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On the Lie Algebra of a Linear Impulsive System

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***Abstract*— Recent investigations have focused on characterizing uniform exponential stability of linear impulsive systems in terms of properties of an associated Lie algebra together with familiar stability criteria. Of interest in this regard is determining whether this Lie algebra is *solvable*. Necessary and sufficient conditions have previously been derived that explicitly involve the system data but are nonconstructive and thus potentially difficult to verify. In this paper, necessary and sufficient conditions for solvability are once again cast directly in terms of the system description, except these new conditions are constructive. Our approach is motivated by a decades-old result that addressed the problem of finding a common eigenvector of a pair of matrices.**

I. INTRODUCTION

In recent work, lie-algebraic methods have emerged as a profitable approach to the stability analysis of switched linear systems ([1], [3], [7]) as well as the synthesis of stabilizing feedback laws [4]. In [5], the focus was on adapting this analysis to a class of linear impulsive systems of the form

$$\begin{aligned} \dot{x}(t) &= A_{\mathcal{C}} x(t) & t &\in \mathbb{R} \setminus \mathcal{T} \\ x(\tau_k) &= A_{\mathcal{I}} x(\tau_k^-) & \tau_k &\in \mathcal{T} \end{aligned} \tag{1}$$

in which the state $x(t)$ evolves in continuous-time as governed by the above differential equation and undergoes instantaneous changes at the impulse times in the set $\mathcal{T}$, assumed to form a strictly increasing sequence.

The first step in this investigation was to specify a Lie algebra associated with this system class in a way that reflects the asymmetric roles played by the linear maps $A_{\mathcal{C}}$ and $A_{\mathcal{I}}$. Specifically, $A_{\mathcal{I}}$ acts as a mapping on the state space, henceforth assumed to be invertible, whereas $A_{\mathcal{C}}$ is the infinitesimal generator of the matrix exponential $e^{A_{\mathcal{C}} t}$. This distinction will be made precise in the next section. It was then established, under natural stability-related hypotheses, that when this Lie algebra is *solvable* the linear impulsive system (1) is exponentially stable uniformly with respect to the set of impulse times.

The paper [6] sought to develop conditions for solvability of this Lie algebra that explicitly involve the system data. These conditions also led to the specification of a constant-parameter quadratic Lyapunov function that demonstrates uniform exponential stability in the affirmative case. Unfortunately, these conditions were nonconstructive and so the aim of this paper is the formulation of conditions that can be checked constructively. The key idea behind the derivation of the main result herein stems from an observation made in the paper [10] that treated the problem of finding a common eigenvector of a pair of matrices.

The remainder of this paper is organized as follows. In Section II, the Lie algebra associated with (1) is specified along with several related constructions that will be used in the sequel. Section III reviews the nonconstructive conditions derived in [6] under which this Lie algebra is solvable and also presents a reformulation that leads to the constructive conditions proposed in Section IV. Finally, concluding remarks are offered in Sections V.

II. LIE-ALGEBRAIC FRAMEWORK

The notation, terminology, and fundamental results on Lie groups and Lie algebras follows standard references on the subject, for example [2], [9]. It will be convenient to work with the (algebraically closed) complex field $\mathbb{C}$ and, in this section, take $\mathcal{X} = \mathbb{C}^n$ as the underlying vector space. Let $\mathsf{GL}(\mathcal{X})$ denote the general linear group on $\mathcal{X}$ for which the group operation is composition and let $\mathfrak{gl}(\mathcal{X})$ denote the associated Lie algebra with bracket given by the commutator $[\cdot, \cdot]$. For $A \in \mathfrak{gl}(\mathcal{X})$, the adjoint operator $\mathrm{ad}_A : \mathfrak{gl}(\mathcal{X}) \to \mathfrak{gl}(\mathcal{X})$ is defined by $\mathrm{ad}_A X = [A, X]$. For $g \in \mathsf{GL}(\mathcal{X})$, the adjoint map $\mathrm{Ad}_g : \mathfrak{gl}(\mathcal{X}) \to \mathfrak{gl}(\mathcal{X})$ is defined by $\mathrm{Ad}_g X = gXg^{-1}$. It follows that $\mathrm{Ad}_g^{-1} X = \mathrm{Ad}_{g^{-1}} X = g^{-1} X g$.

A Lie subalgebra L of $\mathfrak{gl}(\mathcal{X})$ may be generated from a set of elements of $\mathfrak{gl}(\mathcal{X})$

$$\mathfrak{G} = \{A_k \in \mathfrak{gl}(\mathcal{X}), \quad k \geq 0\} \tag{2}$$

by taking the $\mathbb{C}-$linear span of elements of $\mathfrak{gl}(\mathcal{X})$ of the form

$$[A_{k_r}, [A_{k_{r-1}}, [\cdots, [A_{k_2}, A_{k_1}] \cdots]]], \; A_{k_j} \in \mathfrak{G} \tag{3}$$

for $j = 1, \ldots, r$ and $r > 0$.

As noted in the Introduction, the maps $A_{\mathcal{C}}$ and $A_{\mathcal{I}}$ that govern the dynamics of the linear impulsive system (1) play fundamentally different roles. As an invertible mapping on the state space, taken to be $\mathcal{X} = \mathbb{C}^n$, we regard $A_{\mathcal{I}}$ as an element of the Lie group $\mathsf{GL}(\mathcal{X})$. As the infinitesimal generator of the matrix exponential $e^{A_{\mathcal{C}} t}$, we regard $A_{\mathcal{C}}$ as an element of the Lie algebra $\mathfrak{gl}(\mathcal{X})$.

To streamline the notation in the remainder of the paper, we henceforth represent $A_{\mathcal{I}}$ simply by $g \in \mathsf{GL}(\mathcal{X})$ and $A_{\mathcal{C}}$ by $A \in \mathfrak{gl}(\mathcal{X})$. Then, following the analysis in [5], [6], we consider the Lie subalgebra L of $\mathfrak{gl}(\mathcal{X})$ generated by the set

$$\mathfrak{G} = \{\mathrm{Ad}_g^{-k} A = g^{-k} A g^k, \quad k \geq 0\} \tag{4}$$

As a consequence of the Cayley-Hamilton Theorem applied to the the linear map Ad_g, the Lie algebra L is generated by $\mathrm{Ad}_g^{-k} A$, $0 \leq k < d$ for some finite $d \leq n^2$. As noted in [5],

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L can alternatively be characterized as the smallest *subspace* of $\mathfrak{gl}(\mathcal{X})$ that is invariant under the linear maps Ad_g, ad_A and contains $\mathrm{span}\{A\}$.

A. Lie Algebra Homomorphisms and Submodules

A Lie algebra homomorphism of two Lie algebras L, K is a linear map $\varphi : \mathsf{L} \to \mathsf{K}$ that is bracket preserving, i.e.

$$\varphi([X,Y]_{\mathsf{L}}) = [\varphi(X), \varphi(Y)]_{\mathsf{K}}$$

for all $X, Y \in \mathsf{L}$.

