Advanced Mathematics 2 - Linear Algebra

Chapter 5: The vector space R^n

Department of Mathematics
The FPT university

Chapter 5 Introduction

Chapter 5 Introduction

Topics:

Chapter 5 Introduction

Topics:

5.1 Subspaces and spanning

Chapter 5 Introduction

Topics:

5.1 Subspaces and spanning

5.2 Independence and dimension

Topics:

5.1 Subspaces and spanning

5.2 Independence and dimension

5.3 Orthogonality

Chapter 5 Introduction

Topics:

5.1 Subspaces and spanning
5.2 Independence and dimension
5.3 Orthogonality
5.4 Rank of a matrix

Chapter 5 Introduction

Topics:

5.1 Subspaces and spanning

5.2 Independence and dimension

5.3 Orthogonality

5.4 Rank of a matrix

5.1 Subspaces and Spanning

5.1 Subspaces and Spanning

A subset U of R^n is called a subspace of R^n if:

A subset U of R^n is called a subspace of R^n if:

- $\overrightarrow{0} \in U$

A subset U of R^n is called a subspace of R^n if:

- $\overrightarrow{0} \in U$
- (Closed under addition) $\overrightarrow{u}, \overrightarrow{v} \in U \Longrightarrow \overrightarrow{u} + \overrightarrow{v} \in U$

5.1 Subspaces and Spanning

A subset U of R^n is called a subspace of R^n if:

- $\overrightarrow{0} \in U$
- (Closed under addition) $\overrightarrow{u}, \overrightarrow{v} \in U \Longrightarrow \overrightarrow{u} + \overrightarrow{v} \in U$
- (Closed under multiplication by scalars) $\overrightarrow{u} \in U \Longrightarrow k\overrightarrow{u} \in U$.

A subset U of R^n is called a subspace of R^n if:

- $\overrightarrow{0} \in U$
- (Closed under addition) $\overrightarrow{u}, \overrightarrow{v} \in U \Longrightarrow \overrightarrow{u} + \overrightarrow{v} \in U$
- (Closed under multiplication by scalars) $\overrightarrow{u} \in U \Longrightarrow k\overrightarrow{u} \in U$.

Example. Which of the following sets are subspaces?

A subset U of R^n is called a subspace of R^n if:

- $\overrightarrow{0} \in U$
- (Closed under addition) $\overrightarrow{u}, \overrightarrow{v} \in U \Longrightarrow \overrightarrow{u} + \overrightarrow{v} \in U$
- (Closed under multiplication by scalars) $\overrightarrow{u} \in U \Longrightarrow k\overrightarrow{u} \in U$.

Example. Which of the following sets are subspaces?

A subset U of R^n is called a subspace of R^n if:

- $\overrightarrow{0} \in U$
- (Closed under addition) $\overrightarrow{u}, \overrightarrow{v} \in U \Longrightarrow \overrightarrow{u} + \overrightarrow{v} \in U$
- (Closed under multiplication by scalars) $\overrightarrow{u} \in U \Longrightarrow k\overrightarrow{u} \in U$.

Example. Which of the following sets are subspaces?

(a) $U = \{(x, y) | x - 2y = 1\}$

A subset U of R^n is called a subspace of R^n if:

- $\overrightarrow{0} \in U$
- (Closed under addition) $\overrightarrow{u}, \overrightarrow{v} \in U \Longrightarrow \overrightarrow{u} + \overrightarrow{v} \in U$
- (Closed under multiplication by scalars) $\overrightarrow{u} \in U \Longrightarrow k\overrightarrow{u} \in U$.

Example. Which of the following sets are subspaces?

(a) $U = \{(x, y) | x - 2y = 1\}$

(b) $U = \{(x, y) | x - y^2 = 0\}$

A subset U of R^n is called a subspace of R^n if:

- $\overrightarrow{0} \in U$
- (Closed under addition) $\overrightarrow{u}, \overrightarrow{v} \in U \Longrightarrow \overrightarrow{u} + \overrightarrow{v} \in U$
- (Closed under multiplication by scalars) $\overrightarrow{u} \in U \Longrightarrow k\overrightarrow{u} \in U$.

Example. Which of the following sets are subspaces?

(a) $U = \{(x, y)|x - 2y = 1\}$

(b) $U = \{(x, y)|x - y^2 = 0\}$

(c) $U = \{(x, y)|x \geq 0, y \geq 0\}$

5.1 Subspaces and Spanning

A subset U of R^n is called a subspace of R^n if:

- $\overrightarrow{0} \in U$
- (Closed under addition) $\overrightarrow{u}, \overrightarrow{v} \in U \Longrightarrow \overrightarrow{u} + \overrightarrow{v} \in U$
- (Closed under multiplication by scalars) $\overrightarrow{u} \in U \Longrightarrow k\overrightarrow{u} \in U$.

Example. Which of the following sets are subspaces?

