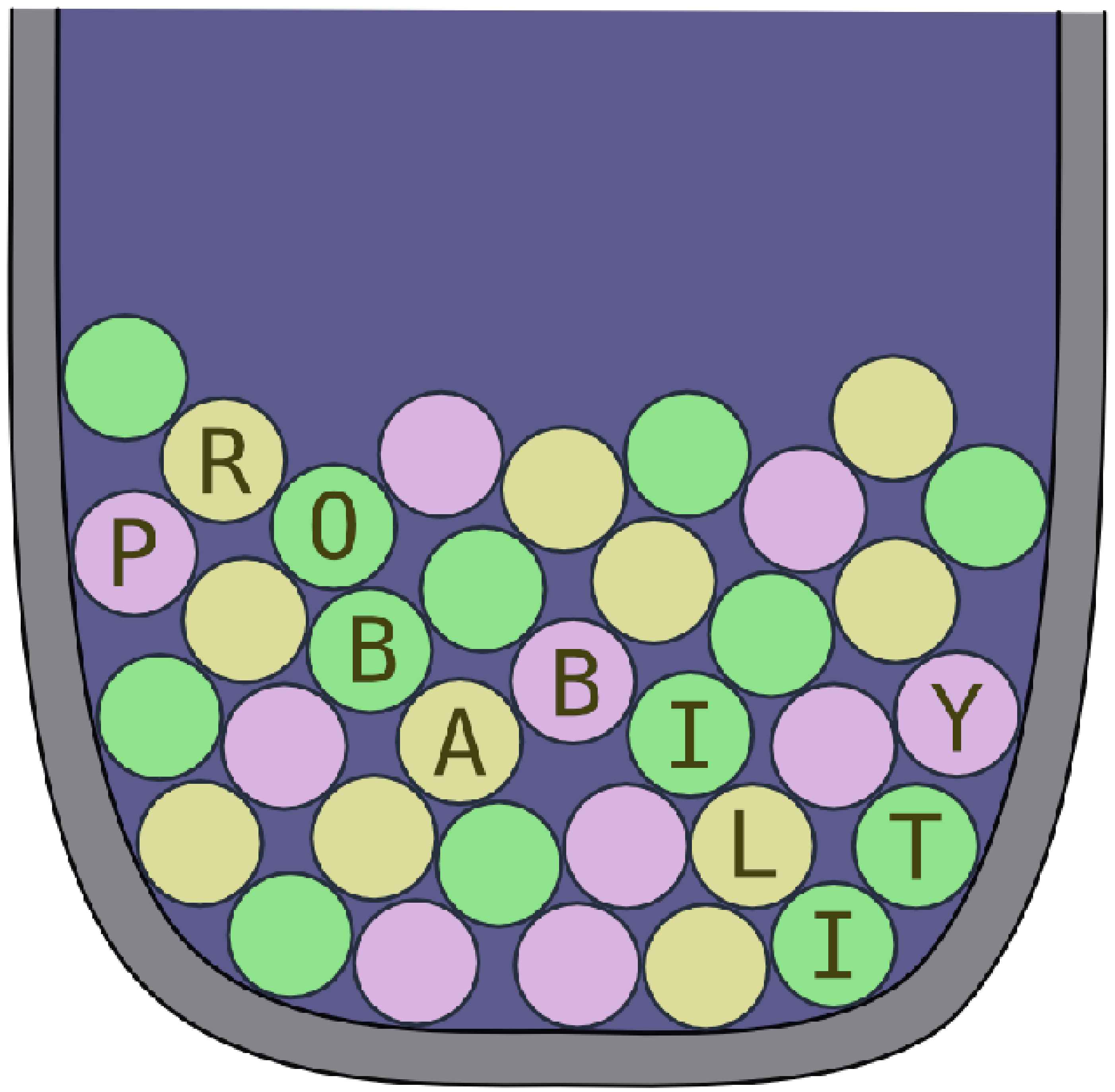
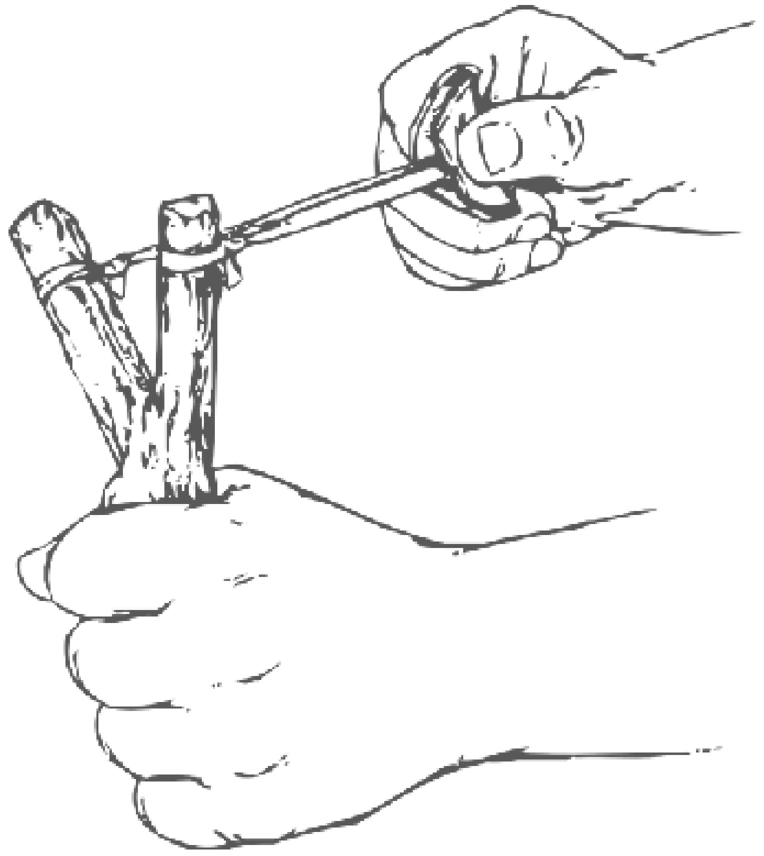


