Hartshorne Solutions

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This document is the a solution of Hartshorne's *Algebraic Geometry* by me during I was learning the AG. During these exercises, I mainly referred to *Algebraic Geometry* and *Étale Cohomology Theory* by Prof. Fu Lei (扶磊), *Algebraic Geometry and Arithmetic Curves* by Prof. Qing Liu (刘青), Prof. Aise Johan de Jong's Stack Project, Andrew Egbert's solution I googled, and all related information in website Stackexchange and Mathoverflow. I also need to thank my friends Zio Khái, Tsjiong ʔit-Miaeng, Kwèi Tsjièng-Biaeng, Wiang Pyin, Ghrà Meng-Dzjin, Lý Bwai-Kon and Mràeng Dzjieng. They helped me a lot when I did these exercises. By the way, ils peuvent également être la mesure pour la recherche de petit ami. Sed bedaŭrinde ili estas ĉiuj malgejaj, so kann jede Dame, die die Absicht von ihnen hat, mich mailen.

At first sheang-je prá ĉiujn solvojn 都 사 hàlái 主要 shyh kràutok mī'self 淆 μαθη' d dzjyghòu tay yō-i sheang-tangnjien originally, 需要 sjý-triò kak- 청조 tsair-hao. Nyan-I-上-t dzjy-ghòu 跟 Biu lao-shy 淆tɕí AG, tā njiàng ngǒ kaj Bwai-Kon 在 1. Okt. chitiền tzyyjii 淆-i-淆 Harshorne-t dih-1-jang béngtshiá bǎ dei-miuk 做-1-做 ankaŭ, ŭ́-shì ngǒ dziòu 做-l-1-hah. Hartshorne-t ghónta liannshyi-tyi dèi-ŭ́ ʃo-ʃīn-ʃa lai-shuo ghón mal-(hao-zù), 在 char top'àn dī dzjyghòu 也 huāshyàn ghónta neng-dzratàu-d top'àn dōu yeou sak^{h}ɯgo or deubuò tre serioze. Gleichzeitig, ngǒ 還在 shanq Looijenga-giáoþụ-d AG-course, 偲lao-d khwà xiān gǎng-l 1-kà 淆-ki d variety theory, kī-mạt d dzjyghòu wǒ damsim 偲 yaw kháu ki-mạt kusi (pitkiàeng 頭-1-nen d shyr-how 偲lảo-d kimạt kháusjý 9.a.m. kara gogogoji mate shiken 습니다, dziòu 做-l-做 dì-1-jang-d dei-miuk again, siáng-triò njiokwá 要把 jychyan ne-trouve-pas-top'àn-d dei-miuk shiee hà-lái, 就 đảthành-l LaTeX.

Nyan-1-上-d shyr-how tshom-kra-l le séminaire du Biu sen-sei, gaang 偲-d nà-pwón AG-livre, hence jiá shiangtang-ŭ́ 淆-l Hartshorne'd dì-2-tsjiang dzen-5-tset tek nội-iong. Nyan-1-hà-d shyr-how 월발-dì jiódé dzìký một 淆doong, một-năng baa Biu laoshy nà-tàu kanzen-dàisriù-d lingo trwyénhrwà-wei jiiher-d sysiáng, jiá một-pháp iòng kiígha-d drikkwan lai sʅkháu moidai. Ŭ́-dzjié 在 shyo♂jaang tek kiàn♂ngyè 下 dziòu khaiʂʅ thatriò Bwai-Kon mann-mann 做 Hartshorne'd tyimuh, swi-njien tha 후래 betray-l ŋǒ.

Kangdzai 也 gǎng-guò, Hartshorne jehbeen-sjio-d deimiuk 對 ʃo-ʃīn-ʃa laisjwiet 很 mal-facile tsuò. Tā baa EGA-SGA1-que-d nwài-jiong nōʃɯk^{h}ɯ 成-l 1-been shū, but kràeng-ta kràeng-drióng'iàu kràeng-ióutshiù d dongshi 偲 dawshyh tuo tshjwit-dz-l shyityi lý, keeyii sjwiet 是 très mwot-ióu tset-tshàu l. 而且, ŋ̥iok^{w}á jyyshyh tímù píkràu mal-facila tekghrwàj yeejiow 算-l, la teksto du livre 裏面 ghrwan ĉiam jín-jiòng deimiuk-t jyegwo, pwiit 1-pen-tsuòdei-1-pen-淆-d huah 근본 淆-bù-l, 1-pen-tsuòdei-1-pen-淆-d huah 근본 做-bú-ghrákhiò. Đis kréndrik 不是 mwot-tset-tshàu-d wen-ntyi, kréndrik keeyii sjwiet 是 sehr muómràiP l. Tsàitsjiásjwiet, Hartshorne jehbeen-sjio-t ghòu-2-

jang xiě-der lwàntshitprettsau-t, 如果 tsjintek sheang 淆 curves and surfaces tek ney-yong jiá-ióu ACGH-d *Geometry of Algebraic Curves* kaj Arnaud Beauville-t *Complex Algebraic Surfaces* keeyii 看, hair pwiitnjio kuoh-사 1-hà biùliok-t 3-tsjiang nèiyóng, tsjìsjiáu 把 shiangjiau lýlwòn duō-사-tém, or giùthéi jiehshaw 1-hah étale topology-t tungsei, shyityi lýmièn 連 elliptic curves d kypwón-gwiin douyaw swán, 사々 chingchuu 又 pwiitghwàj tzeenyanq. Sjwietbraek-l, Hartshorne jehbeen sjio genbeen 不 sjiekghop ʃoŋakʰɯ. 真 'iàu tshrio-淆 AG tekghrwàj kiàn-ngì yonq Biu laoshy nàp-wón AG book, njiokwá 迺 pwiit-jiòng kè gaang s tháulwònpran d 話.

