Constraint Satisfaction Problems

Reference: Prof. Hwee Tou Ng's slide (NUS, Singapore)

Outline

- Constraint Satisfaction Problems (CSP)
- Backtracking search for CSPs
- Local search for CSPs

Constraint satisfaction problems (CSPs)

- Standard search problem:
  - state is a "black box“ – any data structure that supports successor function, heuristic function, and goal test
- CSP:
  - state is defined by variables X_i with values from domain D_i
  - goal test is a set of constraints specifying allowable combinations of values for subsets of variables

Example: Map-Coloring

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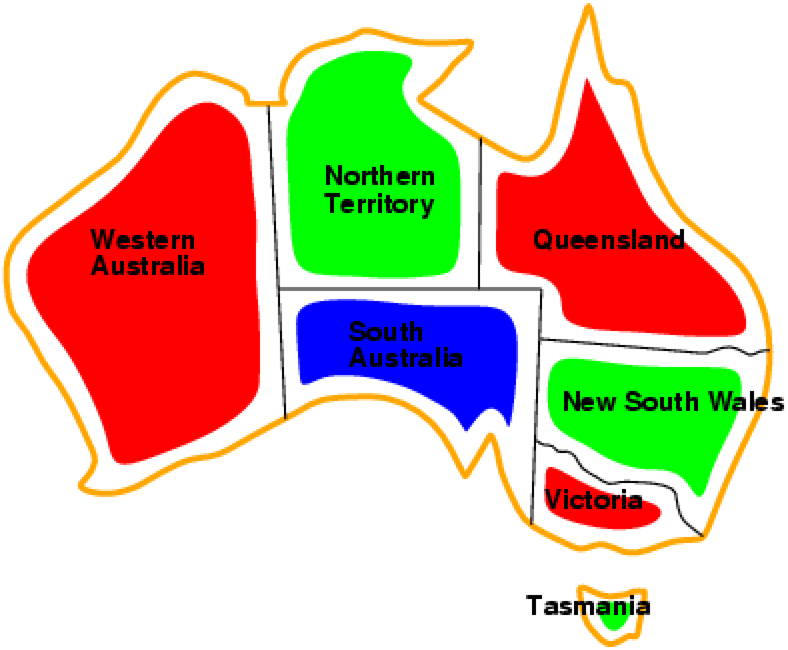
- Variables WA, NT, Q, NSW, V, SA, T
- Domains D_i = {red, green, blue}
- Constraints: adjacent regions must have different colors
  – e.g., WA ≠ NT, or (WA,NT) in {(red, green),(red, blue),(green, red), (green, blue),(blue, red),(blue, green)}

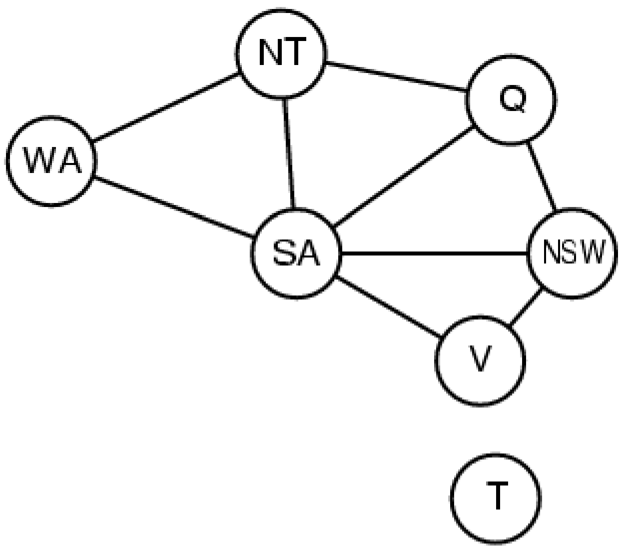
Example: Map-Coloring



- Solutions are complete and consistent assignments, e.g., WA = red, NT = green,Q = red,NSW = green,V = red,SA = blue,T = green

Constraint graph

- Binary CSP: each constraint relates two variables
- Constraint graph: nodes are variables, arcs are constraints



