Advanced Mathematics - Linear Algebra

Chapter 3: Determinants and Diagonalization

Department of Mathematics
The FPT university

Chapter 3 Introduction

Chapter 3 Introduction

Topics:

Chapter 3 Introduction

Topics:

3.1 The cofactor expansion

Chapter 3 Introduction

Topics:

3.1 The cofactor expansion

3.2 Determinants and matrix inverses

Chapter 3 Introduction

Topics:

3.1 The cofactor expansion

3.2 Determinants and matrix inverses

3.3 Diagonalization and Eigenvalues

Chapter 3 Introduction

Topics:

Chapter 3 Introduction

Topics:

3.1 The cofactor expansion

3.2 Determinants and matrix inverses

3.3 Diagonalization and Eigenvalues

3.4 An applications to Linear recurrences

3.1 Cofactor expansion

3.1 Cofactor expansion

3.1 Cofactor expansion

Let $A = [a_{ij}]$ be an $n \times n$ matrix.

3.1 Cofactor expansion

Let $A = [a_{ij}]$ be an $n \times n$ matrix.

- A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith-row and the jth-column.

3.1 Cofactor expansion

Let $A = [a_{ij}]$ be an $n \times n$ matrix.

- A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith-row and the jth-column.
- The (i,j)-cofactor of A is the number

3.1 Cofactor expansion

Let $A = [a_{ij}]$ be an $n \times n$ matrix.

- A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith-row and the jth-column.
- The (i,j)-cofactor of A is the number

$$c_{ij}(A) = (-1)^{i+j} \det(A_{ij}).$$

3.1 Cofactor expansion

Let $A = [a_{ij}]$ be an $n \times n$ matrix.

- A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith-row and the jth-column.
- The (i,j)-cofactor of A is the number

$$c_{ij}(A) = (-1)^{i+j} \det(A_{ij}).$$

Example.

3.1 Cofactor expansion

Let $A = [a_{ij}]$ be an $n \times n$ matrix.

- A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith-row and the jth-column.
- The (i,j)-cofactor of A is the number

$$c_{ij}(A) = (-1)^{i+j} \det(A_{ij}).$$

Example. The $(2,3)$-cofactor of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is

Let $A = [a_{ij}]$ be an $n \times n$ matrix.

- A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith-row and the jth-column.
- The (i, j)-cofactor of A is the number

$$c_{ij}(A) = (-1)^{i+j} \det(A_{ij}).$$

Example. The $(2, 3)$-cofactor of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is

$$c_{23}(A) = (-1)^{2+3} \det \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} = 6$$

Determinants

Determinants

Determinants

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant of A is defined recursively as follows

Determinants

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant of A is defined recursively as follows

$$\det(A) = |A| = a_{11}c_{11}(A) + a_{12}c_{12}(A) + \cdots + a_{1n}c_{1n}(A)$$

Determinants

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant of A is defined recursively as follows

$$\det(A) = |A| = a_{11}c_{11}(A) + a_{12}c_{12}(A) + \cdots + a_{1n}c_{1n}(A)$$

This is called the cofactor expansion along the first row.

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant of A is defined recursively as follows

$$\det(A) = |A| = a_{11}c_{11}(A) + a_{12}c_{12}(A) + \cdots + a_{1n}c_{1n}(A)$$

This is called the cofactor expansion along the first row.

Example.

Determinants

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant of A is defined recursively as follows

$$\det(A) = |A| = a_{11}c_{11}(A) + a_{12}c_{12}(A) + \cdots + a_{1n}c_{1n}(A)$$

This is called the cofactor expansion along the first row.

Example. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} =$

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant of A is defined recursively as follows

$$\det(A) = |A| = a_{11}c_{11}(A) + a_{12}c_{12}(A) + \cdots + a_{1n}c_{1n}(A)$$

This is called the cofactor expansion along the first row.

Example. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} =$

$$= 1(-1)^{1+1}\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3(-1)^{1+3}\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 0$$

Theorem

Theorem

The determinant of a matrix can be calculated using the expansion along any row or any column.