Tswàjghòu, ŋǒ 還是 'iàu gaanshieh Hartshorne honnêtement: īnuèi 在 사 tsjiàphien darann-d references d dzjyghòu ngă shyrtzay njín-pwiit-triù 要 thuódzau, thus mwotbrènpwiap jiow dzáu-l 1-thàu pīnyīn ghèithuòng based on the Kwáng-Wiìn-'yimghèi. Cette Gedanke ŋá pwónkhwa shyrhow 就 yeou-l, 最後 khiakdzjié Hartshorne 치-háu-l mia procrastination.

I am not entirely sure that all my solutions are correct. So if anybody finds any mistakes in my solutions, please e-mails me. My e-mail address is under my name at the beginning of this document, you can see it even if you cannot understand the meaning of the above gibberish.

最後，做了一點微小的工作，謝謝大家！

Content

1 Varieties

1.1 Affine Varieties

Solution 1.1.1. (a) $A(Y) = k[x,y]/(x^2-y) = k[x,x^2] \cong k[x]$.

(b) $A(Z) = k[x,y]/(xy-1) = k[x, \frac{1}{x}] \cong k(T)$. Then we may assume $x = f(T)$ and $\frac{1}{x} = g(T)$ both polynomial, and

$$1 = x \cdot \frac{1}{x} = f(T)g(T)$$

So $\deg(fg) = 0$, i.e. $\deg(f) = \deg(g) = 0$, which makes a contradiction.

(c)Assume that $p = ax^2 + bxy + cy^2 + dx + ey + f$. Then we will consider the following conditions.

(1) If $b^2 = 4ac$, we have $ax^2 + bxy + cy^2 = (\sqrt{a}x + \sqrt{c}y)^2$. If we write $t = \sqrt{a}x + \sqrt{c}y$, we have $p = t^2 + \tilde{d}t + \tilde{e}y + f$, where $\tilde{e}$ might be zero. We take t as $t - \frac{\tilde{d}}{2}$, then $p = t^2 + \tilde{e}y + \tilde{f} = t^2 + s$, where $s = \tilde{e}y + \tilde{f}$.

If $s = 0$ or a constant, then we know that p is line or lines, which is not a conic. If s involved a variable, then

$$A(W) = k[t,s]/(t^2+s) \cong k[t].$$

(2) If $b^2 \neq 4ac$, similarly we can change the coordinate as $p = t^2 + s^2 + g$. If $g = 0$, $p = (t + \sqrt{-1}s)(t - \sqrt{-1}s) = 0$ is two lines. If $g \neq 0$, we may assume $h = -1$ by changing coordinate, and

$$A(W) = k[t,s]/(t^2+s^2-1) = k[t+\sqrt{-1}s, t-\sqrt{-1}s]/(t^2+s^2-1) = k[x,y]/(xy-1) = A(Z).$$

Solution 1.1.2 (The Twisted Cubic Curve)**.** Clearly $I(Y) = (y - x^2, z - x^3)$, and $A(Y) = k[x,y,z]/I(Y) = k[x,x^2,x^3] \cong k[x]$. So $\dim Y = \dim A(Y) = 1$.

Solution 1.1.3. If $x \neq 0$, then $xz - x = 0$ means $z = 1$, so $x^2 - yz = x^2 - y$. Then $k[x,y,z]/(z-1, x^2-y) = k[x,x^2] \cong k[x]$. So we can denote X_1 as $Z(xz - x, x^2 - yz, z - 1)$, then X_1 is irreducible and $X_1 \subsetneq Y$.

If $x = 0$, then $xz - x = 0$ means nothing, and $x^2 - yz = 0$ means $yz = 0$. Then $Z(yz) = Z(y) \cup Z(z)$. Then we may denote $X_2 = Z(x,y)$ and $X_3 = Z(x,z)$, and we have $Y \cap Z(x) = X_2 \cup X_3$. And obviously X_2 and X_3 are irreducible.

Solution 1.1.4. Just need to find a closed set in $\mathbb{A}^2$ which cannot be treated as product of two closed set in $\mathbb{A}^1$. Define X as the zeros of $x = y$ in $\mathbb{A}^2$. Then if $X = Y \times Z$ for some closed sets Y and Z in $\mathbb{A}^1$, we may find $a \neq b \in \mathbb{A}^1$, and $(a,a), (b,b) \in \mathbb{A}^2$, but (a,b) and (b,a) are not in X, which makes a contradiction.

