Solutions of Daniel Bump's *Automorphic Forms & Representations*

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This document is the complete solution of the exercises in *Automorphic Forms and Representations* by Prof. David Bump. During this work, I partially referred to *A First Course in Modular Forms* by Prof. Fred Diamond for the first chapter, and *Automorphic Representations and L-Functions for the General Linear Group* by Prof. Dorian Goldfeld and *Automorphic Forms on GL(2)* by Prof. Hervé Jacquet and Prof. Robert Langlands for the rest chapters. And for some exercises I cannot do and also cannot find the solutions on books and internet, I ask my friend Jiang Lý-Jiang for them. By the way, my friend Jiang Lý-Jiang est un célibataire charmant. Se iu damo ŝatas lerta, serioza, respondeca kaj malmulta ascetika viro, können Sie mir eine E-Mail für einige Informationen schicken.

Automorphic forms tsjiàkà dongshi 吧, ŋǒ pwónlai should learn it en mia universitato. 2016 nian pwónkhwa dah-4 조-l 중과원-d Den Jiá kyōjyu 做 pit-sjiet, Den láuprán grandaj-manoj-1-svingas 就 kyip-l ŋ 17 nian-dii Triang Wií & ʔwiín Tsjy-Wií 2-zūn Dieux 才 pwiat d plej nova solvo ŋ̡iàng wǒ kann. Nah-geh dzjyghòu ŋǒ (1) pwiitghwàj [ag] (2) pwiitghwàj [ar], khàntàu luennwen des dieux d krénkrèi buhfenn d 時候 driktsiep hōkai. Jyeguoo Den láuprán grandaj-manoj-1-svingas yow kyip-l 窩小 1-pwón *Number Theory in Function Fields* by Michael Rosen ŋ̡iàng-ŋá ɸɯjɯjasɯmi ghwai-kra kann. Kràgy kann corps de fonctions 倒是 mran-sriáng-t, 1-hwei ghrokghràu 就 각득 'iàusyí-l.

Kràgy hweilai 先 kaiʃi (ŋé)-buu [ag], pitkiàeng jyoder [g]-d dongshi khànpwiit-túng 還 keeyii khàu 상상, kekka lien scheme dou pwiit-túng-t ŋǒ 1-dzjiàngsjióu 就 khàn stack, 간진 jiowshyh tshjiongphiong 在 tsaksyí-t dih-1-shiann. Khàn-l 1gè-yó shyrtzay khàn(pwiit-túng) 就 keh khàn (af), senmwiìn Jiang Lý-Jiang ŋoo keeyii khàn quel livre 'a, 偲 shuo Den láuprán suisen kann the Bump one, ketkwá wǒ 根本 khàn(pwiit-túng). Krek-l-1-then wǒ yowchiuh 문 Den láuprán, 偲 jyrjie sjwiet: "迺 keeyii driktsiep kàn Bump mra, mwiándzjiàng 수교-chyan taang dzriangdzjiàng fān-1-fān dziòu kann-wãl." 結果 я dokbyet mei-jyetsau-d khàn-l the Gelfand one, tiel daw ghèndzài 還 pwiitghwàj [af].

Mia kara kontrolisto láuTshàj jyoder ŋǒ tzuoh tungsei estas tre malrapida, bràengtàu [af]-t tungsei yow pwiitghwàj, 就 문-ŋá 'iàu-pwiit-'iàu shian haohau khàn ein Buch, 窩小 sjwiet oui~oui~, jyeguoo 就 khaisjý mann-mann 쇄 the Bump. Bump tsjiàsjio bå, ganjyo dzau-tém gen Hartshorne

1-jiàng, mwotŋ̩in day mei ky-tshrió d genbeen srwat(pwiitdúng).

저서 dèi-1-tsjiang shànglái 就 kaktsjióng 고능, kannshanqchiuh ŋ̩iokwá 1-점 quadratic field-d L-function 和 modular form doumeiyeou khankwà-t-ghrwàj 감각 konpwón 刷 búdòng. ʔitpen 간서 'itpen chi-laobeen bǎ ぜんめん 3-4-절 tzuo-hl tsjyghòu 就 개시 srwat-pwiitdúng, 在 Rankin-Selberg her Maass form shàngmián haw-l tshra-pwiit-ta 1개다월-d jiàngtsý, ŋ̩ienɣòu 就 tsiepdzjiáng-l 만악-t $GL(2,\mathbb{R})$-t beaushyh. Dèi-2-tsjiang 1-jeengjang 感覺 취시 kaktsjióng functional analysis 在 liŋŋiak ŋ̩in, ŋǒ pwónlai jiowshyh 淆 [a] 淆-pwiit-ghrákhiò 纔 lai 淆 [nt]-d ʐen, biaengdzjy kráu-kráu L-function-&-ζ-function-d [ca]-d 성질 就 tshra-pwiit-ta yaw ŋǎ 반조명-l, maintenant kráu $GL(2,\mathbb{R})$-shanqmiann-d [a] 진적 simlý 全是 /mʰamaʲpʰi/. Tsjintek, 還 기득 dih-2-jang ióukà deimiuk 要 조 1-geh non-symmetric 판본-d Green function, jyeguoo ngwianlai-t 대칭-t pwiangpwiap 根本 thwaik-wáng b-chulai, 問 您-m kráu-[pde]-t dahlao 來 폭력 swán yee swán-p-tshjwitlai, 真的 dzra-l hao-jiitian 纔 @ móupwónsjio 上 조도, 연후 jiow 발현 deimiuk-t hint genbeen mwot-jiòng, chihder 요사.

Brachmond khiò-Srentung-t 시후 bã dì-3-jāng dì-1-jié Tate's Thesis kráughwan-l jyhow 취 卡在 §3.2-t móu-1-kà tyimuh 상-l. 회래 jyhow Juli l meibannfaa 就 개시 淆 Ænglisch-l, kao-l 2-3-kà-ngwiat-t 雞阿姨 her Toefl 지후 bey láu-Tshàj juajuh 강-l Liou Jio-t seminar tsjyghòu 又 중신 개시 shua Bump. Ketkwá yow 在 dung-1-kà 문제 shanq 卡-l 반천. Kháughwan gresub jyhow 又 신경 jiauliuh 몰법 淆zip 又 kaishyy srwat duo, driktàu 所有 분삭 dou tshjwitlai jyhow 纔 진적 ióu hao 심태 lái srwat Bump, ketkwá 1-개다월 dziòu baa ghòu-2-tsjiang shuawan-l. 不過 pwiittok buhshuo Liou Jio her Wiang Gháu-Ñien nahgeh 과 jende 挺 yeouyonq-t, 불연 Bump 這個 siásjio fangfǎ dèi-3-tsjiang & dèi-4-tsjiang 호상 yiinyonq konpwón have no 열독 순서 t 자료 chu-淆 genbeen khan-pwiittúng. Wiang Gháu-Njien yonq nàpwón *Local Langlands on* GL(2)-t-sjio-t-pwiangpwiap gǎŋɣʷán local-t representation jyhow ŋǒ 再 종신 개시 kann §3.2, tsjyghòu jende huòrán khailáng. Therefore, dàjkài hairshyh 要 anti-Amway 1-shiah tsjiàsjio 真的 pwiit-sjiekghop chu淆 kann baʰ, ghwandzwien b-liijiee láu-Den nàjǒng "수전 phwian-phwian shu 就 khan-ghwan Bump" shyh tsryím-m̩ kráu-t. 진적, shin háulwiè ʔo ŋǒ kio-ŋ̩ien 쇄-l 2-pwón genbeen pwiit-sjiekghop yonqlai ʃɯ-淆 d sjio.

Finally, if anybody finds any mistakes, please e-mail me.

又做了一點微小的工作，謝謝大家！

Content

1 Modular Forms

1.1 Dirichlet L-functions

Solution 1.1.1. Clearly,

$$\tau(\bar{\chi}) = \sum_{n \bmod N} \overline{\chi(n)} e^{2\pi i n/N} = \sum_{(n,N)=1} \overline{\chi(-1)} \cdot \overline{\chi(-n)} \cdot \overline{e^{2\pi i(-n)/N}} = \overline{\chi(-1)} \cdot \overline{\tau(\chi)}$$

Since $\chi(-1) = \pm 1$, we have $\overline{\chi(-1)} = \chi(-1)$, i.e. $\tau(\bar{\chi}) = \chi(-1)\overline{\tau(\chi)}$.

Solution 1.1.2 (Dirichlet)**.** (a) By Hadamard's formula, the radius of convergence of the series$=$ $(\limsup n^{-1/n})^{-1} = 1$. Hence when $|x| < 1$, $-\log(1-x)$ is valid. When $|x| = 1$ but $x \neq 1$, since $n^{-1} \to 0$, and $|\sum_{n=1}^{N} x^n| = |\frac{x^n}{1-x}| = |1-x|^{-1}$ is bounded, by Dirichlet's criterion, $\sum_n \frac{x^n}{n}$ is conditionally convergent.

(b) By Eq. (1.7), we have

$$\begin{aligned} L(1,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} &= \frac{\chi(-1)\tau(\chi)}{N} \sum_{n=1}^{\infty} \sum_{m \bmod N} \frac{\overline{\chi(m)} e^{2\pi i n m/N}}{n} \\ &= \frac{\chi(-1)\tau(\chi)}{N} \sum_{m \bmod N} \left(\overline{\chi(m)} \sum_{n=1}^{\infty} \frac{e^{2\pi i n m/N}}{n} \right) \\ &= -\frac{\chi(-1)\tau(\chi)}{N} \sum_{m \bmod N} \overline{\chi(m)} \log(1 - e^{2\pi i m/N}) \end{aligned}$$

Moreover, if $\chi(-1) = 1$, we have

$$\begin{aligned} 2 \sum_{m \bmod N} \overline{\chi(m)} \log(1 - e^{2\pi i m/N}) &= \sum_{m \bmod N} \overline{\chi(m)} \left(\log(1 - e^{2\pi i m/N}) + \log(1 - e^{-2\pi i m/N}) \right) \\ &= 2 \sum_{m \bmod N} \overline{\chi(m)} \log|1 - e^{2\pi i m/N}| \end{aligned}$$

Hence $L(1,\chi) = -\frac{\tau(\chi)}{N} \sum_{m \bmod N} \overline{\chi(m)} \log|1 - e^{2\pi i m/N}|$. If $\chi(-1) = -1$, we have

$$\begin{aligned} 2 \sum_{m \bmod N} \overline{\chi(m)} \log(1 - e^{2\pi i m/N}) &= \sum_{m \bmod N} \overline{\chi(m)} \left(\log(1 - e^{2\pi i m/N}) - \log(1 - e^{-2\pi i m/N}) \right) \\ &= 2 \sum_{m \bmod N} \overline{\chi(m)} \frac{\pi i m}{N} = \frac{2\pi i}{N} \sum_{m \bmod N} \overline{\chi(m)} m \end{aligned}$$

Hence $L(1,\chi) = \frac{i\pi\tau(\chi)}{N^2} \sum_{m \bmod N} \overline{\chi(m)} m$.

Solution 1.1.3. (a) For the existence, the Legendre symbol $(\frac{\cdot}{p})$ is a quadratic character modulo p. For the uniqueness, since any morphism $\phi \in \mathrm{Hom}((\mathbb{Z}/(p))^*, \{\pm 1\})$, it is determined by $\phi(a)$, where

a is the generator of $(\mathbb{Z}/(p))^*$. Hence there only exist two quadratic character modulo p: a trivial one and the Legendre symbol.

(b) By definition of the Legendre symbol, trivial.

(c) Since $\sum_{m \mod p} e^{2\pi i m/p} = 0$, we have

$$\tau(\chi) = \sum_{m \mod p} \chi(m) e^{2\pi i m/p} = \sum_{m \mod p} (1 + \chi(m)) e^{2\pi i m/p} = \sum_{m=n^2 \mod p} e^{2\pi i n^2/p} = \sum_{n \mod p} e^{2\pi i n^2/p}$$

Solution 1.1.4. Consider the semicircle C consisting of $[-R, R]$ and $Re^{i\theta}$ for $\theta \in [0, \pi]$. When $v > 0$, we have

$$\left| \int_0^\pi (re^{i\theta} - \tau)^{-k} e^{2\pi i v r e^{i\theta}} \mathrm{d}\theta \right| \leq \int_0^\pi |re^{i\theta} - \tau|^{-k} e^{-2\pi v r \sin(\theta)} \mathrm{d}\theta \ \leq \pi |r - |\tau||^{-k} \xrightarrow{r \to \infty} 0$$

Hence when $v > 0$,

$$\hat{f}(v) = \lim_{r \to \infty} \int_{-r}^{r} f(u) e^{2\pi i u v} \mathrm{d}u \ = \lim_{r \to \infty} \int_C f(u) e^{2\pi i u v} \mathrm{d}u \ = 2\pi i \ \mathrm{res}_{u=\tau} e^{2\pi i u v} (u - \tau)^{-k}$$

When $v \leq 0$, changing the coordinate $u \mapsto -u$, $v \mapsto -v$, then

$$\hat{f}(v) = \int_{-\infty}^{\infty} f(u) e^{2\pi i u v} \mathrm{d}u \ = \int_{-\infty}^{\infty} f(-u) e^{2\pi i (-u)(-v)} \mathrm{d} - u \ ) = - \int_{-\infty}^{\infty} f(-u) e^{2\pi i u v} \mathrm{d}u$$

Since τ is in the upper half plane, hence the semicircle C does not contain the singular point $-\tau$. By residue number theorem, $\hat{f}(v) = 0$. So we have

$$\hat{f}(v) = \begin{cases} \dfrac{(2\pi i)^k}{(k-1)!} v^{k-1} e^{2\pi i v \tau} & \text{if } v > 0 \\ 0 & \text{if } v \leq 0 \end{cases}$$

By Poisson's formula, we have $\sum_{n=-\infty}^{\infty} (n - \tau)^{-k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n \tau}$.

Solution 1.1.5 (Quadratic Fields)**.** If γ, δ is another $\mathbb{Z}$-basis, since $\mathfrak{o}_K \cong \mathbb{Z} \oplus \mathbb{Z}$, there exists a $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbb{Z})$ such that $(\alpha, \beta) = (\gamma, \delta) \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and $\left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| = \pm 1$. Since $\alpha' = (a\gamma + c\delta)' = a\gamma' + c\delta'$ and same for β, we have

$$D_K = \left|\begin{smallmatrix} \alpha & \beta \\ \alpha' & \beta' \end{smallmatrix}\right|^2 = \left|\begin{smallmatrix} \gamma & \delta \\ \gamma' & \delta' \end{smallmatrix} \begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right|^2 = \left|\begin{smallmatrix} \gamma & \delta \\ \gamma' & \delta' \end{smallmatrix}\right|^2$$

So this definition is independent of the choice of the basis. Moreover, since clearly $D_K \in \mathfrak{o}_K$, we have

$$D_K' = \left|\begin{smallmatrix} \alpha' & \beta' \\ \alpha & \beta \end{smallmatrix}\right|^2 = D_K$$

i.e. D_K is invariant under the nontrivial Galois automorphism of $K/\mathbb{Q}$. So we have $D_K \in \mathbb{Z}$.

Solution 1.1.6 (Fundamental Discriminants)**.** (a) When $q = p$ is an odd prime, by 1.1.3., there exists only one nontrivial primitive quadratic character, a.k.a. the Legendre symbol. When $q = p^n$, then similarly with 1.1.3., we may take a generator a of $(\mathbb{Z}/(p^n))^*$, then there exists only one nontrivial quadratic character such that $\chi(a) = -1$. But $\chi(a^{p-1}) = 1$ since p is odd, hence this character is not primitive. When $q = 4$, since $(\mathbb{Z}/(4))^* = \{\pm 1\}$, the unique nontrivial primitive quadratic character is $\chi(1) = 1$ and $\chi(-1) = -1$. When $q = 8$, we have

	1	3	5	7
χ_1	1	1	1	1
χ_2	1	1	−1	−1
χ_3	1	−1	−1	1

Hence they exist but are not unique. When $q = 2^n$ for $n \geq 4$, if there exists a nontrivial primitive quadratic character χ, since $(2^{n-2}+1)^2 = 2^{2n-4} + 2^{n-1} + 1 \equiv 2^{n-1} + 1 \bmod 2^n$, we know $\chi(2^{n-1}+1) = \chi(2^{n-2}+1)^2 = 1$. But for any odd number a such that $1 \leq a \leq 2^{n-1} - 1$, we have $(2^{n-1}+1)a = 2^{n-1}a + a \equiv 2^{n-1} + a \bmod 2^n$, hence $\chi(2^{n-1}+a) = \chi(a)\chi(2^{n-1}+1) = \chi(a)$. So $\chi : (\mathbb{Z}/(2^n))^* \to \{\pm 1\}$ must factor through $(\mathbb{Z}/(2^{n-1}))^*$, i.e. not primitive.

(b) By Chinese remainder theorem, trivial. If $d = 2^n p_1 \dots p_k$ is a product of odd primes, 4, or 8, by (a) there exist nontrivial quadratic characters $\chi_p : (\mathbb{Z}/(p))^* \to \{\pm 1\}$, then we just need to define $\chi = \prod_{p|d} \chi_p$. Conversely, if we have a primitive quadratic character $\chi : (\mathbb{Z}/(d))^* \to \{\pm 1\}$, then clearly $\chi_p = \chi|_{(\mathbb{Z}/(p^n))^*}$ is a primitive character. Then by (a), trivial.

(c) By (b), we have the following cases: 1. If $|D| \equiv 1 \bmod 4$, then $\chi(-1) = (\frac{-1}{|D|}) = (-1)^{\frac{|D|-1}{2}} = 1$. Hence $\chi(a) = (\frac{a}{|D|}) = (\frac{a}{D}) = (\frac{D}{a})(-1)^{\frac{D-1}{2}\frac{a-1}{2}} = (\frac{D}{a})$. 2. If $|D| \equiv -1 \bmod 4$, then $\chi(-1) = (\frac{-1}{|D|}) = (-1)^{\frac{|D|-1}{2}} = -1$. Hence $\chi(a) = (\frac{a}{|D|}) = (\frac{|D|}{a})(-1)^{\frac{|D|-1}{2}\frac{a-1}{2}} = (\frac{D}{a})(-1)^{\frac{|D|-1}{2}\frac{a-1}{2}+\frac{a-1}{2}} = (\frac{D}{a})$. 3. If $|D| = 4N$ for some $N \equiv 1 \bmod 4$, then $\chi(-1) = \chi_4(-1)\frac{-1}{N} = -1$. Hence $\chi(a) = \chi_4(a)(\frac{a}{N}) = (-1)^{\frac{a-1}{2}}(\frac{N}{a})(-1)^{\frac{N-1}{2}\frac{a-1}{2}} = (\frac{D}{a})(\frac{-1}{a})(\frac{2}{a})^2(-1)^{\frac{a-1}{2}} = (\frac{D}{a})$. 4. If $|D| = 4N$ for some $N \equiv -1 \bmod 4$, then $\chi(-1) = \chi_4(-1)\frac{-1}{N} = 1$. Hence $\chi(a) = \chi_4(a)(\frac{a}{N}) = (-1)^{\frac{a-1}{2}}(\frac{N}{a})(-1)^{\frac{N-1}{2}\frac{a-1}{2}} = (\frac{D}{a})(\frac{2}{a})^2(-1)^{\frac{N-1}{2}\frac{a-1}{2}+\frac{a-1}{2}} = (\frac{D}{a})$. 5. If $|D| = 8N$ for some $N \equiv 1 mod 4$ and $\chi|_{(\mathbb{Z}/(8))^*} = \chi_2$ as in (a), we have $\chi(-1) = \chi_2(-1)(\frac{-1}{N}) = -1$. Hence $\chi(a) = (-1)^{\frac{a^2-4a+3}{8}}(\frac{a}{N}) = (\frac{N}{a})(-1)^{\frac{a^2-4a+3}{8}+\frac{N-1}{2}\frac{a-1}{2}} = (\frac{D}{a})(\frac{-1}{a})(\frac{2}{a})^3(-1)^{\frac{a^2-4a+3}{8}+\frac{N-1}{2}\frac{a-1}{2}} = (\frac{D}{a})(-1)^{\frac{a^2-4a+3}{8}+\frac{N-1}{2}\frac{a-1}{2}+\frac{a-1}{2}+\frac{a^2-1}{8}} = (\frac{D}{a})$. 6. If $|D| = 8N$ for some $N \equiv -1 \bmod 4$ and $\chi|_{(\mathbb{Z}/(8))^*} = \chi_2$ as in (a), we have $\chi(-1) = \chi_2(-1)(\frac{-1}{N}) = 1$. Hence $\chi(a) = (-1)^{\frac{a^2-4a+3}{8}}(\frac{a}{N}) = (\frac{N}{a})(-1)^{\frac{a^2-4a+3}{8}+\frac{N-1}{2}\frac{a-1}{2}} = (\frac{D}{a})(\frac{2}{a})^3(-1)^{\frac{a^2-4a+3}{8}+\frac{N-1}{2}\frac{a-1}{2}} = (\frac{D}{a})(-1)^{\frac{a^2-4a+3}{8}+\frac{N-1}{2}\frac{a-1}{2}+\frac{a^2-1}{8}} = (\frac{D}{a})$. 7. If $|D| = 8N$ for some $N \equiv 1 \bmod 4$ and $\chi|_{(\mathbb{Z}/(8))^*} = \chi_3$ as in (a), we have $\chi(-1) = \chi_3(-1)(\frac{-1}{N}) = 1$. Hence $\chi(a) = \chi_3(a)(\frac{a}{N}) = (\frac{N}{a})(-1)^{\frac{a^2-1}{8}+\frac{N-1}{2}\frac{a-1}{2}} = (\frac{D}{a})(\frac{2}{a})^3(-1)^{\frac{a^2-1}{8}+\frac{N-1}{2}\frac{a-1}{2}} = (\frac{D}{a})(-1)^{\frac{a^2-1}{8}+\frac{N-1}{2}\frac{a-1}{2}+\frac{a^2-1}{8}} = (\frac{D}{a})$. 8. If $|D| = 8N$ for some $N \equiv -1 \bmod 4$ and $\chi|_{(\mathbb{Z}/(8))^*} = \chi_3$ as in (a), we have $\chi(-1) = \chi_3(-1)(\frac{-1}{N}) = -1$. Hence $\chi(a) = \chi_3(a)(\frac{a}{N}) = (\frac{N}{a})(-1)^{\frac{a^2-1}{8}+\frac{N-1}{2}\frac{a-1}{2}} = (\frac{D}{a})(\frac{-1}{a})(\frac{2}{a})^3(-1)^{\frac{a^2-1}{8}+\frac{N-1}{2}\frac{a-1}{2}} = (\frac{D}{a})(-1)^{\frac{a^2-1}{8}+\frac{N-1}{2}\frac{a-1}{2}+\frac{a-1}{2}+\frac{a^2-1}{8}} = (\frac{D}{a})$.

(d) By Kronecker-Weber's theorem, there exists a (minimal) cyclotomic field $\mathbb{Q}(\zeta_m)$ such that $\mathbb{Q} \subset K \subset \mathbb{Q}(\zeta_m)$. Since $G = \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) = (\mathbb{Z}/(m))^*$, we have $\hat{G} = \{$all primitive Dirichlet character modulo m$\}$. Then there exists a qradratic character χ such that $G/\chi^{-1}(1) \cong \mathrm{Gal}(K/\mathbb{Q})$. So $K =$

$\mathbb{Q}(\sqrt{\chi(-1)m})$. Since χ is a primitive character, K is uniquely corresponding to it, by (c), there exists a unique fundamental discriminant D such that $K = \mathbb{Q}(\sqrt{D})$.

(e) As we've discussed in (c), $D \equiv 0$ or $1 \bmod 4$. If $D \equiv 0 \bmod 4$, we may assume $D = 4N$, then $K = \mathbb{Q}(\sqrt{N})$. Then by (b), N is square-free, by (c), we know $N \equiv 2$ or $3 \bmod 4$. So clearly an integral basis is $(1, \sqrt{N}) = (1, \tau)$. If $D \equiv 1 \bmod 4$, we clearly know that the integral basis is $(1, \frac{\sqrt{D}+1}{2}) = (1, \tau)$ since τ has the minimal polynomial $\tau^2 - \tau = \frac{D-1}{4}$. Moreover, $\mathrm{Disc}(K/\mathbb{Q}) = (\tau - \sigma(\tau))^2 = D$, where $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ is the nontrivial one.

(f) If $\chi_D(p) = 0$, then $p|D$, so p ramifies in $\mathbb{Q}(\sqrt{D}) = K$. If $\chi_D(p) = 1$, by definition in (e), for $\sigma_p \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ we have $\chi_D(\sigma_p) = 1$. Hence $(p) = \mathfrak{P}_1\mathfrak{P}_2$ with $\mathfrak{P}_1 = \sigma(\mathfrak{P}_2)$, i.e. p splits in K. If $\chi_D(p) = -1$, by definition in (e), the σ_p can induce σ, hence p remains prime.

Solution 1.1.7. Just use the normal theta function $\theta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z}$. Then $\theta(t) = t^{-1/2}\theta(1/t)$. And $\int_0^\infty \theta(t) t^{s/2} \frac{\mathrm{d}t}{t}$ is convergent for all $s \neq 1$. Then since $\int_0^\infty e^{-\pi t n^2} t^{s/2} \frac{\mathrm{d}t}{t} = \pi^{-s/2}\Gamma(\frac{s}{2})n^{-s}$, we have $Z(s) = Z(1-s)$, where $Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

Solution 1.1.8 (Riemann (1892))**.** (a) Since there does not exist any pole out of zero, this integral is independent of the radius of the circle around zero. Hence for $\mathrm{Re}(s) > 0$, we have

$$\int_C (-x)^{s-1} e^{-x} \mathrm{d}x = \lim_{r\to 0} \int_{C_r} (-x)^{s-1} e^{-x} \mathrm{d}x = (e^{-\pi i s} - e^{\pi i s}) \int_0^\infty t^{s-1} e^{-t} \mathrm{d}t = -2i\sin(\pi s)\Gamma(s)$$

So for all $s \in \mathbb{C}$, we have $\frac{1}{\Gamma(1-s)} = \frac{i}{2\pi} \int_C (-x)^{s-1} e^{-x} \mathrm{d}x$.

(b) Since for $\mathrm{Re}(x) > 0$, we have

$$\int_C \frac{(-x)^{s-1}}{e^x - 1} \mathrm{d}x = \sum_{n=1}^{\infty} \int_C (-x)^{s-1} e^{nx} \mathrm{d}x = \sum_{n=1}^{\infty} n^{-s} \int_C (-nx)^{s-1} e^{nx} \mathrm{d}(nx) = \sum_{n=1}^{\infty} \frac{-2\pi i n^{-s}}{\Gamma(1-s)}$$

so we have

$$-\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-x)^{s-1}}{e^x - 1} \mathrm{d}x = \sum_{n=1}^{\infty} n^{-s} = \zeta(s)$$

for all $s \in \mathbb{C}$.

(c) By 1.1.7., we have

$$\zeta(n) = \zeta(1-n)\pi^{n-\frac{1}{2}} \cdot \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}$$

If n is a negative even integer, since $1 - s > 1$, we have $\Gamma(\frac{1-s}{2}) > 0$ and $\zeta(1-s) > 0$. But since $\frac{s}{2}$ is a negative integer, $\Gamma(\frac{s}{2})$ is a pole. So $\zeta(n) = 0$.

If n is a positive even integer, we have

$$\zeta(1-n) = -\frac{\Gamma(n)}{2\pi i} \int_C \frac{(-x)^{-n}}{e^x - 1} \mathrm{d}x = -\frac{(n-1)!}{2\pi i} 2\pi i \mathrm{res}_{x=0} \frac{(-x)^{-n}}{e^x - 1} = -(n-1)! \cdot \frac{B_n}{n!} = -\frac{B_n}{n}$$

(d) If n is a positive even integer, we have

$$\zeta(n) = \zeta(1-n)\pi^{n-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} = -\frac{2^{n-1}\pi^n(-1)^{n/2}B_n}{n!}$$

Solution 1.1.9. Is this a question?

Solution 1.1.10. (a) If $\chi(-1) = 1$, we have the functional equation

$$L(s,\chi) = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\tau(\chi)N^{-s}L(1-s,\bar{\chi})$$

So $L(0,\chi) = \pi^{-1/2}\frac{\sqrt{\pi}}{\Gamma(0)}\tau(\chi)L(1,\bar{\chi})$. Since $\tau(\chi)L(1,\bar{\chi})$ is not zero, and Γ has a simple pole at 0, we know $L(0, chi)$ is a simple zero. If $\chi(-1) = -1$, we have the functional equation

$$L(s,\chi) = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s+\epsilon}{2})}{\Gamma(\frac{s+\epsilon}{2})}(-i)\tau(\chi)N^{-s}L(1-s,\bar{\chi})$$

So $L(0,\chi) = \pi^{-1/2}\frac{1}{\sqrt{\pi}}(-i)\tau(\chi)L(1,\bar{\chi}) = -\frac{\tau(\chi)}{\pi}L(1,\bar{\chi})$ is nonzero.

(b) Consider the representation $\rho : \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \to \mathrm{GL}(1,\mathbb{C})$ commutating with the isomorphism $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^*$ and the Dirichlet character χ. So clearly $L(s,\rho) = L(s,\chi)$. Hence by (a), if $\chi(-1) = 1$, $L(s,\rho)$ has a simple zero at $s = 0$, i.e. $r = 1$, if $\chi(-1) = -1$, $L(s,\rho)$ is nonzero at $s = 0$, i.e. $r = 0$.

(c) Since $\sum_{m \bmod N}\overline{\chi(m)} = 0$, we have

$$\begin{aligned} L(1,\chi) &= -\frac{\chi(-1)\tau(\chi)}{N}\sum_{m \bmod N}\overline{\chi(m)}\log(1-e^{2\pi i m/N}) \\ &= -\frac{\tau(\chi)}{N}\sum_{m \bmod N}\overline{\chi(m)}(\log(1-e^{2\pi i m/N}) - \log(1-e^{2\pi i/N})) \\ &= -\frac{\tau(\chi)}{N}\sum_{m \bmod N}\overline{\chi(m)}\log\epsilon_m \end{aligned}$$

So

$$L(1,\chi) - \left(-\frac{\tau(\chi)}{N}\sum_{m \bmod N}\overline{\chi(m)}\log|\epsilon_m|\right) = -\frac{\tau(\chi)}{N}\sum_{m \bmod N}\overline{\chi(m)}\mathrm{Arc}(\epsilon_m) = -\frac{\tau(\chi)}{N}\sum_{m \bmod N}\overline{\chi(m)}\frac{m-1}{N}\pi$$

Since $\chi(-1) = 1$, i.e. $\chi(-m) = \chi(m)$, so we have $2\sum_{m \bmod N}\overline{\chi(m)}\frac{m-1}{N} = \sum_{m \bmod N}\overline{\chi(m)}(\frac{m-1}{N} + \frac{N-m-1}{N}) = \sum_{m \bmod N}\overline{\chi(m)}(\frac{N-2}{N}) = 0$. Hence $L(1,\chi) = -\frac{\tau(\chi)}{N}\sum_{m \bmod N}\overline{\chi(m)}\log|\epsilon_m|$.

1.2 The Modular Group

Solution 1.2.1. Simply,

$$\begin{aligned}
\operatorname{Im}(g(z)) &= \operatorname{Im}\left(\frac{ax+b+iay}{cx+d+icy}\right) = |cz+d|^{-2}\operatorname{Im}((ax+b+iay)(cx+d-icy)) \\
&= |cz+d|^{-2}(ay(cx+d)-cy(ax+b)) = |cz+d|^{-2}(ad-bc)y \\
&= |cz+d|^{-2}y
\end{aligned}$$

Solution 1.2.2. For any $z \in \mathcal{H}$, there exists $g \in \Gamma$ such that $g(z) \in \bar{F}$. Then there exists some i such that $\gamma_i^{-1} g \in \Gamma'$, i.e. $\gamma_i^{-1}(g(z)) \in \bar{F}' = \bigcup \overline{\gamma_i(F)}$. If $z_1, z_2 \in F'$ such that $z_1 = \gamma(z_2)$ for some $\gamma \in \Gamma'$, we may assume $z_1 \in F\gamma_i$ for some i, then $\gamma_i^{-1}(z_1) = \gamma_i^{-1}\gamma(z_2) \in F$. So $z_1 = z_2$ and $\gamma_i^{-1} = \gamma_i^{-1}\gamma$, i.e. $\gamma = I$.

Solution 1.2.3. (a) Clearly $[\Gamma(1) : \Gamma(2)] = 4$, and we can pick a set of coset representatives of $\Gamma(2)\backslash\Gamma(1)$ as $\{I, S, T, TS\}$, where $S = \left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]$, and $T = \left[\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right]$. So by 1.2.2., $F \cup FS \cup FT \cup FTS$ is a fundamental domain of $\Gamma(2)$. But clearly $\overline{F \cup FS \cup FT \cup FT} = \bar{F}'$, where F' is the area we want, we know that F' is the fundamental domain of $\Gamma(2)$.

(b) **(Maybe wrong, because $-I \in \Gamma(2)$ but cannot be generated by these two elements.** (cf. https://mathoverflow.net/questions/29700/generators-for-congruence-group-gamma2) **So we may prove that $\Gamma(2)$ is generated by these two elements and $-I$.)** Denote $\Gamma = \langle U, V, -I \rangle$, where $U = \left[\begin{smallmatrix} 1 & 2 \\ & 1 \end{smallmatrix}\right]$ and $V = \left[\begin{smallmatrix} 1 & \\ 2 & 1 \end{smallmatrix}\right]$. Since $\{\pm I\} \subset \Gamma$, it is sufficient to show that the image $\bar{\Gamma} = \overline{\Gamma(2)}$ in $\mathrm{PSL}(2,\mathbb{R})$. Let $\gamma \in \overline{\Gamma(2)}$, we may find a sequence $\gamma_1, \ldots, \gamma_n \in \Gamma(2)$ such that $\gamma_1(F') = F'$, and $\gamma_n(F') = \gamma(F')$, and each $\gamma_k(F')$ is adjacent to $\gamma_{k+1}(F')$. So $\gamma_1 = I$ and $\gamma_n = n$. Observe that the domains $\gamma F'$ that are adjacent to F' are precisely

$$\begin{aligned}
U(F') &= \{z \in \mathcal{H} \mid 3/2 < x < 7/2,\ |z-3/2| > 1/2,\ |z-5/2| > 1/2,\ |z-7/2| > 1/2\} \\
V(F') &= \{z \in \mathcal{H} \mid x < 1/2,\ |z-1/2| < 1/2,\ |z-1/6| > 1/6,\ |z-11/30| > 1/30,\ |z-3/8| > 1/8\} \\
U^{-1}(F') &= \{z \in \mathcal{H} \mid -5/2 < x < -1/2,\ |z+5/2| > 1/2,\ |z+3/2| > 1/2,\ |z+1/2| > 1/2\} \\
V^{-1}(F') &= \{z \in \mathcal{H} \mid |z-1/2| < 1/2,\ |z-3/8| > 1/8,\ |z-1/6| > 1/6,\ |z-5/6| > 1/6,\ |z-5/8| > 1/8\} \\
UV^{-1}(F') &= \{z \in \mathcal{H} \mid |z-5/2| < 1/2,\ |z-19/8| > 1/8,\ |z-13/6| > 1/6,\ |z-17/6| > 1/6,\ |z-21/8| > 1/8\} \\
VU^{-1}(F) &= \{z \in \mathcal{H} \mid x > 1/2,\ |z-1/2| < 1/2,\ |z-5/6| > 1/6,\ |z-19/30| > 1/30,\ |z-5/8| > 1/8\}
\end{aligned}$$

So since $\gamma_k(F')$ is adjacent to $\gamma_{k+1}(F')$, we must have $\gamma_k^{-1}\gamma_{k+1} = U, V, U^{-1}, V^{-1}$ or VU^{-1}, i.e. $\overline{\Gamma(2)} = \bar{\Gamma}$.

Solution 1.2.4. If e is an elliptic point of Γ, then for some element $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in the stabilizer of e not equal to $\pm I$, we have $\frac{az+b}{cz+d} = z$, i.e. $cz^2 + (d-a)z - b = 0$. Since $z \in \mathcal{H}$, we must have

$$0 > \Delta = (d-a)^2 + 4bc = a^2 + d^2 - 2ad + 4bc = a^2 + d^2 + 2ad - 4 = (a+d)^2 - 4$$

i.e. $|a+d| < 2$. Since Γ is discrete in $\mathrm{SL}(2,\mathbb{R})$, the choice of the stabilizer of e is finite. Then we need to prove that is cyclic. We may denote the stabilizer of e in $\bar{\Gamma}$ as $\bar{\Gamma}_e$. Then for any $\gamma \in \bar{\Gamma}_e$, $z \mapsto \gamma z$ on $\mathcal{H}$ is a conformal mapping. Since the Cayley transform $K(z) = \frac{z-e}{z-\bar{e}}$ is also a conformal mapping, we have a morphism $\phi_\gamma : \mathcal{D} \to \mathcal{D}$, $z \mapsto K \circ \gamma \circ K^{-1}(z)$, which is a conformal mapping on $\mathcal{D}$, hence it is just a rotation with angle θ_γ. So $\{\theta_\gamma \mid \gamma \in \bar{\Gamma}_e\}$ is a finite subgroup of $\mathbb{S}^1$, which must be cyclic.

Solution 1.2.5. (a) Since every element $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{C})$ is a conformal mapping on $\mathbb{C}$, if $\gamma : \mathcal{D} \to \mathcal{D}$, it must map the bound to the bound. So for every $\theta \in [0, 2\pi]$, we have $|\frac{ae^{i\theta}+b}{ce^{i\theta}+d}| = 1$, i.e. $|ae^{i\theta} + b| = |ce^{i\theta} + d|$. Since

$$|ae^{i\theta} + b|^2 = (ae^{i\theta} + b)(\bar{a}e^{-i\theta} + \bar{b}) = |a|^2 + |b|^2 + a\bar{b}e^{i\theta} + \bar{a}be^{-i\theta}$$

and same for $|ce^{i\theta}+d|$. Since θ is ambiguous, we must have $|a|^2+|b|^2 = |c|^2+|d|^2$, $a\bar{b} = c\bar{d}$ and $\bar{a}b = \bar{c}d$. So we have two case: $(c, d) = (a, b)$ or $(\bar{b}, \bar{a})$. But since $\gamma \in \mathrm{SL}(2, \mathbb{C})$, we only have $(c, d) = (\bar{b}, \bar{a})$, i.e. $\gamma \in \mathrm{SU}(1, 1)$. And clearly every element in $\mathrm{SU}(1, 1)$ maps the unit disk onto itself.

(b) Take $g = \left[\begin{smallmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{smallmatrix}\right]$. Then for any $\left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{R})$, we have

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{(x+w)+i(z-y)}{2} & \frac{(y+z)+i(w-x)}{2} \\ \frac{(y+z)+i(x-w)}{2} & \frac{(x+w)+i(y-z)}{2} \end{bmatrix} \in \mathrm{SU}(1, 1)$$

And for any $\left[\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right] \in \mathrm{SU}(1, 1)$, we only need to take $x = \mathrm{Re}(a) - \mathrm{Im}(b)$, $y = \mathrm{Im}(a) + \mathrm{Re}(b)$, $z = -\mathrm{Im}(a) + \mathrm{Re}(b)$, $w = \mathrm{Re}(a) + \mathrm{Im}(b)$, so $\mathrm{SU}(1, 1) = g\mathrm{SL}(2, \mathbb{R})g^{-1}$.

(c) If $\gamma = \left[\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right] \in \mathrm{SU}(1, 1)$ fixes the zero, we have $0 = \gamma(0) = \frac{b}{\bar{a}}$, i.e. $b = 0$, hence a must be some $e^{i\theta}$.

Solution 1.2.6. (a) For any $\left[\begin{smallmatrix} a & b \\ & a^{-1} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} c & d \\ & c^{-1} \end{smallmatrix}\right]$ in B, we have

$$\begin{bmatrix} a & b \\ & a^{-1} \end{bmatrix} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \begin{bmatrix} c & d \\ & c^{-1} \end{bmatrix} = \begin{bmatrix} bc & bd - a/c \\ c/a & d/a \end{bmatrix}$$

So the union of B and BSB is disjoint. For any $\left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{R}) - B$, we only need to pick $a = 1$, $b = x/z$, $c = z$ and $d = w$, then $\left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right] \in BSB$. So $\mathrm{SL}(2, \mathbb{R}) = B \coprod BSB$.

(b) We only need to check this measure is invariant under the action of $\left[\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]$. So, we have $|a^2y|^{-2}\mathrm{d}(a^2x)\, \mathrm{d}(a^2y) = a^{-4}|y|^{-2}a^4\mathrm{d}x\, \mathrm{d}y = |y|^{-2}\mathrm{d}x\, \mathrm{d}y$, $|y|^{-2}\mathrm{d}(x + b)\, \mathrm{d}y = |y|^{-2}\mathrm{d}x\, \mathrm{d}y$, and $|\frac{y}{x^2+y^2}|^{-2}\mathrm{d}\left(-\frac{x}{x^2+y^2}\right) \mathrm{d}\left(\frac{y}{x^2+y^2}\right) = (x^2+y^2)^2|y|^{-2}\left(\frac{x^2-y^2}{(x^2+y^2)^2}\mathrm{d}x + \frac{2xy}{(x^2+y^2)^2}\mathrm{d}y\right)\left(\frac{-2xy}{(x^2+y^2)^2}\mathrm{d}x + \frac{x^2-y^2}{(x^2+y^2)^2}\mathrm{d}y\right) = (x^2 + y^2)^2|y|^{-2}(x^2 + y^2)^{-2}\mathrm{d}x\, \mathrm{d}y = |y|^{-2}\mathrm{d}x\, \mathrm{d}y$.

(c) Since this measure is invariant under the action of $\Gamma(1)$, we have

$$\int_{\Gamma(1)\backslash\mathcal{H}} \frac{\mathrm{d}x\, \mathrm{d}y}{|y|^{-2}} = \int_F \frac{\mathrm{d}x\, \mathrm{d}y}{|y|^{-2}} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\sqrt{1-x^2}}^{\infty} y^{-2}\mathrm{d}y\, \mathrm{d}x = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1 - x^2}}\mathrm{d}x = \frac{\pi}{3}$$

Solution 1.2.7. If $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ fixes a point $z \in \mathbb{P}^1\mathbb{C}$, we have $\frac{az+b}{cz+d} = z$, i.e. $cz^2 + (d-a)z - b = 0$. So $\Delta = (d-a)^2 + 4bc = a^2 + d^2 - 2ad + 4bc = (a+d)^2 - 4$.

(a) If $|\mathrm{tr}(\gamma)| < 2$, $\Delta < 0$, then there exist two fixed complex points $\frac{(a-d)\pm\sqrt{\Delta}}{2c}$ of γ.

(b) If $|\mathrm{tr}(\gamma)| > 2$, $\Delta > 0$, then there exist two fixed real points $\frac{(a-d)\pm\sqrt{\Delta}}{2c}$.

(c) If $|\mathrm{tr}(\gamma)| = 2$, $\Delta = 0$, then there exists only one fixed real points $\frac{a-d}{2c}$. If $\mathrm{tr}(\gamma) = 2$, and λ is an eigenvalue of γ, we have

$$0 = |A - \lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad-bc) = \lambda^2 - 2\lambda + 1$$

So λ must be 1.

Solution 1.2.8. (a) ($\Rightarrow$) If γ is elliptic, there exists a $z \in \mathcal{H}$ fixed by γ. By 1.2.4., we know the stabilizer of γ is finite. Since any γ^n will fix z, so γ must have finite order. ($\Leftarrow$) Suppose γ is not elliptic. We may assume $a + d > 0$ or we just need to change γ to $-\gamma$. So $\mathrm{tr}(\gamma) = a + d \geq 2$. If γ is hyperbolic, the equation $|A - \lambda I| = 0$ has two solutions with $\lambda_1 > 1 > \lambda_2$. So $\mathrm{tr}(\gamma^n) = \lambda_1^n + \lambda_2^n$ could be very big, hence γ does not have finite order. If γ is parabolic, it has only one eigenvalue 1, so there must exists some invertible matrix P such that $\gamma = P\left[\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right]P^{-1}$ since $\gamma \neq I$, then $\gamma^n = P\left[\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right]P^{-1}$ can never be I.

(b) If $\mathrm{tr}(\gamma) = 0$, the fixed point $z = \frac{a-d+2i}{2c} = \frac{a+i}{c}$ if $c > 0$ or $z = \frac{a-d-2i}{2c} = \frac{a-i}{c}$ if $c < 0$. If $c > 0$, we may take $g = \left[\begin{smallmatrix} \sqrt{c}^{-1} & \frac{a}{\sqrt{c}} \\ & \sqrt{c} \end{smallmatrix}\right]$, then $g(i) = z$. So $\gamma = \pm g S g^{-1}$ for $S = \left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]$. If $c < 0$, we may take $g = \left[\begin{smallmatrix} \sqrt{-c}^{-1} & \frac{-a}{\sqrt{-c}} \\ & \sqrt{-c} \end{smallmatrix}\right]$ and the rest are the same.

If $\mathrm{tr}(\gamma) = \pm 1$, the fixed point $z = \frac{a-d+\sqrt{3}i}{2c} = \frac{2a-1+\sqrt{3}i}{2c}$ if $c > 0$ or $z = \frac{a-d-\sqrt{3}i}{2c} = \frac{2a-1-\sqrt{3}i}{2c}$ if $c < 0$. If $c > 0$, we may take $g = \left[\begin{smallmatrix} \sqrt{c}^{-1} & \frac{a}{\sqrt{c}} \\ & \sqrt{c} \end{smallmatrix}\right]$, then $g(\rho) = z$ for $\rho = \frac{-1+\sqrt{3}i}{2}$. So $\gamma = \pm g U g^{-1}$ for $U = \left[\begin{smallmatrix} & -1 \\ 1 & 1 \end{smallmatrix}\right]$. If $c < 0$, we may take $g = \left[\begin{smallmatrix} \sqrt{-c}^{-1} & \frac{-a}{\sqrt{-c}} \\ & \sqrt{-c} \end{smallmatrix}\right]$ and the rest are the same.

