Advanced Mathematics - Linear Algebra

Chapter 2: Matrix algebra

Department of Mathematics
The FPT university

Chapter 2 Introduction

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Topics:

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2.1 Matrix operations: Addition, Scalar multiplication, Transposition

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2.7 Applications

2.1 Matrix addition, Scalar multiplication, and Transposition

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A matrix is an array consisting of m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & a_{ij} & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

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- The number a_{ij} is called the (i, j)-entry of the matrix.
- If $m = n$ then we say A is a square matrix.

Matrix Addition and Scalar multiplication

Addition and Subtraction

Addition and Subtraction

$$\begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} \pm \begin{bmatrix} A & B & C \\ X & Y & Z \end{bmatrix} = \begin{bmatrix} a \pm A & b \pm B & c \pm C \\ x \pm X & y \pm Y & z \pm Z \end{bmatrix}$$

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Scalar multiplication

$$k \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ kx & ky & kz \end{bmatrix}, \quad \text{where } k \text{ is a real number}$$

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Theorem

- $A + B = B + A$
- $A + 0 = 0 + A = A$
- $A + (B + C) = (A + B) + C$
- $k(A + B) = kA + kB$
- $kA + pA = (k + p)A$
- $k(pA) = (kp)A$
- $1.A = A$

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Example. Find A such that:

$$\left(2A^T + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

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Example. The matrix $\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$ is symmetric, but $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ is not.

Question. Is the sum of two symmetric matrices symmetric?

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Examples.

$$\begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} \begin{bmatrix} M & N \\ P & Q \\ U & V \end{bmatrix} = \begin{bmatrix} aM + bP + cU & aN + bQ + cV \\ xM + yP + zU & xN + yQ + zV \end{bmatrix}$$

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$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 3a & 3b & 3c \\ 4a & 4b & 4c \end{bmatrix}$$

The identity matrix I_n is the $n \times n$ matrix with 1 on the diagonal and 0 everywhere else.

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Theorem

Let A, B, C be matrices of sizes such that the indicated operations can be performed. I denotes an identity matrix, and k a real number.

- $AI = IA = A$
- $AB \neq BA$, *in general*
- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $k(AB) = (kA)B = A(kB)$
- $(AB)^T = B^T A^T$

Block multiplication

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$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 1 & 0 & 5 & 6 \\ 0 & 1 & 7 & 8 \end{bmatrix} \begin{bmatrix} 9 & 10 \\ 11 & 12 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} =$$

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$$= \begin{bmatrix} A & 0 \\ I & B \end{bmatrix} \begin{bmatrix} C \\ 0 \end{bmatrix} = \begin{bmatrix} AC \\ C \end{bmatrix} = \begin{bmatrix} 31 & 34 \\ 71 & 78 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

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Matrices and Linear systems

A system of linear equations

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$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & \cdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

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can be written as a matrix equation $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

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Notes.

- A matrix may not have an inverse. For example, the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ does not have an inverse.
- The inverse matrix of a matrix A, if exists, is unique and is denoted by A^{-1}. In this case we say A is invertible.

Inverses of 2×2 matrices

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The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is invertible if and only if* $\det(A) \neq 0$. *In that case,*

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Example. Find inverses of the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

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Consider a system of linear equations $AX = B$. If the matrix A is invertible, then $X = A^{-1}B$ is the solution.

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Let A be an $n \times n$ matrix. The following are equivalent:

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2.5 Matrix transformation

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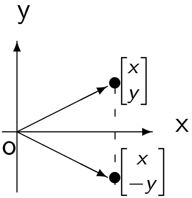

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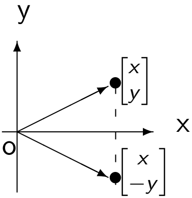
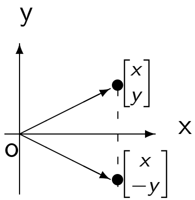
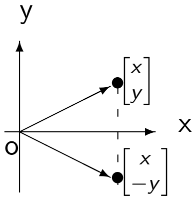




T is reflection in the x-axis

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



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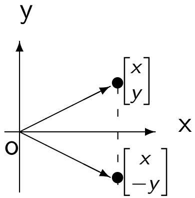