Theorem

The determinant of a matrix can be calculated using the expansion along any row or any column.

Example. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \text{expansion along 2nd column } = 0$

Theorem

The determinant of a matrix can be calculated using the expansion along any row or any column.

Example. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} =$ expansion along 2nd column $= 0$

Note.

Theorem

The determinant of a matrix can be calculated using the expansion along any row or any column.

Example. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \text{expansion along 2nd column} = 0$

Note. When calculating determinants, choose rows or columns that have lots of 0.

Theorem

The determinant of a matrix can be calculated using the expansion along any row or any column.

Example. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \text{ expansion along 2nd column } = 0$

Note. When calculating determinants, choose rows or columns that have lots of 0.

Example.

Theorem

The determinant of a matrix can be calculated using the expansion along any row or any column.

Example. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} =$ expansion along 2nd column $= 0$

Note. When calculating determinants, choose rows or columns that have lots of 0.

Example. Evaluate $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \end{vmatrix}$

Theorem

The determinant of a matrix can be calculated using the expansion along any row or any column.

Example. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \text{ expansion along 2nd column } = 0$

Note. When calculating determinants, choose rows or columns that have lots of 0.

Example. Evaluate $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \end{vmatrix}$

Triangular matrices

Triangular matrices

Upper triangular matrices.

Triangular matrices

Upper triangular matrices.

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

Triangular matrices

Upper triangular matrices.

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

Lower triangular matrices.

Triangular matrices

Upper triangular matrices.

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

Lower triangular matrices.

$$\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$$

Triangular matrices

Upper triangular matrices.

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

Lower triangular matrices.

$$\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$$

Question.

Triangular matrices

Upper triangular matrices.

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

Lower triangular matrices.

$$\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$$

Question. Calculate determinants of triangular matrices.

Determinants and row operations

Determinants and row operations

Theorem

Theorem

Let A be a square matrix.

Theorem

Let A be a square matrix.

- *If A has a zero row or zero column then* $\det(A) = 0$

Determinants and row operations

Theorem

Let A be a square matrix.

- *If A has a zero row or zero column then* $\det(A) = 0$
- *If two rows or two columns are interchanged, the determinant changes sign*

Determinants and row operations

Theorem

Let A be a square matrix.

- *If A has a zero row or zero column then* $\det(A) = 0$
- *If two rows or two columns are interchanged, the determinant changes sign*
- *If two rows or two columns are identical then* $\det(A) = 0$

Determinants and row operations

Theorem

Let A be a square matrix.

- *If A has a zero row or zero column then* $\det(A) = 0$
- *If two rows or two columns are interchanged, the determinant changes sign*
- *If two rows or two columns are identical then* $\det(A) = 0$
- *If a row or a column is multiplied by a number c then the determinant is multiplied by c*

Determinants and row operations

Theorem

Let A be a square matrix.

- *If A has a zero row or zero column then* $\det(A) = 0$
- *If two rows or two columns are interchanged, the determinant changes sign*
- *If two rows or two columns are identical then* $\det(A) = 0$
- *If a row or a column is multiplied by a number c then the determinant is multiplied by c*
- *If a multiple of one row (column) is added to a another row (column), the determinant does not change.*

Determinants and row operations

Theorem

Let A be a square matrix.

- *If A has a zero row or zero column then* $\det(A) = 0$
- *If two rows or two columns are interchanged, the determinant changes sign*
- *If two rows or two columns are identical then* $\det(A) = 0$
- *If a row or a column is multiplied by a number c then the determinant is multiplied by c*
- *If a multiple of one row (column) is added to a another row (column), the determinant does not change.*

Example 1. Suppose $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 6,$

Example 1. Suppose $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 6$, find

$$\begin{vmatrix} a+x & b+y & c+z \\ 3x & 3y & 3z \\ -p & -q & -r \end{vmatrix}$$

Example 1. Suppose $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 6$, find

$$\begin{vmatrix} a+x & b+y & c+z \\ 3x & 3y & 3z \\ -p & -q & -r \end{vmatrix}$$