Solution 1.2.9. Clearly $\Gamma\backslash\mathcal{H}$ is compact $\Leftrightarrow$ $\Gamma\backslash\mathcal{H} = \Gamma\backslash\mathcal{H}^*$ $\Leftrightarrow$ $\mathcal{H}^* = \mathcal{H}$ $\Leftrightarrow$ there exists no parabolic element in Γ by 1.2.7.(c).

Solution 1.2.10. ($\Rightarrow$) If $\gamma = \left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right]$ is parabolic, we have $a = \frac{x-1}{z} \in \mathbb{P}^1(\mathbb{Q})$. ($\Leftarrow$) If $a \in \mathbb{P}^1(\mathbb{Q})$, we may assume $a = \frac{p}{q}$ some p, q such that $(p,q) = 1$. Take $P = pq$ and $Q = q^2$ we have $a = \frac{P}{Q}$. Then $(P+1, Q) = (pq+1, q^2) = 1$. So we may take $x = P+1$, $y = -\frac{P^2}{Q} = -p^2$, $z = Q$ and $w = 1-P$. Then γ fixes a.

Solution 1.2.11. Clearly if we fix the $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma(1)$ is hyperbolic, we just need to take $D = (a+d)^2 - 4$. Then the fixed point $z = \frac{(a-d)\pm\sqrt{D}}{2c}$, and the eigenvalue $\frac{a+d\pm\sqrt{D}}{2}$ are in $K = \mathbb{Q}(\sqrt{D})$.

We may suppose D is a fundamental discriminant here. For any $\alpha \in K$, there exists $\alpha_1, \alpha_2 \in \mathbb{Q}$ such that $\alpha = \alpha_1 + \alpha_2\sqrt{D}$. Take an integer q such that $q\alpha_1^2, 2q\alpha_2 \in \mathbb{Z}$. Then we may assume $\alpha_2 = \frac{p}{2q}$. By theory of Pell's equation, we may take a solution (x,y) of $x^2 - y^2D = 4$ such that $4pq|y$. Denote

$m = \frac{y}{p}$. Take $c = mq$, $n = mp$. Then $n^2D + 4 = m^2p^2D + 4 = y^2D + 4 = x^2$ is a square. Denote $\Delta = n^2D$, then $\sqrt{\Delta + 4}$ is an (even) integer. So we may take $a = \frac{\sqrt{\Delta+4}}{2} + mq\alpha_1$ and $d = \frac{\sqrt{\Delta+4}}{2} - mq\alpha_1$. And

$$b = \frac{ad-1}{c} = \frac{\frac{\Delta+4}{4} - m^2q^2\alpha_1^2 - 1}{mq} = \frac{mp^2D}{4q} - mq\alpha_1^2$$

By choosing of m, it can be divided by $4q$, so b is an integer. Then $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{Z})$ has an eigenvector

$$(\frac{a - d + \sqrt{(a+d)^2 - 4}}{2c}, 1) = (\frac{2mq\alpha_1 + mp\sqrt{D}}{2mq}, 1) = (\alpha_1 + \alpha_2\sqrt{D}, 1)$$

So for any fractional ideal $\mathfrak{a}$ with basis (a_1, a_2), we just need to take $\alpha = \frac{a_1}{a_2}$, and a γ such that $(\alpha, 1)$ is an eigenvector of γ, and so is (a_1, a_2).

If $\mathfrak{a}' = (n)\mathfrak{a}$ is an ideal in the same ideal class of $\mathfrak{a}$, then (na_1, na_2) is a basis of $\mathfrak{a}'$, and γ still has an eigenvector (na_1, na_2). Suppose $g \in \mathrm{GL}(2, \mathbb{Z})$ such that $\gamma' = g\gamma g^{-1} \in \mathrm{SL}(2, \mathbb{Z})$. Then γ' is an eigenvector of $(a_1, a_2)g$, which is a basis of fractional ideal $\mathfrak{a}' = (\det g)\mathfrak{a}$. Hence the conjugation class of γ in $\mathrm{GL}(2, \mathbb{Z})$ is only depending on the ideal class of $\mathfrak{a}$.

1.3 Modular Forms for SL$(2, \mathbb{Z})$

Solution 1.3.1. If $k \geq 4$ is an even integer, then

$$\begin{aligned} E_k(z) &= 2\zeta(k) + 2\sum_{m=1}^{\infty}\left(\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k}\right) \\ &= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{n=1} n^{k-1}e^{2\pi i mnz} \\ &= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i nz} \end{aligned}$$

Since $\sigma_{k-1}(n) \sim \log(n)n^{k-1}$, the radius of convergence of $\sum_{n=1}^{\infty}\sigma_{k-1}(n)x^n$ is $r = \lim_n \left|\frac{\sigma_{k-1}(n)}{\sigma_{k-1}(n+1)}\right| = 1$. Since $z \in \mathcal{H}$, we have $|e^{2\pi i z}| < 1$. So $E_k(z)$ is absolutely convergent for all $z \in \mathcal{H}$.

Solution 1.3.2. (a) Since f has the Γ-invariant, it is actually a meromorphic function on $\mathbb{C}/\Gamma$, which is a compact Riemann surface. So if f has no poles, it must be constant by Liouville's theorem.

(b) Clearly

$$\begin{aligned}\theta(z, w+2z) &= \sum_{n=-\infty}^{\infty} e^{2\pi i(n^2 z+n(w+2z))} \\ &= \sum_{n=-\infty}^{\infty} e^{2\pi i(n^2 z+2nz+z+nw+w-z-w))} \\ &= \sum_{n=-\infty}^{\infty} e^{2\pi i((n+1)^2 z+(n+1)w-(z+w))} = (qx)^{-1}\theta(z,w)\end{aligned}$$

And for $P(z,w)$:

$$\begin{aligned}P(z, w+2z) &= \prod_{n=1}^{\infty}(1+e^{2\pi i((2n-1)z+w+2z)})\prod_{n=1}^{\infty}(1+e^{2\pi i((2n-1)z-w-2z)}) \\ &= \prod_{n=2}^{\infty}(1+e^{2\pi i((2n-1)z+w)})\prod_{n=0}(1+e^{2\pi i((2n-1)z-w)}) \\ &= P(z,w)\cdot\frac{1+e^{2\pi i(-z-w)}}{1+e^{2\pi i(z+w)}} \\ &= P(z,w)\cdot e^{2\pi i(-z-w)} = (qx)^{-1}P(z,w)\end{aligned}$$

(c) For fixed z, if $P(z,w)=0$, then there exists some n such that $1+e^{2\pi i((2n-1)z+w)}=0$ or $1+e^{2\pi i((2n-1)z-w)}=0$, i.e. $w=m+\frac{1}{2}\pm(2n-1)z$ for some m,n, i.e. $w=\frac{1}{2}+z+\lambda$ for some $\lambda\in\Lambda$. If $w=\frac{1}{2}+z+(2mz+n)$, then by (b),

$$\theta(z,\frac{1}{2}+(2m+1)z+n) = \left(\prod_{k=1}^{m} e^{2\pi i(2kz+\frac{1}{2})}\right)^{-1}\theta(z,z+\frac{1}{2})$$

Then by definition

$$\theta(z,z+\frac{1}{2}) = \prod_{k=-\infty}^{\infty} e^{2\pi i(k^2 z+k(z+\frac{1}{2}))} = \prod_{k=-\infty}^{\infty}(-1)^k e^{2\pi iz(k^2+k)} = \prod_{k=-\infty}^{\infty}(-1)^k e^{2\pi iz(k^2-k)}$$

But

$$\theta(z,z+\frac{1}{2}) = \prod_{k=-\infty}^{\infty}(-1)^k e^{2\pi iz(k^2+k)} = \prod_{k=-\infty}^{\infty}(-1)^{k-1} e^{2\pi iz(k^2+k)}$$

So $\theta(z,z+\frac{1}{2})=0$, i.e. $\theta(z,z+\frac{1}{2}+\lambda)=0$ for all $\lambda\in\Lambda$. So by (a), $f(w)=\theta(z,w)/P(z,w)$ is a constant, i.e. the quotient is independent of w. So $\theta(z,w)=\phi(q)P(z,w)$ for some $\phi(q)$ or $\phi(z)$.

(d) Clearly

$$\theta(4z,\frac{1}{2}) = \sum_{k=-\infty}^{\infty} e^{2\pi i(4zk^2+\frac{k}{2})} = \sum_{k=-\infty}^{\infty} e^{2\pi i(z(2k)^2+\frac{2k}{4})} = \sum_{k=-\infty}^{\infty} e^{2\pi i(zk^2+\frac{k}{4})} = \theta(z,\frac{1}{4})$$

And

$$\frac{P(4z,\frac{1}{2})}{P(z,\frac{1}{4})} = \prod_{n=1}^{\infty} \frac{(1+e^{2\pi i((2n-1)4z+\frac{1}{2})})(1+1+e^{2\pi i((2n-1)4z-\frac{1}{2})})}{(1+e^{2\pi i((2n-1)z+\frac{1}{4})})(1+1+e^{2\pi i((2n-1)z-\frac{1}{4})})}$$
$$= \prod_{n=1}^{\infty}(1-e^{2\pi i(4n-2)z})(1-e^{4\pi i(4n-2)z}) = \prod_{n=1}^{\infty}(1-q^{4n-2})(1-q^{8n-4})$$

So since $\theta(z,\frac{1}{4}) = \phi(q)P(z,\frac{1}{4})$ and $\theta(4z,\frac{1}{2}) = \phi(q^4)P(z,\frac{1}{2})$, we have $\phi(q) = \frac{P(4z,\frac{1}{2})}{P(z,\frac{1}{4})}\phi(q^4)$. Fix a q such that $|q| < 1$. Then

$$\begin{aligned}
\phi(q) &= \phi(q^4)\left(\prod_{n=1}(1-q^{2(2n-1)})\right)\left(\prod_{n=1}(1-q^{4(2n-1)})\right) \\
&= \phi(q^{16})\left(\prod_{n=1}(1-q^{2(2n-1)})\right)\left(\prod_{n=1}(1-q^{4(2n-1)})\right)\left(\prod_{n=1}(1-q^{8(2n-1)})\right)\left(\prod_{n=1}(1-q^{16(2n-1)})\right) \\
&= \dots \\
&= \phi(0)\cdot\prod_{k=1}(1-q^{2k})
\end{aligned}$$

So we only need to prove $\phi(0) = 1$. Since $\lim_{q\to 0}\sum_{n=-\infty}^{\infty} q^{n^2} = 1$, and $\lim_{q\to 0}\prod_{n=1}^{\infty}(1+q^{2n-1}) = 1$, we know $\lim_{q\to 0}\phi(q) = \frac{1}{1^2} = 1$. So we've done.

Solution 1.3.3. Since $\gamma\rho = \rho$, we know $f(\rho) = f(\gamma\rho) = (-\rho)^k f(\rho)$. Since $3 \nmid k$, we know $(-\rho)^k \neq 1$, i.e. $f(\rho) = 0$. Similarly if $\gamma = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, $\gamma i = i$. If f is a modular form with weight k such that $4 \nmid k$, we know $f(i) = f(\gamma i) = i^k f(i)$, i.e. $f(i) = 0$.

Solution 1.3.4. If G_4 and G_6 are not algebraically independent, there exists an equation $\sum a_{ij}G_4^iG_6^j = 0$. Since $\mathcal{M}(\Gamma)$ is a graded ring, we may assume this equation is homogeneous, i.e. $f(G_4,G_6) = \sum_{i=0}^n a_iG_4^{3i}G_6^{2(n-i)} = 0$. Since $\mathbb{C}$ is algebraically closed, we know $f(G_4,G_6) = a_n\prod_i(G_4^3 - b_iG_6^2)$. Since $f \equiv 0$, there exists some $b = b_i$ such that $G_4^3 = bG_6^2$. So $\Delta = \frac{1}{1728}(G_4^3 - G_6^2) = \frac{b-1}{1728}G_6^2$. Since $G_6(i) = 0$ by 1.3.4., i.e. $\Delta(i) = 0$, which contradicts with the fact that Δ is everywhere nonzero on $\mathcal{H}$.

Solution 1.3.5. For general f, we still have $f(iy) = (-1)^{k/2}y^{-k}f(\frac{i}{y})$. We denote the constant term of

Fourier expansion of $f(z)$ as a_0. Then

$$\begin{aligned}
\Lambda(f,s) &= \int_0^\infty (f(iy)-a_0)y^s \frac{\mathrm{d}y}{y} \\
&= \int_1^\infty (f(iy)-a_0)y^s \frac{\mathrm{d}y}{y} + \int_0^1 (f(iy)-a_0)y^s \frac{\mathrm{d}y}{y} \\
&= \int_1^\infty (f(iy)-a_0)y^s \frac{\mathrm{d}y}{y} + \int_1^\infty (f(\frac{i}{y})-a_0)y^{-s} \frac{\mathrm{d}y}{y} \\
&= \int_1^\infty (f(iy)-a_0)y^s \frac{\mathrm{d}y}{y} + \int_1^\infty ((-1)^{k/2}y^k f(iy)-a_0)y^{-s} \frac{\mathrm{d}y}{y} \\
&= \int_1^\infty (f(iy)-a_0)y^s \frac{\mathrm{d}y}{y} + (-1)^{k/2}\int_1^\infty (f(iy)-a_0)y^{k-s} \frac{\mathrm{d}y}{y} - \frac{a_0}{s} - \frac{a_0 i^k}{k-s}
\end{aligned}$$

So clearly $\Lambda(f,s) = (-1)^{k/2}\Lambda(f,k-s)$.

Solution 1.3.6. We may suppose $f(z) = \sum_{i=1}^\infty a_i q^i$ is cusp, and $g(z) = \sum_{i=0}^\infty b_i q^i$ is a modular form. Then $f(z)g(\bar{z}) = \sum_{i=1}^\infty c_i q^i$ for some $c_i = \sum_{j=1}^i a_j b_{i-j}$. So $f(z)g(\bar{z}) \to 0$ very rapidly as $z \to i\infty$, hence $\langle f, g\rangle$ is well-defined if only f is cusp.

For the Eisenstein series E_k, we know

$$\begin{aligned}
E_k &= \frac{1}{2}\sum_{(m,n)\neq(0,0)} (mz+n)^{-k} = \frac{1}{2}\sum_{p=1}^\infty \sum_{(m,n)=1} (pmz+pn)^{-k} = \frac{1}{2}\zeta(k)\sum_{(m,n)=1}(mz+n)^{-k} \\
&= \frac{1}{2}\zeta(k)\sum_{\gamma=\left[\begin{smallmatrix} a & b \\ m & n\end{smallmatrix}\right]\in\mathrm{SL}(2,\mathbb{Z})} (mz+n)^{-k}
\end{aligned}$$

Then for any $\gamma = \left[\begin{smallmatrix} a & b \\ m & n\end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{Z})$, we have

$$\overline{f(\gamma z)} = \overline{(mz+n)^k f(z)} = (\overline{mz+n})^k \overline{f(z)} = \left(\frac{|mz+n|^2}{mz+n}\right)^k \overline{f(z)}$$

And if we write $z = x+yi$, then

$$\mathrm{Im}(\gamma z) = \mathrm{Im}\left(\frac{ax+b+iay}{mx+n+imy}\right) = \frac{ay(mx+n)-my(ax+b)}{|mz+n|^2} = \frac{y}{|mz+n|^2}$$

So for any cusp form f,

$$\begin{aligned}
\langle E_k, f\rangle &= \int_{\Gamma\backslash\mathcal{H}} E_k(z)\overline{f(z)}y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \frac{1}{2}\zeta(k)\int_{\Gamma\backslash\mathcal{H}} \sum_{(m,n)=1} (mz+n)^{-k}\overline{f(z)}y^k\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \frac{1}{2}\zeta(k)\int_{\mathcal{H}} \overline{f(z)}y^k\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \frac{1}{2}\zeta(k)\int_0^\infty \left(\sum_{n=-\infty}^{\infty}\int_n^{n+1}\overline{f(z)}\mathrm{d}x\right)y^k\frac{\mathrm{d}y}{y^2} \\
&= \frac{1}{2}\zeta(k)\int_0^\infty \left(\sum_{n=-\infty}^{\infty} 0\right)y^k\frac{\mathrm{d}y}{y^2} = 0
\end{aligned}$$

Solution 1.3.7. (a) Since f has only one simple pole at m, we know $f^{-1}(\infty) = \{m\}$ and m is not a ramified point. So $\deg f = 1$, i.e. f is an isomorphism.

(b) Clearly for any $\gamma \in \mathrm{SL}(2,\mathbb{Z})$,

$$j(\gamma z) = \frac{G_4^3(\gamma z)}{\Delta(\gamma z)} = \frac{(cz+d)^{12}G_4^3(z)}{(cz+d)^{12}\Delta(z)} = j(z)$$

i.e. j is an automorphic function for $\mathrm{SL}(2,\mathbb{Z})$. Since

$$\begin{aligned}
G_4(z) &= 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + 60480q^6 + \dots \\
G_6(z) &= 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 - 1575504q^5 - 4058208q^6 - \dots
\end{aligned}$$

we have

$$\Delta(z) = \frac{1}{1728}(G_4^3 - G_6^2) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 + \dots$$

Then

$$\begin{aligned}
\Delta^{-1}(z) &= q^{-1} + 24 + 324q + 3200q^2 + 25650q^3 + 176256q^4 + \dots \\
j(z) &= q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \dots
\end{aligned}$$

Since $G_4(\rho) = 0$, and $G_6(i) = 0$ by 1.3.3., we know

$$\begin{aligned}
j(i) &= 1728\frac{G_4^3(i)}{G_4^3(i) - G_6^2(i)} = 1728 \\
j(\rho) &= 1728\frac{G_4^3(\rho)}{G_4^3(\rho) - G_6^2(\rho)} = 0
\end{aligned}$$

Since $j(z)$ has only a simple pole at $i\infty$, by (a) we have an isomorphism $\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}^* \cong \mathbb{P}^1(\mathbb{C})$.

Solution 1.3.8. First we may count $[\Gamma(1):\Gamma(n)]$. Since $\Gamma(n) = \ker(\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z}))$, and by Chinese remainder theorem, this map is surjective, we know $[\Gamma(1):\Gamma(n)] = \sharp(\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z}))$. If $N = \prod p_i^{e_i}$, by Chinese remainder theorem, we know $\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z}) \cong \prod \mathrm{SL}(2,\mathbb{Z}/p_i^{e_i}\mathbb{Z})$. Clearly, if $e=1$, we know $\sharp\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z}) = \frac{(p^2-1)(p^2-p)}{p-1} = p^3(1-p^{-2})$. For bigger e, for any $\gamma \in \ker(\mathrm{SL}(2,\mathbb{Z}/p^{e+1}\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/p^e\mathbb{Z}))$, γ has the form $\left[\begin{smallmatrix} 1+ap^e & bp^e \\ cp^e & 1+dp^e \end{smallmatrix}\right]$ with $1 = \det\gamma = 1+(a+d)p^e$ for some $a,b,c,d \in \mathbb{Z}/p\mathbb{Z}$, i.e. $a+d=0$. So $\sharp\ker(\mathrm{SL}(2,\mathbb{Z}/p^{e+1}\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/p^e\mathbb{Z})) = p^3$, i.e. $\sharp\mathrm{SL}(2,\mathbb{Z}/p^e\mathbb{Z}) = p^{3e}(1-p^{-2})$. And $\sharp\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z}) = N^3\prod_{p|N}(1-p^{-2})$.

If $N=2$, we know $\left[\begin{smallmatrix} -1 & \\ & -1 \end{smallmatrix}\right] \in \Gamma(2)$, so $\deg f = [\Gamma(1):\Gamma(N)] = 6$. If $N>2$, we know $\left[\begin{smallmatrix} -1 & \\ & -1 \end{smallmatrix}\right] \notin \Gamma(N)$, so $\deg f = [\Gamma(1):\{\pm1\}\ltimes\Gamma(N)] = \frac{1}{2}[\Gamma(1):\Gamma(N)] = \frac{1}{2}N^3\prod_{p|N}(1-p^{-2})$.

If $z \in \Gamma(N)\backslash\mathcal{H}^*$ ramifies in the covering map $\Gamma(N)\backslash\mathcal{H}^* \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}^*$, it must has a preimage $\bar{z} \in \mathcal{H}^*$ such that $\bar{z}$ ramifies, i.e. $z = i, \rho$ or ∞. For i, we have $\mathrm{Stab}(i) = \langle\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right]\rangle$, so $\sharp f^{-1}(i) = \deg f/(\sharp(\mathrm{Stab}(i)/(\mathrm{Stab}(i)\cap\Gamma(N)\ltimes\{\pm1\}))) = \frac{n}{2}$. For ρ, we have $\mathrm{Stab}(\rho) = \langle\left[\begin{smallmatrix} 1 & 1 \\ -1 & \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & \\ & -1 \end{smallmatrix}\right]\rangle$, so $\sharp f^{-1}(\rho) = \deg f/(\sharp(\mathrm{Stab}(\rho)/(\mathrm{Stab}(\rho)\cap\Gamma(N)\ltimes\{\pm1\}))) = \frac{n}{3}$. For ∞, we have $\mathrm{Stab}(\infty) = \langle\left[\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & \\ & -1 \end{smallmatrix}\right]\rangle$, so $\sharp f^{-1}(\infty) = \deg f/(\sharp(\mathrm{Stab}(\infty)/(\mathrm{Stab}(\infty)\cap\Gamma(N)\ltimes\{\pm1\}))) = \frac{n}{N}$.

By 1.3.7., we know $\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}^* \cong \mathbb{P}^1(\mathbb{C})$, hence has genus 0. So by Hurwitz's theorem, when $N=2$, we have $2g(\Gamma(2)\backslash\mathcal{H}^*)-2 = 6\times(2\times0-2)+\frac{6}{2}\times(2-1)+\frac{6}{3}\times(3-1)+\frac{6}{2}\times(2-1)$, i.e. $g(\Gamma(2)\backslash\mathcal{H}^*) = 0$. And when $N=3$, we have $2g(\Gamma(3)\backslash\mathcal{H}^*)-2 = 12\times(2\times0-2)+\frac{12}{2}\times(2-1)+\frac{12}{3}\times(3-1)+\frac{12}{3}\times(3-1)$, i.e. $g(\Gamma(3)\backslash\mathcal{H}^*) = 0$

Solution 1.3.9. Clearly $\Gamma(11) \subset \Gamma_1(11) \subset \Gamma_0(11) \subset \mathrm{SL}(2,\mathbb{Z})$. Since $\Gamma_1(11)/\Gamma(11) = \{\left[\begin{smallmatrix} 1 & 0 \\ & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} 1 & 10 \\ & 1 \end{smallmatrix}\right]\}$, and $\Gamma_0(11)/\Gamma_1(11) = \{\left[\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 2 & \\ & 6 \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} 10 & \\ & 10 \end{smallmatrix}\right]\}$, we have $[\mathrm{SL}(2,\mathbb{Z}):\Gamma_0(11)] = \frac{[\mathrm{SL}(2,\mathbb{Z}):\Gamma(11)]}{[\Gamma_0(11):\Gamma_1(11)]\cdot[\Gamma_1(11):\Gamma(11)]} = \frac{1320}{10\times11} = 12$. Since $\left[\begin{smallmatrix} -1 & \\ & -1 \end{smallmatrix}\right] \in \Gamma_0(11)$, we have $\deg f = [\mathrm{SL}(2,\mathbb{Z}):\Gamma_0(11)] = 12$, where $f : \Gamma_0(11)\backslash\mathcal{H}^* \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}^*$ is the covering map. Similarly with 1.3.8., the ramification points are $f^{-1}(i)$, $f^{-1}(\rho)$ and $f^{-1}(\infty)$. Moreover, $\sharp f^{-1}(i) = 6$, $e(i) = 2$, $\sharp f^{-1}(\rho) = 4$, $e(\rho) = 3$, $\sharp f^{-1}(\infty) = 2$ and $e(\infty) = 6$. So by Hurwitz's theorem, we have $2g(\Gamma_0(11)\backslash\mathcal{H}^*)-2 = 12(0-2)+6\times(2-1)+4\times(3-1)+2\times(6-1) = 0$, i.e. $g(\Gamma_0(11)\backslash\mathcal{H}^*) = 1$.

Solution 1.3.10 (Picard's theorem)**.** If ϕ is an entire function on $\mathbb{C}$ such that two complex numbers a,b are not in the image, we may compose a linear function f such that $f\circ\phi : \mathbb{C} \to \Gamma(2)\backslash\mathcal{H}$, since $\Gamma(2)$ has only two cusps and $g(\Gamma(2)\backslash\mathcal{H}) = 0$. Denote $\kappa : z \mapsto \frac{z-i}{z+i}$ to be the Caylay transformation. Then $\kappa\circ f\circ\phi$ is a function from $\mathbb{C}$ to $\Gamma(2)\backslash\mathcal{D}$, which is bounded. So by Liouville's theorem, $\kappa\circ f\circ\phi$ is just a constant, i.e. ϕ is constant.

Solution 1.3.11. For any covering $f : V \to U$, we pick $u_0 \in U$ and $v_0 \in f^{-1}(u_0)$. Then for any $\gamma \in \pi^{-1}(V)$, it can be treated as a loop $\gamma : [0,1] \to V$ such that $\gamma(0) = \gamma(1) = v_0$. Then $f(\gamma)$ is a loop in U, i.e. $f(\gamma) \in \pi^{-1}(U)$. So f induces an inclusion $f : \pi^{-1}(V) \to \pi^{-1}(U)$. If we pick another u_0' and v_0', we may consider a path $\pi : [0,1] \to V$ such that $\pi(0) = v_0'$ and $\pi(1) = v_0$ and its inverse π' such that $\pi'(x) = \pi(1-x)$. Then for any loop $\gamma \in \pi_1(V,v_0)$, we know $\pi+\gamma+\pi' \in \pi_1(V,v_0')$. So $f(\pi+\gamma+\pi') \in \pi_1(U,u_0')$ is conjugate with $f(\gamma) \in \pi_1(U,u_0) \cong \pi_1(U,u_0')$. So the map from covering

$f : V \to U$ to the conjugacy class of $\pi_1(V)$ is well-defined. If $\phi : V \to V'$ are two equivalent covering of U, for any loop $\gamma \in V$, $\phi(\gamma)$ is a loop in V'. Obviously the subgroups $\pi_1(V)$ and $\pi_1(V')$ are conjugate in $\pi_1(U)$. Conversely, if we fix a subgroup $\Gamma \in \pi_1(U)$, we may define an action of Γ on the universal covering $\tilde{U}$ of U as follows: for any $g \in \Gamma$ and $\tilde{z} \in \tilde{U}$, we may treat g as a loop $g : [0,1] \to U$ with $g(0) = g(1) = z = \tilde{p}_U(\tilde{z})$, then by path-lifting property, g can be lifted to a path $\tilde{g} : [0,1] \to \tilde{U}$ with $\tilde{g}(0) = \tilde{z}$, then we define $g(\tilde{z}) = \tilde{g}(1)$. So $V_\Gamma = \Gamma\backslash\tilde{U}$ is a covering of U, and this process is inverse to the process from covering to subgroups. So we have a bijection between equivalence classes of coverings of U and conjugacy classes of subgroups of the fundamental group $\pi_1(U)$. By our definition of action of Γ on $\tilde{U}$, we clearly have $\pi(\Gamma\backslash\tilde{U}) \cong \Gamma$.

Solution 1.3.12. ($\Rightarrow$) If p dominates p', for any fixed $v_0 \in V$, $v_0' = q(v_0) \in V'$ and $u_0 = p(v_0) \in U$, we have the inclusions $\pi_1(V, v_0) \hookrightarrow \pi_1(V', v_0') \hookrightarrow \pi_1(U, u_0)$. So Γ is conjugate with a subgroup of Γ' in $\pi_1(U)$.

($\Leftarrow$) We may change Γ to one of its conjugacy such that $\Gamma \subset \Gamma'$. Then by 1.3.11., $V \cong \Gamma\backslash\tilde{U}$ and $V' \cong \Gamma'\backslash\tilde{U}$, so we have a morphism $q : V \to V'$ commute with the covering p and p', i.e. p dominates p'.

Solution 1.3.13. For any $u_0 \in U$ and $v_0, v_0' \in p^{-1}(u_0)$, we may fix a path $\pi : [0,1] \to V$ such that $\pi(0) = v_0$ and $\pi(1) = v_0'$. Then $\pi' = p(\pi)$ is a loop in U, and clearly $\pi_1(V, v_0) = \pi'\pi_1(V, v_0')\pi'^{-1}$. But since G acts transitively on the fiber $p^{-1}(u_0)$, there exists a g such that $g(v_0) = v_0'$. So for any loop $\gamma \in \pi_1(V, v_0')$, $g(\pi' + \gamma + \pi'^{-1})$ is a loop in $\pi_1(V, v_0')$, i.e. $\pi'\pi_1(V, v_0')\pi'^{-1} = \pi_1(V, v_0')$. So $\pi_1(V)$ is normal in $\pi_1(U)$. And clearly we have a morphism $f : \pi_1(U) \to G$, where for any $v \in V$, $u = p(v)$, and $\gamma \in \pi_1(U)$ with $\gamma(0) = \gamma(1) = u$, we define $f(\gamma)(v) = \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is a path in V lifted by γ with $\tilde{\gamma}(0) = v$. Clearly $\ker(f) = \Gamma$, then $G \cong \pi_1(U)/\Gamma$.

Solution 1.3.14. Well, the universal covering is regular, so trivial. For non-trivial way, every covering corresponds to a conjugacy class of subgroup Γ of $\pi_1(U)$, then $\Gamma' = \bigcap g\Gamma g^{-1}$ is a normal subgroup of $\pi_1(U)$ contained in Γ. So by 1.3.12. and 1.3.13., $V' \to U$ corresponding to Γ' is a regular covering which dominates $V \to U$.

Solution 1.3.15. Clearly if we denote $d = \deg f$, then for any point $P \in Y$ not ramifying, we have $\sharp f^{-1}(P) = d$, hence $f|_V$ is finite. Conversely, suppose we have a finite covering $f' : V' \to U$ with degree d. Then V' inherits a complex structure from U by f'. Locally speaking, for any $P \in X - U$, we may take a sufficiently small chart Ω of P with $y : \Omega \to \mathbb{C}$. We may assume $y(P) = 0$. Then $y \circ f'$ on $f'^{-1}(U \cap \Omega)$ is finitely many copies of x^e for some function x defined in each connected component O_i. Then in each copy, by Rudin's theorem we may extend O_i to O_i' by adding a preimage of P. So we can extend $f' : V' \to U$ to a compact X' with holomorphic function $f : X' \to Y$ such that $f' = f|_{V'}$.

Solution 1.3.16. (Seems a little wrong. We may prove that Gal$(F_X/F_Y) \cong \pi_1(U)/\Gamma$.) Clearly one valuation of $F(X)$ is corresponding to a point on X, and same for Y. Then fixing a point $y \in Y$ with valuation v_y and $t \in K(Y)$ such that $v_y(t) = 1$, we have $\deg(F_X/F_Y) = \sum_{x\in X} v_x(f^\sharp(t)) = \sum_{x\in f^{-1}(y)} e_X = \deg f$, where $f^\sharp : F_Y \to F_X$ is the inclusion induced by f. Moreover, if the covering $V \to U$ is regular, $\Gamma = \pi_1(V)$ is a normal subgroup in $\pi_1(U)$. Denote $G = \pi_1(U)/\Gamma$. Then G is a group of automorphism of X/Y which acts transitively on each fiber $f^{-1}(u)$ for every $u \in U$ by 1.3.13. For any $\phi \in F_X - F_Y$, there exists some $x \in X$ such that $v_x(\phi) \neq 0$. Then for any $x' \in f^{-1}(f(x))$, there exists a $g \in G$ such that $g(x) = x'$, then $\phi \neq \phi \circ g$. So F_Y is the fixed field of G acting on F_X, hence F_X/F_Y is Galois. And in this case, $\mathrm{Gal}(F_X/F_Y) \cong \mathrm{Aut}(F_X/F_Y) = G = \pi_1(U)/\Gamma$. Conversely, if F_X/F_Y is Galois, F_Y is the fixed field of $G = \mathrm{Gal}(F_X/F_Y)$. For any $x, x' \in V$ with $y = f(x) = f(x') \in U$, $v_x, v_{x'}$ are two valuations above v_y. Hence there exists a $g \in G$ such that $v_x(\phi) = v_{x'}(g(\phi))$ for all $\phi \in F_X$. So G is actually acting on V transitively on each fiber, i.e. $V \to U$ is regular.

Solution 1.3.17. Denote γ_1 as loop circling the point y_0, and γ_2 as loop circling the point y_1. Then $\pi_1(U) = \mathbb{Z} * \mathbb{Z}$ generated by γ_0 and γ_1. Then consider the normal subgroup Γ generated by γ_0^2, γ_1^2 and $(\gamma_0\gamma_1)^3$. Define $V = \Gamma\backslash\tilde{U}$ and the covering map $f : V \to U$ as in 1.3.11., then we have $\deg f = [\pi_1(U) : \Gamma] = \sharp\{1, \gamma_0, \gamma_1, \gamma_0\gamma_1, (\gamma_0\gamma_1)^2, \gamma_0\gamma_1\gamma_0\} = 6$. Then we extend the covering $V \to U$ onto $X \to Y$ by 1.3.15. So for any $x \in f^{-1}(y_0)$, the stabilizer of x is $\{1, g_0\}$, so there are 3 points in the fiber, each have ramification index 2. And same for y_1. For any $x \in f^{-1}(y_\infty)$, the stabilizer of x is $\{1, g_0g_1, (g_0g_1)^2\}$, so there are 2 points in the fiber, each have ramification index 3. Then by Hurwitz's formula, we have $2g(X) - 2 = 6 \times (0-2) + 2 \times 3 \times (2-1) + 2 \times (3-1) = -2$, i.e. $g(X) = 0$.

Construct a group Γ' generated by $\gamma_0^i, \gamma_1^j, (\gamma_0\gamma_1)^k$ with $i = 1$ or 2, $j = 1$ or 2 and $k = 1$ or 3. Then we have $\Gamma \subset \Gamma' \subset \pi_1(U)$. Then by 1.3.12., there exists a covering $p : W \to U$ dominated by f. Then by 1.3.15., we can extend p to a holomorphic map $p : Z \to Y$ dominated by $f : X \to Y$ with morphism $q : X \to Z$.

Solution 1.3.18. Consider the projection $f : \Gamma(2)\backslash\mathcal{H}^* \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}^*$. Denote $y_0 = 0$, $y_1 = 1728$, $y_\infty = \infty$. Then y_0 and y_1 have 3-point fiber with each ramification index 2, and y_∞ has 2-point fiber with each ramification index 3. Then the morphism $f|_{\Gamma(2)\backslash\mathcal{H}}$ is the covering map as in 1.3.17. Since for the equation $z^3 - zj_0 - 16j_0$ has discriminant $\Delta = 4j_0^3 - 6912j_0^2$, we know this equation has no multiple roots when $j_0 \neq 0$ or 1728. So if we define $Z = \{(z_0, \tau) \in \mathbb{C} \times (\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}) \mid z_0^3 - z_0 j(\tau) - 16j(\tau) = 0\}$ with projection $p_2 : Z \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}$, by 1.3.17., there exists a holomorphic map $q : X \to Z$ such that $f = p_2 \circ q$. Then the projection p_1 is what we need.

Solution 1.3.19. Emmm... I don't want to duplicate the process of 1.3.17. and 1.3.18. Since $\Gamma(3)$ is generated by $T^3 = \left[\begin{smallmatrix} 1 & 3 \\ & 1 \end{smallmatrix}\right], ST^3S^{-1} = \left[\begin{smallmatrix} 1 & \\ -3 & 1 \end{smallmatrix}\right], TST^3S^{-1}T^{-1} = \left[\begin{smallmatrix} -2 & 3 \\ -3 & 4 \end{smallmatrix}\right]$ and $T^{-1}ST^3S^{-1}T = \left[\begin{smallmatrix} 4 & 3 \\ -3 & -2 \end{smallmatrix}\right]$, and we have $\eta(Sz) = \sqrt{-iz}\eta(z)$, $\eta(S^{-1}z) = \sqrt{iz}\eta(z)$, $\eta(Tz) = e^{\pi i/12}\eta(z)$ and $\eta(T^{-1}z) = e^{-\pi i/12}\eta(z)$, we know

$$\begin{aligned}
\eta(\left[\begin{smallmatrix} 1 & 3 \\ & 1 \end{smallmatrix}\right] z) &= e^{\pi i/4}\eta(z) \\
\eta(\left[\begin{smallmatrix} 1 & \\ -3 & 1 \end{smallmatrix}\right] z) &= \sqrt{-i \cdot \frac{3z-1}{z}} \cdot e^{\pi i/4} \cdot \sqrt{iz}\eta(z) = e^{\pi i/4}\sqrt{3z-1}\eta(z) \\
\eta(\left[\begin{smallmatrix} -2 & 3 \\ -3 & 4 \end{smallmatrix}\right] z) &= e^{\pi i/12} \cdot \sqrt{-i \cdot \frac{3z-4}{z-1}} \cdot e^{\pi i/4} \cdot \sqrt{i \cdot (z-1)}e^{-\pi i/12}\eta(z) = e^{\pi i/4}\sqrt{3z-4}\eta(z) \\
\eta(\left[\begin{smallmatrix} 4 & 3 \\ -3 & -2 \end{smallmatrix}\right] z) &= e^{-\pi i/12} \cdot \sqrt{-i \cdot \frac{3z+2}{z+1}} \cdot e^{\pi i/4} \cdot \sqrt{i \cdot (z+1)}e^{\pi i/12}\eta(z) = e^{\pi i/4}\sqrt{3z+2}\eta(z)
\end{aligned}$$

So for $\eta^8(z)$, we have

$$\begin{aligned}
\eta(\left[\begin{smallmatrix} 1 & 3 \\ & 1 \end{smallmatrix}\right] z)^8 &= \eta(z)^8 \\
\eta(\left[\begin{smallmatrix} 1 & \\ -3 & 1 \end{smallmatrix}\right] z)^8 &= (3z-1)^4\eta(z)^8 \\
\eta(\left[\begin{smallmatrix} -2 & 3 \\ -3 & 4 \end{smallmatrix}\right] z)^8 &= (3z-4)^4\eta(z)^8 \\
\eta(\left[\begin{smallmatrix} 4 & 3 \\ -3 & -2 \end{smallmatrix}\right] z)^8 &= (3z+2)^4\eta(z)^8
\end{aligned}$$

So η^8 is a modular form w.r.t. $\Gamma(3)$, hence $\frac{G_4}{\eta^8}$ is an automorphic function on $\Gamma(3)\backslash\mathcal{H}$. And $(\frac{G_4}{\eta^8})^3 = \frac{G_4}{\Delta} = j$.

1.4 Hecke Operators

Solution 1.4.1. (a) Fix a $T \in S$. Clearly T has an eigenvector v with eigenvalue λ, which span a 1-dimensional subspace V of H. So we may extend $\{v\}$ to an orthogonal basis of H as $\{v = v_1, \ldots, v_n\}$. Then T can be represented as matrix

$$\begin{bmatrix} \lambda & 0 & \ldots & 0 \\ a_2 & * & \ldots & * \\ \ldots & \ldots & \ldots & \ldots \\ a_n & * & \ldots & * \end{bmatrix}$$

Then the $(1,1)$-entry of TT^* is $\lambda\bar{\lambda}$, but the $(1,1)$-entry of T^*T is $\lambda\bar{\lambda}+\sum a_i\bar{a}_i$. So $\sum a_i\bar{a}_i = 0$, i.e. all $a_i = 0$. Hence H is invariant in the orthogonal complement of V. Then by induction on the dimension of H, it can be decomposed as a finite direct sum of one-dimensional eigenspaces.

For the eigenspace V of all vector with eigenvalue λ in H, for any $T' \in S$, we know $TT'(v) = T'T(v) = \lambda T'(v)$, hence V is invariant under the action of any operator in S. If V is not an eigenspace of T', then as above process, T' can decompose V into several eigenspace $V = \bigoplus V_i$ with $\dim V_i < \dim V$. Then we know for any $T'' \in S$, it must be invariant in each V_i. Since $\dim H$ is finite, the decomposition process must be stop in finite step, i.e. $H = \bigoplus V_k$ such that V_k is an eigenspace for all $T \in S$.

(b) Clearly $T_m^* = T_{-m}$. Then $T_m T_{-m} = T_{-m} T_m = T_0$, i.e. T_m's are all normal. But they do not have eigenvector in H.

Solution 1.4.2. As in the proof of proposition 1.4.1., the representations of coset are taken as the representations of $(\Gamma(1) \cap \alpha\Gamma(a)\alpha^{-1})\backslash\alpha\Gamma(1)$, hence for any set of α_i with $\Gamma(1)\alpha\Gamma(a) = \bigcup \Gamma(1)\alpha_i$ and any matrix $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, we must have $\Gamma(1)\alpha\Gamma(1) = \bigcup \Gamma(1)\alpha_i\gamma$. So we have a permutation: $\alpha_i\gamma$ is one-to-one corresponding to $\alpha_{s(i)}$. So if $\sigma, \sigma' \in \Gamma(1)\backslash\mathrm{GL}(2,\mathbb{Q})^+$ in the same double coset of $\Gamma(1)\backslash\mathrm{GL}(2,\mathbb{Q}^+)/\Gamma(1)$, there exists a $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ such that $\sigma' = \sigma\gamma$. So if $\sigma \in \Gamma(1)\alpha_i\beta_j$, we have $\sigma' = \sigma\gamma \in \Gamma(1)\alpha_i\beta_j\gamma = \Gamma(1)\alpha_{s'(i)}\beta_{s(j)}$, i.e. $m(\alpha,\beta;\sigma) = m(\alpha,\beta;\sigma')$.

Solution 1.4.3. Simply, if $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2,\mathbb{R})^+$, we have $\alpha^{-1} = \frac{1}{\det\alpha}\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$. So

$$(g|\alpha^{-1})(z) = (\det\alpha)^{-k/2}(-cz+a)^{-k}(\det\alpha)^k g(\alpha^{-1}z) = (\det\alpha)^{k/2}(-cz+a)^{-k}g(\alpha^{-1}z)$$

Then we may take a variate changing as $z \mapsto \alpha z$, we have

$$(g|\alpha^{-1})(\alpha z) = (\det\alpha)^{k/2}(-c\alpha z+a)^{-k}g(z) = (\det\alpha)^{k/2}(\frac{\det\alpha}{cz+d})^{-k}g(z) = (\det\alpha)^{-k/2}(cz+d)^k g(z)$$

And

$$\mathrm{Re}(\alpha z) = \mathrm{Re}\left(\frac{ax+b+iay}{cx+d+icy}\right) = \frac{\det\alpha\cdot\mathrm{Re}(z) + ac|z|^2 + bd}{|cz+d|^2}$$
$$\mathrm{Im}(\alpha z) = \mathrm{Im}\left(\frac{ax+b+iay}{cx+d+icy}\right) = \frac{\det\alpha\cdot y}{|cz+d|^2}$$

So we have

$$\frac{\mathrm{d}(\mathrm{Re}(\alpha z))\ \mathrm{d}(\mathrm{Im}(\alpha z))}{(\mathrm{Im}(\alpha z))^2} = \frac{\mathrm{d}x\ \mathrm{d}y}{y^2}$$

Then

$$\begin{aligned}
&f(\alpha z)\overline{(g|\alpha^{-1})(\alpha z)}\mathrm{Im}(\alpha z)^k\frac{\mathrm{d}(\mathrm{Re}(\alpha z))\ \mathrm{d}(\mathrm{Im}(\alpha z))}{(\mathrm{Im}(\alpha z))^2}\\
=&(\det\alpha)^{-k/2}\overline{(cz+d)}^k f(\alpha z)\overline{g(z)}(\det\alpha)^k y^k|cz+d|^{-2k}\frac{\mathrm{d}x\ \mathrm{d}y}{y^2}\\
=&(\det\alpha)^{k/2}(cz+d)^k f(\alpha z)\overline{g(z)}y^k\frac{\mathrm{d}x\ \mathrm{d}y}{y^2}\\
=&(f|\alpha)(z)\overline{g(z)}y^k\frac{\mathrm{d}x\ \mathrm{d}y}{y^2}
\end{aligned}$$

Hence we have $\langle f|\alpha, g\rangle = \langle f, g|\alpha^{-1}\rangle$.

Solution 1.4.4. (a) Denote $\delta = \frac{d_1}{d_2}$. The direct part is trivial, since for any matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma(1)$, we have g.c.d.$(a,b,c,d) = 1$, then g.c.d.$(d_1a, d_1b, d_2c, d_2d) = d_1$, and left multiplying a matrix in $\Gamma(1)$ will not change the g.c.d. of four entries. For the inverse part, suppose we have a matrix $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{Mat}_2(\mathbb{Z})$ such that $\det\alpha = d_1d_2$ and g.c.d.$(a,b,c,d) = d_2$. We may assume $d_2 = 1$ by taking $d_2^{-1}\alpha$ instead of α. Then consider the lattice $\Lambda_1 = \mathbb{Z}^2$, $\Lambda_2 = \langle(a,b),(c,d)\rangle$. By Elementary

Divisor Theorem, there exist a basis ξ_1, ξ_2 of $\mathbb{Z}^2$ and positive integers D_1, D_2 such that $D_2 | D_1$ such that $D_1\xi_1, D_2\xi_2$ is a basis of Λ_2. Replacing ξ_1 by $-\xi_1$ if necessary, we may assume the matrix $\Xi = \left[\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right]$ has determinant 1, i.e. $\Xi \in \mathrm{SL}(2,\mathbb{Z})$. Then there exists a $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ such that $\gamma\alpha = D\Xi$, where $D = \left[\begin{smallmatrix} D_1 & \\ & D_2 \end{smallmatrix}\right]$, hence $\alpha \in \Gamma(1)D\Gamma(1)$. Then by direct part, we know $D_1 = \text{g.c.d.}(a,b,c,d) = 1$. And clearly $D_2 = \det \alpha$.