Proposition 2.1: If L is a Lie algebra generated by a set $\mathfrak{G}$ and $\varphi(\cdot)$ is a Lie algebra homomorphism defined on L, then the image $\hat{\mathsf{L}} = \varphi(\mathsf{L})$ is the Lie algebra generated by

$$\hat{\mathfrak{G}} = \{\hat{A}_k := \varphi(A_k), \quad k \geq 0\}$$

Proof: The bracket preserving property of the homomorphism $\varphi(\cdot)$ leads to

$$\varphi([A_{k_r}, [A_{k_{r-1}}, [\cdots, [A_{k_2}, A_{k_1}] \cdots]]]) = [\hat{A}_{k_r}, [\hat{A}_{k_{r-1}}, [\cdots, [\hat{A}_{k_2}, A_{k_1}] \cdots]]]$$

for all $A_{k_j} \in \mathfrak{G}$, $j = 1, \dots, r$ and $r > 0$. Using linearity of the homomorphism $\varphi(\cdot)$, it can be shown that any element of the Lie algebra generated by $\hat{\mathfrak{G}}$ lies in $\hat{\mathsf{L}} = \varphi(\mathsf{L})$ and, conversely, any element of $\hat{\mathsf{L}} = \varphi(\mathsf{L})$ lies in the Lie algebra generated by $\hat{\mathfrak{G}}$. ■

A Lie subalgebra L of $\mathfrak{gl}(\mathcal{X})$ gives the vector space $\mathcal{X}$ the structure of a Lie module for L, or $\mathsf{L}-$module. Specifically, the map $\mathsf{L} \times \mathcal{X} \to \mathcal{X}$ that takes (A, x) into $A \cdot x := Ax$ is bilinear and takes $([X,Y], x)$ into $X \cdot (Y \cdot x) - Y \cdot (X \cdot x)$ for all $X, Y \in \mathsf{L}$ and $x \in \mathcal{X}$.

A subspace $\mathcal{V} \subset \mathcal{X}$ that is invariant under every element of the Lie algebra L is called a *submodule* of $\mathcal{X}$ in the sense that for all $v \in \mathcal{V}$, $([X,Y], v)$ maps into $X \cdot (Y \cdot v) - Y \cdot (X \cdot v) \in \mathcal{V}$ for all $X, Y \in \mathsf{L}$.

Proposition 2.2: If L is a Lie algebra generated by a set $\mathfrak{G}$, then $\mathcal{V} \subset \mathcal{X}$ is a submodule if and only if $\mathcal{V}$ is invariant under every element of $\mathfrak{G}$.

Proof: Clearly, if $\mathcal{V}$ is a submodule, then $\mathcal{V}$ is invariant under every element of $\mathfrak{G}$. Conversely, if $\mathcal{V}$ is invariant under every element of $\mathfrak{G}$ then a straightforward induction argument gives that $\mathcal{V}$ is invariant under iterated brackets of the form 3 for all $A_{k_j} \in \mathfrak{G}$, $j = 1, \dots, r$ and $r > 0$. Consequently, $\mathcal{V}$ is invariant under an arbitrary element of L expressed as a linear combination of iterated brackets. Hence, $\mathcal{V}$ is a submodule. ■

For a Lie algebra L generated by a set $\mathfrak{G}$ of the form (4) for fixed $g \in \mathsf{GL}(\mathcal{X})$ and $A \in \mathfrak{gl}(\mathcal{X})$, a subspace $\mathcal{V}$ that is invariant under both g and A is, consequently, invariant under each $\mathrm{Ad}_g^{-k} A$, $k \geq 0$. By Proposition 2.2, $\mathcal{V}$ is a submodule.

A submodule $\mathcal{V}$ gives rise to two Lie algebra homomorphisms on L of interest in the sequel. First, let $\hat{\mathcal{V}}$ be a vector space having the same dimension as $\mathcal{V}$ and let $\Omega_{\mathcal{V}} : \hat{\mathcal{V}} \to \mathcal{X}$ be the insertion map whose image is $\mathcal{V}$. Every element $X \in \mathsf{L}$ has a well-defined restriction to $\mathcal{V}$, denoted $\hat{X}$, that satisfies the commutative relationship

$$X\Omega_{\mathcal{V}} = \Omega_{\mathcal{V}}\hat{X}$$

The mapping that takes $X \in \mathsf{L}$ to $\hat{X} \in \mathfrak{gl}(\hat{\mathcal{V}})$, denoted $\omega_{\mathcal{V}}(\cdot)$, is clearly linear and bracket-preserving and therefore defines a Lie algebra homomorphism on L. If L is a Lie algebra generated by a set $\mathfrak{G}$, then by Proposition 2.1, $\hat{\mathsf{L}}_{\mathcal{V}} = \omega_{\mathcal{V}}(\mathsf{L})$ is the Lie algebra generated by

$$\hat{\mathfrak{G}}_{\mathcal{V}} = \{\omega_{\mathcal{V}}(A_k), \quad k \geq 0\} \tag{5}$$

Next, corresponding to any element $X \in \mathsf{L}$ we let $\bar{X}$ denote the map induced in the quotient space $\mathcal{X}/\mathcal{V}$ by X. In terms of the canonical projection $\Pi_{\mathcal{V}} : \mathcal{X} \to \mathcal{X}/\mathcal{V}$, the induced map satisfies the commutative relationship

$$\Pi_{\mathcal{V}} X = \bar{X} \Pi_{\mathcal{V}}$$

The mapping that takes $X \in \mathsf{L}$ to $\bar{X} \in \mathfrak{gl}(\mathcal{X}/\mathcal{V})$, denoted $\pi_{\mathcal{V}}(\cdot)$, is also linear and bracket-preserving and therefore defines a Lie algebra homomorphism on L.

If L is a Lie algebra generated by a set $\mathfrak{G}$ of the form (2), then again by Proposition 2.1, $\bar{\mathsf{L}}_{\mathcal{V}} = \pi_{\mathcal{V}}(\mathsf{L})$ is the Lie algebra generated by

$$\bar{\mathfrak{G}}_{\mathcal{V}} = \{\pi_{\mathcal{V}}(A_k), \quad k \geq 0\} \tag{6}$$

In the case of a Lie algebra L generated by a set $\mathfrak{G}$ of the form (4) for fixed $g \in \mathsf{GL}(\mathcal{X})$ and $A \in \mathfrak{gl}(\mathcal{X})$, a subspace $\mathcal{V}$ that is invariant under both g and A is, consequently, invariant under each $\mathrm{Ad}_g^{-k} A$, $k \geq 0$. By Propopsition 2.2, $\mathcal{V}$ is a submodule. In addition to the above constructions, g and A have well-defined restrictions to $\mathcal{V}$, denoted $\hat{g}$ and $\hat{A}$, respectively. Elements of the set $\hat{\mathfrak{G}}_{\mathcal{V}}$ in (5) that generate $\hat{\mathsf{L}}_{\mathcal{V}}$ are given by

$$\omega_{\mathcal{V}}(\mathrm{Ad}_g^{-k} A) = \mathrm{Ad}_{\hat{g}}^{-k} \hat{A}, \quad k \geq 0$$

Likewise, in terms of maps denoted $\bar{g}$ and $\bar{A}$ induced in the quotient space $\mathcal{X}/\mathcal{V}$ by g and A, respectively, elements of the set $\bar{\mathfrak{G}}_{\mathcal{V}}$ in (6) that generate $\bar{\mathsf{L}}$ are given by

$$\pi_{\mathcal{V}}(\mathrm{Ad}_g^{-k} A) = \mathrm{Ad}_{\bar{g}}^{-k} \bar{A}, \quad k \geq 0$$

We next consider a pair of submodules $\mathcal{V}_1 \subset \mathcal{V}_2$. Corresponding to the submodule $\mathcal{V}_1$ we have the canonical projection $\Pi_{\mathcal{V}_1} : \mathcal{X} \to \mathcal{X}/\mathcal{V}_1$ and the Lie algebra homomorphism $\pi_{\mathcal{V}_1} : \mathsf{L} \to \mathfrak{gl}(\mathcal{X}/\mathcal{V}_1)$ for which

$$\Pi_{\mathcal{V}_1} X = \pi_{\mathcal{V}_1}(X)\, \Pi_{\mathcal{V}_1}$$

for all $X \in \mathsf{L}$. Upon defining $\mathcal{V}_2/\mathcal{V}_1 := \Pi_{\mathcal{V}_1}\mathcal{V}_2 \subset \mathcal{X}/\mathcal{V}_1$, we see that, for any $X \in \mathsf{L}$ and $\bar{X} = \pi_{\mathcal{V}_1}(X)$, there holds

$$\bar{X}(\mathcal{V}_2/\mathcal{V}_1) = \bar{X}\, \Pi_{\mathcal{V}_1}\mathcal{V}_2 = \Pi_{\mathcal{V}_1} X\mathcal{V}_2 \subset \Pi_{\mathcal{V}_1}\mathcal{V}_2$$

from which we conclude that $\mathcal{X}/\mathcal{V}_1$ is a Lie module for $\bar{\mathsf{L}}_{\mathcal{V}_1} = \pi_{\mathcal{V}_1}(\mathsf{L})$ for which $\mathcal{V}_2/\mathcal{V}_1$ is a submodule.