Example 2. Find all values of x such that the determinant of the matrix

Example 1. Suppose $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 6$, find

$$\begin{vmatrix} a+x & b+y & c+z \\ 3x & 3y & 3z \\ -p & -q & -r \end{vmatrix}$$

Example 2. Find all values of x such that the determinant of the matrix

$$\begin{bmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{bmatrix}$$

Example 1. Suppose $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 6$, find

$$\begin{vmatrix} a+x & b+y & c+z \\ 3x & 3y & 3z \\ -p & -q & -r \end{vmatrix}$$

Example 2. Find all values of x such that the determinant of the matrix

$$\begin{bmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{bmatrix}$$

is 0.

3.2 Determinants and inverses

Properties

Properties

Let A and B be $n \times n$ matrices, I the $n \times n$ identity matrix.

Properties

Let A and B be $n \times n$ matrices, I the $n \times n$ identity matrix.

- $\det(I) = 1$

Properties

Let A and B be $n \times n$ matrices, I the $n \times n$ identity matrix.

- $\det(I) = 1$
- $\det(A^T) = \det(A)$

Properties

Let A and B be $n \times n$ matrices, I the $n \times n$ identity matrix.

- $\det(I) = 1$
- $\det(A^T) = \det(A)$
- $\det(cA) = c^n \det(A)$

Properties

Let A and B be $n \times n$ matrices, I the $n \times n$ identity matrix.

- $\det(I) = 1$
- $\det(A^T) = \det(A)$
- $\det(cA) = c^n \det(A)$
- $\det(AB) = \det(A)\det(B)$

Properties

Let A and B be $n \times n$ matrices, I the $n \times n$ identity matrix.

- $\det(I) = 1$
- $\det(A^T) = \det(A)$
- $\det(cA) = c^n \det(A)$
- $\det(AB) = \det(A)\det(B)$

Theorem

Theorem

A is invertible if and only if $\det(A) \neq 0$.

Theorem

A is invertible if and only if $\det(A) \neq 0$. In this case,

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Theorem

A is invertible if and only if $\det(A) \neq 0$. In this case,

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Example. Let A, B be 3×3 matrices with $\det(A) = 2, \det(B) = 3$.

Theorem

A is invertible if and only if $\det(A) \neq 0$. In this case,

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Example. Let A, B be 3×3 matrices with $\det(A) = 2, \det(B) = 3$. Find $\det(5A^2B^TA^{-1})$.

Theorem

A is invertible if and only if $\det(A) \neq 0$. In this case,

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Example. Let A, B be 3×3 matrices with $\det(A) = 2, \det(B) = 3$. Find $\det(5A^2B^TA^{-1})$.

Adjugate matrix

Adjugate matrix

Adjugate matrix

Let A be an $n \times n$ matrix. The adjugate matrix of A, denoted by adj(A), is defined as

$$adj(A) = [c_{ij}(A)]^T = \begin{bmatrix} c_{11}(A) & c_{12}(A) & \cdots & c_{1n}(A) \\ c_{21}(A) & c_{22}(A) & \cdots & c_{2n}(A) \\ & & \cdots & \\ c_{n1}(A) & c_{n2}(A) & \cdots & c_{nn}(A) \end{bmatrix}^T$$

Adjugate matrix

Let A be an $n \times n$ matrix. The adjugate matrix of A, denoted by adj(A), is defined as

$$adj(A) = [c_{ij}(A)]^T = \begin{bmatrix} c_{11}(A) & c_{12}(A) & \cdots & c_{1n}(A) \\ c_{21}(A) & c_{22}(A) & \cdots & c_{2n}(A) \\ & & \cdots & \\ c_{n1}(A) & c_{n2}(A) & \cdots & c_{nn}(A) \end{bmatrix}^T$$