(b) Denote Clearly for any $\left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right] \in \Gamma(1)$ and any $a \in \mathbb{Z}$, we can do the Euclidean division algorithm

$$\begin{bmatrix} & 1 \\ -1 & a \end{bmatrix}\begin{bmatrix} d_1 x & d_1 y \\ d_2 z & d_2 w \end{bmatrix} = \begin{bmatrix} d_2 z & d_2 w \\ a d_2 z - d_1 x & a d_2 w - d_1 y \end{bmatrix}$$

Then in finite step for left multiplying some matrices $\left[\begin{smallmatrix} & 1 \\ -1 & a \end{smallmatrix}\right]$ for some a, the left lower term will be 0, i.e. $\left[\begin{smallmatrix} d_1 & \\ & d_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right] \in \Gamma(1)\left[\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right]$ for some a, b, d. And since

$$\begin{bmatrix} 1 & n \\ & 1 \end{bmatrix}\begin{bmatrix} a & b \\ & d \end{bmatrix} = \begin{bmatrix} a & b + nd \\ & d \end{bmatrix}$$

$$\begin{bmatrix} -1 & \\ & -1 \end{bmatrix}\begin{bmatrix} a & b \\ & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ & -d \end{bmatrix}$$

for any $n \in \mathbb{Z}$, we may assume b runs over $b \bmod d$ and $a > 0$. Since left multiplying a matrix in $\Gamma(1)$ does not change the g.c.d. of four entries of matrix, we have $\text{g.c.d.}(a,b,d) = \text{g.c.d.}(d_1,d_2) = d_2$. And clearly $ad = \det\left[\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right] = \det d_1 d_2 = d_1 d_2$, we have $d > 0$ and $ad = d_1 d_2$. And if $\left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} a' & b' \\ & d' \end{smallmatrix}\right]$ for some $\left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right] \in \Gamma(1)$, we have

$$\begin{bmatrix} a' & b' \\ & d' \end{bmatrix} = \begin{bmatrix} ax & bx + dy \\ az & dw + bz \end{bmatrix}$$

So $z = 0$. Then $\det\left[\begin{smallmatrix} x & y \\ & w \end{smallmatrix}\right] = xw = 1$, we have $x = w = 1$ and $a = a'$, $d = d'$ since a, d, a', d' are all positive. Moreover, since b, b' are actually in $\mathbb{Z}/d\mathbb{Z}$, they must equal to each other. So we have $\left[\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} a' & b' \\ & d' \end{smallmatrix}\right]$, i.e.

$$\Gamma(1)\left[\begin{smallmatrix} d_1 & \\ & d_2 \end{smallmatrix}\right]\Gamma(1) = \coprod_{\substack{a,d>0,\ ad=d_1 d_2 \\ b \bmod d \\ \text{g.c.d.}(a,b,d)=d_2}} \Gamma(1)\begin{bmatrix} a & b \\ & d \end{bmatrix}$$

(c) By (b), trivial

$$\Delta_n = \coprod_{\substack{d_1 d_2 = n \\ d_2 | d_1}} \coprod_{\substack{a,d>0,\ ad=d_1 d_2 \\ b \bmod d \\ \text{g.c.d.}(a,b,d)=d_2}} \Gamma(1)\begin{bmatrix} a & b \\ & d \end{bmatrix} = \coprod_{\substack{a,d>0, ad=n \\ b \bmod d}} \Gamma(1)\begin{bmatrix} a & b \\ & d \end{bmatrix}$$

Solution 1.4.5. The surjectivity is just from the Chinese Remainder Theorem. And the index of $\Gamma(N)$ in $\Gamma(1)$ have been determined at 1.3.8., which equals to $N^3 \prod_{p|N}(1 - p^2)$.

Solution 1.4.6. For any δ mod N with $(\delta, N) = 1$, we may pick a $\gamma_d = \left[\begin{smallmatrix} * & * \\ * & \delta \end{smallmatrix}\right] \in \Gamma_1(N)$. Then for any Dirichlet character χ modulo N, we can define a operator $\pi_\chi(f) = \phi(N)^{-1} \sum_{\delta \bmod N} \chi(\delta)^{-1} f|\gamma_\delta$ for any $f \in S_k(\Gamma_1(N))$. Then for any $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$, we have

$$\pi_\chi(f)|\gamma = \phi(N)^{-1} \sum_{\delta \bmod N} \chi(\delta)^{-1} f|(\gamma_\delta\gamma) = \chi(d)\phi(N)^{-1} \sum_{\delta \bmod N} \chi(\delta)^{-1}\chi(d)^{-1} f|(\gamma_\delta\gamma) = \chi(d) f$$

So the image of π_χ is $S_k(\Gamma_0(N), \chi)$. Moreover, we have

$$\pi_\chi^2(f) = \phi(N)^{-2} \sum_{\delta \bmod N} \sum_{\delta' \bmod N} \chi(\delta)^{-1}\chi(\delta')^{-1} f|\gamma_\delta\gamma_{\delta'} = \phi(N)^{-2} \sum_{d \bmod N} \phi(N)\chi(d) f|\gamma_d = \pi_\chi(f)$$

So π_χ is actually the projection from $S_k(\Gamma_1(N))$ to $S_k(\Gamma_0(N), \chi)$. And since

$$\sum_\chi \pi_\chi(f) = \phi(N)^{-1} \sum_{\delta \bmod N} (\sum_\chi \overline{\chi(\delta)} f|\gamma_\delta = \phi(N)^{-1} \cdot \phi(N) \cdot f|\gamma_1 = f$$

And for any different χ, χ', we may denote the nontrivial character $\chi_0 = \chi/\chi'$ and have

$$\begin{aligned}
\pi_\chi\pi_{\chi'}(f) &= \phi(N)^{-2} \sum_{\delta \bmod N} \sum_{\delta' \bmod N} \chi(\delta)^{-1}\chi'(\delta')^{-1} f|\gamma_\delta\gamma_{\delta'} \\
&= \phi(N)^{-2} \sum_{\delta \bmod N} \sum_{\delta' \bmod N} \chi_0(\delta)^{-1}\chi'(\delta\delta')^{-1} f|\gamma_\delta\gamma_{\delta'} \\
&= \phi(N)^{-2} \sum_{d \bmod N} \left(\sum_{\delta \bmod N} \overline{\chi_0(\delta)} \right) \chi'(d)^{-1} f|\gamma_d \\
&= 0
\end{aligned}$$

So $S_k(\Gamma_1(N)) = \bigoplus_\chi S_k(\Gamma_0(N), \chi)$. Finally, if $f \in S_k(\Gamma_0(N), \chi)$ and $f' \in S_k(\Gamma_0(N), \chi')$ for some $\chi \neq \chi'$, by 1.4.3. we have

$$\langle f, f' \rangle = \langle f|\gamma_d, f'|\gamma_d \rangle = \chi(d)\chi'(d)^{-1} \langle f, f' \rangle$$

for all d mod N. Since $\chi \neq \chi'$, there exists a d such that $\chi(d) \neq \chi'(d)$, hence $\langle f, f' \rangle = 0$.

Solution 1.4.7. If Γ is a congruence group, f is a modular form for Γ, we know $f|\gamma = f$ for any $\gamma \in \Gamma$. Then $(f|\alpha)|(\alpha^{-1}\gamma\alpha) = f|\alpha$. So we only need to prove that if Γ is a congruence group, there exists an $\alpha \in \mathrm{GL}(2, \mathbb{Q})^+$ such that $\alpha^{-1}\Gamma\alpha^{-1} \supset \Gamma_1(M)$ for some M. Since Γ is congruent, there exists some N such that $\Gamma(N) \subset \Gamma$, i.e. $\alpha^{-1}\Gamma(N)\alpha \subset \alpha^{-1}\Gamma\alpha$. So we only need to prove that for any N, there exists α and M such that $\Gamma_1(M) \subset \alpha^{-1}\Gamma(N)\alpha$. Just take $M = N^2$, $\alpha = \left[\begin{smallmatrix} N^{-1} & \\ & N^{-2} \end{smallmatrix}\right]$. Then for any $\gamma \in \Gamma_1(M)$, it must have the form $\gamma = \left[\begin{smallmatrix} 1+aN^2 & b \\ cN^2 & 1+dN^2 \end{smallmatrix}\right]$. Then $\alpha\gamma\alpha^{-1} = \left[\begin{smallmatrix} 1+aN^2 & bN \\ cN & 1+dN^2 \end{smallmatrix}\right] \in \Gamma(N)$, so $\gamma \in \alpha^{-1}\Gamma(N)\alpha$.

Solution 1.4.8. Suppose $\alpha \in G_0(N)$, by proposition 1.4.3., we have $\alpha = \gamma_1\delta\gamma_2$ for some $\gamma_1, \gamma_2 \in \mathrm{SL}(2, \mathbb{Z})$ and $\delta = \left[\begin{smallmatrix} d_1 & \\ & d_2 \end{smallmatrix}\right]$. Denote m as the l.c.m. of all denominators of entries in α, then $(m, N) = 1$, and $\gamma_1(m\delta)\gamma_2 = m\alpha \in \mathrm{Mat}_2(\mathbb{Z})$. Then by 1.4.4.(a) we know md_1, md_2 are positive integers with $md_2|md_1$, and md_2 =g.c.d. of all entries of $m\alpha$, which must be prime to N. So d_1, d_2 are positive elements of $\mathbb{Z}_\Sigma^\times$ and $d = d_1/d_2$ is a positive integer prime to N. Moreover, since

$$\begin{bmatrix} 1 & \\ -1 & 1 \end{bmatrix}\begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix}\begin{bmatrix} 1 & \\ d & 1 \end{bmatrix} = \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix}$$

For $\gamma_2 = \frac{x}{y}zw$, we have $(z, x) = 1$

$$\begin{bmatrix} 1 & \\ d & 1 \end{bmatrix}^k \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ kdx + z & kdy + w \end{bmatrix}$$

So we may take a k such that $N|kdx + z$ since $(d, N) = 1$. Then we may define $\gamma_2' = \left[\begin{smallmatrix} 1 & \\ d & 1 \end{smallmatrix}\right]^k \gamma_2$ and $\gamma_1' = \gamma_1 \begin{bmatrix} 1 & \\ -1 & 1 \end{bmatrix}^k$ so $\gamma_2' \in \Gamma_0(N)$. And moreover, if $\gamma_1' = \frac{x}{y}zw$, $\gamma_2' = \frac{x'}{y'}z'w'$, we have

$$\gamma_1'\delta\gamma_2' = \begin{bmatrix} * & * \\ d_2(dx'z + z'w) & * \end{bmatrix}$$

Since $(d_2, N) = 1$, and $N|d_2(dx'z + z'w)$, we have $N|dx'z + z'w$, so $N|x'z$. Since $(x', z') = 1$ and $N|z'$, we must have $(x', N) = 1$. So $N|z$, i.e. $\gamma_1' \in \Gamma_0(N)$. So we have a set of coset representatives for $\Gamma_0(N)\backslash G_0(N)/\Gamma_0(N)$ as $\left[\begin{smallmatrix} d_1 & \\ & d_2 \end{smallmatrix}\right]$. And the converse is trivial.

Solution 1.4.9. Clearly we have $\Gamma_0(N)\backslash\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N)\backslash\Gamma_0(N)\alpha\Gamma_0(N)\alpha^{-1}\alpha = (\Gamma_0(N)\cap\alpha\Gamma_0\alpha^{-1})\backslash\Gamma_0(N)\alpha$. Since for any $\gamma = \left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right] \in \Gamma_0(N)$, we have $\alpha\gamma\alpha^{-1} = \left[\begin{smallmatrix} x & dy \\ d^{-1}z & w \end{smallmatrix}\right]$, where $d = \frac{d_1}{d_2}$. So $\Gamma_0(N) \cap \alpha\Gamma_0(N)\alpha^{-1} = \Gamma_N^d = \{\left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right] \in \Gamma(1) \,|\, d|y, N|z\}$. Then clearly we have a right coset representative of $(\Gamma_0(N)\cap\alpha\Gamma_0\alpha^{-1})\backslash\Gamma_0(N)$ as $\{\left[\begin{smallmatrix} 1 & 0 \\ & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} 1 & d-1 \\ & 1 \end{smallmatrix}\right]\}$, i.e. $[\Gamma_0(N)\alpha\Gamma_0(N) : \Gamma_0(N)] = d$. For $\Gamma(1)\backslash\Gamma(1)\alpha\Gamma(1)$, similarly we we know it$\cong (\Gamma(1) \cap \alpha\Gamma(1)\alpha^{-1})\backslash\Gamma(1)\alpha$ and as a set of representatives $\{\left[\begin{smallmatrix} 1 & 0 \\ & 1 \end{smallmatrix}\right]\alpha, \ldots, \left[\begin{smallmatrix} 1 & d-1 \\ & 1 \end{smallmatrix}\right]\alpha\}$. So we may use the same representatives in both cases.

Solution 1.4.10. For any $\Gamma_0(N)\alpha\Gamma_0(N)$ for $\alpha \in G_0(N)$, by 1.4.8., we may assume $\alpha = \left[\begin{smallmatrix} d_1 & \\ & d_2 \end{smallmatrix}\right]$ for some $d_2|d_1$ in the positive part of $\mathbb{Z}_\Sigma^\times$. Then we may take some m coprime to N such that $md_1, md_2 \in \mathbb{Z}^+$. Then by 1.4.9., both $\Gamma_0(N)\backslash\Gamma_0(N)(m\alpha)\Gamma_0(N)$ and $\Gamma(1)\backslash\Gamma(1)(m\alpha)\Gamma(1)$ have representatives as $\{\left[\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right] m\alpha \,|\, 0 \le i < d\}$. So $\Gamma_0(N)\backslash\Gamma_0(N)\alpha\Gamma_0(N)$ and $\Gamma(1)\backslash\Gamma(1)\alpha\Gamma(1)$ have representatives as $\{\left[\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right] \alpha \,|\, 0 \le i < d\}$. T_α equals to the same one in $\mathcal{R}$ for all $\alpha \in G_0(N)$. So $\mathcal{R}_N$ is a subring of $\mathcal{R}$, hence it is commutative.

Solution 1.4.11. By 1.4.3., we have

$$\langle f|_\chi\alpha, g\rangle = \langle \chi(\alpha)^{-1}f|\alpha, g\rangle = \chi(\alpha)^{-1}\langle f, g|\alpha^{-1}\rangle = \chi(\alpha)^{-1}\langle f, \chi(\alpha^{-1})g|_\chi\alpha^{-1}\rangle = \langle f, g|_\chi\alpha^{-1}\rangle$$

So $\langle f|T_\alpha, g\rangle = \langle f, g|T_{\alpha^{-1}}\rangle$, i.e. $T_\alpha^* = T_{\alpha^{-1}}$. So $T_\alpha T_{\alpha^{-1}} = T_{\alpha^{-1}}T_\alpha = T_I$, hence normal w.r.t. the Petersson inner product.

Solution 1.4.12. By 1.4.11. and 1.4.1., trivial.

Solution 1.4.13. Similarly for every positive integer n, we can define

$$T(n) = \sum_{\substack{d_1 d_2 = n, d_2 | d_1 \\ (d_1, N) = 1}} T_{\alpha(d_1, d_2)}$$

So if $(n, N) \neq 1$, we have $T(n) = 0$, which has a little different with the $T(n)$ in $S_k(\Gamma(1))$. By proposition 1.4.4. or just exercise 1.4.4., we know

$$\begin{aligned}
(T(n)f)(z) &= \sum_{ad=n, a,d>0} \sum_{b \bmod d} \left(\frac{a}{d}\right)^{k/2} f\left(\frac{az+b}{d}\right) \\
&= \sum_{ad=n} \sum_{b \bmod d} \left(\frac{a}{d}\right)^{k/2} \sum_{m=1}^{\infty} A(m) e^{amz/d} e^{mb/d} \\
&= \sum_{m=1}^{\infty} \sum_{ad=n, d|m} \left(\frac{a}{d}\right)^{k/2} d e^{amz/d} A(m)
\end{aligned}$$

So the Fourier coefficients of $A(n)f$ are

$$B(m) = \sum_{\substack{ad=n \\ a|m}} \left(\frac{a}{d}\right)^{k/2} dA\left(\frac{md}{a}\right)$$

We may assume f is a normalized Hecke eigenform, and $T(n)f = n^{1-k/2}\lambda(n)f$. Then we have

$$n^{1-k/2}\lambda(n)A(m) = \sum_{\substack{ad=n \\ a|m}} \left(\frac{a}{d}\right)^{k/2} dA\left(\frac{md}{a}\right)$$

So if $(n, m) = 1$ and $(n, N) = 1$, we have $\lambda(n)A(m) = A(mn)$. So

$$L(s, f) = \sum A(n) n^{-s} = \left(\sum{}' A(n) n^{-s}\right) \prod_{p \nmid N} \left(\sum_{r=0}^{\infty} A(p^r) p^{-rs}\right)$$

And similarly we have $A(p^{r+1}) - A(p)A(p^r) + p^{k-1}A(p^{r-1}) = 0$ for any $k \geq 1$ and $p \nmid N$. So

$$L(s, f) = \left(\sum{}' A(n) n^{-s}\right) \prod_{p \nmid N} \left(1 - A(p)P^{-s} + p^{k-1-2s}\right)^{-1}$$

Solution 1.4.14. For any $\gamma = \left[\begin{smallmatrix} a & b \\ 6c & d \end{smallmatrix}\right] \in \Gamma_0(6)$, we know $\left[\begin{smallmatrix} a & 6b \\ c & d \end{smallmatrix}\right] \in \Gamma(1)$. So

$$\Delta(6\gamma z) = \Delta\left(\frac{a(6z)+6b}{c(6z)+d}\right) = \Delta(6z)$$

So $f \in S_{12}(\Gamma_0(6)) = S_{12}(\Gamma_0(6), 1)$. For the Fourier coefficients, we've got

$$\Delta(z) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 + \dots$$

in 1.3.7. Then for $\Delta(6z)$ and f, we have

$$\begin{aligned} \Delta(6z) &= & & q^6 + \dots \\ f(z) &= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - & & 6047q^6 + \dots \end{aligned}$$

i.e. $A(2) = -24$, $A(3) = 252$, $A(6) = -6047$, but $A(6) = A(2)A(3) + 1$.

1.5 Twisting

Solution 1.5.1. For any $\gamma = \left[\begin{smallmatrix} a & b \\ D^2Nc & d \end{smallmatrix}\right] \in \Gamma_0(D^2N)$, we have

$$\begin{aligned} f_\chi|\gamma &= \frac{\chi(-1)\tau(\chi)}{D} \sum_{\substack{m \bmod D \\ (m,D)=1}} \overline{\chi(m)} f \left\| \begin{bmatrix} D & m \\ & D \end{bmatrix} \begin{bmatrix} a & b \\ D^2Nc & d \end{bmatrix} \right. \\ &= \frac{\chi(-1)\tau(\chi)}{D} \sum_{\substack{m \bmod D \\ (m,D)=1}} \overline{\chi(m)} f \left\| \begin{bmatrix} a + DNmc & b + \frac{md}{D} \\ D^2Nc & d \end{bmatrix} \right. \\ &= \frac{\chi(-1)\tau(\chi)}{D} \sum_{\substack{m \bmod D \\ (m,D)=1}} \overline{\chi(m)} f \left\| \begin{bmatrix} a + DNmc & b - Nm^2cd^2 - md\frac{ad-1}{D} \\ D^2Nc & d - DNmcd^2 \end{bmatrix} \begin{bmatrix} D & md^2 \\ & D \end{bmatrix} \right. \\ &= \frac{\chi(-1)\tau(\chi)}{D} \sum_{\substack{m \bmod D \\ (m,D)=1}} \overline{\chi(m)} f \left\| \begin{bmatrix} a + DNmc & b - Nm^2cd^2 - DNmbcd \\ D^2Nc & d - DNmcd^2 \end{bmatrix} \begin{bmatrix} D & md^2 \\ & D \end{bmatrix} \right. \\ &= \frac{\chi(-1)\tau(\chi)}{D} \sum_{\substack{m \bmod D \\ (m,D)=1}} \overline{\chi(m)} \psi(d) f \left\| \begin{bmatrix} D & md^2 \\ & D \end{bmatrix} \right. \\ &= \frac{\chi(-1)\tau(\chi)\chi^2(d)\psi(d)}{D} \sum_{\substack{m \bmod D \\ (m,D)=1}} \overline{\chi(md^2)} f \left\| \begin{bmatrix} D & md^2 \\ & D \end{bmatrix} \right. \\ &= \chi^2(d)\psi(d) f_\chi \end{aligned}$$

So $f_\chi \in S_k(\Gamma_0(D^2N), \chi^2\psi)$.

Solution 1.5.2. On the line $\mathrm{Re}(s) = \pi/2$, we have

$$\left|e^{e^{-i(pi/2+it)}}\right| = \left|e^{e^{t-ipi/2}}\right| = \left|e^{ie^t}\right| = 1$$

And same for $\mathrm{Re}(s) = -\pi/2$. But if $\mathrm{Re}(s) \in (-\pi/2, \pi/2)$, we have

$$\left|e^{e^{-i(s+it)}}\right| = \left|e^{e^{t-is}}\right| = \left|e^{e^t(\cos(-s)+i\sin(-s))}\right| = e^{\cos(s)e^t}$$

which is unbounded.

1.6 The Rankin-Selberg Method

Solution 1.6.1. We may take some $0 < \varepsilon < \frac{y^2}{|x|}$, then we have $|mz+n|^2 = (x^2+y^2)m^2 + 2xmn + n^2 \geq (x^2+y^2)m^2 - 2|x|\cdot|m|\cdot|n| + n^2 \geq (x^2+y^2)m^2 - |x|(((|x|+\varepsilon)m^2 + (|x|+\varepsilon)^{-1}n^2) + n^2 = (y^2 - |x|\varepsilon)m^2 + \frac{\varepsilon}{|x|+\varepsilon}n^2$. So

$$\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} ((mx+n)^2 + (my)^2)^{-s} \leq \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} ((y^2 - |x|\varepsilon)m^2 + \frac{\varepsilon}{|x|+\varepsilon}n^2)^s$$

is convergent for all $\mathrm{Re}(s) > 1$. So $E(z,s)$ is convergent for all $\mathrm{Re}(s) > 1$. For any $\gamma = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] \in \Gamma(1)$, we have

$$\frac{\mathrm{Im}(\gamma z)}{|m(\gamma z)+n|^2} = \frac{\frac{y}{|cz+d|^2}}{\frac{|(am+cn)z+(bm+dn)|^2}{|cz+d|^2}} = \frac{y}{|(am+cn)z+(bm+dn)^2}$$

Since γ makes a permutation of $\mathbb{Z}^2 - \{(0,0)\}$, we have $E(z,s) = E(\gamma z, s)$.

Solution 1.6.2. (a) Since $f(x) = O((1+|x|)^M)$, we can define a function $F(x) = \sum_{m\in\mathbb{Z}^n} f(x+m)$. Then clearly F is periodic with period 1 for all variables x_i, so we have a Fourier expansion

$$F(x) = \sum_{m\in\mathbb{Z}^n} a_m e^{2\pi i\langle m,x\rangle}$$

And

$$a_m = \int_{[0,1]^n} F(x)e^{-2\pi i\langle m,x\rangle}\mathrm{d}x = \int_{[0,1]^n}\sum_{k\in\mathbb{Z}^n} f(x+k)e^{-2\pi i\langle m,x\rangle}\mathrm{d}x = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle m,x\rangle} = \hat{f}(-m)$$

So $\sum_{m\in\mathbb{Z}^n} f(m) = F(0) = \sum_{m\in\mathbb{Z}^n} a_m = \sum_{m\in\mathbb{Z}^n}\hat{f}(m)$.

(b) Consider the function $f_t(m,n) = e^{-\pi(m^2 z\bar{z} + (z+\bar{z})mn + n^2)t/y}$, then the Fourier transform of f_t is

$$\hat{f}_t(p,q) = \int_{\mathbb{R}^2} e^{-\pi(m^2 z\bar{z}+(z+\bar{z})mn+n^2)t/y}e^{2\pi i(mp+nq)}\mathrm{d}m\,\mathrm{d}n = \frac{1}{t}e^{-\pi|qz-p|^2/(ty)} = \frac{1}{t}f_{t^{-1}}(q,-p)$$

So by Poisson summation formula, we have $\Theta(t) = t^{-1}\Theta(t^{-1})$.

(c) Clearly

$$\begin{aligned}
\int_0^\infty (\Theta(t)-1)t^s \frac{\mathrm{d}t}{t} &= \int_0^\infty \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} e^{-\pi|mz+n|^2 t/y} t^s \frac{\mathrm{d}t}{t} \\
&= \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \int_0^\infty e^{-\pi|mz+n|^2 t/y} t^s \frac{\mathrm{d}t}{t} \\
&= \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \pi^{-s}\Gamma(s)\left(\frac{y}{|mz+n|^2}\right)^s \\
&= 2E(z,s)
\end{aligned}$$

So for the functional equation, we have

$$\begin{aligned}
E(z,s) &= \frac{1}{2}\int_0^\infty (\Theta(t)-1)t^s\frac{\mathrm{d}t}{t} \\
&= \frac{1}{2}\left(\int_1^\infty (\Theta(t)-1)t^s\frac{\mathrm{d}t}{t} + \int_0^1 (t^{-1}\Theta(t^{-1})-1)t^s\frac{\mathrm{d}t}{t}\right) \\
&= \frac{1}{2}\left(\int_1^\infty (\Theta(t)-1)t^s\frac{\mathrm{d}t}{t} + \int_1^\infty (\Theta(t)-1)t^{1-s}\frac{\mathrm{d}t}{t} + \int_1^\infty (1-t^{-1})t^{1-s}\frac{\mathrm{d}t}{t}\right) \\
&= \frac{1}{2}\left(\int_1^\infty (\Theta(t)-1)t^s\frac{\mathrm{d}t}{t} + \int_1^\infty (\Theta(t)-1)t^{1-s}\frac{\mathrm{d}t}{t}\right) + \frac{1}{2s(s-1)}
\end{aligned}$$

So $E(z,s) = E(z,1-s)$.

Solution 1.6.3. Here is a typo: the definition of $E_T(z,s)$ may be

$$E_T(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)\sum_{\gamma\in\overline{\Gamma_\infty}\backslash\overline{\Gamma(1)}} y_T(\gamma(z))^s$$

with a $\zeta(2s)$ in the equation.

(a) If K is compact, there exist positive constants C_0, C_1 such that $C_0 < y < C_1$ for all $z \in K$. Then for all $\gamma \in \overline{\Gamma_\infty}\backslash\overline{\Gamma(1)}$ represented by $\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] \in \Gamma(1)$, we have $\mathrm{Im}(\gamma z) = \frac{y}{|cz+d|^2}$. If $c = 0$, we may assume $|d| \geq 1$, then $\frac{y}{|cz+d|^2} = \frac{y}{d^2} \leq C_1$. If $c \neq 0$, we have $|c| \geq 1$, then $|cz+d| = |cx+d+icy| \geq |c|y$, then $\frac{y}{|cz+d|^2} \leq \frac{y}{|c|y} = |c|^{-1} \leq 1$. So we only need to take $T_0 = \max\{C_0, 1\}$, then for all $T > T_0$, we have $y_T(\gamma z) = \mathrm{Im}(\gamma z)$, i.e. $E_T(z,s) = E(z,s)$.

(b) Simply,

$$\begin{aligned}
\int_{\Gamma(1)\backslash\mathcal{H}} E_T(z,s)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} &= \pi^{-s}\Gamma(s)\zeta(2s)\sum_{\gamma\in\overline{\Gamma_\infty}\backslash\overline{\Gamma(1)}}\int_{\Gamma(1)\backslash\mathcal{H}} y_T(\gamma z)^s\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \pi^{-s}\Gamma(s)\zeta(2s)\int_{\overline{\Gamma_\infty}\backslash\mathcal{H}} y_T(z)^s\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \pi^{-s}\Gamma(s)\zeta(2s)\int_{-1/2}^{1/2}\int_0^T (y)^{s-2}\mathrm{d}y\,\mathrm{d}x \\
&= \pi^{-s}\Gamma(s)\zeta(2s)\frac{T^{s-1}}{s-1}
\end{aligned}$$

(c) If z is in the fundamental domain, we know $|z| > 1$, and $-1/2 < x < 1/2$. Then for any $\gamma = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] \in \Gamma(1)$, if $c = 0$, we have $d^2 \geq 1$, and $\frac{y}{|cz+d|^2} = \frac{y}{d^2}$. So if $y < T$, we have $\frac{y}{d^2} < T$, i.e. $y_T(\gamma z) = \mathrm{Im}(\gamma z)$. And for the $c \neq 0$ part, we have $\frac{y}{|cz+d|^2} \leq 1$ as in (a), so $y_T(\gamma z) = \mathrm{Im}(\gamma z)$ since $T > 1$. Then when $y < T$, we have $E_T(z,s) = E(z,s)$. And when $kT \leq y < (k+1)T$, we have

$$E_T(z,s) - E(z,s) = -\pi^{-s}\Gamma(s)\zeta(2s)y^s \cdot \sum_{i=1}^{k} i^{-2}$$

(d) Fix a sufficiently small ε. Denote C as a circle around 1 of radius ε. Then

$$\int_C\int_{\Gamma(1)\backslash\mathcal{H}} |E_T(z,s)|\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}\mathrm{d}s \leq \int_C |\pi^{-s}\Gamma(s)\zeta(2s)\frac{T^{s-1}}{s-1}|\mathrm{d}s \leq 2\pi\frac{T^\varepsilon}{\varepsilon}\pi^{-1+\varepsilon}\Gamma(1+\varepsilon)\zeta(2+2\varepsilon)$$

So this integration is convergent absolutely, i.e. the two integration can commute. So the integration can interchange with residue. So

$$\begin{aligned}
\frac{1}{2}\mathrm{vol}(\Gamma(1)\backslash\mathcal{H}) &= \mathrm{res}_{s=1}\int_{\Gamma(1)\backslash\mathcal{H}} E(z,s)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \int_{\Gamma(1)\backslash\mathcal{H}} \mathrm{res}_{s=1}E(z,s)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \int_{\Gamma(1)\backslash\mathcal{H}} \mathrm{res}_{s=1}E_T(z,s)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \mathrm{res}_{s=1}\int_{\Gamma(1)\backslash\mathcal{H}} E_T(z,s)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \mathrm{res}_{s=1}\pi^{-s}\Gamma(s)\zeta(2s)\frac{T^{s-1}}{s-1} = \pi^{-1}\Gamma(1)\zeta(2)T^0 = \frac{\pi}{6}
\end{aligned}$$

Solution 1.6.4. For any cusp eigenform g, we have $\int_{-1/2}^{1/2} g(z)\mathrm{d}x = 0$. So

$$\begin{aligned}\langle E_k, g\rangle &= \int_{\Gamma(1)\backslash\mathcal{H}} E_k(z)\overline{g(z)}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\ &= \int_{\Gamma_\infty\backslash\mathcal{H}} \mathrm{Im}(z)\overline{g(z)}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\ &= \int_0^\infty \left(\int_{-1/2}^{1/2}\overline{g(z)}\mathrm{d}x\right)\frac{\mathrm{d}y}{y} = 0\end{aligned}$$

Solution 1.6.5. (a) Since $\phi(s)$ is an analytic function on half real axis $s > \sigma$, it must be convergent at the point $s = \sigma - \varepsilon$ for some small ε. Then we may denote $\phi_n(s) = \sum_{k=1}^n a_k k^{-s}$. And we have

$$\phi_{m,n} = \sum_{k=m+1}^{n} \frac{a_k}{k^{\sigma-\varepsilon}}\frac{1}{k^{s-\sigma+\varepsilon}} = \frac{\phi_n(\sigma-\varepsilon)}{n^{s-\sigma+\varepsilon}} - \frac{\phi_m(\sigma-\varepsilon)}{m^{s-\sigma+\varepsilon}} + \sum_{k=m}^{n-1}\phi_k(\sigma-\varepsilon)\psi_k$$

where $\psi_k = k^{-(s-\sigma+\varepsilon)} - (k+1)^{-(s-\sigma+\varepsilon)} = (s-\sigma+\varepsilon)\int_k^{k+1} x^{-(s-\sigma+\varepsilon+1)}\mathrm{d}x$. Since $|\phi_n(\sigma-\varepsilon)| < C$ for some constant C, for any s satisfying $\mathrm{Re}(s) \geq \sigma$, we have

$$\begin{aligned}\left|\sum_{k=m}^{n-1}\phi_k(\sigma-\varepsilon)\psi_k\right| &\leq C\sum_{k=m}^{n-1}|\psi_k| \leq C|s-\sigma+\varepsilon|\sum_{k=m}^{n-1}\int_k^{k+1} x^{-(\mathrm{Re}(s)-\sigma+\varepsilon+1)}\mathrm{d}x \\ &\leq C|s-\sigma|(\mathrm{Re}(s)-\sigma+\varepsilon)^{-1}\left(m^{-(\mathrm{Re}(s)-\sigma+\varepsilon)} - n^{-(\mathrm{Re}(s)-\sigma+\varepsilon)}\right) \\ &\leq C|s-\sigma|(\mathrm{Re}(s)-\sigma+\varepsilon)^{-1}m^{-(\mathrm{Re}(s)-\sigma+\varepsilon)} \longrightarrow 0\end{aligned}$$

So $\phi(s) = \lim_{n\to\infty} f_{1,n}$ is convergent for all s satisfying $\mathrm{Re}(s) \geq \sigma$.

(b) By theorem 1.6.2., $\Lambda(s, f\times f)$ has holomorphic continuation on the field $\mathbb{C} - \{k, k-1\}$, so $\phi(s) = \sum a(n)^2 n^{-s}$ has holomorphic continuation on this field. Then for any $\varepsilon > 0$, $\phi(s)$ has analytic continuation on a neighbourhood of the half plane $\mathrm{Re}(s) \geq k+\varepsilon$. By (a) we know $\phi(s)$ is convergent for all s satisfying $\mathrm{Re}(s) \geq k+\varepsilon$. Since ε is ambiguous, $\phi(s)$ is convergent for all s with $\mathrm{Re}(s) > k$.

(c) By (b), we know the abscissa of convergence of $\psi(s) = \sum a(n)^2 n^{-s}$ satisfies

$$\sigma' = \limsup_{N\to\infty}\frac{\log\left(\sum_{n=1}^N a(n)^2\right)}{\log(N)} \leq k$$

For any $\lambda > 0$, we have $|a(n)| \leq \max\{n^\lambda, n^{-\lambda}|a(n)|^2\}$ and $|a(n)| \geq \min\{n^\lambda, n^{-\lambda}|a(n)|^2\}$. Then the abscissa of convergence of $\phi(s) = \sum |a(n)|n^{-s}$ is

$$\begin{aligned}
\sigma &= \limsup_{N\to\infty} \frac{\log\left(\sum_{n=1}^N a(n)\right)}{\log(N)} \le \limsup_{N\to\infty} \frac{\log\left(\sum_{n=1}^N \max\{n^\lambda, n^{-\lambda}|a(n)|^2\}\right)}{\log(N)} \\
&\le \max\left\{\limsup_{N\to\infty} \frac{\log\left(\sum_{n=1}^N n^{-\lambda}a(n)^2\right)}{\log(N)},\ \limsup_{N\to\infty} \frac{\log\left(\sum_{n=1}^N n^\lambda\right)}{\log(N)}\right\} \\
&= \max\left\{\limsup_{N\to\infty} \frac{\log N^{-\lambda} + \log\left(\sum_{n=1}^N a(n)^2\right)}{\log(N)},\ \limsup_{N\to\infty} \frac{\log \frac{N^{\lambda+1}}{\lambda+1}}{\log(N)}\right\} \\
&\le \max\{\sigma' - \lambda, \lambda + 1\}
\end{aligned}$$

And similarly we have

$$\sigma \ge \limsup_{N\to\infty} \frac{\log\left(\sum_{n=1}^N \min\{n^\lambda, n^{-\lambda}|a(n)|^2\}\right)}{\log(N)} \ge \min\{\sigma' - \lambda, \lambda + 1\}$$

So when $\lambda = \frac{\sigma'-1}{2}$, the bound $\{\sigma' - \lambda, \lambda + 1\}$ is just a point $\frac{\sigma'+1}{2}$, i.e. $\sigma = \frac{\sigma'+1}{2} \le \frac{k+1}{2}$. So $\sum a(n)n^{-s}$ converge absolutely on the half plane $\mathrm{Re}(s) > \frac{k+1}{2}$.

1.7 Hecke Characters and Hilbert Modular Forms

Solution 1.7.1. Clearly,

$$\begin{aligned}
L(s,\psi)L(s,\psi\chi_D) &= \left(\prod_p (1-\psi(p)p^{-s})^{-1}\right)\left(\prod_p (1-\psi(p)\chi_D(p)p^{-s})^{-1}\right) \\
&= \left(\prod_{\chi_D(p)=1} (1-\psi(p)p^{-s})^{-2}\right)\left(\prod_{\chi_D(p)=-1} (1-\psi(p)^2p^{-2s})^{-1}\right)\left(\prod_{\chi_D(p)=0} (1-\psi(p)p^{-s})^{-1}\right) \\
&= L(s, \psi \circ N)
\end{aligned}$$

Solution 1.7.2. For every Schwartz function f, we define a function $F(x) = \sum_{a\in\mathfrak{o}} f(x+a)$. Then F is clearly $\mathfrak{o}$-periodic, hence it has a Fourier expansion $F(x) = \sum_{\nu\in\mathfrak{o}} A(\nu)e^{2\pi i \mathrm{tr}(\nu x/d)}$. And

$$A(\nu) = \frac{1}{\sqrt{D}}\int_{\mathfrak{o}\backslash\mathbb{R}^n} F(x)e^{-2\pi i\mathrm{tr}(\nu x/d)}\mathrm{d}x \ = \frac{1}{\sqrt{D}}\int_{\mathbb{R}^n} f(x)e^{-2\pi i\mathrm{tr}(\nu x/d)}\mathrm{d}x \ = \hat{f}(\nu)$$

So

$$\sum_{\alpha\in\mathfrak{o}} f(\alpha) = F(0) = \sum_{\alpha\in\mathfrak{o}} A(\alpha) = \sum_{\alpha\in\mathfrak{o}} \hat{f}(\alpha)$$

Solution 1.7.3. If $\mathfrak{a} = \gamma\mathfrak{o} + \delta\mathfrak{o}$, there exists $\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] \in \mathrm{SL}(2,\mathfrak{o})$ such that $\gamma = a\alpha + b\beta$, $\delta = c\alpha + d\beta$. So

$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right](\frac{\alpha}{\beta}) = \frac{\gamma}{\delta}$, i.e. they are in the same $\mathrm{SL}(2, \mathfrak{o})$-orbit. So ξ_A is independent on the choice of α, β. If $\mathfrak{a} = \mathfrak{a}' \cdot (f)$ for some $f \in \mathfrak{o}$, so if $\mathfrak{a}' = \alpha\mathfrak{o} + \beta\mathfrak{o}$, we have $\mathfrak{a} = f\alpha\mathfrak{o} + f\beta\mathfrak{o}$, and $\frac{\alpha}{\beta} = \frac{f\alpha}{f\beta}$. So ξ_A is independent on the choice of $\mathfrak{a}$, hence we have a map $\mathrm{Cl}(F) \to \Gamma\backslash\mathbb{P}^1(F)$. This map is clearly surjective. For injectivity, if there exists two ideals $\mathfrak{a} = \alpha\mathfrak{o} + \beta\mathfrak{o}$ and $\mathfrak{a}' = \alpha'\mathfrak{o} + \beta'\mathfrak{o}$ such that $\frac{\alpha}{\beta} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right](\frac{\alpha'}{\beta'})$ for some $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathfrak{o})$, we know $\mathfrak{a}' = \alpha'\mathfrak{o} + \beta'\mathfrak{o} = (a\alpha' + b\beta')\mathfrak{o} + (c\alpha' + d\beta')\mathfrak{o}$, so we may assume $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right]$. Then $\frac{\alpha}{\beta} = \frac{\alpha'}{\beta'}$, so clearly $\mathfrak{a}$ and $\mathfrak{a}'$ are in the same class of $\mathrm{Cl}(F)$.

Solution 1.7.4. For any $\alpha \in \mathrm{GL}(2, F)^+$, similarly with lemma 1.4.1., there exists an ideal $\mathfrak{m} \subset \mathfrak{o}$ such that $\alpha\Gamma\alpha^{-1} \supset \Gamma(\mathfrak{m}) = \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathfrak{o}) \mid c \in \mathfrak{m}\}$. So we know $\Gamma\backslash\Gamma\alpha\Gamma \cong (\Gamma \cap \alpha\Gamma\alpha^{-1})\backslash\Gamma\alpha$ is a finite set, i.e. we may write

$$\Gamma\alpha\Gamma = \bigcup_{i=1}^{N} \Gamma\alpha_i$$

for some $\alpha_i \in \mathrm{GL}(2, F)^+$. Then for any $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathbb{R})^+ \cap \mathrm{Mat}(2, \mathfrak{o})$, we can define

$$(f|\gamma)(z) = (N(\det\gamma))^{k/2}\left(\prod(c_j z_j + d_j)\right)^{-k} f(\gamma(z))$$

Then we can define the Hecke operator T_α as

$$f|T_\alpha = \sum f|\alpha_i$$

Then all T_α form a Hecke algebra $\mathcal{R}_F$, because the associativity and the commutativity are the same with the case $F = \mathbb{Q}$ since we've assumed that F is a totally real field with narrow class number 1. Since we may define a Peterson inner product for cusps Hilbert modular forms f, g as

$$\langle f, g\rangle = \int_{\Gamma\backslash\mathcal{H}^n} f(z)\overline{g(z)}\mathbb{N}(y)^k \prod_i \frac{\mathrm{d}x_i\,\mathrm{d}y_i}{y_i^2}$$

And similarly, the action of $\mathcal{R}_F$ is self-adjoint in the Hermite space of cusp forms. So if f is an T_α-eigenform, f must be eigen for all operator T in $\mathcal{R}_F$.

Then for any $\nu \in \mathfrak{o}^+$, we may define

$$T(\nu) = \sum_{\substack{\delta_1,\delta_2 \gg 0 \\ \delta_1/\delta_2 \in \mathfrak{o} \\ \delta_1\delta_2 = \nu}} T_{\alpha(\delta_1,\delta_2)}$$

where $\alpha(\delta_1, \delta_2) = \left[\begin{smallmatrix} \delta_1 & \\ & \delta_2 \end{smallmatrix}\right]$. So we have

$$f|T(\nu) = \sum_{\substack{\alpha,\delta \gg 0 \\ \alpha\delta = \nu \\ \beta \bmod \delta}} f|\left[\begin{smallmatrix} \alpha & \beta \\ & \delta \end{smallmatrix}\right]$$

So

$$\begin{aligned}(T(\nu)f)(z) &= \sum_{\alpha\delta=\nu}\sum_{\beta \bmod \delta} \mathbb{N}(\alpha/\delta)^{k/2} f\left(\frac{\alpha z+\beta}{\delta}\right)\\ &= \sum_{\alpha\delta=\nu}\sum_{\beta \bmod \delta} \mathbb{N}(\alpha/\delta)^{k/2} \sum_{\mu\in\mathfrak{o}_+} A(\mu) e^{2\pi i \mathrm{tr}(\alpha z\nu/d\delta)} e^{2\pi i \mathrm{tr}(\beta\nu/d\delta)}\\ &= \sum_{\mu\in\mathfrak{o}_+}\sum_{\substack{\alpha\delta=\nu\\ \delta|\mu}} \mathbb{N}(\alpha/\delta)^{k/2}\mathbb{N}(\delta) e^{2\pi i \mathrm{tr}(\alpha\mu z/d\delta)} A(\mu)\end{aligned}$$

So if f is a Hecke eigenform, we may define $\lambda(\nu)$ as $T(\nu)f = \mathbb{N}(\nu)^{1-k/2}\lambda(\nu)f$. Then we have

$$\mathbb{N}(\nu)^{1-k/2}\lambda(\nu)A(\mu) = \sum_{\substack{\alpha\delta=\nu\\ \alpha|\mu}} \mathbb{N}(\alpha/\delta)^{k/2}\mathbb{N}(\delta)A(\mu\delta/\alpha)$$

So if $(\mu,\nu) = 1$, we have $\lambda(\nu)A(\mu) = A(\mu\nu)$. Then if $\mu = 1$, we have $\lambda(\nu)A(1) = A(\nu)$. So if f is normalized, i.e. $A(1) = 1$, we have $A(\mu)A(\nu) = A(\mu\nu)$ if $(\mu,\nu) = 1$. And for the non-coprime case, if $\mu = \pi$ and $\nu = \pi^r$ for some prime $\pi \in \mathfrak{o}_+$, then we have

$$A(\pi^{r+1}) - A(\pi)A(\pi^r) + \mathbb{N}(\pi)^{k-1}A(p^{r-1}) = 0$$

Then

$$L(s,f) = \sum_{\mu\in\mathfrak{o}_+} A(\mu)\mathbb{N}(\mu)^{-s} = \prod_{\mathfrak{p}\text{ prime}}\sum_{n=0}^{\infty} A(\mathfrak{p}^n)\mathbb{N}(\mathfrak{p}^n)^{-s} = \prod\left(1 - A(\mathfrak{p})\mathbb{N}(\mathfrak{p})^{-s} + \mathbb{N}(\mathfrak{p})^{k-1-2s}\right)^{-1}$$

Solution 1.7.5. By definition we clearly know $\mathrm{Cl}^+(F) = 1 \Rightarrow \mathrm{Cl}(F) = 1$. And clearly $\mathfrak{o}_+^\times \supset \{u^2 \mid u \in \mathfrak{o}^\times\}$, so we clearly know every totally positive unit is a square$\Leftrightarrow [\mathfrak{o}^\times : \mathfrak{o}_+^\times] = 2^{n-1}$. So we may assume $\mathrm{Cl}(F) = 1$ and prove that $\mathrm{Cl}^+(F) = 1 \Leftrightarrow [\mathfrak{o}^\times : \mathfrak{o}_+^\times] = 2^n$.