Varieties of CSPs

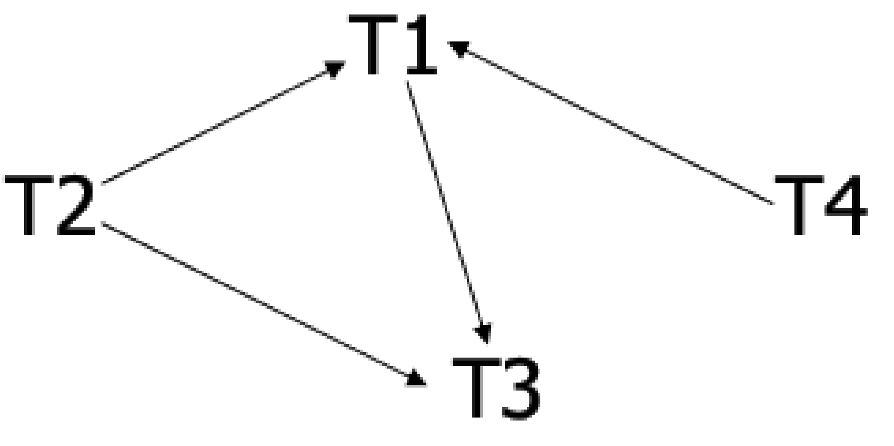
- Discrete variables
  - finite domains
  - infinite domains
- Continuous variables

Varieties of constraints

- Unary constraints involve a single variable,
  - e.g., SA ≠ green
- Binary constraints involve pairs of variables,
  - e.g., SA ≠ WA
- Higher-order constraints involve 3 or more variables,
  - e.g., cryptarithmetic column constraints

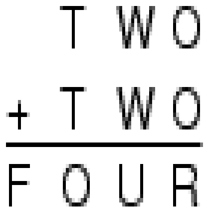
Example: Task Scheduling

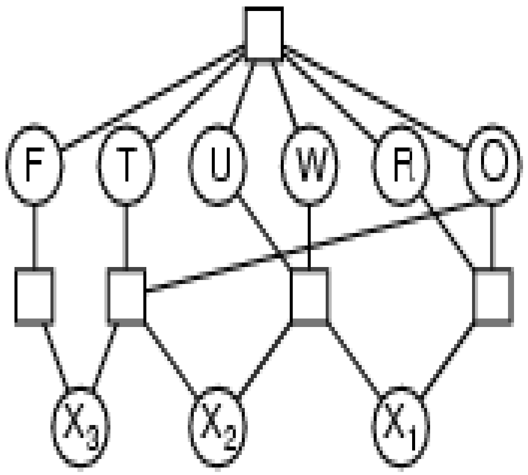
- T1 must be done during T3
- T2 must be achieved before T1 starts
- T2 must overlap with T3
- T4 must start after T1 is complete



Example: Cryptarithmetic

- Variables: F T U W R O X_1 X_2 X_3
- Domains: {0,1,2,3,4,5,6,7,8,9}
- Constraints: Alldiff (F,T,U,W,R,O)
  - $O + O = R + 10 \cdot X_1$
  - $X_1 + W + W = U + 10 \cdot X_2$
  - $X_2 + T + T = O + 10 \cdot X_3$
  - $X_3 = F$, $T \neq 0$, $F \neq 0$





Real-world CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetabling problems
  - e.g., which class is offered when and where?
- Transportation scheduling
- Factory scheduling

