COMPUTING MOMENTS OF COMPOUND DISTRIBUTIONS

R. Kaas University of Amsterdam
M.J. Goovaerts K.U. Leuven & University of Amsterdam

ABSTRACT. The first few moments of compound distributions may be obtained by conditioning on the number of terms. It is shown how this method can be adapted to construct a recursive scheme for computing higher order moments of compound distributions.

1. INTRODUCTION

Suppose that the random variable S, representing e.g. the total amount of claims on an insurance portfolio in a certain year, may be written as

$$S = \sum_{i=1}^{N} X_i \tag{1}$$

where $X_1, X_2, \ldots$ are i.i.d. random variables (claim-sizes), independent of the random variable N (the number of claims). We know two algorithms to compute the moments of S when N is Poisson distributed. When λ is the Poisson parameter and $p_j = EX^j$, according to Shiu (1977) we have

$$E(S-\lambda p_1)^k = k!\left\{\lambda \frac{p_k}{k!} + \frac{\lambda^2}{2!}\sum \frac{p_l p_m}{l!m!} + \frac{\lambda^3}{3!}\sum \frac{p_l p_m p_n}{l!m!n!} + \ldots\right\}$$

(the summations are over all values of l, m, n, ... satisfying l, m, n, ... $\geq$ 2 as well as l+m+n+...=k), and in Goovaerts et al. (1984) the following useful recursive formula may be found (page 12):

$$E(S-\lambda p_1)^k = \lambda \sum_{t=0}^{k-2} \binom{k-1}{t} E(S-\lambda p_1)^t p_{k-t} \tag{2}$$

We will present in section 2 a recursive scheme to compute moments $E(S^k)$ when the distribution of N is arbitrary. The number of arithmetic operations required for computing $E(S^k)$ increases with k^3, the storage needed is proportional to k^2.

M. Goovaerts et al. (eds.), Insurance and Risk Theory, 381–384.

In section 3 we show how by the same algorithm the moments of the ruin probability function ψ can be computed. One of the main reasons one might be interested in computing moments of compound distributions can be found in a lot of results recently obtained for calculating actuarial quantities such as stop-loss premiums, ruin probabilities and claimsize distributions in case only incomplete information is available, such as some moments of the claimsize distribution per accident. The interested reader is referred to Kaas & Goovaerts (1985) and Goovaerts & Kaas (1985).

2. ALGORITHM

First we will compute conditional expectations of S given N=n. Observe that by symmetry, Newton's Binomial Theorem and independence, for all n=0,1,...

$$
\begin{aligned}
E\Big(\sum_{i=1}^{n} X_i\Big)^k &= \sum_{i=1}^{n} EX_i\Big(\sum_{j=1}^{n} X_j\Big)^{k-1} \\
&= nEX_n\Big(\sum_{j=1}^{n} X_j\Big)^{k-1} \\
&= nEX_n \sum_{t=0}^{k-1}\binom{k-1}{t} X_n^t\Big(\sum_{j=1}^{n-1} X_j\Big)^{k-1-t} \\
&= n\sum_{t=0}^{k-1}\binom{k-1}{t} p_{t+1} E\Big(\sum_{j=1}^{n-1} X_j\Big)^{k-1-t} \qquad (3)
\end{aligned}
$$

Letting $n^{!k}=n(n-1)\ldots(n-k+1)$, we will show that coefficients a_{jk}, $j=1,2,\ldots,k$; $k=1,2,\ldots$ exist, such that for all $n=1,2,\ldots$

$$
E\Big(\sum_{i=1}^{n} X_i\Big)^k = \sum_{j=1}^{k} a_{jk} n^{!j} \qquad (4)
$$

Indeed, suppose that such a_{jl} have been computed for $l < k$, taking of course $a_{11}=p_1$, then by (3)