(a) $U = \{(x, y) | x - 2y = 1\}$

(b) $U = \{(x, y) | x - y^2 = 0\}$

(c) $U = \{(x, y) | x \geq 0, y \geq 0\}$

A subset U of R^n is called a subspace of R^n if:

- $\overrightarrow{0} \in U$
- (Closed under addition) $\overrightarrow{u}, \overrightarrow{v} \in U \Longrightarrow \overrightarrow{u} + \overrightarrow{v} \in U$
- (Closed under multiplication by scalars) $\overrightarrow{u} \in U \Longrightarrow k\overrightarrow{u} \in U$.

Example. Which of the following sets are subspaces?

(a) $U = \{(x, y) | x - 2y = 1\}$

(b) $U = \{(x, y) | x - y^2 = 0\}$

(c) $U = \{(x, y) | x \geq 0, y \geq 0\}$

(d) $U = \{(x, y) | x - 2y = 0\}$

A subset U of R^n is called a subspace of R^n if:

- $\overrightarrow{0} \in U$
- (Closed under addition) $\overrightarrow{u}, \overrightarrow{v} \in U \Longrightarrow \overrightarrow{u} + \overrightarrow{v} \in U$
- (Closed under multiplication by scalars) $\overrightarrow{u} \in U \Longrightarrow k\overrightarrow{u} \in U$.

Example. Which of the following sets are subspaces?

(a) $U = \{(x, y)|x - 2y = 1\}$

(b) $U = \{(x, y)|x - y^2 = 0\}$

(c) $U = \{(x, y)|x \geq 0, y \geq 0\}$

(d) $U = \{(x, y)|x - 2y = 0\}$

(e) $U = \{(r, s^3, t)|r, s, t \in R\}$

A subset U of R^n is called a subspace of R^n if:

- $\overrightarrow{0} \in U$
- (Closed under addition) $\overrightarrow{u}, \overrightarrow{v} \in U \Longrightarrow \overrightarrow{u} + \overrightarrow{v} \in U$
- (Closed under multiplication by scalars) $\overrightarrow{u} \in U \Longrightarrow k\overrightarrow{u} \in U$.

Example. Which of the following sets are subspaces?

(a) $U = \{(x, y) | x - 2y = 1\}$

(b) $U = \{(x, y) | x - y^2 = 0\}$

(c) $U = \{(x, y) | x \geq 0, y \geq 0\}$

(d) $U = \{(x, y) | x - 2y = 0\}$

(e) $U = \{(r, s^3, t) | r, s, t \in R\}$

(f) $U = \{(r, s^2, t) | r, s, t \in R\}$

A subset U of R^n is called a subspace of R^n if:

- $\overrightarrow{0} \in U$
- (Closed under addition) $\overrightarrow{u}, \overrightarrow{v} \in U \Longrightarrow \overrightarrow{u} + \overrightarrow{v} \in U$
- (Closed under multiplication by scalars) $\overrightarrow{u} \in U \Longrightarrow k\overrightarrow{u} \in U$.

Example. Which of the following sets are subspaces?

(a) $U = \{(x, y)|x - 2y = 1\}$

(b) $U = \{(x, y)|x - y^2 = 0\}$

(c) $U = \{(x, y)|x \geq 0, y \geq 0\}$

(d) $U = \{(x, y)|x - 2y = 0\}$

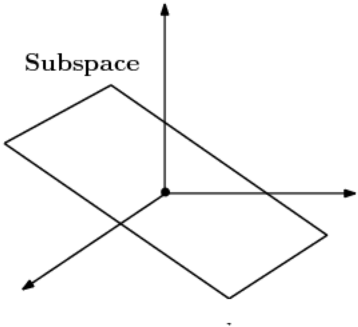
(e) $U = \{(r, s^3, t)|r, s, t \in R\}$

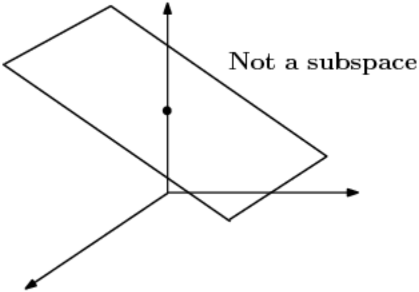
(f) $U = \{(r, s^2, t)|r, s, t \in R\}$

Note.

Note. Lines and planes that go through the origin are subspaces of R^3. Those that do not go through the origin are not subspaces.

Note. Lines and planes that go through the origin are subspaces of R^3. Those that do not go through the origin are not subspaces.





Spanning sets

Spanning sets

Spanning sets

- A linear combination of $\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}$ is a vector

$$t_1\overrightarrow{X_1} + t_2\overrightarrow{X_2} + \cdots + t_k\overrightarrow{X_k},$$

where $t_1, t_2, \ldots, t_n$ are real numbers.