Note: many real-world problems involve real-valued variables

Standard search formulation

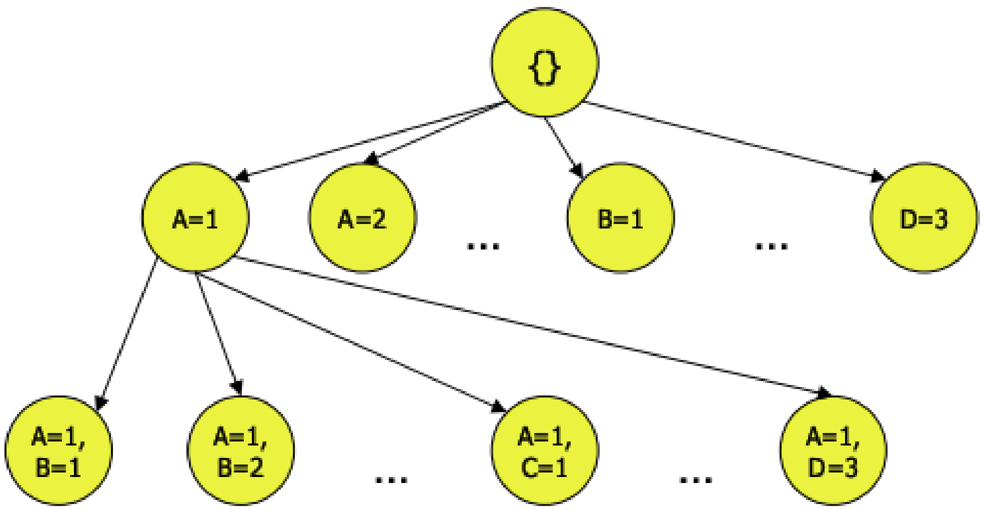
- Let's start with the straightforward approach, then fix it
- States are defined by the values assigned so far
  - Initial state: the empty assignment { }
  - Successor function: assign a value to an unassigned variable that does not conflict with current assignment

    → fail if no legal assignments
  - Goal test: the current assignment is complete

1. This is the same for all CSPs
2. Every solution appears at depth n with n variables
   → use depth-first search
3. The path to the goal is not important

CSP Search tree size

$b = (n - \ell)d$ at depth ℓ, hence $n! \cdot d^n$ leaves



Variables: A,B,C,D
Domains: 1,2,3

Depth 1: 4 variables x 3 domains = 12 states

Depth 2: 3 variables x 3 domains = 9 states

Depth 3: 2 variables x 3 domains = 6 states

Depth 4: 1 variable x 3 domains = 3 states (leaf level)

Backtracking search

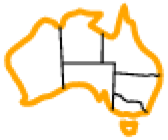
- Variable assignments are commutative}, i.e.,
  - [WA = red then NT = green] same as [NT = green then WA = red]
- Only need to consider assignments to a single variable at each node
  - b = d and there are d^n leaves
- Depth-first search for CSPs with single-variable assignments is called backtracking search
  - Backtracking search is the basic uninformed algorithm for CSPs
  - Can solve n-queens for n ≈ 25

Backtracking search

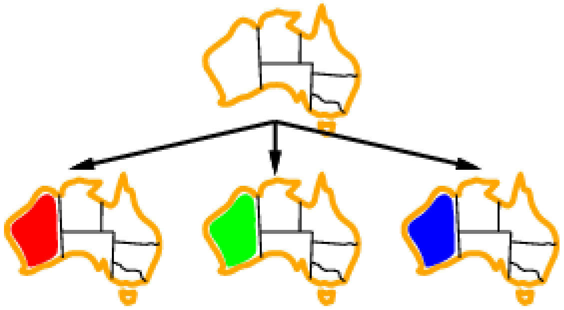
```
function BACKTRACKING-SEARCH( csp) returns a solution, or failure
   return RECURSIVE-BACKTRACKING({}, csp)

function RECURSIVE-BACKTRACKING( assignment,csp) returns a solution, or
failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE( Variables[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment according to Constraints[csp] then
         add { var = value } to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failue then return result
         remove { var = value } from assignment
   return failure
```

Backtracking example



Backtracking example



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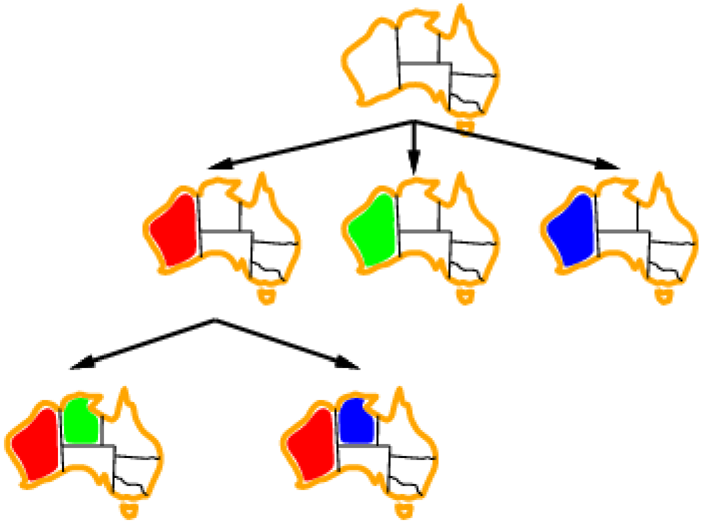