$$
\begin{aligned}
E\Big(\sum_{i=1}^{n} X_i\Big)^k &= n\left\{p_k + \sum_{t=0}^{k-2}\binom{k-1}{t} p_{t+1} \sum_{j=1}^{k-1-t} a_{j,k-1-t}(n-1)^{!j}\right\} \\
&= np_k + \sum_{t=0}^{k-2}\binom{k-1}{t} p_{t+1} \sum_{j=1}^{k-1-t} a_{j,k-1-t} n^{!(j+1)}
\end{aligned}
$$

$$= np_k + \sum_{j=1}^{k-1} n^{!(j+1)} \sum_{t=0}^{k-1-j} \binom{k-1}{t} p_{t+1} a_{j,k-1-t}$$

$$= np_k + \sum_{j=2}^{k} n^{!j} \sum_{t=0}^{k-j} \binom{k-1}{t} p_{t+1} a_{j-1,k-1-t}$$

$$= \sum_{j=1}^{k} a_{jk} n^{!j} \tag{5}$$

if we take $a_{1k}=p_k$, and for $j=2,3,\ldots,k$

$$a_{jk} = \sum_{t=0}^{k-j} \binom{k-1}{t} p_{t+1} a_{j-1,k-1-t} \tag{6}$$

Using (4) we directly obtain

$$E(S^k) = \sum_{n=0}^{\infty} P(N=n) E(S^k | N=n)$$

$$= \sum_{n=0}^{\infty} P(N=n) \sum_{j=1}^{k} a_{jk} n^{!j}$$

$$= \sum_{j=1}^{k} a_{jk} E(N^{!j}) \tag{7}$$

The coefficients a_{jk} in (7) are computed using (6); the factorial moments of N can be computed from the ordinary moments, but in fact often are more easily calculated themselves. In Janardan (1984) one finds expressions for factorial and ordinary moments of many counting distributions, including those used in actuarial work.

3. APPLICATION

It is well known that in a compound Poisson process the probability of non-ruin $\psi^*(u)$ with initial reserve u may be written as a compound geometric distribution. If θ is the safety loading, we have

$$\psi^*(u) = \frac{\theta}{1+\theta} \sum_{n=0}^{\infty} (1+\theta)^{-n} H^{*n}(u) \tag{8}$$

with the distribution function H defined by

$$H(x) = \int_0^x \frac{1-P(t)}{p_1}\,dt \quad \text{(for } x>0\text{, 0 elsewhere)} \tag{9}$$

For a proof, see e.g. Beekman (1974), p. 67/68.

We may apply the algorithm of the preceding section to compute the moments of ψ^*. The moments of H can be obtained by partial integration:

$$\begin{aligned}\int_0^\infty x^j dH(x) &= \frac{1}{p_1}\int_0^\infty x^j (1-P(x))\,dx \\ &= \frac{1}{p_1}\left\{\frac{x^{j+1}}{j+1}(1-P(x))\Big|_0^\infty + \int_0^\infty \frac{x^{j+1}}{j+1}\,dP(x)\right\} \\ &= \frac{p_{j+1}}{p_1(j+1)}\end{aligned} \tag{10}$$

Using the techniques of Kaas & Goovaerts (1985) it is then possible to determine bounds on the probability of ruin given some moments of the claim distribution and the Poisson parameter.

REFERENCES

Beekman, J.A. (1974). *Two stochastic processes*. Almqvist and Wiksell, Stockholm.

Goovaerts, M.J. & Kaas, R. (1985). Application of the problem of moments to derive integrals with integral constraints. *Insurance: Mathematics & Economics*.

Goovaerts, M.J., de Vylder, F. & Haezendonck, J. (1984). *Insurance premiums*. North-Holland.

Janardan, K.G. (1984). Moments of certain series distributions and their applications. *SIAM J. Appl. Math.*, 854-868.

Kaas, R. & Goovaerts, M.J. (1985). Best bounds for positive distributions with fixed moments. *Insurance: Mathematics & Economics*.

Shiu, E.S.W. (1977). Moments of two distributions in collective risk theory. *Scand. Act. J.*, 185-187.