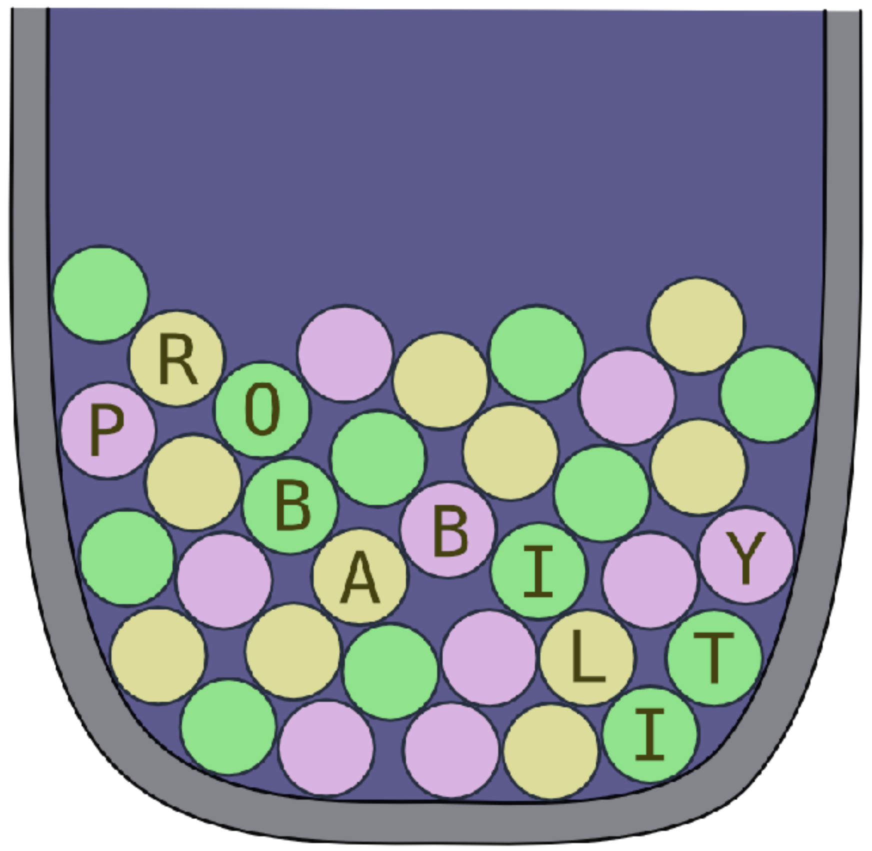
DATA ANALYSIS

PROBABILITY BASICS

LEARNING GOALS



- become familiar with the notion of probability
  - axiomatic definition & interpretation
  - joint, marginal & conditional probability
- Bayes rule
- random variables
- probability distributions in R
- probability distributions as approximated by samples



Probability

ELEMENTARY OUTCOMES AND EVENTS

- a random process has **elementary outcomes** $\Omega = \{\omega_1, \omega_2, \ldots\}$
  - elementary outcomes are mutually exclusive
  - exhausts the space of possibilities

> **Example.** The set of elementary outcomes of a single coin flip is $\Omega_{\text{coin flip}} = \{\text{heads}, \text{tails}\}$. The elementary outcomes of tossing a six-sided die is $\Omega_{\text{standard die}} = \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\}$.[27]

- any $A \subseteq \Omega$ is an event
  - standard set-theoretic notation for negation, conjunction, disjunction etc.
  - example "rolling an odd number" $A = \{⚀, ⚂, ⚄\}$

PROBABILITY DISTRIBUTION

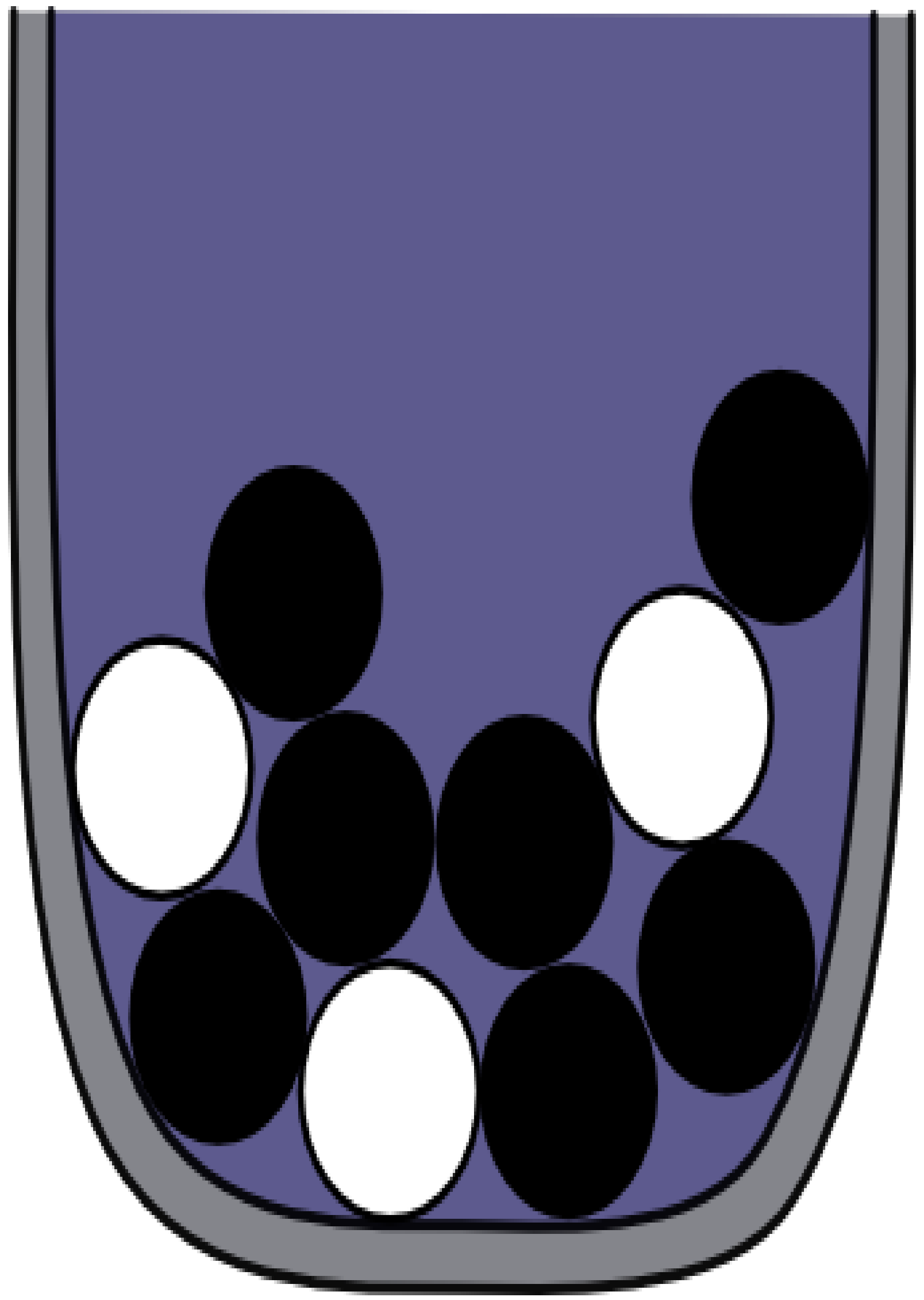
A **probability distribution** P over Ω is a function $P : \mathfrak{P}(\Omega) \rightarrow \mathbb{R}$ that assigns to all events $A \subseteq \Omega$ a real number (from the unit interval, see A1 below), such that the following (so-called Kolmogorov axioms) are satisfied:

A1. $0 \leq P(A) \leq 1$

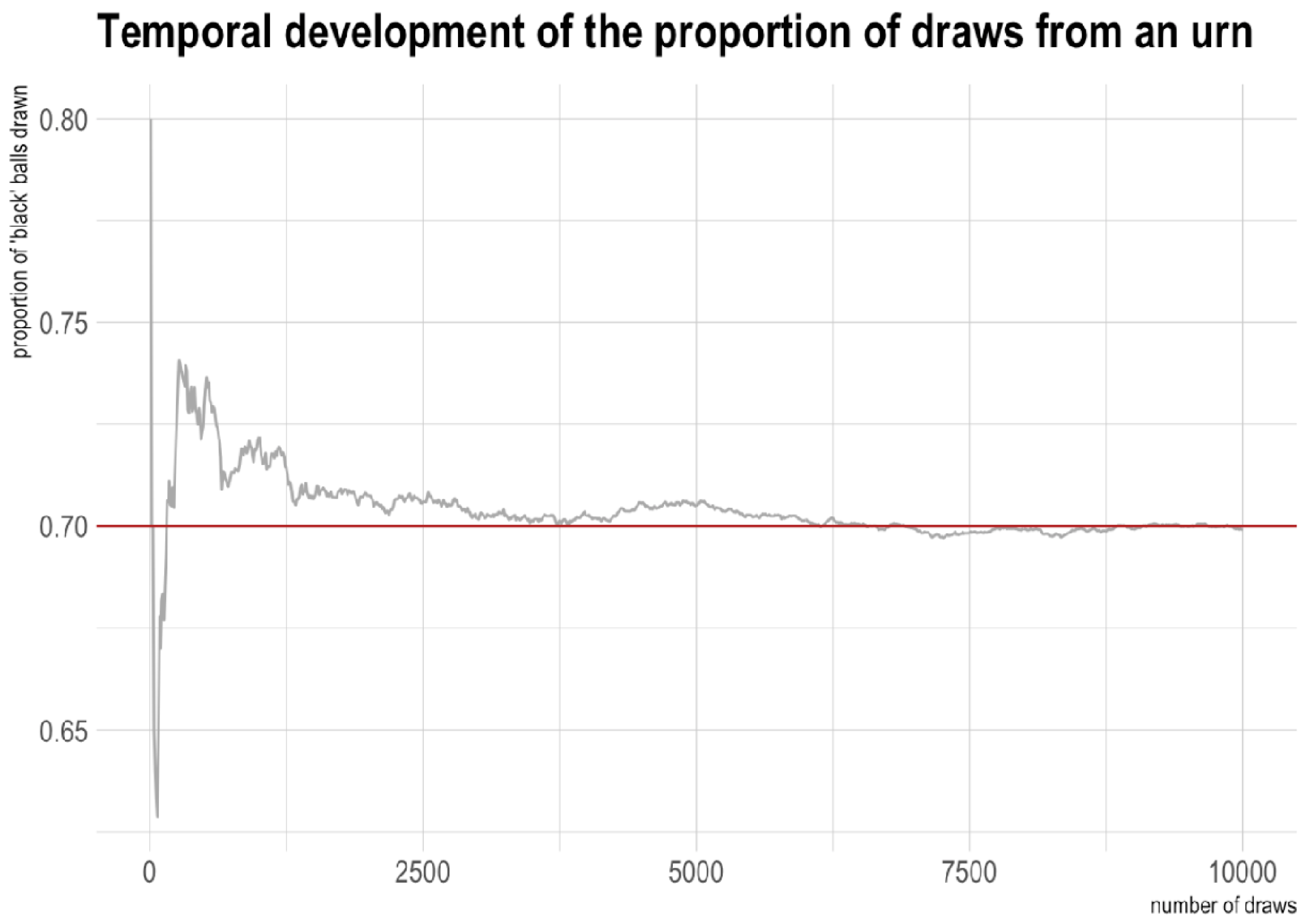
A2. $P(\Omega) = 1$

A3. $P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$ whenever $A_1, A_2, A_3, \ldots$ are mutually exclusive

