7. Schemes №1, №2 of LTI Plants Parameterization

7.1. Scheme №1. Output parameterization

Plant

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = b_m u^{(m)} + \ldots + b_0 u, \qquad (7.1)$$

where a_i, b_j, $i = \overline{0, n-1}$, $j = \overline{0, m}$ are the constant parameters.

7. Schemes №1, №2 of LTI Plants Parameterization

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where $a_i, b_j,\ i = \overline{0, n-1},\ j = \overline{0, m}$ are the constant parameters.

Apply transfer function

$$H(s) = \frac{1}{K(s)} = \frac{1}{s^n + k_{n-1}s^{n-1} \ldots + k_0}$$

with a Hurwitz polynomial $K(s) = s^n + k_{n-1}s + \ldots + k_0$ to (7.1) assuming initial conditions $y(0), \ldots, y^{(n-1)}(0)$ zero .

7. Schemes №1, №2 of LTI Plants Parameterization

7.1. Scheme №1. Output parameterization

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with a Hurwitz polynomial $K(s) = s^n + k_{n-1} s + \ldots + k_0$ to (7.1) assuming initial conditions $y(0), \ldots, y^{(n-1)}(0)$ zero . Using *Example 6.3* we have

7.1. Scheme №1. Output parameterization

Parameterization

$$y=\left(k_{n-1}-a_{n-1}\right)\frac{s^{n-1}}{K(s)}[y]+\ldots+\left(k_1-a_1\right)\frac{s}{K(s)}[y]+\left(k_0-a_0\right)\frac{1}{K(s)}[y]+$$

$$b_m\frac{s^m}{K(s)}[u]+\ldots+b_1\frac{s}{K(s)}[u]+b_0\frac{1}{K(s)}[u],$$

7.1. Scheme №1. Output parameterization

Parameterization

$$y=\underbrace{(k_{n-1}-a_{n-1})}_{\theta_n}\underbrace{\frac{s^{n-1}}{K(s)}[y]}_{\xi_n}+\ldots+\underbrace{(k_1-a_1)}_{\theta_2}\underbrace{\frac{s}{K(s)}[y]}_{\xi_2}+\underbrace{(k_0-a_0)}_{\theta_1}\underbrace{\frac{1}{K(s)}[y]}_{\xi_1}+$$

$$\underbrace{b_m}_{\theta_{n+m+1}}\underbrace{\frac{s^m}{K(s)}[u]}_{\nu_{m+1}}+\ldots+\underbrace{b_1}_{\theta_{n+2}}\underbrace{\frac{s}{K(s)}[u]}_{\nu_2}+\underbrace{b_0}_{\theta_{n+1}}\underbrace{\frac{1}{K(s)}[u]}_{\nu_1},$$

$$y=\theta^T\omega, \tag{7.2}$$

where $\theta=col(\theta_1,\theta_2,\cdots,\theta_{n+m+1})\in\mathbb{R}^{n+m+1}$,

$\omega=col(\xi_1,\xi_2,\cdots,\xi_n,\nu_1,\nu_2,\cdots,\nu_{m+1})\in\mathbb{R}^{n+m+1}$.

7.1. Scheme №1. Output parameterization

Parameterization

$$y = \underbrace{(k_{n-1} - a_{n-1})}_{\theta_n} \underbrace{\frac{s^{n-1}}{K(s)}[y]}_{\xi_n} + \ldots + \underbrace{(k_1 - a_1)}_{\theta_2} \underbrace{\frac{s}{K(s)}[y]}_{\xi_2} + \underbrace{(k_0 - a_0)}_{\theta_1} \underbrace{\frac{1}{K(s)}[y]}_{\xi_1} +$$

$$\underbrace{b_m}_{\theta_{n+m+1}} \underbrace{\frac{s^m}{K(s)}[u]}_{\nu_{m+1}} + \ldots + \underbrace{b_1}_{\theta_{n+2}} \underbrace{\frac{s}{K(s)}[u]}_{\nu_2} + \underbrace{b_0}_{\theta_{n+1}} \underbrace{\frac{1}{K(s)}[u]}_{\nu_1},$$

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \quad \ldots \\ \dot{\xi}_{n-1} = \xi_n \\ \dot{\xi}_n = -k_0\xi_1 - k_1\xi_2 - \cdots - k_{n-1}\xi_n + y \end{cases} \qquad (7.3)$$

$$\begin{cases} \dot{\nu}_1 = \nu_2 \\ \dot{\nu}_2 = \nu_3 \\ \quad \ldots \\ \dot{\nu}_{n-1} = \nu_n \\ \dot{\nu}_n = -k_0\nu_1 - k_1\nu_2 - \cdots - k_{n-1}\nu_n + u \end{cases} \qquad (7.4)$$

7.1. Scheme №1. Output parameterization

Parameterization

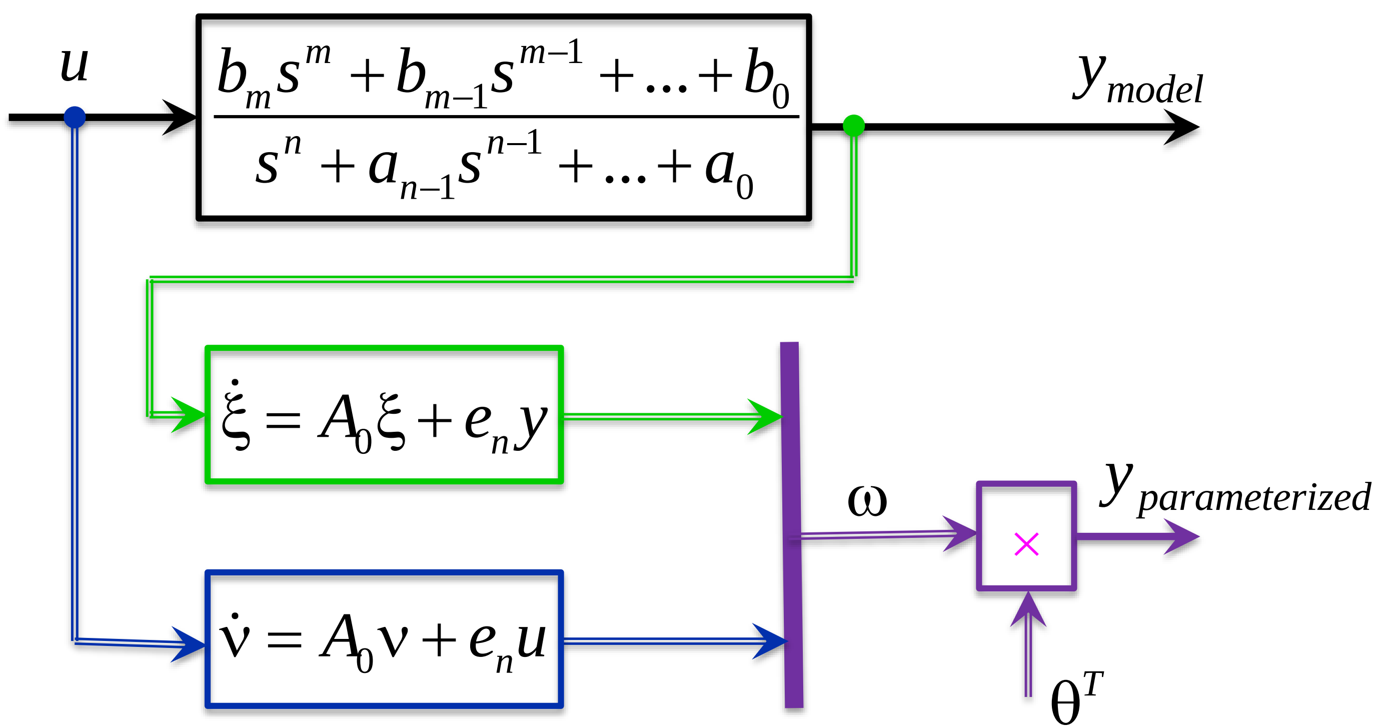
$$y = \underbrace{(k_{n-1} - a_{n-1})}_{\theta_n}\underbrace{\frac{s^{n-1}}{K(s)}[y]}_{\xi_n} + \ldots + \underbrace{(k_1 - a_1)}_{\theta_2}\underbrace{\frac{s}{K(s)}[y]}_{\xi_2} + \underbrace{(k_0 - a_0)}_{\theta_1}\underbrace{\frac{1}{K(s)}[y]}_{\xi_1} +$$

$$\underbrace{b_m}_{\theta_{n+m+1}}\underbrace{\frac{s^m}{K(s)}[u]}_{\nu_{m+1}} + \ldots + \underbrace{b_1}_{\theta_{n+2}}\underbrace{\frac{s}{K(s)}[u]}_{\nu_2} + \underbrace{b_0}_{\theta_{n+1}}\underbrace{\frac{1}{K(s)}[u]}_{\nu_1},$$

$$\begin{cases} \dot{\xi} = A_0\xi + e_n y, & (7.3) \\ \dot{\nu} = A_0\nu + e_n u & (7.4) \end{cases}$$

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -k_0 & -k_1 & -k_2 & \cdots & -k_{n-1} \end{bmatrix}, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

7.1. Scheme №1. Output parameterization



$$y_{parameterized} = \theta^T \omega$$

7.2. Scheme №2. State parameterization

Plant is given in canonical form

$$\begin{cases} \dot{x} = Ax + bu, \\ y = c^T x, \end{cases} \qquad (7.5)$$

where

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \; b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix}, \; c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$a_i, b_j, \; i = \overline{0, n-1}, \; j = \overline{0, m}$ are the constant parameters.

7.2. Scheme №2. State parameterization

Apply transfer matrix

$$\Phi(s) = \left(I_{n\times n}s - A_0^*\right)^{-1} \tag{7.6}$$

with a Hurwitz matrix

$$A_0^* = \begin{bmatrix} -k_{n-1} & 1 & 0 & \cdots & 0 \\ -k_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_1 & 0 & 0 & \cdots & 1 \\ -k_0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

to (7.5) assuming initial conditions $x(0)$ zero:

$$\left(I_{n\times n}s - A_0^*\right)^{-1}[\dot{x}] = \left(I_{n\times n}s - A_0^*\right)^{-1} A[x] + \left(I_{n\times n}s - A_0^*\right)^{-1} b[u]$$

$$\Downarrow$$

7.2. Scheme №2. State parameterization

$$\left(I_{n\times n}s - A_0^*\right)^{-1}\cdot s\,[x] = \left(I_{n\times n}s - A_0^*\right)^{-1} A[x] + \left(I_{n\times n}s - A_0^*\right)^{-1} b[u]$$

7.2. Scheme №2. State parameterization

$$\left(I_{n\times n}s - A_0^*\right)^{-1}\cdot s \overset{\pm A_0^*}{}[x] = \left(I_{n\times n}s - A_0^*\right)^{-1} A[x] + \left(I_{n\times n}s - A_0^*\right)^{-1} b[u]$$

$$\Downarrow$$

$$\left(I_{n\times n}s - A_0^*\right)^{-1}\left(I_{n\times n}s \pm A_0^*\right)[x] = \left(I_{n\times n}s - A_0^*\right)^{-1} A[x] + \left(I_{n\times n}s - A_0^*\right)^{-1} b[u]$$

7.2. Scheme №2. State parameterization

$$\left(I_{n\times n}s - A_0^*\right)^{-1}\cdot s \overset{\pm A_0^*}{} [x] = \left(I_{n\times n}s - A_0^*\right)^{-1} A[x] + \left(I_{n\times n}s - A_0^*\right)^{-1} b[u]$$

$$\Downarrow$$

$$\left(I_{n\times n}s - A_0^*\right)^{-1}\left(I_{n\times n}s \pm A_0^*\right)[x] = \left(I_{n\times n}s - A_0^*\right)^{-1} A[x] + \left(I_{n\times n}s - A_0^*\right)^{-1} b[u]$$

$$\Downarrow$$

$$x = \left(I_{n\times n}s - A_0^*\right)^{-1}\left(A - A_0^*\right)[x] + \left(I_{n\times n}s - A_0^*\right)^{-1} b[u]$$

7.2. Scheme №2. State parameterization

$\pm A_0^*$

$$\left(I_{n\times n}s - A_0^*\right)^{-1}\cdot s\,[x] = \left(I_{n\times n}s - A_0^*\right)^{-1}A[x] + \left(I_{n\times n}s - A_0^*\right)^{-1}b[u]$$

$$\Downarrow$$

$$\left(I_{n\times n}s - A_0^*\right)^{-1}\left(I_{n\times n}s \pm A_0^*\right)[x] = \left(I_{n\times n}s - A_0^*\right)^{-1}A[x] + \left(I_{n\times n}s - A_0^*\right)^{-1}b[u]$$

$$\Downarrow$$

$$x = \left(I_{n\times n}s - A_0^*\right)^{-1}\underbrace{\left(A - A_0^*\right)}[x] + \left(I_{n\times n}s - A_0^*\right)^{-1}b[u]$$

$$\begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} -k_{n-1} & 1 & 0 & \cdots & 0 \\ -k_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_1 & 0 & 0 & \cdots & 1 \\ -k_0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} k_{n-1} - a_{n-1} \\ \vdots \\ k_1 - a_1 \\ k_0 - a_0 \end{bmatrix}\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

7.2. Scheme №2. State parameterization

$\pm A_0^*$

$$\left(I_{n\times n}s - A_0^*\right)^{-1}\cdot s\,[x] = \left(I_{n\times n}s - A_0^*\right)^{-1}A[x] + \left(I_{n\times n}s - A_0^*\right)^{-1}b[u]$$

$$\Downarrow$$

$$\left(I_{n\times n}s - A_0^*\right)^{-1}\left(I_{n\times n}s \pm A_0^*\right)[x] = \left(I_{n\times n}s - A_0^*\right)^{-1}A[x] + \left(I_{n\times n}s - A_0^*\right)^{-1}b[u]$$

$$\Downarrow$$

$$x = \left(I_{n\times n}s - A_0^*\right)^{-1}\left(A - A_0^*\right)[x] + \left(I_{n\times n}s - A_0^*\right)^{-1}b[u]$$

$$\begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} -k_{n-1} & 1 & 0 & \cdots & 0 \\ -k_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_1 & 0 & 0 & \cdots & 1 \\ -k_0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} k_{n-1} - a_{n-1} \\ \vdots \\ k_1 - a_1 \\ k_0 - a_0 \end{bmatrix} c^T$$

7.2. Scheme №2. State parameterization

$$\left(I_{n\times n}s-A_0^*\right)^{-1}\cdot \overset{\pm A_0^*}{s}\,[x]=\left(I_{n\times n}s-A_0^*\right)^{-1}A[x]+\left(I_{n\times n}s-A_0^*\right)^{-1}b[u]$$

$$\Downarrow$$

$$\left(I_{n\times n}s-A_0^*\right)^{-1}\left(I_{n\times n}s\pm A_0^*\right)[x]=\left(I_{n\times n}s-A_0^*\right)^{-1}A[x]+\left(I_{n\times n}s-A_0^*\right)^{-1}b[u]$$

$$\Downarrow$$

$$x=\left(I_{n\times n}s-A_0^*\right)^{-1}\begin{bmatrix}k_{n-1}-a_{n-1}\\ \vdots\\ k_1-a_1\\ k_0-a_0\end{bmatrix}c^T[x]+\left(I_{n\times n}s-A_0^*\right)^{-1}b[u]$$

7.2. Scheme №2. State parameterization

$$\left(I_{n\times n}s-A_0^*\right)^{-1}\cdot s\overset{\pm A_0^*}{\vphantom{x}}[x]=\left(I_{n\times n}s-A_0^*\right)^{-1}A[x]+\left(I_{n\times n}s-A_0^*\right)^{-1}b[u]$$

$$\Downarrow$$

$$\left(I_{n\times n}s-A_0^*\right)^{-1}\left(I_{n\times n}s\pm A_0^*\right)[x]=\left(I_{n\times n}s-A_0^*\right)^{-1}A[x]+\left(I_{n\times n}s-A_0^*\right)^{-1}b[u]$$

$$\Downarrow$$

$$x=\left(I_{n\times n}s-A_0^*\right)^{-1}\begin{bmatrix}k_{n-1}-a_{n-1}\\ \vdots\\ k_1-a_1\\ k_0-a_0\end{bmatrix}[y]+\left(I_{n\times n}s-A_0^*\right)^{-1}\begin{bmatrix}0\\ \vdots\\ 0\\ b_m\\ \vdots\\ b_0\end{bmatrix}[u]$$

7.2. Scheme №2. State parameterization

$$\begin{bmatrix} k_{n-1}-a_{n-1} \\ \vdots \\ k_1-a_1 \\ k_0-a_0 \end{bmatrix} = \sum_{i=0}^{n-1}(k_i-a_i)e_{n-i}, \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix} = \sum_{j=0}^{m} b_j e_{m+1-j},$$

where $e_i = col\left(0,\ldots,0,\underset{i}{1},0,\ldots,0\right)$.

