^2 - 2e^2 + f^2 + d^2\}]$[Common bracket is removed]

$= 3e^2 - [d^2 - 4\ \{2f^2 - 2e^2 + d^2\}]$

$= 3e^2 - [d^2 - 8f^2 + 8e^2 - 4d^2]$[Curly bracket is removed]

$= 3e^2 - [-3d^2 - 8f^2 + 8e^2]$

$= 3e^2 + 3d^2 + 8f^2 - 8e^2$[Rectangular bracket is removed]

$= 3d^2 - 5e^2 + 8f^2$