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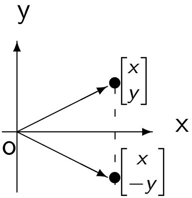

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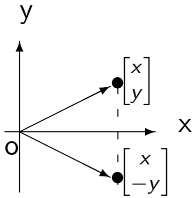

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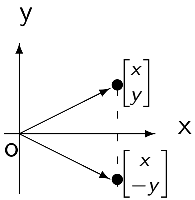

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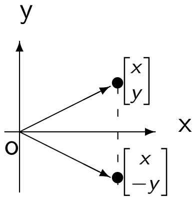

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Question. Find the matrix of the following transformation in the plane:

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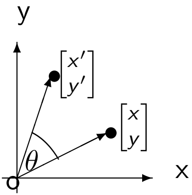
(b) Projection on the x-axis


y
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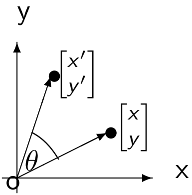
θ
O
x



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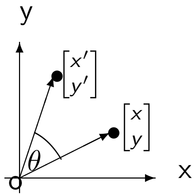

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Property of Linear Transformation

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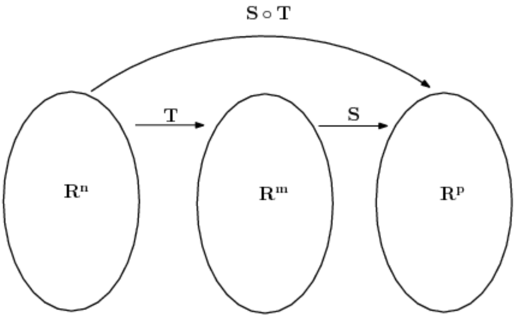

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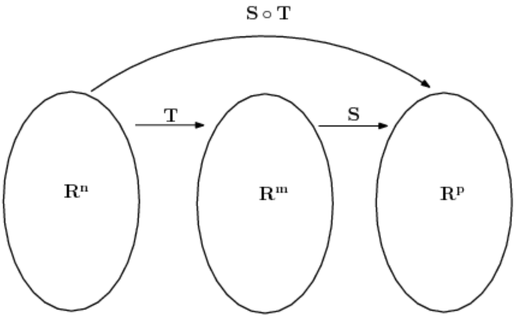

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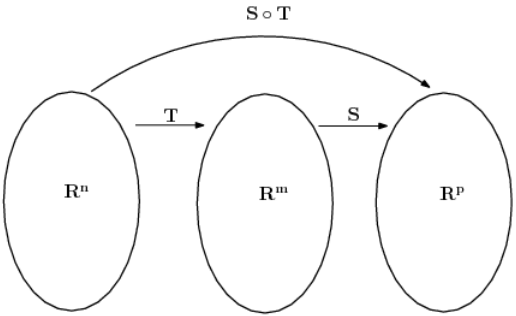

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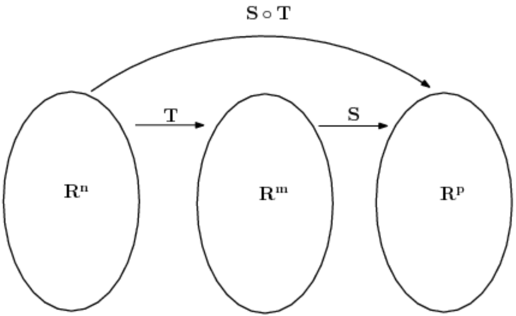

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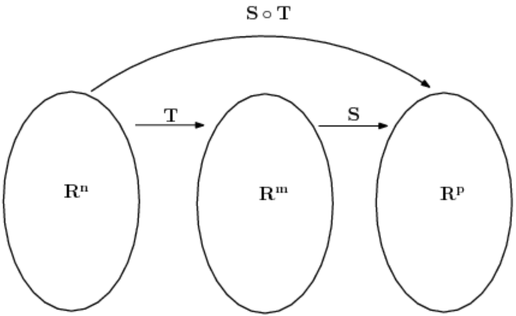

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Solutions P to the equation $(I - E)P = 0$ are called equilibrium price structures.