7.2. Scheme №2. State parameterization

$$\begin{bmatrix} k_{n-1}-a_{n-1} \\ \vdots \\ k_1-a_1 \\ k_0-a_0 \end{bmatrix} = \sum_{i=0}^{n-1}(k_i-a_i)e_{n-i}, \qquad \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix} = \sum_{j=0}^{m} b_j e_{m+1-j},$$

where $e_i = col\Big(0,\ldots,0,\underset{i}{1},0,\ldots,0\Big)$.

$$x = \left(I_{n\times n}s - A_0^*\right)^{-1}\begin{bmatrix} k_{n-1}-a_{n-1} \\ \vdots \\ k_1-a_1 \\ k_0-a_0 \end{bmatrix}[y] + \left(I_{n\times n}s - A_0^*\right)^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix}[u]$$

7.2. Scheme №2. State parameterization

$$\begin{bmatrix} k_{n-1}-a_{n-1} \\ \vdots \\ k_1-a_1 \\ k_0-a_0 \end{bmatrix} = \sum_{i=0}^{n-1}(k_i-a_i)e_{n-i}, \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix} = \sum_{j=0}^{m} b_j e_{m+1-j},$$

where $e_i = col\left(0,\ldots,0,\underset{i}{1},0,\ldots,0\right).$

$$x = \sum_{i=0}^{n-1}(k_i-a_i)\left(I_{n\times n}s-A_0^*\right)^{-1}e_{n-i}\left[y\right]+\sum_{j=0}^{m}b_j\left(I_{n\times n}s-A_0^*\right)^{-1}e_{m+1-j}\left[u\right]$$

7.2. Scheme №2. State parameterization

$$\begin{bmatrix} k_{n-1}-a_{n-1} \\ \vdots \\ k_1-a_1 \\ k_0-a_0 \end{bmatrix} = \sum_{i=0}^{n-1}(k_i-a_i)e_{n-i}, \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix} = \sum_{j=0}^{m} b_j e_{m+1-j},$$

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$$x = \sum_{i=0}^{n-1}(k_i-a_i)\left(I_{n\times n}s-A_0^*\right)^{-1}e_{n-i}[y] + \sum_{j=0}^{m} b_j\left(I_{n\times n}s-A_0^*\right)^{-1}e_{m+1-j}[u]$$

$$x = \sum_{i=0}^{n-1}\theta_{i+1}\left(I_{n\times n}s-A_0^*\right)^{-1}e_{n-i}[y] + \sum_{j=0}^{m}\theta_{j+1+n}\left(I_{n\times n}s-A_0^*\right)^{-1}e_{m+1-j}[u] \quad (7.7)$$

$$\theta = col\left(k_0-a_0, k_1-a_1, \cdots, k_{n-1}-a_{n-1}, b_0, b_1, \cdots, b_m\right) \in \mathbb{R}^{n+m+1}$$

7.2. Scheme №2. State parameterization

$$x=\sum_{i=0}^{n-1}\theta_{i+1}\left(I_{n\times n}s-A_0^*\right)^{-1}e_{n-i}\left[y\right]+\sum_{j=0}^{m}\theta_{j+1+n}\left(I_{n\times n}s-A_0^*\right)^{-1}e_{m+1-j}\left[u\right]$$

Parameterization with sets of filters

$$x=\sum_{i=1}^{n}\theta_i\xi_i^*+\sum_{j=1}^{m+1}\theta_{j+n}\nu_j^*, \qquad (7.8)$$

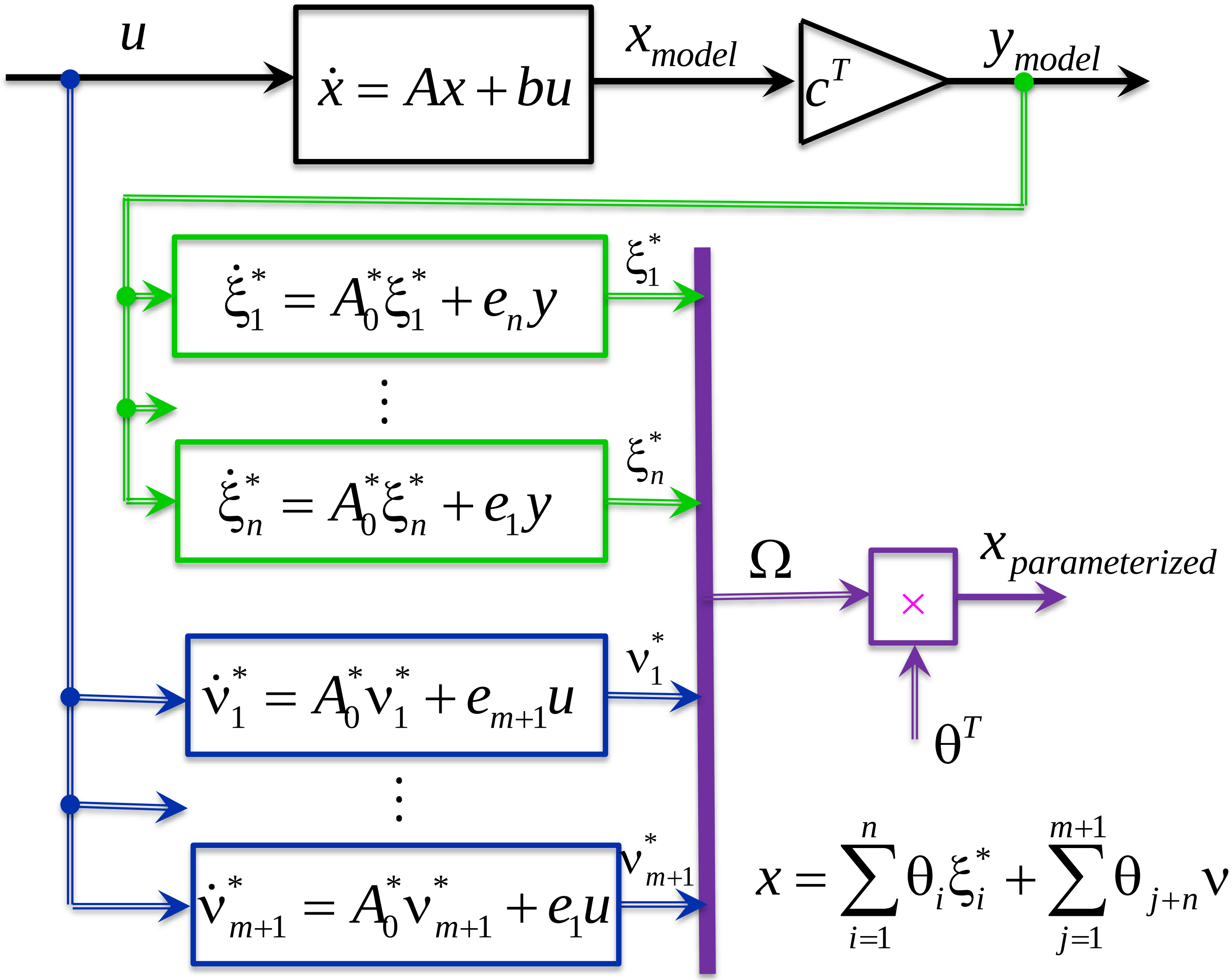
where $e_i=col\Big(0,\ldots,0,\underset{i}{1},0,\ldots,0\Big)$,

$$\begin{cases}\dot{\xi}_i^*=A_0^*\xi_i^*+e_{n+1-i}y, \quad i=\overline{1,n}, & (7.9)\\ \dot{\nu}_j^*=A_0^*\nu_j^*+e_{m+2-j}u, \; j=\overline{1,m+1}, & (7.10)\end{cases}$$

$$A_0^*=\begin{bmatrix}-k_{n-1} & 1 & 0 & \cdots & 0\\ -k_{n-2} & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -k_1 & 0 & 0 & \cdots & 1\\ -k_0 & 0 & 0 & \cdots & 0\end{bmatrix},$$

$$\theta=col\left(k_0-a_0,k_1-a_1,\cdots,k_{n-1}-a_{n-1},b_0,b_1,\cdots,b_m\right)\in\mathbb{R}^{n+m+1}$$

7.2. Scheme №2. State parameterization



$$x = \sum_{i=1}^{n} \theta_i \xi_i^* + \sum_{j=1}^{m+1} \theta_{j+n} \nu_j^* = \Omega\theta$$

8. Adaptive observer design

Problem statement

Plant

$$\begin{cases} \dot{x} = Ax + bu, \\ y = c^T x, \end{cases} \tag{8.1}$$

where $x \in \mathbb{R}^n$ is the unmeasurable state vector of known dimension,

$u,\ y \in \mathbb{R}$ are the measurable input and output, respectively,

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix},\ c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$a_i, b_j,\ i = \overline{0, n-1},\ j = \overline{0, m}$ are the unknown constant parameters.

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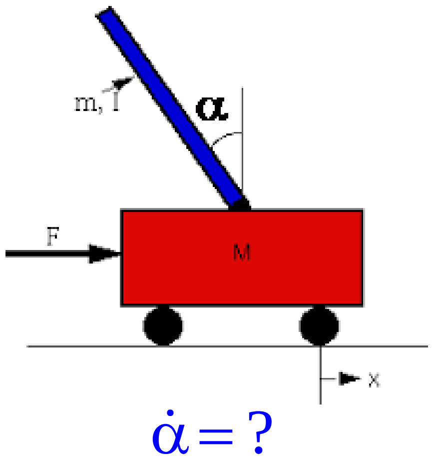

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix},\ c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$a_i, b_j,\ i = \overline{0, n-1},\ j = \overline{0, m}$ are the unknown constant parameters.

Problem statement

Assumption 8.1. Matching conditions (general case): There exists such a Hurwitz matrix A_0^* and a vector of unknown parameters θ that

$$A_0^* = A + \theta c^T. \tag{8.2}$$

Objective to design and estimator that generates the vector $\hat{x}(t)$ satisfying limiting equality

$$\lim_{t\to\infty} \left\| x(t) - \hat{x}(t) \right\| = 0. \tag{8.3}$$

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Assumption 8.2. Signal $u(t)$ is persistently excited, i.e. contains a number of harmonics sufficient for identification of $(n+m+1)/2$ parameters.

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Solution

Obstacles

Unmeasurable $\boldsymbol{x}$

Unknown parameters

Solution

1. The obstacle of unmeasurable state is resolved by using parameterization

Scheme#2 of parameterization

$$x=\sum_{i=1}^{n}\theta_i\xi_i^* + \sum_{j=1}^{m+1}\theta_{j+n}\nu_j^*,$$

where $\theta = col\left(k_0-a_0, k_1-a_1, \cdots, k_{n-1}-a_{n-1}, b_0, b_1, \cdots, b_m\right)\in\mathbb{R}^{n+m+1}$,

$$\begin{cases}\dot{\xi}_i^* = A_0^*\xi_i^* + e_{n+1-i}y, \quad i=\overline{1,n}, & (8.5)\\ \dot{\nu}_j^* = A_0^*\nu_j^* + e_{m+2-j}u, \; j=\overline{1,m+1}, & (8.6)\end{cases}$$

$$e_i = col\Big(0,\ldots,0,\underset{i}{1},0,\ldots,0\Big),$$

$$A_0^* = \begin{bmatrix} -k_{n-1} & 1 & 0 & \cdots & 0\\ -k_{n-2} & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -k_1 & 0 & 0 & \cdots & 1\\ -k_0 & 0 & 0 & \cdots & 0\end{bmatrix},$$

Solution

and replacement of the vector θ by its estimate $\hat{\theta}$:

$$\hat{x} = \sum_{i=1}^{n} \hat{\theta}_i \xi_i^* + \sum_{j=1}^{m+1} \hat{\theta}_{j+n} \nu_j^*, \tag{8.4}$$

where

$$\begin{cases} \dot{\xi}_i^* = A_0^* \xi_i^* + e_{n+1-i} y, \quad i = \overline{1,n}, & (8.5) \\ \dot{\nu}_j^* = A_0^* \nu_j^* + e_{m+2-j} u, \; j = \overline{1,m+1}, & (8.6) \end{cases}$$

$$e_i = col\Big(0,\ldots,0,\underset{i}{1},0,\ldots,0\Big),$$

$$A_0^* = \begin{bmatrix} -k_{n-1} & 1 & 0 & \cdots & 0 \\ -k_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_1 & 0 & 0 & \cdots & 1 \\ -k_0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

Solution

2. The obstacle of unknown parameters is resolved using parameterization

$$\boxed{\begin{array}{c}\textit{Scheme}\,\#1\;\; \textit{of}\\ \textit{parameterization}\end{array}}\qquad y = \theta^T \omega,$$

where $\omega = col\left(\xi_1, \xi_2, \cdots, \xi_n, \nu_1, \nu_2, \cdots, \nu_{m+1}\right) \in \mathbb{R}^{n+m+1},$

$$\begin{cases} \dot{\xi} = A_0 \xi + e_n y, & (8.7) \\ \dot{\nu} = A_0 \nu + e_n u, & (8.8) \end{cases}$$

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -k_0 & -k_1 & -k_2 & \cdots & -k_{n-1} \end{bmatrix}, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Solution

2. The obstacle of unknown parameters is resolved using parameterization

$\boxed{\begin{array}{c} \textit{Scheme\#1 of} \\ \textit{parameterization} \end{array}}$

$$y = \theta^T \omega, \tag{8.9}$$

where $\omega = col\left(\xi_1, \xi_2, \cdots, \xi_n, \nu_1, \nu_2, \cdots, \nu_{m+1}\right) \in \mathbb{R}^{n+m+1}$.