Example. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix}$. Then

Adjugate matrix

Let A be an $n \times n$ matrix. The adjugate matrix of A, denoted by $\mathrm{adj}(A)$, is defined as

$$adj(A) = [c_{ij}(A)]^T = \begin{bmatrix} c_{11}(A) & c_{12}(A) & \cdots & c_{1n}(A) \\ c_{21}(A) & c_{22}(A) & \cdots & c_{2n}(A) \\ & & \cdots & \\ c_{n1}(A) & c_{n2}(A) & \cdots & c_{nn}(A) \end{bmatrix}^T$$

Example. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix}$. Then

$$adj(A) = \begin{bmatrix} c_{11}(A) & c_{12}(A) & c_{13}(A) \\ c_{21}(A) & c_{22}(A) & c_{23}(A) \\ c_{31}(A) & c_{32}(A) & c_{33}(A) \end{bmatrix}^T = \begin{bmatrix} 1 & -11 & 2 \\ -5 & -5 & 5 \\ -2 & 7 & -4 \end{bmatrix}$$

Formula for inverses

Formula for inverses

Formula for inverses

- The square matrix A is invertible if and only if $\det(A) \neq 0$.

Formula for inverses

- The square matrix A is invertible if and only if $\det(A) \neq 0$.
- In this case,

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Formula for inverses

- The square matrix A is invertible if and only if $\det(A) \neq 0$.
- In this case,

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Example 1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix}$. Then

Formula for inverses

- The square matrix A is invertible if and only if $\det(A) \neq 0$.
- In this case,
$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Example 1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix}$. Then

$$A^{-1} = \frac{1}{\det(A)} adj(A) = \frac{-1}{15} \begin{bmatrix} 1 & -11 & 2 \\ -5 & -5 & 5 \\ -2 & 7 & -4 \end{bmatrix}$$

Formula for inverses

- The square matrix A is invertible if and only if $\det(A) \neq 0$.
- In this case,
$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Example 1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix}$. Then

$$A^{-1} = \frac{1}{\det(A)} adj(A) = \frac{-1}{15} \begin{bmatrix} 1 & -11 & 2 \\ -5 & -5 & 5 \\ -2 & 7 & -4 \end{bmatrix}$$

Example 2. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 3 & 1 \\ 2 & 1 & -4 \end{bmatrix}$.

- The square matrix A is invertible if and only if $\det(A) \neq 0$.
- In this case,
$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Example 1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix}$. Then

$$A^{-1} = \frac{1}{\det(A)} adj(A) = \frac{-1}{15} \begin{bmatrix} 1 & -11 & 2 \\ -5 & -5 & 5 \\ -2 & 7 & -4 \end{bmatrix}$$

Example 2. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 3 & 1 \\ 2 & 1 & -4 \end{bmatrix}$. Find the $(2,3)$-entry of A^{-1}

Cramer's rule

Cramer's rule

Consider the linear system $AX = B$. If A is invertible, then

$$x_i = \frac{\det(A_i)}{\det(A)},$$

$A_i =$ the matrix obtained from A by replacing the ith-column with B.

Consider the linear system $AX = B$. If A is invertible, then

$$x_i = \frac{\det(A_i)}{\det(A)},$$

$A_i =$ the matrix obtained from A by replacing the ith-column with B.

Example. Solve for x_2 from the system

Cramer's rule

Consider the linear system $AX = B$. If A is invertible, then

$$x_i = \frac{\det(A_i)}{\det(A)},$$

$A_i =$ the matrix obtained from A by replacing the ith-column with B.

Example. Solve for x_2 from the system

$$\begin{cases} x_1 + 2x_2 - 3x_3 = 1 \\ 3x_1 - 2x_2 + x_3 = 2 \\ 2x_1 + x_2 + 2x_3 = 3 \end{cases}$$

Cramer's rule

Consider the linear system $AX = B$. If A is invertible, then

$$x_i = \frac{\det(A_i)}{\det(A)},$$

$A_i =$ the matrix obtained from A by replacing the ith-column with B.

Example. Solve for x_2 from the system

$$\begin{cases} x_1 + 2x_2 - 3x_3 = 1 \\ 3x_1 - 2x_2 + x_3 = 2 \\ 2x_1 + x_2 + 2x_3 = 3 \end{cases}$$

Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

- Let A be a square matrix.