Solution 1.1.5. If B is a finitely generated algebra over k, it can be written as $k[x_1, \ldots, x_n]/\mathfrak{a}$ for some integer n and ideal $\mathfrak{a} \in k[x_1, \ldots, x_n]$. Since B has no nilpotent, then $\mathfrak{a}$ is radical. Denoting $Y = Z(\mathfrak{a})$, we have $A/I(Y) = A/\sqrt{\mathfrak{a}} = A/\mathfrak{a} = B$.

Conversely, If $B = k[x_1, \ldots, x_n]/I(Y)$ for some n and Y, then B is clearly finitely generated. Since $I(Y)$ is radical, B is reduced.

Solution 1.1.6. (a) Firstly, $X = \bar{Y} \cup (X - Y)$. Since X is irreducible, we have $X = \bar{Y}$ or $X = X - Y$. Since Y is non-empty, we have $X = \bar{Y}$, i.e. Y is dense in X. Secondly, if $Y = (Y \cap Y_1) \cup (Y \cap Y_2)$ for some closed Y_1 and Y_2, we know that $X = Y_1 \cup Y_2 \cup (X - Y)$. Similarly, $X = Y_1$ or $X = Y_2$ or $X = X - Y$. Hence $X = Y_1$ or $X = Y_2$, i.e. Y is irreducible.

(b) If $\bar{Y} = Y_1 \cup Y_2$ for some closed Y_1 and Y_2, then we have $Y = (Y \cap Y_1) \cup (Y \cap Y_2)$. Since Y is irreducible, we have $Y = Y \cap Y_1$ or $Y = Y \cap Y_2$. We may assume $Y = Y \cap Y_1$, then $Y \subset Y_1$, i.e. $\bar{Y} \subset Y_1$. Hence $\bar{Y} = Y_1$, which means $\bar{Y}$ is irreducible.

Solution 1.1.7. (a) The (i$\Leftrightarrow$iii) and (ii$\Leftrightarrow$iv) is trivial. Then we only need to prove the (i$\Leftrightarrow$ii).

(i⇒ii): If $\{X_\alpha\}_{\alpha\in I}$ is a set of closed subset of X, we can pick some X_1 in it. If X_1 is minimal, it's proved. If not, we may find some $X_2 \subset X_1$, then we do the same discuss of X_2 and so on. Then we may find a chain $X_1 \supset X_2 \supset \dots$. Since X is noetherian, we have some $X_n = X_{n+1} = \dots$, then X_n is minimal.

(ii⇒i): If $X_1 \supset X_2 \supset \dots$, then $\{X_i\}$ is a set of closed subset of X, then $\{X_i\}$ has minimal one, namely X_n. Then $X_n \supset X_{n+1} \supset \dots X_n$ implies $X_n = X_{n+1} = \dots$, i.e. X is noetherian.

(b) If X has an open covering $\{X_\alpha\}_{\alpha\in I}$, then we may define $Y = \{\bigcup_{finitecovering} X_i | X_i \in \{X_\alpha\}\}$. Since Y is a set of open subsets of X, then Y has a maximal element, i.e. $U = X_1 \cup \dots \cup X_n$. Since U is maximal, if $U \subsetneq X$, we can find some $X' \in \{X_\alpha\}_{\alpha\in I}$ such that $X' \cap (X - U) \neq \emptyset$. Then $U' = X_1 \cup \dots \cup X_n \cup X' \supsetneq U$, which is contradict with the maximality of U. Hence $U = X$, i.e. X has a finite covering.

(c) If $Y \subset X$ is a subset, and $Y \cap U_0 \subset Y \cap U_1 \supset \dots$ is an ascending chain of open sets in Y for some U_i upen in X. Since X is noetherian, we have $U_n = U_{n+1} = \dots$, then $Y \cap U_n = Y \cap U_{n+1} = \dots$, then Y is noetherian.

(d) We firstly prove that X has a discrete topology. For any closed subset $Y \subset X$, we know $X - Y$ is open. Then for any $y \in Y$ and $z \in X - Y$, we have some open sets U_{yz} and V_{yz} such that $y \in U_{yz}$, $z \in V_{yz}$ and $U_{yz} \cap V_{yz} = \emptyset$. We may assume $V_{yz} \in X - Y$, since we may change V_{yz} as $V_{yz} \cap (X - Y)$. Then $X - Y = \bigcup_{z\in X-Y} V_{yz}$ for this fixed y, then this covering has a finite subcovering, i.e. $X - Y = \bigcup_i V_{yz_i}$. We may define an open set $U_y = \bigcap_i U_{yz_i}$, then we have $U_y \cap (X - Y) = 0$, i.e. $U_y \subset Y$. Then $Y = \bigcup_{y\in Y} U_y$ is an open covering, hence it has a finite subcovering, i.e. Y is an union of finite open set, namely Y is open. Hence every closed set is open, X has a discrete topology.

Secondly we will prove X is finite. Clearly if X is not finite, we have an infinite ascending chain $\{x_1\} \subset \{x_1, x_2\} \subset \dots$, which is contradict with that X is noetherian.