Introduce the error of identification (see Example 6.3)

$$e = y - \hat{\theta}^T \omega, \tag{8.10}$$

which after replacement of (8.9) gives the static error model (see Lecture 6.1)

$$e = \tilde{\theta}^T \omega \tag{8.11}$$

with the vector of parametric errors $\tilde{\theta} = \theta - \hat{\theta}$.

Solution

The static error model motivates design of adaptation algorithm

$$\dot{\hat{\theta}} = \gamma \omega e \tag{8.12}$$

with a positive gain γ.

Solution Summary

Adjustable parameterized model of the plant

$$\hat{x} = \sum_{i=1}^{n} \hat{\theta}_i \xi_i^* + \sum_{j=1}^{m+1} \hat{\theta}_{j+n} \nu_j^* \quad (8.4)$$

Filters

$$\begin{cases} \dot{\xi}_i^* = A_0^* \xi_i^* + e_{n+1-i} y, \quad i = \overline{1, n}, & (8.5) \\ \dot{\nu}_j^* = A_0^* \nu_j^* + e_{m+2-j} u, \; j = \overline{1, m+1} & (8.6) \end{cases}$$

Identification error

$$e = y - \hat{\theta}^T \omega, \quad (8.10)$$

$$\omega = col\left(\xi_1, \xi_2, \cdots, \xi_n, \nu_1, \nu_2, \cdots, \nu_{m+1}\right)$$

Filters

$$\begin{cases} \dot{\xi} = A_0 \xi + e_n y, & (8.7) \\ \dot{\nu} = A_0 \nu + e_n u, & (8.8) \end{cases}$$

Adaptation algorithm

$$\dot{\hat{\theta}} = \gamma \omega e \quad (8.12)$$

Solution Summary

Adjustable parameterized model of the plant

$$\hat{x} = \sum_{i=1}^{n} \hat{\theta}_i \xi_i^* + \sum_{j=1}^{m+1} \hat{\theta}_{j+n} \nu_j^* \qquad (8.4)$$

Filters

$$\begin{cases} \dot{\xi}_i^* = A_0^* \xi_i^* + e_{n+1-i} y, \quad i = \overline{1, n}, & (8.5) \\ \dot{\nu}_j^* = A_0^* \nu_j^* + e_{m+2-j} u, \; j = \overline{1, m+1} & (8.6) \end{cases}$$

Identification error

$$e = y - \hat{\theta}^T \omega, \qquad (8.10)$$

$$\omega = col\left(\xi_1, \xi_2, \cdots, \xi_n, \nu_1, \nu_2, \cdots, \nu_{m+1}\right)$$

Filters

$$\begin{cases} \dot{\xi} = A_0 \xi + e_n y, & (8.7) \\ \dot{\nu} = A_0 \nu + e_n u, & (8.8) \end{cases}$$

Adaptation algorithm

$$\dot{\hat{\theta}} = \gamma \omega e \qquad (8.12)$$

Homework Simplify the Structure

Solution Summary

Properties

1. If u is bounded, all the signals in the system are bounded;
2. If the signal u contains at least *(n+m+1)/2* harmonics,
   then the norms $\|x(t)-\hat{x}(t)\|$, $\|\tilde{\theta}(t)\|$ approach zero exponentially;
3. If the signal u contains at least *(n+m+1)/2* harmonics, there exists an optimal γ, for which the rate of convergence of $\|\tilde{\theta}(t)\|$ is maximum.

Solution Summary

Properties

1. If u is bounded, all the signals in the system are bounded;

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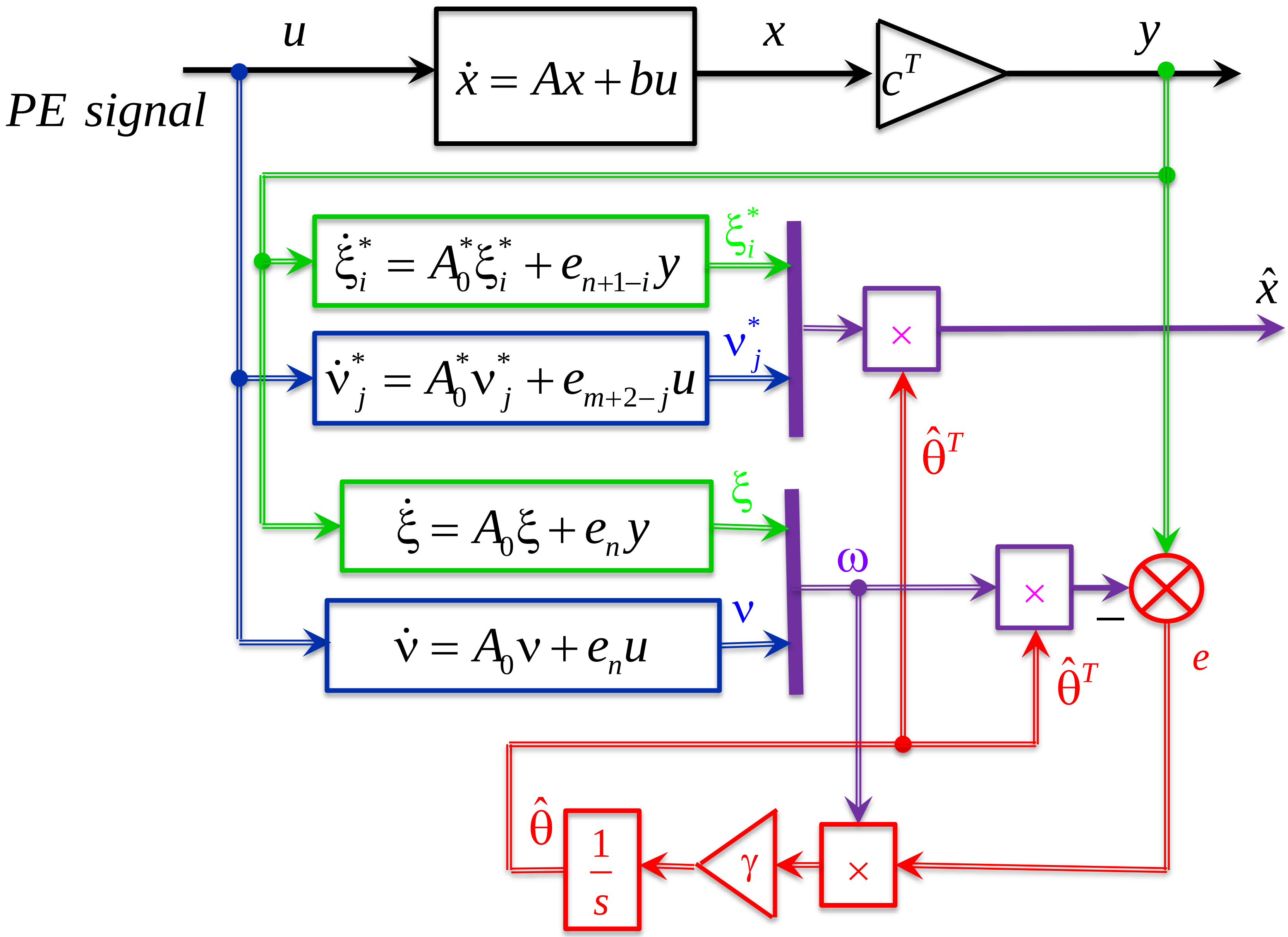
3. If the signal u contains at least $(n+m+1)/2$ harmonics, there exists an optimal γ, for which the rate of convergence of $\|\tilde{\theta}(t)\|$ is maximum.

Significant restriction of practical implementation


PE signal
u
$\dot{x} = Ax + bu$
x
c^T
y
$\dot{\xi}_i^* = A_0^* \xi_i^* + e_{n+1-i} y$
ξ_i^*
$\dot{\nu}_j^* = A_0^* \nu_j^* + e_{m+2-j} u$
ν_j^*
$\hat{x}$
$\hat{\theta}^T$
$\dot{\xi} = A_0 \xi + e_n y$
ξ
ω
$\dot{\nu} = A_0 \nu + e_n u$
ν
$\hat{\theta}^T$
e
$\hat{\theta}$
$\frac{1}{s}$
γ

Simulation results

Plant

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-a_1 & 1\\ -a_0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}b_1\\ b_0\end{bmatrix}u$$

Unknown parameters

$$a_0=-1,\ a_1=-2,$$
$$b_0=3,\quad b_1=4$$

Filters

$$\begin{bmatrix}\dot{\xi}^*_{11}\\ \dot{\xi}^*_{12}\end{bmatrix}=\begin{bmatrix}-5 & 1\\ -6 & 0\end{bmatrix}\begin{bmatrix}\xi^*_{11}\\ \xi^*_{12}\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}y,\quad \begin{bmatrix}\dot{\nu}^*_{11}\\ \dot{\nu}^*_{12}\end{bmatrix}=\begin{bmatrix}-5 & 1\\ -6 & 0\end{bmatrix}\begin{bmatrix}\nu^*_{11}\\ \nu^*_{12}\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u,$$

$$\begin{bmatrix}\dot{\xi}^*_{21}\\ \dot{\xi}^*_{22}\end{bmatrix}=\begin{bmatrix}-5 & 1\\ -6 & 0\end{bmatrix}\begin{bmatrix}\xi^*_{21}\\ \xi^*_{22}\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}y,\quad \begin{bmatrix}\dot{\nu}^*_{21}\\ \dot{\nu}^*_{22}\end{bmatrix}=\begin{bmatrix}-5 & 1\\ -6 & 0\end{bmatrix}\begin{bmatrix}\nu^*_{21}\\ \nu^*_{22}\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u,$$

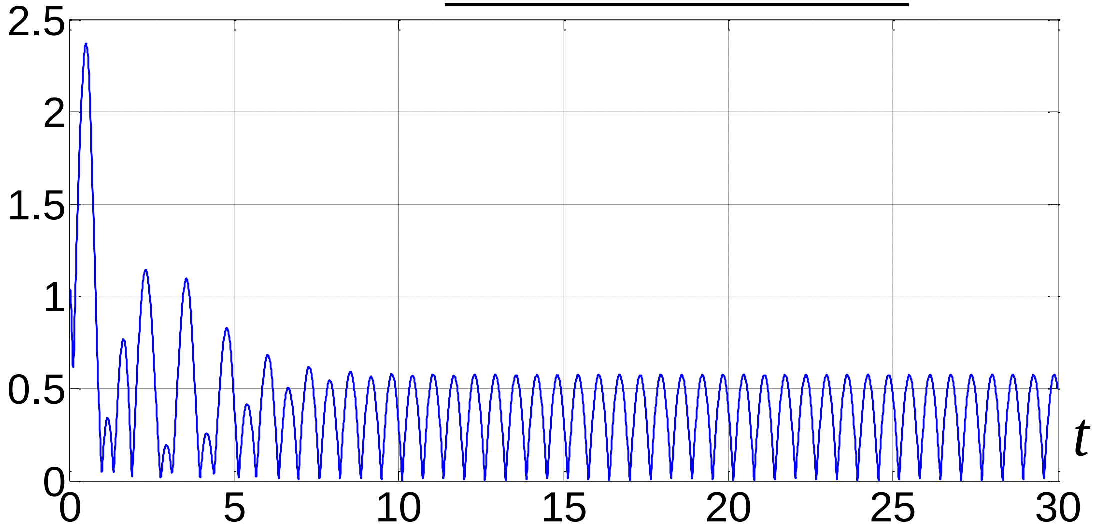
$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -6 & -5\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}y,\quad \begin{bmatrix}\dot{\nu}_1\\ \dot{\nu}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -6 & -5\end{bmatrix}\begin{bmatrix}\nu_1\\ \nu_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