We follow previous constructions to obtain an insertion map $\Omega_{\mathcal{V}_2/\mathcal{V}_1} : \hat{\mathcal{V}}_2 \to \mathcal{X}/\mathcal{V}_1$ with image $\mathcal{V}_2/\mathcal{V}_1$ and a Lie algebra homomorphism $\omega_{\mathcal{V}_2/\mathcal{V}_1} : \bar{\mathsf{L}}_{\mathcal{V}_1} \to \mathfrak{gl}(\hat{\mathcal{V}}_2)$ in terms of which

$$\bar{X}\Omega_{\mathcal{V}_2/\mathcal{V}_1} = \Omega_{\mathcal{V}_2/\mathcal{V}_1}\, \omega_{\mathcal{V}_2/\mathcal{V}_1}(\bar{X})$$

for all $\bar{X} \in \bar{\mathsf{L}}_{\mathcal{V}_1}$. We set $\hat{\mathsf{L}}_{\mathcal{V}_2/\mathcal{V}_1} = \omega_{\mathcal{V}_2/\mathcal{V}_1}(\bar{\mathsf{L}}_{\mathcal{V}_1}) = (\omega_{\mathcal{V}_2/\mathcal{V}_1} \circ \pi_{\mathcal{V}_1})(\mathsf{L})$.

As a consequence of Proposition 2.1, if L is generated by a set $\mathfrak{G}$ of the form (2), then $\bar{\mathsf{L}}_1$ is generated by the set

$$\bar{\mathfrak{G}}_{\mathcal{V}_1} = \{\pi_{\mathcal{V}_1}(A_k), \quad k \geq 0\}$$

and $\hat{\mathsf{L}}_{\mathcal{V}_2/\mathcal{V}_1}$ is generated by the set

$$\hat{\mathfrak{G}}_{\mathcal{V}_2/\mathcal{V}_1} = \{(\omega_{\mathcal{V}_2/\mathcal{V}_1} \circ \pi_{\mathcal{V}_1})(A_k), \quad k \geq 0\}$$

Corresponding to the submodule $\mathcal{V}_2$ we have the canonical projection $\Pi_{\mathcal{V}_2} : \mathcal{X} \to \mathcal{X}/\mathcal{V}_2$ and the Lie algebra homomorphism $\pi_{\mathcal{V}_2} : \mathsf{L} \to \mathfrak{gl}(\mathcal{X}/\mathcal{V}_2)$ for which

$$\Pi_{\mathcal{V}_2} X = \pi_{\mathcal{V}_2}(X)\, \Pi_{\mathcal{V}_2}$$

for all $X \in \mathsf{L}$. There exists a surjective map $\Delta_{\mathcal{V}_2/\mathcal{V}_1} : \mathcal{X}/\mathcal{V}_1 \to \mathcal{X}/\mathcal{V}_2$ for which $\Delta_{\mathcal{V}_2/\mathcal{V}_1}\Pi_{\mathcal{V}_1} = \Pi_{\mathcal{V}_2}$ and $\mathrm{Ker}\, \Delta_{\mathcal{V}_2/\mathcal{V}_1} = \mathcal{V}_2/\mathcal{V}_1$. The mapping that takes $\bar{X} \in \bar{\mathsf{L}}_{\mathcal{V}_1}$ to $\bar{\bar{X}} \in \bar{\bar{\mathsf{L}}}_{\mathcal{V}_2}$ according to

$$\Delta_{\mathcal{V}_2/\mathcal{V}_1} \bar{X} = \bar{\bar{X}} \Delta_{\mathcal{V}_2/\mathcal{V}_1}$$

defines a Lie algebra homomorphism $\delta_{\mathcal{V}_2/\mathcal{V}_1} : \bar{\mathsf{L}}_{\mathcal{V}_1} \to \bar{\mathsf{L}}_{\mathcal{V}_2}$ yielding the factorization $\pi_{\mathcal{V}_2}(\cdot) = (\delta_{\mathcal{V}_2/\mathcal{V}_1} \circ \pi_{\mathcal{V}_1})(\cdot)$.

For a Lie algebra L generated by a set $\mathfrak{G}$ of the form (4) for fixed $g \in \mathsf{GL}(\mathcal{X})$ and $A \in \mathfrak{gl}(\mathcal{X})$, subspaces $\mathcal{V}_1 \subset \mathcal{V}_2$ that are invariant under both g and A are submodules. Consequently, some of the preceding constructions involving elements of L apply to g and A directly. For $i = 1, 2$, we may define maps $\bar{g}_i$, $\bar{A}_i$ induced in the quotient space $\mathcal{X}/\mathcal{V}_i$ by g and A, respectively. When applied to elements of $\mathfrak{G}$ given in (4), the Lie algebra homomorphisms $\pi_{\mathcal{V}_i}(\cdot)$, $i = 1, 2$ produce

$$\pi_{\mathcal{V}_i}(\mathrm{Ad}_g^{-k} A) = \mathrm{Ad}_{\bar{g}_i}^{-k} \bar{A}_i, \quad i = 1, 2, \quad k \geq 0$$

The subspace $\mathcal{V}_2/\mathcal{V}_1 \subset \mathcal{X}/\mathcal{V}_1$ is invariant under both $\bar{g}_1$ and $\bar{A}_1$ and so these induced maps have restrictions to $\mathcal{V}_2/\mathcal{V}_1$, denoted $\hat{g}_2$, $\hat{A}_2$, respectively. The homomorphisms $\omega_{\mathcal{V}_2/\mathcal{V}_1}(\cdot), \delta_{\mathcal{V}_2/\mathcal{V}_1}(\cdot)$ satisfy

$$\begin{aligned} (\omega_{\mathcal{V}_2/\mathcal{V}_1} \circ \pi_{\mathcal{V}_1})(\mathrm{Ad}_g^{-k} A) &= \mathrm{Ad}_{\hat{g}_2}^{-k} \hat{A}_2 \\ (\delta_{\mathcal{V}_2/\mathcal{V}_1} \circ \pi_{\mathcal{V}_1})(\mathrm{Ad}_g^{-k} A) &= \mathrm{Ad}_{\bar{g}_2}^{-k} \bar{A}_2, \quad k \geq 0 \end{aligned}$$

B. Abelian, Nilpotent, and Solvable Lie Algebras

An *ideal* I of a Lie algebra L is a Lie subalgebra of L that, in addition, satisfies $[\mathsf{I}, \mathsf{L}] \subset \mathsf{I}$ ($[X, Y] \in \mathsf{I}$ for all $X \in \mathsf{I}$ and $Y \in \mathsf{L}$). We will be interested in several properties of a Lie algebra L that are defined in terms of certain ideals. The *derived Lie algebra* $\mathsf{L}' = [\mathsf{L}, \mathsf{L}]$ is an ideal of L by construction. The *lower central series* of ideals is defined as

$$\mathsf{L}^0 \supset \mathsf{L}^1 \supset \mathsf{L}^2 \supset \cdots$$

in which $\mathsf{L}^0 := \mathsf{L}$ and $\mathsf{L}^{k+1} := [\mathsf{L}^k, \mathsf{L}]$, $k \geq 0$. The *derived series* of ideals of L is given by

$$\mathsf{L}^{(0)} \supset \mathsf{L}^{(1)} \supset \mathsf{L}^{(2)} \supset \cdots$$

in which $\mathsf{L}^{(0)} := \mathsf{L}$ and $\mathsf{L}^{(k+1)} := [\mathsf{L}^{(k)}, \mathsf{L}^{(k)}]$, $k \geq 0$.