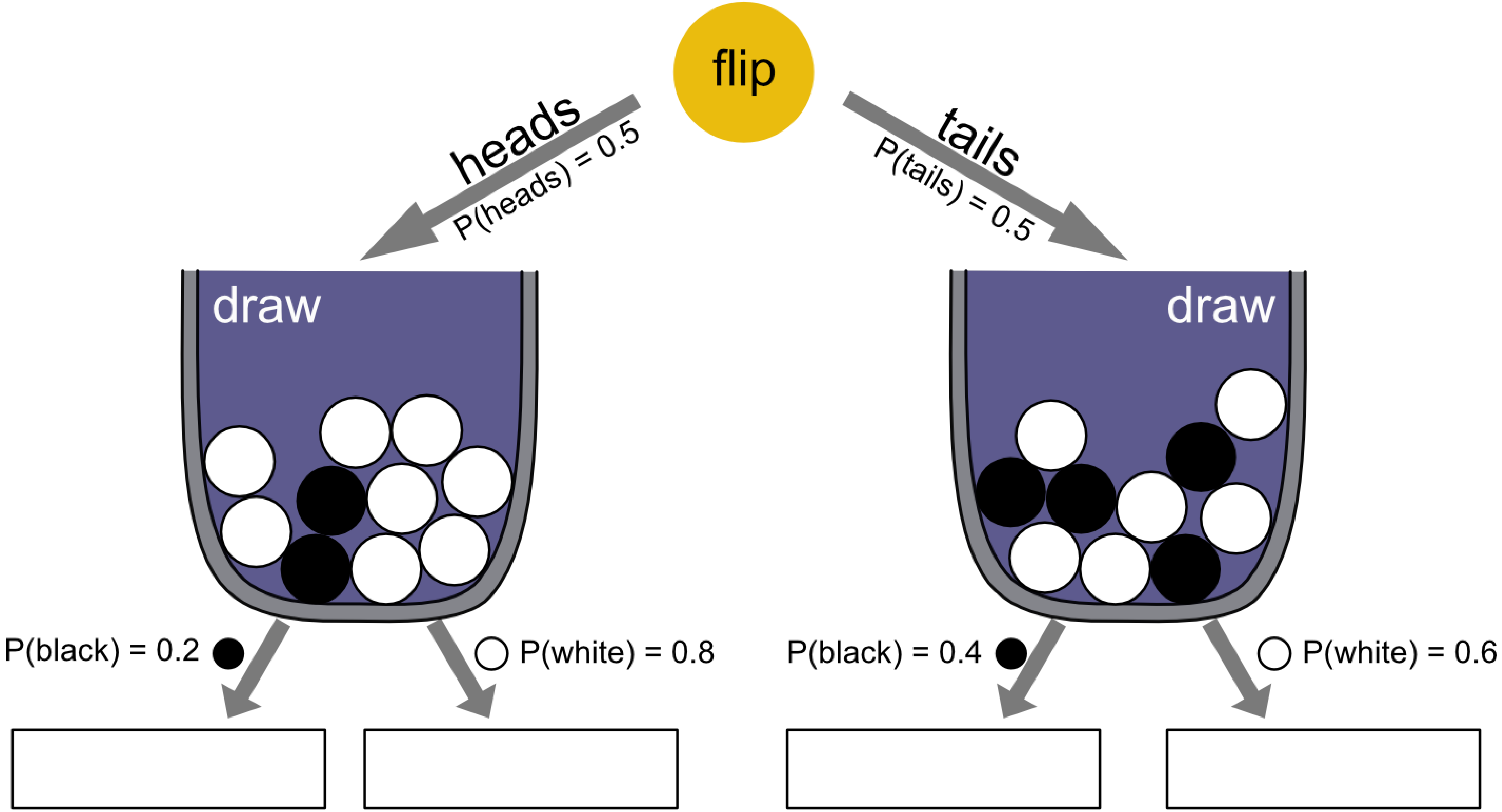
Think of an **urn** as a container with balls of different colors with different proportions (see Figure 7.1). In the simplest case, there is a number of $N > 1$ balls of which $k > 0$ are black and $N - k > 0$ are white. (There are at least one black and one white ball.) For a single random draw from our urn we have: $\Omega_{\text{our urn}} = \{\text{white}, \text{black}\}$. We now draw from this urn with replacement. That is, we shake the urn, draw one ball, observe its color, take note of the color, and put it back into the urn. Each ball has the same chance of being sampled. If we imagine an infinite sequence of single draws from our urn with replacement, the limiting proportion with which we draw a black ball is $\frac{k}{N}$. This statement about frequency is what motivates saying that the probability of drawing a black ball on a single trial is (or should be[35]) $P(\text{black}) = \frac{k}{N}$.



If every ball has an equal probability of being drawn, what is the probability of drawing a black ball?


Temporal development of the proportion of draws from an urn
proportion of 'black' balls drawn
0.80
0.75
0.70
0.65
0
2500
5000
7500
10000
number of draws