Adaptation gain

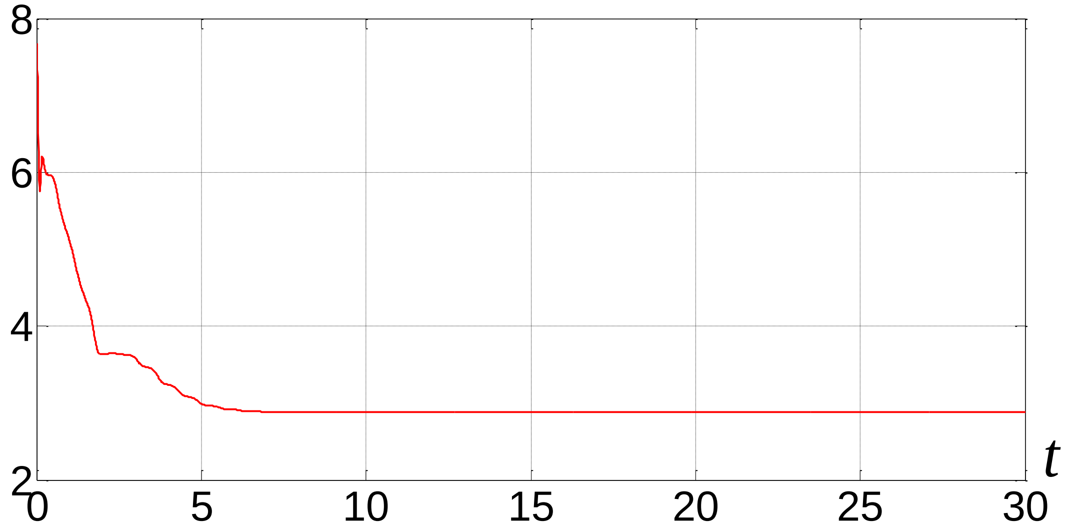
$$\gamma=1000$$

Simulation results

$\|x(t)-\hat{x}(t)\|$



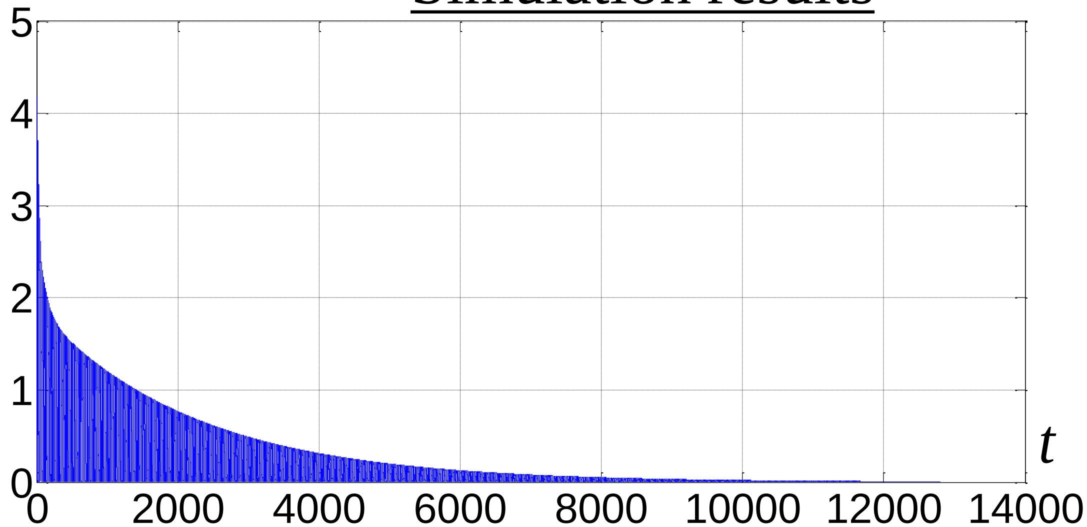
$\|\tilde{\theta}(t)\|$



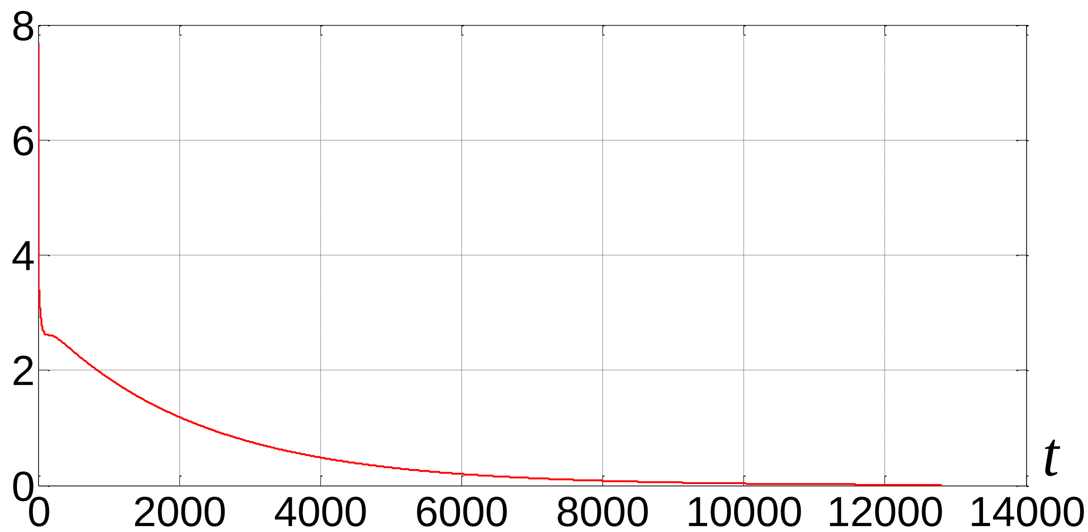
Input (control) signal $u(t)=\sin(t)$

Simulation results

$\|x(t)-\hat{x}(t)\|$



$\|\tilde{\theta}(t)\|$



Input (control) signal $u(t)=\sin(t)+\cos(5t)$

9. Scheme №3 of LTI plants parameterization

Application of adaptive observers in the output control can deteriorate the closed-loop system performance due to the dependence of identification process from PE condition.

9. Scheme №3 of LTI plants parameterization

Application of adaptive observers in the output control can deteriorate the closed-loop system performance due to the dependence of identification process from PE condition.

In was observed in **(K. Åstrom and B. Wittenmark, On Self-tuning Regulators, Automatica, Vol. 9, pp. 185-199, 1973.)** that it is not necessary to identify the plant parameters, but instead it is sufficient to directly tune the controller parameters and overcome the problem of PE condition.

9. Scheme №3 of LTI plants parameterization

Plant

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = b_m u^{(m)} + \ldots + b_0 u, \tag{9.1}$$

where $a_i, b_j,\ i = \overline{0, n-1},\ j = \overline{0, m}$ are the constant parameters.

Transfer function representation

$$y(t) = \frac{b_m s^m + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}\left[u(t)\right] = \frac{b(s)}{a(s)}\left[u(t)\right] \tag{9.2}$$

with coprime polynomials $a(s), b(s)$.

Assume initial conditions $y(0), \ldots, y^{(n-1)}(0)$ zero.

9. Scheme №3 of LTI plants parameterization

Lemma (R. V. Monopoli, 1974, *direct control parameterization lemma* **)**
There exists a constant vector $\theta \in \mathbb{R}^{2n-1}$ such that

$$y(t) = \frac{1}{\delta_M(s)}\left[\theta^T \omega(t) + b_m u(t)\right], \qquad (9.3)$$

where $\omega = col(y, \nu_1, \nu_2) \in \mathbb{R}^{2n-1}$ is the regressor vector,

$\delta_M(s)$ is an arbitrary Hurwitz polynomial of degree $(n-m)$ (the same as the plant relative degree),

$\nu_1, \nu_2 \in \mathbb{R}^{n-1}$ are the vectors generated by the stable filters

$$\begin{cases} \dot{\nu}_1 = \Lambda \nu_1 + e_{n-1} y, & (9.4) \\ \dot{\nu}_2 = \Lambda \nu_2 + e_{n-1} u, & (9.5) \end{cases} \quad \Lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\gamma_0 & -\gamma_1 & -\gamma_2 & \cdots & -\gamma_{n-2} \end{bmatrix}, \; e_{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

10. Model Reference Adaptive Control (MRAC) Schemes

Problem statement

Plant

$$y(t)=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\ldots+a_0}[u(t)]=\frac{b(s)}{a(s)}[u(t)] \qquad (10.1)$$

with **unknown** parameters $a_i, b_j,\ i=\overline{0,n-1},\ j=\overline{0,m}$, unmeasurable state , the measurable input u and output y , the known order n and the relative degree $\rho=n-m$.

10. Model Reference Adaptive Control (MRAC) Schemes

Problem statement

Plant

$$y(t)=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\ldots+a_0}[u(t)]=\frac{b(s)}{a(s)}[u(t)] \qquad (10.1)$$

with **unknown** parameters $a_i, b_j,\ i=\overline{0,n-1},\ j=\overline{0,m}$, unmeasurable state , the measurable input u and output y , the known order n and the relative degree $\rho=n-m$.

The objective is to design a control $u(t)$ such that

$$\lim_{t\to\infty}\left\|y_M(t)-y(t)\right\|=0, \qquad (10.2)$$

where y_M is the output of reference model with PWC reference signal g :

$$y_M^{\rho}+a_{M\,n-1}y_M^{\rho-1}+\ldots+a_{M\,0}y_M=a_{M\,0}g \qquad (10.3)$$

10. Model Reference Adaptive Control (MRAC) Schemes

Problem statement

Plant

$$y(t)=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\ldots+a_0}\left[u(t)\right]=\frac{b(s)}{a(s)}\left[u(t)\right] \qquad (10.1)$$

with **unknown** parameters $a_i, b_j,\ i=\overline{0,n-1},\ j=\overline{0,m}$, unmeasurable state , the measurable input u and output y , the known order n and the relative degree $\rho=n-m$.

The objective is to design a control $u(t)$ such that

$$\lim_{t\to\infty}\left\|y_M(t)-y(t)\right\|=0, \qquad (10.2)$$

where y_M is the output of reference model with PWC reference signal g :

$$y_M(t)=\frac{a_{M0}}{s^\rho+a_{M\rho-1}s^{\rho-1}+a_{M\rho-2}s^{\rho-2}+\cdots+a_{M0}}\left[g(t)\right]=\frac{a_{M0}}{\delta_M(s)}\left[g(t)\right] \quad (10.3)$$

Problem statement

Assumption 10.1. Plant is controllable ;

Assumption 10.2. Plant is minimum phase, i.e. The polynomial

$$b(s) = b_m s^m + \ldots + b_1 s + b_0$$ is Hurwitz;

Assumption 10.3. b_m is known (for the sake of simplicity and without loss of generality).

Solution

Obstacles

Unmeasurable

$\dot{y}, \ldots, y^{(n-1)}$

Unknown parameters

Solution

1. Obstacle of unmeasurable state.

Introduce the output tracking error

$$\varepsilon = y_M - y \tag{10.4}$$

Solution

1. Obstacle of unmeasurable state.

Introduce the output tracking error

$$\varepsilon = y_M - y \tag{10.4}$$

and using *direct control parameterization lemma* we replace y by (9.3) and then replace y_M by (10.3):

$$y = \frac{1}{\delta_M(s)}\left[\theta^T\omega + b_m u\right],$$

$$y_M = \frac{a_{M0}}{\delta_M(s)}[g]$$

Solution

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Introduce the output tracking error

$$\varepsilon = y_M - y \tag{10.4}$$

and using *direct control parameterization lemma* we replace y by (9.3) and then replace y_M by (10.3):

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1. Obstacle of unmeasurable state.

Introduce the output tracking error

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$$\Downarrow$$

$$\varepsilon = \frac{1}{\delta_M(s)}\left[a_{M0}g - \theta^T\omega - b_m u\right] \tag{10.5}$$

See also Example 6.6. in page 6.50 (second order system) with similar notations

Solution

2. Obstacle of unknown parameters.

Error equation

$$\varepsilon = \frac{1}{\delta_M(s)}\left[a_{M0}g - \theta^T\omega - b_m u\right]$$

Certainty equivalence principle

Adjustable control

$$u = \frac{1}{b_m}\left(a_{M0}g - \hat{\theta}^T\omega\right) \qquad u = \frac{1}{b_m}\delta_M(s)\left[\frac{a_{M0}}{\delta_M(s)}[g] - \hat{\theta}^T\bar{\omega}\right]$$

$$\Downarrow \qquad \Downarrow$$

Error model

$$\varepsilon = -\frac{1}{\delta_M(s)}\left[\tilde{\theta}^T\omega\right]$$

$$\varepsilon = -\tilde{\theta}^T\bar{\omega},$$

$$\bar{\omega} = \frac{1}{\delta_M(s)}[\omega]$$

R. V. Monopoli, 1974,

A. Feuer and A.S. Morse, 1978

A. S. Morse, 1994

Augmented Error Solution

2. Obstacle of unknown parameters.

Based on the error equation

$$\varepsilon = \frac{1}{\delta_M(s)}\left[a_{M0}g - \theta^T\omega - b_m u\right] \tag{10.5}$$

and the certainty equivalence principle we design adjustable control

$$u = \frac{1}{b_m}\left(a_{M0}g - \hat{\theta}^T\omega\right) \tag{10.6}$$

that after replacement to (10.5) gives the **dynamic** error model

(see Lecture 6, Pages 6.42-6.57)

$$\varepsilon = -\frac{1}{\delta_M(s)}\left[\tilde{\theta}^T\omega\right]. \tag{10.7}$$

Augmented Error Solution

2. Obstacle of unknown parameters.

Model (10.7) motivates the design of *augmented error* adaptation algorithm (see Lecture 6, Pages 6.42-6.57)

$$\dot{\hat{\theta}} = -\gamma\bar{\omega}\hat{\varepsilon}, \tag{10.8}$$

$$\hat{\varepsilon} = \varepsilon + \hat{\theta}^T\bar{\omega} - \frac{1}{\delta_M(s)}\left[\hat{\theta}^T\omega\right], = -\tilde{\theta}^T\bar{\omega} \tag{10.9}$$

$$\bar{\omega} = \frac{1}{\delta_M(s)}[\omega],$$

where γ is the normalized adaptation gain defined as

$$\gamma(t) = \frac{\gamma_0}{1+\bar{\omega}^T\bar{\omega}}, \quad \gamma_0 = const > 0. \tag{10.10}$$

Augmented Error Solution Summary

Adjustable control $$u = \frac{1}{b_m}\left(a_{M0}g - \hat{\theta}^T\omega\right) \quad (10.6)$$

Regressor $$\omega = col(y, \nu_1, \nu_2)$$

Filters $$\begin{cases} \dot{\nu}_1 = \Lambda\nu_1 + e_{n-1}y, & (9.4) \\ \dot{\nu}_2 = \Lambda\nu_2 + e_{n-1}u & (9.5) \end{cases}$$

Adaptation Algorithm $$\dot{\hat{\theta}} = -\gamma\bar{\omega}\hat{\varepsilon} \quad (10.8)$$

Augmented error $$\hat{\varepsilon} = \varepsilon + \hat{\theta}^T\bar{\omega} - \frac{1}{\delta_M(s)}\left[\hat{\theta}^T\omega\right] \quad (10.9)$$

Error $$\varepsilon = y_M - y$$

Filtered regressor $$\bar{\omega} = \frac{1}{\delta_M(s)}[\omega]$$

Normalized adaptation gain $$\gamma(t) = \frac{\gamma_0}{1 + \bar{\omega}^T(t)\bar{\omega}(t)}, \quad \gamma_0 = const > 0. \quad (10.10)$$

Augmented Error Solution Summary

Adjustable control $$u = \frac{1}{b_m}\left(a_{M0} g - \hat{\theta}^T \omega\right) \qquad (10.6)$$

Regressor $$\omega = col(y, \nu_1, \nu_2)$$

Filters $$\begin{cases} \dot{\nu}_1 = \Lambda \nu_1 + e_{n-1} y, & (9.4) \\ \dot{\nu}_2 = \Lambda \nu_2 + e_{n-1} u & (9.5) \end{cases}$$

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Augmented Error Solution Summary

Apply the Swapping Lemma

Adaptation Algorithm $\dot{\hat{\theta}} = -\gamma\bar{\omega}\hat{\varepsilon}$ (10.8)

Augmented error $\hat{\varepsilon} = \varepsilon + \hat{\theta}^T\bar{\omega} - \frac{1}{\delta_M(s)}\left[\hat{\theta}^T\omega\right]$ (10.9)

Filtered regressor $\bar{\omega} = \frac{1}{\delta_M(s)}[\omega]$

Normalized adaptation gain $\gamma(t) = \frac{\gamma_0}{1+\bar{\omega}^T(t)\bar{\omega}(t)}$ (10.10)

Augmented Error Solution Summary

Apply the Swapping Lemma

Adaptation Algorithm $\dot{\hat{\theta}} = -\gamma \bar{\omega} \hat{\varepsilon}$ (10.8)

Augmented error $\hat{\varepsilon} = \varepsilon + W_c(s)\left[W_b(s)\left[\omega^T\right]\dot{\hat{\theta}}\right]$ (10.9)

$$W_C(s) = c^T (Is - A)^{-1},\; W_b(s) = (Is - A)^{-1} b$$

Filtered regressor $\bar{\omega} = \dfrac{1}{\delta_M(s)}[\omega] = c^T (Is - A)^{-1} b$

