Graphs and Matlab

Graphs are introduced along with their adjacency matrix.
Paths are defined and an algorithm presented for counting paths of length n.

Prerequisites:
Matrix multiplication.
A little knowledge of the transpose of a matrix.

We use the Matlab file graph.m.

Graph :
- a finite collection of vertices or nodes together with
- a set of edges which join certain pairs of vertices.

Think of the edges of a graph as :
- two-way streets along which traffic that can flow in either direction.

The graph shown in Figure 1 has six vertices or nodes: P1, P2, P3, P4, P5, and P6.

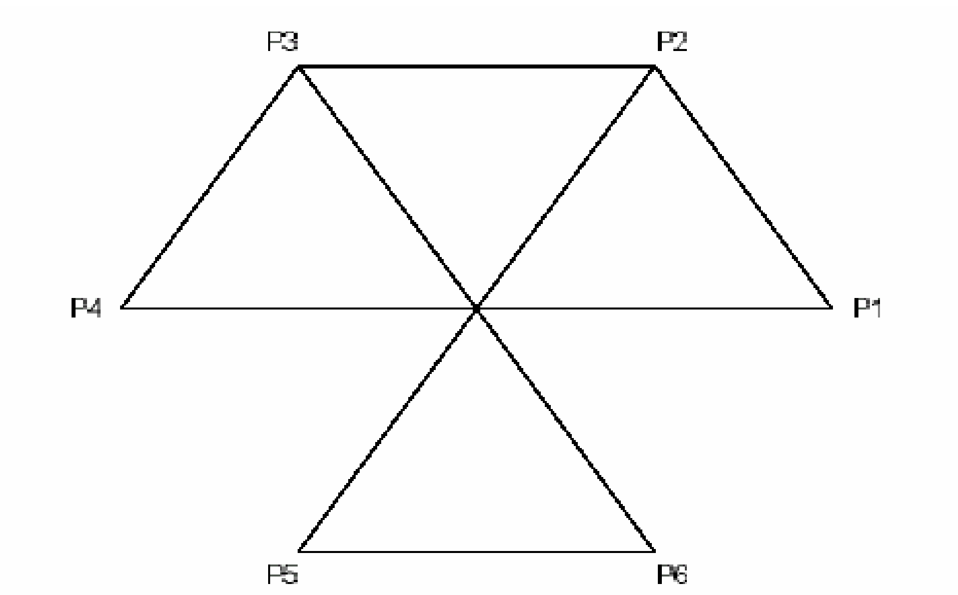


Figure 1. An undirected graph.

The edges of the graph in Figure 1 are the segments connecting :
P1 and P2,
P1 and P4,
P2 and P3,
P2 and P5,
P3 and P4, and
P5 and P6.

1.1 The Adjacency Matrix

Associated with each graph is an adjacency matrix A = (aij) which is constructed according to the following rule:

aij =
1 ,if there is an edge connecting Pj to Pi:
0 ,otherwise

The graph in Figure 1 has six vertices: P1, P2, P3, P4, P5, and P6.

Therefore, the adjacency matrix will have six rows and six columns.

On a notebook:
Create a blank template for the adjacency matrix.

Label the columns of the matrix with the names of the vertices.

Similarly, label the rows of the matrix with the names of the vertices, but place these on the right side of the matrix.

from						
P_1	P_2	P_3	P_4	P_5	P_6	to
						P_1
						P_2
						P_3
						P_4
						P_5
						P_6

Ones:

In Figure 1 there is an edge connecting P2 to P5, so a52 = 1.
This same edge connects P5 to P2, so a25 = 1 also.

Zeros:

There is no edge connecting P3 to P1, so a13 = 0.
There is no edge connecting P6 to P6, so a66 = 0.

from						
P_1	P_2	P_3	P_4	P_5	P_6	to
		0				P_1
				1		P_2
						P_3
						P_4
	1					P_5
					0	P_6

Use the graph in Figure 1 to fill in the remainder of the adjacency matrix as follows.

from						
P_1	P_2	P_3	P_4	P_5	P_6	to
0	1	0	1	0	0	P_1
1	0	1	0	1	0	P_2
0	1	0	1	0	1	P_3
1	0	1	0	0	0	P_4
0	1	0	0	0	1	P_5
0	0	1	0	1	0	P_6

Therefore, the adjacency matrix for the digraph in Figure 2 is :

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Note that the matrix A is symmetric with respect to its main diagonal.

This is so because
- an edge connecting Pi to Pj implies that there is also an edge connecting Pj to Pi.

You can easily check for symmetry by taking the transpose of the matrix.

If $A^T = A$, then the matrix A has to be symmetric
.