Definition 2.3: The Lie algebra L is:

1) *Abelian* if $\mathsf{L}' = 0$;
2) *nilpotent* if $\mathsf{L}^r = 0$ for some finite integer r;
3) *solvable* if $\mathsf{L}^{(r)} = 0$ for some finite integer r. □

As a consequence of the definitions, an Abelian Lie algebra is nilpotent, and a nilpotent Lie algebra is solvable. Moreover, a Lie algebra is solvable if and only if its derived Lie algebra is nilpotent. The characterization of solvability utilized in [6] involves the following construction.

Definition 2.4: A chain of subspaces

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_n, \quad \dim \mathcal{S}_j = j, \quad j = 1, \ldots, n \tag{7}$$

is called a flag of the vector space $\mathcal{X}$. If each subspace $\mathcal{S}_j$ is invariant under every element of the Lie algebra L, then the flag is said to be invariant under L, or $\mathsf{L}-$invariant. □

The following result is known as Lie's theorem.

Proposition 2.5: A Lie algebra $\mathsf{L} \subset \mathfrak{gl}(\mathcal{X})$ is solvable if and only if there exists a flag of $\mathcal{X}$ that is $\mathsf{L}-$invariant. □

III. Nonconstructive Conditions

In [6], solvability of the Lie algebra L generated by the set $\mathfrak{G}$ in (4) was linked to the existence of another chain of subspaces

$$0 = \mathcal{W}_0 \subset \mathcal{W}_1 \subset \mathcal{W}_2 \subset \cdots \subset \mathcal{W}_\ell = \mathcal{X} \tag{8}$$

in general coarser than a flag (7), in which each member is invariant under both g and A.

Under the assumption that such a subspace chain exists, we associate with each subspace $\mathcal{W}_i$, $i = 1, \ldots, \ell$ the map $\bar{g}_i$ (respectively, $\bar{A}_i$) induced in the quotient space $\mathcal{X}/\mathcal{W}_i$ by the map g (respectively, A). In terms of the canonical projection $\Pi_{\mathcal{W}_i} : \mathcal{X} \to \mathcal{X}/\mathcal{W}_i$ these induced maps satisfy the commutative relationships

$$\Pi_{\mathcal{W}_i} g = \bar{g}_i \Pi_{\mathcal{W}_i}, \quad \Pi_{\mathcal{W}_i} A = \bar{A}_i \Pi_{\mathcal{W}_i}, \quad i = 1, \ldots, \ell$$

Additionally, it follows from $g-$invariance (respectively, $A-$invariance) of $\mathcal{W}_i$ that $\mathcal{W}_i/\mathcal{W}_{i-1} := \Pi_{\mathcal{W}_{i-1}}\mathcal{W}_i$ is a subspace of $\mathcal{X}/\mathcal{W}_{i-1}$ that is invariant under the induced map $\bar{g}_{i-1}$ (respectively, $\bar{A}_{i-1}$). As in Section II-A, we let $\Omega_{\mathcal{W}_i/\mathcal{W}_{i-1}} : \hat{\mathcal{W}}_i \to \mathcal{X}/\mathcal{W}_{i-1}$ be the insertion map with image $\mathcal{W}_i/\mathcal{W}_{i-1}$. Letting $\hat{g}_i$ (respectively, $\hat{A}_i$) denote restriction of $\bar{g}_{i-1}$ (respectively, $\bar{A}_{i-1}$) to $\mathcal{W}_i/\mathcal{W}_{i-1}$, we arrive at the commutative relationships

$$\begin{aligned} \bar{g}_{i-1}\Omega_{\mathcal{W}_i/\mathcal{W}_{i-1}} &= \Omega_{\mathcal{W}_i/\mathcal{W}_{i-1}}\hat{g}_i \\ \bar{A}_{i-1}\Omega_{\mathcal{W}_i/\mathcal{W}_{i-1}} &= \Omega_{\mathcal{W}_i/\mathcal{W}_{i-1}}\hat{A}_i, \quad i = 1, \ldots, \ell \end{aligned} \tag{9}$$

We may now recall the main result from [6].

Theorem 3.1: ([6, Theorem 3.9]) The Lie algebra L generated by the set (4) is solvable if and only if there exists a chain of subspaces (8) for which

1) Each $\mathcal{W}_i$ is invariant under both g and A;
2) For the linear maps $\hat{g}_i$ and $\hat{A}_i$ defined according to (9) there exists a cyclic generator $\hat{w}_i$ for $\hat{\mathcal{W}}_i$ relative to $\hat{g}_i$ for which $\{\hat{w}_i, \hat{g}_i\hat{w}_i, \ldots, \hat{g}_i^{\kappa_i-1}\hat{w}_i\}$ is an eigenbasis for $\hat{A}_i$;
3) $q_i = \min\{1 \leq q \leq \kappa_i \mid \mathrm{Ad}_{\hat{g}_i}^{-q} \hat{A}_i = \hat{A}_i\}$ is well-defined and divides κ_i. □

By joint invariance under g and A, each subspace $\mathcal{W}_i$, $i = 1, \ldots, \ell$ in the chain (8) is a submodule of the $\mathsf{L}-$module $\mathcal{X}$. Consequently, various Lie-algebraic constructions presented in Section II may be applied, mutatis mutandis, to the present context.

Associated with each $\mathcal{W}_i$ as a submodule, we have the Lie algebra homomorphism $\pi_{\mathcal{W}_i} : \mathsf{L} \to \mathfrak{gl}(\mathcal{X}/\mathcal{W}_i)$ in terms of which $\bar{\mathsf{L}}_{\mathcal{W}_i} := \pi_{\mathcal{W}_i}(\mathsf{L})$ is the Lie subalgebra of $\mathfrak{gl}(\mathcal{X}/\mathcal{W}_i)$ generated by the set

$$\bar{\mathfrak{G}}_{\mathcal{W}_i} = \{\pi_{\mathcal{W}_i}(\mathrm{Ad}_g^{-k} A) = \mathrm{Ad}_{\bar{g}_i}^{-k} \bar{A}_i, \quad k \geq 0\}$$