Denote the group of all principal fractional ideals as P, and the group of totally positive principal fractional ideals as P_+, then we have $0 \to P/P_+ \to \mathrm{Cl}^+(F) \to \mathrm{Cl}(F) \to 0$. Then we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{o}_+^\times \arrow[r] \arrow[d] & F_+^\times \arrow[r] \arrow[d] & P_+ \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathfrak{o}^\times \arrow[r] & F^\times \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

So by snake lemma, we know $0 \to \mathfrak{o}^\times/\mathfrak{o}_+^\times \to F^\times/F_+^\times \to P/P_+ \to 0$. Since we have a morphism $\psi : F^\times \to \prod \mathbb{R}^\times/(\mathbb{R}^\times)^2 \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus n}$, and by weak approximation theorem, ψ is a surjection, we have

$\ker\psi = F_+^\times$. So we have

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{o}^\times/\mathfrak{o}_+^\times \arrow[r, "f"] \arrow[dr] & F^\times/F_+^\times \arrow[r] \arrow[d, "\cong"', "\psi"] & P/P_+ \arrow[r] & 0 \\ & & (\mathbb{Z}/2\mathbb{Z})^{\oplus n} & & \end{tikzcd}$$

So we have $\mathrm{Cl}(F) = \mathrm{Cl}^+(F) \Leftrightarrow f$ is surjective$\Leftrightarrow [\mathfrak{o}^\times : \mathfrak{o}_+^\times] = 2^n$.

Solution 1.7.6. When $p \equiv 3 \bmod 4$, the integer ring of $\mathbb{Q}(\sqrt{p})$ is $\mathbb{Z}[\sqrt{p}]$. Then if we have $a^2 - pb^2 = \pm 1$ for some integer a, b, we know $a \pm b\sqrt{p}$ is a unit in $\mathbb{Z}[\sqrt{p}]^\times$. Since $a^2, b^2 \equiv 0, 1 \bmod 4$, and $p \equiv 3 \bmod 4$, we know $a^2 - pb^2 \equiv 0, 1, 2 \bmod 4$, hence $a^2 - pb^2 = 1$. So for any $a > 0$, $a \pm b\sqrt{p} \in \mathbb{Z}[\sqrt{p}]^\times$, we have $a \pm b\sqrt{p} > 0$, i.e. it is totally positive. Hence $\mathbb{Z}[\sqrt{p}]^\times : \mathbb{Z}[\sqrt{p}]_+^\times] = 2 < 4$. Then by 1.7.5., $\mathrm{Cl}^+(\mathbb{Q}(\sqrt{p})$ cannot be 1.

1.8 Artin L-Functions and Langlands Functoriality

Solution 1.8.1. It seems to ask us to prove the equation 8.12. in general case according to the context. And actually I think the equation 8.10. need to quote the exercise 1.3.3. in section 1.3. of Serge Lang's *Complex Analysis*.

In the general case, for any $E/K/F$, and any $\mathfrak{p} \subset \mathfrak{o}_F$, we may assume $\mathfrak{p}\mathfrak{o}_K = (\mathfrak{P}_1 \dots \mathfrak{P}_g)^e$, and each $\mathfrak{P}_i$ has residue class degree f. Then $efg = n = \deg(K/F)$. We may pick a set of representatives $\{\gamma_1 = 1, \gamma_2, \dots, \gamma_n\}$ of $\mathrm{Gal}(E/F)/\mathrm{Gal}(E/K)$. Then for each $\sigma \in \mathrm{Gal}(E/F)$, if $\sigma \in \mathrm{Gal}(E/K)$, we can define

$$\tau'(\sigma) = \begin{bmatrix} \tau(\sigma) & & & \\ & \tau(\gamma_2^{-1}\sigma\gamma_2) & & \\ & & \cdots & \\ & & & \tau(\gamma_n^{-1}\sigma\gamma_n) \end{bmatrix}$$

For each $\sigma \in \mathrm{Gal}(E/F)$ not in $\sigma \in \mathrm{Gal}(E/K)$, we may assume $\gamma_i\sigma \in \mathrm{Gal}(E/K)$ for some i, then we may define $\tau'(\sigma) = (T_{kl})_{1 \le k, l \le n}$, where

$$T_{kl} = \begin{cases} \tau(\gamma_{s_i(k)}\sigma\gamma_{s_i(k+1)}^{-1}) & \text{if } l = s_i(k) \\ 0 & \text{if not} \end{cases}$$

where $s_i \in S_n$ is the permutation mapping k to the index of $\gamma_i\gamma_k$. So since $g(\mathfrak{p}) = g$, we know

$\gamma_i^g \in \mathrm{Gal}(E/K)$. Then $\tau'(\sigma)^{kg}$ is diagonal and has the form

$$\tau'(\sigma)^{kg} = \begin{bmatrix} \tau(\sigma)^{kg} & & & \\ & \tau(\gamma_2\sigma\gamma_2^{-1})^{kg} & & \\ & & \cdots & \\ & & & \tau(\gamma_n\sigma\gamma_n^{-1})^{kg} \end{bmatrix}$$

Since $\gamma_i, \gamma_i^2, \dots, \gamma_i^{g-1}$ interchange $\mathfrak{P}_1$ to $\mathfrak{P}_2, \dots, \mathfrak{P}_g$, we know if $\alpha_{1,1}, \dots, \alpha_{1,n}$ are the eigenvalues of $\tau(\sigma)$, then $\alpha_{k,1}, \dots, \alpha_{k,n}$ are the eigenvalues of $\tau(\gamma_k\sigma\gamma_k^{-1})$. And moreover, since $\mathrm{tr}(\tau'(\sigma)^{kg}) = n\mathrm{tr}(\tau(\sigma)^{kg}) = n\sum a_i^{kg}$ for all k, and for any $g \nmid k$, we have $\mathrm{tr}(\tau'(\sigma)^k) = 0$. So the eigenvalues of $\tau'(\sigma)$ are $\zeta_f^k a_{i,j}^{1/f}$ for all i, j, k. So

$$L_{\mathfrak{p}}(s, \tau') = \prod_{i,j}\prod_k (1 - \zeta_f^k a_{i,j}^{1/f}\mathbb{N}(\mathfrak{p})^{-s})^{-1} = \prod_{i,j}(1 - a_{i,j}\mathbb{N}(\mathfrak{p})^{-fs})^{-1} \prod_{i,j}(1 - a_{i,j}\mathbb{N}(\mathfrak{P}_i)^{-s})^{-1} = \prod_i L_{\mathfrak{P}_i}(s, \tau)^e$$

Solution 1.8.2. (a) We may assume $H\backslash G = \{g_1, \dots, g_n\}$. Then for any $g \in G$, we may assume $g = hg_i$ for some i and $h \in H$. So $(\rho_H \otimes \tau)^G(g) = g_i \cdot (\rho_H \otimes \tau)(h) = g_i \cdot (\rho \otimes \tau^G)(h) = (\rho \otimes \tau^G)(g)$ since $\rho(h)$ is just a nonzero complex number. So $(\rho_H \otimes \tau)^G \cong \rho \otimes \tau^G$.

(b) If $E/K/F$ are Galois extensions, we know $\mathrm{Gal}(E/K)$ is a subgroup of $\mathrm{Gal}(E/F)$. Then for any Galois representation $(\rho, \mathrm{Gal}(E/F))$ and $(\tau, \mathrm{Gal}(E/K))$, we know

$$L(s, (\rho_{\mathrm{Gal}(E/K)} \otimes \tau)^{\mathrm{Gal}(E/F)}) = L(s, (\rho \otimes \tau^{\mathrm{Gal}(E/F)})$$

1.9 Maass Forms

Solution 1.9.1. Clearly $\mathrm{SL}(2, \mathbb{R})$ is generated by matrices U_a, T_x and S, where $U_a = \left[\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right]$, $T_x = \left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right]$ and $S = \left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]$. So we only need to prove that Δ is invariant under these matrices. Then for any smooth function f on $\mathcal{H}$, by Riemann-Cauchy theorem we know $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$, then

$$\begin{aligned}
\Delta(f(\left[\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right] z)) &= \Delta(f(a^2 z)) = \Delta(f)(a^2 z) \\
\Delta(f(\left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right] z)) &= \Delta(f(z + x)) = \Delta(f)(z + x) \\
\Delta(f(\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right] z)) &= -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(f \circ \left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right])(z) = -y^2 \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f \circ \left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right])(z) \\
&= -y^2 \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x}(\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right] z) \cdot |z|^{-2} - i\frac{\partial f}{\partial y}(\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right] z) \cdot |z|^{-2}\right) \\
&= -\mathrm{Im}(\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right] z)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(f)(\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right] z) = \Delta(f)(\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right] z)
\end{aligned}$$

Solution 1.9.2. Since $a_0 = \int_0^1 f(z)\mathrm{d}x$, we have

$$\Delta a_0 = \int_0^1 \Delta(f)(z)\mathrm{d}x \ = \nu \int_0^1 f(z)\mathrm{d}x \ = \nu a_0$$

where $\nu = s(1-s)$. So a_0 satisfies the differential equation $\Delta a_0 = s(1-s)a_0$, which has two solutions y^s and y^{1-s}. So a_0 is the linear combination of these two solutions.

Solution 1.9.3. Since f is a Maass form, we may pick a C and an N such that $f(x+iy) < Cy^N$ for all $x \in [0,1]$. Then since f is cuspidal, we have $\int_0^1 f(z+x)\mathrm{d}x \ = 0$ for all z, so $0 = \lim_{y\to\infty} \int_0^1 f(x+iy)\mathrm{d}x \ = \lim_{y\to\infty} f(iy)$. So $N = 0$ when f is cusp. Then for $n > 0$,

$$|a_n| \sqrt{y} K_\nu(2\pi ny)e^{-2\pi ny} = \left| \int_0^1 f(x+iy)e^{-2\pi inx}\mathrm{d}x \right| \leq \int_0^1 |f(x+iy)|\mathrm{d}x \ = O(1)$$

Since f is $\mathrm{SL}(2,\mathbb{Z})$-invariant, we know when $y \to 0$, we also have this $O(1)$ estimation. So if we take $y = \frac{1}{n}$, we have $|a_n|n^{-1/2}K_\nu(2\pi)e^{-2\pi} = O(1)$, i.e. $|a_n| = O(n^{1/2})$.

Solution 1.9.4 (Hecke theory for Maass forms)**.** The commutativity with the Laplacian operator is just 1.9.1. And the property of self-adjoint in the Petersonn inner product is the same with that in modular form w.r.t. $\mathrm{SL}(2,\mathbb{Z})$. So the space Maass cusp form has a diagonal decomposition. If $f = \sum a_n \sqrt{y}K_\nu(2\pi|n|y)e^{2\pi inx}$ is an eigenfunction, we have

$$\begin{aligned}
(T(n)f)(z) &= \sum_{ad=n} \sum_{b \bmod d} f\left(\frac{az+b}{d}\right) \\
&= \sum_{ad=n} \sum_{b \bmod d} \sum_{m\neq 0} a_m \sqrt{\frac{ay}{d}} K_\nu\left(\frac{2\pi|m|ay}{d}\right) e^{2\pi imax/d} e^{2\pi imb/d} \\
&= \sum_{m\neq 0} \sum_{\substack{ad=n \\ d|m}} da_m \sqrt{\frac{ay}{d}} K_\nu\left(\frac{2\pi|m|ay}{d}\right) e^{2\pi imax/d} \\
&= \sum_{m\neq 0} \sum_{d|(m,n)} a_m \sqrt{ny} K_\nu(2\pi|m|ny/d^2) e^{2\pi imnx/d^2}
\end{aligned}$$

So since f is eigenfunction, we may assume $T_n f = \sqrt{n}\lambda_n f$, then we have

$$\sqrt{n}\lambda_n a_k = \sum_{\substack{k=mn/d^2 \\ d|(n,m)}} a_m \sqrt{n}$$

Then if $(k,n) = 1$, the summation just has one term $d = n$, $m = k$, then $\lambda_n a_k = a_{kn}$. When $k = 1$, we have $\lambda_n a_1 = a_n$. So if f is normalized, i.e. $a_1 = 1$, we have $\lambda_n = a_n$. And if $(n,m) = 1$, we have $a_n a_m = a_{nm}$. For the non-coprime case, if $k = p$ and $n = p^r$, the summation has two terms, the first is $d = p^r$ and $m = p^{r+1}$, and the second is $d = p^{r-1}$ and $m = p^{r-1}$. So we have $a_{p^r}a_p = a_{p^{r-1}} + a_{p^{r+1}}$, i.e.

$$(1 - a_p X + X^2)\left(\sum_{r=0}^{\infty} a_{p^r} X^r\right) = 1$$

So we have

$$L(s,f) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p \left(\sum_{r=0}^{\infty} a_{p^r} p^{-rs}\right) = \sum_p (1 - a_p p^{-s} + p^{-2s})^{-1}$$

Solution 1.9.5. In the case $\sigma(-1) = (-1)^\epsilon$, f_σ is an odd form. So we have

$$\begin{aligned} f_\sigma &= \sum_{\substack{m \bmod p \\ p\nmid m}} \sum_{n\neq 0} \overline{\sigma(m)} a(n) \sqrt{y} K_\nu(2\pi|n|y) e^{2\pi inx} e^{2\pi inm/p} \\ &= \tau(\overline{\sigma}) \sum_{n\neq 0} \sigma(n) a(n) \sqrt{y} K_\nu(2\pi|n|y) e^{2\pi inx} \\ &= \frac{1}{i^\epsilon} \tau(\overline{\sigma}) \sum_{\mathfrak{a}} \sigma(\mathbb{N}(\mathfrak{a}))\psi(\mathfrak{a}) \sqrt{y} K_\nu(2\pi\mathbb{N}(\mathfrak{a})y) \sin(2\pi\mathbb{N}(\mathfrak{a})x) \end{aligned}$$

So we may consider the function g_σ as

$$g_\sigma(z) = \frac{1}{2\pi i}\frac{\partial f_\sigma}{\partial x}(z) = \frac{1}{i^\epsilon}\tau(\overline{\sigma}) \sum_{\mathfrak{a}} \mathbb{N}(\mathfrak{a})\sigma(\mathbb{N}(\mathfrak{a}))\psi(\mathfrak{a}) \sqrt{y} K_\nu(2\pi\mathbb{N}(\mathfrak{a})y) \cos(2\pi\mathbb{N}(\mathfrak{a})x)$$

Then we have

$$\begin{aligned} \int_0^\infty g(iy) y^{s+1/2} \frac{\mathrm{d}y}{y} &= \frac{1}{4i^\epsilon}\tau(\overline{\sigma}) \sum_{\mathfrak{a}} \mathbb{N}(\mathfrak{a})\sigma(\mathbb{N}(\mathfrak{a}))\psi(\mathfrak{a}) 2^{s-1} \Gamma\left(\frac{s+\nu+1}{2}\right)\Gamma\left(\frac{s-\nu+1}{2}\right) \\ &= \frac{1}{4i^\epsilon}\tau(\overline{\sigma})\pi^{-s}\Gamma\left(\frac{s+\nu+1}{2}\right)\Gamma\left(\frac{s-\nu+1}{2}\right) L(s,(\sigma\circ N)\psi) \\ &= \frac{1}{4i^\epsilon}\tau(\overline{\sigma}) D^{-s/2}\Lambda(s,\sigma\circ N)\psi) \end{aligned}$$

So by Mellin transform, we have

$$g_\sigma(iy) = \frac{1}{8\pi i^{\epsilon+1}\sqrt{y}}\tau(\overline{\sigma}) \int_{\sigma-i\infty}^{\sigma+i\infty} D^{-s/2}\Lambda(s,(\sigma\circ N)\psi) y^{-s}\mathrm{d}s$$

Since we have the functional equation

$$\Lambda(s,(\sigma\circ N)\psi) = (-1)^\epsilon \sigma(D)\frac{\tau(\sigma)^2}{p} p^{1-2s}\Lambda(1-s,\overline{(\sigma\circ N)\psi})$$

So

$$\begin{aligned} g_\sigma(iy) &= (-1)^\epsilon \sigma(-D) \frac{1}{8\pi i^{\epsilon+1}} \sqrt{1/p^2Dy^3}\tau(\sigma) \int_{\sigma-i\infty}^{\sigma+i\infty} D^{-s/2}\Lambda(s, \overline{(\sigma\circ N)\psi})(p^2Dy)^s \mathrm{d}s \\ &= (-1)^\epsilon\sigma(-D)\frac{1}{y^2} g_{\overline{\sigma}}\left(\frac{i}{p^2Dy}\right) \end{aligned}$$

Then by lemma 9.2., we obtain Eq. (9.24) in the case $\sigma(-1) = (-1)^\epsilon$.

1.10 Base Change

Solution 1.10.1. In this case $\mathfrak{o} = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$. Denote $\Gamma \subset \mathrm{SL}(2, \mathfrak{o})$ as the group generated by Γ_∞ and w. For any $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathfrak{o})$, if $a = 0$, we may change γ to γw and assume $a \neq 0$. Since $\mathbb{Q}(\sqrt{5})$ is Euclidean, we may assume $b = ra + s$ such that $\mathbb{N}(s) < \mathbb{N}(a)$. Then we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & s \\ c & d' \end{bmatrix} \begin{bmatrix} 1 & r \\ & 1 \end{bmatrix}$$

For some $d' = d - rc \in \mathfrak{o}$. If $s = 0$, we know $\left[\begin{smallmatrix} a & \\ c & d' \end{smallmatrix}\right] = w^{-1}\left[\begin{smallmatrix} d' & -c \\ & a \end{smallmatrix}\right] w$. So $\gamma \in \Gamma$. If $s \neq 0$, since $\left[\begin{smallmatrix} a & s \\ c & d' \end{smallmatrix}\right] = \left[\begin{smallmatrix} s & -a \\ d' & -c \end{smallmatrix}\right]$, we may repeat the process. Since $\mathbb{N}(s) < \mathbb{N}(a)$ in each process, this will stop in finite steps. So $\gamma \in \Gamma$.

Solution 1.10.2. Suppose F is a totally real field with narrow class number 1, because the more general case will involve the adelic language quoted to Van der Geer's *Hilbert Modular Surface*. Let $A(\nu)$ be a function on $\mathfrak{o}$ satisfying $A(\nu) = A(\epsilon\nu)$ for every $\epsilon \in \mathfrak{o}_+^\times$. Assume $A(\nu) = O(\mathbb{N}(\nu)^k)$ for some constant k, and $A(0) = 0$. Define $f(z) = \sum_{\nu\in\mathfrak{o}^+} A(\nu)e^{2\pi i \mathrm{tr}(\nu z/d)}$. Then we may have

$$L(s, f) = \sum_{\nu \in \mathfrak{o}_+^\times \backslash \mathfrak{o}_+} A(\nu)\mathbb{N}(\nu)^{-s}$$

$$\Lambda(s, f) = D^s(2\pi)^{-ns}\Gamma(s)^n L(s, f)$$

Then for twisted case, for any Hecke character ψ of F with trivial conductor, we may define $L(s, f, \psi) = \sum_{\nu\in\mathfrak{o}_+^\times\backslash\mathfrak{o}_+} \psi(\nu)A(\nu)\mathbb{N}(\nu)^{-s}$. If $\psi(z) = \psi_{\mathfrak{f}}(z)\psi_\infty(z) = \psi_{\mathfrak{f}}(z)\,(\mathrm{sgn}(z^{\epsilon_i}|z|^{\nu_i})$, we may define

$$\Lambda(s, f, \psi) = D^s(2\pi)^{-ns}\prod_j \Gamma\left(\frac{s-\nu_j}{2}\right) L(s, f, \psi)$$

Then the following two conditions are equivalent:

(1) f satisfies $f(\frac{-1}{z}) = \mathbb{N}(z)^k f(z)$;

(2) The functions $\Lambda(s, f, \psi)$ has analytic continuation to all s, are bounded in vertical strips, and satisfy the functional equation

$$\Lambda(s, f, \psi) = \Lambda(k - s, f, \psi^{-1})$$

The part (1⇒2) is easy. Since we have

$$
\begin{aligned}
&\int_{\mathfrak{o}_+^\times\backslash\mathbb{R}_+^n} f(iy)\prod_j y_j^{s-\nu_j}\prod_j\frac{\mathrm{d}y_j}{y_j}\\
=&\int_{\mathfrak{o}_+^\times\backslash\mathbb{R}_+^n}\sum_{\nu\in\mathfrak{o}+} A(\nu)e^{-2\pi\mathrm{tr}(\nu y/d}\prod_j y_j^{s-\nu_j}\prod_j\frac{\mathrm{d}y_j}{y_j}\\
=&\sum_{\alpha\in\mathfrak{o}_+^\times\backslash\mathfrak{o}_+} A(\alpha\mathfrak{o})\sum_{\nu\in\mathfrak{o}_+^\times}\int_{\mathfrak{o}_+^\times\backslash\mathbb{R}_+^n} e^{-2\pi\mathrm{tr}(\alpha y/d}\prod_j y_j^{s-\nu_j}\prod_j\frac{\mathrm{d}y_j}{y_j}\\
=&\sum_{\alpha\in\mathfrak{o}_+^\times\backslash\mathfrak{o}_+} A(\alpha\mathfrak{o})\int_{\mathbb{R}_+^n} e^{-2\pi\mathrm{tr}(\alpha y/d)}\prod_j y_j^{s-\nu_j}\prod_j\frac{\mathrm{d}y_i}{y_i}\\
=&\sum_{\alpha\in\mathfrak{o}_+^\times\backslash\mathfrak{o}_+} A(\alpha\mathfrak{o})\prod_j\left((2\pi)^{-s}\left(\frac{d^{(j)}}{\alpha^{(j)}}\right)^s\Gamma(s-\nu_j)\right)\\
=&\sum_{\mathfrak{a}} A(\mathfrak{a})\mathbb{N}(\mathfrak{a})^{-s}\psi(a)D^s(2\pi)^{-ns}\psi_\infty(d)^{-1}\\
=&D^s(2\pi)^{-ns}i^{-\mathrm{tr}(\epsilon)}\Lambda(s,f,\psi)
\end{aligned}
$$

So since $f(-1/z)=\mathbb{N}(z)^k f(z)$, we get the functional equation of Λ. And the convergence is the same with the quadratic case.

For the part (2⇒1), we need to construct a the fundamental domain for $\mathfrak{o}_+^\times\backslash\mathbb{R}_+^n$. Since $\log(\mathfrak{o}_+^\times)\subset\mathbb{R}^n$ is a $(n-1)$-rank lattice Λ, we know $\log(\mathfrak{o}_+^\times\backslash\mathbb{R}_+^n)=\mathbb{R}\times(\Lambda\backslash\mathbb{R}^{n-1})$. So we may choose a fundamental domain F of $\Lambda\backslash\mathbb{R}^{n-1}$) which is centrosymmetric at the original point. Then

$$
\Lambda(s,f,\psi)=2\cdot i^{\mathrm{tr}\epsilon}\int_0^\infty\int_F f(y_0e^{t_1},\dots,y_0e^{t_{n-1}},y_0e^{-(t_1+\dots+t_{n-1})})e^{-2\mathrm{tr}(t\nu)}\mathrm{d}t_1\ \dots\mathrm{d}t_{n-1}\ y_0^{2s}\frac{\mathrm{d}y_0}{y_0}
$$

Then by the Mellin inversion formula, we have

$$
\begin{aligned}
&\int_F f(y_0e^{t_1},\dots,y_0e^{t_{n-1}},y_0e^{-(t_1+\dots+t_{n-1})})e^{-2\mathrm{tr}(t\nu)}\mathrm{d}t_1\ \dots\mathrm{d}t_{n-1}\\
=&\frac{(-1)^{\mathrm{tr}\epsilon}}{4\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Lambda\left(\frac{s}{2},f,\psi\right)y_0^{-s}\mathrm{d}s\\
=&\frac{(-1)^{\mathrm{tr}\epsilon}}{4\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Lambda\left(\frac{2k-s}{2},f,\psi^{-1}\right)y_0^{-s}\mathrm{d}s\\
=&-\frac{(-1)^{\mathrm{tr}\epsilon}}{4\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Lambda\left(\frac{s}{2},f,\psi^{-1}\right)y_0^{s-2k}\mathrm{d}s\\
=&\int_F f(y_0^{-1}e^{-t_1},\dots,y_0e^{-t_{n-1}},y_0e^{t_1+\dots+t_{n-1}})y_0^{-2k}e^{-2\mathrm{tr}(t\nu)}\mathrm{d}t_1\ \dots\mathrm{d}t_{n-1}
\end{aligned}
$$

So we may define a function

$$g(t_1,\ldots,t_{n-1}) = f(y_0e^{t_1},\ldots,y_0e^{t_{n-1}},y_0e^{-(t_1+\ldots+t_{n-1})}) - f(y_0^{-1}e^{-t_1},\ldots,y_0e^{-t_{n-1}},y_0e^{t_1+\ldots+t_{n-1}})y_0^{-2k}$$

which is a periodic function of $t_1,\ldots,t_{n-1}$. Clearly every Fourier coefficients of g vanish, we have $g=0$. So f satisfies the modularity.

Solution 1.10.3. We may define the function

$$f_{t,k}(m,n) = (mDz+n)^k e^{-\pi|mDz+n|^2t/y}$$

Then we have

$$\int_0^\infty f_{t,k}(m,n)t^{s+1/2}\frac{\mathrm{d}t}{t} = \pi^{-s-1/2}\Gamma(s+1/2)\frac{(mDz+n)^k y^{s+1/2}}{|mDz+n|^{2s+1}}$$

So if we define $\Theta_k(t,\chi_D) = \sum_{(m,n)\in\mathbb{Z}^2}\chi_D(n)f_{t,k}(m,n)$, we must have

$$\int_0^\infty (\Theta_k(t,\chi_D)-1)t^{s+1/2}\frac{\mathrm{d}t}{t} = E^*(z,s)$$

For the functional equation, we may consider the Poisson summation formula. So we firstly compute the Fourier transformation of $f_{t,k}(m,n)$ as

$$\begin{aligned}
\hat{f}_{t,0}(p,q) &= \int_{-\infty}^\infty f_{t,0}(m,n)e^{2\pi i(mp+nq)}\mathrm{d}m\,\mathrm{d}n = \frac{1}{Dt}e^{-\pi|qDz-p|^2/(D^2ty)} = \frac{1}{Dt}f_{1/D^2y,0}(q,-p) \\
\hat{f}_{t,k}(p,q) &= (-2\pi i)^{-k}\left(Dz\frac{\partial}{\partial p}+pdifq\right)^k\hat{f}_{t,0} = \left(\frac{-(qDz-p)}{Dt}\right)^k\hat{f}_{t,0} \\
&= (-1)^k(qDz-p)^k(Dt)^{-k-1}e^{-\pi|qDz-p|^2/(D^2ty)} = (-1)^k(Dt)^{-k-1}f_{1/D^2t,k}(q,-p)
\end{aligned}$$

Hence by twisted Poisson's formula, we have

$$\begin{aligned}
\Theta_k(t,\chi_D) &= \sum_{m,n}\chi_D(n)f_{t,k}(m,n) \\
&= \tau(\overline{\chi_D})^{-1}\sum_{p,q}\overline{\chi_D(p)}(-1)^k(Dt)^{-k-1}f_{1/D^2t,k}(q/D,-p/D) \\
&= \tau(\overline{\chi_D})^{-1}(-1)^k(Dt)^{-k-1}\Theta_k(1/D^4t,\overline{\chi_D})
\end{aligned}$$

So we have

$$\begin{aligned}
E^*(z,s) &= \int_0^\infty (\Theta_k(t,\chi_D)-1)t^{s+1/2}\frac{\mathrm{d}t}{t} \\
&= \int_{1/D^2}^\infty (\Theta_k(t,\chi_D)-1)t^{s+1/2}\frac{\mathrm{d}t}{t} + \int_0^{1/D^2} (\Theta_k(t,\chi_D)-1)t^{s+1/2}\frac{\mathrm{d}t}{t} \\
&= \int_{1/D^2}^\infty (\Theta_k(t,\chi_D)-1)t^{s+1/2}\frac{\mathrm{d}t}{t} + \int_0^{1/D^2} (\tau(\overline{\chi_D})^{-1}(-1)^k(Dt)^{-k-1}\Theta_k(1/D^4t,\overline{\chi_D})-1)t^{s+1/2}\frac{\mathrm{d}t}{t} \\
&= \int_{1/D^2}^\infty (\Theta_k(t,\chi_D)-1)t^{s+1/2}\frac{\mathrm{d}t}{t} + \int_{1/D^2}^\infty (\tau(\overline{\chi_D})^{-1}(-1)^k(D^3t)^{k+1}\Theta_k(t,\overline{\chi_D})-1)t^{-s-1/2}D^{-4s-2}\frac{\mathrm{d}t}{t} \\
&= \int_{1/D^2}^\infty (\Theta_k(t,\chi_D)-1)t^{s+1/2}\frac{\mathrm{d}t}{t} + \tau(\overline{\chi_D})^{-1}(-1)^k\int_{1/D^2}^\infty (\Theta_k(t,\overline{\chi_D})-1)t^{k-s+1/2}D^{3k-4s+1}\frac{\mathrm{d}t}{t} \\
&\quad + \int_{1/D^2}^\infty (\tau(\overline{\chi_D})^{-1}(-1)^k(D^3t)^{k+1}-1)t^{-s-1/2}D^{-4s-2}\frac{\mathrm{d}t}{t} \\
&= \int_{1/D^2}^\infty (\Theta_k(t,\chi_D)-1)t^{s+1/2}\frac{\mathrm{d}t}{t} + \tau(\overline{\chi_D})^{-1}(-1)^k\int_{1/D^2}^\infty (\Theta_k(t,\overline{\chi_D})-1)t^{k-s+1/2}D^{k-2s}\frac{\mathrm{d}t}{t} \\
&\quad + \frac{\tau(\overline{\chi_D})(-1)^k}{(s-k-3/2)D^{2k-3s-1/2}} - \frac{1}{(s-1/2)D^{3s+5/2}}
\end{aligned}$$

So we have $E^*(z,s) = z^kD^{2k-3s}E^*(-1/Dz, k-s)$.

Plus, the method using L_k as the hint in the book seems not work. Since by definition we know if $\chi_D(-1) = (-1)^{k+1}$, we have $E_k(z,s) = 0$. So $L_{k+1}\circ L_k$ must act as zero on E_k.

Solution 1.10.4. (a) Clearly for sufficiently large σ,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} f(u)g(s-u)\mathrm{d}u &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_0^\infty f(u)\psi(y)y^{s-u}\frac{\mathrm{d}y}{y}\mathrm{d}u \\
&= \frac{1}{2\pi i}\int_0^\infty \left(\int_{\sigma-i\infty}^{\sigma+i\infty} f(u)y^{-u}\mathrm{d}u\right)\psi(y)y^s\frac{\mathrm{d}y}{y} = \int_0^\infty \phi(y)\psi(y)y^s\frac{\mathrm{d}y}{y} = h(s)
\end{aligned}$$

(b) Since we have $\int_0^\infty e^{-y} y^s \frac{\mathrm{d}y}{y} = \Gamma(s)$, and $\int_0^\infty K_\nu(y) y^s \frac{\mathrm{d}y}{y} = 2^{-(s-2)}\Gamma(\frac{s+\nu}{2})\Gamma(\frac{s-\nu}{2})$, then we know

$$
\begin{aligned}
\int_0^\infty K_\nu(y) e^{-y} y^s \frac{\mathrm{d}y}{y} &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} 2^{u-2}\Gamma\left(\frac{u+\nu}{2}\right)\Gamma\left(\frac{u-\nu}{2}\right)\Gamma(s-u)\mathrm{d}u \\
&= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} 2^{u-2}\Gamma\left(\frac{u+\nu}{2}\right)\Gamma\left(\frac{u-\nu}{2}\right)2^{s-u-1}\pi^{-1/2}\Gamma\left(\frac{s-u}{2}\right)\Gamma\left(\frac{s-u+1}{2}\right)\mathrm{d}u \\
&= \frac{2^{s-2}}{\pi^{3/2} i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma\left(\frac{u+\nu}{2}\right)\Gamma\left(\frac{u-\nu}{2}\right)\Gamma\left(\frac{s-u}{2}\right)\Gamma\left(\frac{s-u+1}{2}\right)\mathrm{d}\frac{u}{2} \\
&= \frac{2^{s-2}}{\pi^{3/2} i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma\left(u+\frac{\nu}{2}\right)\Gamma\left(u-\frac{\nu}{2}\right)\Gamma\left(\frac{s}{2}-u\right)\Gamma\left(\frac{s+1}{2}-u\right)\mathrm{d}u \\
\text{Barnes' Lemma:} \quad &= \frac{2^{s-2}}{\pi^{1/2}}\frac{\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s+\nu+1}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s-\nu+1}{2}\right)}{\Gamma(s+1/2)} \\
&= \frac{2^{s-2}}{\pi^{1/2}}\frac{\pi^{1/2}}{2^{s+\nu-1}}\frac{\pi^{1/2}}{2^{s-\nu-1}}\frac{\Gamma(s+\nu)\Gamma(s-\nu)}{\Gamma(s+1/2)} \\
&= \sqrt{\pi}2^{-s}\frac{\Gamma(s+\nu)\Gamma(s-\nu)}{\Gamma(s+1/2)}
\end{aligned}
$$

2 Automorphic Forms and Representations of GL$(2, \mathbb{R})$

2.1 Maass Forms and the Spectral Problem

Solution 2.1.1. Clearly,

$$R_k\Delta_k = -R_kL_{k+2}R_k - \frac{k}{2}\left(1+\frac{k}{2}\right)R_k = \left(\Delta_{k+2} - \frac{k+2}{2}\left(1-\frac{k+2}{2}\right)\right)R_k - \frac{k}{2}\left(1+\frac{k}{2}\right)R_k = \Delta_{k+2}R_k$$
$$L_k\Delta_k = -L_kR_{k-2}L_k + \frac{k}{2}\left(1-\frac{k}{2}\right)L_k = \left(\Delta_{k-2} + \frac{k-2}{2}\left(1+\frac{k-2}{2}\right)\right)L_k + \frac{k}{2}\left(1-\frac{k}{2}\right)L_k = \Delta_{k-2}L_k$$

Solution 2.1.2. For any $b = \left[\begin{smallmatrix} u & \\ & u\end{smallmatrix}\right]\left[\begin{smallmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2}\end{smallmatrix}\right]$ and $c = \left[\begin{smallmatrix} v & \\ & v\end{smallmatrix}\right]\left[\begin{smallmatrix} w^{1/2} & zw^{-1/2} \\ & w^{-1/2}\end{smallmatrix}\right]$, we have

$$\begin{aligned}
\mathrm{d}_L(cb) &= \mathrm{d}_L\left(\left[\begin{smallmatrix} uv & \\ & uv\end{smallmatrix}\right]\left[\begin{smallmatrix} (yw)^{1/2} & (wx+z)(yw)^{-1/2} \\ & (yw)^{-1/2}\end{smallmatrix}\right]\right) = \frac{\mathrm{d}(wx+z)\,\mathrm{d}(yw)}{(yw)^2}\frac{\mathrm{d}(uv)}{(uv)} \\
&= \frac{w\mathrm{d}x\ \cdot w\mathrm{d}y}{y^2w^2}\frac{v\mathrm{d}u}{uv} = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}\frac{\mathrm{d}u}{u} = \mathrm{d}_Lb \\
\mathrm{d}_R(bc) &= \mathrm{d}_R\left(\left[\begin{smallmatrix} uv & \\ & uv\end{smallmatrix}\right]\left[\begin{smallmatrix} (yw)^{1/2} & (x+yz)(yw)^{-1/2} \\ & (yw)^{-1/2}\end{smallmatrix}\right]\right) = \mathrm{d}x + yz\,\frac{\mathrm{d}(yw)}{(yw)}\frac{\mathrm{d}(uv)}{(uv)} \\
&= (\mathrm{d}x\ + z\mathrm{d}y\ )\frac{\mathrm{d}y}{y}\frac{\mathrm{d}u}{u} = \mathrm{d}x\ \frac{\mathrm{d}y}{y}\frac{\mathrm{d}u}{u} = \mathrm{d}_Rb
\end{aligned}$$

Solution 2.1.3 (Iwasawa Decomposition)**.** For any $g = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] \in \mathrm{GL}(2,\mathbb{R})^+$, we have

$$\begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta)\end{bmatrix} = \begin{bmatrix} a\cos\theta + b\sin\theta & b\cos\theta - a\sin\theta \\ c\cos\theta + d\sin\theta & d\cos\theta - c\sin\theta\end{bmatrix}$$

So there exists a unique θ modulo 2π such that $c\cos\theta + d\sin\theta = 0$. So $g = \left[\begin{smallmatrix} a' & b' \\ & d'\end{smallmatrix}\right]\kappa_\theta$ for some a', b', d'. Since $g \in \mathrm{GL}(2,\mathbb{R})^+$, we know $\left[\begin{smallmatrix} a' & b' \\ & d'\end{smallmatrix}\right] \in \mathrm{GL}(2,\mathbb{R})^+$, i.e. $a'd' > 0$. So we may take $u = \sqrt{a'd'}$, $y = \frac{a'}{d'}$, and $x = \sqrt{b'}d'$. So $g = \left[\begin{smallmatrix} u & \\ & u\end{smallmatrix}\right]\left[\begin{smallmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2}\end{smallmatrix}\right]\kappa_\theta$ for the unique $x \in \mathbb{R}, y, u \in \mathbb{R}^+$ and $\theta \in \mathbb{R}/(2\pi)$.

Solution 2.1.4. The notation is confused here. Bump did not define the notation $\rho(g)\rho(g')f$ to be either $\rho(g)\,(\rho(g')f)$ or $\rho(g')\,(\rho(g)f)$. So if we use the first definition, for all $x \in G$, we have $(\rho(g)\rho(g')f)(x) = (\rho(g)f)(xg') = f(xg'g) = (\rho(g'g)f)(x)$, i.e. $\rho(g'g)f = \rho(g)\rho(g')f$.

Solution 2.1.5. Clearly for $v \in H_k$ and $v' \in H_{k'}$ with different k and k', we have $\langle v, v'\rangle = \langle \rho(\kappa_\theta)v, \rho(\kappa_\theta)v'\rangle = e^{i(k-k')\theta}\langle v, v'\rangle$, so $\langle v, v'\rangle = 0$, i.e. the spaces H_k are orthogonal. Suppose $\bigoplus H_k \subsetneq H$. We may pick a nonzero v not in $\bigoplus H_k$. For any $\delta > 0$, since ρ is continuous, there exists a small $\epsilon > 0$ such that $\|\rho(\kappa_\theta)v - v\| < \delta$ for any $\theta \in U = [-\epsilon, \epsilon]$. And there exists an impulse function $F(\theta) = \sum_{i=-N}^N a_i e^{in\theta}$ such that $\frac{1}{2\pi}\int_0^{2\pi} F(\theta)\mathrm{d}\theta\ = 1$, and for all $\theta \notin U$, we have $F(\theta) < \delta'$ for some small δ'. Moreover, since if $u = \int_0^{2\pi} e^{in\theta}\rho(\kappa_\theta)v\mathrm{d}\theta$, we know

$$\rho(\kappa_{\theta_0})u = \int_0^{2\pi} e^{in\theta}\rho(\kappa_{\theta+\theta_0})v\mathrm{d}\theta \ = e^{-in\theta_0}u$$

which means u is in H_{-n}. So we know $\frac{1}{2\pi}\int_0^{2\pi} F(\theta)\rho(\kappa_\theta)v\mathrm{d}\theta \ \in \bigoplus H_k$. And moreover we have

$$\begin{aligned}
\|\frac{1}{2\pi}\int_0^{2\pi} F(\theta)\rho(\kappa_\theta)v\mathrm{d}\theta \ - v\| &= \|\frac{1}{2\pi}\left(\int_0^{2\pi} F(\theta)\rho(\kappa_\theta)v\mathrm{d}\theta \ - \int_0^{2\pi} F(\theta)v\mathrm{d}\theta\right)\| \\
&\le \frac{1}{2\pi}\left(\int_U |F(\theta)|\cdot\|\rho(\kappa_\theta)v - v\|\mathrm{d}\theta \ + \int_{\mathbb{S}^1-U}|F(\theta)|\cdot\|\rho(\kappa_\theta)v - v\|\mathrm{d}\theta\right) \\
&\le \frac{1}{2\pi}\left(\int_U |F(\theta)|\cdot\delta\mathrm{d}\theta \ + \int_{\mathbb{S}^1-U}\delta'\cdot(2\|v\|)\mathrm{d}\theta\right) \\
&\le \frac{1}{2\pi}\left(\delta + 4\pi\delta'\|v\|\right)
\end{aligned}$$

Since $\|v\|$ is fixed and δ, δ' is ambiguous and sufficiently small, so v has distance 0 to $\bigoplus H_k$, and this makes a contradiction. So $H = \bigoplus H_k$.

Solution 2.1.6. We have two assertions in the proof. The first assertion: $C_c(\mathcal{F})$ is dense in $L^2(\mathcal{F})$. The second assertion: we can find a function ϕ_2 that is smooth, nonnegative and has compact support contained in U and satisfies $\int_G \phi_2(h)\mathrm{d}h \ = 1$.

For the first one, clearly for any $f \in L^2(\mathcal{F})$ and any $\epsilon > 0$, there exists g to be a finite linear combination of simple measurable function such that $|f - g| < \frac{\epsilon}{2}$. And moreover, by Lusin's theorem, there exists a smooth function f' supported compactly such that $|f' - g| < \frac{\epsilon}{2}$, so $|f - f'| < \epsilon$, i.e. $C_c(\mathcal{F})$ is dense in $L^2(\mathcal{F})$.

For the second one, we may fix an $x \in U$, and there exists a neighbourhood $V \subset U$ around x_0 such that V is homeomorphic to an open subset of $\mathbb{R}^n$. Shrinking V, we may assume $V \cong B_{x_0}(r)$ for some radius $r > 0$. Then we may define a function g on G as

$$g = \begin{cases} e^{-|x-x_0|^2} & \text{if } x \in V \\ 0 & \text{if } x \notin V \end{cases}$$

So we may consider some λ such that $\phi_2 = \lambda g$ such that $\int_G \phi_2(h)\mathrm{d}h \ = 1$. Then clearly ϕ_2 is a smooth, non-negative, supported compactly on U function.

Solution 2.1.7 (Holomorphic modular forms as Maass forms)**.** (a) Clearly for any $\gamma = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] \in \Gamma$, we have

$$\left(y^{k/2}f(z)\right)|_k\gamma = \left(\frac{c\bar{z}+d}{|cz+d|}\right)^k \mathrm{Im}(\gamma z)^{k/2}f(\gamma z) = \left(\frac{c\bar{z}+d}{|cz+d|}\right)^k \frac{y^{k/2}}{|cz+d|^k}(cz+d)^k f(z) = y^k f(z)$$

And for Δ_k, we know

$$\begin{aligned}\Delta_k(y^{\frac{k}{2}}f(z)) &= -y^2\left(y^{k/2}\frac{\partial^2 f}{\partial x^2}(z)+\frac{k}{2}(\frac{k}{2}-1)y^{\frac{k}{2}-2}f(z)+ky^{\frac{k}{2}-1}\frac{\partial f}{\partial y}(z)+y^{\frac{k}{2}}\frac{\partial^2 f}{\partial y^2}(z)\right)+iky^{\frac{k}{2}+1}\frac{\partial f}{\partial x}(z)\\ &= -y^{\frac{k}{2}+2}\Delta_e(f)(z)+ky^{\frac{k}{2}+1}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)+\frac{k}{2}\left(1-\frac{k}{2}\right)y^{\frac{k}{2}}f(z)\\ &= \frac{k}{2}\left(1-\frac{k}{2}\right)y^{\frac{k}{2}}f(z)\end{aligned}$$

So $y^{k/2}f(z)$ is a Maass form with eigenvalue $\frac{k}{2}(1-\frac{k}{2})$. Moreover,

$$L_k(y^{\frac{k}{2}}f(z)) = -iy^{\frac{k}{2}+1}\frac{\partial f}{\partial x}(z)+\frac{k}{2}\cdot y^{\frac{k}{2}}f(z)+y^{\frac{k}{2}+1}\frac{\partial f}{\partial y}(z)-\frac{k}{2}y^{\frac{k}{2}}f(z)=y^{\frac{k}{2}+1}\left(-i\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)=0$$

(b) If g is a Maass form with weight k such that $L_k g = 0$, we may prove that $y^{-k/2}g$ is a modular form with weight k. Clearly we have $f(\gamma z) = (cz+d)^k f(z)$ for any $\gamma = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] \in \Gamma$. And since $y^{\frac{k}{2}+1}(-i\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}) = L_k(y^{\frac{k}{2}}f(z)) = L_k(g(z)) = 0$, we know f is holomorphic. Hence f is a modular form.