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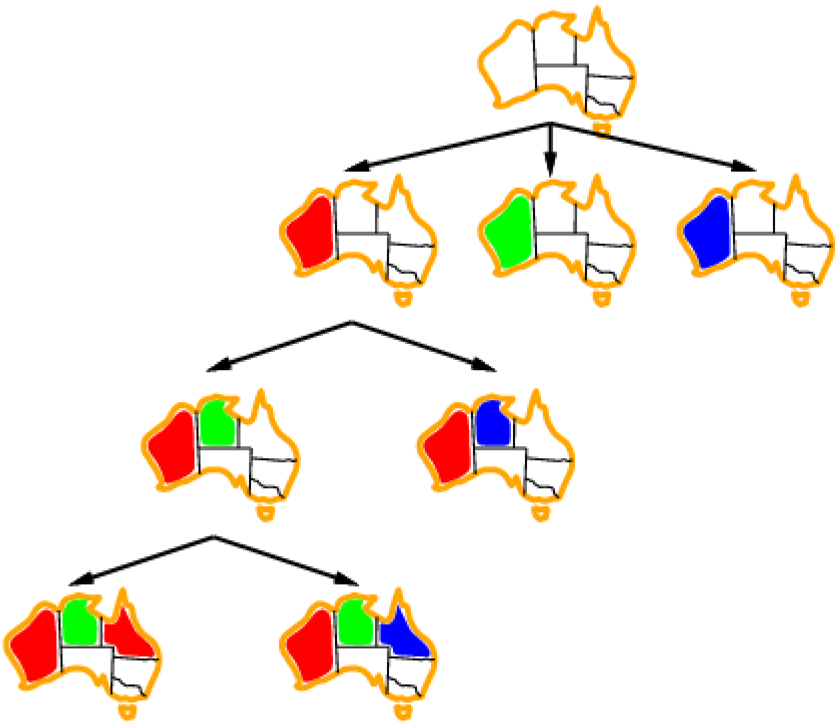
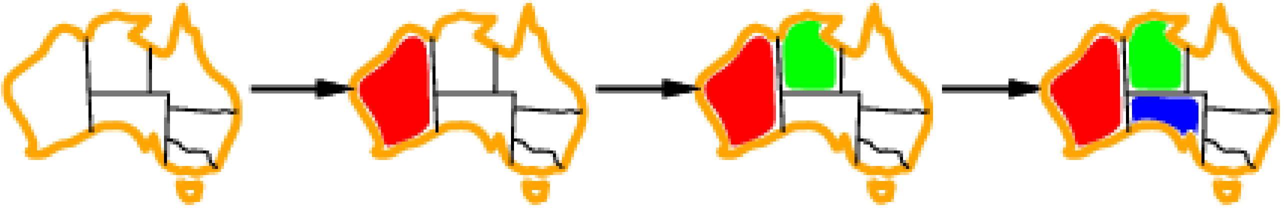


Improving backtracking efficiency

- General-purpose methods can give huge gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?

Most constrained variable

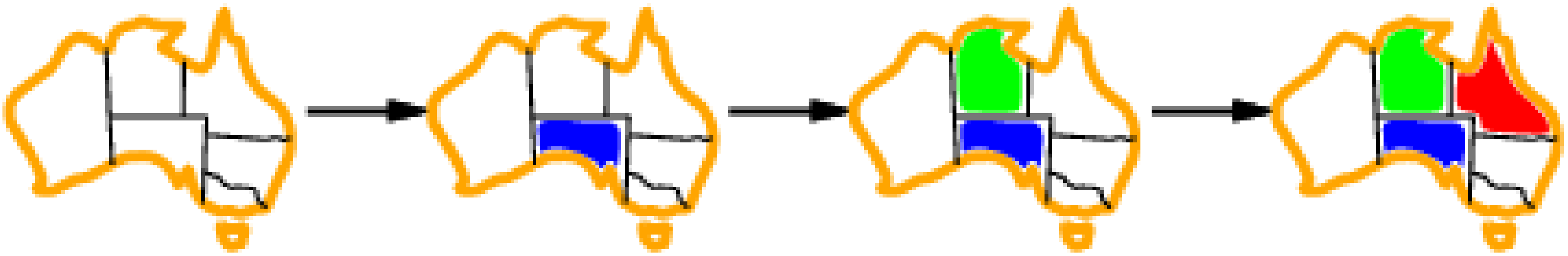
- Most constrained variable: choose the variable with the fewest legal values



