INVERTIBILITY OF R_i

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Exercise 2.1. Write an explicit formula fo the differential d below

$$R_i \otimes R_i' = (0 \to U_i\{1\}) \xrightarrow{d} A_n \oplus U_i\{-1\} \xrightarrow{d} U_i\{-1\} \to 0)$$

and check that the complex decomposes into a direct sum

$$(0 \to U_i\{1\} \xrightarrow{\mathbf{1}} U_i\{1\} \to 0) \oplus (0 \to A_n \to 0) \oplus (0 \to U_i\{-1\} \xrightarrow{\mathbf{1}} U_i\{-1\} \to 0)$$

The first and the last summands are null-homotopic, implying that

$$R_i \otimes R_i' \cong (0 \to A_n \to 0) = A_n.$$

Proof. Note that $R_i = (0 \to U_i\{1\} \xrightarrow{\beta_i} A_n \to 0)$ and $R_i' = (0 \to A_n \xrightarrow{\gamma_i} U_i\{-1\} \to 0)$ where $U_i = P_i \otimes_{\mathbb{Z}} {}_iP$, β_i takes $x \otimes y \in P_i \otimes_{\mathbb{Z}} {}_iP$ to $xy \in A_n$, and γ_i takes 1 to $(i-1|i) \otimes (i|i-1) + (i+1|i) \otimes (i|i+1) + X_i \otimes (i) + (i) \otimes X_i$. Subsequently, the double complex corresponding to $R_i \otimes R_i'$ has the form

$$\begin{tikzcd}
 & 0 & 0 & \\
0 \arrow[r] & U_i \otimes U_i \arrow[u] \arrow[r, "\beta_i \otimes \mathbf{1}"] & A_n \otimes U_i\{-1\} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & U_i\{1\} \otimes A_n \arrow[u, "\mathbf{1} \otimes \gamma_i"] \arrow[r, "\beta_i \otimes \mathbf{1}"] & A_n \otimes A_n \arrow[u, "\mathbf{1} \otimes \gamma_i"] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Therefore, we get the following total complex

$$R_i \otimes R_i' = 0 \to U_i\{1\} \otimes A_n \to A_n \otimes A_n \oplus U_i \otimes U_i \to A_n \otimes U_i\{-1\} \to 0$$

Note that

$$\begin{tikzcd}
0 \arrow[r] & U_i\{1\} \otimes A_n \arrow[r, "\begin{bmatrix} \beta_i \otimes \mathbf{1} \\ \mathbf{1} \otimes \gamma_i \end{bmatrix}"] \arrow[d, "\cong"] & A_n \otimes A_n \oplus U_i \otimes U_i \arrow[r, "\begin{bmatrix} -\mathbf{1} \otimes \gamma_i & \beta_i \otimes \mathbf{1} \end{bmatrix}"] \arrow[d, "\cong"] & A_n \otimes U_i\{-1\} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & U_i\{1\} \arrow[r, "d_1"] & A_n \oplus U_i\{1\} \oplus U_i\{-1\} \arrow[r, "d_2"] & U_i\{-1\} \arrow[r] & 0
\end{tikzcd}$$

The first vertical isomorphism from the left is followed from the fact that $U_i\{1\}$ is an A_n-bimodule. Similar arguments applied to the third vertical isomorphism. The second vertical isomorphism is obtained from Claim 1.7 by using the definition of U_i and A_n are tensored over itself.

To see that those isomorphisms are actually chain map, we need to define d_1 and d_2 accordingly. First, d_1 is a composition of maps

$$\begin{aligned}
x \otimes y \in U_i\{1\} &\mapsto x \otimes y \otimes 1 \\
&\mapsto \beta_i(x \otimes y) \otimes 1 + x \otimes y \otimes \gamma_i(1) \\
&\quad = xy \otimes 1 \\
&+ x \otimes (i) \otimes ((i-1|i) \otimes (i|i-1) + (i+1|i) \otimes (i|i+1) + X_i \otimes (i) + (i) \otimes X_i)y \\
&\mapsto xy + x \otimes X_i \otimes y + x \otimes (i) \otimes X_i y \\
&\mapsto xy + x \otimes y + x \otimes X_i y \in A_n \oplus U_i\{1\} \oplus U_i\{-1\},
\end{aligned}$$

Date: April 1, 2018.

in other words, $d_1 = \begin{bmatrix} \beta_i \\ \mathbf{1} \\ x\otimes y \mapsto x \otimes X_i y \end{bmatrix}$.