Enter the matrix A.

```
<<A=[0 1 0 1 0 0;1 0 1 0 1 0;0 1 0 1 0 1;1 0 1 0 0 0;0 1 0 0 0 1;0 0 1 0 1 0]
```

A=

```
0     1     0     1     0     0
1     0     1     0     1     0
0     1     0     1     0     1
1     0     1     0     0     0
0     1     0     0     0     1
0     0     1     0     1     0
```

Take the transpose of the matrix A.

<<A.
ans=

0	1	0	1	0	0
1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	0	0
0	1	0	0	0	1
0	0	1	0	1	0

Note that matrix A^T is identical to the matrix A.

This is always the case when the matrix is symmetric with respect to its main diagonal.

1.2 Checking with the M-file graph.m

Using a Matlab M-file called graph.m.

If you
- provided the adjacency matrix as input,

then
- the function graph.m will draw the associated graph.

The vertices are arranged on the unit circle to simplify the programming of graph.m.

The first vertex is placed at (1; 0) and the remaining vertices are spaced at equal increments along the unit circle in a counter-clockwise direction.

In practice, you can arrange your nodes or vertices in any geometric pattern you wish.

In fact, if your vertices represent major airports in the United States, you might want to get a map of the country and place each node atop the city where the airport is located.

It is easy to use the file graph.m to draw a graph based on the adjacency matrix A.

<<graph(A)

2. Paths

A sequence of edges connecting vertex Pi to Pj is called a path.

The length of a path is the number of edges in the sequence.

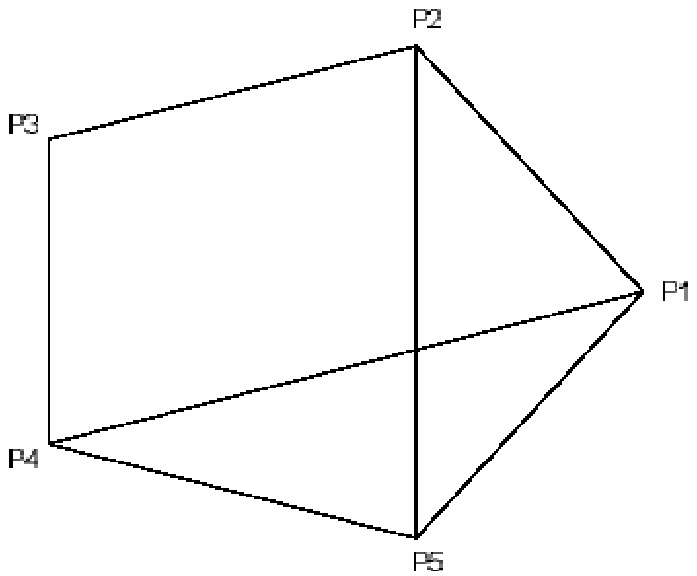


Figure 2.

In Figure 2, the single edge P1P2 connecting P1 to P2 is a path of length one.

The path from P1 to P2 via the vertex at P5 uses the edges P1P5 and P5P2 and has length two.

The path from P1 to P2 via vertices P4 and P3 uses the edges P1P4, P4P3, and P3P2 and has length 3.

A path is called a simple path if no edge is traversed more than once.

2.1 Counting Paths of Length Two

The adjacency matrix for the graph in Figure 2 is :

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Use Matlab to compute A^2

.

```
<<A=[0 1 0 1 1;1 0 1 0 1;0 1 0 1 0;1 0 1 0 1;1 1 0 1 0]
A=

0    1    0    1    1
1    0    1    0    1
0    1    0    1    0
1    0    1    0    1
1    1    0    1    0

<<A^2

ans =

     3    1    2    1    2
     1    3    0    3    1
     2    0    2    0    2
     1    3    0    3    1
     2    1    2    1    3
```

Note that there are 2 paths of length two from P1 to P5:

-the path from P1 to P2 to P5, and

-the path from P1 to P4 to P5. (see graph)

Note that the element: fifth row, first column of A^2 is 2, the number of paths of length two from P1 to P5.

There are 3 paths of length two from P2 to P2:
-the path from P2 to P1 to P2,
-the path from P2 to P3 to P2, and
-the path from P2 to P5 to P2 (see graph).

The element in the second row, second column of A^2 is 3, the number of paths of length 2 from P2 to P2.

It is no coincidence that the number of paths of length two can be found be examining the square of the adjacency matrix.

In general, if $a^{(2)}_{ij}$ is the element in the ith row and jth column of A^2, then $a^{(2)}_{ij}$ is the number of paths of length two from Pj to Pi.

2.2 Counting the Paths of Length N

In general, if $a^{(N)}_{ij}$ is the element in the ith row, jth column of matrix A^N, then $a^{(N)}_{ij}$ is the number of paths of length N from Pj to Pi.