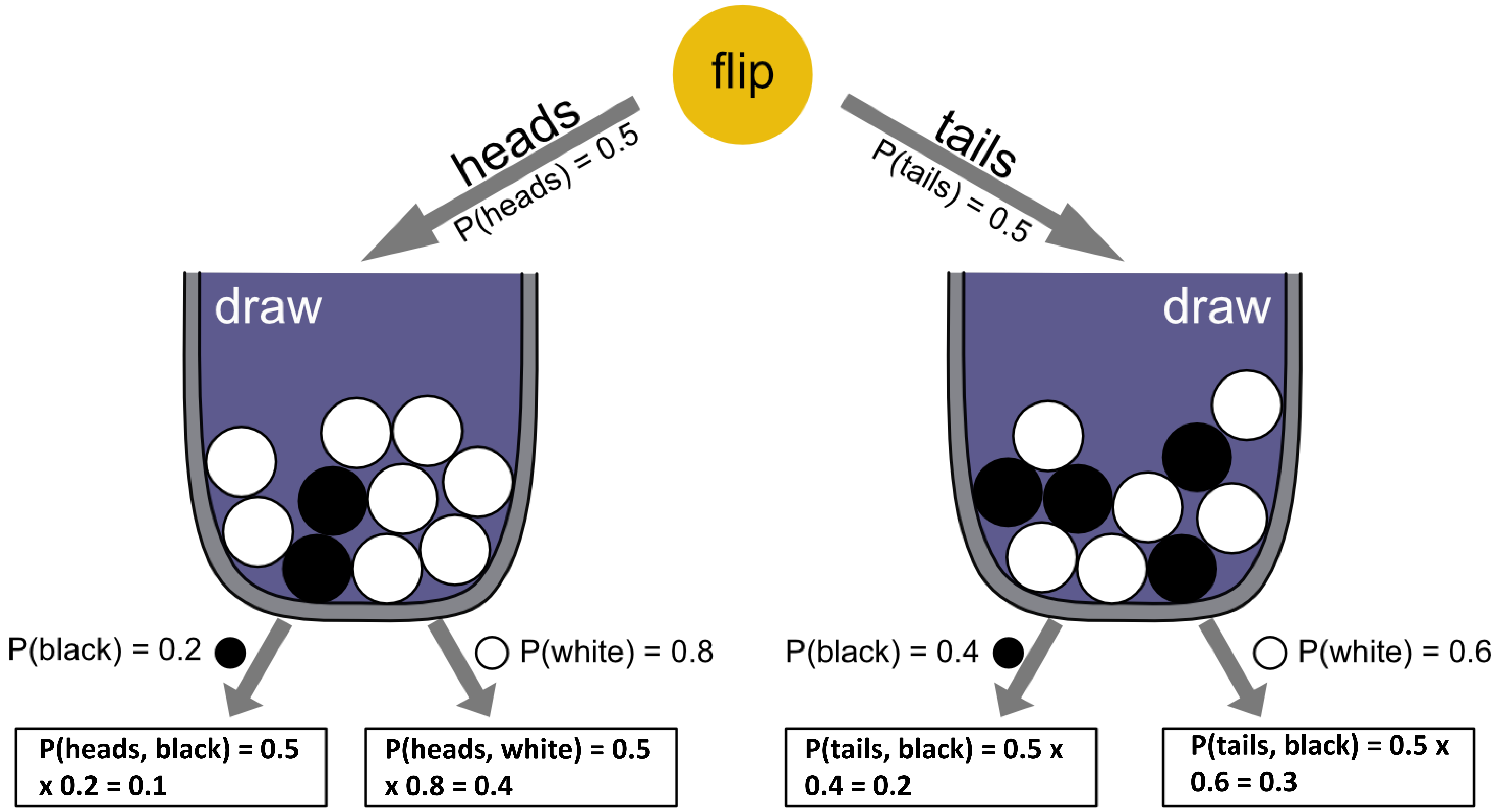
Structured events


flip
heads
P(heads) = 0.5
tails
P(tails) = 0.5
draw
draw
P(black) = 0.2
P(white) = 0.8
P(black) = 0.4
P(white) = 0.6

JOINT PROBABILITY DISTRIBUTIONS

- Set of all possible outcomes:
  - $\Omega_{flip-\&-draw} = \{\langle \text{heads}, \text{black} \rangle, \langle \text{heads}, \text{white} \rangle, \langle \text{tails}, \text{black} \rangle, \langle \text{tails}, \text{white} \rangle\}$
- Structured elementary outcomes: $\Omega_{flip-\&-draw} = \Omega_{flip} \times \Omega_{draw}$
  - shorthand notation $P(\text{heads}, \text{black})$ instead of $P(\langle \text{heads}, \text{black} \rangle)$

	heads	**tails**
black	$0.5 \times 0.2 = 0.1$	$0.5 \times 0.4 = 0.2$
white	$0.5 \times 0.8 = 0.4$	$0.5 \times 0.6 = 0.3$


flip
heads
P(heads) = 0.5
tails
P(tails) = 0.5
draw
draw
P(black) = 0.2
P(white) = 0.8
P(black) = 0.4
P(white) = 0.6
P(heads, black) = 0.5 x 0.2 = 0.1
P(heads, white) = 0.5 x 0.8 = 0.4
P(tails, black) = 0.5 x 0.4 = 0.2
P(tails, black) = 0.5 x 0.6 = 0.3

MARGINAL DISTRIBUTIONS

- if $\Omega = \Omega_1 \times \ldots \Omega_n$ and $A_i \subseteq \Omega_i$, the **marginal probability** of A_i is:

$$P(A_i) = \sum_{A_1 \subseteq \Omega_1, \ldots, A_{i-1} \subseteq \Omega_{i-1}, A_{i+1} \subseteq \Omega_{i+1}, \ldots, A_n \subseteq \Omega_n} P(A_1, \ldots, A_{i-1}, A_i, A_{i+1}, \ldots A_n)$$

	heads	tails	$\sum$
black	$0.5 \times 0.2 = 0.1$	$0.5 \times 0.4 = 0.2$	$P(\text{black}) = 0.3$
white	$0.5 \times 0.8 = 0.4$	$0.5 \times 0.6 = 0.3$	$P(\text{white}) = 0.7$
$\sum$	$P(\text{heads}) = 0.5$	$P(\text{tails}) = 0.5$	

Conditional probability & Bayes rule

CONDITIONAL PROBABILITY

- the conditional probability of A given B is: $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

$$P(\text{black} \mid \text{heads}) = \frac{P(\text{black, heads})}{P(\text{heads})} = \frac{0.1}{0.5} = 0.2$$

	heads	tails	Σ
black	$0.5 \times 0.2 = 0.1$	$0.5 \times 0.4 = 0.2$	$P(\text{black}) = 0.3$
white	$0.5 \times 0.8 = 0.4$	$0.5 \times 0.6 = 0.3$	$P(\text{white}) = 0.7$
Σ	$P(\text{heads}) = 0.5$	$P(\text{tails}) = 0.5$	

BAYES RULE

- Bayes rule follows straightforwardly from the definition of conditional probability:

$$P(B \mid A) = \frac{P(A \mid B)\ P(B)}{P(A)}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(B \mid A)\ P(A)$$

$$P(B \cap A) = P(A \mid B) \cdot P(B)$$

PREVIEW::BAYES FOR DATA ANALYSIS

$$P(B \mid A) = \frac{P(A \mid B)\, P(B)}{P(A)}$$

likelihood of data prior over parameters

$$\underbrace{P(\theta \mid D)}_{\text{posterior over parameters}} = \frac{\overbrace{P(D \mid \theta)}^{\text{likelihood of data}}\, \overbrace{P(\theta)}^{\text{prior over parameters}}}{\underbrace{P(D)}_{\text{marginal likelihood of data}}}$$

posterior over parameters

marginal likelihood of data

Random variables

RANDOM VARIABLES

- a random variable is a function: $X : \Omega \to \mathbb{R}$
  - if range of X is countable, we speak of a discrete random variable
  - otherwise, we speak of a continuous random variable
- think: distribution of a summary statistic
- notation:
  - shorthand notation $P(X = x)$ instead of $P(\{\omega \in \Omega \mid X(\omega) = x\})$
  - similarly write stuff like $P(X \leq x)$ or $P(1 \leq X \leq 2)$

RANDOM VARIABLE :: EXAMPLES

Example. For a single flip of a coin we have $\Omega_{\text{coin flip}} = \{\text{heads}, \text{tails}\}$. A usual way of mapping this onto numerical outcomes is to define $X_{\text{coin flip}} : \text{heads} \mapsto 1; \text{tails} \mapsto 0$. Less trivially, consider flipping a coin two times. Elementary outcomes should be individuated by the outcome of the first flip and the outcome of the second flip, so that we get:

$$\Omega_{\text{two flips}} = \{\langle \text{heads}, \text{heads} \rangle, \langle \text{heads}, \text{tails} \rangle, \langle \text{tails}, \text{heads} \rangle, \langle \text{tails}, \text{tails} \rangle\}$$

Consider the random variable $X_{\text{two flips}}$ that counts the total number of heads. Crucially, $X_{\text{two flips}}(\langle \text{heads}, \text{tails} \rangle) = 1 = X_{\text{two flips}}(\langle \text{tails}, \text{heads} \rangle)$. We assign the same numerical value to different elementary outcomes.