Next, corresponding to the submodules $\mathcal{W}_{i-1} \subset \mathcal{W}_i$, it follows that $\mathcal{X}/\mathcal{W}_{i-1}$ is a Lie module for $\bar{\mathsf{L}}_{\mathcal{W}_i}$ with $\mathcal{W}_i/\mathcal{W}_{i-1}$ as a submodule. In addition, we have the Lie algebra homomorphism $\omega_{\mathcal{W}_i/\mathcal{W}_{i-1}} : \bar{\mathsf{L}}_{\mathcal{W}_i} \to \mathfrak{gl}(\hat{\mathcal{W}}_i)$ in terms of which $\hat{\mathsf{L}}_{\mathcal{W}_i/\mathcal{W}_{i-1}} := \omega_{\mathcal{W}_i/\mathcal{W}_{i-1}}(\bar{\mathsf{L}}_{\mathcal{W}_{i-1}})$ is the Lie subalgebra of $\mathfrak{gl}(\hat{\mathcal{W}}_i)$ generated by the set

$$\hat{\mathfrak{G}}_{\mathcal{W}_i/\mathcal{W}_{i-1}} = \{\omega_{\mathcal{W}_i/\mathcal{W}_{i-1}}(\mathrm{Ad}_{\bar{g}_{i-1}}^{-k} \bar{A}_{i-1}) = \mathrm{Ad}_{\hat{g}_i}^{-k} \hat{A}_i, \quad k \geq 0\}$$

Each $\hat{\mathsf{L}}_{\mathcal{W}_i/\mathcal{W}_{i-1}}$ may be obtained directly from L by composing the preceding homomorphisms according to $\hat{\mathsf{L}}_{\mathcal{W}_i/\mathcal{W}_{i-1}} = (\omega_{\mathcal{W}_i/\mathcal{W}_{i-1}} \circ \pi_{\mathcal{W}_{i-1}})(\mathsf{L})$.

Finally, for the surjective map $\Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}} : \mathcal{X}/\mathcal{W}_{i-1} \to \mathcal{X}/\mathcal{W}_i$ that satisfies $\Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}}\Pi_{\mathcal{W}_{i-1}} = \Pi_{\mathcal{W}_i}$ with $\mathrm{Ker}\,\Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}} = \mathcal{W}_i/\mathcal{W}_{i-1}$ and yields

$$\begin{aligned}&\Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}}\bar{g}_{i-1} = \bar{g}_i\Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}},\\ &\Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}}\bar{A}_{i-1} = \bar{A}_i\Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}}, \quad i = 1, \ldots, \ell \end{aligned} \tag{10}$$

we have the Lie algebra homomorphism $\delta_{\mathcal{W}_i/\mathcal{W}_{i-1}} : \bar{\mathsf{L}}_{\mathcal{W}_{i-1}} \to \bar{\mathsf{L}}_{\mathcal{W}_i}$ that yields the factorization $\pi_{\mathcal{W}_i}(\cdot) = (\delta_{\mathcal{W}_i/\mathcal{W}_{i-1}} \circ \pi_{\mathcal{W}_{i-1}})(\cdot)$, $i = 1, \ldots, \ell$.

With these constructions in place, we may state and prove the following key consequence of Theorem 3.1.

Lemma 3.2: Suppose the Lie algebra L generated by the set (4) is solvable and let $\mathcal{W}_i$, $i = 1, \ldots, \ell$ denote the chain of subspaces with the properties stated in Theorem 3.1. Then for $i = 1, \ldots, \ell$, the Lie algebra $\hat{\mathsf{L}}_{\mathcal{W}_i/\mathcal{W}_{i-1}} = (\omega_{\mathcal{W}_i/\mathcal{W}_{i-1}} \circ \pi_{\mathcal{W}_{i-1}})(\mathsf{L})$ is *Abelian*.

Proof: The second and third conditions of Theorem 3.1 imply that

$$\hat{A}_i\hat{g}_i^{r-1}\hat{w}_i = \lambda_{i,r}\hat{g}_i^{r-1}\hat{w}_i$$

for *all* $r \geq 1$ with $\lambda_{i,r} \in \sigma(\hat{A}_i)$. For integers $k \geq 0$ and $1 \leq m \leq \kappa_i$

$$\begin{aligned}(\mathrm{Ad}_{\hat{g}_i}^{-k} \hat{A}_i)(\hat{g}_i^{m-1}\hat{w}_i) &= (\hat{g}_i^{-k}\hat{A}_i\hat{g}_i^{k})(\hat{g}_i^{m-1}\hat{w}_i)\\ &= \lambda_{i,k+m}\hat{g}_i^{m-1}\hat{w}_i\end{aligned}$$

and so, for nonnegative integers $j, k \geq 0$ and $1 \leq m \leq \kappa_i$,

$$\begin{aligned}&\big[\mathrm{Ad}_{\hat{g}_i}^{-j} \hat{A}_i,\, \mathrm{Ad}_{\hat{g}_i}^{-k} \hat{A}_i\big](\hat{g}_i^{m-1}\hat{w}_i)\\ &\quad= (\lambda_{i,j+m}\lambda_{i,k+m} - \lambda_{i,k+m}\lambda_{i,j+m})\hat{g}_i^{m-1}\hat{w}_i = 0\end{aligned}$$

Thus, because $\{\hat{w}_i, \hat{g}_i\hat{w}_i, \ldots, \hat{g}_i^{\kappa_i-1}\hat{w}_i\}$ is a basis for the vector space $\hat{\mathcal{W}}_i$

$$\big[\mathrm{Ad}_{\hat{g}_i}^{-j} \hat{A}_i,\, \mathrm{Ad}_{\hat{g}_i}^{-k} \hat{A}_i\big] = 0 \tag{11}$$

In particular, for $j = 0$,

$$\mathrm{ad}_{\hat{A}_i} \mathrm{Ad}_{\hat{g}_i}^{-k} \hat{A}_i = \big[\hat{A}_i,\, \mathrm{Ad}_{\hat{g}_i}^{-k} \hat{A}_i\big] = 0$$

from which it follows that

$$\mathrm{span}\{\mathrm{Ad}_{\hat{g}_i}^{-k} \hat{A}_i,\, k \geq 0\} = \mathrm{span}\{\mathrm{Ad}_{\hat{g}_i}^{-k} \hat{A}_i,\, 0 \leq k < q_i\}$$

is invariant under both $\mathrm{Ad}_{\hat{g}_i}$, $\mathrm{ad}_{\hat{A}_i}$ and contains $\mathrm{span}\{\hat{A}_i\}$ and thus directly specifies the Lie algebra $\hat{\mathsf{L}}_{\mathcal{W}_i/\mathcal{W}_{i-1}}$. Moreover, $\hat{\mathsf{L}}_{\mathcal{W}_i/\mathcal{W}_{i-1}}$ is Abelian as a consequence of (11). ■

The preceding lemma allows the conditions of Theorem 3.1 to be repackaged into terms that serve to motivate our approach to the development of constructive conditions for solvability of the Lie algebra L presented in the next section.

Theorem 3.3: The Lie algebra L generated by the set (4) is solvable if and only if there exists a chain of subspaces $\mathcal{W}_i$, $i = 1, \ldots, \ell$ for which

1) Each $\mathcal{W}_i$ is invariant under both g and A;
2) Each Lie algebra $\hat{\mathsf{L}}_{\mathcal{W}_i/\mathcal{W}_{i-1}} = (\omega_{\mathcal{W}_i/\mathcal{W}_{i-1}} \circ \pi_{\mathcal{W}_{i-1}})(\mathsf{L})$ is Abelian.