Solution 1.1.8. We may assume $H = I(f)$, and Z is an irreducible component of $Y \cap H$. Denote $\bar{f}$ as the image of f under $A \twoheadrightarrow A/I(Y)$, then $Y \not\subseteq H$ means $(f) \not\subseteq I(Y)$, i.e. $f \notin I(Y)$, hence $\bar{f}$ is not a zero-divisor. Since we have prime ideal $\mathfrak{p}$ in $A(Y)$ containing $\bar{f}$, then if $\bar{f}$ is a unit, then $\mathfrak{p} = (1)$, which makes a contradiction. Thus $Y \cap H \neq \emptyset$.

Since $I(Z)$ is prime ideal in A, then the image of $I(Z)$ under $A \to A(Y)$, as we denote as $\mathfrak{p}$, is a prime, and contains $\bar{f}$. By irreducibility of Z, we have that $\mathfrak{p}$ is minimal over $(\bar{f})$. Then by Hauptidealsatz, height $\mathfrak{p} = 1$, i.e. $\dim z = \dim A(Y)/\mathfrak{p} = \dim A(Y) - \text{height } \mathfrak{p} = r - 1$.

Solution 1.1.9. If $Z(\mathfrak{a}) = \bigcup Y_i$, then we only need to prove that height $I(Y_i) \leq r$, where we know that $I(Y_i)$ is the minimal prime ideal over $\mathfrak{a}$. Then by Krull's height lemma, we have height $I(Y_i) \leq r$. where we used that $\mathfrak{a}$ is generated by r elements.

Solution 1.1.10. (a) Any chain in Y is also a chain in X, trivial.

(b) By (a), we just need to prove that $\dim X \leq \sup \dim U_i$. If $X_0 \subset X_1 \subset \dots \subset X_n$ is a chain in X, where $n = \dim X$. So X_0 is just a point, then there must have some U_i such that $X_0 \subset U_i$. Thus for every X_j, we have $X_j \cap U_i \neq \emptyset$. Then $X_j \cap U_i$ is irreducible and dense in X_j. So $X_0 \cap U_i \subset \dots X_n \cap U_i$ is a chain in U_i, then $\dim X \leq \dim U_i \leq \sup \dim U_i$.

(c) $X = \{0, 1\}$ with open sets $\emptyset, \{1\}, \{0, 1\}$. Then $\dim X = 2$, but $\dim\{1\} = 1$.

(d) If $Y \neq X$, we know any chain $Y_0 \subset Y_1 \subset \dots \subset Y_n$ in Y can be expressed as $Y_0 \subset \dots \subset Y_n \subset X$ in X, which is contracted with that $\dim Y = \dim X$.

(e) Take $X = \mathbb{Z}$, and closed set of X are all finite subsets. Then $\{0\} \subset \{0, 1\} \subset \dots$ means that $\dim X = \infty$. And every descent closed chain of X has a finite beginning, thus the chain if finite.

Solution 1.1.11. Firstly we will prove that $\dim Y = 1$. As we can define a homomorphism $\varphi : \mathbb{A}^1 \to Y$, $s \mapsto (s^3, s^4, s^5)$. If $\varphi(s) = \varphi(t)$, then clearly $s = t$, i.e. φ is a bijection. Thus $\dim Y = \dim \mathbb{A}^1 = 1$.

Secondly we will prove that $I(Y)$ cannot be generated by two elements. If $f \in A$, and $f(t^3, t^4, t^5) = 0$, we may assume $f = \sum a_{ijk} x^i y^j z^k$. Then

$$0 = \sum a_{ijk} t^{3i+4j+5k},$$

hence $\sum_{3i+4j+5k=s} a_{ijk} = 0$. So all polynomial satisfying this make up $I(Y)$. When $s = 5$, we have $a_{001} = 0$. When $s = 10$, we have $a_{210} + a_{002} = 0$. So by calculation, $y^2 - xz, x^3 - yz, x^2y - z^2 \in I(Y)$, and they cannot be generated by two elements.

Solution 1.1.12. Let $f = (x^2 - 1)^2 + y^2$. Then f is irreducible in $\mathbb{R}[x, y]$. But $Z(f)$ is two points.

1.2 Projective Varieties

Solution 1.2.1. If $Z(\mathfrak{a}) = \varnothing$, we know that $Z(\mathfrak{a})$ in $\mathbb{A}^{n+1}$ is $\varnothing$ or $\{0\}$. Both cases are trivial. If $Z(\mathfrak{a}) \neq \varnothing$, we know in $\mathbb{A}^{n+1}$, $Z(\mathfrak{a})$ is a cone of projective $Z(\mathfrak{a})$. If $f(p) = 0$ for all p $\in$projective $Z(\mathfrak{a})$, then in affine $Z(\mathfrak{a})$, $f(\lambda p) = 0$ for all p $\in$affine $Z(\mathfrak{a})$ and $\lambda \in k$. Thus $f(p) = 0$ for all $Z(\mathfrak{a}) - \{0\}$. But when f is homogeneous and $Z(\mathfrak{a}) \neq \varnothing$ in $\mathbb{P}^n$, f has no constant term, thus $f(0) = 0$ in $\mathbb{A}^{n+1}$, which means f is vanishing in $Z(\mathfrak{a})$. Thus by affine Nullstellensatz, we have $f^q \in \mathfrak{a}$ for some q, which means the homogeneous Nullstellensatz.