CUMULATIVE DISTRIBUTION & PROBABILITY MASS:: DISCRETE RVs

For a discrete random variable X, the **cumulative distribution function** F_X associated with X is defined as:

$$F_X(x) = P(X \leq x) = \sum_{x' \in \{\mathrm{Rng}(X) | x' \leq x\}} P(X = x)$$

The **probability mass function** f_x associated with X is defined as:

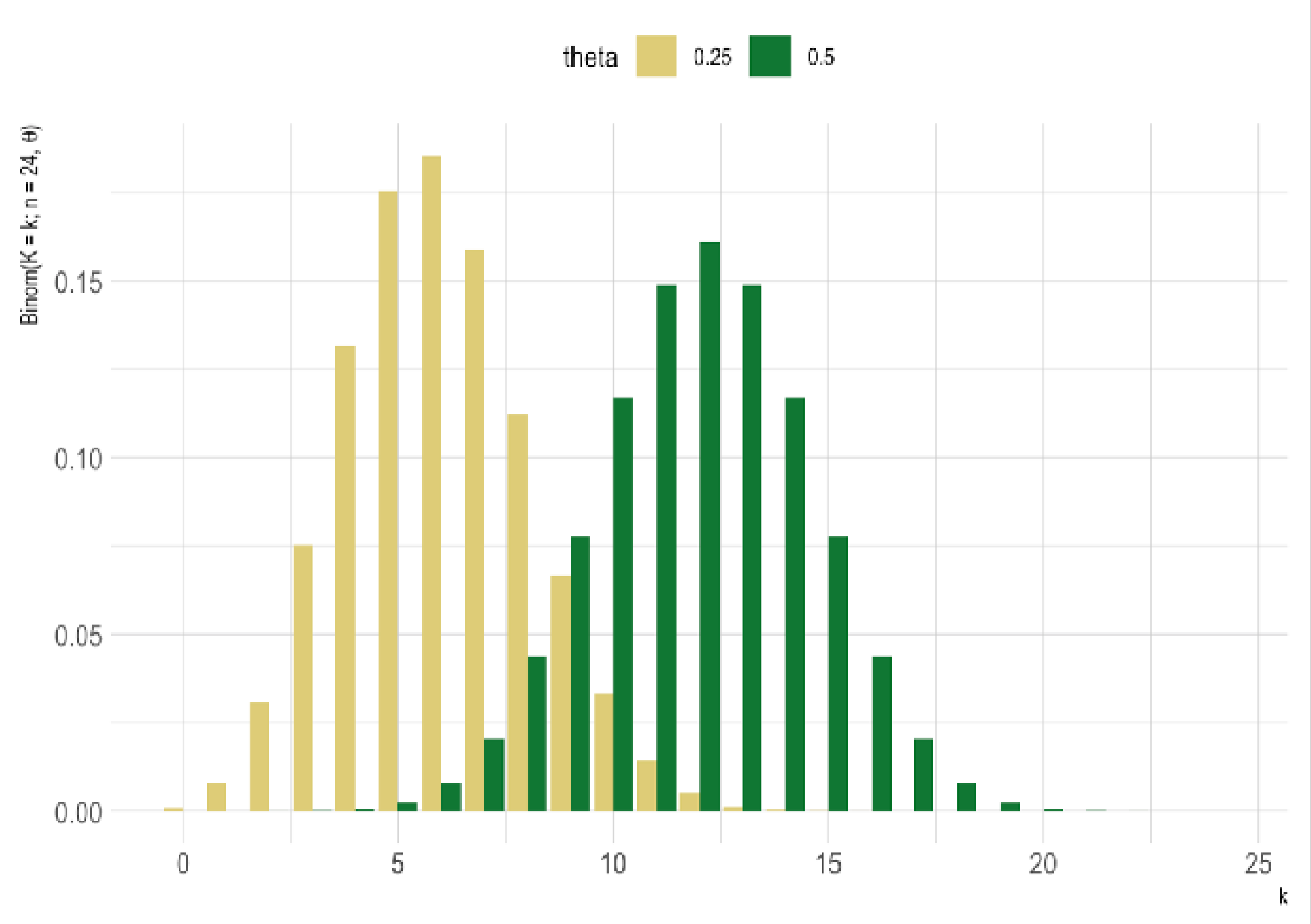
$$f_X(x) = P(X = x)$$

Example. Suppose we flip a coin with a bias of θ towards heads n times. What is the probability that we will see heads k times? If we map the outcome of heads to 1 and tails to 0, this probability is given by the Binomial distribution, as follows:

$$\mathrm{Binom}(K = k; n, \theta) = \binom{n}{k} \theta^k (1 - \theta)^{n-k}$$

Here $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient,

$$\mathrm{Binom}(K = k; n, \theta) = \binom{n}{k} \theta^k (1 - \theta)^{n-k}$$



probability mass function

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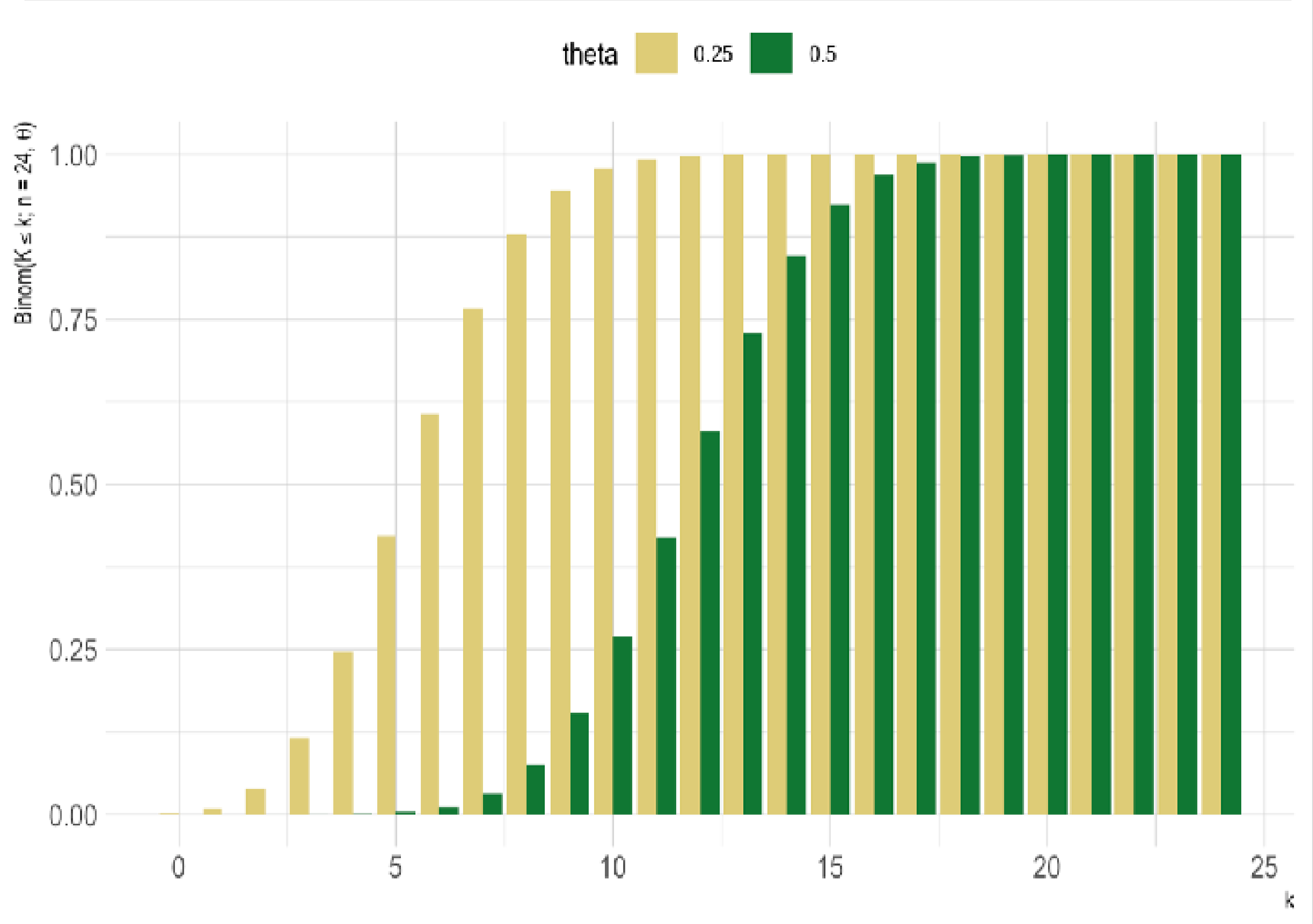


cumulative probability function

CUMULATIVE DISTRIBUTION & PROBABILITY MASS:: CONTINUOUS RVs

For a continuous random variable X, the probability $P(X = x)$ will usually be zero: it is virtually impossible that we will see precisely the value x realized in a random event that can realize uncountably many numerical values of X. However, $P(X \leq x)$ does take workable values and so we define the cumulative distribution function F_X associated with X as:

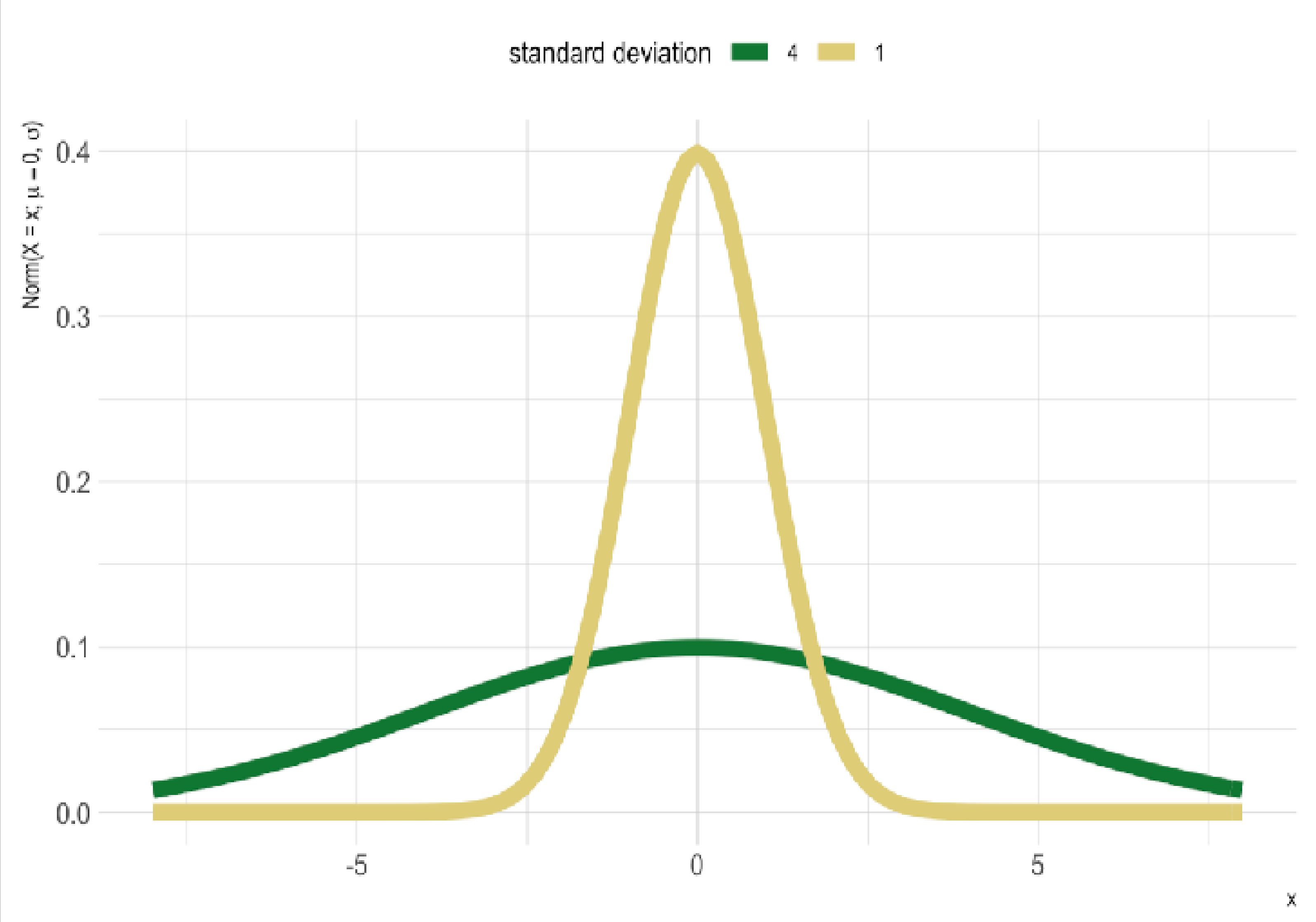
$$F_X(x) = P(X \leq x)$$

Instead of a probability **mass** function, we derive a **probability density function** from the cumulative function as:

$$f_X(x) = F'(x)$$

A probability density function can take values greater than one, unlike a probability mass function.