Normalized adaptation gain $\gamma(t) = \dfrac{\gamma_0}{1 + \bar{\omega}^T(t)\bar{\omega}(t)}$ (10.10)

Augmented Error Solution Summary

Apply the Swapping Lemma

Adaptation Algorithm $$\dot{\hat{\theta}} = -\gamma \bar{\omega} \hat{\varepsilon} \quad (10.8)$$

Augmented error $$\hat{\varepsilon} = \varepsilon + W_c(s)\left[W_b(s)\left[\omega^T\right]\dot{\hat{\theta}}\right] \quad (10.9)$$

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Normalized adaptation gain $$\gamma(t) = \frac{\gamma_0}{1 + \bar{\omega}^T(t)\bar{\omega}(t)} \quad (10.10)$$

Augmented Error Solution Summary

Augmented error
$$\hat{\varepsilon} = \varepsilon - W_c(s)\left[W_b(s)\left[\omega^T\right]\frac{\gamma_0\bar{\omega}\hat{\varepsilon}}{1+\bar{\omega}^T\bar{\omega}}\right] \qquad (10.9)$$

$$W_C(s) = c^T\left(Is - A\right)^{-1},\ W_b(s) = \left(Is - A\right)^{-1} b$$

Filtered regressor
$$\bar{\omega} = \frac{1}{\delta_M(s)}\left[\omega\right]$$

Augmented Error Solution Summary

Goes to zero

Bounded

Augmented error $$\hat{\varepsilon} = \varepsilon - \gamma_0 W_c(s)\left[W_b(s)\left[\omega^T\right]\frac{\bar{\omega}\hat{\varepsilon}}{1+\bar{\omega}^T\bar{\omega}}\right] \quad (10.9)$$

$$W_C(s) = c^T\left(Is - A\right)^{-1},\; W_b(s) = \left(Is - A\right)^{-1} b$$

Filtered regressor $$\bar{\omega} = \frac{1}{\delta_M(s)}[\omega]$$

Augmented Error Solution Summary

Properties of the closed-loop system (See Lecture 6.3):

1. All the signals in the system are bounded;
2. The error $\hat{\varepsilon}(t)$ approaches zero asymptotically;
3. The $\|\tilde{\theta}(t)\|$ approaches zero asymptotically if ω satisfies the Persistent Excitation condition;
4. The error $\varepsilon(t)$ approaches zero asymptotically.

Augmented Error Solution Summary

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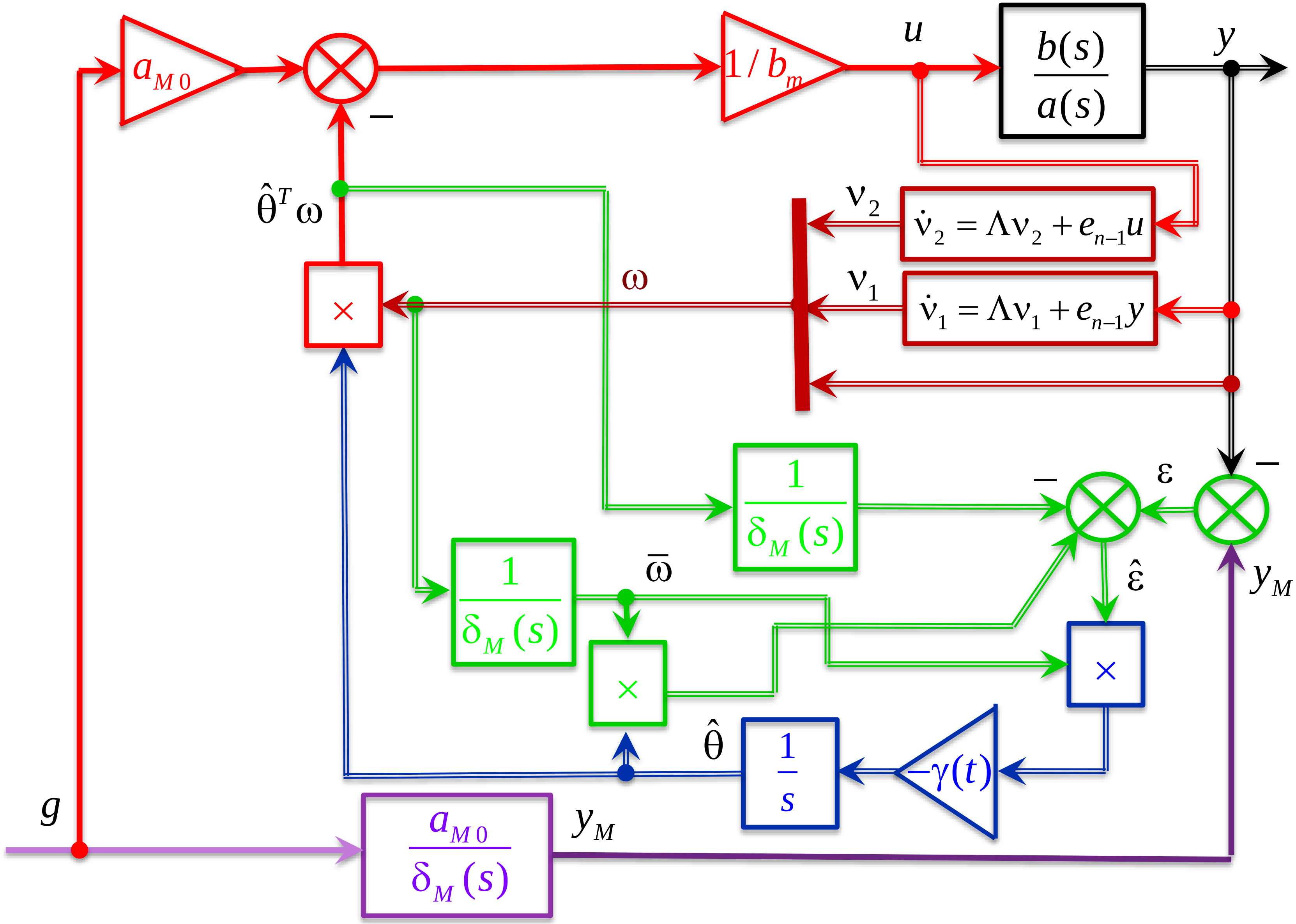
1. All the signals in the system are bounded;
2. The error $\hat{\varepsilon}(t)$ approaches zero asymptotically;
3. The $\|\tilde{\theta}(t)\|$ approaches zero asymptotically if ω satisfies the Persistent Excitation condition;
4. The error $\varepsilon(t)$ approaches zero asymptotically;
5. Rate of parametric convergence is bounded due to normalization factor

$$\dot{\tilde{\theta}} = -\gamma_0 \frac{\bar{\omega}\bar{\omega}^T}{1+\bar{\omega}^T\bar{\omega}}\tilde{\theta}$$


a_{M0}
$-$
$1/b_m$
u
$\frac{b(s)}{a(s)}$
y
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ν_2
$\dot{\nu}_2 = \Lambda\nu_2 + e_{n-1}u$
ω
ν_1
$\dot{\nu}_1 = \Lambda\nu_1 + e_{n-1}y$
$\times$
$\frac{1}{\delta_M(s)}$
$-$
ε
$-$
$\frac{1}{\delta_M(s)}$
$\bar{\omega}$
$\hat{\varepsilon}$
y_M
$\times$
$\times$
$\hat{\theta}$
$\frac{1}{s}$
$-\gamma(t)$
g
$\frac{a_{M0}}{\delta_M(s)}$
y_M

Simulation results

See Lecture 6, pages 6.55-6.57 (second order system)

High Order Tuner Solution

2. Obstacle of unknown parameters.

Based on the error equation

$$\varepsilon = \frac{1}{\delta_M(s)}\left[a_{M0}g - \theta^T\omega - b_m u\right] \qquad (10.11)$$

and the certainty equivalence principle we design adjustable control

$$u = \frac{1}{b_m}\delta_M(s)\left[\frac{a_{M0}}{\delta_M(s)}[g] - \hat{\theta}^T\bar{\omega}\right] \qquad (10.12)$$

that after replacement to (10.11) gives the **static** error model

(see Lecture 6, Pages 6.1-6.14)

$$\varepsilon = -\tilde{\theta}^T\bar{\omega} \qquad (10.13)$$

with filtered regressor $\bar{\omega} = 1/\delta_M(s)[\omega]$.

Two problems arise immediately

Control depends on the high order time derivatives $\dot{\hat{\theta}},\ldots,\hat{\theta}^{(\rho-1)}$ hidden in the term

$$\delta_M(s)\left[\hat{\theta}^T\bar{\omega}\right]$$

$$u=\frac{1}{b_m}\delta_M(s)\left[\frac{a_{M0}}{\delta_M(s)}[g]-\hat{\theta}^T\bar{\omega}\right]$$

Two problems arise immediately

Control depends on the high order time derivatives $\dot{\hat{\theta}},\ldots,\hat{\theta}^{(\rho-1)}$ hidden in the term

$$\delta_M(s)\left[\hat{\theta}^T\bar{\omega}\right]$$

$$u=\frac{1}{b_m}\delta_M(s)\left[\frac{a_{M0}}{\delta_M(s)}[g]-\hat{\theta}^T\bar{\omega}\right]$$

Adaptation algorithm

$$\dot{\hat{\theta}}=-\gamma\bar{\omega}\varepsilon$$

based on the static error model $\varepsilon=-\tilde{\theta}^T\bar{\omega}$ is unable to generate $\dot{\hat{\theta}},\ldots,\hat{\theta}^{(\rho-1)}$ for control

High Order Tuner Solution

2. Obstacle of unknown parameters.

The High Order Tuner **(A. S. Morse, 1994)**

$$\begin{cases} \dot{\hat{\psi}}_i = -\bar{\omega}_i \varepsilon, & (10.14) \\ \dot{\eta}_i = (1+\mu\bar{\omega}^T\bar{\omega})(\bar{A}\eta_i + \bar{b}\hat{\psi}_i), & (10.15) \\ \hat{\theta}_i = \bar{c}^T\eta_i, & (10.16) \end{cases}$$

where $\bar{\omega}_i, \hat{\psi}_i, \hat{\theta}_i$ $i = \overline{1, 2n-1}$ are the elements of vectors $\bar{\omega}, \hat{\psi}, \hat{\theta}$, respectively,

$$\mu \geq \frac{(2n-1)}{2}\left\| \bar{c} - P\bar{A}^{-1}\bar{b} \right\|^2$$

is a constant gain, $P = P^T \succ 0$ is the solution of Lyapunov equation $\bar{A}^T P + P\bar{A} = -2I_{\rho-1\times\rho-1}$, $(\bar{c}^T, \bar{A}, \bar{b})$ is the matrices triple being the minimal realization of stable transfer function $\alpha(0)/\alpha(s)$.

High Order Tuner Solution

2. Obstacle of unknown parameters.

The High Order Tuner **(A. S. Morse, 1994)**

$$\begin{cases} \dot{\hat\psi}_i = -\bar\omega_i\varepsilon, & (10.14) \\ \dot\eta_i = (1+\mu\bar\omega^T\bar\omega)(\bar A\eta_i + \bar b\hat\psi_i), & (10.15) \\ \hat\theta_i = \bar c^T\eta_i, & (10.16) \end{cases}$$

Since $\bar c^T\bar A^{i-1}\bar b = 0,\ i = \overline{1,\rho},$ we obtain for the case $\rho = 2:$

$$\bar c^T\bar A^{-1}\bar b = -1$$

$$\dot{\hat\theta}_i = \bar c^T\dot\eta_i = \bar c^T(1+\mu\bar\omega^T\bar\omega)(\bar A\eta_i + \bar b\hat\psi_i)$$

$$\ddot{\hat\theta}_i = 2\bar c^T\mu\bar\omega^T\dot{\bar\omega}(\bar A\eta_i + \bar b\hat\psi_i) + \bar c^T(1+\mu\bar\omega^T\bar\omega)(\bar A\dot\eta_i + \bar b\dot{\hat\psi}_i) =$$

$$2\bar c^T\mu\bar\omega^T\dot{\bar\omega}(\bar A\eta_i + \bar b\hat\psi_i) + (1+\mu\bar\omega^T\bar\omega)^2(\bar c^T\bar A^2\eta_i + \bar c^T\bar A\bar b\hat\psi_i) - (1+\mu\bar\omega^T\bar\omega)\bar c^T\bar b\bar\omega_i\varepsilon$$

Hello from classical control theory

High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner

Introduce auxiliary signals

$$z_i = \eta_i + \bar{A}^{-1}\bar{b}\,\hat{\psi}_i \tag{10.17}$$

High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner

Introduce auxiliary signals

$$z_i = \eta_i + \bar{A}^{-1}\bar{b}\hat{\psi}_i \tag{10.17}$$

and calculate their time derivatives:

$$\dot{z}_i = \dot{\eta}_i + \bar{A}^{-1}\bar{b}\dot{\hat{\psi}}_i$$

$$\dot{\eta}_i = (1+\mu\bar{\omega}^T\bar{\omega}\ )(\bar{A}\eta_i + \bar{b}\hat{\psi}_i) \quad (10.15) \qquad \dot{\hat{\psi}}_i = -\bar{\omega}_i\varepsilon \quad (10.14)$$

High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner

Introduce auxiliary signals

$$z_i = \eta_i + \bar{A}^{-1}\bar{b}\hat{\psi}_i \tag{10.17}$$

and calculate their time derivatives:

$$\dot{z}_i = (1+\mu\bar{\varpi}^T\bar{\varpi}\ )(\bar{A}\eta_i + \bar{b}\hat{\psi}_i) - \bar{A}^{-1}\bar{b}\bar{\varpi}_i\varepsilon$$

High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner

Introduce auxiliary signals

$$z_i = \eta_i + \bar{A}^{-1}\bar{b}\hat{\psi}_i \tag{10.17}$$

and calculate their time derivatives

$$\dot{z}_i = (1 + \mu\bar{\omega}^T\bar{\omega})\bar{A}z_i - \bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon. \tag{10.18}$$

High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner

Introduce auxiliary signals

$$z_i = \eta_i + \bar{A}^{-1}\bar{b}\hat{\psi}_i \qquad (10.17)$$

and calculate their time derivatives

$$\dot{z}_i = (1+\mu\bar{\omega}^T\bar{\omega})\bar{A}z_i - \bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon. \qquad (10.18)$$