Solution 1.2.2. (i$\Rightarrow$ii): $Z(\mathfrak{a}) = \varnothing$, then in $\mathbb{A}^{n+1}$ we have $Z(\mathfrak{a}) = \varnothing$ or $\{0\}$. Then in the first case, $\mathfrak{a} = S$, i.e. $\sqrt{\mathfrak{a}} = S$. In the second case we have $\sqrt{\mathfrak{a}} = I(\{0\}) = S_+$.

(ii$\Rightarrow$iii): If $\sqrt{\mathfrak{a}} = S$, then $\mathfrak{a} = S$, then trivial. If $\sqrt{\mathfrak{a}} = S_+$, then for every x_i, we have $x_i^{r_i} \in \mathfrak{a}$. Thus taking $r = \max\{r_i\}$, we know that $x_i^r \in \mathfrak{a}$, i.e. $S_{r(n+1)} \subset \mathfrak{a}$ by pigeonhole principle.

(iii$\Rightarrow$i): If $\mathfrak{a} \supset S_d \supset \{x_0^d, \dots, x_n^d\}$, then $Z(\mathfrak{a}) \subset Z(x_0^d, \dots, x_n^d) = \varnothing$.

Solution 1.2.3. (a) If $p \in Z(T_2)$, then T_2 vanishes on p, i.e. T_1 vanishes on p, hence $p \in Z(T_1)$.

(b) If $f \in I(Y_2)$, then f vanishes on Y_2, i.e. vanishes on Y_1, hence $f \in I(Y_1)$.

(c) By (b), we have $I(Y_1 \cup Y_2) \subset I(Y_1) \cap I(Y_2)$. Conversely, if $f \in I(Y_1) \cap I(Y_2)$, then f vanishes on both Y_1 and Y_2, i.e. f vanishes on $Y_1 \cup Y_2$. Hence $I(Y_1) \cap I(Y_2) \subset I(Y_1 \cup Y_2)$.

(d) If $Z(\mathfrak{a}) \neq \varnothing$, then we know that $Z(\mathfrak{a})$ in $\mathbb{A}^{n+1}$ is not only $\{0\}$, i.e. $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

(e)Obviously, $Y \subset Z(I(Y))$, and $Z(I(Y))$ is closed, then $\bar{Y} \subset Z(I(Y))$. Conversely, if $W \supset Y$ is closed and $W = Z(\mathfrak{a})$, we have $\mathfrak{a} \subset I(Z(\mathfrak{a})) \subset I(Y)$, i.e. $W \supset Z(I(Y))$. Hence we have $Z(I(Y)) = \bar{Y}$.

Solution 1.2.4. (a) If Y is closed, then $Y = \bar{Y} = Z(I(Y))$. Conversely, if $\mathfrak{a}$ is a radical homogeneous ideal, and if $Z(\mathfrak{a}) \neq \varnothing$. Then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$. If $Z(\mathfrak{a}) = \varnothing$, then $\mathfrak{a} = S$ or S_+. But $\mathfrak{a} \neq S_+$ by hypothesis, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$.

(b) ($\Rightarrow$): If $f, g \in I(Y)$, then $Z(fg) = Z(f) \cup Z(g) \supset Y$. Then $Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$. Since Y is irreducible, we may assume $Y = Y \cap Z(f)$, i.e. $Y \subset Z(f)$, then $f \in I(Y)$.

($\Leftarrow$): If $Y = Y_1 \cup Y_2$, then $I(Y) = I(Y_1) \cap I(Y_2)$. Since $I(Y)$ is prime, we may assume $I(Y) = I(Y_1)$, i.e. $Y = Y_1$.

(c) Since $I(\mathbb{P}^n) = (0)$ is prime, obviously.

Solution 1.2.5. (a) If $Y_1 \supset Y_2 \supset \dots$ is a descent chain in $\mathbb{P}^n$, then $I(Y_1) \subset I(Y_2) \subset \dots$ in S is an ascent chain, hence stable.

(b) Write S as all irreducible projective varieties which cannot be written as union of finite irreducible varieties in $\mathbb{P}^n$, then S has a minimal element since $\mathbb{P}^n$ is noetherian, namely Y. Y is reducible, thus $Y = Y_1 \cup Y_2$ for some variety Y_1 and Y_2 such that $Y_1 \subsetneq Y$ and $Y_2 \subsetneq Y$. Then $Y_1, Y_2 \notin S$. So Y_1, Y_2 can be written as a union of finite irreducible varieties, so does Y, which makes a contradiction.

Solution 1.2.6. Consider $\varphi_i : U_i \to \mathbb{A}^n$ in (2.2), then $\varphi_i(U_i \cap Y)$ is an affine variety. Then if $x_i \in I(Y)$, then $x_i = 0$ in $S(Y)$. So $S(Y)_{x_i}$ is trivial. If $x_i \notin I(Y)$, we know that $S(Y)_{x_i}$ consists of all elements $\frac{f}{x_i^t}$ for some homogeneous polynomial f and positive integer t. Then $(S(Y)_{x_i})$ consists of all $f(\frac{x_0}{x_i}, \dots \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i})$. So we can define an $\alpha : (S(Y)_{x_i})_0 \to A(Y_i)$ and $\alpha^{-1}(f) = x_i^{-\deg f} f$. Thus $A(Y_i) \cong (S(Y)_{x_i})_0$. Then

$$S(Y)_{x_i} = (S(Y)_{x_i})_0[x_i, x_i^{-1}] \cong A(Y_i)[x_i, x_i^{-1}].$$

Thus $\dim S(Y) = \deg K(S(Y)) = \deg K(S(Y)_{x_i}) = \deg K(S(Y)_{x_i})_0 + 1 = 1 + \deg K(A(Y_i)) = 1 + \dim Y \cap U_i = 1 + \dim Y$.