$$\mathcal{N}(X = x; \mu, \sigma) = \frac{1}{\sqrt{2\sigma^2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$



probability density function

CUMULATIVE DISTRIBUTION & PROBABILITY MASS:: CONTINUOUS RVs

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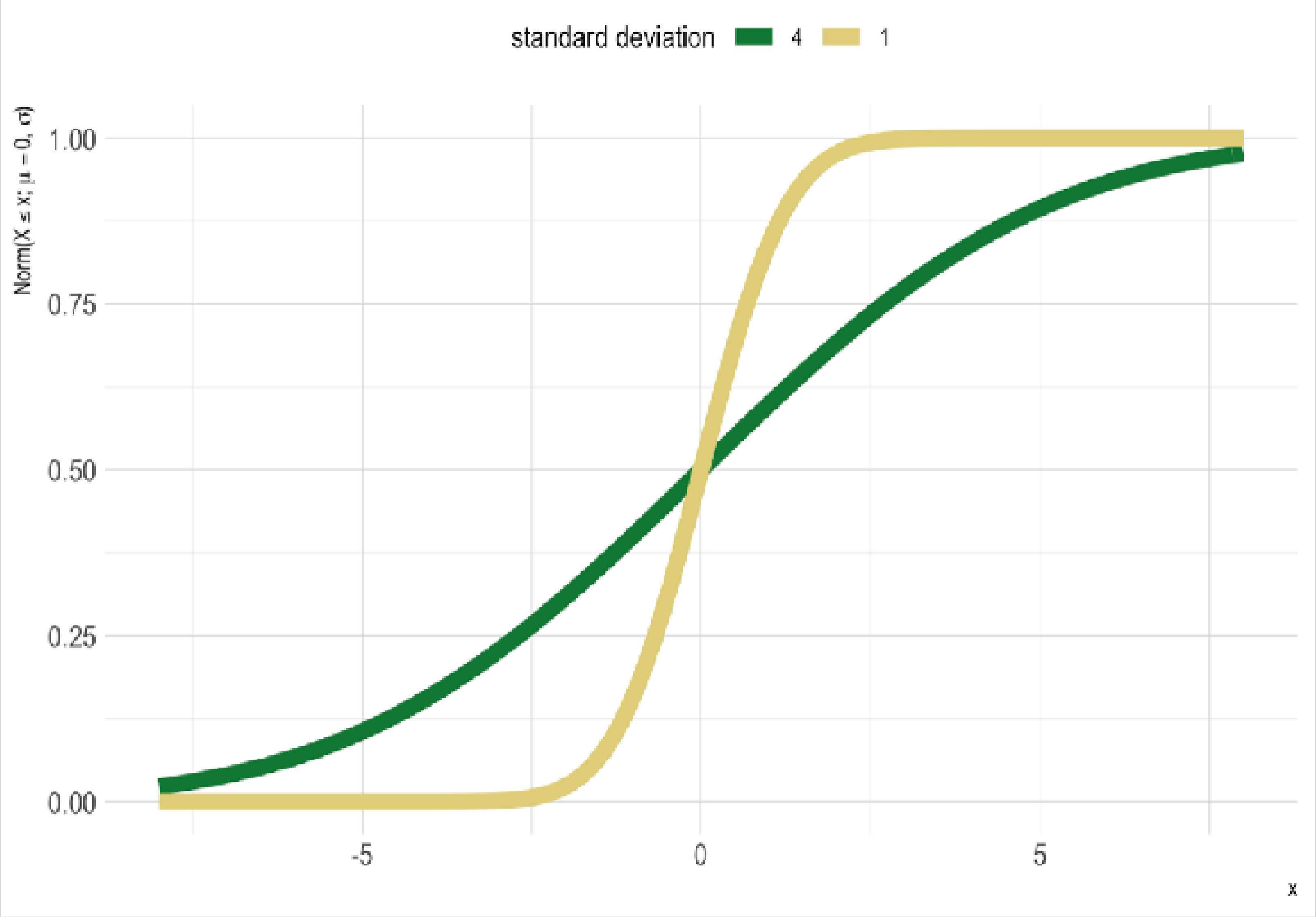

cumulative probability function

EXPECTED VALUED OF A RANDOM VARIABLE

- the expected value of random variable $X : \Omega \rightarrow \mathbb{R}$ is:

  If X is discrete: $\mathbb{E}_X = \sum_x x \times f_X(x)$

  If X is continuous: $\mathbb{E}_X = \int x \times f_X(x) \, \mathrm{d}x$

- think: mean of a representative sample of

VARIANCE OF A RANDOM VARIABLE

- the variance of random variable $X : \Omega \to \mathbb{R}$ is:

  If X is discreet: $\mathrm{Var}(X) = \sum_x (\mathbb{E}_X - x)^2 \times f_X(x) = \mathbb{E}_{X^2} - \mathbb{E}_X^2$

  If X is continuous: $\mathrm{Var}(X) = \int (\mathbb{E}_X - x)^2 \times f_X(x)\,\mathrm{d}x = \mathbb{E}_{X^2} - \mathbb{E}_X^2$

- think: variance of a representative sample of

Probability distributions in R

PROBABILITY DISTRIBUTIONS IN R

- for each distribution `mydist`, there are four types of functions
  - **`d`**`mydist(x, ...)` density function gives the (mass/density) $f(x)$ for `x`
  - **`p`**`mydist(x, ...)` cumulative probability function gives cumulative distribution $F(x)$ for `x`
  - **`q`**`mydist(p, ...)` quantile function gives value x with `p = pmydist(x, ...)`
  - **`r`**`mydist(n, ...)` random sample function returns `n` samples from the distribution

EXAMPLE :: NORMAL DISTRIBUTION

```
# density of standard normal at x = 1
dnorm(x = 1, mean = 0, sd = 1)
```

```
## [1] 0.2419707
```

```
# cumulative density of standard normal at q = 0
pnorm(q = 0, mean = 0, sd = 1)
```

```
## [1] 0.5
```

```
# point where the cumulative density of standard normal is p = 0
qnorm(p = 0.5, mean = 0, sd = 1)
```

```
## [1] 0
```

```
# n = 3 random samples from a standard normal
rnorm(n = 3, mean = 0, sd = 1)
```

```
## [1]  0.5382749 -0.1837154 -0.3165524
```