Next, d_2 is a composition of maps

$$\begin{aligned} a + x_1 \otimes_{\mathbb{Z}} y_1 + x_2 \otimes_{\mathbb{Z}} y_2 &\mapsto 1 \otimes_{A_n} a + x_1 \otimes_{\mathbb{Z}} X_i \otimes_{A_n} y_1 + x_2 \otimes_{\mathbb{Z}} (i) \otimes_{A_n} y_2 \\ &\mapsto 1 \otimes_{A_n} a + x_1 \otimes (i|i-1) \otimes (i-1|i) \otimes y_1 + x_2 \otimes (i) \otimes (i) \otimes y_2 \\ &\mapsto -a \otimes \gamma_i(1) + x_1 X_i \otimes (i) \otimes y_1 + x_2 \otimes (i) \otimes y_2 \\ &\mapsto -a\gamma_i(1) + x_1 X_i \otimes y_1 + x_2 \otimes y_2, \end{aligned}$$

in other words, $d_2 = \begin{bmatrix} -\gamma_i & x \otimes y \mapsto xX_i \otimes y & \mathbf{1} \end{bmatrix}$. They are chain map because

$$\begin{aligned} d_2 d_1 (x \otimes y) &= \begin{bmatrix} -\gamma_i & x \otimes y \mapsto xX_i \otimes y & \mathbf{1} \end{bmatrix} \begin{bmatrix} \beta_i \\ \mathbf{1} \\ x\otimes y \mapsto x \otimes X_i y \end{bmatrix} (x \otimes y) \\ &= \begin{bmatrix} -\gamma_i & x \otimes y \mapsto xX_i \otimes y & \mathbf{1} \end{bmatrix} \begin{bmatrix} xy \\ x \otimes y \\ x \otimes X_i y \end{bmatrix} \\ &= -x\gamma_i(1)y + xX_i \otimes y + x \otimes X_i y \\ &= -x((i-1|i) \otimes (i|i-1) + (i+1|i) \otimes (i|i+1) + X_i \otimes (i) + (i) \otimes X_i)y \\ &\quad + xX_i \otimes y + x \otimes X_i y \\ &= 0 \qquad \text{by the definition of } x \text{ and } y \end{aligned}$$

After that, we need to check the existence of isomorphism between the chain complexes so that the complex decomposes into a direct sum as described before.

$$\begin{tikzcd}[ampersand replacement=\&] 0 \arrow[r] \& U_i\{1\} \arrow[r, "\begin{bmatrix} \beta_i \\ \mathbf{1} \\ x\otimes y \mapsto x\otimes X_i y \end{bmatrix}"] \arrow[d, "\cong"] \& A_n \oplus U_i\{1\} \oplus U_i\{-1\} \arrow[r, "\begin{bmatrix} -\gamma_i & x\otimes y \mapsto xX_i\otimes y & \mathbf{1} \end{bmatrix}"] \arrow[d, "\cong"] \& U_i\{-1\} \arrow[r] \arrow[d, "\cong"] \& 0 \\ 0 \arrow[r] \& U_i\{1\} \arrow[r, "\begin{bmatrix} 0 \\ \mathbf{1} \\ 0 \end{bmatrix}"'] \& A_n \oplus U_i\{1\} \oplus U_i\{-1\} \arrow[r, "\begin{bmatrix} 0 & 0 & \mathbf{1} \end{bmatrix}"] \& U_i\{-1\} \arrow[r] \& 0 \end{tikzcd}$$

It can be verified that we can take the first and third map as the identity map $\mathbf{1}$ and the second map as $\begin{bmatrix} \mathbf{1} & -\beta_i & 0 \\ 0 & \mathbf{1} & 0 \\ -\gamma_i & x \otimes y \mapsto xX_i \otimes y & \mathbf{1} \end{bmatrix}$, so that the two chain complexes are isomorphic to each other.

Finally, since the second map is a direct sum of three chain complexes, namely $(0 \to U_i\{1\} \xrightarrow{\mathbf{1}} U_i\{1\} \to 0)$, $(0 \to A_n \to 0)$, and $(0 \to U_i\{-1\} \xrightarrow{\mathbf{1}} U_i\{-1\} \to 0)$. But, the first and the last chain complexes are null-homotopic. Consequently, $R_i \otimes R_i' \cong A_n$.