- a.k.a. minimum remaining values (MRV) heuristic

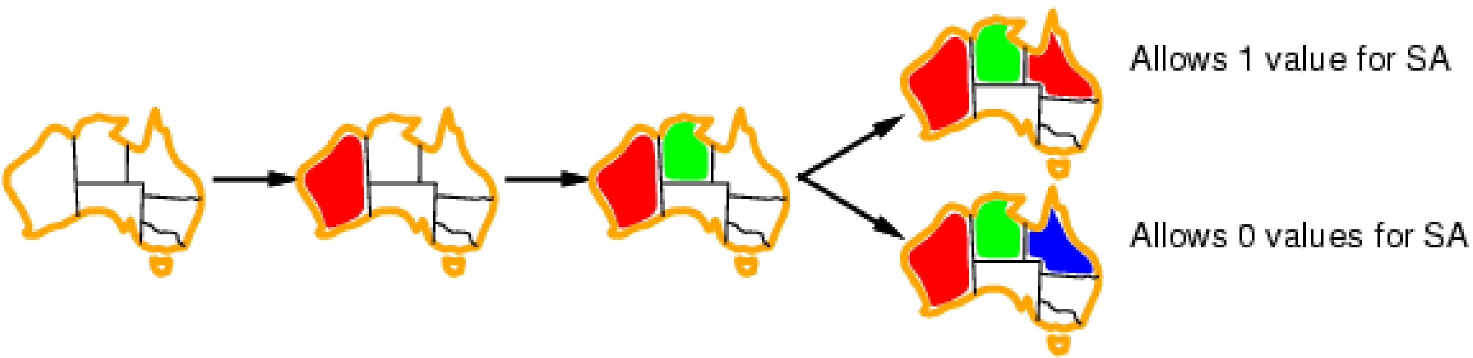
Most constraining variable

- Most constraining variable:
  - choose the variable with the most constraints on remaining variables



Least constraining value

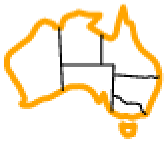
- Given a variable, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables

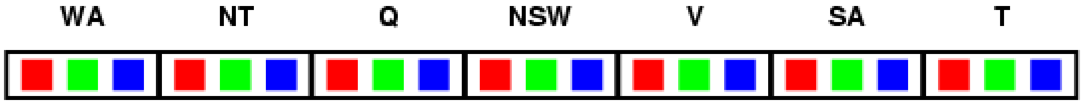


- Combining these heuristics makes 1000 queens feasible

Forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values





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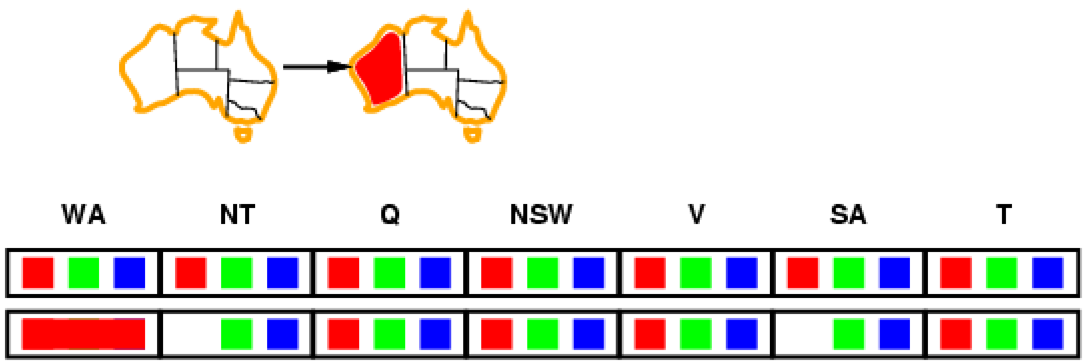