Spanning sets

- A linear combination of $\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}$ is a vector
$$t_1\overrightarrow{X_1} + t_2\overrightarrow{X_2} + \cdots + t_k\overrightarrow{X_k},$$
where $t_1, t_2, \ldots, t_n$ are real numbers.
- $\text{Span}\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ = the set of all such linear combinations

- A linear combination of $\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}$ is a vector

$$t_1\overrightarrow{X_1} + t_2\overrightarrow{X_2} + \cdots + t_k\overrightarrow{X_k},$$

where $t_1, t_2, \ldots, t_n$ are real numbers.
- $\text{Span}\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ = the set of all such linear combinations

Example.

Spanning sets

- A linear combination of $\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}$ is a vector
$$t_1\overrightarrow{X_1} + t_2\overrightarrow{X_2} + \cdots + t_k\overrightarrow{X_k},$$
where $t_1, t_2, \ldots, t_n$ are real numbers.
- $\text{Span}\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ = the set of all such linear combinations

Example. Show that

Spanning sets

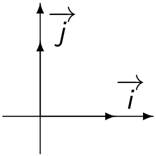
- A linear combination of $\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}$ is a vector

$$t_1\overrightarrow{X_1} + t_2\overrightarrow{X_2} + \cdots + t_k\overrightarrow{X_k},$$

  where $t_1, t_2, \ldots, t_n$ are real numbers.
- $\text{Span}\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ = the set of all such linear combinations

Example. Show that

$$span\{\overrightarrow{i}, \overrightarrow{j}\} = R^2$$



Theorem

Theorem

Span$\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ is a subspace.

Theorem

$\text{Span}\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ is a subspace.

Example 1.

Theorem

Span$\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ is a subspace.

Example 1. Check if the vector $[1, 2, 0]$ is in span$\{[1, 0, 1], [2, 1, 1]\}$

Theorem

$\text{Span}\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ is a subspace.

Example 1. Check if the vector $[1, 2, 0]$ is in span$\{[1, 0, 1], [2, 1, 1]\}$

Example 2.

Theorem

Span$\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ is a subspace.

Example 1. Check if the vector $[1, 2, 0]$ is in span$\{[1, 0, 1], [2, 1, 1]\}$

Example 2. Show that

$$span\{\overrightarrow{u}, \overrightarrow{v}\} = span\{\overrightarrow{u}, \overrightarrow{v}, 2\overrightarrow{u} + 3\overrightarrow{v}\}$$

Theorem

Span$\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ is a subspace.

Example 1. Check if the vector $[1, 2, 0]$ is in span$\{[1, 0, 1], [2, 1, 1]\}$

Example 2. Show that

$$span\{\overrightarrow{u}, \overrightarrow{v}\} = span\{\overrightarrow{u}, \overrightarrow{v}, 2\overrightarrow{u} + 3\overrightarrow{v}\}$$

5.2 Independence and dimension

The set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ is called linearly independent if, whenever

$$t_1\overrightarrow{X_1} + t_2\overrightarrow{X_2} + \cdots + t_n\overrightarrow{X_k} = 0$$

then $t_1 = t_2 = \cdots = t_k = 0$.

The set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \dots, \overrightarrow{X_k}\}$ is called linearly independent if, whenever

$$t_1\overrightarrow{X_1} + t_2\overrightarrow{X_2} + \cdots + t_n\overrightarrow{X_k} = 0$$

then $t_1 = t_2 = \cdots = t_k = 0$.

Examples. Which sets are linearly independent?

The set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ is called linearly independent if, whenever
$$t_1\overrightarrow{X_1} + t_2\overrightarrow{X_2} + \cdots + t_n\overrightarrow{X_k} = 0$$
then $t_1 = t_2 = \cdots = t_k = 0$.

Examples. Which sets are linearly independent?

(a) $\{[1, 0, -2], [2, 1, 0], [1, 1, 2]\}$

The set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ is called linearly independent if, whenever

$$t_1\overrightarrow{X_1} + t_2\overrightarrow{X_2} + \cdots + t_n\overrightarrow{X_k} = 0$$

then $t_1 = t_2 = \cdots = t_k = 0$.

Examples. Which sets are linearly independent?

(a) $\{[1, 0, -2], [2, 1, 0], [1, 1, 2]\}$

(b) $\{[1, 0, -2, 5], [2, 1, 0, -1], [1, 1, 2, 1]\}$

The set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ is called linearly independent if, whenever
$$t_1\overrightarrow{X_1} + t_2\overrightarrow{X_2} + \cdots + t_n\overrightarrow{X_k} = 0$$
then $t_1 = t_2 = \cdots = t_k = 0$.

Examples. Which sets are linearly independent?

(a) $\{[1, 0, -2], [2, 1, 0], [1, 1, 2]\}$

(b) $\{[1, 0, -2, 5], [2, 1, 0, -1], [1, 1, 2, 1]\}$

The set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ is called linearly independent if, whenever

$$t_1\overrightarrow{X_1} + t_2\overrightarrow{X_2} + \cdots + t_n\overrightarrow{X_k} = 0$$

then $t_1 = t_2 = \cdots = t_k = 0$.

Examples. Which sets are linearly independent?