Multiply (10.17) by $\bar{c}^T$:

$$\bar{c}^T z_i = \underbrace{\bar{c}^T\eta_i}_{} + \underbrace{\bar{c}^T\bar{A}^{-1}\bar{b}}_{}\hat{\psi}_i$$

$$\hat{\theta}_i = \bar{c}^T\eta_i \qquad (10.16)$$

Hello from classical control theory

$$\bar{c}^T\bar{A}^{-1}\bar{b} = -\alpha(0)/\alpha(s)\Big|_{s=0} = -1$$

High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner

Introduce auxiliary signals

$$z_i = \eta_i + \bar{A}^{-1}\bar{b}\hat{\psi}_i \tag{10.17}$$

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$$\dot{z}_i = (1 + \mu\bar{\omega}^T\bar{\omega})\bar{A}z_i - \bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon. \tag{10.18}$$

Multiply (10.17) by $\bar{c}^T$:

$$\bar{c}^T z_i = \hat{\theta}_i - \hat{\psi}_i$$

High Order Tuner Solution

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$$z_i = \eta_i + \bar{A}^{-1}\bar{b}\hat{\psi}_i \tag{10.17}$$

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Multiply (10.17) by $\bar{c}^T$:

$$\bar{c}^T z_i = \hat{\theta}_i - \hat{\psi}_i$$

$$\tilde{\theta}_i = \theta_i - \hat{\theta}_i \qquad \tilde{\psi}_i = \theta_i - \hat{\psi}_i$$

High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner

Introduce auxiliary signals

$$z_i = \eta_i + \bar{A}^{-1}\bar{b}\hat{\psi}_i \tag{10.17}$$

and calculate their time derivatives

$$\dot{z}_i = (1+\mu\bar{\omega}^T\bar{\omega}\ )\bar{A}z_i - \bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon. \tag{10.18}$$

Multiply (10.17) by $\bar{c}^T$:

$$\bar{c}^T z_i = \tilde{\psi}_i - \tilde{\theta}_i$$

High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner

Introduce auxiliary signals

$$z_i = \eta_i + \bar{A}^{-1}\bar{b}\hat{\psi}_i \tag{10.17}$$

and calculate their time derivatives

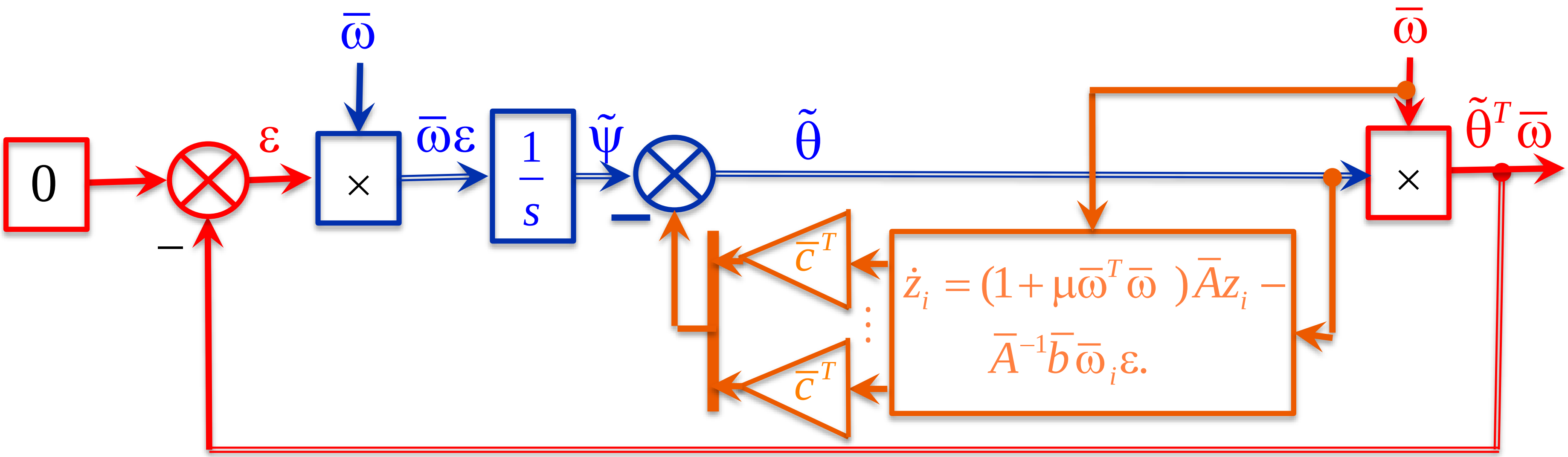
$$\dot{z}_i = (1+\mu\bar{\omega}^T\bar{\omega})\bar{A}z_i - \bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon. \tag{10.18}$$

Multiply (10.17) by $\bar{c}^T$:

$$\tilde{\theta}_i = \tilde{\psi}_i - \bar{c}^T z_i \tag{10.19}$$

High Order Tuner Solution

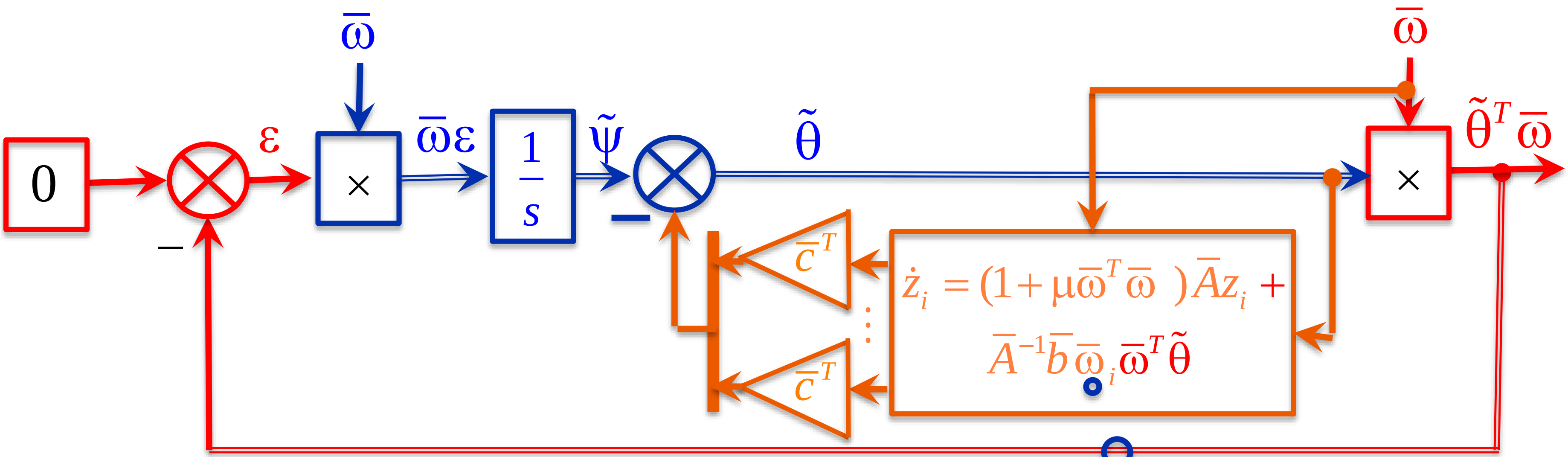
Remark 10.1. One-syllable words about high order tuner



Error model	$\varepsilon = -\tilde{\theta}^T\bar{\omega}$	(10.13)
Auxiliary dynamics	$\dot{z}_i = (1+\mu\bar{\omega}^T\bar{\omega})\bar{A}z_i - \bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon$	(10.18)
Parametric error model	$\tilde{\theta}_i = \tilde{\psi}_i - \bar{c}^T z_i$	(10.19)
	$\dot{\tilde{\psi}}_i = \bar{\omega}_i\varepsilon$	(10.14)

High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner



Error model $\varepsilon = -\tilde{\theta}^T\bar{\omega}$ (10.13)

Auxiliary dynamics $\dot{z}_i = (1+\mu\bar{\omega}^T\bar{\omega})\bar{A}z_i - \bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon$ (10.18)

Parametric error model $\tilde{\theta}_i = \tilde{\psi}_i - \bar{c}^T z_i$ (10.19)

$\dot{\tilde{\psi}}_i = \bar{\omega}_i\varepsilon$ (10.14)

Filters with expanding cutoff frequency

Lyapunov function?

Error model	$\varepsilon = -\tilde{\theta}^T \bar{\omega}$	(10.13)
Auxiliary dynamics	$\dot{z}_i = (1 + \mu\bar{\omega}^T \bar{\omega})\bar{A}z_i - \bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon$	(10.18)
Parametric error model	$\tilde{\theta}_i = \tilde{\psi}_i - \bar{c}^T z_i$	(10.19)
	$\dot{\tilde{\psi}}_i = \bar{\omega}_i\varepsilon$	(10.14)

$$V = \frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P z_i + \frac{1}{2}\sum_{i=1}^{2n-1} \tilde{\psi}_i^2 \qquad (10.20)$$

Error model $\varepsilon = -\tilde{\theta}^T \bar{\omega}$ (10.13)

Auxiliary dynamics $\dot{z}_i = (1 + \mu \bar{\omega}^T \bar{\omega})\bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon$ (10.18)

Parametric error model

$\tilde{\theta}_i = \tilde{\psi}_i - \bar{c}^T z_i$ (10.19)

$\dot{\tilde{\psi}}_i = \bar{\omega}_i \varepsilon$ (10.14)

$$V = \frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P z_i + \frac{1}{2}\sum_{i=1}^{2n-1} \tilde{\psi}_i^2 \qquad (10.20)$$

$$\dot{V} = \frac{1}{2}\sum_{i=1}^{2n-1} \dot{z}_i^T P z_i + \frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P \dot{z}_i + \sum_{i=1}^{2n-1} \tilde{\psi}_i \dot{\tilde{\psi}}_i$$

Error model $\varepsilon = -\tilde{\theta}^T \bar{\omega}$ (10.13)

Auxiliary dynamics $\dot{z}_i = (1+\mu\bar{\omega}^T\bar{\omega})\bar{A} z_i - \bar{A}^{-1}\bar{b}\,\bar{\omega}_i \varepsilon$ (10.18)

Parametric error model $\tilde{\theta}_i = \tilde{\psi}_i - \bar{c}^T z_i$ (10.19)

$\dot{\tilde{\psi}}_i = \bar{\omega}_i \varepsilon$ (10.14)

$$V = \frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P z_i + \frac{1}{2}\sum_{i=1}^{2n-1} \tilde{\psi}_i^2 \qquad (10.20)$$

$$\dot{V} = \frac{1}{2}\sum_{i=1}^{2n-1} \dot{z}_i^T P z_i + \frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P \dot{z}_i + \sum_{i=1}^{2n-1} \tilde{\psi}_i \dot{\tilde{\psi}}_i =$$

$$\frac{1}{2}\sum_{i=1}^{2n-1}\left((1+\mu\bar{\omega}^T\bar{\omega})\bar{A}z_i - \bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon\right)^T P z_i + \frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P\left((1+\mu\bar{\omega}^T\bar{\omega})\bar{A}z_i - \bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon\right) +$$

$$\sum_{i=1}^{2n-1} \tilde{\psi}_i \bar{\omega}_i \varepsilon$$

Error model $\qquad \varepsilon = -\tilde{\theta}^T\bar{\omega} \qquad (10.13)$

Auxiliary dynamics $\qquad \dot{z}_i = (1+\mu\bar{\omega}^T\bar{\omega})\bar{A}z_i - \bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon \qquad (10.18)$

Parametric error model $\qquad \tilde{\theta}_i = \tilde{\psi}_i - \bar{c}^T z_i \qquad (10.19)$

$$\dot{\tilde{\psi}}_i = \bar{\omega}_i\varepsilon \qquad (10.14)$$

$$V=\frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P z_i+\frac{1}{2}\sum_{i=1}^{2n-1}\tilde{\psi}_i^2 \qquad (10.20)$$

$$\dot{V}=\frac{1}{2}\sum_{i=1}^{2n-1}\dot{z}_i^T P z_i+\frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P \dot{z}_i+\sum_{i=1}^{2n-1}\tilde{\psi}_i\dot{\tilde{\psi}}_i=$$

$$\frac{1}{2}\sum_{i=1}^{2n-1}\left((1+\mu\bar{\omega}^T\bar{\omega})\bar{A}z_i-\bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon\right)^T P z_i+\frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P\left((1+\mu\bar{\omega}^T\bar{\omega})\bar{A}z_i-\bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon\right)+$$

$$\sum_{i=1}^{2n-1}\left(\tilde{\theta}_i\bar{\omega}_i\varepsilon+\bar{c}^T z_i\bar{\omega}_i\varepsilon\right)$$

Error model $\varepsilon=-\tilde{\theta}^T\bar{\omega}$ (10.13)

Auxiliary dynamics $\dot{z}_i=(1+\mu\bar{\omega}^T\bar{\omega})\bar{A}z_i-\bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon$ (10.18)

Parametric error model $\tilde{\theta}_i=\tilde{\psi}_i-\bar{c}^T z_i$ (10.19)

$\dot{\tilde{\psi}}_i=\bar{\omega}_i\varepsilon$ (10.14)

$$V=\frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P z_i+\frac{1}{2}\sum_{i=1}^{2n-1}\tilde{\psi}_i^2 \qquad (10.20)$$

$$\dot{V}=\frac{1}{2}\sum_{i=1}^{2n-1}\dot{z}_i^T P z_i+\frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P\dot{z}_i+\sum_{i=1}^{2n-1}\tilde{\psi}_i\dot{\tilde{\psi}}_i=$$

$$\frac{1}{2}\sum_{i=1}^{2n-1}\left((1+\mu\bar{\omega}^T\bar{\omega})\bar{A}z_i-\bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon\right)^T P z_i+\frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P\left((1+\mu\bar{\omega}^T\bar{\omega})\bar{A}z_i-\bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon\right)+$$

$$\sum_{i=1}^{2n-1}\left(\tilde{\theta}_i\bar{\omega}_i\varepsilon+\bar{c}^T z_i\bar{\omega}_i\varepsilon\right)=(1+\mu\bar{\omega}^T\bar{\omega})\frac{1}{2}\sum_{i=1}^{2n-1} z_i^T\left(\overbrace{\bar{A}^T P+P\bar{A}}^{-2I}\right)z_i-$$

$$\frac{1}{2}\sum_{i=1}^{2n-1}\left[\left(\bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon\right)^T P z_i+z_i^T P\left(\bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon\right)\right]-\varepsilon^2+\sum_{i=1}^{2n-1}\bar{c}^T z_i\bar{\omega}_i\varepsilon$$

$$=$$

Error model $\qquad \varepsilon=-\tilde{\theta}^T\bar{\omega}$ $\qquad$ (10.13)

Auxiliary dynamics $\qquad \dot{z}_i=(1+\mu\bar{\omega}^T\bar{\omega})\bar{A}z_i-\bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon$ $\qquad$ (10.18)

Parametric error model $\qquad \tilde{\theta}_i=\tilde{\psi}_i-\bar{c}^T z_i$ $\qquad$ (10.19)