Eigenvalues and Eigenvectors

- Let A be a square matrix. A number λ is called an eigenvalue of A if $AX = \lambda X$ for some column vector $X \neq 0$.

Eigenvalues and Eigenvectors

- Let A be a square matrix. A number λ is called an eigenvalue of A if $AX = \lambda X$ for some column vector $X \neq 0$.
- Such a vector X is called an eigenvector corresponding to the eigenvalue λ, or a λ-eigenvector.

Eigenvalues and Eigenvectors

- Let A be a square matrix. A number λ is called an eigenvalue of A if $AX = \lambda X$ for some column vector $X \neq 0$.
- Such a vector X is called an eigenvector corresponding to the eigenvalue λ, or a λ-eigenvector.

Example.

3.3 Diagonalization and Eigenvalues

Eigenvalues and Eigenvectors

- Let A be a square matrix. A number λ is called an eigenvalue of A if $AX = \lambda X$ for some column vector $X \neq 0$.
- Such a vector X is called an eigenvector corresponding to the eigenvalue λ, or a λ-eigenvector.

Example. Let $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ and $X = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

Eigenvalues and Eigenvectors

- Let A be a square matrix. A number λ is called an eigenvalue of A if $AX = \lambda X$ for some column vector $X \neq 0$.
- Such a vector X is called an eigenvector corresponding to the eigenvalue λ, or a λ-eigenvector.

Example. Let $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ and $X = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

Then 4 is an eigenvalue for A and $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 4$.

Eigenvalues and Eigenvectors

- Let A be a square matrix. A number λ is called an eigenvalue of A if $AX = \lambda X$ for some column vector $X \neq 0$.
- Such a vector X is called an eigenvector corresponding to the eigenvalue λ, or a λ-eigenvector.

Example. Let $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ and $X = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

Then 4 is an eigenvalue for A and $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 4$.

How to find eigenvalues and eigenvectors?

How to find eigenvalues and eigenvectors?

How to find eigenvalues and eigenvectors?

Let A be a square matrix.

How to find eigenvalues and eigenvectors?

Let A be a square matrix.

- The characteristic polynomial of A is $c_A(x) = \det(xI - A)$.

How to find eigenvalues and eigenvectors?

Let A be a square matrix.

- The characteristic polynomial of A is $c_A(x) = \det(xI - A)$.
- Eigenvalues of A are the solutions of the equation $c_A(x) = 0$.

How to find eigenvalues and eigenvectors?

Let A be a square matrix.

- The characteristic polynomial of A is $c_A(x) = \det(xI - A)$.
- Eigenvalues of A are the solutions of the equation $c_A(x) = 0$.
- To find λ-eigenvectors, solve the system $(\lambda I - A)X = 0$.

How to find eigenvalues and eigenvectors?

Let A be a square matrix.

- The characteristic polynomial of A is $c_A(x) = \det(xI - A)$.
- Eigenvalues of A are the solutions of the equation $c_A(x) = 0$.
- To find λ-eigenvectors, solve the system $(\lambda I - A)X = 0$.

Example.

How to find eigenvalues and eigenvectors?

Let A be a square matrix.

- The characteristic polynomial of A is $c_A(x) = \det(xI - A)$.
- Eigenvalues of A are the solutions of the equation $c_A(x) = 0$.
- To find λ-eigenvectors, solve the system $(\lambda I - A)X = 0$.

Example. Find eigenvalues and eigenvectors of the matrices

How to find eigenvalues and eigenvectors?

Let A be a square matrix.

- The characteristic polynomial of A is $c_A(x) = \det(xI - A)$.
- Eigenvalues of A are the solutions of the equation $c_A(x) = 0$.
- To find λ-eigenvectors, solve the system $(\lambda I - A)X = 0$.

Example. Find eigenvalues and eigenvectors of the matrices

$$\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix},$$

How to find eigenvalues and eigenvectors?