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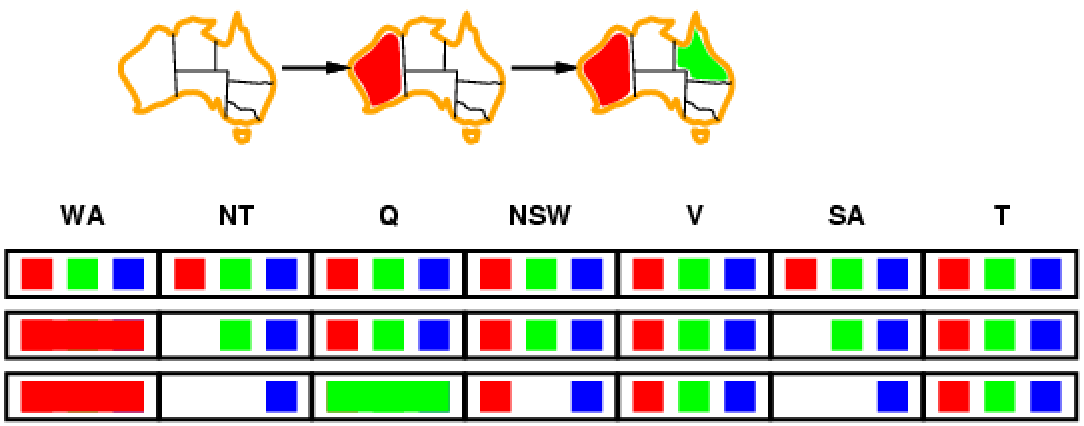

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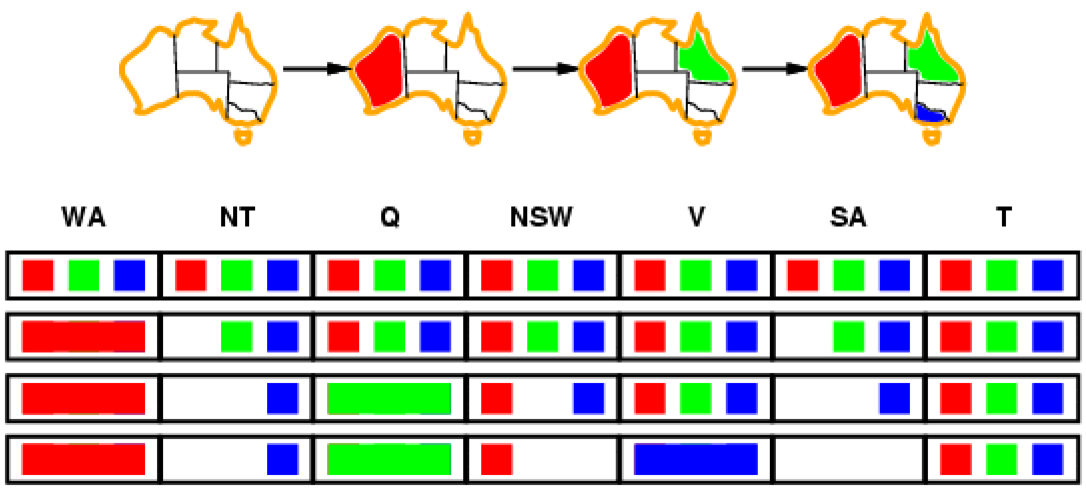
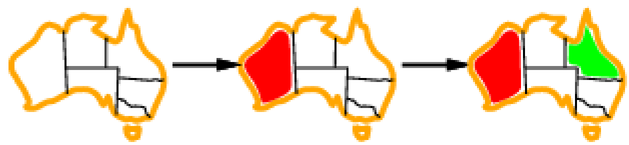