Proof: Necessity follows as a consequence of Theorem 3.1 and Lemma 3.2. To demonstrate sufficiency, we claim that for all $X, Y \in \mathsf{L}$ there holds

$$[\pi_{\mathcal{W}_i}(X), \pi_{\mathcal{W}_i}(Y)]^{\ell-i} = 0, \quad i = 0, \ldots, \ell \tag{12}$$

in which the $[\cdot,\cdot]$ denotes the bracket on $\mathfrak{gl}(\mathcal{X}/\mathcal{W}_i)$ that is inherited by $\bar{\mathsf{L}}_{\mathcal{W}_i}$ and the exponentiation takes place in $\mathfrak{gl}(\mathcal{X}/\mathcal{W}_i)$ as an associative algebra. Because $\Pi_0 : \mathcal{X} \to \mathcal{X}/\mathcal{W}_0$ is an isomorphism, $\pi_{\mathcal{W}_0}(\cdot)$ is a Lie algebra isomorphism and so $\big[X, Y\big]^{\ell} = 0$. Thus, the derived Lie algebra $\mathsf{L}' = [\mathsf{L}, \mathsf{L}]$ is spanned by nilpotent maps and is therefore a nilpotent Lie subalgebra of $\mathfrak{gl}(\mathcal{X})$ by Engel's theorem. This, in turn, implies that the Lie algebra L is solvable.

We prove the claim by induction on $\ell \geq i \geq 0$. For $i = \ell$, we have from $\mathcal{W}_\ell = \mathcal{X}$ that $\Omega_{\mathcal{W}_\ell/\mathcal{W}_{\ell-1}}$ is an isomorphism between $\hat{\mathcal{W}}_\ell$ and $\mathcal{W}_\ell/\mathcal{W}_{\ell-1} = \mathcal{X}/\mathcal{W}_{\ell-1}$. It follows that $\omega_{\mathcal{W}_\ell/\mathcal{W}_{\ell-1}}(\cdot)$ is a Lie algebra isomorphism between $\bar{\mathsf{L}}_{\mathcal{W}_{\ell-1}} = \pi_{\mathcal{W}_{\ell-1}}(\mathsf{L})$ and $\hat{\mathsf{L}}_{\mathcal{W}_\ell/\mathcal{W}_{\ell-1}}$. Because the latter Lie algebra is Abelian, for any $X, Y \in \mathsf{L}$ we have that

$$\hat{X} := (\omega_{\mathcal{W}_\ell/\mathcal{W}_{\ell-1}} \circ \pi_{\mathcal{W}_{\ell-1}})(X),\; \hat{Y} := (\omega_{\mathcal{W}_\ell/\mathcal{W}_{\ell-1}} \circ \pi_{\mathcal{W}_{\ell-1}})(Y)$$

are elements of $\hat{\mathsf{L}}_{\mathcal{W}_\ell/\mathcal{W}_{\ell-1}}$ and so

$$[\pi_{\mathcal{W}_{\ell-1}}(X), \pi_{\mathcal{W}_{\ell-1}}(Y)] = \omega_{\hat{\mathcal{W}}_\ell}^{-1}\big([\hat{X}, \hat{Y}]\big) = 0$$

Now, suppose that (12) holds for all $X, Y \in \mathsf{L}$ for some $\ell > i > 0$. We let $\bar{X} = \pi_{\mathcal{W}_{i-1}}(X)$, $\bar{\bar{X}} = \pi_{\mathcal{W}_i}(X)$ so that $\bar{\bar{X}} = \delta_{\mathcal{W}_i/\mathcal{W}_{i-1}}(\bar{X})$. Similarly, with $\bar{Y} = \pi_{\mathcal{W}_{i-1}}(Y)$, $\bar{\bar{Y}} = \pi_{\mathcal{W}_i}(Y)$ we have $\bar{\bar{Y}} = \delta_{\mathcal{W}_i/\mathcal{W}_{i-1}}(\bar{Y})$. Consequently,

$$\Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}}\bar{X} = \bar{\bar{X}}\Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}},\; \Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}}\bar{Y} = \bar{\bar{Y}}\Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}}$$

and it follows easily that

$$\Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}}[\bar{X}, \bar{Y}]^r = [\bar{\bar{X}}, \bar{\bar{Y}}]^r\Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}}, \quad r \geq 0.$$

Now, for any $\bar{x} \in \mathcal{X}/\mathcal{W}_{i-1}$ the induction hypothesis gives

$$\Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}}[\bar{X}, \bar{Y}]^{\ell-i}\bar{x} = [\bar{\bar{X}}, \bar{\bar{Y}}]^{\ell-i}\Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}}\bar{x} = 0$$

so that

$$[\bar{X}, \bar{Y}]^{\ell-i}\bar{x} \in \mathrm{Ker}\ \Delta_{\mathcal{W}_i/\mathcal{W}_{i-1}} = \mathcal{W}_i/\mathcal{W}_{i-1} = \mathrm{Im}\ \Omega_{\mathcal{W}_i/\mathcal{W}_{i-1}}$$

and we may write $[\bar{X},\bar{Y}]^{\ell-i}\bar{x} = \Omega_{\mathcal{W}_i/\mathcal{W}_{i-1}}\hat{w}$ for some $\hat{w} \in \hat{\mathcal{W}}_i$. It follows that, because the Lie algebra $\hat{\mathsf{L}}_{\mathcal{W}_i/\mathcal{W}_{i-1}}$ is Abelian,

$$\begin{aligned}[\bar{X},\bar{Y}]^{\ell-i+1}\bar{x} &= [\bar{X},\bar{Y}]\Omega_{\mathcal{W}_i/\mathcal{W}_{i-1}}\hat{w} \\ &= \Omega_{\mathcal{W}_i/\mathcal{W}_{i-1}}[\omega_{\mathcal{W}_i/\mathcal{W}_{i-1}}(\bar{X}),\omega_{\mathcal{W}_i/\mathcal{W}_{i-1}}(\bar{Y})]\hat{w} \\ &= 0\end{aligned}$$

from which $[\pi_{\mathcal{W}_{i-1}}(X),\pi_{\mathcal{W}_{i-1}}(Y)]^{\ell-i+1} = 0$ which concludes the induction argument and the proof. ■

IV. Constructive Conditions

The necessary and sufficient conditions for solvability of the Lie algebra L presented in both Theorems 3.1 and 3.3 are nonconstructive because the chain of subspaces (8) having the requisite properties is not specified explicitly.

Our approach to formulating constructive conditions is motivated by the paper [10] by Shemesh that treats the problem of determining common eigenvectors of a pair of $n \times n$ matrices A, B. The main result [10, Theorem 3.1] establishes that a common eigenvector exists if and only if

$$\mathcal{N} = \bigcap_{j,k=1}^{n-1} \operatorname{Ker}[A^j, B^k] \neq 0$$

Following the proof of this theorem, Shemesh observes that $\mathcal{N}$ is the *largest* subspace that is invariant under both A, B on which A and B commute.

Here we seek to adapt this result to the context of the Lie algebra L generated by the set $\mathfrak{G}$ in (4) and, in particular, make contact with the conditions given in Theorem 3.3 under which L is solvable. We begin by defining the subspace

$$\mathcal{L} = \bigcap_{j,k\geq 0} \operatorname{Ker}[\operatorname{Ad}_g^{-j} A, \operatorname{Ad}_g^{-k} A] \tag{13}$$

that involves the elements of the generating set $\mathfrak{G}$ in (4). The maps $\operatorname{Ad}_g^{-j} A$, $\operatorname{Ad}_g^{-k} A$ commute when acting on vectors in $\mathcal{L}$.