Let A be a square matrix.

- The characteristic polynomial of A is $c_A(x) = \det(xI - A)$.
- Eigenvalues of A are the solutions of the equation $c_A(x) = 0$.
- To find λ-eigenvectors, solve the system $(\lambda I - A)X = 0$.

Example. Find eigenvalues and eigenvectors of the matrices

$$\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

How to find eigenvalues and eigenvectors?

Let A be a square matrix.

- The characteristic polynomial of A is $c_A(x) = \det(xI - A)$.
- Eigenvalues of A are the solutions of the equation $c_A(x) = 0$.
- To find λ-eigenvectors, solve the system $(\lambda I - A)X = 0$.

Example. Find eigenvalues and eigenvectors of the matrices

$$\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

Computing A^k

Computing A^k

If $A = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, a diagonal matrix, then

Computing A^k

If $A = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, a diagonal matrix, then

$$A^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

Computing A^k

If $A = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, a diagonal matrix, then

$$A^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

In general, if we can find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$, then $A = PDP^{-1}$, which implies that

Computing A^k

If $A = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, a diagonal matrix, then

$$A^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

In general, if we can find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$, then $A = PDP^{-1}$, which implies that

$$A^k = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1}) = PD^kP^{-1}.$$

Computing A^k

If $A = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, a diagonal matrix, then

$$A^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

In general, if we can find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$, then $A = PDP^{-1}$, which implies that

$$A^k = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1}) = PD^kP^{-1}.$$

Since D^k is easy to compute, A^k is now computable.

Diagonalization Algorithm

Diagonalization Algorithm

Diagonalization

Diagonalization Algorithm

Diagonalization

Let A be a square matrix. If there is an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ then we say the matrix A is diagonalizable.

Diagonalization

Let A be a square matrix. If there is an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ then we say the matrix A is diagonalizable.

Diagonalization Algorithm.

Diagonalization Algorithm

Diagonalization

Let A be a square matrix. If there is an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ then we say the matrix A is diagonalizable.

Diagonalization Algorithm.

Diagonalization Algorithm

Diagonalization

Let A be a square matrix. If there is an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ then we say the matrix A is diagonalizable.

Diagonalization Algorithm. Let A be an $n \times n$ matrix.

Diagonalization Algorithm

Diagonalization

Let A be a square matrix. If there is an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ then we say the matrix A is diagonalizable.

Diagonalization Algorithm. Let A be an $n \times n$ matrix.

- Find eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A.

Diagonalization Algorithm

Diagonalization

Let A be a square matrix. If there is an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ then we say the matrix A is diagonalizable.

Diagonalization Algorithm. Let A be an $n \times n$ matrix.

- Find eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A.
- For each eigenvalue compute basic eigenvectors.

Diagonalization Algorithm

Diagonalization

Let A be a square matrix. If there is an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ then we say the matrix A is diagonalizable.

Diagonalization Algorithm. Let A be an $n \times n$ matrix.

- Find eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A.
- For each eigenvalue compute basic eigenvectors.
- If there are exactly a total of n basic eigenvectors, then A is diagonalizable.

Diagonalization Algorithm

Diagonalization

Let A be a square matrix. If there is an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ then we say the matrix A is diagonalizable.

Diagonalization Algorithm. Let A be an $n \times n$ matrix.

- Find eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A.
- For each eigenvalue compute basic eigenvectors.
- If there are exactly a total of n basic eigenvectors, then A is diagonalizable.
- In this case D is the diagonal matrix with $\lambda_1, \lambda_2, \dots, \lambda_n$ in the diagonal, and P is the matrix whose columns are the basic eigenvectors.

Example. Diagonalize the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Example. Diagonalize the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Eigenvalues. $c_A(x) = (x-2)^2(x+1)$. Eigenvalues $2, -1, -1$.

Example. Diagonalize the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Eigenvalues. $c_A(x) = (x-2)^2(x+1)$. Eigenvalues $2, -1, -1$.

Eigenvectors.