For the case of R_k, if g is a Maass form annihilated by R_k, we may consider the function $f = y^{k/2}g$. So

$$0 = R_k(y^{-\frac{k}{2}}f(z)) = iy^{-\frac{k}{2}+1}\frac{\partial f}{\partial x}(z)-\frac{k}{2}y^{\frac{k}{2}}f(z)+y^{-\frac{k}{2}+1}\frac{\partial f}{\partial y}(z)+\frac{k}{2}y^{\frac{k}{2}}f(z)=y^{-\frac{k}{2}+1}\left(i\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)$$

So f is an antiholomorphic function with the modularity:

$$f(\gamma z) = \mathrm{Im}(\gamma z)^{k/2}g(\gamma z) = \frac{y^{k/2}}{|cz+d|^k}g(z)\left(\frac{|cz+d|}{c\bar{z}+d}\right)^k = (c\bar{z}+d)^{-k}f(z)$$

Solution 2.1.8 (Positivity of the Laplacian)**.** (a) Clearly $\langle \delta f, f\rangle = \langle -R_{-2}L_0 f, f\rangle = \langle L_0 f, L_0 f\rangle \geq 0$. And the equality equals to $L_0 f = 0$, i.e. f is a modular form with weight 0, i.e. f is a constant function.

(b) If f is an eigenfunction of Δ_1 with eigenvalue λ, we have $\langle \Delta_1 f, f\rangle = \lambda\langle f, f\rangle$. And on the other side, we have

$$\langle \Delta_1 f, f\rangle = \langle -R_{-1}L_1 f, f\rangle + \frac{1}{4}\langle f, f\rangle = \langle L_1 f, L_1 f\rangle + \frac{1}{4}\langle f, f\rangle \geq \frac{1}{4}\langle f, f\rangle$$

So we have $\lambda \geq \frac{1}{4}$.

(c) Consider the induction. Fix an eigenform f with eigenvalue λ. If $L_k f$ is not zero, we know that $\Delta_{k-2}L_k f = L_k\Delta_k f = \lambda L_k f$. So the eigenvalue λ of f is the eigenvalue of $L_k f$. Then by reduction, $\lambda = \frac{l}{2}(1-\frac{l}{2})$ for some $1 \leq l < k$ such that $l \equiv k \bmod 2$, or $\lambda \geq 0$. If $L_k f$ is zero, then $\Delta_k f = -R_{k-2}L_k f + \frac{k}{2}(1-\frac{k}{2})f = \frac{k}{2}(1-\frac{k}{2})f$. So λ just can have the form $\frac{l}{2}(1-\frac{l}{2})$ for some $1 \leq l \leq k$ such that

$l \equiv k \bmod 2$, or $\lambda \geq 0$.

When k is odd, the induction has a problem. When $L_k f = 0$, by 2.1.7.(b) we know $f = y^{k/2} g$ for some modular form g with weight k. But since k is an odd number, g must be zero, i.e. $f = 0$. So in this case, λ of f is just the eigenvalue of $L_k(f)$. So by induction and (b), $\lambda \geq \frac{1}{4}$.

Solution 2.1.9. Fix a $(g_1, v_1) \in G \times H$. For any $\epsilon > 0$, by definition there exists a δ such that for any $v_2 \in H$ such that $|v_1 - v_2| < \delta$, we have $|\pi(g_1)v_1 - \pi(g_1)v_2| < \epsilon/2$. Fix a v_2 in this small neighbourhood, there exists a neighbour G_{v_2} of g_1 such that for any $g_2 \in G_{v_2}$, we always have $|\pi(g_1)v_2 - \pi(g_2)v_2| < \epsilon/2$. So there exists an open neighbourhood $U = \bigcup_{|v_2 - v_1| < \delta} \{v_2\} \times G_{v_2} \subset G \times H$ of (g_1, v_1) such that for any $(g_2, v_2) \in U$, we always have

$$|\pi(g_1)v_1 - \pi(g_2)v_2| \leq |\pi(g_1)v_1 - \pi(g_1)v_2| + |\pi(g_1)v_2 - \pi(g_2)v_2| < \epsilon$$

So $(g, v) \mapsto \pi(g)v$ is continuous, then clearly π is a representation.

2.2 Basic Lie Theory

Solution 2.2.1 (The adjoint representations of G and $\mathfrak{g}$)**.** Actually we have

$$\begin{aligned}
(\mathrm{ad}X)(Y) &= \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ad}(\exp(tX))(Y)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \exp(tX) Y \exp(-tX)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} (I + tX + t^2X^2 + \ldots) Y (I - tX + t^2X^2 - \ldots)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} (I + tX + t^2X^2 + \ldots) Y (I - tX + t^2X^2 - \ldots)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} (Y + t(XY - YX) + t^2(X^2Y - XYX + YX^2) + \ldots)\Big|_{t=0} \\
&= XY - YX = [X, Y]
\end{aligned}$$

Solution 2.2.2. (a) Clearly, $\mathfrak{h} = \{X \in \mathfrak{gl}(2, \mathbb{R}) \mid \det \exp(X) = 0\} = \{X \in \mathfrak{gl}(2, \mathbb{R}) \mid \det(\exp(X)) = 1\}$. But $\det(\exp(X)) = \exp(\mathrm{tr}(X))$. So $\mathfrak{h} = \{X \in \mathfrak{gl}(2, \mathbb{R}) \mid \det \exp(X) = 0\} = \{X \in \mathfrak{gl}(2, \mathbb{R}) \mid \mathrm{tr}(X) = 0\}$.

(b) Clearly if $A = x\hat{R} + y\hat{H}$, we have

$$\exp(A) = \begin{bmatrix} e^y & \frac{x}{2y}(e^{2y} - 1)e^{-y} \\ & e^{-y} \end{bmatrix}$$

So $\exp(\mathfrak{h}) = H$, i.e. $\mathfrak{h}$ is the Lie algebra of H.

(c) Clearly if $A = x\hat{R}$, we have $\exp(A) = \left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right]$. So $\mathfrak{h}$ is the Lie algebra of H.

(d) If $A = x\hat{H}$, we have $\exp(A) = \left[\begin{smallmatrix} e^x & \\ & e^{-x} \end{smallmatrix}\right]$. So $\exp(\mathfrak{h}) = H$, i.e. $\mathfrak{h}$ is the Lie algebra of H.

(e) For the final one, if $A = xW$, we have $\exp(A) = \left[\begin{smallmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{smallmatrix}\right]$. So $\mathfrak{h}$ is the Lie algebra of H.

Solution 2.2.3. Simply,

$$\begin{aligned}
\Delta =& -\frac{1}{4}\Big(H^2 + 2RL + 2LR\Big) \\
=& -\frac{1}{4}\Bigg(\Big(-i\frac{\partial}{\partial\theta}\Big)^2 + 2e^{2i\theta}\Big(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial\theta}\Big)\circ e^{-2i\theta}\Big(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial\theta}\Big) \\
&+2e^{-2i\theta}\Big(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial\theta}\Big)\circ e^{2i\theta}\Big(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial\theta}\Big)\Bigg) \\
=& -\frac{1}{4}\Bigg(-\frac{\partial^2}{\partial\theta^2} + 2\Big(y^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2} - \frac{y}{2}\frac{\partial^2}{\partial x\,\partial\theta} - \frac{y}{2i}\frac{\partial^2}{\partial y\,\partial\theta}\Big) \\
&- 2\Big(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial\theta}\Big) - i\Big(-iy\frac{\partial^2}{\partial\theta\,\partial x} + y\frac{\partial^2}{\partial\theta\,\partial y} - \frac{1}{2i}\frac{\partial^2}{\partial\theta^2}\Big) \\
&+ 2\Big(y^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2} - \frac{y}{2}\frac{\partial^2}{\partial x\,\partial\theta} + \frac{y}{2i}\frac{\partial^2}{\partial y\,\partial\theta}\Big) \\
&-2\Big(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial\theta}\Big) + i\Big(iy\frac{\partial^2}{\partial\theta\,\partial x} + y\frac{\partial^2}{\partial\theta\,\partial y} + \frac{1}{2i}\frac{\partial^2}{\partial\theta^2}\Big)\Bigg) \\
=& -\frac{1}{4}\Big(-\frac{\partial^2}{\partial\theta^2} + 4y^2\frac{\partial^2}{\partial x^2} + 4y^2\frac{\partial^2}{\partial y^2} - 2y\frac{\partial^2}{\partial x\,\partial\theta} - 2y\frac{\partial^2}{\partial x\,\partial\theta} + \frac{\partial^2}{\partial\theta^2}\Big) \\
=& -y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) + y\frac{\partial^2}{\partial x\,\partial\theta}
\end{aligned}$$

Solution 2.2.4. (a) If $[X, Y] = 0$, we have $XY = YX$. Then

$$\begin{aligned}
\exp(X+Y) &= I + (X+Y) + \frac{1}{2!}(X+Y)^2 + \frac{1}{3!}(X+Y)^3 + \dots \\
&= I + (X+Y) + \frac{1}{2!}(X^2 + XY + YX + Y^2) + \frac{1}{3!}(X^3 + 3X^2Y + 3XY^2 + Y^3) + \dots \\
&= \Big(I + X + \frac{1}{2!}X^2 + \dots\Big)\Big(I + Y + \frac{1}{2!}Y^2 + \dots\Big) \\
&= \exp(X)\exp(Y)
\end{aligned}$$

(b) For every X and every t, we have $\det(\exp(tX)) = \exp(t\cdot\mathrm{tr}(X)) \neq 0$. So this map has image in $\mathrm{GL}(n,\mathbb{R})$, i.e. this is a homomorphism $\mathbb{R} \to \mathrm{GL}(n,\mathbb{R})$.

(c) Suppose we have a homomorphism $f : \mathbb{R} \to \mathrm{GL}(n,\mathbb{R})$. Since the Lie algebra of $\mathbb{R}$ is just $\mathbb{R}$, we know the homomorphism f induces a homomorphism $\mathfrak{f} : \mathbb{R} \to \mathfrak{gl}(n,\mathbb{R})$ between Lie algebras. Clearly $\mathfrak{f}$ is a linear morphism of vector space, we know $\mathfrak{f}(t) = tX$ for some $X \in \mathfrak{gl}(n,\mathbb{R})$. So $f(t) = \exp(tX)$.

Solution 2.2.5. By 2.2.1., we know $(\mathrm{ad}X)(D) = [X, D] = XD - DX = 0$. So for any $g \in G$ we have $\mathrm{Ad}(g)(D) = D$ for any D in the center of $U(\mathfrak{g})$.

Solution 2.2.6. (a) For any skew-symmetric matrix A in $\mathrm{Mat}(n,\mathbb{R})$, we know $\exp(A)^{-1} = \exp(-A) =$

$\exp(A^T) = \exp(A)^T$, i.e. $\exp(A) \in \mathrm{SO}(n)$. And clearly their have the same dimension, hence the Lie algebra of $\mathrm{SO}(n)$ is the vector space of all skew-symmetric matrix in $\mathrm{Mat}(n, \mathbb{R})$.

(b) When $n = 2$, every $A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \in \mathrm{SO}(2)$ has the preimage $\begin{bmatrix} & \theta \\ -\theta & \end{bmatrix}$. So the case $n = 2$ is over. For higher n, for any $\kappa \in \mathrm{SO}(n)$, clearly if λ is an eigenvalue of κ, we have $\det(\kappa - \lambda I) = 0$. So

$$0 = \det(\kappa - \lambda I) = \det(\kappa - \lambda\kappa\kappa^T) = \det(\kappa)\det(I - \lambda\kappa^T) = \det(\kappa - \lambda^{-1} I) \cdot (-\lambda)^n$$

So λ^{-1} is also an eigenvalue. And, since $\kappa\kappa^T = I$, we obviously know every λ has length 1. Moreover, if λ is a real eigenvalue, we know for every $v \in \mathbb{R}^n$, we have $0 < v^T\kappa v = \lambda v^T v$, i.e. $\lambda > 0$. So by the Jordan decomposition, there exists a $\tau \in \mathrm{SO}(n)$ such that

$$\tau\kappa\tau^{-1} = \begin{bmatrix} \kappa_1 & & & \\ & \cdots & & \\ & & \kappa_r & \\ & & & I_{n-2r} \end{bmatrix}$$

for some $\kappa_i = \kappa_{\theta_i} \in \mathrm{SO}(2)$, where θ_i is the arc of the pairwise eigenvalue λ_i and λ_i^{-1}. So there exists X_i for $1 \le i \le r$ such that $\exp(X_i) = \kappa_i$. Then

$$\tau\kappa\tau^{-1} = \begin{bmatrix} \kappa_1 & & & \\ & \cdots & & \\ & & \kappa_r & \\ & & & I_{n-2r} \end{bmatrix} = \exp\left(\begin{bmatrix} X_1 & & & \\ & \cdots & & \\ & & X_r & \\ & & & 0_{n-2r} \end{bmatrix}\right)$$

i.e.

$$\kappa = \exp\left(\tau \begin{bmatrix} X_1 & & & \\ & \cdots & & \\ & & X_r & \\ & & & 0_{n-2r} \end{bmatrix} \tau^{-1}\right)$$

2.3 Discreteness of the Spectrum

Solution 2.3.1. The question here has a typo, the equation might be $\frac{|\gamma z - \zeta|}{|\gamma z - \bar{\zeta}|} < r$ according to the context. Fix the $\zeta = \xi + i\eta$ for some $\xi, \eta \in \mathbb{R}$ with $\eta > 0$. Then the field $\mathcal{R} = \{z = x + iy \in \mathbb{C} \mid \left|\frac{z-\zeta}{z-\bar{\zeta}}\right| < r$ is a round with equation

$$(x - \xi)^2 - \left(y - \eta \cdot \frac{1 + r^2}{1 - r^2}\right)^2 < \left(\frac{2r\eta}{1 - r^2}\right)^2$$

So we have $\mathcal{R} \cap \mathcal{H} = \mathcal{R}$ when $0 < r < 1$. Then

$$S(\mathcal{R}) = \int_R \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \pi \cdot \left(\frac{1 + r^2}{\sqrt{1 + r^2 + r^4}} - 1\right) \sim \frac{\pi}{2} \cdot \frac{r^2}{1 - r^2}$$

So $\sharp\{\gamma \mid C_\zeta(\gamma z) < r\} < C \cdot \frac{r^2}{1-r^2}$ for some C determined by the density of Γ.

Solution 2.3.2. Now we think about the Δ_k. For any $f \in C^\infty(\Gamma\backslash\mathcal{H}, \chi, k)$, we have

$$\langle \Delta_k f, f\rangle = \langle L_k f, L_k f\rangle + \frac{k}{2}\left(1 - \frac{k}{2}\right)\langle f, f\rangle$$

So if we define $Lf = \Delta_k f + sf$ for any $s > \frac{k}{2}(\frac{k}{2} - 1)$, the spectrum of L is bounded away from zero. Then we need to construct a Green function $g(z, \zeta)$.

We may define $\hat{g}(z,\zeta) = \left(\frac{z-\bar{\zeta}}{\zeta-\bar{z}}\right)^{k/2} g(z,\zeta)$. Then we have $(\hat{\Delta}_k + s)\hat{g}(z, i) = 0$ for $\hat{\Delta}_k = \left(\frac{z+i}{i-\bar{z}}\right)^{k/2} \cdot \Delta_k \cdot \left(\frac{z+i}{i-\bar{z}}\right)^{-k/2}$. Then if we denote $w = \frac{z-i}{z+i}$, and change $\hat{g}(z, i)$ into a function $h(w) = \hat{g}(z, i) : \mathcal{D} \to \mathbb{C}$. For this w, we may assume $w = \sqrt{\frac{\sigma-1}{\sigma}}e^{i\theta}$ for $1 \le \sigma < \infty$ and $-\infty < \theta < \infty$. If we assume $\hat{g}(z,i) = h(\sigma)$, then the PDE of $\hat{g}(z, i)$ becomes

$$h''(\sigma) + \left(\frac{1}{\sigma} + \frac{1}{\sigma - 1}\right)h'(\sigma) + \left(\frac{k^2}{4\sigma} + s\right)\frac{h(\sigma)}{\sigma(\sigma-1)} = 0$$

So by proposition 2.3.3., there exists a solution $h(\sigma)$ such that h has a unique logarithmic singularity at $\sigma = 1$ and vanishes at infinity up to scalar. Then we have

$$\hat{g}(z,\zeta) = h\left(\frac{1}{4}\frac{|z-\zeta|^2}{y\eta} + 1\right)$$

Then we have $\hat{g}(z,\zeta) = \overline{\hat{g}(\zeta, z)}$, i.e. $g(z,\zeta) = \overline{g(\zeta, z)}$. And $Lg(z,\zeta) = 0$. And moreover, for any $\gamma, \gamma' \in \Gamma$, we have

$$g(\gamma z, \gamma'\zeta) = j(\gamma, z)^{-1} j(\gamma', \zeta) g(z, \zeta)$$

where $j(\gamma, z) = \frac{(cz+d)^k}{|cz+d|^k}$ for $\gamma = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$. Then we may define

$$G(z,\zeta) = \frac{1}{2}\sum_{\gamma\in\Gamma}\chi(\gamma) j(\gamma, z) g(z, \gamma\zeta)$$

and we will have

$$\int_{\mathcal{H}} g(z,\zeta) f(\zeta)\mathrm{d}\zeta = \int_{\mathcal{F}} G(z,\zeta) f(\zeta)\mathrm{d}\zeta$$

Then we have $\int_{\mathcal{F}} |G(z,\zeta)|^2 \mathrm{d}\zeta < \infty$, and $Lf(z) = \int_{\mathcal{F}} G(z,\zeta) f(\zeta)\mathrm{d}\zeta$. So we know L is a Hilbert-Schmidt operation. So if $\{\lambda'\}$ is the set of eigenvalues of L, we have $\sum \lambda'^{-2} < \infty$. Since $\lambda' = \lambda - s$, we have $\sum(\lambda + s)^{-2} < \infty$, i.e. $\sum \lambda^{-2} < \infty$.

2.4 Basic Representation Theory

Solution 2.4.1. (a) By theorem 2.4.1., both H_1 and H_2 are finitely dimensional. Then we may pick an orthogonal normal basis $\{w_1, \ldots, w_n\}$ of H_2, and we may define a projection morphism $\pi : H_1 \to H_2$ as $\pi(v) = \sum \langle v, w_i \rangle w_i$. Since H_2 is irreducible, we have only two cases. The first is $\mathrm{Im}(\pi) = 0$, which means H_1 is orthogonal to H_2, and we've done. The second is $\mathrm{Im}(\pi) = H_2$, then clearly $H_1 \supset H_2$. Since H_1 is also irreducible, we have $H_1 = H_2$ or $H_2 = 0$, and each of these cases are impossible.

(b) By (a), the invariance is trivial. And clearly $\bigoplus H(\sigma) \subset H$. If $\bigoplus H(\sigma) \subsetneq H$, we may take the orthogonal complement H' of $\bigoplus H(\sigma)$. Then $(\pi|_{H'}, H)$ is a unitary representation on K. So it has irreducible subrepresentation, which is contained in $H(\sigma)$ for some σ and makes a contradiction to our construction.

(c) In this question we may assume $H = H(\sigma)$. If we have a copy $H' \subset H$ isomorphic to σ, then $H = H' \oplus H'^{\perp}$. Then $\dim H = \dim H'^{\perp} + \dim \sigma$. If $\dim H$ is finite, then the equality is proven by induction. If $\dim H$ is infinite, we know $H'^{\perp}$ is also infinitely dimensional. So in this process we may find infinitely many copies of σ and each of them are orthogonal to each other, hence $\sharp\{$copies of $\sigma\} = \infty = \dim H(\sigma)/\dim \sigma$.

Solution 2.4.2. (a) Consider $\phi(g,t) = f(ge^{tX}) - \sum_{n=0}^{\infty} \frac{t^n}{n!}(\mathrm{d}X^n\, f)(g)$. We clearly have $\phi(g,0) = 0$ for all $g \in G$. And moreover,

$$\begin{aligned}
\mathrm{d}X\, \phi(g,t) &= \frac{\mathrm{d}}{\mathrm{d}s} f(ge^{tX}e^{sX})|_{s=0} - \sum_{n=0}^{\infty} \frac{t^n}{n!}(\mathrm{d}X^{(n+1)} f)(g) \\
&= \frac{\partial}{\partial t} f(ge^{tX}) - \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!}(\mathrm{d}X^{\,n} f)(g) \\
&= \frac{\partial}{\partial t}\phi(g,t)
\end{aligned}$$

So by lemma 2.2.2., we have $\phi(g,t) \equiv 0$, i.e. $f(ge^X) = \sum_{n=0}^{\infty} \frac{1}{n!}(\mathrm{d}X^{\,n} f)(g)$.

(b) For any $\phi \in H$, we know $\langle \pi(e^{tX})f, \phi \rangle$ is a smooth function of t, then by (a) we have

$$\langle \pi(e^{tX})f, \phi \rangle = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{d}X^{\,n} \langle f, \phi \rangle = \sum_{n=0}^{\infty} \frac{t^n}{n!} \langle X^n f, \phi \rangle$$

i.e. $\langle \pi(e^{tX})f - \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n f, \phi \rangle = 0$.

Solution 2.4.3 (Gelfand Pairs)**.** ($\Rightarrow$) Trivial. ($\Leftarrow$) If (π, V) is an irreducible unitary representation of G, for any $\xi, \eta \in V^H$ and any $g \in G$, we may define $\phi_{\xi,\eta}(g) = \overline{\langle \pi(g)\xi, \eta \rangle}$, which is a K-bi-invariant function of G into $\mathbb{C}$, i.e. $\phi_{\xi,\eta} \in R$. For any $\theta \in V$, we have

$$\langle \pi(\phi_{\xi,\eta})v, \theta \rangle = \sum_{g \in G} \langle \pi(g)v, \theta \rangle \overline{\langle \pi(g)\xi, \eta \rangle} = \langle v, \xi \rangle \langle \eta, \theta \rangle$$

So we have $\pi(\phi_{\xi,\eta})v = \langle v, \xi\rangle\eta$. Moreover, we have

$$\pi(\phi_{\eta,\xi} * \phi_{\xi,\eta})v = \pi(\phi_{\eta,\xi}) \circ \pi(\phi_{\xi,\eta})v = \langle v, \xi\rangle\langle \eta, \eta\rangle\xi$$

And clearly if $v = \xi$, we have $\pi(\phi_{\eta,\xi} * \phi_{\xi,\eta})\xi \neq 0$. So the image of $\pi(\phi_{\eta,\xi} * \phi_{\xi,\eta})$ is $\mathbb{C}\eta$. Since R is commutative, we have $\mathbb{C}\xi = \mathbb{C}\eta$, i.e. V^H has dimension 1.

Solution 2.4.4. For any $X \in \mathfrak{g}$, $Y \in \mathfrak{k}$ and $f \in V$, if Λ is any linear functional on V, we may define $\phi : K \times \mathbb{R} \to \mathbb{C}$ as

$$\phi(g, t) = \Lambda\left(\pi(ge^{tY})X\pi(e^{-tY}g^{-1})f - \pi(g)(\mathrm{Ad}(e^{tY}X)\pi(g^{-1})f\right)$$

In lemma 2.2.2., we only need the connectedness of G, so it's also stated for $K = \mathrm{SO}(n, \mathbb{R})$. So by lemma 2.2.2., we need to verify two thing. The first one is $\phi(g, 0) = \Lambda(\pi(g)X\pi(g^{-1})f - \pi(g)X\pi(g^{-1})f) = 0$. For the second one, we have

$$\begin{aligned}\frac{\partial}{\partial t}\phi(g, t) &= \Lambda\left(\pi(g)Y \circ X \circ (-Y)\pi(g^{-1})f - \pi(g)((\mathrm{ad}Y)X)\pi(g^{-1})f\right) \\ &= \mathrm{d}Y\ \phi(g, t)\end{aligned}$$

So by lemma 2.2.2., we have $\phi(g, t) \equiv 0$. By 2.2.6.(b), we know the exponential map $\mathfrak{k} \to K$ is surjective, i.e. all ge^{tY} will cover the whole K, so we always have $\pi(g)\pi(X)\pi(g^{-1})f = \pi(\mathrm{Ad}(g)X)f$ for any $g \in K$ and $X \in \mathfrak{g}$.

Solution 2.4.5 (Induction from a subgroup of finite index)**.** Clearly we have $K/K_0 = K/(G_0 \cap K) \cong KG_0/G_0 \cong G/G_0$, where we have $KG_0 = G$ since K is maximal and every connected component of G contains a unique connected component of K. So we have $[G : G_0] = [K : K_0]$. Then we will verify the three conditions in the definition.

(i) Clearly $V = (K/K_0) \ltimes V_0$. And since V_0 is an algebraic direct sum of finite dimensional invariant subspaces under the action of K_0, and so does V of K.

(ii) Since $\mathfrak{k}_0 = \mathfrak{k}$, we have

$$(\pi(X)F)(k) = \pi_0(\mathrm{Ad}(k)X)F(k) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\pi_0(\mathrm{Ad}(k)e^{tX})F(k)\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}(\pi(e^{tX})F)(k)\right|_{t=0}$$

for any $X \in \mathfrak{k}$ and $F \in V$.

(iii) For any $g \in K$ and $X \in \mathfrak{g}$, we have

$$\begin{aligned}(\pi(\mathrm{Ad}(g)X)F)(k) &= \pi_0(\mathrm{Ad}(kg)X)F(k) \\ &= \pi_0(\mathrm{Ad}(k)g)\pi_0(\mathrm{Ad}(k)X)\pi_0(\mathrm{Ad}(k)g^{-1})F(k) \\ &= (\pi(g)\pi(X)\pi(g^{-1})F)(k)\end{aligned}$$

So V is a $(\mathfrak{g}, K)$-module.

Solution 2.4.6 (Schur's Lemma for $(\mathfrak{g}, K)$-Modules)**.** (a) Clearly if V and W are not isomorphic, then clearly $\ker \lambda$ is a subrepresentation of V, hence $\ker \lambda = 0$ or V by irreducibility of V. If $\ker \lambda = 0$, V is a subrepresentation of W, which is impossible. If $\ker \lambda = V$, we know $\lambda = 0$. So without lost of generality, we may assume $V = W$. Suppose ρ is an irreducible finite-dimensional representation of K such that $V(\rho)$ is nonzero. Then as representations of K, $\lambda : V \to V$ must maps $V(\rho)$ to $V(\rho)$ by the same reason. If t is an eigenvalue of λ on $V(\rho)$, then $(\lambda - t \cdot \mathrm{id}_V)$ commutes with $\pi(g)$ and $\pi(X)$ for all $g \in K$ and $X \in \mathfrak{g}$. So $\lambda = t$, i.e. λ acts by scalar. Hence $\mathrm{Hom}(V, V)$ has at most dimension 1.

(b) We may fix a $w \in W$ and a basis $\{v_i\}$ of V, then for any two bilinear forms $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$, we may define $\lambda_i = \frac{\langle v_i, w\rangle_1}{\langle v_i, w\rangle_2}$. Then we consider the linear map $\lambda(v) = \lambda(\sum a_i v_i) = \sum a_i \lambda_i v_i$. So clearly $\lambda : V \to V$ commutes with the actions of $\mathfrak{g}$ and K by (4.12) and (4.13), so by (a) we know λ is just a scalar, i.e. λ_i are all the same. So $\langle\ ,\ \rangle_1 = \lambda \langle\ ,\ \rangle_2$, i.e. the space of such bilinear maps has at most one dimension.

(c) All things are the same with (b).

2.5 Irreducible $(\mathfrak{g}, K)$-Modules for GL$(2, \mathbb{R})$

Solution 2.5.1. For any irreducible admissible representation (π', H) of $G = GL(2, \mathbb{R})^+$, we firstly assume it is infinitesimally equivalent to $(\pi_{s_1, s_2, \epsilon}, H(s_1, s_2, \epsilon))$ for some s_1, s_2 and ϵ of type (i) in theorem 2.5.4. Since for any $u \in \mathbb{R}^+$, $\left[\begin{smallmatrix} u & \\ & u \end{smallmatrix}\right] = e^{X_u}$ for some $X_u = \left[\begin{smallmatrix} \log u & \\ & \log u \end{smallmatrix}\right]$, and for any $u \in \mathbb{R}^-$, $\left[\begin{smallmatrix} u & \\ & u \end{smallmatrix}\right] = e^{X_u}$ for some $X_u = \left[\begin{smallmatrix} \log -u & \pi \\ -\pi & \log -u \end{smallmatrix}\right]$. Then for any $v \in H^{\mathrm{fin}}$, by 2.4.2.(b) we know $\pi'(\left[\begin{smallmatrix} u & \\ & u \end{smallmatrix}\right])v = \sum_{n=0}^{\infty} \frac{1}{n!} X_u^n v = \pi_{s_1, s_2, \epsilon} v = \mathrm{sgn}(u)^\epsilon |u|^{s_1 + s_2} v = \mathrm{sgn}(u)^\epsilon |u|^\mu v$. So we may define $\pi = \pi' \otimes \det^{-t}$ for $t = \frac{\mu}{2}+$, then $\pi'(\left[\begin{smallmatrix} u & \\ & u \end{smallmatrix}\right])v = v$ for any $u \in \mathbb{R}^+$, and $\pi' = \pi \otimes \det^t$. And trivially, π' is irreducible admissible iff so is π. Moreover, this condition is obviously equivalent to $\mu = 0$ according to above. For type (ii), all things are the same and we will omit it.

Solution 2.5.2. Clearly $\mathfrak{sl}(2, \mathbb{R})$ consists with all 2×2-matrix with zero trace, so it is generated by H, R, L but no Z. Then proposition 2.5.2., theorem 2.5.1. and theorem 2.5.2. are also standing for $(\mathfrak{g}, K)$-module for $\mathrm{SL}(2, \mathbb{R})$, i.e. we have

(1) If λ is not of the form $\frac{k}{2}(1 - \frac{k}{2})$ with some integer k with same parity of ϵ, then there exists at most one isomorphism class of $(\mathfrak{g}, K)$-modules on which Δ acts by λ.

(2) If $\lambda = \frac{k}{2}(1 - \frac{k}{2})$ with some integer k with same parity of ϵ, then the K-types of an irreducible admissible $(\mathfrak{g}, K)$-module is one of $\Sigma^+(k), \Sigma^-(k)$ and $\Sigma^0(k)$. And there exists at most one isomorphism class of irrducible admissible $(\mathfrak{g}, K)$-module with a such given K-type.

And the construction is the same with the case $\mathrm{GL}(2, \mathbb{R})^+$. For any $\epsilon \in \{0, 1\}$ and any $s \in \mathbb{C}$, we may define

$$H^\infty_{\epsilon, s} = \left\{ f \in C^\infty(G) \mid f(\left[\begin{smallmatrix} u & t \\ & u^{-1} \end{smallmatrix}\right] g) = \mathrm{sgn}(u)^\epsilon |u|^{\nu+1} f(g) \right\}$$

where $\nu = 2s - 1$. Then we may completize $H^\infty_{\epsilon,s}$ into a Hilbert space $H_{\epsilon,s}$ for the norm associated with $\langle f, g\rangle = \langle f|_K, g|_K\rangle_{L^2(K)}$. So we have a representation of $\mathrm{SL}(2, \mathbb{R})$ on $H_{\epsilon,s}$, and $H^\infty_{\epsilon,s}$ is a $(\mathfrak{g}, K)$-

module. Moreover, $H_{\epsilon,s}$ is generated by the functions of the form $f_k\left(\begin{bmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2}\end{bmatrix}\kappa_\theta\right) = y^s e^{ik\theta}$, which satisfies $Hf_k = kf_k$, $Rf_k = (s+\frac{k}{2})f_{k+2}$, $Lf_k = (s-\frac{k}{2})f_{k-2}$ and $\Delta f_k = \lambda f_k$. So we finish the construction.

Solution 2.5.3. (a) Since $\mathrm{O}(2) = \langle \mathrm{SO}(2), \eta\rangle$, so the $\mathrm{O}(2)$-representation structure of V is uniquely determined by specifying the $\mathrm{SO}(2)$-representation structure together with the action of η. And moreover, (5.28) are from (4.11) directly since $\eta = \eta^{-1}$.

(b) Conversely, if we have a $(\mathfrak{g}, K)$-module $\pi : \mathrm{SO}(2) \to \mathrm{End}(V)$ and $\eta \in \mathrm{End}(V)$ with $\eta^2 = 1$, we can define a representation $\pi' : \mathrm{O}(2) \to \mathrm{End}(V)$ as $\pi'|_{\mathrm{SO}(2)} = \pi$, and for any $g \in \mathrm{O}(2) - \mathrm{SO}(2)$, we may define $\pi'(g) = \pi(g \cdot \left[\begin{smallmatrix} -1 & \\ & 1\end{smallmatrix}\right])\eta$. Since $\eta^2 = 1$ and $\mathrm{Ad}(\eta)Z = Z$, the representation π' is well-defined. And adding the rest three equalities of (5.28), the $\mathrm{O}(2)$-representation π' is compatible with the action of π on $\mathfrak{g}$, so π' is a $(\mathfrak{g}, \mathrm{O}(2))$-module.

Solution 2.5.4. (a) By 2.5.3., we only need to show that there exist at most two η's. By (5.28), we know for any $v \in V(k)$ for some fixed k, we have $H(\eta v) = \eta(-Hv) = -k\eta v$, i.e. $\eta v \in V(-k)$. If $\epsilon = 0$, we have $\eta : V(0) \to V(0)$. Since $V(0)$ has dimension 1, there exists at most two η such that $\eta v = \pm v$ on $V(0)$. If $\epsilon = 1$, we have $\eta : V(1) \to V(-1)$. Consider the two-dimensional subspace $W = V(1) \oplus V(-1)$, and H acts like $\left[\begin{smallmatrix} 1 & \\ & -1\end{smallmatrix}\right]$. If η acts like $\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]$ then we have two equations:

$$\begin{bmatrix} a & b \\ c & d\end{bmatrix}^2 = \begin{bmatrix} 1 & \\ & 1\end{bmatrix}, \qquad \begin{bmatrix} a & b \\ -c & -d\end{bmatrix} = \begin{bmatrix} -a & b \\ -c & d\end{bmatrix}$$

Hence we have solutions $\eta = \left[\begin{smallmatrix} & b \\ b^{-1} & \end{smallmatrix}\right]$. But if we fix a base $\{v_1, v_{-1}\}$ of W, we have $(\eta R\eta)(v_1) = b^2 v_{-1}$, but $Lv_1 = v_{-1}$. So b has only two solutions, a.k.a. $b = \pm 1$. So η have at most two form: $\left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} & -1 \\ -1 & \end{smallmatrix}\right]$.

(b) If the condition (iii) in the definition of $(\mathfrak{g}, K)$-module were omitted, the equations in (5.28) will degenerate to only one: $\eta^2 = 1$. Then for any k, we can define an η_k as $\eta(V(k)) = V(-k)$, $\eta(V(-k)) = V(k)$, and $\eta|_{V(l)} = \mathrm{id}_{V(l)}$ for any $l \neq \pm k$. So we have infinite number of $(\mathfrak{g}, \mathrm{O}(2))$-module structures on V.

Solution 2.5.5. Deduced from proposition 2.5.5. directly.

Solution 2.5.6. Similarly for any l such that $|l| \geq k$ and $v \in V(l)$, we have $H(\eta v) = \eta(-Hv) = -l\eta v$, so $\eta v \in V(-l)$. Then we may choose $v_k \in V(k)$ and $v_{-k} \in V(-k)$, and define $v_{k+2n} = R^n v_k$, $v_{-k-2n} = L^n v_{-k}$. And we may assume $\eta v_l = \delta_l v_{-l}$. Since $\eta^2 = 1$, we have $\delta_l \delta_{-l} = 1$ for all l. And since $\eta R \eta = L$, for any $l > 0$ we have

$$\left(-\lambda + \frac{l}{2}\left(1 - \frac{l}{2}\right)\right) v_{l-2} = Lv_l = \eta R \eta v_l = \delta_l \delta_{-l+2} \cdot \left(-\lambda + \frac{l}{2}\left(1 - \frac{l}{2}\right)\right) v_{l-2}$$

So we have $\delta_k = \delta_{k+2} = \ldots$, we may denote this as δ. So we have $\eta v_l = \delta^{\mathrm{sgn}(l)} v_{-l}$. So different induced representations of $\mathcal{D}^+_\mu(k) \oplus \mathcal{D}^-_\mu(k)$ are differed by scalars $\delta \oplus \delta^{-1}$ on each part, hence they are isomorphic.

Solution 2.5.7 (Finite-dimensional representations of $\mathrm{GL}(2,\mathbb{R})^+$ and $\mathrm{GL}(2,\mathbb{R})$)**.** For $\hat{H}, \hat{R}, \hat{L}, \hat{Z}$, we directly have

$$\begin{aligned}
\mathrm{d}\hat{H}\,\hat{\xi}_r &= \frac{\mathrm{d}}{\mathrm{d}t}\pi(e^{t\hat{H}})\hat{\xi}_r\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}e^{tr}e^{-t(h-r)}\hat{\xi}_r\Big|_{t=0} = (2r-h)\hat{\xi}_r \\
\mathrm{d}\hat{R}\,\hat{\xi}_r &= \frac{\mathrm{d}}{\mathrm{d}t}\pi(e^{t\hat{R}})\hat{\xi}_r\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\bigvee_r e_1\right)\vee\left(\bigvee_{h-r}(te_1+e_2)\right)\Bigg|_{t=0} = (h-r)\hat{\xi}_{r+1} \\
\mathrm{d}\hat{L}\,\hat{\xi}_r &= \frac{\mathrm{d}}{\mathrm{d}t}\pi(e^{t\hat{L}})\hat{\xi}_r\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\bigvee_r (e_1+te_2)\right)\vee\left(\bigvee_{h-r} e_2\right)\Bigg|_{t=0} = r\hat{\xi}_{r-1} \\
\mathrm{d}\hat{Z}\,\hat{\xi}_r &= \frac{\mathrm{d}}{\mathrm{d}t}\pi(e^{t\hat{Z}})\hat{\xi}_r\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}e^{th}\hat{\xi}_r\Big|_{t=0} = h\hat{\xi}_r
\end{aligned}$$

Then by (2.16) we have the same three of H, R, L, Z.

For the map $\phi : \xi_r \to (-1)^r f_{2r-k+2}$, we have

$$\begin{aligned}
\phi(\mathrm{d}H\,\xi_r) &= \phi((2r-h)\xi_r) = (2r-h)(-1)^r f_{2r-k+2} = \mathrm{d}H\, f_{2r-k+2} \\
\phi(\mathrm{d}R\,\xi_r) &= \phi((h-r)\xi_{r+1}) = (h-r)(-1)^{r+1} f_{2r-k+4} = \mathrm{d}R\, f_{2r-k+2} \\
\phi(\mathrm{d}L\,\xi_r) &= \phi(r\xi_{r-1}) = r(-1)^{r-1} f_{2r-k} = \mathrm{d}L\, f_{2r-k+2} \\
\phi(\mathrm{d}Z\,\xi_r) &= \phi(h\xi_r) = h(-1)^r f_{2r-k+2} = \mathrm{d}H\, f_{2r-k+2}
\end{aligned}$$

So ϕ is a homomorphism of representation.

For $\pi = \pi_h$ and quasicharacter χ, since $e^{t\hat{H}}$, $e^{t\hat{R}}$ and $e^{t\hat{L}}$ all have determinant 1, so tensoring a χ does not change the action of H, R, L on ξ_r. But for Z, we have

$$(\pi\otimes\chi)(\mathrm{d}\hat{Z}\,)\hat{\xi}_r = \frac{\mathrm{d}}{\mathrm{d}t}\chi(e^{2t})\pi(e^{t\hat{Z}})\hat{\xi}_r\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\chi(e^{2t})e^{th}\hat{\xi}_r\Big|_{t=0} = (h+2\chi'(1))\xi_r$$

Solution 2.5.8. By 2.4.4., we only need to verify that for any $X \in \mathfrak{k}$, we have $\hat{\pi}(X)\Lambda = \frac{\mathrm{d}}{\mathrm{d}t}\hat{\pi}(\exp(tX))\Lambda|_{t=0}$. For any $v \in V$, we have

$$\langle v, X\Lambda\rangle = -\langle Xv, \Lambda\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\langle \exp(tX)v, \Lambda\rangle\Big|_{t=0} = -\frac{\mathrm{d}}{\mathrm{d}t}\langle v, \exp(-tX)\Lambda\rangle\Big|_{t=0} = \langle v, \frac{\mathrm{d}}{\mathrm{d}t}\exp(tX)\Lambda\Big|_{t=0}\rangle$$

So this is a $(\mathfrak{g}, K)$-module.

If V is irreducible, by theorem 2.5.4. we know $V = \bigoplus_{k\in\Sigma} V(k)$. Then if we pick a nonzero $v_k \in V(k)$ for every $k \in \Sigma$ as a basis, we can define Λ_k as $\langle v_l, \Lambda_k\rangle = \delta_{kl}$. Then clearly Λ is a vector space with basis $\{\Lambda_k \mid k \in \Sigma\}$. Then we can define an isomorphism $\phi : \hat{V} \to V$ between vector spaces as $\phi(v_k) = \Lambda_k$. Then for any $g \in K$, we have

$$\langle v, \hat{\pi}(g)\Lambda\rangle = \langle \pi(g^{-1})v, \Lambda\rangle = \langle v, (\pi(g^{-1}))^*\Lambda\rangle$$

And for any $X \in \mathfrak{g}$, we have

$$\langle v, X\Lambda\rangle = \langle (-X)v, \Lambda\rangle = \langle v, (-X)^*\Lambda\rangle$$

So we know $\hat{\pi}(g)$ acts by $\pi({}^T g^{-1})$ under the isomorphism ϕ. So $\hat{\pi}$ is isomorphic to π with the automorphism $g \mapsto {}^T g^{-1}$ of $\mathrm{GL}(2,\mathbb{R})^+$.

2.6 Unitaricity and Intertwining Integrals

Solution 2.6.1. Clearly for any $f_{k,s}$ we have $\|f_{k,s}\|^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta}e^{-ik\theta}\mathrm{d}\theta = 1$. But by proposition 2.6.3., we have

$$\|M(s)f_{k,s}\| = \sqrt{\pi}\left|\frac{\Gamma(s)\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(s+\frac{k}{2}\right)\Gamma\left(s-\frac{k}{2}\right)}\right| \neq 1$$

So $M(s)$ is not isometry. Moreover, by lemma 1.9.1., we have

$$\begin{aligned}
\Gamma\left(s+\frac{k}{2}\right)\Gamma\left(s-\frac{k}{2}\right) &= -4\int_0^\infty u^{2s-1}K_k(2u)\mathrm{d}u \\
&= -4\int_0^\infty u^{2s-1}\int_0^\infty v^k e^{-(v+v^{-1})u}\frac{\mathrm{d}v}{v}\mathrm{d}u \\
&= -4\int_0^\infty u^{2s-1}\left(\int_0^1 v^k e^{-(v+v^{-1})u}\frac{\mathrm{d}v}{v} + \int_1^\infty v^k e^{-(v+v^{-1})u}\frac{\mathrm{d}v}{v}\right)\mathrm{d}u \\
&= -4\int_0^\infty u^{2s-1}\left(\int_1^\infty v^{-k} e^{-(v+v^{-1})u}\frac{\mathrm{d}v}{v} + \int_1^\infty v^k e^{-(v+v^{-1})u}\frac{\mathrm{d}v}{v}\right)\mathrm{d}u \\
&= -4\int_0^\infty u^{2s-1}\int_1^\infty \left(v^k + v^{-k}\right)e^{-(v+v^{-1})u}\frac{\mathrm{d}v}{v}\mathrm{d}u
\end{aligned}$$

So when $k \to \infty$, the function $\Gamma\left(s+\frac{k}{2}\right)\Gamma\left(s-\frac{k}{2}\right) \to \infty$ with a exponential speed. So there exists a constant ϵ such that $\sum n^{2\epsilon}$ is convergent but $\sum |\sqrt{\pi}\frac{\Gamma(s)\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(s+\frac{k}{2}\right)\Gamma\left(s-\frac{k}{2}\right)}|^{-1}n^{2\epsilon}$ is not convergent. So we have $\sum n^\epsilon f_{k,1-s} \in \overline{H^\infty(s_2,s_1,\epsilon)} = H(s_2,s_1,\epsilon)$ but it has no preimage in $H(s_1,s_2,\epsilon)$.

Solution 2.6.2. By proposition 2.6.5., we may assume $\mu = it$ for some $t > 0$. Then we consider the case $s_1 = \frac{1}{2}(k-1+\mu)$, $s_2 = \frac{1}{2}(1-k+\mu)$. So $\lambda = \frac{k}{2}(1-\frac{k}{2})$. Then we consider the space $H^\infty(s_1,s_2,\epsilon)$ with the G-invariant norm $\langle f, f'\rangle = \int_K f(\kappa)\overline{i^\epsilon M(s)f'(\kappa)}\mathrm{d}\kappa$ as in theorem 2.6.4. Since we have $k > 1$, we know $s = \frac{1}{2}(k+\mu)$ has real part greater than $\frac{1}{2}$. Then for any $l = k + 2n$, we have

$$\begin{aligned}
\langle f_{l,s}, f_{l,s}\rangle &= (-1)^{\frac{l}{2}}\sqrt{\pi}\frac{\Gamma(s)\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(s+\frac{l}{2}\right)\Gamma\left(s-\frac{l}{2}\right)} \\
&= (-1)^{\frac{k}{2}+n}\sqrt{\pi}\frac{\Gamma\left(\frac{k+\mu}{2}\right)\Gamma\left(\frac{k+\mu-1}{2}\right)}{\Gamma\left(k+n+\frac{\mu}{2}\right)\Gamma\left(-n+\frac{\mu}{2}\right)} \\
&= (-1)^{\frac{k}{2}+n}\pi^{2n-\mu}\frac{\Gamma\left(\frac{k+\mu}{2}\right)\Gamma\left(\frac{k+\mu-1}{2}\right)}{\Gamma\left(k+n+\frac{\mu}{2}\right)\Gamma\left(n+\frac{1-\mu}{2}\right)}\frac{\zeta(-2n+\mu)}{\zeta(1+2n-\mu)}
\end{aligned}$$

So clearly if we take t sufficiently closed to zero, this $\langle f_{l,s}, f_{l,s}\rangle$ are all positive. So this is an inner product, hence $\mathcal{D}_\mu^+(k)$ has a unitary representation. And so does $\mathcal{D}_\mu^-(k)$.

