Advanced Mathematics - Linear Algebra

Chapter 1: Systems of linear equations

Department of Mathematics
The FPT university

Course Introduction

Course Introduction

Course name: MAE101-Advanced Mathematics: Linear Algebra

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Textbook: *Linear Algebra with applications*, K. Nicholson

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- Chapter 3: Determinants and Diagonalization

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Topics:

Chapter 1 Introduction

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1.1 Solutions and Elementary operations

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- In this example, a solution is $(1, 2, 3)$, or we can write $x = 1, y = 2, z = 3$.
- If a system has no solution we say it is inconsistent. If a system has at least a solution we say it is consistent.

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Multiply the first equation by -3 then add to the second equation we get

$$-7y = 14$$

which implies that $y = -2$.

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- Interchange two equations
- Multiply one equation by a nonzero number
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Note. There are possibly more than one echelon form for a given matrix.

Example 1. Solve the system

$$\begin{cases} x_1 - 2x_2 - x_3 + 3x_4 = 1 \\ 2x_1 - 4x_2 + x_3 = 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 = 4 \end{cases}$$

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- Non-leading variables are free parameters

Example 2.

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- has a unique solution
- has infinitely many solutions.

Reduced row-echelon form

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True or False?

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True or False?

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True or False?

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Theorem

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Consider a system of m linear equations in n variables. Suppose the rank of the augmented matrix is r. If the system has a solution then the number of free parameters in the solution is $n - r$.

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For a homogeneous linear system, if the number of variables is bigger than the number of equations, then the system has infinitely many solutions, or equivalently, it has a non-trivial solution.

1.4 Application: Network flow

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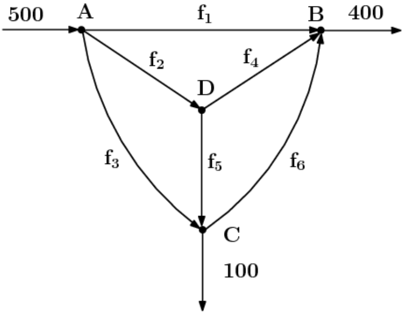

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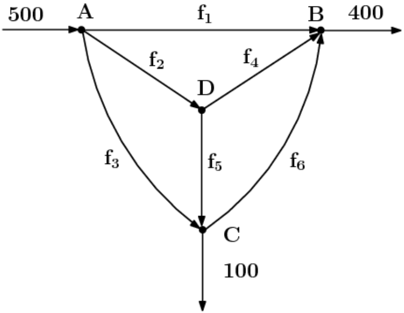
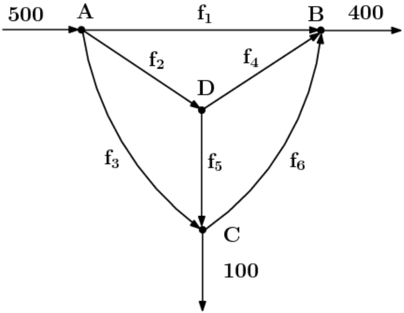


A network of one-way street is shown the diagram. The rate of flow of cars into intersection A is 500 cars per hour, and 400 and 100 cars per hour emerge from B and C. Find the possible flows along each street.



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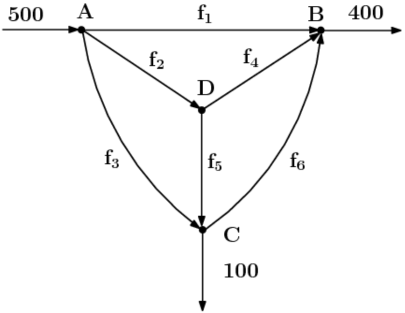

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$$\begin{cases} f_1 + f_2 + f_3 = 500 \\ f_1 + f_4 + f_6 = 400 \\ f_3 + f_5 - f_6 = 100 \\ f_2 - f_4 - f_5 = 0 \end{cases}$$