(a) $\{[1, 0, -2], [2, 1, 0], [1, 1, 2]\}$
(b) $\{[1, 0, -2, 5], [2, 1, 0, -1], [1, 1, 2, 1]\}$

Note.

Note.

- The set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ is linearly independent if and only if none of these vectors is a linear combination of the others.

Note.

- The set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ is linearly independent if and only if none of these vectors is a linear combination of the others.
- The set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_k}\}$ is linearly independent if and only if the system
$$\begin{bmatrix} \overrightarrow{X_1} & \overrightarrow{X_2} & \cdots & \overrightarrow{X_k} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_k \end{bmatrix} = 0$$
has only the trivial solution $t_1 = t_2 = \cdots = t_k = 0$.

Theorem

Theorem

Let A be a square matrix. The following are equivalent:

Theorem

Let A be a square matrix. The following are equivalent:

- *A is invertible*

Theorem

Let A be a square matrix. The following are equivalent:

- *A is invertible*
- *The columns of A are linearly independent*

Theorem

Let A be a square matrix. The following are equivalent:

- *A is invertible*
- *The columns of A are linearly independent*
- *The rows of A are linearly independent*

Theorem

Let A be a square matrix. The following are equivalent:

- *A is invertible*
- *The columns of A are linearly independent*
- *The rows of A are linearly independent*

Theorem

Let A be a square matrix. The following are equivalent:

- *A is invertible*
- *The columns of A are linearly independent*
- *The rows of A are linearly independent*

Example. Determine whether the set $(1,2,1),(-1,1,2),(3,2,4)$ is linearly independent.

Theorem

Let A be a square matrix. The following are equivalent:

- *A is invertible*
- *The columns of A are linearly independent*
- *The rows of A are linearly independent*

Example. Determine whether the set $(1, 2, 1), (-1, 1, 2), (3, 2, 4)$ is linearly independent.

Fundamental Theorem

Theorem

Let A be a square matrix. The following are equivalent:

- *A is invertible*
- *The columns of A are linearly independent*
- *The rows of A are linearly independent*

Example. Determine whether the set $(1,2,1),(-1,1,2),(3,2,4)$ is linearly independent.

Fundamental Theorem

Let $U = span\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$. Then there are at most m linearly independent vectors in U.

Theorem

Let A be a square matrix. The following are equivalent:

- *A is invertible*
- *The columns of A are linearly independent*
- *The rows of A are linearly independent*

Example. Determine whether the set $(1,2,1),(-1,1,2),(3,2,4)$ is linearly independent.

Fundamental Theorem

Let $U = span\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$. Then there are at most m linearly independent vectors in U.

Basis

Basis

Basis

Let U be a subspace of R^n. A set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$ is called a basis of U if:

Basis

Let U be a subspace of R^n. A set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$ is called a basis of U if:

- $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$ is linearly independent.

Let U be a subspace of R^n. A set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$ is called a basis of U if:

- $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$ is linearly independent.
- $U = span\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$

Basis

Let U be a subspace of R^n. A set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$ is called a basis of U if:

- $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$ is linearly independent.
- $U = span\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$

Example 1. $\{\overrightarrow{i}, \overrightarrow{j}\}$ is a basis of R^2 because it is independent and $R^2 = span\{\overrightarrow{i}, \overrightarrow{j}\}$. It is called the standard basis of R^2.

Basis

Let U be a subspace of R^n. A set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$ is called a basis of U if:

- $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$ is linearly independent.
- $U = span\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$

Example 1. $\{\overrightarrow{i}, \overrightarrow{j}\}$ is a basis of R^2 because it is independent and $R^2 = span\{\overrightarrow{i}, \overrightarrow{j}\}$. It is called the standard basis of R^2.

Example 2. Show that the set $\{(1,0),(1,1)\}$ is another basis for R^2.

Basis

Let U be a subspace of R^n. A set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \dots, \overrightarrow{X_m}\}$ is called a basis of U if:

- $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \dots, \overrightarrow{X_m}\}$ is linearly independent.
- $U = span\{\overrightarrow{X_1}, \overrightarrow{X_2}, \dots, \overrightarrow{X_m}\}$

Example 1. $\{\overrightarrow{i}, \overrightarrow{j}\}$ is a basis of R^2 because it is independent and $R^2 = span\{\overrightarrow{i}, \overrightarrow{j}\}$. It is called the standard basis of R^2.

Example 2. Show that the set $\{(1,0), (1,1)\}$ is another basis for R^2.

Theorem-Definition

Theorem-Definition

- Any two bases of a subspace U have the same number of vectors.

Dimension

Theorem-Definition

- Any two bases of a subspace U have the same number of vectors.
- The dimension of U, denoted by $\dim U$, is the number of vectors in any basis of U.

Dimension

Theorem-Definition

- Any two bases of a subspace U have the same number of vectors.
- The dimension of U, denoted by $\dim U$, is the number of vectors in any basis of U.