Solution 2.6.3. If (π, V) and (π', V') are two infinitesimal equivalent irreducible unitary representations of $G = \mathrm{GL}(2,\mathbb{R})$, we have an isomorphism $\phi : V^\infty \cong V'^\infty$. Then for any $v \in V^\infty$ and $v' \in V'^\infty$, we can define a bilinear form $\langle\cdot,\cdot\rangle_1 : V^\infty \times V'^\infty \to \mathbb{C}$ as $\langle v, v'\rangle_1 = (v, \phi^{-1}(v'))_V$. Then for any $g \in K$ and $X \in \mathfrak{g}$, we have $\langle \pi(g)v, \pi'(v')\rangle_1 = (\pi(g)v, \phi^{-1}(\pi'(g)v'))_V = \pi(g)v, \pi(g)\phi^{-1}(v'))_V = (v, \phi^{-1}(v'))_V = \langle v, v'\rangle_1$, and $\langle \pi(X)v, v'\rangle_1 = (\pi(X)v, \phi^{-1}(v'))_V = -(v, \pi(X)\phi^{-1}(v'))_V = -(v, \phi^{-1}(\pi'(X)v'))_V = -\langle v, \pi'(X)v'\rangle_1$. And similarly we can define another bilinear form as $\langle v, v'\rangle_2 = (\phi(v), v')_{V'}$ and it has the same property. So by 2.4.6.(b), we have $\langle\cdot,\cdot\rangle_1 = \lambda\langle\cdot,\cdot\rangle_2$ for some fixed λ. So we have $(v,v)_V = \langle v, \phi(v)\rangle_1 = \lambda\langle v, \phi(v)\rangle_2 = \lambda(\phi(v), \phi(v))_{V'}$. Hence ϕ is an isometry, which can be extended to $\phi : V \to V'$. For any $g \in \mathrm{GL}(2,\mathbb{R})$, if $g \in \mathrm{GL}(2,\mathbb{R})^+$, for the same reason with theorem 2.6.6., we have $\phi(\pi(g)v) = \pi'(g)\phi(v)$ for any $v \in V$. And for other g, we have $g = g\left[\begin{smallmatrix}-1&\\&1\end{smallmatrix}\right]\left[\begin{smallmatrix}-1&\\&1\end{smallmatrix}\right]$ with $g\left[\begin{smallmatrix}-1&\\&1\end{smallmatrix}\right] \in \mathrm{GL}(2,\mathbb{R})^+$ and $\left[\begin{smallmatrix}-1&\\&1\end{smallmatrix}\right] \in K$. Then $\phi(\pi(g)v) = \phi(\pi(g\left[\begin{smallmatrix}-1&\\&1\end{smallmatrix}\right])\pi(\left[\begin{smallmatrix}-1&\\&1\end{smallmatrix}\right])v) = \pi'(g\left[\begin{smallmatrix}-1&\\&1\end{smallmatrix}\right])\phi(\pi(\left[\begin{smallmatrix}-1&\\&1\end{smallmatrix}\right])v) = \pi'(g\left[\begin{smallmatrix}-1&\\&1\end{smallmatrix}\right])\pi'(\left[\begin{smallmatrix}-1&\\&1\end{smallmatrix}\right])\phi(v) = \pi'(g)\phi(v)$. So ϕ is an isomorphism between these two representations.

Solution 2.6.4. By theorem 2.5.5., for any fixed λ and μ, there exist at most two infinitesimal equivalent classes of irreducible admissible representations on which Z and Δ act by λ and μ. So if in any infinitesimal equivalent class there exists more than one unitary representations, by 2.6.3., they are isomorphic. So for any fixed λ and μ, there exist at most two unitary irreducible admissible representations on which Z and Δ act by λ and μ.

2.7 Representations and the Spectral Problem

No Exercise Here.

2.8 Whittaker Modules

No Exercise Here.

2.9 A Theorem of Harish-Chandra

No Exercise Here.

3 Automorphic Representations

3.1 Tate's Thesis

Solution 3.1.1. (a) Denote the morphism $G \to \mathrm{GL}(m, \mathbb{C})$ by ϕ. Since $\mathrm{GL}(m, \mathbb{C})$ is a connected Lie group, it does not have nontrivial open subgroup. So For any sufficiently small neighbourhood N of 1 which does not contain any subgroup of $\mathrm{GL}(m, \mathbb{C})$, we know $\phi^{-1}(N)$ is open. For any open compact subgroup $H \subset \phi^{-1}(N)$, we know $\phi(H)$ is also a group contained in N, then $\phi(H) = 1$, i.e. $H \subset \ker \phi$.

(b) By (a), there exists $m \subset \mathbb{Z}$ such that ψ is trivial on $\mathfrak{p}^m$. Since ψ is not trivial, ψ is not trivial on $\mathfrak{p}^m$ for all $m \in \mathbb{Z}$. So we may take a minimal one, which is called the conductor.

(c) By Schur's lemma, since k is abelian as a finite group, there exists $\sharp k$ 1-dimensional representation of k. Clearly any nontrivial additive character ψ is a 1-dimensional representation of k. Then we may construct all other representation of k as $\psi_a(x) = \psi(ax)$. Then these are all 1-dimensional representation of k, i.e. all additive character of k. So any other additive character has the form ϕ_a.

(d) Let $\mathfrak{p}^{-m_1}$ be the conductor of ψ_1. We may use the induction. When $N = m_1$, for any $x \in \mathfrak{p}^{-N}$, we have $\psi_1(x) = 1 = \psi(0)$. So we may define $a_N = 0$. When N increase, for any $x \in \mathfrak{p}^{-N}$, by (b) there exists some a_N such that $\psi_1(x) = \phi(a_N x)$. So clearly we have $a_N - a_{N-1} \in \mathfrak{p}^{N-1-m_1}$. So $\{a_N\}$ is a Cauchy sequence, we may take $a = \lim a_N$. Then $\psi_1(x) = \psi(ax)$ for any x.

(e) By (d) we know $a \mapsto \psi_a$ is a 1-1 correspondence between F and $\hat{F}$. For any open subset $b + \mathfrak{p}^N \in F$, for any bounded subset $A \subset F$ and any $a = b + b' \in b + \mathfrak{p}^N$, we have $\psi_a(x) = \psi(ax) = \psi(bx)\psi(b'x)$ for any $x \in A$, then $\sup_{x \in A} |\psi_a(x)| \leq \sup_{x \in A} |\psi(bx)| \sup_{x \in A} |\psi(b'x)|$. For the last part, since $b' \in \mathfrak{p}^N$ and A is bounded, this supreme will be take by one x. So this 1-1 correspondence is a topological isomorphism.

(f) In the case of $\mathbb{R}$, for any $\psi' \in \hat{\mathbb{R}}$, we may assume $\psi'(1) = e^{2\pi i a}$ with some $a \in \mathbb{R}$. Then $\psi'(n) = \psi'(1)^n = e^{2\pi i n a}$, for any rational number $\psi'(p/q)^q = \psi'(p) = e^{2\pi i p a}$, i.e. $\psi'(p/q)^q = e^{2\pi i (p/q) a}$ since this morphism is continuous. Then for any $x \in \mathbb{R}$, we must have $\psi'(x) = e^{2\pi i x a}$ by continuity. So $\psi' = \psi_a$, i.e. $a \mapsto \psi_a$ is a 1-1 correspondence between $\mathbb{R}$ and $\hat{\mathbb{R}}$. Since the morphism $\psi_a(x) = e^{2\pi i a x}$ is continuous of both two variants a and x, the 1-1 correspondence is a topological isomorphism.

In the case of $\mathbb{C}$, for any ψ' we may assume $\psi'(1) = e^{2\pi i a}$ for some $a \in \mathbb{R}$. Then by the same method with $\mathbb{R}$, we know $\psi'(x) = e^{2\pi i x a}$ for all $x \in \mathbb{R}$. For the imaginary part, we may assume $\psi'(i) = e^{4\pi i b}$, then clearly $\psi'(ix) = e^{4\pi i x b}$ for all $x \in \mathbb{R}$. Then for any $x + iy \in \mathbb{C}$, we have $\psi'(x + iy) = e^{4\pi i (xa + yb)} = e^{4\pi i \mathrm{Re}((x+iy)(a-ib))}$. So $a \mapsto \psi_a$ is a 1-1 correspondence between $\mathbb{C}$ and $\hat{\mathbb{C}}$. And the topological isomorphism is the same.

Solution 3.1.2. (a) Clearly ψ is an additive character, and clearly $\psi(1) = 1$. So $\psi(\mathbb{Z}) = 1$. Hence we only need to prove that $\psi(1/n) = 1$ for every $n \in \mathbb{Z}_+$. We may assume $n = \prod p_i^{n_i}$ for finitely many primes p_i and $n_i > 0$. Then clearly $\psi_{v_i}(1/n) = e^{-2\pi i a_i / p_i^{n_i}}$ for some $a_i \in \mathbb{Z}$ such that $(np_i^{-n_i})a_i \equiv 1 \bmod p_i^{n_i}$. So, $n \cdot \left(\sum_i a_i p_i^{-n_i}\right) \equiv n \cdot \left(a_i p_i^{-n_i}\right) \equiv 1 \bmod p_i^{n_i}$. Since all p_i are different, we know $n \cdot \left(\sum_i a_i p_i^{-n_i}\right) \equiv$

1 mod n, i.e. $\sum_i a_i p_i^{-n_i} \in \frac{1}{n} + \mathbb{Z}$. So $\psi(1/n) = e^{2\pi i \sum_i a_i p_i^{-n_i}} \cdot e^{-2\pi i/n} = 1$.

(b) For any place v of F, we may assume v is over v' for some place v' of $\mathbb{Q}$. Then we may define $\psi_v(x) = \psi_{v'}(\mathbf{Tr}_{F_v/\mathbb{Q}_v} x)$. Then trivial.

Solution 3.1.3. (a) We may assume $\psi = \prod \psi_v$. Then for every $\psi' = \prod_{\psi_v'}$, there exists some $a_v \in F_v$ such that $\psi_v'(x) = \psi_v(a_v x)$ for every $x \in F_v$ by 3.1.1.(d) and (f). So $\psi' = \psi_a$ for $a = \prod a_v \in \mathbb{A}$.

(b) By 3.1.1.(e) and (f), since the topology basis of $\mathbb{A}$ is the restrict product of the basis of all place F_v, the 1-1 correspondence above is a topological isomorphism.

(c) ($\Leftarrow$) Trivial. ($\Rightarrow$) If $a \notin F$, we know the set $a \cdot F$ is dense in $\mathbb{A}/F$. Since ψ is continuous, if ψ_a is trivial on F we know ψ is trivial on the whole $\mathbb{A}$, which makes a contradiction.

(d) By (c) we have $\widehat{\mathbb{A}/F} \cong F$, then by Pontriagin's duality we know $\hat{F} = \widehat{\widehat{\mathbb{A}/F}} \cong \mathbb{A}/F$.

Solution 3.1.4. We write $\mathbb{A}^\times = \mathbb{A}_{\text{fin}}^\times \times (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$. Then for any ideal $\mathfrak{f} \subset \mathfrak{o}$, we define $S_0(\mathfrak{f}) = \{$ finite $v \mid v(\mathfrak{f}) > 0\}$ and S_1 the set of all other finite places. Then we define $U_v(\mathfrak{f}) = \{x \in \mathfrak{o}_v \mid x - 1 \in \mathfrak{f}\mathfrak{o}_v\}$ for all $v \in S_0(\mathfrak{f})$, and $U_{\text{fin}}(\mathfrak{f}) = \prod_{v \in S_0(\mathfrak{f})} U_v(\mathfrak{f}) \times \prod_{v \in S_1(\mathfrak{f})} \mathfrak{o}_v^\times$, $U(\mathfrak{f}) = U_{\text{fin}}(\mathfrak{f}) \times (\mathbb{R}_+^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$. Then every character χ of $\mathfrak{A}^\times$ is trivial on an open subgroup of $\mathfrak{A}_{\text{fin}}^\times$. When $\mathfrak{f}$ varies, all $U_{\text{fin}}(\mathfrak{f})$ form a basis of neighbourhoods of the identity in $\mathfrak{A}_{\text{fin}}^\times$. So there exists a maximal $\mathfrak{f}$ such that $\chi|_{U_{\text{fin}}(\mathfrak{f})} \equiv 1$. And χ on each $\mathbb{R}^\times$ and $\mathbb{C}^\times$ has the form $\text{sgn}(x)^\epsilon |x|^\lambda$ and $|z|^\lambda \cdot (\frac{z}{\bar{z}})^\mu$ for some $\epsilon = 0$ or 1, purely imaginary λ and real μ. So we may define $\chi_1(x) = \chi(x) \prod_{i=1}^{r_1} |x|_i^{-\lambda_i} \prod_{j=1}^{r_2} |z|_i^{-\lambda_j} (\frac{z}{\bar{z}})^{-\mu_j}$.

Denote $V(\mathfrak{f}) = \left(\prod_{v \in S_0(\mathfrak{f})} U_v(\mathfrak{f})\right) \times \left(\prod_{v \in S_1(\mathfrak{f})} \mathfrak{o}_v^\times\right) \times (\mathbb{R}_+^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$. Then by approximation theorem we have $\mathbb{A}^\times = F^\times V(\mathfrak{f})$, i.e. $\mathbb{A}^\times/F^\times = V(\mathfrak{f})/(F^\times \cap V(\mathfrak{f}))$. Then we may consider the restriction of χ_1 on $V(\mathfrak{f})$. Since χ_1 is trivial on $U(\mathfrak{f})(F^\times \cap V(\mathfrak{f}))$, we may prove that $V(\mathfrak{f})/U(\mathfrak{f})(F^\times \cap V(\mathfrak{f})) \cong (\mathfrak{o}/\mathfrak{f})^\times$. Define $I_\mathfrak{f}$ to be the set of all fractional ideals prime to $\mathfrak{f}$, and $P_\mathfrak{f}$ to be principal fractional ideals $\alpha\mathfrak{o}$ such that $\alpha \in F^\times \cap V(\mathfrak{f})$. Then $(\mathfrak{o}/\mathfrak{f})^\times \cong I_\mathfrak{f}/P_\mathfrak{f}$. On the other hand, for any $a = (a_v) \in \mathfrak{A}^\times$, we may define a fractional ideal $\iota(a) = \prod_v \mathfrak{p}_v^{\text{ord}_v(a_v)}$. So we have a morphism $V(\mathfrak{f}) \to I_\mathfrak{f} \to I_\mathfrak{f}/P_\mathfrak{f}$. It is clearly surjective with kernel $U(\mathfrak{f})(F^\times \cap V(\mathfrak{f}))$. So we know every Groessencharakter $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ is induced from a Hecke character and some infinity part characters.

Conversely, for any Groessencharaktercharacter $\chi_0 = \chi_{0,\text{fin}}\chi_{0,\infty} : (\mathfrak{o}/\mathfrak{f})^\times \to \mathbb{C}^\times$, there exists a Hecke character $\chi_{\text{fin}} : \mathbb{A}^\times/F^\times \cong V(\mathfrak{f})/(F^\times \cap V(\mathfrak{f})) \to V(\mathfrak{f})/U(\mathfrak{f})(F^\times \cap V(\mathfrak{f})) \cong (\mathfrak{o}/\mathfrak{f})^\times \to \mathbb{C}^\times$. Then $\chi = \chi_{\text{fin}}\chi_{0,\infty}$. And clearly $\chi_0(p_v) = \chi(\mathfrak{p}_v)$.

Solution 3.1.5. (a) Similarly with 3.1.4., for any Hecke character $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$, there exists a $U_{\text{fin}}(\mathfrak{f})$ such that $\chi|U_{\text{fin}}$ is trivial. So χ_{fin} is a quasicharacter of $V(\mathfrak{f})/U(\mathfrak{f})(F^\times \cap V(\mathfrak{f}))$, i.e. finite order. So χ_{fin} is a character, then we just need to consider the infinite places. For every $F_v = \mathbb{R}$, we have $\chi_v(x) = \text{sgn}(x)^\epsilon |x|^\lambda$ for some $\lambda \in \mathbb{C}$. For every $F_v = \mathbb{C}$, we have $\chi_v(z) = |z|^\lambda \text{Arc}(z)^n$ for some $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}_+$. So if we define $\chi_v'(x) = |x|^{\Re(\lambda)}$ if $F_v = \mathbb{R}$, and $\chi_v'(z) = |z|^{\Re(\lambda)}$ if $F_v = \mathbb{C}$, we know $\chi \cdot \chi'^{-1}$ is a character. And since χ' is a character of $\mathbb{A}^\times/F^\times$, we know for every $x \in F^\times$, we have $\prod |x^{(j)}|^{\Re(\lambda_j)} = 1$. So we must have $\Re(\lambda_j)$ are all the same, i.e. χ' is actually in the form $|\cdot|^s$ for some $s \in \mathbb{R}$.

(b) Since F is non-Archimedean, $F^\times$ has a basis of open subsets $1 + \mathfrak{p}^n$ around 1. Since χ is continuous, there exists a minimal n such that $\chi|_{1+\mathfrak{f}^n}$ is trivial. So $\chi|_{\mathbb{O}^\times}$ is actually a quasicharacter of $\mathbb{O}^\times/(1+\mathfrak{p}^n)$, which is a finite group, hence $\chi|_{\mathbb{O}^\times}$ is a character. Since $F^\times = \mathbb{O}^\times \cdot p^{\mathbb{Z}}$, where p is the local parameter of F. So we may take an $s \in \mathbb{R}$ such that $\chi(p)|p|^s \in \mathbb{S}^1$, then $\chi(x)|x|^s$ is a character.

Solution 3.1.6. Denote the group of all fractional ideals of F as I and the group of all principal fractional ideals of F as P. We define a map $\iota : \mathbb{A}^\times \to I$ as $(a) \mapsto \prod_{v\nmid\infty} \mathfrak{p}_v^{\mathrm{ord}_v a_v}$. Then we have a character $\chi : A^\times \to I \to C \to \mathbb{C}^\times$. For any $a \in F^\times$, we clearly know that $\iota(a) \in P$. So χ induces a character $\chi : \mathbb{A}^\times/F^\times \to I/P \cong C \to \mathbb{C}^\times$. And clearly, $\chi_v(\varpi_v) = \chi_0(\mathfrak{p}_v)$.

Solution 3.1.7. We may assume $\Phi(x) = \prod_v \Phi_v(x_v)$ for some Schwartz-Bruhat functions Φ_v and most of Φ_v for finite v are characteristic functions of $\mathbb{O}_v$. Denote $S = \{\text{all infinite places}\} \cup \{\text{finite places } v \text{ such that } \Phi_v \neq \mathbf{1}_{\mathbb{O}_v^\times}\}$. So if $\Phi(x) \neq 0$, we know $x_v \in \mathbb{O}_v$ for all $v \notin S$. So for every $a \in F$, we know $\Phi(x+a) = \prod \Phi_v(x_v+a)$. If $v \notin S$, we know $\Phi(x_v+a) \neq 0$ iff $x_v+a \in \mathbb{O}_v$, i.e. $a \in \mathbb{O}_v$. Hence $\Phi(x+a) \neq 0$ iff $a \in \mathbb{O}_v$ for all $v \notin S$. So there exists some $M \in \mathbb{O}$ such that $\sum_{a\in F} \Phi(x+a) = \sum_{n\in\mathbb{O}} \Phi(x+\frac{n}{M})$. Since for all infinite v, we know $\Phi_v(x+\frac{n}{M})$ decay rapidly, we know $\sum_{n\in\mathbb{O}} \Phi(x+\frac{n}{M})$ is convergent absolutely and uniformly in any compact subset.

Solution 3.1.8. When $F_v = \mathbb{R}$ and $\chi = 1$, we take $\Phi_v(x) = e^{-\pi x^2}$, then $\hat{\Phi}_v(x) = \int_{\mathbb{R}} e^{-\pi y^2} e^{2\pi i x y} \mathrm{d}y = e^{-\pi x^2}$. So

$$\begin{aligned}
\zeta_v(s,\chi_v,\Phi_v) &= \int_{\mathbb{R}^\times} \Phi_v(x)\chi_v(x)|x|^s \frac{\mathrm{d}x}{x} = 2\int_0^\infty e^{-\pi x^2} x^{s-1}\mathrm{d}x = \pi^{-s/2}\Gamma(s/2)\\
L_v(s,\chi_v) &= \pi^{-s/2}\Gamma(s/2)\\
\epsilon_v(s,\chi_v,\psi_v) &= \frac{\zeta_v(1-s,\chi_v,\hat{\Phi}_v)}{L_v(1-s,\chi_v^{-1})} \cdot \frac{L_v(s,\chi_v)}{\zeta_v(s,\chi_v,\Phi_v)} = 1
\end{aligned}$$

When $F_v = \mathbb{R}$ and $\chi = \mathrm{sgn}$, we take $\Phi_v(x) = xe^{-\pi x^2}$, then $\hat{\Phi}_v(x) = \int_{\mathbb{R}} xe^{-\pi y^2} e^{2\pi i x y}\mathrm{d}y = ixe^{-\pi x^2}$. So

$$\begin{aligned}
\zeta_v(s,\chi_v,\Phi_v) &= \int_{\mathbb{R}^\times} \Phi_v(x)\chi_v(x)|x|^s \frac{\mathrm{d}x}{x} = 2\int_0^\infty e^{-\pi x^2} x^{s}\mathrm{d}x = \pi^{-(s+1)/2}\Gamma((s+1)/2)\\
L_v(s,\chi_v) &= \pi^{-(s+1)/2}\Gamma((s+1)/2)\\
\epsilon_v(s,\chi_v,\psi_v) &= \frac{\zeta_v(1-s,\chi_v,\hat{\Phi}_v)}{L_v(1-s,\chi_v^{-1})} \cdot \frac{L_v(s,\chi_v)}{\zeta_v(s,\chi_v,\Phi_v)} = i
\end{aligned}$$

When $F_v = \mathbb{C}$ and $\chi = |x|_v^v \mathrm{Arc}(x)^k$, we have $L_v(s,\chi_v) = 2\cdot(2\pi)^{s+v+\frac{|k|}{2}}\Gamma\left(s+v+\frac{|k|}{2}\right)$. Then we may take $\Phi_v(x) = \bar{x}^k e^{-2\pi|x|}$ when $k \geq 0$, and $\Phi_v(x) = x^{-k}e^{-2\pi|x|}$ when $k < 0$. When $k \geq 0$, we have

$$\begin{aligned}
\hat{\Phi}_v(x) &= i^k x^{-k} e^{-2\pi|x|}\\
\zeta_v(s,\chi_v,\Phi_v) &= \int_0^\infty r^{2s+2v-1}\cdot\frac{1}{2\pi}\int_0^{2\pi} \Phi_v(re^{i\theta})e^{ik\theta}\mathrm{d}\theta\,\mathrm{d}r = \int_0^\infty r^{2s+2v-1}\cdot\frac{1}{2\pi}\int_0^{2\pi} r^k e^{-ik\theta}e^{-2\pi r^2}e^{ik\theta}\mathrm{d}\theta\,\mathrm{d}r\\
&= \int_0^\infty e^{-2\pi r^2} r^k r^{2s+2v-1}\mathrm{d}r = \frac{1}{2}\cdot(2\pi)^{s+v+\frac{k}{2}}\Gamma\left(s+v+\frac{k}{2}\right)
\end{aligned}$$

And when $k < 0$, we similarly have

$$\begin{aligned}
\hat{\Phi}_v(x) &= i^{-k}\bar{x}^k e^{-2\pi|x|} \\
\zeta_v(s,\chi_v,\Phi_v) &= \int_0^\infty r^{2s+2v-1}\cdot\frac{1}{2\pi}\int_0^{2\pi}\Phi_v(re^{i\theta})e^{ik\theta}\mathrm{d}\theta\,\mathrm{d}r \;=\int_0^\infty r^{2s+2v-1}\cdot\frac{1}{2\pi}\int_0^{2\pi} r^{-k}e^{-ik\theta}e^{-2\pi r^2}e^{ik\theta}\mathrm{d}\theta\,\mathrm{d}r \\
&= \int_0^\infty e^{-2\pi r^2}r^{-k}r^{2s+2v-1}\mathrm{d}r \;=\frac{1}{2}\cdot(2\pi)^{s+v+\frac{|k|}{2}}\Gamma\left(s+v+\frac{|k|}{2}\right)
\end{aligned}$$

So we have

$$\begin{aligned}
\epsilon(s,\chi_v,\psi_v) &= \frac{\zeta_v(1-s,\chi_v,\hat{\Phi}_v)}{L_v(1-s,\chi_v^{-1})}\cdot\frac{L_v(s,\chi_v)}{\zeta_v(s,\chi_v,\Phi_v)} \\
&= \frac{i^{|k|}\frac{1}{2}(2\pi)^{1-s+v+\frac{|k|}{2}}\Gamma(1-s+v+\frac{|k|}{2})}{2(2\pi)^{1-s+v+\frac{|k|}{2}}\Gamma(1-s+v+\frac{|k|}{2})}\cdot\frac{2(2\pi)^{s+v+\frac{|k|}{2}}\Gamma(s+v+\frac{|k|}{2})}{\frac{1}{2}(2\pi)^{s+v+\frac{|k|}{2}}\Gamma(s+v+\frac{|k|}{2}} \\
&= i^{|k|}
\end{aligned}$$

Solution 3.1.9. Take $\Phi(x) = \mathbf{1}_{1+\mathfrak{p}^N}$ for sufficiently large N. Then we have $\chi_v(1+\mathfrak{p}^N) = 1$. So

$$\zeta_v(s,\chi_v,\Phi_v) = \int_{F_v^\times}\Phi_v(x)\chi_v(x)|x|_v^s\frac{\mathrm{d}x}{x} = \int_{1+\mathfrak{p}^N}|x|_v^s\frac{\mathrm{d}x}{x} = \mathrm{vol}(1+\mathfrak{p}^N)$$

For $\hat{\Phi}_v$, we have

$$\hat{\Phi}_v(x) = \int_F \Phi(y)\psi(xy)\mathrm{d}y \;= \int_{1+\mathfrak{p}^N}\psi(xy)\mathrm{d}y$$

So

$$\begin{aligned}
\zeta_v(s,\chi_v^{-1},\hat{\Phi}_v) &= \int_{F_v^\times}\int_{1+\mathfrak{p}^N}\psi(xy)\chi_v^{-1}(x)|x|_v^s\mathrm{d}y\;\frac{\mathrm{d}x}{x} = \int_{1+\mathfrak{p}^N}\int_{F_v^\times}\psi(xy)\chi_v^{-1}(x)|x|_v^s\frac{\mathrm{d}x}{x}\mathrm{d}y \\
&= \int_{1+\mathfrak{p}^N}\int_{F_v^\times}\psi(x)\chi_v^{-1}(xy^{-1})|xy^{-1}|_v^s\frac{\mathrm{d}x}{x}\mathrm{d}y \;=\mathrm{vol}(1+\mathfrak{p}^N)\int_{F_v^\times}\psi(x)\chi_v^{-1}(x)|x|_v^s\frac{\mathrm{d}x}{x}
\end{aligned}$$

So we have

$$\gamma_v(s,\chi_v,\psi_v) = \frac{\zeta_v(1-s,\chi_{nu}^{-1},\hat{\Phi}_v)}{\zeta_v(s,\chi_v,\Phi_v} = \int_{F_v^\times}|x|^{-s}\chi_v^{-1}(x)\psi_v(x)\mathrm{d}x$$

Since $\int_{F_v^\times} = \sum_{n\in\mathbb{Z}} \int_{\varpi_v^n \mathbb{O}^\times}$. For $n \ll 0$, we denote $m = n - C(\psi) + C(\chi)$, then

$$\begin{aligned}\int_{\varpi_v^n \mathbb{O}_v^\times} |x|^{-s}\chi_v^{-1}(x)\psi_v(x)\mathrm{d}x &= |\varpi_v|^{-ns} \int_{\mathbb{O}_v^\times} \chi_v^{-1}(\varpi_v^n x)\psi_v(\varpi_v^n x)\frac{\mathrm{d}x}{|\varpi_v^n|} \\ &= |\varpi_v^{-n(s+1)}\chi_v(\varpi_v)^{-n} \int_{\mathbb{O}_v^\times/(1+\mathfrak{p}^{C(\chi)})} \chi^{-1}(x) \int_{1+\mathfrak{p}^{C(\chi)}} \psi(xy\varpi_v^n)\mathrm{d}y\, \mathrm{d}x \\ &= |\varpi_v^{-n(s+1)}\chi_v(\varpi_v)^{-n} \int_{\mathbb{O}_v^\times/(1+\mathfrak{p}^{C(\chi)})} \chi^{-1}(x)\psi(x\varpi_v^n) \int_{\mathfrak{p}^{C(\chi)}} \psi(xz\varpi_v^n)\mathrm{d}z\, \mathrm{d}x\end{aligned}$$

Since $n \ll 0$, we know $m \ll 0$, so $\int_{\mathfrak{p}^{C(\chi)}} \psi(xz\varpi_v^n)\mathrm{d}z \ = 0$ since the conductor of ψ is bigger than m. So the whole integrand $\int_{\varpi_v^n \mathbb{O}_v^\times}$ is zero for $n << 0$. So if N is sufficiently large, we have

$$\gamma_v(s, \chi_v, \psi_v) = \int_{F_v^\times} |x|^{-s}\chi_v^{-1}(x)\psi_v(x)\mathrm{d}x \ = \int_{\mathfrak{p}^{-N}} |x|^{-s}\chi_v^{-1}(x)\psi_v(x)\mathrm{d}x$$

3.2 Classical Automorphic Forms and Representations

Solution 3.2.1. (a) Take $\mathcal{F}$ to be the open fundamental domain of Γ in G/Z. Then we may extend elements of $C_c(\mathcal{F})$ to element of $C(\Gamma\backslash G, \chi, \omega)$. And clearly we may identify $L^2(\Gamma\backslash G, \chi, \omega)$ with $L^2(\mathcal{F})$. Then since Γ has finite volume, $C_c(\mathcal{F})$ is dense in $L^2(\mathcal{F})$.

(b) By (a) and 2.1.9., we only need to prove that for any $f \in C_c(\Gamma\backslash G, \chi, \omega)$, the morphism $G \to L^2(\Gamma\backslash G, \chi, \omega)$, $g \mapsto \rho(g)f$ is continuous. Since f support compactly, it is uniformly continuous. So $g \mapsto \rho(g)f$ is continuous.

Solution 3.2.2. If f is of moderate growth at ∞, we know $f(x+iy) \ll y^N$ when $y \to \infty$ for some $N \in \mathbb{N}$. Then we may pick a $\mathcal{G}_{c,d}$ as in proposition 3.2.2. Then for any $g = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \in \mathcal{G}_{c,d}$, we know $gi = x + iy \in \mathcal{F}_{c,d}$. So $|F(g)| = |f(gi)| = |f(x+iy)| \ll y^N$. Since $x + yi = \frac{\alpha i+\beta}{\gamma i+\delta} = \frac{\alpha\gamma+\beta\delta}{\gamma^2+\delta^2} + \frac{\det g}{\gamma^2+\delta^2}i$, we know $y = \frac{\det g}{\gamma^2+\delta^2} \ll \|g\|^{-2}$. So $|F(g)| \ll \|g\|^{-2N}$. Conversely, if we have $|F(g)| \ll \|g\|^N$ for some N, we have $|f(x+yi)| = |F(\left[\begin{smallmatrix} y & x \\ & 1 \end{smallmatrix}\right])| \ll \|\left[\begin{smallmatrix} y & x \\ & 1 \end{smallmatrix}\right]\|^N = (x^2+y^2+y^{-2}+1)^{N/2} \ll y^N$ since $x + yi \in \mathcal{F}_{c,d}$.

Solution 3.2.3. Consider the open continuous morphism $\psi : K \times B \times K \to G$, $(k_1, b, k_2) \mapsto k_1bk_2$. Then take $\Omega' = p_2(\phi^{-1}(\Omega))$. For any open covering $\Omega' = \bigcup U_i$, we know KU_iK is open in G. Then $KU_iK \cap \Omega$ is open in Ω. So $\Omega = \bigcup(KU_iK \cap \Omega)$ is a open covering. Since Ω is compact, this covering is a finite covering, so the covering of Ω' is also finite, i.e. Ω' is compact.

Solution 3.2.4. Let Σ be the set of all sets S of irreducible invariant subspaces of $L_0^2(\Gamma\backslash G, \chi, \omega)$ such that the elements of S are mutually orthogonal. By Zorn's lemma, there exists a maximal element S in Σ. Denote $\mathfrak{h}$ as the orthogonal complement of the closure of the direct sum of the elements of S. If $\mathfrak{h} \neq 0$, we can take a nonzero element f. Then we may choose a ϕ such that $\rho(\phi)$ is self-adjoint, and $\rho(\phi)f \neq 0$ by proposition 3.2.3.(ii). So $\rho(\phi)$ has an eigenvalue λ, and L the corresponding finite-dimensional eigenspace. Denote $L_0 \subset L$ as a nonzero subspace which is minimal with respect to the property that L_0 may be expressed as the intersection of L with a nonzero invariant subspace. Since L is finite-dimensional, the essence of L_0 is clear. Take V as the intersection of all closed invariant subspaces W of $\mathfrak{h}$ such that $L_0 = L \cap W$. If V is not irreducible, we have $V = V_1 \oplus V_2$. If $0 \neq f_0 \in L_0$, we may write $f_0 = f_1 + f_2$ for some $f_i \in V_i$. Then clearly V_i's are invariant under $\rho(\phi)$. So $\rho(\phi)f_i - \lambda f_i \in V_i$ for $i = 1, 2$. Since $(\rho(\phi)f_1 - \lambda f_1) + (\rho(\phi)f_2 - \lambda f_2) = (\rho(\phi)f_0 - \lambda f_0) = 0$, we have $\rho(\phi)f_1 = \lambda f_1$ and $\rho(\phi)f_2 = \lambda f_2$. So $f_1 \neq 0$ or $f_2 \neq 0$ is a nonzreo eigenvector of $\rho(\phi)$, which makes a contradiction with the minimality of L_0. So $\mathfrak{h} = 0$, i.e. $L_0^2(\Gamma\backslash G, \chi, \omega)$ can decompose into a Hilbert space direct sum of subspaces that are invariant and irreducible under the right regular representation ρ.

Solution 3.2.5. (a) For any $X \in \mathfrak{g}$, we have $(X\rho(\phi)f)(g) = \rho(\phi_X)f(g)$, which is well-defined. So $\rho(\phi)f \in C^1(\Gamma\backslash G, \chi, \omega$. Then by induction, we have $\rho(\phi)f \in C^\infty(\Gamma\backslash G, \chi, \omega)$. Moreover, if $g \in \mathcal{G}_{c,d}$, we have

$$(\rho(\phi)f)(g) = \int_{\Gamma_\infty Z(\mathbb{R})\backslash G} K'(g,h)f(h)\mathrm{d}h \ll \int_{\Gamma_\infty Z(\mathbb{R})\backslash G} y^{-N} f(h)\mathrm{d}h \leq y^{-N} \int_{\bar{\mathcal{G}}_{B^{-1}c,d}} |f(h)|\mathrm{d}h$$

For any $N > 0$ by equation (2.27). Since $\mathcal{G}_{B^{-1}c,d}$ has finite volume and $f \in L_0^2(\Gamma\backslash G, \chi, \omega)$, the integrand above is convergent. So $(\rho(\phi)f)(g) \ll y^{-N}$ for every $N > 0$, i.e. is rapidly decreasing.

(b) Clearly, if $f_i \in C_c^\infty(\Gamma\backslash G, \chi, \omega, k_i)$ and $k_i \neq k_j$, we clearly have $\langle f_1, f_2 \rangle = 0$. And clearly Δ preserve $C_c^\infty(\Gamma\backslash G, \chi, \omega, k)$. So we may assume $f_1, f_2 \in C_c^\infty(\Gamma\backslash G, \chi, \omega, k)$ for a same k. In $C_c^\infty(\Gamma\backslash G, \chi, \omega, k)$, for any f, f' we have $f(g)\overline{f'(g)}$ is invariant under $Z(\mathbb{R})$-action, we may assume ω is trivial. Similarly with proposition 2.1.8., we know $L_0^2(\Gamma\backslash G, \chi, k)$ is isomorphic to $L_0^2(\Gamma\backslash\mathcal{H}, \chi, k)$, i.e. $\langle f, f \rangle = \langle F, F' \rangle$ for $f(g) = F(gi)$. And by equation (1.30) in chapter II, we have $\langle \Delta f_1, f_2 \rangle = \langle \Delta_k F_1, F_2 \rangle = \langle F_1, \Delta_k F_2 \rangle = \langle f_1, \Delta f_2 \rangle$.

(c) Statement: The space $L_0^2(\Gamma\backslash G, \chi, k)$ decomposes into a Hilbert space direct sum of eigenspaces for Δ.

For every character ξ is $C_c^\infty(K\backslash G/K, \sigma)$, we denote $H(\xi) = \{f \in L_0^2(\Gamma\backslash G, \chi, k) \,|\, \rho(\phi)f = \xi(\phi)f, \ \forall\phi \in C_c^\infty(K\backslash G/K, \sigma)\}$. Then similarly with theorem 2.3.4., $H(\xi)$ has finite dimension, and for any other character η, $H(\xi)$ is orthogonal to $H(\eta)$, and $L_0^2(\Gamma\backslash G, \chi, k)$ is the Hilbert space direct sum of all $H(\xi)$. Since $\rho(\phi)$ commutes with Δ, clearly we know Δ is a self-adjoint operator on $H(\xi)$ for all ξ. Since $H(\xi)$ is finite-dimensional, it must be the direct sum of Δ-eigenspaces. So $L_0^2(\Gamma\backslash G, \chi, k)$ is the Hilbert space direct sum of all eigenspaces for Δ.

Solution 3.2.6. Clearly $\mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$ is stable under $U(\mathfrak{g}_\mathbb{C})$. Suppose now $f \in \mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$ satisfies $|f(g)| \ll \|g\|^N$ for some N. Then $\rho(G)f$ is a subspaces in $L_0^2(\Gamma\backslash G, \chi, \omega)$ and invariant under the ρ. So it is an admissible $(\mathfrak{g}, K)$-module contained in $\mathcal{A}_0(\Gamma\backslash G, \chi, omega)$ by theorem 3.2.2., hence $U(\mathfrak{g}_\mathbb{C})f$ is an admissible $(\mathfrak{g}, K)$-module. Moreover, we may pick an $\alpha \in C^\infty(G)$ such that $\alpha(kgk^{-1}) = \alpha(k)$ for all $k \in K$ such that $f = f * \alpha$. Then for any $X \in \mathfrak{g}_\mathbb{C}$, we have $\mathrm{d}X\ (f * \alpha) = f * (\mathrm{d}X\ \alpha)$. So $\mathrm{d}X\ f$ satisfies the same estimation with f.

3.3 Automorphic Representations of $\mathbf{GL}(n)$

Solution 3.3.1. Firstly, since $\mathrm{GL}(n,F)\backslash\mathrm{GL}(n,\mathbb{A})$ is locally compact, we know $C_c(\mathrm{GL}(n,F)\backslash\mathrm{GL}(n,\mathbb{A}),\omega)$ is dense in $L^2(\mathrm{GL}(n,F)\backslash\mathrm{GL}(n,\mathbb{A}),\omega)$. So by 2.1.9., we only need to prove that for any $f\in C_c(\mathrm{GL}(n,F)\backslash\mathrm{GL}(n,\mathbb{A}),\omega)$, the morphism $\mathrm{GL}(n,\mathbb{A})\to L^2(\mathrm{GL}(n,F)\backslash\mathrm{GL}(n,\mathbb{A}),\omega)$, $g\mapsto\rho(g)f$ is continuous. Since f has compact support, we know f is uniformly continuous on $\mathbb{A}_\infty$, hence for any $\varepsilon>0$, there exists a small neighbourhood $U_\infty\in\mathrm{GL}(n,\mathbb{A}_\infty)$ of 1 such that $|\rho(g)f-f|<\varepsilon$. So we may take a $U_{\text{fin}}\subset\mathrm{GL}(n,\mathbb{A}_{\text{fin}}$ such that $f(x+t)=f(x)$ for any $x\in\mathrm{GL}(n,\mathbb{A}$ and $t\in U_{\text{fin}}$. Then we define $U=U_{\text{fin}}\times U_\infty$ then for any $g=g_{\text{fin}}g_\infty\in U$ we have $|\rho(g)f-f|=|\rho(g_\infty)f-f|<\varepsilon$, i.e. $\mathrm{GL}(n,\mathbb{A})\to L^2(\mathrm{GL}(n,F)\backslash\mathrm{GL}(n,\mathbb{A}),\omega)$ is continuous.

Solution 3.3.2. For any K-finite function f, we denote the vector space spanned by $\rho(k)f$ for all $k\in K$ by W, which is finite dimensional. Denote $W'=\{Xv\mid X\in\mathfrak{g},v\in W\}$, which is clearly finite dimensional. Then for any $Xv\in W'$ and $k\in K$, we know $\rho(k)(Xv)=(\mathrm{Ad}(k)X)(\rho(k)v)\in W'$. So W' is K-invariant. Obviously, $\rho(K)(Xf)\subset W'$, which means Xf is K-finite.

Solution 3.3.3. (a) Clearly for any subset $X\subset\mathrm{Mat}_n(F)$ and $A\in\mathrm{GL}(n,\mathbb{F})$, we have $\mathrm{vol}(AX)=\det(A)^n\mathrm{vol}(X)$ by simply considering the Borel set. So clearly the measure $|\det(x)|^{-n}\mathrm{d}x$ is a left and right Haar measure on $\mathrm{GL}(n,F)$, and $\mathrm{GL}(n,F)$ is unimodular.

(b) For every local piece F_v, there exists a Haar measure $\mu_v(x_v)=|\det(x_v)|^{-n}\mathrm{d}x_v$ by (a). Then $\mu\left(\prod x_v\right)=\bigotimes\mu_v(x_v)$ is a left and right Haar measure of $\mathrm{GL}(n,\mathbb{A})$. For any $x=\prod x_v\in\mathbb{A}$, we have $\det(x_v)=1$ for almost all v, hence this global measure is well-defined. So $\mathrm{GL}(n,\mathbb{A})$ is unimodular.

Solution 3.3.4. If F is an arbitrary global field, we need to consider an $N\in O$ instead of an integer. Then by proposition 3.3.1. and theorem 3.3.1., we know $\mathrm{vol}(Z(\mathbb{A})\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A})/K_0(N))\le\mathrm{vol}(\Gamma_0(N)\backslash\mathrm{SL}(2,F))\cdot\mathrm{Cl}(F)$, which is finite. Here, the most important modification is that we need to consider the class number of F.

Solution 3.3.5. We denote two kinds of restriced tensor product as V_x and V_y. Then we may set an isomorphism $V_x\to V_y$ as $\prod z_v\mapsto\prod z_v$. We only need to clarify the well-definedness of this morphism. For any $\prod z_v$, for almost all v we have $z_v=x_v^\circ$. So for almost all v we have $z_v=x_v^\circ=y_v^\circ$, which means $\prod z_v\in V_y$. Hence this isomorphism is obvious.

Solution 3.3.6. We may suppose π is irreducible or we just need to decompose it as a product of irreducible representations. By theorem 3.3.4., $(\pi^{\text{fin}},V^{\text{fin}})$ is admissible, hence $\pi^{\text{fin}}=\bigotimes\pi_v^{\text{fin}}$ in V^{fin} by theorem 3.3.3. So clearly we know the v-component of V_χ^{fin} is $\chi_v\otimes\pi_v^{\text{fin}}$. Since V^{fin} is dense in V, so the v-component of V_χ is $\chi_v\otimes\pi_v$.

Solution 3.3.7. For any non-archimedean place v, the corresponding $\hat\pi_v\cong\omega_v^{-1}\otimes\pi_v$ is in the theorem 4.2.3. For archimedean place, the corresponding has two part: $\hat\pi_v(k)=\omega_v^{-1}(\det(k))\cdot\pi_v(k)$ for any $k\in K$, and $\hat\pi_v(X)=\omega_v^{-1}(e^{\mathrm{tr}(X)})\cdot\pi_v(X)$ for any $X\in\mathfrak{g}$. The prove is similar with the theorem 4.2.3. if we admit the exercise 2.5.8. For the adele, clearly $\hat\pi=\bigotimes\hat\pi_v=\bigotimes\left(\omega_v^{-1}\otimes\pi_v\right)=\omega^{-1}\otimes\pi$.