$\lambda = 2 : X = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Example. Diagonalize the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Eigenvalues. $c_A(x) = (x-2)^2(x+1)$. Eigenvalues $2, -1, -1$.

Eigenvectors.

$$\lambda = 2 : X = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\lambda = -1 : X = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Example. Diagonalize the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Eigenvalues. $c_A(x) = (x-2)^2(x+1)$. Eigenvalues $2, -1, -1$.

Eigenvectors.

$\lambda = 2 : X = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$\lambda = -1 : X = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Note

Note

If there are not enough basic eigenvectors then the matrix is not diagonalizable.

Note

If there are not enough basic eigenvectors then the matrix is not diagonalizable.

Example.

Note

If there are not enough basic eigenvectors then the matrix is not diagonalizable.

Example. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Note

If there are not enough basic eigenvectors then the matrix is not diagonalizable.

Example. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $c_A(x) = (x-1)^2$.

Note

If there are not enough basic eigenvectors then the matrix is not diagonalizable.

Example. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.Then $c_A(x) = (x-1)^2$.Thus $\lambda = 1$ is the only eigenvalue.

Note

If there are not enough basic eigenvectors then the matrix is not diagonalizable.

Example. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.Then $c_A(x) = (x-1)^2$.Thus $\lambda = 1$ is the only eigenvalue.

For $\lambda = 1$, there is only one basic eigenvector $X = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

Note

If there are not enough basic eigenvectors then the matrix is not diagonalizable.

Example. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.Then $c_A(x) = (x-1)^2$.Thus $\lambda = 1$ is the only eigenvalue.

For $\lambda = 1$, there is only one basic eigenvector $X = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, therefore A is not diagonalizable.

Linear Dynamical System

Linear Dynamical System

A sequence of columns $V_0, V_1, \ldots,$ is called a linear dynamical system if

$$V_{k+1} = AV_k$$

for all $k = 0, 1, \ldots,$ where A is a square matrix.

Linear Dynamical System

A sequence of columns $V_0, V_1, \ldots,$ is called a linear dynamical system if

$$V_{k+1} = AV_k$$

for all $k = 0, 1, \ldots,$ where A is a square matrix.

This implies that $V_k = A^k V_0$ for all k.

Linear Dynamical System

A sequence of columns $V_0, V_1, \ldots,$ is called a linear dynamical system if

$$V_{k+1} = AV_k$$

for all $k = 0, 1, \ldots,$ where A is a square matrix.

This implies that $V_k = A^k V_0$ for all k.

Problem: Compute V_k.

Linear Dynamical System

A sequence of columns $V_0, V_1, \ldots,$ is called a linear dynamical system if

$$V_{k+1} = AV_k$$

for all $k = 0, 1, \ldots,$ where A is a square matrix.

This implies that $V_k = A^k V_0$ for all k.

Problem: Compute V_k. This is equivalent to computing A^k.

Linear Dynamical System

A sequence of columns $V_0, V_1, \ldots,$ is called a linear dynamical system if

$$V_{k+1} = AV_k$$

for all $k = 0, 1, \ldots,$ where A is a square matrix.

This implies that $V_k = A^k V_0$ for all k.

Problem: Compute V_k. This is equivalent to computing A^k.

Theorem

Let $P = [X_1 \quad X_2 \cdots X_n]$ be an invertible matrix such that $P^{-1}AP = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Let $P^{-1}V_0 = [b_1 \quad b_1 \cdots b_n]^T$. Then

Linear Dynamical System

A sequence of columns $V_0, V_1, \ldots,$ is called a linear dynamical system if

$$V_{k+1} = AV_k$$

for all $k = 0, 1, \ldots,$ where A is a square matrix.

This implies that $V_k = A^k V_0$ for all k.

Problem: Compute V_k. This is equivalent to computing A^k.

Theorem

Let $P = [X_1 \quad X_2 \cdots X_n]$ be an invertible matrix such that $P^{-1}AP = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Let $P^{-1}V_0 = [b_1 \quad b_1 \cdots b_n]^T$. Then

$$V_k = b_1\lambda_1^k X_1 + b_2\lambda_2^k X_2 + \cdots + b_n\lambda_n^k X_n$$

3.4 An application to Linear recurrences

Problem.