Theorem-Definition

- Any two bases of a subspace U have the same number of vectors.
- The dimension of U, denoted by $\dim U$, is the number of vectors in any basis of U.

Examples.

Dimension

Theorem-Definition

- Any two bases of a subspace U have the same number of vectors.
- The dimension of U, denoted by $\dim U$, is the number of vectors in any basis of U.

Examples.

- $\dim R^2 = 2, \dim R^3 = 3, \dim R^n = n.$

Theorem-Definition

- Any two bases of a subspace U have the same number of vectors.
- The dimension of U, denoted by $\dim U$, is the number of vectors in any basis of U.

Examples.

- $\dim R^2 = 2, \dim R^3 = 3, \dim R^n = n.$

Theorem

Dimension

Theorem-Definition

- Any two bases of a subspace U have the same number of vectors.
- The dimension of U, denoted by $\dim U$, is the number of vectors in any basis of U.

Examples.

- $\dim R^2 = 2, \dim R^3 = 3, \dim R^n = n.$

Theorem

- Any largest set of independent vectors in U is a basis for U.

Dimension

Theorem-Definition

- Any two bases of a subspace U have the same number of vectors.
- The dimension of U, denoted by $\dim U$, is the number of vectors in any basis of U.

Examples.

- $\dim R^2 = 2, \dim R^3 = 3, \dim R^n = n$.

Theorem

- Any largest set of independent vectors in U is a basis for U.
- Dimension of U is the maximum number of linearly independent vectors in U.

Dimension

Theorem-Definition

- Any two bases of a subspace U have the same number of vectors.
- The dimension of U, denoted by $\dim U$, is the number of vectors in any basis of U.

Examples.

- $\dim R^2 = 2, \dim R^3 = 3, \dim R^n = n$.

Theorem

- Any largest set of independent vectors in U is a basis for U.
- Dimension of U is the maximum number of linearly independent vectors in U.

Theorem

Theorem

Let U be a subspace of R^n. Then:

Theorem

Let U be a subspace of R^n. Then:

- *U has a basis and* $\dim U \leq n$.

Theorem

Let U be a subspace of R^n. Then:

- *U has a basis and $\dim U \leq n$.*
- *Any independent set in U can be enlarged to a basis of U.*

Theorem

Let U be a subspace of R^n. Then:

- *U has a basis and* $\dim U \leq n$.
- *Any independent set in U can be enlarged to a basis of U.*
- *Let $U = span\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$.*

Theorem

Let U be a subspace of R^n. Then:

- *U has a basis and* $\dim U \leq n$.
- *Any independent set in U can be enlarged to a basis of U.*
- *Let $U = span\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$. If the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$ is dependent, then we can delete some vectors from this set to obtain a basis of U.*

Theorem

Let U be a subspace of R^n. Then:

- *U has a basis and $\dim U \leq n$.*
- *Any independent set in U can be enlarged to a basis of U.*
- *Let $U = span\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$. If the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$ is dependent, then we can delete some vectors from this set to obtain a basis of U.*

Theorem

Let U be a subspace of R^n. Then:

- *U has a basis and $\dim U \leq n$.*
- *Any independent set in U can be enlarged to a basis of U.*
- *Let $U = span\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$. If the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$ is dependent, then we can delete some vectors from this set to obtain a basis of U.*

Example. Find a basis and the dimension of the subspace

Example. Find a basis and the dimension of the subspace

(1) $U = Span\{[1, 1, 2], [1, -1, 0], [3, -1, 2]\}$

Example. Find a basis and the dimension of the subspace

(1) $U = Span\{[1, 1, 2], [1, -1, 0], [3, -1, 2]\}$

(2) $U = \{[a + 2b + c, a + 2b, c] |\ a, b, c \in R\}$

Example. Find a basis and the dimension of the subspace

(1) $U = Span\{[1,1,2],[1,-1,0],[3,-1,2]\}$

(2) $U = \{[a+2b+c, a+2b, c] |\ a,b,c \in R\}$

(3) $U = \{(x,y,z) |\ x+3y-z=0, x-y-z=0\}$

Example. Find a basis and the dimension of the subspace

(1) $U = Span\{[1, 1, 2], [1, -1, 0], [3, -1, 2]\}$

(2) $U = \{[a + 2b + c, a + 2b, c] |\ a, b, c \in R\}$

(3) $U = \{(x, y, z) |\ x + 3y - z = 0, x - y - z = 0\}$

(4) $U = \{(x, y, z, w) |\ x + 3y - z + w = 0\}$

Example. Find a basis and the dimension of the subspace

(1) $U = Span\{[1, 1, 2], [1, -1, 0], [3, -1, 2]\}$
(2) $U = \{[a + 2b + c, a + 2b, c] |\ a, b, c \in R\}$
(3) $U = \{(x, y, z) |\ x + 3y - z = 0, x - y - z = 0\}$
(4) $U = \{(x, y, z, w) |\ x + 3y - z + w = 0\}$

5.3 Orthogonality

5.3 Orthogonality

- A set $\{X_1, X_2, \dots X_k\}$ of vectors in R^n is called an orthogonal set if $X_i \neq 0$ for all i,

5.3 Orthogonality

- A set $\{X_1, X_2, \ldots X_k\}$ of vectors in R^n is called an orthogonal set if $X_i \neq 0$ for all i, and $X_i \cdot X_j = 0$ for all $i \neq j$.