3.4 The Tensor Product Theorem

Solution 3.4.1. (a) Clearly we have two isomorphisms: 1. $V \cong \hat{V}$, $y \mapsto \langle -, y \rangle$; 2. $V \otimes \hat{V} \to \mathrm{End}(V)$, $(x \otimes f) \mapsto f(\cdot)x$. Combining these two morphisms we get the ϕ, which is an isomorphism. Moreover,

$$\phi(x \otimes y) \circ \phi(z \otimes w)(v) = \phi(x \otimes y)(\langle v, w \rangle z) = \langle v, w \rangle \phi(x \otimes y) z = \langle v, w \rangle \langle z, y \rangle x = \langle z, y \rangle \phi(x \otimes w)(v)$$

(b) Just calculate

$$\begin{aligned}
(\Phi(x \otimes y) * \Phi(z \otimes w))(g) &= \int_K \Phi(x \otimes y)(k) \Phi(z \otimes w)(gk^{-1}) \mathrm{d}k \\
&= \int_K \dim(V)^2 \langle x, \pi(k) y \rangle \langle z, \pi(gk^{-1}) w \rangle \mathrm{d}k \\
&= \int_K \dim(V)^2 \langle x, \pi(k) y \rangle \langle \pi(k) z, \pi(g) w \rangle \mathrm{d}k \\
\text{(Schur ortho. relation)} &= \dim(V) \langle z, y \rangle \langle x, \pi(g) w \rangle \\
&= \langle z, y \rangle \Phi(x \otimes w)(g)
\end{aligned}$$

So we have a morphism $\theta : \mathrm{End}(V) \to V \otimes V \to C(K)$.

(c) For any image $\Phi(x \otimes y)$ in $\theta : \mathrm{End}(V) \to C(K)$, clearly we know it is smooth by the construction. And moreover, we have $\Phi(x \otimes y)(kg) = \dim(V) \langle x, \pi(kg) y \rangle = \dim(V) \langle \pi(k^{-1}) x, \pi(g) y \rangle = \theta(\phi(\pi(k^{-1}) x \otimes y))(g)$ and same for the right action. Then since V has finite dimension, $\Phi(x \otimes y)$ is K-finite. So the image of θ is actually in the Hecke algebra $\mathcal{H}_K$. Hence we can construct a morphism $\Theta : \bigoplus \mathrm{End}_{\mathbb{C}}(V_i) \to \mathcal{H}_K$, where the direct sum is taken over all irreducible representation classes of K. Then we need to prove this is isomorphic.

Obviously the morphism Θ is injective. Then we need to prove for any $f \in \mathcal{H}_K$, the finitely dimensional linear space W_f spanned by $f(\cdot k)$ for all $k \in K$ is contained in $\mathrm{Im}\Theta$. Since W_f is a representation of K, it can be decomposed into a series of irreducible representations. Then it suffices to prove that for any irreducible representation (π, V) and morphism $u : V \to \mathcal{H}_K$, we have $\mathrm{Im} u \subset \mathrm{Im}\Theta$. For any $f \in L^2(K)$, $v \in V$, we denote $\tilde{f}(k) = \overline{f(g^{-1})}$. Then

$$\begin{aligned}
(f * u(v))(k) = \int_K f(g) u(v)(kg^{-1}) \mathrm{d}g &= \int_K f(g^{-1}) L_k(u(v))(g) \mathrm{d}g = \int_K \overline{\tilde{f}(g)} L_k(u(v))(g) \mathrm{d}g \\
&= \langle L_k(u(v)), \tilde{f} \rangle = \langle u(\pi(k^{-1}) v), \tilde{f} \rangle = \langle \pi(k)^{-1} v, u^* \tilde{f} \rangle = \pi(\langle \cdot, u^* \tilde{f} \rangle)(k)
\end{aligned}$$

which means $f * u(v) \in \mathrm{Im}\theta_V$. Since we can pick a series $\{f_n\}$ of functions in $L^2(K)$ such that $f_n * u(v) \to u(v)$, we have $u(v) \in \overline{\mathrm{Im}\theta_V}$. Since $\mathrm{Im}\theta_V$ has finite dimension, it must be closed in $\mathcal{H}_K$, so $u(v) \in \mathcal{H}_K$. So we've done.

Solution 3.4.2. ($\Rightarrow$) Trivial. ($\Leftarrow$) If we've had a representation (π, V) of $\mathcal{H}_K$. By Peter-Weyl theorem, we have $\widehat{\mathcal{H}_K} = L^2(K)$. So π induces a morphism $\pi : L^2(K) \to \mathrm{End}(V)$. Since π is smooth, we can pick a series $\{f_n\}$ such that $f_n \to \delta_1$ and we define $\pi'(k) = \lim \pi(L_k f_n)$, which is a representation (π', V) of K.

Solution 3.4.3. Here we may need that M is smooth.

($\Leftarrow$) Suppose $N \subset M$ is a nontrivial submodule. Since $M = \bigcup_{e\in\mathcal{E}} M[e]$, we may find a sufficiently large $e \in \mathcal{E}^\circ$ such that $N[e] \neq 0$. So $M[e]$ is not simple, which contradicts with the assumption.

($\Rightarrow$) If $M[e]$ is not simple for some $e \in \mathcal{E}^\circ$, we may assume $N \subset M[e]$ is a nontrivial submodule. Then we need to prove $HN \cap M[e] = N$. For any $\sum r_i n_i \in HN \cap M[e]$, we know $\sum r_i n_i = e(\sum r_i n_i) = \sum e \sum r_i e^{-1} \cdot e n_i = \sum e \sum r_i e^{-1} \cdot n_i \in H[e]N = N$. And the converse one is trivial. So HN is a nontrivial submodule of M, which contradicts to the assumption that M is simple.

Solution 3.4.4. ($\Leftarrow$) Trivial. ($\Rightarrow$) We denote $\phi_e : M[e] \to N[e]$. For any $m \in M$, since M is admissible, there exists some $e \in \mathcal{E}^\circ$ such that $m \in M[e]$. So we can define $\phi(m) = \phi_e(m) \in N[e] \subset N$. Suppose there exists two e, e' such that $m \in M[e]$ and $M[e']$. By definition, there exists e'' such that $e'' > e$ and e', so $\phi(m) = \phi_{e''}(m) = \phi_e(m) = \phi_{e'}(m)$. So the map ϕ is well-defined. For any $m, m' \in M$, there exists $e, e' \in \mathcal{E}^\circ$ such that $m \in M[e]$ and $m' \in M[e']$. Then similarly we can find a $e'' > e$ and e' such that $e, e' \in M[e'']$. So $\phi(m + m') = \phi_{e''}(m + m') = \phi_{e''}(m) + \phi_{e''}(m') = \phi(m) + \phi(m')$. And obviously the action of R is clear. So $\phi : M \to N$ is a morphism. And trivially, it is an isomorphism.

Solution 3.4.5. Suppose $\dim_\Omega M > 1$. Then we may find a nontrivial Ω-subspace N of M. For any $r \in R$, we define $f : M \to M$ as $f(m) = rm$. Then for any $s \in R$ we have $f(sm) = rsm = srm = s \cdot f(m)$, hence $f \in \mathrm{End}_R(M)$. By Schur's lemma, we have $\mathrm{End}_R(M) \cong \Omega$, so $f(m) = km$ for some $k \in \Omega$. Then for any $n \in N$, we have $an = f(n) = kn \in N$. So the action of R on N is into itself, which means N is a R-submodule of M, which contradicts with the assumption that M is simple.

Solution 3.4.6. 1. No! The most important part of the proof is the special case that all e_v° are spherical idempotent, in this case, we strongly depend on the theorem 3.4.3. to make the isomorphism between the original M and the tensor product we constructed. 2. If we assume instead that $\mathcal{H}_v[e_v^\circ]$ is commutative for almost all v, the theorem is still true since by definition commutativity implies a trivial antiinvolution $\iota = \mathrm{id}$ which makes those e° be spherical.

3.5 Whittaker Models and Automorphic Forms

No Exercise Here.

3.6 Adelization of Classical Automorphic Forms

No Exercise Here.

3.7 Eisenstein Series and Intertwining Integrals

Solution 3.7.1. Since $E^\sharp(g,f) = L(2s,\xi_1\xi_2^{-1})E(g,f) = E^*(g,f)\cdot\prod_{v\in S}L_v(2s,\xi_{1,v},\xi_{2,v}^{-1})$, and the S is finite, so the poles of $E^\sharp(g,f)$ is the sum of the poles of $E^*(g,f)$ and $L_v(2s,\xi_{1,v}\xi_{2,v}^{-1})$ for all $v\in S$.

Solution 3.7.2. In the complex case, the Iwasawa decomposition makes us decompose any $g\in \mathrm{GL}(2,\mathbb{C})$ as

$$g=\begin{bmatrix}\zeta & \\ & \zeta\end{bmatrix}\begin{bmatrix}\eta^{1/2} & \xi\eta^{-1/2}\\ & \eta^{-1/2}\end{bmatrix}\begin{bmatrix}s & t\\ -\bar{t} & \bar{s}\end{bmatrix}$$

where $|s|^2+|t|^2=1$, $\zeta,\eta,\xi\in\mathbb{C}$. This decomposition may be not unique. But consider the $K=SU(2)$, since any one-dimensional representation of K is the trivial representation, we must have $f|_K\equiv 1$. So

$$f(g)=(\chi_1^{-1}\chi_2)(\eta^{1/2})|\eta|^{-1}(\chi_1\chi_2)(\zeta)f\left(w_0\begin{bmatrix}1 & \eta^{-1}(x+\xi)\\ & 1\end{bmatrix}\right)$$

So we are reduced to prove the analytic continuation of $\int_{\mathbb{C}} f(w_0\left[\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right])\psi(\eta x)\mathrm{d}x$. Since

$$w_0\begin{bmatrix}1 & x\\ & 1\end{bmatrix}=\begin{bmatrix}\Delta^{-1} & -x\Delta^{-1}\\ & \Delta\end{bmatrix}\begin{bmatrix}\overline{-x\Delta^{-1}} & (x/|x|)^2\Delta^{-1}\\ -\overline{(x/|x|)^2\Delta^{-1}} & -x\Delta^{-1}\end{bmatrix}$$

where $\Delta=-\sqrt{1+|x|^2}\cdot\frac{x}{|x|}$. So the integral becomes $\int_{\mathbb{C}}(1+|x|^2)^{-s}e^{i\eta(\lambda x+\mu\bar{x})}\mathrm{d}x$ and the rest is the same as we've done in the real case.

3.8 The Rankin-Selberg Method

No Exercise Here.

3.9 The Global Langlands Conjectures

Solution 3.9.1. If $G=\mathrm{SO}(n)$ or $\mathrm{Sp}(2n)$, $\phi: {}^LG\to {}^L\mathrm{GL}(m,\mathbb{C})$ is the L-groups inclusion and m is $2n$ or $2n+1$ as we need, any π is a lift from the representation π' on G, by Langlands we have $L_S(s,\pi,r)=L_S(s,\pi',r\circ\phi)$ for any $r: {}^L\mathrm{GL}(m,\mathbb{C})$. Since for any kind of G, we have an automorphism $g\to g^{-T}$ in G, so $\pi'\cong\hat{\pi}'$, which means $L_S(s,\pi',r\circ\phi)=L_S(s,\hat{\pi}',r\circ\phi)$. And by comparing the Satake parameters, we know $\hat{\pi}$ is a lift from $\hat{\pi}'$, so $L_S(s,\hat{\pi},r)=L_S(s,\hat{\pi}',r\circ\phi)$. Then we have $L_S(s,\pi,r)=L_S(s,\hat{\pi},r)$ for any r. So by "strong" multiplicity theorem on $\mathrm{GL}(m)$, we have $\pi\cong\hat{\pi}$, i.e. π is self-contragredient.

Suppose $G=\mathrm{SO}(2n)$ or $\mathrm{Sp}(2n)$, $\phi: {}^LG\to {}^L\mathrm{GL}(m,\mathbb{C})$, and the rest notations are the same. Then $L_S(s,\pi\times\hat{\pi})$ is the L-function of the representation $\pi\times\hat{\pi}$, which is lifted from $\pi'\times\hat{\pi}'$. So we only need to consider the representation $\pi'\times\hat{\pi}'$ on ${}^L(G)$. Since $-I\in G$, clearly $L_S(s,\pi',\vee^2)$ has a poly at $s=1$, which induces the pole of $L_S(s,\pi,\wedge^2)$ at $s=1$. In the case of $G=\mathrm{SO}(2n+1)$, clearly $L_S(s,\pi',\vee^2)$ has no pole since $-I\notin G$. So $L_S(s,\pi',\wedge^2)$ has a pole at $s=1$, which induces the pole of $L_S(s,\pi,\wedge^2)$.

3.10 The Triple Convolution

No Exercise Here.

4 Representations of $\mathrm{GL}(2)$ Over a p-adic Field

4.1 $\mathrm{GL}(2)$ Over a Finite Field

Solution 4.1.1 (Frobenius reciprocity)**.** (a) We denote the (1.3) as $\Phi : \mathrm{Hom}_G(U, V^G) \to \mathrm{Hom}_H(U_H, V)$ and $\Psi : \mathrm{Hom}_H(U_H, V) \to \mathrm{Hom}_G(U, V^G)$. Then for any $\phi \in \mathrm{Hom}_G(U, V^G)$, we have $\Psi \circ \Phi(\phi)(u) = \Psi(\phi(u)(1) = (g \mapsto \phi(gu)(1)) = (g \mapsto \phi(u)(g)) = \phi(u)$. And conversely, for any $\phi' \in \mathrm{Hom}_H(U_H, V)$, we have $\Phi \circ \Psi(\phi')(u) = \Phi(g \mapsto \phi'(gu)) = \phi'(u)$. So Φ is an isomorphism.

(b) We first consider the case that π_1 and π_2 are both irreducible. If $\pi_1 = \pi_2$, then for any x, we have $\chi_1(x)\overline{\chi_2(x)} = 1$. So $\langle \chi_1, \chi_2 \rangle_G = \frac{1}{|G|} \sum_{x\in G} 1 = 1$. If $\pi_1 \neq \pi_2$, we know $\chi_1\chi_2^{-1} \neq 1$. So $\sum_{x\in G} \chi_1(x)\overline{\chi_2(x)} = 0$, i.e. $\langle \chi_1, \chi_2 \rangle_G = 0$.

So if $\pi_1 = \bigoplus \rho_i^{k_{i,1}}$ and $\pi_2 = \bigoplus \rho_i^{k_{i,2}}$ for a series of irreducible representations ρ_i, then

$$\langle \chi_1, \chi_2 \rangle_G = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{a=1}^{k_{n,1}} \sum_{b=1}^{k_{n,2}} \langle \chi_{\rho_i}, \chi_{\rho_j} \rangle_G = \sum_{i=1}^{n} k_{i,1} k_{i,2} = \dim \mathrm{Hom}_G(\pi_1, \pi_2)$$

(c) Clearly $\langle \chi^G, \sigma \rangle_G = \dim \mathrm{Hom}_G(V^G, U) = \dim \mathrm{Hom}_H(V, U_H) = \langle \chi, \sigma_H \rangle_H$.

Solution 4.1.2 (Mackey's theorem)**.** (a) Clearly $\Delta(x_i) \circ \pi_{1,i}(s) = \Delta(x_i) \circ \pi_1(x_i^{-1} s x_i) = \Delta(s x_i) = \pi_2(s) \circ \Delta(x_i) = \pi_{2,i}(s)\Delta(x_i)$.

(b) ($\subset$) By (a) and proposition 4.1.2., trivial. ($\supset$) For any $(\Delta_i)_{i=1}^r \in \bigoplus_{i=1}^n \mathrm{Hom}_{S_i}(\pi_{1,i}, \pi_{2,i})$, we can define $\Delta(g)(v_1) = \pi_{i,2}(h_2) \circ \Delta_i(s_i) \circ \pi_{i,1}(h_1)(v_i)$, where $g = h_2 s_i h_1$. Then by proposition 4.1.2., we've done.

Solution 4.1.3. As the hint, we only need to consider the case that F is algebraic closed. In this case, for any $A \in \mathrm{GL}(n, F)$, there exists $P \in O(n, F)$ such that $PAP^{-1} = \mathrm{diag}\{J_1, \ldots, J_k\}$, where J_i has the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{bmatrix}$$

Take

$$B_i = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$$

and $B = \mathrm{diag}\{B_1, \ldots, B_k\}$, clearly we have $B^2 = 1$, and $BJB^{-1} = J^T$. SO $P^{-1}BPAP^{-1}B^{-1}P = A^T$.

Solution 4.1.4. (1.20) $\langle \bar{x_1}\bar{x_2}, \bar{y} \rangle = \chi_0(x_1x_2yx_2^{-1}x_1^{-1}y^{-1}) = \chi_0(x_1yx_1^{-1}y^{-1}yx_1y^{-1}x_2yx_2^{-1}y^{-1}yx_1^{-1}y^{-1}) = \chi_0(x_1yx_1^{-1}y^{-1})\chi_0(yx_1y^{-1})\chi_0(x_2yx_2^{-1}y^{-1})\chi_0(yx_1^{-1}y^{-1}) = \chi_0(x_1yx_1^{-1}y^{-1})\chi_0(x_2yx_2^{-1}y^{-1}) = \langle \bar{x_1}, \bar{y} \rangle \langle \bar{x_2}, \bar{y} \rangle$.

(1.21) Similar with above.

(1.22) Trivial.

(1.23) $\langle \bar{x}, \bar{y}\rangle\langle \bar{y}, \bar{x}\rangle = \chi_0(xyx^{-1}y^{-1}yxy^{-1}x^{-1}) = \chi_0(1) = 1.$

Solution 4.1.5. By the hint, we only need to show $\bar{A}\bar{B}$ is isotropic. For any $\bar{y}, \bar{y}' \in \bar{A}$ and $\bar{x}^n, \bar{x}^m \in \bar{B}$, we have $\langle \bar{y}\bar{x}^n, \bar{y}'\bar{x}^m\rangle = \langle \bar{y}, \bar{y}'\rangle \cdot \langle \bar{y}, \bar{x}\rangle^m \cdot \langle \bar{x}, \bar{y}'\rangle^n \cdot \langle \bar{x}, \bar{x}\rangle^{nm} = 1$. So $\bar{A}\bar{B}$ is isotropic, which contradicts with the maximality of A.

Solution 4.1.6. For any $a, b \in A$, we have $1 = \langle \bar{a}, \bar{b}\rangle = \chi_0(aba^{-1}b^{-1})$. So $aba^{-1}b^{-1} \in Z_0$, i.e. $(aZ_0)(bZ_0) = (bZ_0)(aZ_0)$, which means A/Z_0 is abelian.

Solution 4.1.7. (a) For any $A \supset Z$, we know for any $x \in H$ and $a \in A$, we have $\bar{x}\bar{a} = \bar{a}\bar{x}$. So $xa = az \cdot xz'$ for some $z, z' \in Z$. Since Z is the center, we know $xa = (azz')x$, i.e. $xA = Ax$ which means A is normal.

Since $(\chi_B/\chi_A)(s)$ is a character of $s \in A \cap B$, there exists a unique x modulo $(A\cap B)^\perp = A^\perp B^\perp = AB$ such that $(\chi_B/\chi_A)(s) = \langle s, x\rangle = \langle s^{-1}, x^{-1}\rangle$. So $\chi_B(s) = \langle s^{-1}, x^{-1}\rangle \cdot \chi_A(s) = \chi_0(s^{-1}x^{-1}sx) \cdot \chi_A(s) = \chi_A(s^{-1}x^{-1}sx) \cdot \chi_A(s) = \chi_A(x^{-1}sx)$. So there exists only one double coset BxA such that $\chi_B(s) = \chi_A(x^{-1}sx)$ for any $s \in A \cap B$.

(b) By (a) and 4.1.2.(b), $\dim \mathrm{Hom}_H(\pi_A, \pi_B) = \sum \dim \mathrm{Hom}_{S_i}(\pi_{1,i}, \pi_{2,i}) = 1$.

Solution 4.1.8 (The Stone-Von Neumann Theorem)**.** By 4.1.7.(b) we clearly know that the representation π is independent on the choice of A, so there exists a unique isomorphism class of irreducible representations of H with central character χ_0.

Solution 4.1.9. Suppose σ is a coboundary, and $\tilde{G}$ is the corresponding extension. Then we can define a morphism $\theta : G \times A \to \tilde{G}$ as $\theta(g, a) = (g, a\phi(g)^{-1})$. Since $(g, a\phi(g)^{-1})(g', a'\phi(g')^{-1}) = (gg', a\phi(g)^{-1})a'\phi(g')^{-1}\phi(g)\phi(g')\phi(gg')^{-1}) = (gg', aa'\phi(gg')^{-1}$, this morphism is well-defined. And θ clearly has an inversion $\theta^{-1} : (g, a) \mapsto (g, a\phi(g))$. So θ is an isomorphism, i.e. $G \times A$ is equivalent to $\tilde{G}$. So the extension is independent of coboundaries. Conversely, if a cocycle σ defines a trivial extension $\tilde{G}$, which means $\theta : G\times A \to \tilde{G}$, $(g, a) \to (g, a\phi(g)^{-1})$ is an isomorphism, we have $\sigma(g, g') = \phi(g)\phi(g')\phi(gg')^{-1}$, hence it is a coboundary. So we have a bijection between the classes of central extensions of G by A and the elements of $H^2(G, A)$.

Solution 4.1.10. Since $\omega(g)$ is an intertwining map and unique up to constant, we have $\pi({}^g h)\omega(g) = \omega(g)\pi(h)$, and $\omega(g) \in \mathrm{PGL}(n, \mathbb{C})$. So we got a map $g \mapsto \omega(g)$, we need to prove this is a homomorphism. Since $\pi({}^{gg'}h) = \omega(gg')\pi(h)\omega(gg')^{-1}$, and in the other side $\pi(gg'h) = \omega(g)\pi({}^{g'}h)\omega(g)^{-1} = \omega(g)\omega(g')\pi(h)\omega(g')^{-1}\omega(g)^{-1}$, by Schur's lemma, we know $\omega(g)\omega(g') = \omega(gg')$. So ω is a projective representation.

Solution 4.1.11. First show that χ_0 is generic. For any $(v, v', 0), (w, w', 0) \in H$, we have $\langle (v, v', 0), (w, w', 0)\rangle = \chi_0\left((v, v', 0)(w, w', 0)(v, v', 0)^{-1}(w, w', 0)^{-1}\right) = \chi_0(0, 0, 2B(v, w')-2B(v', w)) = \psi(x)(2(B(v, w')-B(v', w)))$. Since B is non-degenerated, so χ_0 is generic. Second we prove that A is polarizing. This is trivial, since A is clearly maximal isotropic, by 4.1.5., it is polarizing.

Solution 4.1.12. For any $\Phi(v)$, we can define a $\phi(v', v, x) = \phi((v', 0, x - B(v', v)) \cdot (0, v, 0)) = \chi(v', 0, x - B(v', v))\Phi(v) = \psi(x - B(v', v))\Phi(v)$. Clearly it is unique. Moreover,

$$\begin{aligned}\pi(u,0,0)\Phi(v) &= \phi((0,v,0)(u,0,0)) = \phi(u,v,-B(u,v))\\ &= \phi((u,0,-2B(u,v)(0,v,0)) = \psi(-2B(u,v))\Phi(v)\\ \pi(0,u,0)\Phi(v) &= \phi((0,v,0)(0,u,0)) = \phi(0,u+v,0) = \Phi(u+v)\end{aligned}$$

Solution 4.1.13. We only need to verify the action of ω_1 on the generators $(w,0,0)$, $(0,w,0)$ and $(0,0,x)$. For the first equation, $\left[\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right](w,0,0) = (w,0,0)$ and $\left[\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right](0,0,x) = (0,0,x)$, so these two is trivial. For the third one,

$$\begin{aligned}(\omega_1(\left[\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right])\pi(0,w,0)\omega_1(\left[\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right]^{-1})\Phi)(v) &= \psi(-xB(v,v))(\omega_1(\left[\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right])\pi(0,w,0)\Phi)(v)\\ &= \psi(-xB(v,v))(\omega_1(\left[\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right])\Phi)(v+w)\\ &= \psi(xB(w,w) + 2xB(v,w))\Phi(v+w) = \pi(xw,w,0)\Phi(v)\end{aligned}$$

So (1.31) is verified. For (1.32), since $\left[\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\right](0,0,x) = (0,0,x)$, we only need to do the other two.

$$\begin{aligned}(\omega_1(\left[\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\right])\pi(w,0,0)\omega_1(\left[\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\right])^{-1}\Phi)(v) &= \chi(a^{-1})^{\dim V}(\omega_1(\left[\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\right])\pi(w,0,0)\Phi)(av)\\ &= \psi(-2aB(v,w))\chi(a^{-1})^{\dim V}(\omega_1(\left[\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\right])\Phi)(av)\\ &= \psi(-2aB(v,w))\Phi(v) = \pi(aw,0,0)\Phi(v)\\ (\omega_1(\left[\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\right])\pi(0,w,0)\omega_1(\left[\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\right])^{-1}\Phi)(v) &= \chi(a^{-1})^{\dim V}(\omega_1(\left[\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\right])\pi(0,w,0)\Phi)(av)\\ &= \chi(a^{-1})^{\dim V}(\omega_1(\left[\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\right])\Phi)(av+w)\\ &= \Phi(v + a^{-1}w) = \pi(0,a^{-1}w,0)\Phi(v)\end{aligned}$$

So (1.32) is verified. For the last one, still,

$$\begin{aligned}(\omega_1(\left[\begin{smallmatrix} & 1\\ -1 & \end{smallmatrix}\right])\pi(w,0,0)\Phi)(v) &= \omega(\left[\begin{smallmatrix} & 1\\ -1 & \end{smallmatrix}\right])\Phi(v)\psi(-2B(v,w))\\ &= \epsilon q^{-\dim V/2}\sum_{u\in V}\Phi(u)\psi(2B(u,v))\psi(-2B(u,w)) = \hat{\Phi}(v-w)\\ &= (\pi(0,-w,0)\hat{\Phi})(v) = (\pi(0,-w,0)\omega_1(\left[\begin{smallmatrix} & 1\\ -1 & \end{smallmatrix}\right])\Phi)(v)\\ (\omega_1(\left[\begin{smallmatrix} & 1\\ -1 & \end{smallmatrix}\right])\pi(0,w,0)\Phi)(v) &= \omega(\left[\begin{smallmatrix} & 1\\ -1 & \end{smallmatrix}\right])\Phi(v+w)\\ &= \epsilon q^{-\dim V/2}\sum_{u\in V}\Phi(u+w)\psi(2B(u,v))\\ &= \epsilon q^{-\dim V/2}\sum_{u\in V}\Phi(u+w)\psi(2B(u+w,v))\psi(-2B(w,v))\\ &= \psi(2B(-w,v))\hat{\Phi}(v) = (\pi(-w,0,0)\hat{\Phi})(v) = (\pi(-w,0,0)\omega_1(\left[\begin{smallmatrix} & 1\\ -1 & \end{smallmatrix}\right])\Phi)(v)\end{aligned}$$

Solution 4.1.14. (a) Since $a \neq 0$, we know

$$\sum_{x\in F}\psi(ax^2)=\sum_{y\in F}(1+\chi(y))\psi(ay)=G+\sum_{y\in F}\psi(ay)=G$$

So by 1.1.3., we know $|G|=\sqrt{q}$. So we only need to prove $G\cdot\chi(a)^{-1}\cdot q^{-1/2}$ is a fourth root of unity. As in the hint, we have

$$\bar{G}=\sum_{y\in F}\chi(y)\psi(-ay)=\sum_{y\in F}\chi(-y)\psi(ay)=\chi(-1)\sum_{y\in F}\chi(y)\psi(ay)=\chi(-1)G$$

So $\epsilon_0^2=\chi(-1)$, which is ± 1. Hence ϵ_0 is a fourth root of unity.

(b) Since B is symmetric, we may pick an orthogonal basis $\{v_1,\dots,v_n\}$ of V such that $B(v_i,v_j)=\delta_{ij}$, where $n=\dim V$. Then

$$\sum_{v\in V}\psi(aB(v,v))=\sum_{k_1\in F}\dots\sum_{k_n\in F}\psi\left(a\sum_i k_i^2\right)=\prod_i\sum_{x\in F}\psi(ax^2)=\epsilon_0^n\chi(a)^n q^{n/2}$$

So we only need to define $\epsilon=\epsilon_0^n\chi(a)^{n-1}$. And clearly $\epsilon^2=\epsilon_0^{2n}\chi(a)^{2n-2}=\epsilon_0^{2n}=\chi(-1)^n$.

Solution 4.1.15. We only need to verify the relations as in lemma 4.1.2.

$$\begin{aligned}
&(\omega_1(\left[\begin{smallmatrix}a&\\&a^{-1}\end{smallmatrix}\right])\omega_1(\left[\begin{smallmatrix}a'&\\&a'^{-1}\end{smallmatrix}\right])\Phi)(v)\\
&\quad=\chi(a)^{\dim V}\chi(a')^{\dim V}\Phi(aa'v)=\chi(aa')^{\dim V}\Phi(aa'v)=(\omega_1(\left[\begin{smallmatrix}aa'&\\&(aa')^{-1}\end{smallmatrix}\right])\Phi)(v)\\
&(\omega_1(\left[\begin{smallmatrix}1&x\\&1\end{smallmatrix}\right])\omega_1(\left[\begin{smallmatrix}1&x'\\&1\end{smallmatrix}\right])\Phi)(v)\\
&\quad=\psi(xB(v,v))\psi(x'B(v,v))\Phi(v)=\psi((x+x')B(v,v))\Phi(v)=(\omega_1(\left[\begin{smallmatrix}1&x+x'\\&1\end{smallmatrix}\right])\Phi)(v)\\
&(\omega_1(\left[\begin{smallmatrix}a&\\&a^{-1}\end{smallmatrix}\right])^{-1}\omega_1(\left[\begin{smallmatrix}1&x\\&1\end{smallmatrix}\right])\omega_1(\left[\begin{smallmatrix}a&\\&a^{-1}\end{smallmatrix}\right])\Phi)(v)\\
&\quad=\chi(a)^{\dim V}(\omega_1(\left[\begin{smallmatrix}a&\\&a^{-1}\end{smallmatrix}\right])^{-1}\omega_1(\left[\begin{smallmatrix}1&x\\&1\end{smallmatrix}\right])\Phi)(av)=\chi(a)^{\dim V}\psi(xB(av,av))(\omega_1(\left[\begin{smallmatrix}a&\\&a^{-1}\end{smallmatrix}\right])^{-1}\Phi)(av)\\
&\quad=\psi(a^2xB(v,v))\Phi(v)=\omega_1(\left[\begin{smallmatrix}1&a^2x\\&1\end{smallmatrix}\right])\Phi)(v)\\
&(\omega_1(\left[\begin{smallmatrix}&1\\-1&\end{smallmatrix}\right])\omega_1(\left[\begin{smallmatrix}a&\\&a^{-1}\end{smallmatrix}\right])\omega_1(\left[\begin{smallmatrix}&1\\-1&\end{smallmatrix}\right])\Phi)(v)\\
&\quad=(\omega_1(\left[\begin{smallmatrix}&1\\-1&\end{smallmatrix}\right])\omega_1(\left[\begin{smallmatrix}a&\\&a^{-1}\end{smallmatrix}\right])\hat{\Phi})(v)=\chi(a)^{\dim V}(\omega_1(\left[\begin{smallmatrix}&1\\-1&\end{smallmatrix}\right])\hat{\Phi})(av)\\
&\quad=\chi(a)^{\dim V}\epsilon^2q^{-\dim V}\sum_{u\in V}\Phi(u)\sum_{w\in V}\psi(2B(u,aw))\psi(2B(v,w))\\
&\quad=\chi(a)^{\dim V}\epsilon^2q^{-\dim V}\sum_{u\in V}\Phi(u)\sum_{w\in V}\psi(2B(u+a^{-1}v,w))\\
&\quad=\chi(a)^{\dim V}\epsilon^2q^{-\dim V}\Phi(-a^{-1}v)\sum_{w\in V}\psi(2B(0,w))\\
&\quad=\chi(-a^{-1})^{\dim V}\Phi(-a^{-1}v)=(\omega_1(\left[\begin{smallmatrix}-a^{-1}&\\&-a\end{smallmatrix}\right])\Phi)(v)
\end{aligned}$$

And the last one is the most difficult:

$$
\begin{aligned}
&(\omega_1(\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right])\omega_1(\left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right])\omega_1(\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right])\Phi)(v) \\
&\quad = (\omega_1(\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right])\omega_1(\left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right])\hat{\Phi})(v) = \epsilon q^{-\dim V/2}\sum_{u\in V}\Phi(u)\omega_1(\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right])\psi(2B(u,v))\psi(xB(v,v)) \\
&\quad = \epsilon q^{-\dim V/2}\sum_{u\in V}\Phi(u)\sum_{w\in V}\psi(2B(u+v,w))\psi(xB(w,w)) \\
&(\omega_1(\left[\begin{smallmatrix} 1 & -x^{-1} \\ & 1 \end{smallmatrix}\right])\omega_1(\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right])\omega_1(\left[\begin{smallmatrix} 1 & -x \\ & 1 \end{smallmatrix}\right])\omega_1(\left[\begin{smallmatrix} -x^{-1} & \\ & -x \end{smallmatrix}\right])\Phi)(v) \\
&\quad = \chi(-x^{-1})^{\dim V}(\omega_1(\left[\begin{smallmatrix} 1 & -x^{-1} \\ & 1 \end{smallmatrix}\right])\omega_1(\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right]))\Phi(-x^{-1}v)\psi(-xB(v,v)) \\
&\quad = \chi(-x^{-1})^{\dim V}(\omega_1(\left[\begin{smallmatrix} 1 & -x^{-1} \\ & 1 \end{smallmatrix}\right]))\sum_{u\in V}\Phi(-x^{-1}u)\psi(-xB(u,u))\psi(2B(u,v)) \\
&\quad = \chi(-x^{-1})^{\dim V}\sum_{u\in V}\Phi(-x^{-1}u)\psi(-xB(u,u))\psi(2B(u,v))\psi(-x^{-1}B(v,v))
\end{aligned}
$$

And similarly with the equation (1.17), we have

$$\sum_{w\in V}\psi(2B(u+v,w))\psi(xB(w,w)) = \epsilon q\psi(-x^{-1}B(u+v,u+v))$$

Then we've done.

Solution 4.1.16. Compare 4.1.15 and proposition 4.1.3., trivial.

Solution 4.1.17. ($\Rightarrow$) If π is a cusp representation of $\mathrm{GL}(n)$, and ρ is a cusp representation of M_λ, Then $\mathrm{Ind}_{M_\lambda}^{\mathrm{GL}(n)}\rho$ is contained in the induced representation $\mathrm{Ind}_{U_\lambda}^{\mathrm{GL}(n)}1$. So by Frobenius reciprocity, $\langle\pi, \mathrm{Ind}_{U_\lambda}^{\mathrm{GL}(n)}1\rangle = \langle\pi|_{U_\lambda}, 1\rangle = \int_{U_\lambda}\mathrm{tr}(\pi(g))\mathrm{d}g \ = 0$. So π is orthogonal to any representation parabolically induced from a cuspidal representation of $M_\lambda(F)$.

($\Leftarrow$) Clearly we have $\langle\pi|_{U_\lambda}, 1\rangle = \langle\pi, \mathrm{Ind}_{U_\lambda}^{\mathrm{GL}(n)}1\rangle = 0$ for all proper λ. So $\int_{U_\lambda}\pi(g)\mathrm{d}g \ = 0$, i.e. π is cuspidal.

Solution 4.1.18 (The Bruhat decomposition for GL(n))**.** (a) For any $g \in \mathrm{GL}(n)$, we may assume l is the smallest number such that $g_{n,l}$ is nonzero. Then there exists a $b_1 \in B$ such that

$$gb_1 = \begin{bmatrix} g'_{11} & \cdots & g'_{1l} & \cdots & g'_{1n} \\ \vdots & & \vdots & & \vdots \\ g'_{n-1,1} & \cdots & g'_{n-1,l} & \cdots & g'_{n-1,n} \\ 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}$$

So there exists $b_1' \in B$ such that

$$b_1' g b_1 = \begin{bmatrix} g_{11}'' & \dots & 0 & \dots & g_{1n}' \\ \vdots & & \vdots & & \vdots \\ g_{n-1,1}'' & \dots & 0 & \dots & g_{n-1,n}' \\ 0 & \dots & 1 & \dots & 0 \end{bmatrix}$$

Then by induction, clearly there exists two series b_i and b_i' such that $b_n' \dots b_1' g b_1 \dots b_n$ is a permutation matrices, so $\mathrm{GL}(n) = \bigcup_{w \in W} BwB$. And clearly this union is disjoint.

(b) Clearly the morphism $W \to P_\mu \backslash \mathrm{GL}(n)/P_\lambda$ is surjective. Then clearly $P_\mu \backslash \mathrm{GL}(n)/P_\lambda \cong (W \cap P_\mu)\backslash W/(W \cap P_\lambda) = W_\mu \backslash W/W_\lambda$.

Solution 4.1.19. By Mackey's theorem, we only need to cumpute the dimension of the space of functions $\Delta : \mathrm{GL}(n, F) \to \mathrm{Hom}_{\mathbb{C}}(V_\pi, V_\theta)$ with the property $\Delta(p_\mu g p_\lambda) = \theta(p_\mu) \cdot \Delta(g) \cdot \pi(p_\lambda)$ for any $p_\mu \in P_\mu(F)$ and $p_\lambda \in P_\lambda(F)$. So by 4.1.18., we only need to consider all function $\Delta = \mathbb{1}_{P_\mu(F)wP_\lambda(F)}$ for some $w \in W$. For some $\phi : V_\pi \to V_\theta$ corresponding to $\mathbb{1}_{P_\mu(F)wP_\lambda(F)}$, we only need to prove $M_\mu(F) \subset wM_\lambda(F)w^{-1}$. If not, $M_\mu(F) \cap wU_\lambda(F)w^{-1}$ is the unipotent radical of the parabolic subgroup $M_\mu(F) \cap wP_\lambda(F)w^{-1}$ of $M_\mu(F)$. For any $n \in M_\mu(F) \cap wU_\lambda(F)w^{-1}$ and $v \in V_\pi$, we have

$$\theta(n)\phi(v) = \theta(n)\phi(\pi(w^{-1}nw)v) = \theta(n)\Delta(w)\pi(w^{-1}n^{-1}w)v = \phi(v)$$

Since θ is cuspidal, we have $\phi(v) = 0$, i.e. $\phi \equiv 0$, which contradicts with our construction. So $M_\mu(F) \subset w^{-1}M_\lambda(F)w$. And similarly $M_\lambda(F) \subset w^{-1}M_\mu(F)w$, i.e. $M_\mu(F) = wM_\lambda(F)w^{-1}$. Hence we have a permutation $\sigma \in S_h$ such that $\lambda_{\sigma(i)} = \mu_i$. So this isomorphism gives us a $M_\lambda(F)$-module struture of V_θ. And by definition of ϕ, it is an $M_\lambda(F)$-module homomorphism. By Schur's lemma, $\pi_{\sigma(i)} \cong \theta_i$.

Solution 4.1.20. (a) If (Π, V_Π) is not irreducible, we have $\dim \mathrm{Hom}(V_\Pi, V_\Pi) > 0$. So by 4.1.19., there exists a nontrivial $\phi : V_\lambda \to V_\lambda$, so there exists at least two indices i and j such that $\pi_i \cong \pi_j$ as $\mathrm{GL}(\lambda_i, F)$-modules.

(b) If π and θ are both irreducible, by (a), there exists no same π_i and π_j, or θ_i and θ_j. So if $\dim \mathrm{Hom}(V_\pi, V_\theta) > 0$, by 4.1.19., π is a rearrangement of θ.

4.2 Smooth And Admissible Representations

Solution 4.2.1. For any

$$b' = \begin{bmatrix} 1 & x_{12}' & \dots & x_{1n}' \\ & 1 & \dots & x_{2n}' \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1' & & & \\ & y_2' & & \\ & & \ddots & \\ & & & y_n' \end{bmatrix}$$

Then we have

$$
\begin{aligned}
bb' &= \begin{bmatrix} 1 & x_{12} & \dots & x_{1n} \\ & 1 & \dots & x_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_n \end{bmatrix} \cdot \begin{bmatrix} 1 & x'_{12} & \dots & x'_{1n} \\ & 1 & \dots & x'_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} y'_1 & & & \\ & y'_2 & & \\ & & \ddots & \\ & & & y'_n \end{bmatrix} \\
&= \begin{bmatrix} 1 & x_{12} & \dots & x_{1n} \\ & 1 & \dots & x_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x'_{12} \cdot \frac{y_1}{y_2} & \dots & x'_{1n} \cdot \frac{y_1}{y_n} \\ & 1 & \dots & x'_{2n} \cdot \frac{y_2}{y_n} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_n \end{bmatrix} \cdot \begin{bmatrix} y'_1 & & & \\ & y'_2 & & \\ & & \ddots & \\ & & & y'_n \end{bmatrix} \\
&= \begin{bmatrix} 1 & t_{12} & \dots & t_{1n} \\ & 1 & \dots & t_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 y'_1 & & & \\ & y_2 y'_2 & & \\ & & \ddots & \\ & & & y_n y'_n \end{bmatrix}
\end{aligned}
$$

where

$$t_{ij} = x_{ij} + x'_{ij} \cdot \frac{y_i}{y_j} + \sum_{k=i+1}^{j-1} x_{i,k} x'_{k,j} \cdot \frac{y_k}{y_j}$$

So

$$\mathrm{d}_R bb' = \prod \mathrm{d}(x_{ij} + x'_{ij} \cdot \frac{y_i}{y_j} + \sum_{k=i+1}^{j-1} x_{i,k} x'_{k,j} \cdot \frac{y_k}{y_j}) \cdot \prod \mathrm{d}^\times y_i y'_i = \prod \mathrm{d} x_{ij} \cdot \prod \mathrm{d}^\times y_i = \mathrm{d}_R b$$

And for the left Haar measure, we have

$$
\begin{aligned}
b'b &= \begin{bmatrix} 1 & x'_{12} & \dots & x'_{1n} \\ & 1 & \dots & x'_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x_{12} \cdot \frac{y'_1}{y'_2} & \dots & x_{1n} \cdot \frac{y'_1}{y'_n} \\ & 1 & \dots & x_{2n} \cdot \frac{y'_2}{y'_n} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} y'_1 & & & \\ & y'_2 & & \\ & & \ddots & \\ & & & y'_n \end{bmatrix} \cdot \begin{bmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_n \end{bmatrix} \\
&= \begin{bmatrix} 1 & s_{12} & \dots & s_{1n} \\ & 1 & \dots & s_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 y'_1 & & & \\ & y_2 y'_2 & & \\ & & \ddots & \\ & & & y_n y'_n \end{bmatrix}
\end{aligned}
$$

where

$$s_{ij} = x'_{ij} + x_{ij} \cdot \frac{y'_i}{y'_j} + \sum_{k=i+1}^{j-1} x'_{i,k} x_{k,j} \cdot \frac{y'_k}{y'_j}$$

So

$$
\begin{aligned}
\mathrm{d}_L b'b &= \prod |y_i y_{i'}|^{2i-n-1} \cdot \prod \mathrm{d}(x'_{ij} + x_{ij} \cdot \frac{y'_i}{y'_j} + \sum_{k=i+1}^{j-1} x'_{i,k} x_{k,j} \cdot \frac{y'_k}{y'_j}) \cdot \prod \mathrm{d}^\times y_i y'_i \\
&= \prod |y_i|^{2i-n-1} \cdot \prod |y'_i|^{2i-n-1} \cdot \prod \left| \frac{y'_i}{y'_j} \right| \mathrm{d}x_{ij} \cdot \prod \mathrm{d}^\times y_i \\
&= \prod |y_i|^{2i-n-1} \cdot \prod \mathrm{d}x_{ij} \cdot \prod \mathrm{d}^\times y_i = \mathrm{d}_L b
\end{aligned}
$$

Solution 4.2.2. ($\Rightarrow$) For any open subgroup H of G, we take a set of representation $\{g_i\}$ of cosets of G/H. Then $G = \bigcup g_i H$. Since every $g_i H$ is also open, and G is compact, so we have $G = \coprod g_i H$ for finitely g_i. So $\{g_i\}$ is a finite set, i.e. $[G : H] < \infty$.

($\Leftarrow$) Conversely, for any subgroup H of finite index, we know $G - H$ is a disjoint union of finite copies of H, which is closed. So H is open.

Solution 4.2.3. Denote the modular function as $\delta : G \to \mathbb{R}_+^*$. Then δ is continuous. Since G is compact, δ is bounded. Since $\mathrm{Im}\delta$ must be a subgroup of $\mathbb{R}_+^*$ and bounded, it must be $\{1\}$. So δ is constantly one, i.e. G is unimodular.

Solution 4.2.4. For any $g' = (g'_{ij}) \in \mathrm{GL}(n, F)$, we have

$$
\begin{aligned}
\mathrm{d}gg' &= |\det(gg')|^{-n} \mathrm{d}_a gg' = |\det(g)|^{-n} |\det(g')|^{-n} \cdot \prod_{i,j} \mathrm{d}_a \left(\sum_{k=1}^n g_{ik} g'_{kj} \right) \\
&= |\det(g)|^{-n} |\det(g')|^{-n} \cdot \left(\sum_{\sigma \in S_n} \left((-1)^{\mathrm{sgn}(\sigma)} \prod g'_{i,\sigma(i)} \right) \right)^n \prod_{i,j} \mathrm{d}_a g_{ij} \\
&= |\det(g)|^{-n} \prod_{i,j} \mathrm{d}_a g_{ij} = \mathrm{d}g \\
\mathrm{d}g'g &= |\det(g'g)|^{-n} \mathrm{d}_a g'g = |\det(g)|^{-n} |\det(g')|^{-n} \cdot \prod_{i,j} \mathrm{d}_a \left(\sum_{k=1}^n g'_{ik} g_{kj} \right) \\
&= |\det(g)|^{-n} |\det(g')|^{-n} \cdot \left(\sum_{\sigma \in S_n} \left((-1)^{\mathrm{sgn}(\sigma)} \prod g'_{\sigma(i),i} \right) \right)^n \prod_{i,j} \mathrm{d}_a g_{ij} \\
&= |\det(g)|^{-n} |\det(g')|^{-n} |\det(g'^T)|^n \prod_{i,j} \mathrm{d}_a g_{ij} = \mathrm{d}g \\
&= |\det(g)|^{-n} \prod_{i,j} \mathrm{d}_a g_{ij} = \mathrm{d}g
\end{aligned}
$$

Solution 4.2.5. For any $x = (x_{ij})_{i<j}, x' = (x'_{ij})_{i<j} \in N(F)$, we have $xx' = t = (t_{ij})_{i<j}$, where

$$
t_{ij} = x_{ij} + x'_{ij} + \sum_{k=i+1}^{j-1} x_{ik} x'_{kj}
$$

So
$$\mathrm{d}xx' = \prod_{i<j} \mathrm{d}\left(x_{ij} + x'_{ij} + \sum_{k=i+1}^{j-1} x_{ik}x'_{kj}\right) = \prod_{i<j} \mathrm{d}x_{ij} = \mathrm{d}x$$
And similarly we have $\mathrm{d}x'x = \mathrm{d}x$. So $N(F)$ is unimodular.