Solution 1.2.7. (a) $S(\mathbb{P}^n) = S$, then by $\dim S = n+1$ we have $\dim \mathbb{P}^n = n$.

(b) Denoting $Y_i = Y \cap U_i$, we have $\bar{Y}_i = \bar{Y} \cap U_i$, then $\dim Y = \dim Y_i = \dim \bar{Y}_i = \dim \bar{Y}$.

Solution 1.2.8. ($\Leftarrow$): If $Y = Z(f)$, then $S(Y) = S/(f)$. By Hauptidealsatz we have height $(f) = 1$, then $\dim S(Y) = \dim S - \text{height } (f) = n$, hence $\dim Y = n-1$.

($\Rightarrow$): If $\dim Y = n-1$, then $\dim S(Y) = n$, and $I(Y)$ is prime. Then height $I(Y) = \dim S - \dim S(Y) = 1$, i.e. $I(Y)$ has a single non-constant generator g, then $I(Y) = (g)$, i.e. $Y = Z(g)$.

Solution 1.2.9 (Projective Closure of Affine Variety)**.** (a) If $f \in I(\bar{Y})$, we may define $g = f(1, x_1, \dots, x_n)$, i.e. $f = \beta(g)$, which means $g \in I(Y)$. Thus $I(\bar{Y}) \subset \beta(I(Y))$. Conversely, if $g \in I(Y)$, then trivially $f = \beta(g) \in I(\bar{Y})$.

(b) In 1.1.2.(a) we have proved that $I(Y) = (y - x^2, z - x^3)$. Then we may denote $f_1 = y - x^2$, $f_2 = z - x^3$. Clearly, $\bar{Y} = \{(s^3, s^2t, st^2, t^3\} \subset \mathbb{P}^3$. Then $x^3 = zw^2$, $y^3 = z^2w$. We have that $I(\bar{Y}) \supset (z^2w - y^3)$. But $\beta(f_1) = yw - x^2$, $\beta(f_2) = zw^2 - x^3$, and $z^2w - y^3 \notin (yw - x^2, zw^2 - x^3)$.

Solution 1.2.10 (The Cone Over a Projective Variety)**.** (a) Clearly $C(Y)$ is also the zero of $I(Y)$, thus an algebraic set in $\mathbb{A}^{n+1}$, and ideal is also $I(Y)$.

(b) ($\Rightarrow$) If $Y = Y_1 \cup Y_2$, we know that $C(Y) = C(Y_1 \cup Y_2) = C(Y_1) \cup C(Y_2)$. If $C(Y) = C(Y_1)$ then $\theta(C(Y)) = \theta(C(Y_1))$, i.e. $Y = Y_1$.

($\Leftarrow$) If Y is irreducible, then $I(Y)$ is prime. So $C(Y)$ is irreducible.

(c) $\dim C(Y) = \dim S(Y) = \dim Y + 1$.

Solution 1.2.11 (Linear Varieties in $\mathbb{P}^n$)**.** (a) ($\Rightarrow$) If $I(Y) = (f_1, \dots, f_r)$, then every point in Y satisfies f_i, i.e. Y is the intersection of hyperplane $Z(f_i)$.

($\Leftarrow$) If Y is the intersection of $Y_1, \dots, Y_r$, then we have $I(Y) = (Z(Y_1), \dots, Z(Y_r))$, and $Z(Y_i)$ is an ideal generated by a single linear polynomial.

(b) Write $Z(Y_i) = (f_i)$. Since $f_1, \dots, fr$ are linear independent. Then height $I(Y) = r$, which means $\dim Y = n - r$.

(c) If $r+s-n \geq 0$, we may consider the cone $C(Y)$ and $C(Z)$ in $\mathbb{A}^{n+1}$, and $\dim C(Y) = r+1$, $\dim C(Z) = s+1$. Then we have $\dim(C(Y) \cap C(Z)) \geq r+s+1-n \geq 1$. Then $C(Y) \cap C(Z) \neq \varnothing$ with dimension ≥ 1, i.e $C(Y) \cap C(Z)$ is not only $\{0\}$, hence $Y \cap Z \neq \varnothing$.

Solution 1.2.12 (The d-Uple Embedding)**.** (a) Since the image of θ is all polynomial in $k[x_0^d, \dots, x_n^d]$, which is a domain. Thus kernal of θ is prime.

(b) If $k \in \ker \phi$, then $f(M_0, \dots, M_N) = 0$. Thus $\mathrm{Im}\rho_d \subset Z(\mathfrak{a})$. If $f \in I(\mathrm{Im}(\rho_d))$, we know $f(x) = 0$ for all $x \in \mathrm{Im}(\rho_d)$, i.e. $f(M_0, \dots, M_N) = 0$. Thus $I(\mathrm{Im}(\rho_d)) \subset \ker \phi$.