5.3 Orthogonality

- A set $\{X_1, X_2, \ldots X_k\}$ of vectors in R^n is called an orthogonal set if $X_i \neq 0$ for all i, and $X_i \cdot X_j = 0$ for all $i \neq j$.
- A set $\{X_1, X_2, \ldots X_k\}$ is called an orthonormal set if it is orthogonal and $||X_i|| = 1$ for all i.

- A set $\{X_1, X_2, \dots X_k\}$ of vectors in R^n is called an orthogonal set if $X_i \neq 0$ for all i, and $X_i \cdot X_j = 0$ for all $i \neq j$.
- A set $\{X_1, X_2, \dots X_k\}$ is called an orthonormal set if it is orthogonal and $||X_i|| = 1$ for all i.

Example.

5.3 Orthogonality

- A set $\{X_1, X_2, \ldots X_k\}$ of vectors in R^n is called an orthogonal set if $X_i \neq 0$ for all i, and $X_i \cdot X_j = 0$ for all $i \neq j$.
- A set $\{X_1, X_2, \ldots X_k\}$ is called an orthonormal set if it is orthogonal and $||X_i|| = 1$ for all i.

Example. Is the set $\{[2, 1, 0], [1, -2, 0], [0, 0, 3]\}$ orthogonal?

- A set $\{X_1, X_2, \ldots X_k\}$ of vectors in R^n is called an orthogonal set if $X_i \neq 0$ for all i, and $X_i \cdot X_j = 0$ for all $i \neq j$.
- A set $\{X_1, X_2, \ldots X_k\}$ is called an orthonormal set if it is orthogonal and $||X_i|| = 1$ for all i.

Example. Is the set $\{[2, 1, 0], [1, -2, 0], [0, 0, 3]\}$ orthogonal? Is it orthonormal?

- A set $\{X_1, X_2, \ldots X_k\}$ of vectors in R^n is called an orthogonal set if $X_i \neq 0$ for all i, and $X_i \cdot X_j = 0$ for all $i \neq j$.
- A set $\{X_1, X_2, \ldots X_k\}$ is called an orthonormal set if it is orthogonal and $||X_i|| = 1$ for all i.

Example. Is the set $\{[2, 1, 0], [1, -2, 0], [0, 0, 3]\}$ orthogonal? Is it orthonormal? If it is not orthonormal, normalize it to obtain an orthonormal set.

- A set $\{X_1, X_2, \ldots X_k\}$ of vectors in R^n is called an orthogonal set if $X_i \neq 0$ for all i, and $X_i \cdot X_j = 0$ for all $i \neq j$.
- A set $\{X_1, X_2, \ldots X_k\}$ is called an orthonormal set if it is orthogonal and $||X_i|| = 1$ for all i.

Example. Is the set $\{[2, 1, 0], [1, -2, 0], [0, 0, 3]\}$ orthogonal? Is it orthonormal? If it is not orthonormal, normalize it to obtain an orthonormal set.

Pythagoras' Theorem

Pythagoras' Theorem

Let $\{X_1, X_2, \ldots X_k\}$ be an orthogonal set.

Pythagoras' Theorem

Let $\{X_1, X_2, \ldots X_k\}$ be an orthogonal set. Then

Pythagoras' Theorem

Let $\{X_1, X_2, \ldots X_k\}$ be an orthogonal set. Then

$$||X_1 + X_2 + \cdots + X_k||^2 = ||X_1||^2 + ||X_2||^2 + \cdots + ||X_k||^2$$

Pythagoras' Theorem

Let $\{X_1, X_2, \dots X_k\}$ be an orthogonal set. Then

$$||X_1 + X_2 + \cdots + X_k||^2 = ||X_1||^2 + ||X_2||^2 + \cdots + ||X_k||^2$$

Theorem

Pythagoras' Theorem

Let $\{X_1, X_2, \ldots X_k\}$ be an orthogonal set. Then

$$||X_1 + X_2 + \cdots + X_k||^2 = ||X_1||^2 + ||X_2||^2 + \cdots + ||X_k||^2$$

Theorem

Every orthogonal set is independent.

Pythagoras' Theorem

Let $\{X_1, X_2, \ldots X_k\}$ be an orthogonal set. Then

$$||X_1 + X_2 + \cdots + X_k||^2 = ||X_1||^2 + ||X_2||^2 + \cdots + ||X_k||^2$$

Theorem

Every orthogonal set is independent.

Expansion Theorem

Pythagoras' Theorem

Let $\{X_1, X_2, \ldots X_k\}$ be an orthogonal set. Then

$$||X_1 + X_2 + \cdots + X_k||^2 = ||X_1||^2 + ||X_2||^2 + \cdots + ||X_k||^2$$

Theorem

Every orthogonal set is independent.