Solution 4.2.6. No, or how could we research the so-called "irreducible admissible representations"???

Solution 4.2.7. Denote $e = (e_n)$ for $e_0 = 1$, and $e_i = 0$ for any other i. Then for any $a = (a_n) \in V$, we have $a = \sum_{n\in\mathbb{Z}} a_n\rho(n)(e)$. Since for almost all n, we have $a_n = 0$, this summation is well-defined, so any nonzero vector in V can be generated by e. So V just has two invariant subspaces: the whole space V and the zero space $\{0\}$.

Solution 4.2.8. By theorem 4.2.2., $\hat{\pi}$ differs with π by a scalar $\omega(\det(g))^{-1}$. So every invariant subspace of π is also a invariant subspace of $\hat{\pi}$, and vice versa. So π is irreducible iff $\hat{\pi}$ is irreducible.

Solution 4.2.9. As the same argument in 3.1.1.(a), $\ker\pi$ contains an open subgroup H of G. Then it clearly contains $\bigcap_{g\in G} gHg^{-1}$, which is also an open subgroup of G. So we may change H by $\bigcap_{g\in G} gHg^{-1}$ and assume H is open normal. Then by the hint, we need to prove $H \supset \mathrm{SL}(2, F)$.

Since H is open, we may assume $K(\varpi^n) \subset H$ for some n. Then since for any $g \in G$, we have $gHg^{-1} \subset H$. So for any $x \in F$, there exists an $a \in F$ such that $\nu(ax) > n$. Then
$$\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} = \begin{bmatrix} a^{-1} & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & ax \\ & 1 \end{bmatrix}\begin{bmatrix} a & \\ & 1 \end{bmatrix}$$
So $N \subset H$. And similarly $N_- = \{\left[\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right]\} \subset H$. Since N and N_- generate the whole $\mathrm{SL}(2, F)$, we know $\mathrm{SL}(2, F) \subset H$. Since π factor through $\mathrm{GL}(2, F) \to \mathrm{GL}(2, F)/\mathrm{SL}(2, F) \cong F^* \to V$, and π is irreducible, it must be one-dimensional because F^* is abelian.

Solution 4.2.10. Suppose (π, V) is such a representation. Since $\pi(a)$ for all $a \in F^*$ are commutative for each other, we can find a basis of V such that $\pi(a)$ is in the form of Jordan matrix for all a. Since π is two-dimensional and indecomposable, we may assume $\pi(a) = \left[\begin{smallmatrix} \omega(a) & \delta(a) \\ & \omega(a) \end{smallmatrix}\right]$. Tensoring with ω^{-1}, we may assume $\pi(a)$ has the form $\left[\begin{smallmatrix} 1 & \delta(a) \\ & 1 \end{smallmatrix}\right]$ for all a. Then since $\delta(ab) = \delta(a) + \delta(b)$ for all $a, b \in F^*$, we must have $\delta(a) = C \cdot \log|a|$.

4.3 Distributions and Sheaves

Solution 4.3.1. For any open set $U \in \mathfrak{T}$, we define $\mathcal{F}'(U) = \{s : U \to \hat{\mathcal{F}} \mid s(x) \in \mathcal{F}_x$ and for any $x \in U$, there exists a $V \in \mathfrak{T}_0$ such that $x \in V \subset U$, and a section $s' \in \mathcal{F}(V)$ such that $s|_V = s'\}$. Clearly since $\mathcal{F}$ is a sheaf, $\mathcal{F}'$ is also a sheaf and for any $U \in \mathfrak{T}_0$ we have $\sigma_U : \mathcal{F}(U) \to \mathcal{F}'(U)$ is an isomorphism, and also, σ_U intertwines with the restrict morphism.

Solution 4.3.2. Clearly $G = \coprod G_i / \sim$ for the equivalence $\sim$ defined as $x_i \sim x_j$ iff there exists a $k > i, j$ such that $\phi_{ik}(x_i) = \phi_{jk}(x_j)$. And $\phi_i : G_i \to G$ factors through $G_i \to \coprod G_i \to G$. Then if $\phi_i(g_i) = 0$, we know $x_i \sim 0$ for some $0 \in G_j$. If $j > i$, then we've done. If else, there exists a $k > i, j$ such that $\phi_{ik}(x_i) = \phi_{jk}(0) = 0 \in G_k$.

Solution 4.3.3. For any $x \in U$, by definition of stalk and 4.3.2., there exists a open set U_x such that $x \in U_x \subset U$ satisfying $\rho_{U,U_x}(f) = 0$. Since all U_x form form an open covering of U, by the axiom of sheaf, we know $f = 0$.

Solution 4.3.4. (a) For any $f \in \mathcal{F}_c$ and $s \in C_c^\infty(X)$, we can canonically define sf as $sf(x) = s(x)f(x)$ for any $x \in X$. Then we need to verify that sf is an element in $\mathcal{F}_c$. For any $x \in X$, there exists an open neighbourhood U of x and a $f' \in \mathcal{F}(U)$ such that $f(y) = f'(y)$ for any $y \in U$. And since s is smooth, there exists a open neighbourhood U' of x such that $s|_{U'}$ is a constant. So for any open set $V \subset U \cap U'$, we know sf on V is the same with $s|_V \cdot f'|_V$. And clearly sf support compactly, hence $sf \in \mathcal{F}_c$. And since f is compactly supported, we may assume $\operatorname{supp}(f) \subset U$ for some open set U. Then clearly $\mathbf{1}_U \cdot f = f$. So $\mathcal{F}_c$ is cosmooth.

(b) For any $m \in M$, since M is cosmooth, there exists an open compact set U such that $\nVdash_U \cdot m = m$, hence $m \in \mathcal{M}(u)$. So $m \in \mathcal{M}_c$. Conversely, for any $m \in \mathcal{M}_c$, there exists an open compact subset U such that $m \in \mathcal{M}(U)$. Then we can extend m into $\mathcal{M}(X)$ with zero outside U, so $m \in M$.

(c) Denote the sheaf associated with $\mathcal{F}_c$ as $\mathcal{G}$. For any $U \in \mathfrak{T}_0$, we can define $\mathcal{G}(U) = \{f \in \mathcal{F}_c \mid \mathbf{1}_U \cdot f = f\}$. So clearly $\mathcal{G}(U) = \mathcal{F}(U)$, i.e. $\mathcal{G} = \mathcal{F}$.

Solution 4.3.5. (a) This question is just 4.2.2.

(b) Suppose $\operatorname{vol}(H) = 0$. For any open compact subgroup G' of G, we denote $H' = H \cap G'$. Then $\operatorname{vol}(H') \leq \operatorname{vol}(H)$, which implies $\operatorname{vol}(H') = 0$. Since $[G' : H'] < \infty$ by (a), we have $\operatorname{vol}(G') = [G' : H'] \cdot \operatorname{vol}(H') = 0$. So every open compact group has volume 0, then the left Haar measure is 0, which makes a contradiction.

4.4 Whittaker Models and the Jacquet Functor

Solution 4.4.1. By the proof of theorem 4.4.1., we only need to find an analogue of theorem 4.4.2. We will discuss the two cases partially.

First we may think $T = \{\left[\begin{smallmatrix} * & \\ & * \end{smallmatrix}\right]\}$ is the split torus. In this case, we will prove that if Δ is a distribution on $\mathrm{GL}(2, F)$ such that $\lambda(t)\Delta = \Delta$ and $\rho(t)\Delta = \Delta$ for any $t \in T$, and $\Delta = -{}^{\iota}\Delta$ for $\iota(g) = \left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right]^T g \left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right]$, then we must have $\Delta = 0$. Similarly we construct a group $G = (T \times T) \rtimes \{1, \mathcal{I}\}$, where $\mathcal{I}^2 = 1$ with $\mathcal{I}(t_1, t_2)\mathcal{I} = ({}^{\iota}t_2^{-1}, {}^{\iota}t_1^{-1})$. And we have a character χ such that $\chi|_{T \times T} = 1$, and $\chi(\mathcal{I}) = -1$. Then the conditions above are equivalent to $\sigma(g)\Delta = \chi(g)\Delta$.

Then we consider $D = \Delta|_{BwB}$. Since BwB consists with all matrices $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $c \neq 0$, we know $T \backslash BwB / T$ has a set of representations as $S = \{\left[\begin{smallmatrix} 1 & b \\ 1 & 1 \end{smallmatrix}\right] \mid b \neq 1\} \cup \{\left[\begin{smallmatrix} & 1 \\ 1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 1 & \end{smallmatrix}\right], \left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right]\}$. So by proposition

4.3.2., we have

$$D(f) = c\int_S \int_{T\times T} f(t_1 s t_2)\mathrm{d}t_1\, \mathrm{d}t_2\, \mathrm{d}s$$

Clearly this distribution is invariant under ι, so we must have $c = 0$. Hence $\Delta|_{BwB} = 0$. Since we have $0 \to \mathfrak{D}(B(F)) \to \mathfrak{D}(\mathrm{GL}(2,F)) \to \mathfrak{D}(BwB) \to 0$, we know $\Delta \in \mathfrak{D}(B(F))$. In this case, we have $T\backslash B/T = \{\left[\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}\right], \left[\begin{smallmatrix}1 & 1\\ & 1\end{smallmatrix}\right]\}$. So for each fibre, the integral $\int_T f(t\delta)\mathrm{d}t$ is invariant under the action of ι, where $\delta = \left[\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}1 & 1\\ & 1\end{smallmatrix}\right]$. So $\Delta = 0$ by our assumption, hence we've done.

Secondly we may think T is a nonsplit torus, we know $T = \{\left[\begin{smallmatrix}x & y\\ Dy & x\end{smallmatrix}\right]\}$ for some $D \in F^* - (F^*)^2$. Then we only need to change the ι above as

$$\iota(g) = \left[\begin{smallmatrix}-1 & \\ & D\end{smallmatrix}\right]^T g \left[\begin{smallmatrix}-1 & \\ & D^{-1}\end{smallmatrix}\right]$$

Then the rest is almost all the same.

Solution 4.4.2. For any $\gamma = \left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right]$ with $c \neq 0$, we denote H as the group generated by γ and $N(F)$. Clearly $H \in \mathrm{SL}(2,F)$. Then since

$$\begin{bmatrix} & -c^{-1}\\ c & \end{bmatrix} = \begin{bmatrix}1 & -a/c\\ & 1\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1 & -d/c\\ & 1\end{bmatrix}$$

we know $\left[\begin{smallmatrix} & -c^{-1}\\ c & \end{smallmatrix}\right] \in H$. Then since

$$\begin{bmatrix} & 1\\ -1 & \end{bmatrix} = \begin{bmatrix}1 & -c^2\\ & 1\end{bmatrix}\cdot\begin{bmatrix} & -c^{-1}\\ c & \end{bmatrix}\cdot\begin{bmatrix}1 & (c-1)/c^4\\ & 1\end{bmatrix}\cdot\begin{bmatrix} & -c^{-1}\\ c & \end{bmatrix}^{-1}\cdot\begin{bmatrix}1 & c\\ & 1\end{bmatrix}\cdot\begin{bmatrix} & -c^{-1}\\ c & \end{bmatrix}\cdot\begin{bmatrix}1 & (1-c)/c^3\\ & 1\end{bmatrix}\cdot\begin{bmatrix} & -c^{-1}\\ c & \end{bmatrix}^{-2}$$

So $w = \left[\begin{smallmatrix} & 1\\ -1 & \end{smallmatrix}\right] \in H$. So $\mathrm{SL}(2,F) = \langle w, N(F)\rangle \subset H$. Hence $\mathrm{SL}(2,F)$ is generated by γ and $N(F)$.

4.5 The Principal Series Representations

Solution 4.5.1. For any compact subgroup $K \subset G$, for any lattice $\Lambda' \in F^n$, we may construct a subgroup $K' = \{g \in K \mid g\Lambda' \subset \Lambda'\}$, which is open in K. Since K is compact, we may find finitely many $g_i \in K$ such that $K = \bigcup g_i K'$. So $\Lambda = \sum g_i \Lambda'$ is a K-invariant lattice.

If K is a compact subgroup containing $\mathrm{GL}(n,\mathfrak{o})$, then there exists a lattice $\Lambda \in F^n$ such that Λ is K-finite. Then we may normalize Λ by multiple a scalar to make $\Lambda \subset \mathfrak{o}^n$ but not in $(\varpi\mathfrak{o})^n$. Suppose $v \in \Lambda$ such that $v \notin (\varpi\mathfrak{o})^n$, then clearly $\Lambda \subset \mathrm{GL}(n,\mathfrak{o}) \cdot v = \mathfrak{o}^n$, i.e. $\Lambda = \mathfrak{o}^n$. And clearly the only compact group stabilizing the $\mathfrak{o}^n$ is $\mathrm{GL}(n,\mathfrak{o})$, so $K \subset \mathrm{GL}(n,\mathfrak{o})$.

Conversely, for any maximal compact subgroup K, Since K stabilizes a lattice Λ, we may write $\Lambda = A \cdot (\mathfrak{o}^n)$ for some $A \in \mathrm{GL}(2,F)$, then clearly $K \subset A\mathrm{GL}(n,\mathfrak{o})A^{-1}$. So we must have $K = A\mathrm{GL}(n,\mathfrak{o})A^{-1}$.

Solution 4.5.2. (a) Since this case is the contragredient case of (b), we only need to prove the (b).

(b) By the construction of $(\sigma(\chi_1,\chi_2),W)$, we know it is the vector space generated by all $f-\rho(n)f$ for all $f\in\mathcal{B}(\chi_1,\chi_2)$ and $n\in N$. And this vector space clearly has no nontrivial G-invariant subspace. So $\sigma(\chi_1,\chi_2)$ is irreducible by the same argument of lemma 4.5.1., or we will make $\delta^{1/2}\chi(\left[\begin{smallmatrix} y & \\ & -y\end{smallmatrix}\right])$ to be another number.

Solution 4.5.3. First we will show that $J(V)$ has at most dimension two, which is equivalently to show the linear functional on $J(V)$ has at most dimension two, i.e. the linear functional on V satisfying $L(\pi(n)\phi)=L(\phi)$ has at most dimension two. We define a distribution Δ on $\mathrm{GL}(2,F)$ by $\Delta(\phi)=L(P(\phi))$. Denote $D=\Delta|_{B(F)}$. Then by proposition 4.3.2., D is given by

$$D(\phi)=c\int_T\int_N \phi(tn)(\delta^{1/2}\chi^{-1})(t)\mathrm{d}n\ \mathrm{d}t$$

So it has one dimension. Denote $D'=\Delta|_{\mathrm{GL}(2,F)-B(F)}$, by proposition 4.3.2., D' is given by

$$D'(\phi)=c'\int_B\int_N \phi(bw_0n^{-1})(\delta^{1/2}\chi^{-1})(b)\mathrm{d}b\ \mathrm{d}n$$

So it has one dimension more. By the exactness of $0\to\mathfrak{D}(B(F))\to\mathfrak{D}(\mathrm{GL}(2,F))\to\mathfrak{D}(\mathrm{GL}(2,F)-B(F))\to 0$, we know this kind of linear functionals has dimension at most two.

Then we will construct two linear functionals as follows: $L_1(\phi)=\phi(1)$, and

$$L_2(\phi)=\int_F\left(\phi\left(\begin{bmatrix} & -1\\ 1 & \end{bmatrix}\begin{bmatrix}1 & x\\ & 1\end{bmatrix}\right)-h(x)\phi(1)\right)\mathrm{d}x$$

where $h(x)=|x|^{-1}(\chi_1^{-1}\chi_2)(x)$ if $|x|>1$ and $h(x)=0$ otherwise. Suppose $\chi_1\neq\chi_2$. Then we may consider $f_2(\left[\begin{smallmatrix} y_1 & z\\ & y_2\end{smallmatrix}\right]\left[\begin{smallmatrix} & -1\\ 1 & \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x\\ & 1\end{smallmatrix}\right])=|y_1/y_2|^{1/2}\chi_1(y_1)\chi_2(y_2)$, and we have

$$L_2(f_2)=\int_F(1-h(x))\mathrm{d}x\ =\mathrm{vol}(\mathfrak{o})+\int_{|x|>1}1-|x|^{-1}(\chi_1^{-1}\chi_2)(x)=1$$

For the case $\chi_1=\chi_2$, we may consider the flat family f_{2,s_1,s_2}, the corresponding $L_{2,s_1,s_2}(f_{2,s_1,s_2})=0$. So it will be continuous to make $L_2(f_2)=0$ in this subtle case. For L_1, we may construct $f_1(bk)=(\delta^{1/2}\chi)(b)$ for $b\in B(F)$, $k\in K_1(\mathfrak{a})$ and $f_1=0$ for other case. Then $f_1(1)=(\delta^{1/2}\chi)(1)=1$, so $L_1(f_1)=1$. But $L_2(f_1)=-\int_F h(x)\mathrm{d}x\ =0$. So $J(V)$ has dimension two.

Solution 4.5.4. $J(\pi(\chi_1,\chi_2))$ is clearly one-dimensional, and $T(F)$ acts on it by multiplying $\delta^{1/2}\chi$.

For the special one, by the exactness of Jacquet module we know $\dim J(\sigma(\chi_1, \chi_2) = 1$. And clearly

$$\begin{aligned} L_2(\rho(t)f_2) &= \int_F \left(\rho\left(\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\begin{bmatrix} t_1 & \\ & t_2 \end{bmatrix}\right) - h(x)\rho\left(\begin{bmatrix} t_1 & \\ & t_2 \end{bmatrix}\right)\right) \mathrm{d}x \\ &= \int_F \left(\rho\left(\begin{bmatrix} t_2 & \\ & t_1 \end{bmatrix}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & t_1^{-1}t_2 x \\ & 1 \end{bmatrix}\right) - h(x)\rho\left(\begin{bmatrix} t_1 & \\ & t_2 \end{bmatrix}\right)\right) \mathrm{d}x \\ &= \chi_1(t_2)\chi_2(t_1)|t_1/t_2|^{1/2} L_2(f_2) = (\delta^{1/2}\chi')(t) L_2(f_2) \end{aligned}$$

For any image $Mf \in \mathcal{B}(\chi_1, \chi_2)$, by equation 5.20., we have

$$\rho(t)(Mf) = M(\rho(t)f) = (\delta^{1/2}\chi)(t) \cdot M(f)$$

So it must fall in $\pi(\chi_2, \chi_1)$. Conversely, for any $Mf \in \mathcal{B}(\chi_2, \chi_1)$, we have

$$\rho_t(Mf) = M(\rho(t)f) = (\delta^{1/2}\chi')(t) \cdot M(f)$$

So Mf is falling in the subrepresentation isomorphic to $\sigma(\chi_1, \chi_2)$.

Solution 4.5.5 (Simple Mackey Theory)**.** (a) For any $u \in U_1$ and $g \in G$, we can define $f_{g,u} \in V_1$ such that $f_{g,u}(x) = \pi(h)u$ if $x = hg$ for some $h \in H_1$, and $f_{g,u}(x) = 0$ otherwise. Clearly every element of V_1 is linearly generated by these $f_{g,u}$. For any $L \in \mathrm{Hom}_{H_2}(V_1, U_2)$, we can define $\Delta : G \to \mathrm{Hom}(U_1, U_2)$ by $\Delta(g)(u) = L(f_{g^{-1},u})$. For the inverse map, we can define $L(f) = \Delta * f = \sum_{g \in H_1 \backslash G} \Delta(g^{-1})(f(g))$. So for any $L(f) = \Delta * f$, we have

$$L(f_{g^{-1},u}) = \Delta * f_{g^{-1},u} = \sum_{k \in H_1 \backslash G} \Delta(k^{-1})(f_{g^{-1},u}(k)) = \Delta(g)(u)$$

Conversely, for any $\Delta(g)(u) = L(f_{g^{-1},u})$, we only need to check the fundamental elements $f_{g,u}$, which is trivial. So we have $\mathrm{Hom}_{H_2}(V_1, U_2)$ is isomorphic to the space of all functions $\Delta : G \to \mathrm{Hom}(U_1, U_2)$ satisfying equation 1.4.

(b) The proof is totally isomorphic to 4.1.2.

(c) Since $S^\gamma, \sigma_1^\gamma, \sigma_2^\gamma$ are just the conjugate of $S_\gamma, \sigma_{1,\gamma}, \sigma_{2,\gamma}$ by γ, trivial.

Solution 4.5.6. Suppose $\langle \cdot, \cdot \rangle$ is an H-invariant positive difinite Hermitian inner product on V_0. Then we may define a positive definite Hermitian inner product on V as

$$\langle\langle f_1, f_2 \rangle\rangle = \int_G \langle f_1(g), f_2(g) \rangle \mathrm{d}g$$

So clearly for any $k \in G$,

$$\langle\langle \pi(k)f_1, \pi(k)f_2 \rangle\rangle = \int_G \langle f_1(gk), f_2(gk) \rangle \mathrm{d}g = \int_G \langle f_1(g), f_2(g) \rangle \mathrm{d}g = \langle\langle f_1, f_2 \rangle\rangle$$

Hence the π on G is unitary.

Solution 4.5.7. For any $f \in \text{c-Ind}_{N(F)}^{B_1(F)} \psi_N$, we have $f(\left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} a & b \\ & 1 \end{smallmatrix}\right]) = \psi_N(x) f(\left[\begin{smallmatrix} a & b \\ & 1 \end{smallmatrix}\right])$ and f is smooth. So for $\phi(x) = f(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right])$, we have $\pi(\left[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right])\phi(x) = f(\left[\begin{smallmatrix} ax & \\ & 1 \end{smallmatrix}\right]) = \phi(ax)$, and $\pi(\left[\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right])\phi(x) = f(\left[\begin{smallmatrix} x & bx \\ & 1 \end{smallmatrix}\right]) = f(\left[\begin{smallmatrix} 1 & bx \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right]) = \psi_N(bx)\phi(x)$, which means $\phi \in \mathcal{K}$. So we have a map $\text{c-Ind}_{N(F)}^{B_1(F)} \psi_N \to \mathcal{K}$. Conversely, for any $\phi \in K$, we can define $f(\left[\begin{smallmatrix} a & x \\ & 1 \end{smallmatrix}\right]) = f(\left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right]) = \psi_N(x)\phi(a)$, which is clearly an element in $\text{c-Ind}_{N(F)}^{B_1(F)} \psi_N$. So these two spaces are isomorphic to each other.

Solution 4.5.8 (Transitivity of induction)**.** We can define a map $V^G \to (V^{H_2})^G$ as $f \mapsto \phi$, where $\phi(g)(h_2) = f(h_2 g)$, and conversely, a map $(V^{H_2})^G \to V^G$ as $\phi \mapsto f$, where $f(g) = \phi(g)(1)$. Clearly these two maps are inverse to each others, so we have $(V^{H_2})^G \cong V^G$. For the compact induction case, we need a condition that H_2 is cocompact to make the definition $\phi(g)(h_2) = f(h_2 g)$ is well-defined. So in this additional condition, we have $(V_c^{H_2})_c^G \cong V_c^G$.

Solution 4.5.9 (Twisting)**.** Clearly we have

$$(\chi \otimes \pi)(\left[\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right] g)v = \chi(ad \cdot \det(g))\pi(\left[\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right] g)v = \chi(a)\chi_1(a)\chi(d)\chi_2(d)\chi(\det(g))\pi(g)v$$

Hence we have an inclusion $\chi \otimes \mathcal{B}(\chi_1, \chi_2) \to \mathcal{B}(\chi\chi_1, \chi\chi_2)$. And conversely we have $\chi^{-1} \otimes \mathcal{B}(\chi\chi_1, \chi\chi_2) \hookrightarrow \mathcal{B}(\chi_1, \chi_2)$, which means $\mathcal{B}(\chi\chi_1, \chi\chi_2) \hookrightarrow \chi \otimes \mathcal{B}(\chi_1, \chi_2)$. Hence we have the isomorphism $\chi \otimes \mathcal{B}(\chi_1, \chi_2) \cong \mathcal{B}(\chi\chi_1, \chi\chi_2)$.

Solution 4.5.10. We may assume $f_{s_1,s_2} \in V_{s_1,s_2}(\rho)$ for some irreducible admissible representation ρ of $K = \mathrm{GL}(2, \mathfrak{o})$, and the general case is the linear combination of these f. By the definition of induced representation, we clearly know $M : V_{s_1,s_2}(\rho) \to V_{s_2,s_1}(\rho)$. So we have $M f_{s_1,s_2} \in V_{s_2,s_1}(\rho)$. Since $\mathcal{B}(\chi_2, \chi_1)$ is admissible, the space $V_{s_2,s_1}(\rho)$ has a basis $\{f_{s_2,s_1}^j \mid 1 \le j \le n\}$. So $M f_{s_1,s_2} = \sum_j \phi_j(s_1, s_2) f_{s_2,s_1}^j$, where $\phi_j(s_1, s_2) = \int_K M f_{s_1,s_2}(k) f_{s_2,s_1}^j(k) \mathrm{d}k$.

4.6 Spherical Representations

Solution 4.6.1. Since all set $K \left[\begin{smallmatrix} \varpi^{n_1} & \\ & \varpi^{n_2} \end{smallmatrix}\right] K$ are disjoint, we only need to prove the "only if" part. Suppose $g = k_1 \left[\begin{smallmatrix} \varpi^{n_1} & \\ & \varpi^{n_2} \end{smallmatrix}\right] k_2$ for some $k_1, k_2 \in K$. Clearly we know $v(\det(g)) = n_1 n_2$. Suppose $k_i = \left[\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right]$. Then we have

$$g = \begin{bmatrix} a_1 a_2 \varpi^{n_1} + b_1 c_2 \varpi^{n_2} & a_1 b_2 \varpi^{n_1} + b_1 d_2 \varpi^{n_2} \\ c_1 a_2 \varpi^{n_1} + d_1 c_2 \varpi^{n_2} & c_1 b_2 \varpi^{n_1} + d_1 d_2 \varpi^{n_2} \end{bmatrix}$$

Since all coefficients of k_1, k_2 are in $\mathfrak{o}$, the ideal generated by the coefficients of g must be contained in $\mathfrak{p}^{n_2}$. Conversely, since

$$c_1(a_1 a_2 \varpi^{n_1} + b_1 c_2 \varpi^{n_2}) - a_1(c_1 a_2 \varpi^{n_1} + d_1 c_2 \varpi^{n_2}) = -\det(k_1) c_2 \varpi^{n_2}$$
$$c_1(a_1 b_2 \varpi^{n_1} + b_1 d_2 \varpi^{n_2}) - a_1(c_1 b_2 \varpi^{n_1} + d_1 d_2 \varpi^{n_2}) = -\det(k_1) d_2 \varpi^{n_2}$$

Since one of c_2, d_2 must be in $\mathfrak{o}^*$, and $\det(k_1) \in \mathfrak{o}^*$, we know the ideal generated by the coefficients of g must contain the $\mathfrak{p}^{n_2}$, hence it is $\mathfrak{p}^{n_2}$.

Solution 4.6.2. Just calculate

$$\begin{aligned}
w'_m &= (T(\mathfrak{p})W_0)(\left[\begin{smallmatrix}\varpi^m & \\ & 1\end{smallmatrix}\right]) = \int_{K\left[\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\right]K} W_0(\left[\begin{smallmatrix}\varpi^m & \\ & 1\end{smallmatrix}\right] g)\mathrm{d}g \\
&= \int_{\left[\begin{smallmatrix}1 & \\ & \varpi\end{smallmatrix}\right]K} W_0(\left[\begin{smallmatrix}\varpi^m & \\ & 1\end{smallmatrix}\right] g)\mathrm{d}g \ + \sum_{b \bmod \mathfrak{p}} \int_{\left[\begin{smallmatrix}\varpi & b \\ & 1\end{smallmatrix}\right]K} W_0(\left[\begin{smallmatrix}\varpi^m & \\ & 1\end{smallmatrix}\right] g)\mathrm{d}g \\
&= W_0(\left[\begin{smallmatrix}\varpi^m & \\ & \varpi\end{smallmatrix}\right]) + \sum_{b \bmod \mathfrak{p}} W_0(\left[\begin{smallmatrix}\varpi^{m+1} & b\varpi^m \\ & 1\end{smallmatrix}\right]) \\
&= W_0(\left[\begin{smallmatrix}\varpi & \\ & \varpi\end{smallmatrix}\right]\left[\begin{smallmatrix}\varpi^{m-1} & \\ & 1\end{smallmatrix}\right]) + \sum_{b \bmod \mathfrak{p}} W_0(\left[\begin{smallmatrix}1 & b\varpi^m \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}\varpi^{m+1} & \\ & 1\end{smallmatrix}\right]) \\
&= \alpha_1\alpha_2 W_0(\left[\begin{smallmatrix}\varpi^{m-1} & \\ & 1\end{smallmatrix}\right]) + \sum_{b \bmod \mathfrak{p}} \psi(b\varpi^m) W_0(\left[\begin{smallmatrix}\varpi^{m+1} & \\ & 1\end{smallmatrix}\right]) \\
&= \alpha_1\alpha_2 w_{n-1} + q w_{m+1}
\end{aligned}$$

So by equation (6.6), we have

$$w_{m+1} = q^{-1/2}(\alpha_1 + \alpha_2)w_m - q^{-1}\alpha_1\alpha_2 w_{m-1}$$

Since $w_{-1} = 0$, we have $w_m = q^{-m/2}\frac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1 - \alpha_2} \cdot w_0$.

4.7 Local Functional Equations

Solution 4.7.1. (a) Clearly we have an exact sequence $0 \to \mathfrak{D}(F)_0 \to \mathfrak{D}(F) \to \mathfrak{D}(F^*)$, where $\mathfrak{D}(F)_0$ is the set of all distribution supported on 0. This exact sequence induces another one $0 \to \mathfrak{D}(F)_0(\chi) \to \mathfrak{D}(F)(\chi) \to \mathfrak{D}(F^*)(\chi)$.

Firstly, for any $D \in \mathfrak{D}(F)_0$, there exists a $\phi_0 \in C_c^\infty(F)$ such that $D(\phi_0) \neq 0$. So clearly $\phi_0(0) \neq 0$. Then we may write $D(\phi_0) = c\phi_0(0)$ for some $c \neq 0$. Then for any other $\phi \in C_c^\infty(F)$, we have $\phi - \frac{\phi(0)}{\phi_0(0)}\phi_0$ is equaly to 0 in a neighbourhood of 0, so $D(\phi) = D(\frac{\phi(0)}{\phi_0(0)}\phi_0) = c\phi(0)$. Hence we know $D = c\delta$, hence $\mathfrak{D}(F)_0$ has dimension 1. And clearly, for any nontrivial character χ, we have $\mathfrak{D}(F)_0(\chi) = 0$.

Secondly, for any χ, we may assume χ is trivial on $1 + \mathfrak{p}^n$ for some n. Then for any $D \in \mathfrak{D}(F^*)(\chi)$, and any $\phi \in C_c^\infty(F^*)$, we may assume $\phi = \sum_{i=1}^N \phi(a_i)\mathbf{1}_{a_i U_m}$, where $U_m = 1 + \mathfrak{p}^m$. Then

$$\begin{aligned}
D(\phi) &= \sum_{i=1}^N \phi(a_i)\chi(a_i)D(\mathbf{1}_{U_m}) = D(\mathbf{1}_{U_n})[\mathfrak{o}^* : U_n] \sum \phi(a_i)\chi(a_i)[\mathfrak{o}^* : U_m]^{-1} \\
&= D(\mathbf{1}_{U_n})[\mathfrak{o}^* : U_n] \int_{F^*} \phi(x)\chi(x)\mathrm{d}^\times x
\end{aligned}$$

And $D(\mathbf{1}_{U_n})[\mathfrak{o}^* : U_n]$ is a constant independent of ϕ. Hence we know $\mathfrak{D}(F^*)(\chi)$ has dimension 1, which means that $\mathfrak{D}(F)(\chi)$ has at most two dimension.

If $\mathfrak{D}(F)(\chi)$ has dimension two, we know χ must be trivial. In this case, we need to prove the morphism $\mathfrak{D}(F)(\chi) \to \mathfrak{D}(F^*)(\chi)$ is the zero morphism and make a contradiction. For any $D \in \mathfrak{D}(F)$ be a preimage of $\mathrm{d}^\times x$ defined by $D(\phi) = \int_{F^*}(\phi(x) - \phi(0)\mathbf{1}_{\mathfrak{o}}(x))\mathrm{d}^\times x$. For any $a \in F^*$, we have

$$\begin{aligned}(\rho(a)D)(\phi) = D(\rho(a^{-1})\phi) &= \int_{F^*}(\phi(a^{-1}x) - \phi(0)\mathbf{1}_{\mathfrak{o}}(a^{-1}x))\mathrm{d}^\times x \\ &= D(\phi) + \phi(0)\int_{F^*}(\mathbf{1}_{\mathfrak{o}}(a^{-1}x) - \mathbf{1}_{\mathfrak{o}}(x))\mathrm{d}^\times x = D(\phi) - \phi(0)\mathrm{ord}(a)\end{aligned}$$

So $\rho(a)D = D - \mathrm{ord}(a)\delta$. So if there exists a $D' \in \mathfrak{D}(F)(\chi)$ as a preimage of $\mathrm{d}^\times x$, we know $D - D' = c\delta$ is invariant under F^*, which makes a contradiction. So $\mathfrak{D}(F)(\chi)$ has dimension one.

(b) **I totally cannot understand why this exercise is relative to proposition 2.1.3. and where the numerator and denominator of equation (1.13) in chapter 2 are? Hence I think maybe this question is asking me to prove proposition 3.1.5., and I will regard the numerator and denominator of equation (1.15) in chapter 3 as distributions.**

We only need to prove that $\Phi \mapsto \zeta(s,\chi,\Phi)$ and $\Phi \mapsto \zeta(1-s,\chi^{-1},\hat{\Phi})$ are two distribution in $\mathfrak{D}(F)(\chi\cdot|\,|^s)$. Then by (a), their are linear, which means there exists a constant $\gamma(s,\chi,\psi)$ independent of Φ such that $\zeta(1-s,\chi^{-1},\hat{\Phi}) = \gamma(s,\chi,\psi)\zeta(s,\chi,\Phi)$. Clearly, we know $\rho(a^{-1})\hat{\Phi} = |a|\rho(a)\hat{\Phi}$, then

$$\begin{aligned}\rho(a)\zeta(s,\chi,\Phi) &= \int_{F^*}\Phi(a^{-1}x)\chi(x)|x|^s\mathrm{d}^\times x \\ &= \int_{F^*}\Phi(x)\chi(ax)|ax|^s\mathrm{d}^\times x = \chi(a)|a|^s\zeta(s,\chi,\Phi) \\ \rho(a)\zeta(1-s,\chi^{-1},\hat{Phi}) &= \int_{F^*}|a|\hat{\Phi}(ax)\chi(x)^{-1}|x|^{1-s}\mathrm{d}^\times x \\ &= \int_{F^*}|a|\hat{\Phi}(x)\chi(a^{-1}x)^{-1}|a^{-1}x|^{1-s}\mathrm{d}^\times x = \chi(a)|a|^s\zeta(1-s,\chi^{-1},\hat{\Phi})\end{aligned}$$

So we've done.

Solution 4.7.2. (a) Suppose ϕ is a finite function on F^*. We may denote V as the vector space spanned by $\phi(ax)$ for all $a \in F^*$. Then for every $a \in F^*$, we may treat $\rho(a)$ as a matrix on $\mathrm{End}(V)$. Since all $\rho(a)$ are commutative with each other, there exists a $P \in \mathrm{End}(V)$ such that $P\rho(a)P$ is in the form of Jordan matrix $\mathrm{diag}\{J_1(a),\ldots,J_k(a)\}$ for all a. We can change the basis of V such that $P = 1$. Then suppose $J_i(a)$ is a $n_i \times n_i$ matrix, since $J_i(a)J_i(b) = J_i(ab)$, we know n_i cannot be greater than 2. So if $n_i = 1$, the $J_i(a) = \lambda_i(a)$ is a character of F^*, which corresponds to a function $\phi(x)$ such that $\phi(ax) = \lambda_i(a)\phi(x)$, so $\phi(x) = \lambda_i(x)$. If $n_i = 2$, $J_i(a) = \left[\begin{smallmatrix}\lambda_i(a) & \lambda_i(a)\log|a|^n \\ & \lambda_i(a)\end{smallmatrix}\right]$, so this corresponds to two functions ϕ_1, ϕ_2 such that $\phi_1(ax) = \lambda_i(a)\phi_i(x) + \lambda_i(a)\log|a|^n \cdot \phi_2(x)$ and $\phi_2(ax) = \lambda_i(a)\phi_2(x)$. So $\phi_2(x) = \lambda_i(x)$, and $\phi_1(x) = \lambda_i(x)\log|x|^n$.

(b) First suppose F is non-archimedean. Clearly $\Omega(y)$ is locally constant with compact support. By theorem 4.7.2., there exists a $\mathfrak{p}^n$ such that $\Omega(y) = c_1|t|^{1/2}\chi_1(t)\mathbf{1}_{\mathfrak{p}^n}(t) + c_2|t|^{1/2}\chi_2(t)\mathbf{1}_{\mathfrak{p}^n}(t)$ or $c_1|t|^{1/2}\chi_1(t)\mathbf{1}_{\mathfrak{p}^n}(t) + c_2 v(t)|t|^{1/2}\chi_1(t)\mathbf{1}_{\mathfrak{p}^n}(t)$. And out of $\mathfrak{p}^n$, Ω is a Schwartz function. So we've done.

Then we may consider the real case. Since $\Omega(y) = W(\left[\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right]) = W(\left[\begin{smallmatrix} |y|^{1/2} & \\ & |y|^{1/2}\end{smallmatrix}\right]\left[\begin{smallmatrix} |y|^{1/2} & \\ & |y|^{-1/2}\end{smallmatrix}\right]\left[\begin{smallmatrix} \mathrm{sgn}(y) & \\ & \mathrm{sgn}(y)\end{smallmatrix}\right]) = |y|^{\mu/2}\mathrm{sgn}(y)^k w(|y|)$, where $w(|y|) = W(\left[\begin{smallmatrix} |y|^{1/2} & \\ & |y|^{-1/2}\end{smallmatrix}\right])$. And $w(t)$ satisfies $w'' + (-\frac{1}{4} + \frac{k}{2t} + \frac{\lambda}{y^2})w = 0$. Since W satisfies the condition of moderate growth, w is a Schwartz function. But $|y|^{\mu/2}\mathrm{sgn}(y)^k$ is a finite function, so we've done.

The complex case is similar with the real case.

(c) First suppose $\xi(y) = \chi(y)$ and F is non-archimedean. We may assume χ is unramified or we just change the variation y to ay for some $a \in F^*$. Then we may separate the integral into two parts $\int_{|y|<1}$ and $\int_{|y|\geq 1}$. The part $\int_{|y|\geq 1}$ is clearly convergent for $\mathrm{Re}(s) \gg 0$. And for the other part, we have

$$\begin{aligned}
\int_{|y|<1} \xi(y)\phi(y)|y|^s \mathrm{d}^\times y &= \int_{\substack{F^*/\mathfrak{o}^* \\ |y|<1}} \left[\sum_{\alpha\in\mathfrak{o}^*} \phi(\alpha y)\right]\xi(y)|y|^s \mathrm{d}^\times y \\
&= \int_{\substack{F^*/\mathfrak{o}^* \\ |y|<1}} \left[\sum_{\alpha\in\mathfrak{o}} \phi(\alpha y)\right]\xi(y)|y|^s \mathrm{d}^\times y - \phi(0)\int_{\substack{F^*/\mathfrak{o}^* \\ |y|<1}} \xi(y)|y|^s \mathrm{d}^\times y \\
&= \int_{\substack{F^*/\mathfrak{o}^* \\ |y|<1}} \left[\sum_{\alpha\in\mathfrak{o}} \phi(\alpha/y)\right]\xi(y)|y|^{s-1} \mathrm{d}^\times y - \phi(0)\int_{\substack{F^*/\mathfrak{o}^* \\ |y|<1}} \cdots \\
&= \int_{\substack{F^*/\mathfrak{o}^* \\ |y|>1}} \left[\sum_{\alpha\in\mathfrak{o}} \phi(\alpha y)\right]\xi(y)^{-1}|y|^{1-s} \mathrm{d}^\times y - \phi(0)\int_{\substack{F^*/\mathfrak{o}^* \\ |y|<1}} \cdots \\
&= \int_{|y|>1} \phi(y)\xi(y)^{-1}|y|^{1-s}\mathrm{d}^\times y + \hat{\phi}(0)\int_{\substack{F^*/\mathfrak{o}^* \\ |y|>1}} \xi(y)^{-1}|y|^{1-s}\mathrm{d}^\times y - \phi(0)\int_{\substack{F^*/\mathfrak{o}^* \\ |y|<1}} \cdots
\end{aligned}$$

So this part is also convergent for $\mathrm{Re}(s) \gg 0$ and we've done.

Suppose $\xi(y) = \chi(y)\log|y|$ and F is non-archimedean. Then the only one different point is at the first equality of above equation, $\xi(\alpha y) = \chi(\alpha)\xi(y) + \log|\alpha|\xi(a)$. So the integral equals to the "inverse" one and more, a $\log|\alpha|$-part, which is convergent by the above case. So we've done for non-archimedean case. And the archimedean case is easy since ϕ is Schwartz.

Solution 4.7.3. (a) The hint is very complete, we only need to prove that if $B_{0,N} = 0$, then $B_0 = 0$. For any $\phi_2 \in V_2 - V_{2,N}$, and any $\phi_1 - \pi_1(\left[\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right])\phi_1 \in V_{1,N}$, we have

$$\begin{aligned}
B_0(\phi_1 - \pi_1(\left[\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right])\phi_1, \phi_2) &= B_0(\phi_1 - \pi_1(\left[\begin{smallmatrix} 1 & x/2 \\ & 1\end{smallmatrix}\right])\phi_1 + \pi_1(\left[\begin{smallmatrix} 1 & x/2 \\ & 1\end{smallmatrix}\right])\phi_1 - \pi_1(\left[\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right])\phi_1, \phi_2) \\
&= B_0(\phi_1 - \pi_1(\left[\begin{smallmatrix} 1 & x/2 \\ & 1\end{smallmatrix}\right])\phi_1, \phi_2 - \pi_2(\left[\begin{smallmatrix} 1 & -x/2 \\ & 1\end{smallmatrix}\right])\phi_2) = 0
\end{aligned}$$

So $B|_{V_{1,N}\times V_2} = 0$. And same for $B|_{V_1\times V_{2,N}}$, which means B_0 actually factors through the projection $V_1 \times V_2 \to J(V_1) \times J(V_2)$. Then since $J(V_1), J(V_2)$ are both finitely dimensional, and π_1, π_2 are not the same, we must have $B(J(V_1), J(V_2)) = 0$. So B is zero.

(b) Since $\mathrm{GL}(2,F) = Z(F)N(F)T_1(F)\mathrm{GL}(2,\mathfrak{o})$, the integral may be taken over $T_1(F)\mathrm{GL}(2,\mathfrak{o})$. For the $\mathrm{GL}(2,\mathfrak{o})$ part, since it is compact, we can decompose it into finitely many disjoint open compact subset to do the integral. And in every part, the integral of $\int_{T_1} W_1(g)W_2(g)f_s(g)\mathrm{d}^\times g$ is a integral of the linear combination of the product of finite function and Schwartz function, by 4.7.2.(c) it is convergent for $\mathrm{Re}(s) \gg 1$. Hence the definition of $Z(W_1, W_2, f_s)$ is well-defined for $\mathrm{Re}(s) \gg 0$, and it has a meromorphic continuation to the whole s-plane. Then clearly $Z(\pi_1(g)W_1, \pi_2(g)W_2, \pi_0(g)f_s) = Z(W_1, W_2, f_s)$ by just changing variable. And clearly it is trilinear.

(c) Since $M(s)\pi_0(g) = \pi_0'(g)M(s)$ for all g, then it is easy to see that $Z(W_1, W_2, M(s)f_s)$ is another trilinear form by the same reason. Then since this space of trilinear form has dimension 1, there exists a $\gamma(s)$ independent of the choice of W_1, W_2 and f_s. And clearly since $Z(W_1, W_2, f_s)/Z(W_1, W_2, M(s)f_s)$ are both meromorphic, then so is the $\gamma(s)$.

4.8 Supercuspidals and the Weil Representation

Solution 4.8.1. Clearly for any $h^* \in H^*$, we have

$$\begin{aligned}\mathcal{F}(\phi \circ \alpha)(h^*) &= \int_H (\phi \circ \alpha)(h)\overline{h^*(h)}\mathrm{d}h \ = \int_H \phi(\alpha h)\overline{h^*(h)}\mathrm{d}h \\ &= \int_H \phi(h)\overline{h^*(\alpha^{-1}h)}|\alpha|^{-1}\mathrm{d}h \ = |\alpha|^{-1}(\mathcal{F}\phi) \circ \alpha^{*-1}(h^*)\end{aligned}$$

So clearly $|\alpha^*| = |\alpha|$.

4.9 The Local Langlands Correspondence

No exercise here.