Advanced Mathematics - Linear Algebra

Chapter 3: Determinants and Diagonalization

Department of Mathematics The FPT university

Topics:

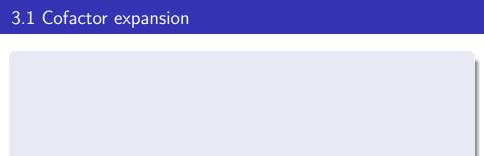
3.1 The cofactor expansion

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- 3.2 Determinants and matrix inverses

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$$c_{23}(A) = (-1)^{2+3} \det \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} = 6$$

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$$= 1(-1)^{1+1} \left| \begin{array}{cc} 5 & 6 \\ 8 & 9 \end{array} \right| + 2(-1)^{1+2} \left| \begin{array}{cc} 4 & 6 \\ 7 & 9 \end{array} \right| + 3(-1)^{1+3} \left| \begin{array}{cc} 4 & 5 \\ 7 & 8 \end{array} \right| = 0$$

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Triangular matrices

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Question. Calculate determinants of triangular matrices.



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- $\bullet \ \det(A^T) = \det(A)$

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$$adj(A) = [c_{ij}(A)]^T = egin{bmatrix} c_{11}(A) & c_{12}(A) & \cdots & c_{1n}(A) \ c_{21}(A) & c_{22}(A) & \cdots & c_{2n}(A) \ & & & & & \ c_{n1}(A) & c_{n2}(A) & \cdots & c_{nn}(A) \end{bmatrix}^T$$

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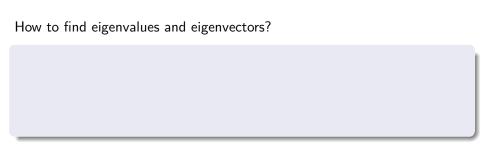
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If
$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$
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Since D^k is easy to compute, A^k is now computable.

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- For each eigenvalue compute basic eigenvectors.
- If there are exactly a total of n basic eigenvectors, then A is diagonalizable.
- In this case D is the diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ in the diagonal, and P is the matrix whose columns are the basic eigenvectors.

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For $\lambda=1$, there is only one basic eigenvector $X=t\begin{bmatrix}1\\0\end{bmatrix}$, therefore A is not diagonalizable.

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Linear Dynamical System

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Theorem

Let $P = [X_1 \quad X_2 \cdots X_n]$ be an invertible matrix such that $P^{-1}AP = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Let $P^{-1}V_0 = [b_1 \quad b_1 \cdots b_n]^T$. Then

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$$V_k = b_1 \lambda_1^k X_1 + b_2 \lambda_2^k X_2 + \dots + b_n \lambda_n^k X_n$$

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Problem. Let $\{x_k\}$ be the sequence defined recursively as $x_0 = 1, x_1 = 1$ and $x_{k+2} = x_{k+1} + 6x_k$ for all $k = 0, 1, \ldots$ Find x_k .

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