Expansion Theorem

Let $\{F_1, F_2, \ldots, F_m\}$ be an orthogonal basis of a subspace U.

Pythagoras' Theorem

Let $\{X_1, X_2, \ldots X_k\}$ be an orthogonal set. Then

$$||X_1 + X_2 + \cdots + X_k||^2 = ||X_1||^2 + ||X_2||^2 + \cdots + ||X_k||^2$$

Theorem

Every orthogonal set is independent.

Expansion Theorem

Let $\{F_1, F_2, \ldots, F_m\}$ be an orthogonal basis of a subspace U. Let X be any vector in U.

Pythagoras' Theorem

Let $\{X_1, X_2, \ldots X_k\}$ be an orthogonal set. Then

$$||X_1 + X_2 + \cdots + X_k||^2 = ||X_1||^2 + ||X_2||^2 + \cdots + ||X_k||^2$$

Theorem

Every orthogonal set is independent.

Expansion Theorem

Let $\{F_1, F_2, \ldots, F_m\}$ be an orthogonal basis of a subspace U. Let X be any vector in U. Then:

Pythagoras' Theorem

Let $\{X_1, X_2, \dots X_k\}$ be an orthogonal set. Then

$$||X_1 + X_2 + \cdots + X_k||^2 = ||X_1||^2 + ||X_2||^2 + \cdots + ||X_k||^2$$

Theorem

Every orthogonal set is independent.

Expansion Theorem

Let $\{F_1, F_2, \dots, F_m\}$ be an orthogonal basis of a subspace U. Let X be any vector in U. Then:

$$X = \frac{X \cdot F_1}{F_1 \cdot F_1} F_1 + \frac{X \cdot F_2}{F_2 \cdot F_2} F_2 + \cdots + \frac{X \cdot F_m}{F_m \cdot F_m} F_m$$

5.4 Rank of a matrix

Problem.

Problem. Let $U = span\{\overrightarrow{X_1}, \overrightarrow{X_2}, \dots, \overrightarrow{X_m}\}$.

Problem. Let $U = span\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$. Find dim U and a basis for U.

Problem. Let $U = span\{\overrightarrow{X_1}, \overrightarrow{X_2}, \dots, \overrightarrow{X_m}\}$. Find dim U and a basis for U.

This is equivalent to determine a maximum set of independent vectors in the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \dots, \overrightarrow{X_m}\}$

Column space and Row space

Column space and Row space

- The column space of a matrix A, denoted by $\text{col}(A)$, is the subspace spanned by the columns of A.

Column space and Row space

- The column space of a matrix A, denoted by $\text{col}(A)$, is the subspace spanned by the columns of A.
- The row space of a matrix A, denoted by $\text{row}(A)$, is the subspace spanned by the rows of A.

Column space and Row space

- The column space of a matrix A, denoted by $\mathrm{col}(A)$, is the subspace spanned by the columns of A.
- The row space of a matrix A, denoted by $\mathrm{row}(A)$, is the subspace spanned by the rows of A.

- The column space of a matrix A, denoted by $\mathrm{col}(A)$, is the subspace spanned by the columns of A.
- The row space of a matrix A, denoted by $\mathrm{row}(A)$, is the subspace spanned by the rows of A.

Problem.

Column space and Row space

- The column space of a matrix A, denoted by $\mathrm{col}(A)$, is the subspace spanned by the columns of A.
- The row space of a matrix A, denoted by $\mathrm{row}(A)$, is the subspace spanned by the rows of A.

Problem. Determine dimensions and bases of the row and column spaces of a matrix.

- The column space of a matrix A, denoted by $\mathrm{col}(A)$, is the subspace spanned by the columns of A.
- The row space of a matrix A, denoted by $\mathrm{row}(A)$, is the subspace spanned by the rows of A.

Problem. Determine dimensions and bases of the row and column spaces of a matrix.

Algorithm for finding bases of column and row spaces

Algorithm for finding bases of column and row spaces

Algorithm for finding bases of column and row spaces

Let $A \stackrel{Echelon}{\Longrightarrow} R$. Then:

Algorithm for finding bases of column and row spaces

Let $A \overset{Echelon}{\Longrightarrow} R$. Then:

- The rows of R containing the leading 1 are a basis of the row space of A.

Let $A \overset{Echelon}{\Longrightarrow} R$. Then:

- The rows of R containing the leading 1 are a basis of the row space of A.
- The columns of A corresponding to the columns of R that contain the leading 1 are a basis of the column space of A.