Constraint propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

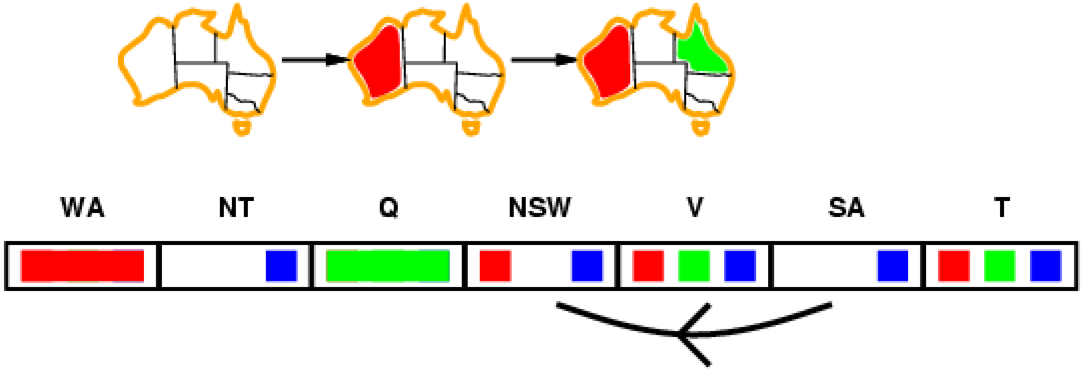


- NT and SA cannot both be blue!
- Constraint propagation repeatedly enforces constraints locally

Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff

  for every value x of X there is some values y of Y allowed



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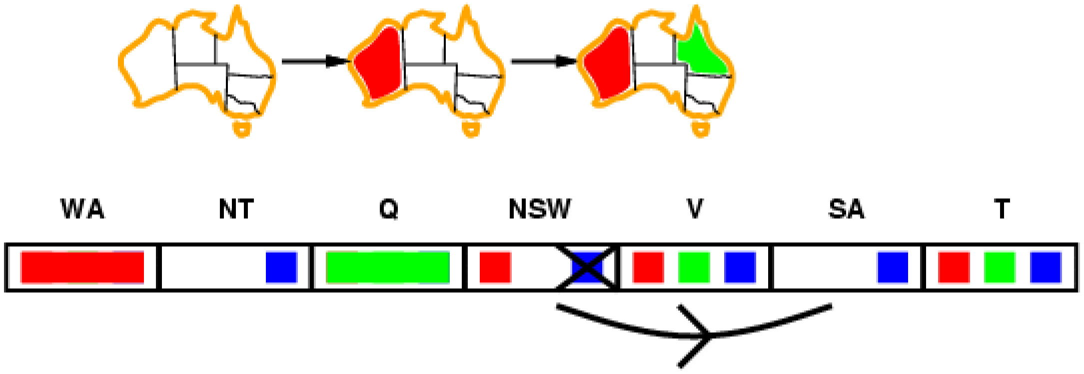

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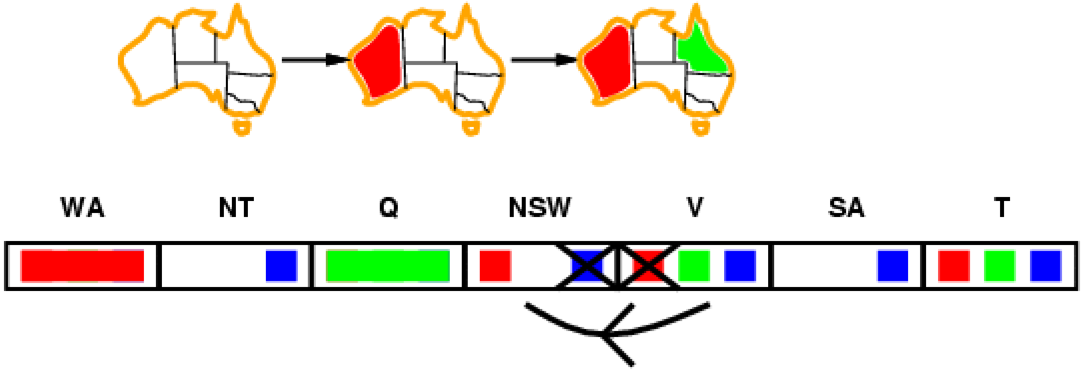