For example, use Matlab to compute A^3.

```
<<
A^3

ans =

     4     7     2     7     5
     7     2     6     2     7
     2     6     0     6     2
     7     2     6     2     7
     5     7     2     7     4
```

The element in the first row, fourth column of A3 is 7.

Therefore, there are **7** paths of length three from P4 to P1:

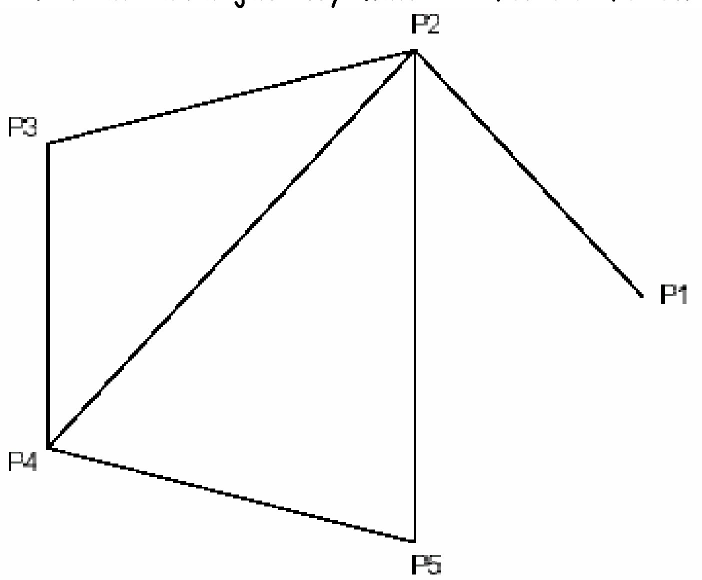
- $4 \rightarrow 3 \rightarrow 2 \rightarrow 1$
- $4 \rightarrow 3 \rightarrow 4 \rightarrow 1$
- $4 \rightarrow 1 \rightarrow 4 \rightarrow 1$
- $4 \rightarrow 5 \rightarrow 4 \rightarrow 1$
- $4 \rightarrow 5 \rightarrow 2 \rightarrow 1$

(the path from P4 to P3 to P2 to P1.)

There are two more paths of length three. Can you find them?

3. Homework

.1. Create an adjacency matrix A for the following graph.



Check your result with graph.m.

Check that matrix A is symmetric with respect to its main diagonal.

Show that A and A^T are equal.

.2. Draw (notebook) the graph associated with the adjacency matrix :

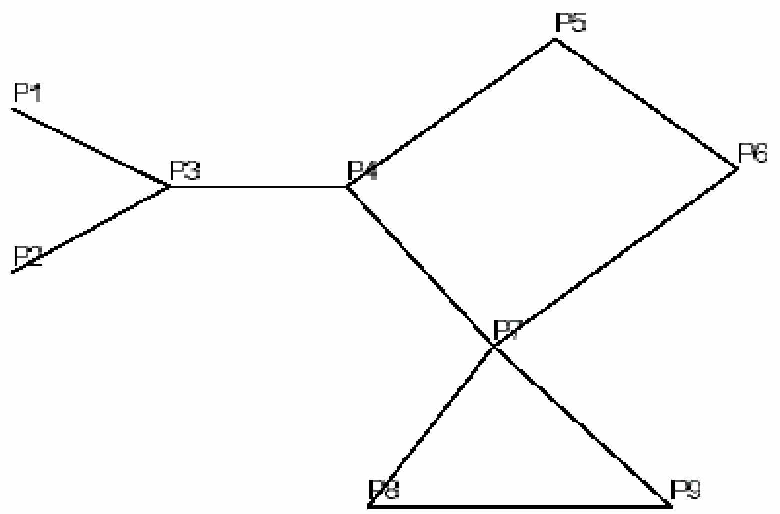
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Use the M-file graph.m to check your result.

.3. What are the remaining two paths of length three in Figure 2 from P4 to P1?

.4. Use Matlab to compute the number of paths of length four from P2 to P3 in Figure 2. List each of these paths.

.5. Find the number of paths of length 9 from P1 to P9 in the following graph.



.6. If A is an adjacency matrix for a graph and $R = A + A^2 + A^3$, then r_{ij} represents the number of paths from P_j to P_i of length 1, 2, or 3.

(a) Let A be the adjacency matrix for the graph in Figure 2. Use Matlab to compute $R = A + A^2 + A^3$.

(b) What is r_{25}?

(c) The element r_{25} should represent the number of paths of length 1, 2, or 3 from P5 to P2:

(c1) List all paths of length 1 from P5 to P2.

(c2) List all paths of length 2 from P5 to P2.

(c3) List all paths of length 3 from P5 to P2.

(c4) Is the total number of paths of length 1, 2, or 3 from parts i, ii, and iii equal to r_{25}?