QED

Lemma 0.1 (Gaussian elimination)**.** *Let X, Y, Z, W, U, V be six objects in an additive category and consider a complex*

$$\cdots \to U \xrightarrow{u} X \oplus Y \xrightarrow{f} Z \oplus W \xrightarrow{v} V \to \cdots$$

where $f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and u, v are arbitrary morphisms. If $D : Y \to W$ is an isomorphism, then the complex above is homotopic to a complex

$$\cdots \to U \xrightarrow{u} X \xrightarrow{A - BD^{-1}C} Z \xrightarrow{v|_Z} V \to \cdots$$

Proof. The key is the following commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & U \arrow[r, "u"] \arrow[d, "id"] & X \oplus Y \arrow[r, "f"] \arrow[d, "\alpha"] & Z \oplus W \arrow[r, "v"] \arrow[d, "\beta"] & V \arrow[r] \arrow[d, "id"] & \cdots \\ \cdots \arrow[r] & U \arrow[r, "\alpha u"] & X \oplus Y \arrow[r, "g"] & Z \oplus W \arrow[r, "v\beta^{-1}"] & V \arrow[r] & \cdots \end{tikzcd}$$

where

$$g = \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \quad \alpha = \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & -BD^{-1} \\ 0 & 1 \end{pmatrix}.$$

The vertical map of complexes is a homotopy equivalence with

$$\alpha^{-1} = \begin{pmatrix} 1 & 0 \\ -D^{-1}C & 1 \end{pmatrix} \quad \text{and} \quad \beta^{-1} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix}.$$

It is straightforward to check that the bottom row is homotopic to

$$\cdots \to U \xrightarrow{u} X \xrightarrow{A-BD^{-1}C} Z \xrightarrow{v|_Z} V \to \cdots,$$

namely,

$$\begin{tikzcd}
\cdots \arrow[r] & U \arrow[r, "\alpha u"] \arrow[d, "id"] & X \oplus Y \arrow[r, "g"] \arrow[d, "{\begin{bmatrix} 1 & 0 \end{bmatrix}}"] \arrow[ddl, dotted, "0"] & Z \oplus W \arrow[r, "v\beta^{-1}"] \arrow[d, "{\begin{bmatrix} 1 & 0 \end{bmatrix}}"] \arrow[ddl, dotted, "s"] & V \arrow[r] \arrow[d, "id"] \arrow[ddl, dotted, "0"] & \cdots \\
\cdots \arrow[r] & U \arrow[r, "u"] \arrow[d, "id"] & X \arrow[r, "A-BD^{-1}C"] \arrow[d, "{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}"] \arrow[ddl, dotted, "0"] & Z \arrow[r, "v|_Z"] \arrow[d, "{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}"'] \arrow[ddl, dotted, "0"] & V \arrow[r] \arrow[d, "id"] \arrow[ddl, dotted, "0"] & \cdots \\
\cdots \arrow[r] & U \arrow[r, "\alpha u"] \arrow[d, "id"] & X \oplus Y \arrow[r, "g"] \arrow[d, "{\begin{bmatrix} 1 & 0 \end{bmatrix}}"'] & Z \oplus W \arrow[r, "v\beta^{-1}"] \arrow[d, "{\begin{bmatrix} 1 & 0 \end{bmatrix}}"'] & V \arrow[r] \arrow[d, "id"] & \cdots \\
\cdots \arrow[r] & U \arrow[r, "u"] & X \arrow[r, "A-BD^{-1}C"] & Z \arrow[r, "v|_Z"] & V \arrow[r] & \cdots
\end{tikzcd}$$

where $s = \begin{bmatrix} 0 & 0 \\ 0 & -D^{-1} \end{bmatrix}$. To see this, note that

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - id = s \circ g + \alpha u \circ 0 = \begin{bmatrix} 0 & 0 \\ 0 & -D^{-1} \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix}$$

and

$$0 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - id = 0 \circ (A - BD^{-1}C) + u \circ 0,$$

as desired.

QED

Second proof of Exercise 2.1. In the light of lemma 6.1, the complex

$$0 \longrightarrow U_i\{1\} \xrightarrow{\begin{bmatrix} \beta_i \\ \mathbf{1} \\ x\otimes y \mapsto x \otimes X_i y \end{bmatrix}} A_n \oplus U_i\{1\} \oplus U_i\{-1\} \xrightarrow{\begin{bmatrix} -\gamma_i & x\otimes y \mapsto xX_i \otimes y & \mathbf{1} \end{bmatrix}} U_i\{-1\} \longrightarrow 0$$

is homotopic to

$$0 \to A_n \to 0.$$

QED