(c) Clearly ρ_d is injective. And in (b) we proved that $\mathrm{Im}\rho_d = Z(\mathfrak{a})$. Thus ρ_d is homomorphism.

(d) All monomials in degree 3 in 2 variables are $x_0^3, x_0^2x_1, x_0x_1^2, x_1^3$. Thus $\rho_d(x_0, x_1) = (x_0^3, x_0^2x_1, x_0x_1^2, x_1^3)$.

Solution 1.2.13. If a curve in $\mathbb{P}^2$ defined by $f(x,y,z) = 0$, maps into Y, then we know that f^2 can be treated as a polynomial of $x^2, y^2, z^2, xy, yz, zx$, which we denote as $g(x^2, y^2, z^2, xy, yz, zx)$. Thus $Z = \rho_2(Z(f)) = Y \cap Z(g) = Y \cap V$.

Solution 1.2.14 (The Segre Embedding)**.** Write points of $\mathbb{P}^N$ as c_{ij} as order to write ψ as $c_{ij} = a_ib_j$. Then the equations of $\mathrm{Im}\psi$ is $c_{ij} \cdot c_{kl} = c_{il} \cdot c_{kj}$ for all i, j, k, l. Hence the image of ψ is a subvariety of $\mathbb{P}^N$.

Solution 1.2.15 (The Quadric Surface)**.** (a) We know $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, $(a_0, a_1, b_0, b_1) \mapsto (a_0b_0, a_1b_1, a_0b_1, a_1b_0)$. Then by 1.2.14. we know that the equation of $\mathrm{Im}\psi$ is just $xy - zw = 0$.

(b) $\{L_t\}$ are given by all $\psi(\mathbb{P}^1 \times \{p\}$ for all $p \in \mathbb{P}^1$. And similarly $\{M_t\}$ are given by all $\psi(\{p\} \times \mathbb{P}^1)$.

(c) Denote the curve $x - y = 0$ in Q as Y, thus the same as 1.1.4.

Solution 1.2.16. (a) If $p = (x, y, z, w) \in Q_1 \cap Q_2$, then P satisfies $x^2 = yw$ and $xy = zw$. So $y^2w = x^2y = xzw$, i.e. $x = w = 0$ or $y^2 = xz$, which is a union of a line and a cubic curve.

(b) If $p = (x, y, z) \in C \cap L$. Then p satisfies $x^2 - yz = 0$ and $y = 0$. So $x = 0$ and $z = 1$. Thus p has only one solution, i.e. $I(p) = (x, y)$. Moreover, $I(C) = (x^2 - yz)$, $I(L) = (y)$, and $I(C) + I(L) = (x^2, y) \neq I(p)$.

Solution 1.2.17 (Complex intersections). (a) We treat $\mathfrak{a}$ as a ideal in $k[x_0, \dots, x_n]$, then it defines a $Y_{\mathfrak{a}}$ in $\mathbb{A}^{n+1}$ as the cone of Y. By 1.1.9., we have that $\dim Y_{\mathfrak{a}} \geq n + 1 - q$, i.e. $\dim Y = \dim Y_{\mathfrak{a}} - 1 \geq n - q$.

(b) If $I(Y) = (f_1, \dots, f_r)$, we may denote $Y_i = Z(f_i)$. Firstly, if $p \in Y$, we know that $I(Y)$ on p is 0. Then $f_i(p) = 0$, i.e. $p \in \bigcap Z(f_i)$. Secondly, if $p \in \bigcap Z(f_i)$, clearly we have $f_i(p) = 0$, i.e. $p \in Y$.

(c) Let Y be all $(s^3, s^2t, st^2, t^3) \in \mathbb{P}^3$. We have $I(Y) = (xy - zw, zw^2 - x^3, xz - y^2)$. And let $H_1 = Z(xy - zw)$, and $H_2 = Z(zw^2 - x^3)$, we have $Y = H_1 \cap H_2$.

1.3 Morphisms

Solution 1.3.1. (a) In 1.1.1.(c) we have proved that for any conic W in $\mathbb{A}^2$, we have $A(W) \cong A(Y)$ or $A(Z)$, where $Y \cong \mathbb{A}^1$ and $Z \cong \mathbb{A}^1 - \{0\}$. Then by corollary 3.7., trivially.

(b) If Y is a proper open subset of $\mathbb{A}^1$, we may assume $Y = \mathbb{A}^1 - \{y_1, \dots, y_n\}$. Defining $Z = Z(x \cdot \prod(y - y_i) - 1)$, we have $Y \cong Z$. Then $A(Z) = k[x, y]/(x \cdot \prod(y - y_i) - 1) = k[y, \frac{1}{\prod(y - y_i)}]$. But $\frac{1}{\prod(y - y_i)} \notin k[y] \cong A(\mathbb{A}^1)$.