- If X loses a value, neighbors of X (Z->X) need to be rechecked

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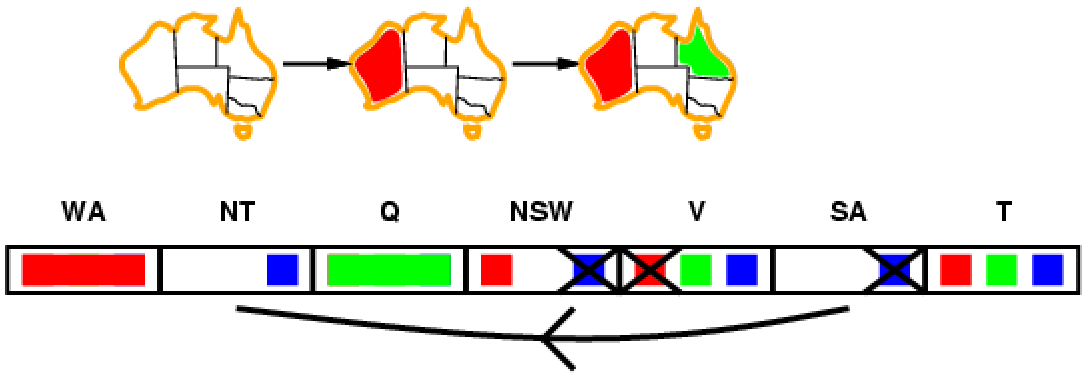


- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

Arc consistency algorithm AC-3

- Time complexity: $O(n^2d^3)$

```
function AC-3( csp) returns the CSP, possibly with reduced domains
    inputs: csp, a binary CSP with variables {X_1, X_2, ..., X_n}
    local variables: queue, a queue of arcs, initially all the arcs in csp

    while queue is not empty do
        (X_i, X_j) ← REMOVE-FIRST(queue)
        if RM-INCONSISTENT-VALUES(X_i, X_j) then
            for each X_k in NEIGHBORS[X_i] do
                add (X_k, X_i) to queue
```

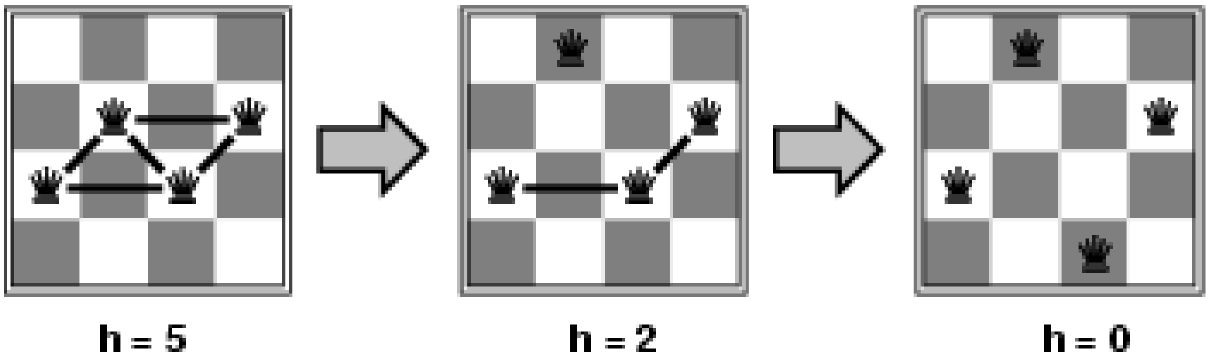
```
function RM-INCONSISTENT-VALUES( X_i, X_j) returns true iff remove a value
    removed ← false
    for each x in DOMAIN[X_i] do
        if no value y in DOMAIN[X_j] allows (x,y) to satisfy constraint(X_i, X_j)
            then delete x from DOMAIN[X_i];  removed ← true
    return removed
```

Local search for CSPs

- Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned
- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators reassign variable values
- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts heuristic:
  - choose value that violates the fewest constraints
  - i.e., hill-climb with $h(n)$ = total number of violated constraints

Example: 4-Queens

- States: 4 queens in 4 columns (4^4 = 256 states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: h(n) = number of attacks



- Given random initial state, can solve *n*-queens in almost constant time for arbitrary *n* with high probability (e.g., *n* = 10,000,000)