Algorithm for finding bases of column and row spaces

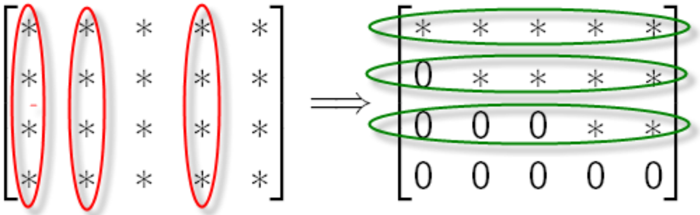
Let $A \overset{Echelon}{\Longrightarrow} R$. Then:

- The rows of R containing the leading 1 are a basis of the row space of A.
- The columns of A corresponding to the columns of R that contain the leading 1 are a basis of the column space of A.

Algorithm for finding bases of column and row spaces

Let $A \xRightarrow{Echelon} R$. Then:

- The rows of R containing the leading 1 are a basis of the row space of A.
- The columns of A corresponding to the columns of R that contain the leading 1 are a basis of the column space of A.



Example. Let $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$. Find bases and dimensions for col(A) and row(A).

Example. Let $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$. Find bases and dimensions for col(A) and row(A).

Solution. $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix} \xrightarrow{Echelon} \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Example. Let $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$. Find bases and dimensions for col(A) and row(A).

Solution. $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix} \overset{Echelon}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A basis of row space: $\{(1, 2, 2, -1), (0, 0, 1, -3)\}$

Example. Let $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$. Find bases and dimensions for col(A) and row(A).

Solution. $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix} \overset{Echelon}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A basis of row space: $\{(1, 2, 2, -1), (0, 0, 1, -3)\}$

A basis of column space: $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$.

Example. Let $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$. Find bases and dimensions for col(A) and row(A).

Solution. $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix} \xrightarrow{Echelon} \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A basis of row space: $\{(1, 2, 2, -1), (0, 0, 1, -3)\}$

A basis of column space: $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$.

$\dim(col(A)) = \dim(row(A)) = 2 = rank(A)$.

Example. Let $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$. Find bases and dimensions for col(A) and row(A).

Solution. $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix} \overset{Echelon}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A basis of row space: $\{(1, 2, 2, -1), (0, 0, 1, -3)\}$

A basis of column space: $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$.

$\dim(col(A)) = \dim(row(A)) = 2 = rank(A)$.

Null space and Image space

Null space and Image space

Null space and Image space

Let A be an $m \times n$ matrix.

Null space and Image space

Let A be an $m \times n$ matrix.

- The null space of a matrix A, denoted by $\text{null}(A)$, is the space of solutions of the system $AX = 0$

Null space and Image space

Let A be an $m \times n$ matrix.

- The null space of a matrix A, denoted by $\text{null}(A)$, is the space of solutions of the system $AX = 0$
- The image of a matrix A, denoted by $\text{im}(A)$, is the subspace consisting of vectors of the form AX, where $X \in R^n$.

Null space and Image space

Let A be an $m \times n$ matrix.

- The null space of a matrix A, denoted by $\text{null}(A)$, is the space of solutions of the system $AX = 0$
- The image of a matrix A, denoted by $\text{im}(A)$, is the subspace consisting of vectors of the form AX, where $X \in R^n$.

Note

Null space and Image space

Let A be an $m \times n$ matrix.

- The null space of a matrix A, denoted by $\text{null}(A)$, is the space of solutions of the system $AX = 0$
- The image of a matrix A, denoted by $\text{im}(A)$, is the subspace consisting of vectors of the form AX, where $X \in R^n$.

Note

Im(A) is the column space of A.

Null space and Image space

Let A be an $m \times n$ matrix.

- The null space of a matrix A, denoted by $\mathrm{null}(A)$, is the space of solutions of the system $AX = 0$
- The image of a matrix A, denoted by $\mathrm{im}(A)$, is the subspace consisting of vectors of the form AX, where $X \in R^n$.

Note

Im(A) is the column space of A.

Example. Let $A = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & -4 & 1 & 0 & 1 \end{bmatrix}$.

Example. Let $A = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & -4 & 1 & 0 & 1 \end{bmatrix}$.

We have

Example. Let $A = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & -4 & 1 & 0 & 1 \end{bmatrix}$.

We have

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & -4 & 1 & 0 & 1 \end{bmatrix} \xRightarrow{Echelon} \begin{bmatrix} 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example. Let $A = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & -4 & 1 & 0 & 1 \end{bmatrix}$.

We have

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & -4 & 1 & 0 & 1 \end{bmatrix} \xRightarrow{Echelon} \begin{bmatrix} 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $\dim(null(A)) = 2$ and $\dim(im(A)) = 3$.

Theorem

Theorem

Let A be an $m \times n$ matrix.

Theorem

Let A be an $m \times n$ matrix. Suppose rank(A)=r. Then:

Theorem

Let A be an $m \times n$ matrix. Suppose rank(A)=r. Then:

- $\dim(row(A)) = \dim(col(A)) = \dim(im(A)) = r$

Theorem

Let A be an $m \times n$ matrix. Suppose rank(A)=r. Then:

- $\dim(row(A)) = \dim(col(A)) = \dim(im(A)) = r$
- $\dim(null(A)) = n - r$