Consider a subspace $\mathcal{N}$ that is invariant under both g and A and contained in $\mathcal{L}$. We again follow the development in Section II-A and let $\hat{\mathcal{N}}$ denote a vector space whose dimension matches that of the subspace $\mathcal{N}$ and let $\Omega_{\mathcal{N}} : \hat{\mathcal{N}} \to \mathcal{X}$ be the insertion map whose image is $\mathcal{N}$. We then let $\hat{g}$ (respectively, $\hat{A}$) denote the restriction of g (respectively, A) to $\mathcal{N}$ in terms of which we have the commutative relationships

$$g\Omega_{\mathcal{N}} = \Omega_{\mathcal{N}}\hat{g}, \quad A\Omega_{\mathcal{N}} = \Omega_{\mathcal{N}}\hat{A}$$

By virtue of the joint invariance under g and A, the subspace $\mathcal{N}$ is a submodule of the $\mathsf{L}-$module $\mathcal{X}$ and there exists a Lie algebra homomorphism $\omega_{\mathcal{N}} : \mathsf{L} \to \mathfrak{gl}(\hat{\mathcal{N}})$ that satisfies

$$X\Omega_{\mathcal{N}} = \Omega_{\mathcal{N}}\,\omega_{\mathcal{N}}(X)$$

for all $X \in \mathsf{L}$. The image $\hat{\mathsf{L}}_{\mathcal{N}} := \omega_{\mathcal{N}}(\mathsf{L})$ is the Lie subalgebra of $\mathfrak{gl}(\hat{\mathcal{N}})$ generated by the set

$$\hat{\mathfrak{G}}_{\mathcal{N}} = \{\omega(\operatorname{Ad}_g^{-k} A) = \operatorname{Ad}_{\hat{g}}^{-k} \hat{A}, \quad k \geq 0\}$$

We may now state a generalization of Shemesh's result.

Lemma 4.1: Let $\mathcal{N} \subset \mathcal{X}$ be a subspace that is invariant under both g and A and let $\hat{\mathsf{L}}_{\mathcal{N}} := \omega_{\mathcal{N}}(\mathsf{L})$. Then $\hat{\mathsf{L}}_{\mathcal{N}}$ is Abelian if and only if $\mathcal{N} \subset \mathcal{L}$.

Proof: First, suppose that $\mathcal{N} \subset \mathcal{L}$. For any pair of elements in $\hat{\mathfrak{G}}_{\mathcal{N}}$

$$\Omega_{\mathcal{N}}[\operatorname{Ad}_{\hat{g}}^{-j} \hat{A}, \operatorname{Ad}_{\hat{g}}^{-k} \hat{A}] = [\operatorname{Ad}_g^{-j} A, \operatorname{Ad}_g^{-k} A]\Omega_{\mathcal{N}} = 0$$

from which $[\operatorname{Ad}_{\hat{g}}^{-j} \hat{A}, \operatorname{Ad}_{\hat{g}}^{-k} \hat{A}] = 0$ because the insertion map $\Omega_{\mathcal{N}}$ is injective. Thus, $\hat{\mathsf{L}}_{\mathcal{N}} = \operatorname{span}\{\operatorname{Ad}_{\hat{g}}^{-k} \hat{A},\ k \geq 0\}$ and is Abelian.

For the converse, suppose that $\hat{\mathsf{L}}_{\mathcal{N}} = \omega_{\mathcal{N}}(\mathsf{L})$ is Abelian. As $\operatorname{Ad}_g^{-j} A$, $\operatorname{Ad}_g^{-k} A \in \mathsf{L}$ for all $j,k \geq 0$

$$0 = \Omega_{\mathcal{N}}[\operatorname{Ad}_{\hat{g}}^{-j} \hat{A}, \operatorname{Ad}_{\hat{g}}^{-k} \hat{A}] = [\operatorname{Ad}_g^{-j} A, \operatorname{Ad}_g^{-k} A]\Omega_{\mathcal{N}}$$

so that $\mathcal{N} = \operatorname{Im}\Omega_{\mathcal{N}} \subset \operatorname{Ker}[\operatorname{Ad}_g^{-j} A, \operatorname{Ad}_g^{-k} A]$ for all $j,k \geq 0$ and thus $\mathcal{N} \subset \mathcal{L}$. ■

Based on [8], the largest subspace $\mathcal{N}$ that is invariant under both g and A and contained in the subspace $\mathcal{L}$ exists, is unique, and can be computed via a subspace algorithm that is easily implemented using standard matrix computations.

The constructive conditions we seek involve a chain of subspaces

$$0 = \mathcal{N}_0 \subset \mathcal{N}_1 \subset \mathcal{N}_2 \subset \cdots \subset \mathcal{N}_r = \mathcal{X} \tag{14}$$

in which, once again, each subspace will be required to be invariant under both g and A but otherwise differs from the chain (8) studied in Section III. As such, we will once again adapt the constructions in Section II in order to establish the main result of the paper. To proceed, we define on each quotient space $\mathcal{X}/\mathcal{N}_i$ the subspace

$$\bar{\mathcal{L}}_i = \bigcap_{j,k\geq 0} \operatorname{Ker}\pi_{\mathcal{N}_i}([\operatorname{Ad}_g^{-j} A, \operatorname{Ad}_g^{-k} A]), \quad i = 0,\ldots,r \tag{15}$$

Theorem 4.2: The Lie algebra L generated by the set $\mathfrak{G}$ in (4) is solvable if and only if there exists a chain of subspaces $\mathcal{N}_i$, $i = 1,\ldots,r$ for which

1) Each $\mathcal{N}_i$ is invariant under both g and A;
2) $\mathcal{N}_i/\mathcal{N}_{i-1}$ is the largest subspace of $\mathcal{X}/\mathcal{N}_{i-1}$ that is invariant under the induced maps $\bar{g}_{i-1}$, $\bar{A}_{i-1}$ and is contained in $\bar{\mathcal{L}}_{i-1}$.

Proof: For sufficiency, the conditions of the Theorem lead to, following the constructions in Section II-A, the specification of Lie subalgebras $\bar{\mathsf{L}}_{\mathcal{N}_{i-1}} = \pi_{\mathcal{N}_{i-1}}(\mathsf{L}) \subset \mathfrak{gl}(\mathcal{X}/\mathcal{N}_{i-1})$ and $\hat{\mathsf{L}}_{\mathcal{N}_i/\mathcal{N}_{i-1}} = \omega_{\mathcal{N}_i/\mathcal{N}_{i-1}}(\bar{\mathsf{L}}_{\mathcal{N}_{i-1}}) \subset \mathfrak{gl}(\hat{\mathcal{N}}_i)$. With $\mathcal{N}_i/\mathcal{N}_{i-1}$ assumed to be a subspace of $\mathcal{X}/\mathcal{N}_{i-1}$ that is invariant under the induced maps $\bar{g}_{i-1}$, $\bar{A}_{i-1}$ and contained in $\bar{\mathcal{L}}_{i-1}$ (in fact, the largest such subspace), we conclude from Lemma 4.1 that $\hat{\mathsf{L}}_{\mathcal{N}_i/\mathcal{N}_{i-1}}$ is Abelian. Consequently, the chain of subspaces $\mathcal{N}_i$, $i = 1,\ldots,r$ satisfies the conditions of Theorem 3.3 from which it follows that the Lie algebra L generated by the set $\mathfrak{G}$ in (4) is solvable.