3.4 An application to Linear recurrences

Problem. Let $\{x_k\}$ be the sequence defined recursively as $x_0 = 1, x_1 = 1$ and $x_{k+2} = x_{k+1} + 6x_k$ for all $k = 0, 1, \ldots$. Find x_k.

3.4 An application to Linear recurrences

Problem. Let $\{x_k\}$ be the sequence defined recursively as $x_0 = 1, x_1 = 1$ and $x_{k+2} = x_{k+1} + 6x_k$ for all $k = 0, 1, \ldots$. Find x_k.

Let $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$.

3.4 An application to Linear recurrences

Problem. Let $\{x_k\}$ be the sequence defined recursively as $x_0 = 1, x_1 = 1$ and $x_{k+2} = x_{k+1} + 6x_k$ for all $k = 0, 1, \ldots$. Find x_k.

Let $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$. Then $V_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and

3.4 An application to Linear recurrences

Problem. Let $\{x_k\}$ be the sequence defined recursively as $x_0 = 1, x_1 = 1$ and $x_{k+2} = x_{k+1} + 6x_k$ for all $k = 0, 1, \ldots$. Find x_k.

Let $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$. Then $V_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and

$$V_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ 6x_k + x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = AV_k.$$

3.4 An application to Linear recurrences

Problem. Let $\{x_k\}$ be the sequence defined recursively as $x_0 = 1, x_1 = 1$ and $x_{k+2} = x_{k+1} + 6x_k$ for all $k = 0, 1, \ldots$. Find x_k.

Let $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$. Then $V_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and

$$V_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ 6x_k + x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = AV_k.$$

<u>Diagonalizing A</u>: $D = diag(3, -2), P = [X_1 \quad X_2] = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$.

3.4 An application to Linear recurrences

Problem. Let $\{x_k\}$ be the sequence defined recursively as $x_0 = 1, x_1 = 1$ and $x_{k+2} = x_{k+1} + 6x_k$ for all $k = 0, 1, \ldots$. Find x_k.

Let $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$. Then $V_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and

$$V_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ 6x_k + x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = AV_k.$$

Diagonalizing A: $D = diag(3, -2), P = [X_1 \quad X_2] = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$.

We compute $P^{-1}V_0 = \begin{bmatrix} 3/5 \\ -2/5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Then

Problem. Let $\{x_k\}$ be the sequence defined recursively as $x_0 = 1, x_1 = 1$ and $x_{k+2} = x_{k+1} + 6x_k$ for all $k = 0, 1, \ldots$. Find x_k.

Let $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$. Then $V_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and

$$V_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ 6x_k + x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = AV_k.$$

<u>Diagonalizing A</u>: $D = diag(3, -2), P = [X_1 \quad X_2] = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$.

We compute $P^{-1}V_0 = \begin{bmatrix} 3/5 \\ -2/5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Then

$$V_k = b_1\lambda_1^k X_1 + b_2\lambda_2^k X_2 = \frac{3}{5}3^k \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{-2}{5}(-2)^k \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

3.4 An application to Linear recurrences

Problem. Let $\{x_k\}$ be the sequence defined recursively as $x_0 = 1, x_1 = 1$ and $x_{k+2} = x_{k+1} + 6x_k$ for all $k = 0, 1, \ldots$. Find x_k.

Let $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$. Then $V_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and

$$V_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ 6x_k + x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = AV_k.$$

Diagonalizing A: $D = diag(3, -2), P = [X_1 \quad X_2] = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$.

We compute $P^{-1}V_0 = \begin{bmatrix} 3/5 \\ -2/5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Then

$$V_k = b_1\lambda_1^k X_1 + b_2\lambda_2^k X_2 = \frac{3}{5}3^k \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{-2}{5}(-2)^k \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Therefore $x_k = \frac{1}{\ }\left(3^{k+1} \quad (-2)^{k+1}\right)$