$\dot{\tilde{\psi}}_i=\bar{\omega}_i\varepsilon$ $\qquad$ (10.14)

$$V=\frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P z_i+\frac{1}{2}\sum_{i=1}^{2n-1}\tilde{\psi}_i^2 \qquad (10.20)$$

$$\dot V=-\left(1+\mu\bar{\omega}^T\bar{\omega}\right)\sum_{i=1}^{2n-1} z_i^T z_i-\sum_{i=1}^{2n-1}\left[z_i^T P\bar{A}^{-1}\bar{b}\,\bar{\omega}_i\varepsilon\right]-\varepsilon^2+\sum_{i=1}^{2n-1}\bar{c}^T z_i\bar{\omega}_i\varepsilon$$

Error model $\quad\varepsilon=-\tilde{\theta}^T\bar{\omega}$ (10.13)

Auxiliary dynamics $\quad\dot z_i=\left(1+\mu\bar{\omega}^T\bar{\omega}\right)\bar{A}z_i-\bar{A}^{-1}\bar{b}\,\bar{\omega}_i\varepsilon$ (10.18)

Parametric error model $\quad\tilde{\theta}_i=\tilde{\psi}_i-\bar{c}^T z_i$ (10.19)

$\dot{\tilde{\psi}}_i=\bar{\omega}_i\varepsilon$ (10.14)

$$V=\frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P z_i+\frac{1}{2}\sum_{i=1}^{2n-1}\tilde{\psi}_i^2 \tag{10.20}$$

$$\dot{V}=-(1+\mu\bar{\omega}^T\bar{\omega})\sum_{i=1}^{2n-1} z_i^T z_i-\sum_{i=1}^{2n-1}\left[z_i^T P\bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon\right]-\varepsilon^2+\sum_{i=1}^{2n-1}\bar{c}^T z_i\bar{\omega}_i\varepsilon\le$$

$$-\sum_{i=1}^{2n-1} z_i^T z_i-\mu\sum_{i=1}^{2n-1}\bar{\omega}_i^2\|z_i\|^2+\sum_{i=1}^{2n-1}\|z_i\|\left\|\bar{c}-P\bar{A}^{-1}\bar{b}\right\|\left|\bar{\omega}_i\right|\left|\varepsilon\right|-\frac{\varepsilon^2}{2}-\frac{\varepsilon^2}{2}$$

Error model $\varepsilon=-\tilde{\theta}^T\bar{\omega}$ (10.13)

Auxiliary dynamics $\dot{z}_i=(1+\mu\bar{\omega}^T\bar{\omega})\bar{A}z_i-\bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon$ (10.18)

Parametric error model $\tilde{\theta}_i=\tilde{\psi}_i-\bar{c}^T z_i$ (10.19)

$\dot{\tilde{\psi}}_i=\bar{\omega}_i\varepsilon$ (10.14)

$$V=\frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P z_i+\frac{1}{2}\sum_{i=1}^{2n-1}\tilde{\psi}_i^2 \qquad (10.20)$$

$$\dot{V}=-(1+\mu\bar{\omega}^T\bar{\omega})\sum_{i=1}^{2n-1} z_i^T z_i-\sum_{i=1}^{2n-1}\left[z_i^T P\bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon\right]-\varepsilon^2+\sum_{i=1}^{2n-1}\bar{c}^T z_i\bar{\omega}_i\varepsilon\le$$

$$-\sum_{i=1}^{2n-1} z_i^T z_i-\mu\sum_{i=1}^{2n-1}\bar{\omega}_i^2\|z_i\|^2+\sum_{i=1}^{2n-1}\|z_i\|\left\|\bar{c}-P\bar{A}^{-1}\bar{b}\right\||\bar{\omega}_i||\varepsilon|-\frac{\varepsilon^2}{2}-\frac{\varepsilon^2}{2}=$$

$$-\sum_{i=1}^{2n-1} z_i^T z_i-\frac{\varepsilon^2}{2}-\sum_{i=1}^{2n-1}\left(\mu\bar{\omega}_i^2\|z_i\|^2-\left\|\bar{c}-P\bar{A}^{-1}\bar{b}\right\|\|z_i\||\bar{\omega}_i||\varepsilon|+\frac{\varepsilon^2}{2(2n-1)}\right)$$

Error model	$\varepsilon=-\tilde{\theta}^T\bar{\omega}$	(10.13)
Auxiliary dynamics	$\dot{z}_i=(1+\mu\bar{\omega}^T\bar{\omega})\bar{A}z_i-\bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon$	(10.18)
Parametric error model	$\tilde{\theta}_i=\tilde{\psi}_i-\bar{c}^T z_i$	(10.19)
	$\dot{\tilde{\psi}}_i=\bar{\omega}_i\varepsilon$	(10.14)

$$V = \frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P z_i + \frac{1}{2}\sum_{i=1}^{2n-1} \tilde{\psi}_i^2 \tag{10.20}$$

$$\dot{V} = -(1+\mu\bar{\omega}^T\bar{\omega})\sum_{i=1}^{2n-1} z_i^T z_i - \sum_{i=1}^{2n-1}\left[z_i^T P \bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon\right] - \varepsilon^2 + \sum_{i=1}^{2n-1}\bar{c}^T z_i \bar{\omega}_i \varepsilon \leq$$

$$-\sum_{i=1}^{2n-1} z_i^T z_i - \mu\sum_{i=1}^{2n-1}\bar{\omega}_i^2\|z_i\|^2 + \sum_{i=1}^{2n-1}\|z_i\|\left\|\bar{c} - P\bar{A}^{-1}\bar{b}\right\||\bar{\omega}_i||\varepsilon| - \frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{2} =$$

$$-\sum_{i=1}^{2n-1} z_i^T z_i - \frac{\varepsilon^2}{2} - \sum_{i=1}^{2n-1}\left(\mu\bar{\omega}_i^2\|z_i\|^2 - \left\|\bar{c} - P\bar{A}^{-1}\bar{b}\right\|\|z_i\||\bar{\omega}_i||\varepsilon| + \frac{\varepsilon^2}{2(2n-1)}\right)$$

What about parameter μ?

$$V = \frac{1}{2}\sum_{i=1}^{2n-1} z_i^T P z_i + \frac{1}{2}\sum_{i=1}^{2n-1} \tilde{\psi}_i^2 \qquad (10.20)$$

$$\dot{V} = -\left(1+\mu\bar{\omega}^T\bar{\omega}\right)\sum_{i=1}^{2n-1} z_i^T z_i - \sum_{i=1}^{2n-1}\left[z_i^T P\bar{A}^{-1}\bar{b}\,\bar{\omega}_i\varepsilon\right] - \varepsilon^2 + \sum_{i=1}^{2n-1}\bar{c}^T z_i\bar{\omega}_i\varepsilon \le$$

$$-\sum_{i=1}^{2n-1} z_i^T z_i - \mu\sum_{i=1}^{2n-1}\bar{\omega}_i^2\|z_i\|^2 + \sum_{i=1}^{2n-1}\|z_i\|\left\|\bar{c} - P\bar{A}^{-1}\bar{b}\right\|\|\bar{\omega}_i\||\varepsilon| - \frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{2} =$$

$$-\sum_{i=1}^{2n-1} z_i^T z_i - \frac{\varepsilon^2}{2} - \sum_{i=1}^{2n-1}\left(\mu\bar{\omega}_i^2\|z_i\|^2 - \left\|\bar{c} - P\bar{A}^{-1}\bar{b}\right\|\|z_i\|\|\bar{\omega}_i\||\varepsilon| + \frac{\varepsilon^2}{2(2n-1)}\right)$$

If

$$\mu \ge \frac{(2n-1)}{2}\left\|\bar{c} - P\bar{A}^{-1}\bar{b}\right\|^2,$$

then

$$\dot{V} \le -\sum_{i=1}^{2n-1} z_i^T z_i - \frac{\varepsilon^2}{2} < 0. \qquad (10.21)$$

High Order Tuner Solution Summary

Adjustable control

$$u = \frac{1}{b_m}\delta_M(s)\left[\frac{a_{M0}}{\delta_M(s)}[g] - \hat{\theta}^T\bar{\omega}\right] \quad (10.12)$$

Regressor

$$\omega = col(y, \nu_1, \nu_2)$$

Filters

$$\begin{cases} \dot{\nu}_1 = \Lambda\nu_1 + e_{n-1}y, & (9.4) \\ \dot{\nu}_2 = \Lambda\nu_2 + e_{n-1}u & (9.5) \end{cases}$$

Adaptation Algorithm

$$\begin{cases} \dot{\hat{\psi}}_i = -\bar{\omega}_i\varepsilon, & (10.14) \\ \dot{\eta}_i = (1+\mu\bar{\omega}^T\bar{\omega})(\bar{A}\eta_i + \bar{b}\hat{\psi}_i), & (10.15) \\ \hat{\theta}_i = \bar{c}^T\eta_i & (10.16) \end{cases}$$

Error

$$\varepsilon = y_M - y$$

Filtered regressor

$$\bar{\omega} = \frac{1}{\delta_M(s)}[\omega]$$

Gain

$$\mu \geq \frac{(2n-1)}{2}\left\|\bar{c} - P\bar{A}^{-1}\bar{b}\right\|^2$$

High Order Tuner Solution Summary

Adjustable control

$$u = \frac{1}{b_m}\delta_M(s)\left[\frac{a_{M0}}{\delta_M(s)}[g] - \hat{\theta}^T\bar{\omega}\right] \qquad (10.12)$$

Regressor

$$\omega = col(y, \nu_1, \nu_2)$$

Filters

$$\begin{cases} \dot{\nu}_1 = \Lambda\nu_1 + e_{n-1}y, & (9.4) \\ \dot{\nu}_2 = \Lambda\nu_2 + e_{n-1}u & (9.5) \end{cases}$$

Adaptation Algorithm

$$\begin{cases} \dot{\hat{\psi}}_i = -\bar{\omega}_i\varepsilon, \\ \dot{\eta}_i = (1+\mu\bar{\omega}^T\bar{\omega})(\bar{A}\eta_i + \bar{b}\hat{\psi}_i), \\ \hat{\theta}_i = \bar{c}^T\eta_i \end{cases}$$

High Order Time Derivatives ($\rho = 2$)

$$\dot{\hat{\theta}}_i = \bar{c}^T\dot{\eta}_i = \bar{c}^T(1+\mu\bar{\omega}^T\bar{\omega})(\bar{A}\eta_i + \bar{b}\hat{\psi}_i)$$

$$\ddot{\hat{\theta}}_i = 2\bar{c}^T\mu\bar{\omega}^T\dot{\bar{\omega}}(\bar{A}\eta_i + \bar{b}\hat{\psi}_i) + (1+\mu\bar{\omega}^T\bar{\omega})^2\bar{c}^T(\bar{A}^2\eta_i + \bar{A}\bar{b}\hat{\psi}_i - \bar{b}\bar{\omega}_i\varepsilon)$$

Error

$$\varepsilon = y_M - y$$

Filtered regressor

$$\bar{\omega} = \frac{1}{\delta_M(s)}[\omega]$$

Gain $\mu \geq \frac{(2n-1)}{2}\left\|\bar{c} - P\bar{A}^{-1}\bar{b}\right\|^2$

High Order Tuner Solution Summary

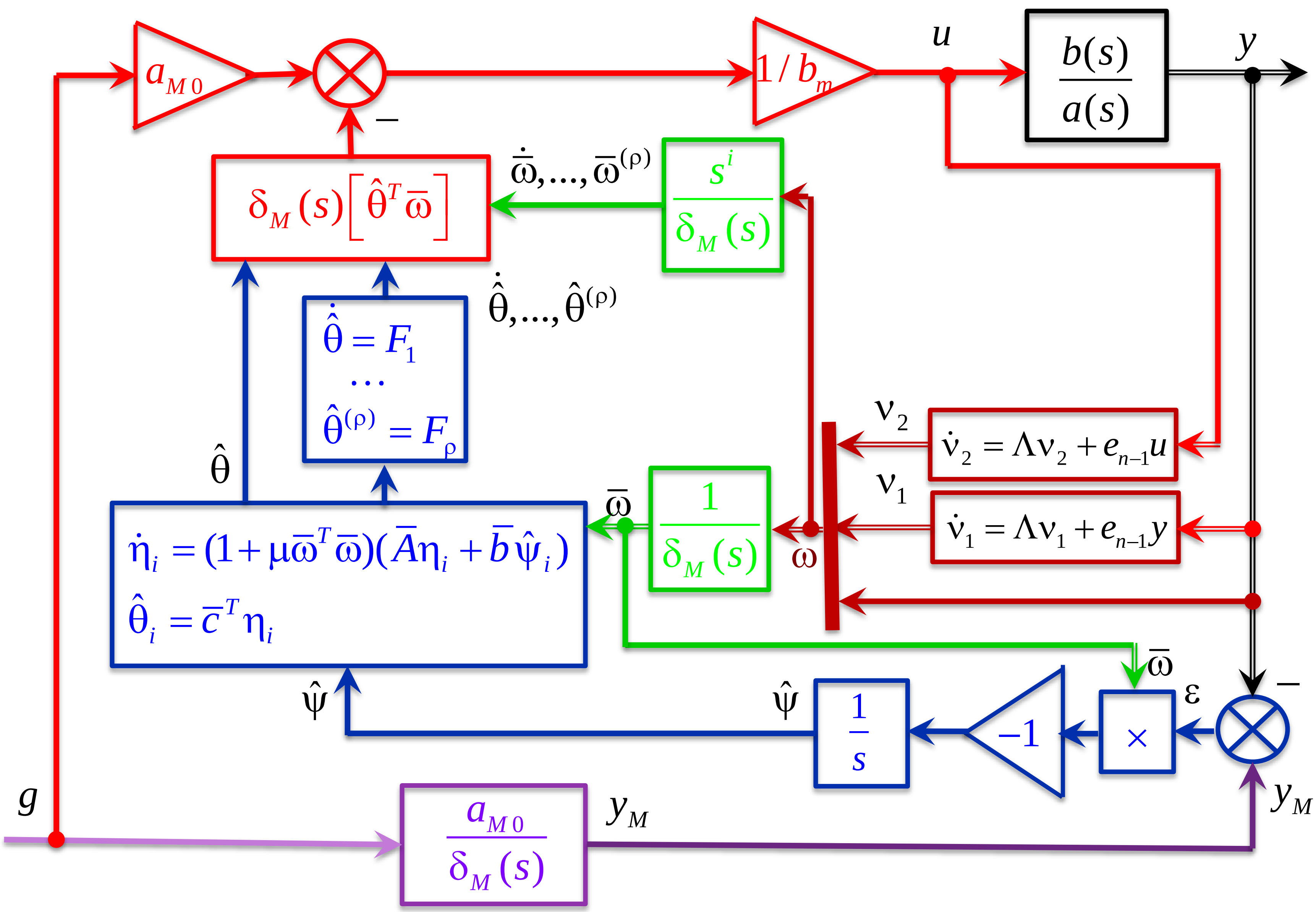
Properties of the closed-loop system:

1. All the signals in the system are bounded;
2. The norm $\| z(t) \|$ approaches zero asymptotically;
3. The error $\varepsilon(t)$ approaches zero asymptotically;
4. The $\| \tilde{\theta}(t) \|$ approaches zero asymptotically if ω satisfies

the Persistent Excitation condition **(R. Ortega, 1995)** ;

5. The adaptation algorithm does not have normalization factor restricting the transient performance of the adaptation law

$$\begin{cases} \dot{\hat{\psi}}_i = -\bar{\omega}_i \varepsilon, \\ \dot{\eta}_i = (1 + \mu \bar{\omega}^T \bar{\omega})(\bar{A}\eta_i + \bar{b}\hat{\psi}_i), \\ \hat{\theta}_i = \bar{c}^T \eta_i. \end{cases}$$


a_{M0}
$-$
$1/b_m$
u
$\frac{b(s)}{a(s)}$
y
$\delta_M(s)\left[\hat{\theta}^T\bar{\omega}\right]$
$\dot{\bar{\omega}},\ldots,\bar{\omega}^{(\rho)}$
$\frac{s^i}{\delta_M(s)}$
$\dot{\hat{\theta}},\ldots,\hat{\theta}^{(\rho)}$
$\dot{\hat{\theta}} = F_1$
$\ldots$
$\hat{\theta}^{(\rho)} = F_\rho$
ν_2
$\dot{\nu}_2 = \Lambda\nu_2 + e_{n-1}u$
$\hat{\theta}$
ν_1
$\dot{\nu}_1 = \Lambda\nu_1 + e_{n-1}y$
$\bar{\omega}$
$\frac{1}{\delta_M(s)}$
ω
$\dot{\eta}_i = (1+\mu\bar{\omega}^T\bar{\omega})(\bar{A}\eta_i + \bar{b}\hat{\psi}_i)$
$\hat{\theta}_i = \bar{c}^T\eta_i$
$\bar{\omega}$
ε
$-$
$\hat{\psi}$
$\hat{\psi}$
$\frac{1}{s}$
-1
$\times$
g
$\frac{a_{M0}}{\delta_M(s)}$
y_M
y_M

Simulation results (conditions from Lecture 6.3)

Plant

$$\ddot{y} + a_1\dot{y} + a_0 y = u$$

Unknown parameters

$$a_0 = 1,\ a_1 = 2$$

Reference model

$$\ddot{y}_M + 5\dot{y}_M + 6y_M = 6g$$

$$W_M(s) = \frac{6}{s^2 + 5s + 6} = \frac{6}{\delta_M(s)}$$

Regressor

$$\omega = col\left(y, \frac{1}{s+8}[y], \frac{1}{s+8}[u] \right)$$

Filtered regressor

$$\bar{\omega} = \frac{1}{s^2 + 5s + 6}[\omega] = \frac{1}{\delta_M(s)}[\omega]$$

Augmented Error Solution. Simulation results

Augmented error $$\hat{\varepsilon} = \varepsilon + \hat{\theta}^T \bar{\omega} - \frac{1}{s^2 + 5s + 6}\left[\hat{\theta}^T \omega\right]$$

Error $$\varepsilon = y_M - y$$

Adjustable control $$u = -\hat{\theta}^T \omega + 6g$$

Adaptation algorithm $$\dot{\hat{\theta}} = -\frac{100}{1 + \bar{\omega}^T(t)\bar{\omega}(t)}\bar{\omega}\hat{\varepsilon}$$

Gain $$\gamma = 100$$

High Order Tuner Solution. Simulation results

Adjustable control

$$u = -\ddot{\hat{\theta}}^T\bar{\omega} - 2\dot{\hat{\theta}}^T\dot{\bar{\omega}} - \hat{\theta}^T\ddot{\bar{\omega}} - 5\dot{\hat{\theta}}^T\bar{\omega} - 5\hat{\theta}^T\dot{\bar{\omega}} - 6\hat{\theta}^T\bar{\omega} + 6g$$

Adaptation algorithm

$$\begin{cases} \dot{\hat{\psi}}_i = -\bar{\omega}_i\varepsilon, \; i = 1,2,3, \\ \dot{\eta}_i = (1+\mu\bar{\omega}^T\bar{\omega})(-5\eta_i + 5\hat{\psi}_i), \\ \hat{\theta}_i = \eta_i \end{cases}$$

High order time derivatives

$$\dot{\hat{\theta}}_i = (1+\mu\bar{\omega}^T\bar{\omega})(-5\eta_i + 5\hat{\psi}_i)$$

$$\ddot{\hat{\theta}}_i = 2\mu\bar{\omega}^T\dot{\bar{\omega}}(-5\eta_i + 5\hat{\psi}_i) + (1+\mu\bar{\omega}^T\bar{\omega})^2\bar{c}^T(25\eta_i - 25\hat{\psi}_i - 5\bar{\omega}_i\varepsilon)$$

Filtered regressor and its derivatives

$$\bar{\omega} = \frac{1}{s^2+5s+6}[\omega],$$

$$\dot{\bar{\omega}} = \frac{s}{s^2+5s+6}[\omega], \quad \ddot{\bar{\omega}} = \frac{s^2}{s^2+5s+6}[\omega]$$

Error $\varepsilon = y_M - y$

Gain $\mu = 54$

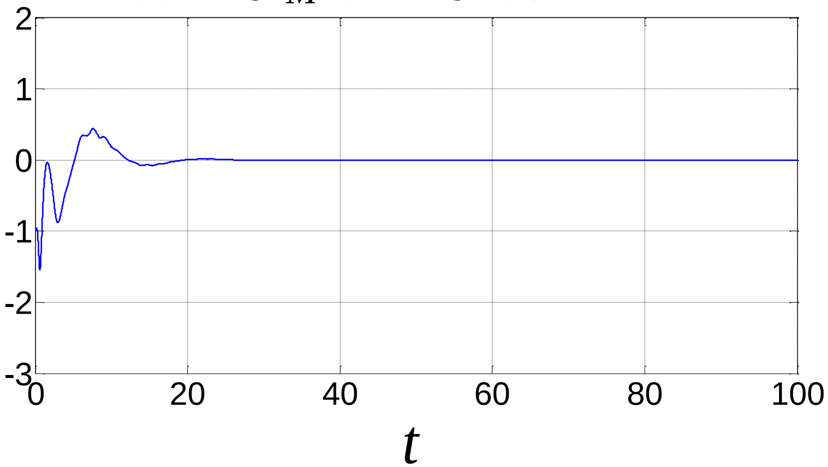
Simulation results

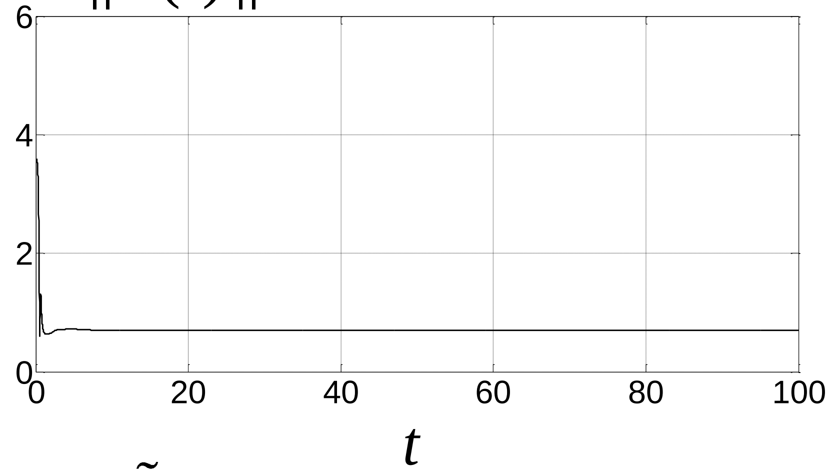
$g(t) = 15\sin(2t)$ (*NOT persistently exciting*)

$\varepsilon(t) = y_M(t) - y(t)$

$\|\tilde{\theta}(t)\|$

AE

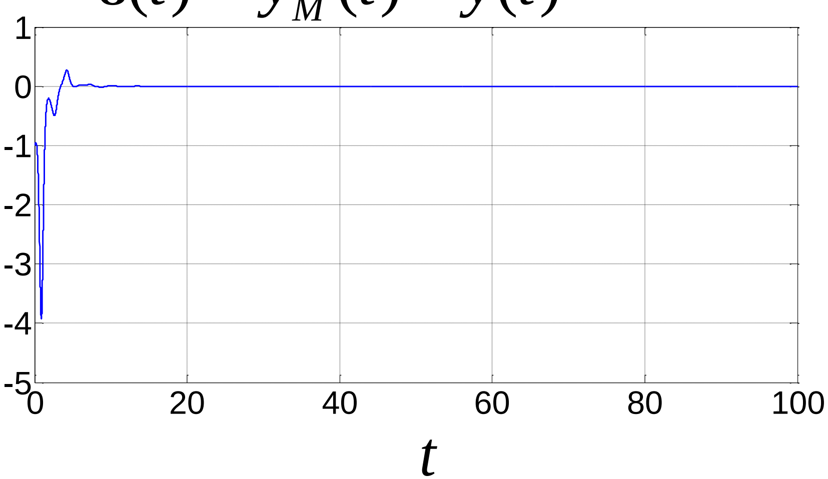


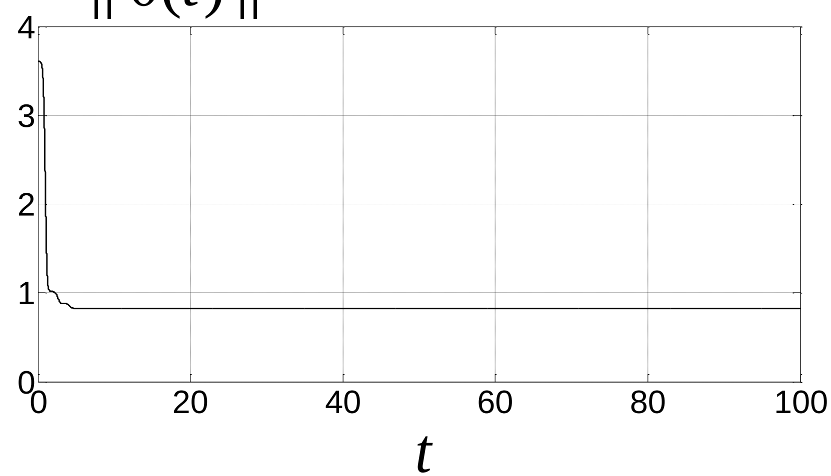


$\varepsilon(t) = y_M(t) - y(t)$

$\|\tilde{\theta}(t)\|$

HOT



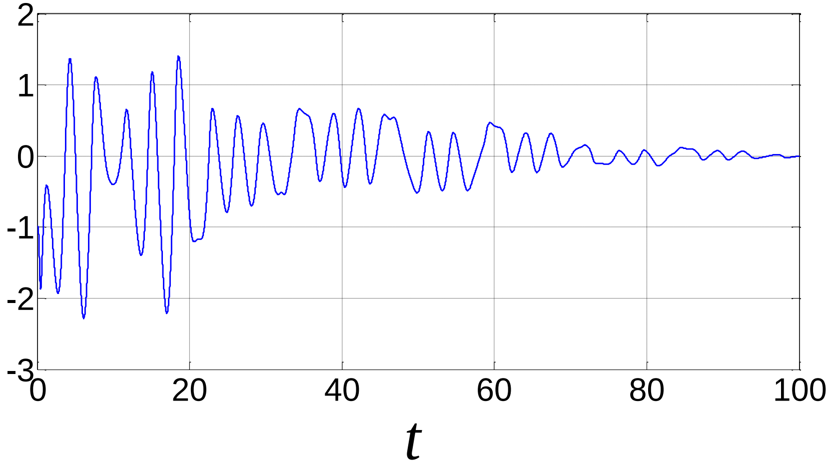


Simulation results

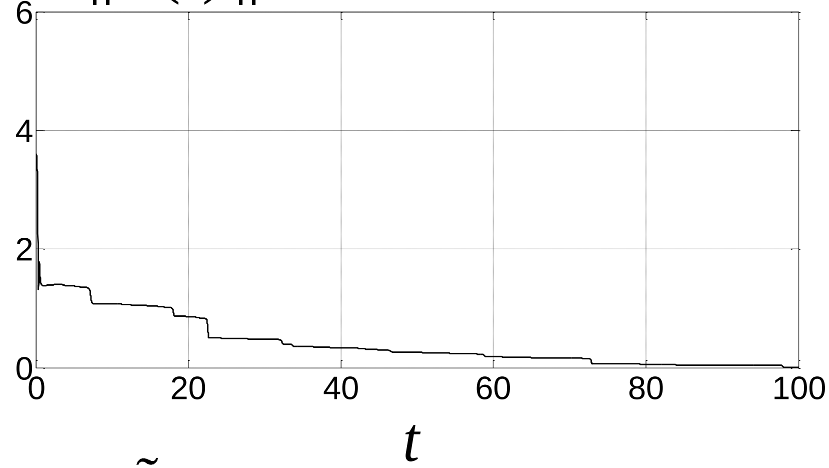
$$g(t) = 15\sin(2t) + 10\cos(0.25t) \; (\textit{persistently exciting})$$

AE

$$\varepsilon(t) = y_M(t) - y(t)$$

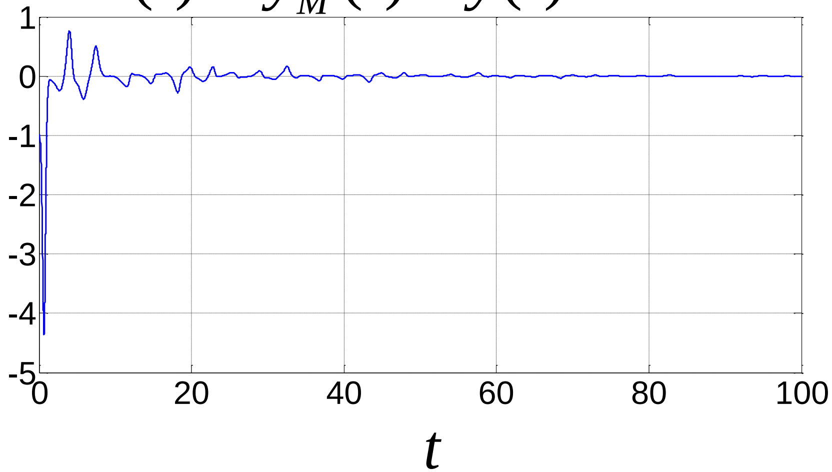


$$\|\tilde{\theta}(t)\|$$



HOT

$$\varepsilon(t) = y_M(t) - y(t)$$



$$\|\tilde{\theta}(t)\|$$