For necessity, suppose the Lie algebra L is solvable. We proceed by mathematical induction to specify a chain of subspaces (14) with the requisite properties. In terms of a chain of subspaces (8) having the properties stated in Theorem 3.3, it follows from $\mathcal{W}_0 = 0$ that $\hat{\mathsf{L}}_{\mathcal{W}_1} \simeq \hat{\mathsf{L}}_{\mathcal{W}_1/\mathcal{W}_0}$ is Abelian. By Lemma 4.1, the nonzero subspace $\mathcal{W}_1$, also being invariant under g and A, is contained in $\mathcal{L}$. The largest such subspace, denoted $\mathcal{N}_1$, is therefore nonzero. Because $\mathcal{N}_0 = 0$ by construction, $\Pi_{\mathcal{N}_0} : \mathcal{X} \to \mathcal{X}/\mathcal{N}_0$ is an isomorphism. Thus, $\mathcal{N}_1/\mathcal{N}_0 \simeq \mathcal{N}_1$, $\bar{g}_0 \simeq g$, $\bar{A}_0 \simeq A$, and $\bar{\mathcal{L}}_0 \simeq \mathcal{L}$ and it follows that the second condition of the theorem is satisfied for $i = 1$.

If $\mathcal{N}_1 = \mathcal{X}$, the process terminates with $r = 1$. Otherwise, in terms of the homomorphism $\pi_{\mathcal{N}_1} : \mathsf{L} \to \mathfrak{gl}(\mathcal{X}/\mathcal{N}_1)$, solvability of L implies solvability of $\bar{\mathsf{L}}_{\mathcal{N}_1} = \pi_{\mathcal{N}_1}(\mathsf{L})$ and so the above argument can be repeated on the vector space $\mathcal{X}/\mathcal{N}_1$. Reasoning similar to that above involving an application of Theorem 3.3 to the solvable Lie algebra $\bar{\mathsf{L}}_{\mathcal{N}_1}$ allows us to conclude that the largest subspace of $\mathcal{X}/\mathcal{N}_1$ that is invariant under both $\bar{g}_1$ and $\bar{A}_1$ and contained in $\bar{\mathcal{L}}_{\mathcal{N}_1}$, denoted $\bar{\mathcal{N}}_2$, is nonzero. We then set $\mathcal{N}_2 = \Pi^{-1}_{\mathcal{N}_1}\bar{\mathcal{N}}_2$ from which it follows that $\mathcal{N}_1 = \text{Ker}\,\Pi_{\mathcal{N}_1} \subset \mathcal{N}_2$. To argue $g-$invariance of $\mathcal{N}_2$, we have

$$\Pi_{\mathcal{N}_1} g \mathcal{N}_2 = \bar{g}_1 \Pi_{\mathcal{N}_1}\mathcal{N}_2 \subset \Pi_{\mathcal{N}_1}\mathcal{N}_2$$

from which

$$g\mathcal{N}_2 \subset \Pi^{-1}_{\mathcal{N}_1}\Pi_{\mathcal{N}_1}\mathcal{N}_2 = \mathcal{N}_2 + \text{Ker}\,\Pi_{\mathcal{N}_1} = \mathcal{N}_2$$

$A-$invariance of $\mathcal{N}_2$ can be argued similarly and so the first condition of the theorem is satisfied for $i = 2$. Because the canonical projection $\Pi_{\mathcal{N}_1}$ is surjective, $\mathcal{N}_2/\mathcal{N}_1 := \Pi_{\mathcal{N}_1}\mathcal{N}_2 = \bar{\mathcal{N}}_2$, and the second condition of the theorem is satisfied for $i = 2$.

Suppose for $q > 2$, there exist subspaces

$$0 = \mathcal{N}_0 \subset \mathcal{N}_1 \subset \cdots \subset \mathcal{N}_q$$

that satisfy the conditions of the theorem for $i = 1, \ldots, q$. If $\mathcal{N}_q = \mathcal{X}$, the process terminates with $r = q$. Otherwise, in terms of the homomorphism $\pi_{\mathcal{N}_q} : \mathsf{L} \to \mathfrak{gl}(\mathcal{X}/\mathcal{N}_q)$, solvability of L implies solvability of the $\bar{\mathsf{L}}_{\mathcal{N}_q} = \pi_{\mathcal{N}_q}(\mathsf{L})$ and so the above argument can be repeated on the vector space $\mathcal{X}/\mathcal{N}_q$. In particular, the largest subspace of $\mathcal{X}/\mathcal{N}_q$ that is invariant under both $\bar{g}_q$ and $\bar{A}_q$ and contained in $\bar{\mathcal{L}}_{\mathcal{N}_q}$, denoted $\bar{\mathcal{N}}_{q+1}$, is nonzero. We then set $\mathcal{N}_{q+1} = \Pi^{-1}_{\mathcal{N}_q}\bar{\mathcal{N}}_{q+1}$ from which it follows that $\mathcal{N}_q = \text{Ker}\,\Pi_{\mathcal{N}_q} \subset \mathcal{N}_{q+1}$. Invariance of $\mathcal{N}_{q+1}$ under g and A can be argued as above for the $i = 2$ case and so the first condition of the theorem is satisfied for $i = q+1$. Because the canonical projection $\Pi_{\mathcal{N}_q}$ is surjective, $\mathcal{N}_{q+1}/\mathcal{N}_q := \Pi_{\mathcal{N}_q}\mathcal{N}_{q+1} = \bar{\mathcal{N}}_{q+1}$ so that the second condition of the theorem is satisfied for $i = q+1$. This concludes the induction argument thereby establishing the necessity of the conditions in the theorem statement. ■

Example 4.3: We consider $\mathcal{X} = \mathbb{C}^4$ and maps having the following matrix representations with respect to the standard basis $\{e_1, e_2, e_3, e_4\}$

$$g = \begin{bmatrix} -1 & 2 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ -1 & 3 & 1 & 2 \\ 0 & 1 & -1 & 1 \end{bmatrix},\ A = \begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & -1 & 0 \\ -1 & 2 & 0 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

We apply Theorem 4.2 to assess whether the Lie algebra L generated by the set $\mathfrak{G}$ in (4) is solvable. For $i = 1$, we find that $\mathcal{L} = \text{span}\{e_1 + e_3, e_2 - e_4\}$ is a $2-$dimensional subspace that is itself invariant under g and A and thus coincides with $\mathcal{N}_1$. The associated canonical projection is given by

$$\Pi_{\mathcal{N}_1} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

This yields induced maps

$$\bar{g}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

for which $\bar{\mathcal{L}}_1 = \mathcal{X}/\mathcal{N}_1$. Consequently, we take $\bar{\mathcal{N}}_2 = \mathcal{X}/\mathcal{N}_1$ from which $\mathcal{N}_2 = \Pi^{-1}_{\mathcal{N}_1}\bar{\mathcal{N}}_2 = \mathcal{X}$. Thus, the conditions of Theorem 4.2 are satisfied and we conclude that the associated Lie algebra is solvable. We may represent $\mathcal{N}_2$ as $\mathcal{N}_2 = \mathcal{N}_1 \oplus \text{span}\{e_3, e_4\}$ yielding a basis for $\mathbb{C}^4$ adapted to $\mathcal{N}_1 \subset \mathcal{N}_2$ in which g and A have the matrix representations

$$g = \left[\begin{array}{cc|cc} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right], \quad A = \left[\begin{array}{cc|cc} -1 & 0 & -1 & 2 \\ 0 & -2 & 1 & 0 \\ \hline 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The submatrices in g and A on each block diagonal position generate an Abelian Lie algebra. The linear impulsive system (1) specified by $A_{\mathcal{C}} = A$, $A_{\mathcal{I}} = g$ satisfies the conditions of [5, Theorem 4.4] and is therefore uniformly exponentially stable.

V. Concluding Remarks

This paper has furthered the investigation of the Lie algebra associated with a linear impulsive system. Necessary and sufficient conditions have been formulated under which this Lie algebra is solvable, which is of interest in connection with analyzing stability characteristics and, perhaps, other system properties such as reachability and observability. In contrast to previous work, the conditions derived herein are constructive and can be checked using basic matrix computations.

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