(c) Since the automorphism group of $\mathbb{P}^2$ acts transitively on any sets of 3 points which are not on a line, we may assume that the conic contains $(0, 0, 1), (0, 1, 0), (1, 0, 0)$, i.e. the conic is of the form like $axy + byz + czx = 0$ for some a, b, c. So changing $x \mapsto \frac{x}{ac}, y \mapsto \frac{y}{ab}, z \mapsto \frac{z}{bc}$, we have $xy + yz + zx = 0$, which is the image of 2-uple embedding of $\mathbb{P}^1$, i.e. all conics are isomorphic to $\mathbb{P}^1$.

(d) We will prove that any two curves in $\mathbb{P}^2$ have nonempty intersection in 1.3.7., but it's clearly wrong in $\mathbb{A}^2$.

(e) Denote this variety by Y, then $\mathscr{O}(Y) = k$. But if Yhas more than one point. Then we have a function wich can differ those two points, hence more than only k. Thus Y has only one point.

Solution 1.3.2. (a) We have $\varphi^{-1}(x, y) = t = \frac{y}{x}$, and $\varphi^{-1}(0, 0) = 0$. Then clearly φ is bijection and bicontinuous. But φ^{-1} cannot be written as a polynomial of x and y, thus cannot be an isomorphism.

(b) If injection of φ follows from the definition and the character p of the field. And the surjection of φ follows from the perfectness of k, but this inverse function cannot be an isomorphism.

Solution 1.3.3. (a) If f is regular at $\varphi(P)$, then f is regular at a neighborhood $\varphi(P) \in U$. So $f \circ \varphi$ is regular at neighborhood $P \in \varphi^{-1}(U)$, which means φ induces a homomorphism $\varphi^* : \mathscr{O}_{\varphi(P),Y} \to \mathcal{P}_{P,X}$.

(b) ($\Rightarrow$) If φ is an isomorphism, it clearly a homeomorphism. So φ and φ^{-1} induce φ^* and $\varphi^{*\,-1}$ like in (a), hence a isomorphism.

($\Leftarrow$) Take $\psi_i : (x_1, \dots, x_n) \mapsto x_i$, then φ^{-1} is defined by all $\psi_i \circ (\varphi^*)^{-1}$.

(c) If $\varphi_P^*(f) = 0$, then $f|_{\varphi(x) \cap V} = 0$. So by density of φ, we have $f = 0$.

Solution 1.3.4. Just construct the inverse, for $x_0 \neq 0$,

$$\begin{aligned} \rho_d^{-1} &: \rho_d(\mathbb{P}^n) \to \mathbb{P}^n \\ (M_0, \dots, M_N) &\mapsto (M_{i^{d-1},0}, M_{i^{d-1},1}, \dots, M_{i^{d-1},n}) \text{ for some } i \text{ such that there are not all } 0 \end{aligned}$$

where $M_{i^{d-1},j}$ is the term corresponding to $x_i^{d-1}x_j$. Since there must exist an i such that $M_{i^{d-1},0}, M_{i^{d-1},1}, \dots, M_{i^{d-1},n}$ are not all zero, and for $i \neq j$ satisfying this, we must have $(M_{j^{d-1},0}, M_{j^{d-1},1}, \dots, M_{j^{d-1},n}) = (M_{j^{d-1},0}, M_{j^{d-1},1}, \dots, M_{j^{d-1},n})$ because they are the same up to a nonzero scalar $(x_i/x_j)^{d-1}$, this map is well-defined. And clearly on each affine piece $\rho_d^{-1}(x_i \neq 0)$ this is a morphism. So ρ_d is an isomorphism. Then ρ_d is regular at $x_0 = 1$.

Solution 1.3.5. STEP 1. We prove that $\mathbb{P}^n - H$ is affine, where H is a hyperplane.

Under some coordinates changing, we may assume that $H = (x_0 = 0)$, then trivial.

STEP 2. By STEP 1 and 1.3.4., we know that for any homogenious polynomial h of degree d, then $\mathbb{P}^n$ is isomorphic to $\rho^d(\mathbb{P}^n) \subset \mathbb{P}^N$. So $\mathbb{P}^n - H$ is isomorphic to $\rho^d(\mathbb{P}^n - H) = \rho^d(\mathbb{P}^n) \cap \rho^d(H)$. Since H is defined by (h), we know that $\rho^d(H)$ is a hyperplane, then $\rho^d(\mathbb{P}^n - H)$ is isomorphic to an affine variety.

Solution 1.3.6. If $h = \frac{f}{g} \in \mathscr{O}(X)$, and f, g are coprime, then g can only vanish at $(0, 0)$. But $\dim Z(g) = 1$, and $\dim\{(0, 0)\} = 0$, so g has no zeros, i.e. $g =$constantneq0. Hence $\mathscr{O} = k[x, y]$. Since $\mathscr{O}(\mathbb{A}^2) = k[x, y]$, we have $\mathrm{id} \in \mathrm{Hom}(\mathscr{O}(X), \mathscr{O}(\mathbb{A}^2))$. Then $\mathrm{id} \in \mathrm{Hom}(\mathbb{A}^2, X)$. But $\mathrm{id}((0, 0)) \notin X$, which makes a contradiction.

Solution 1.3.7. (a) If X, Y are two curves in $\mathbb{P}^2$, we have $\dim X + \dim Y - 2 = 0 \geq 0$. Then by 1.2.11.(c), X, Y has non-empty intersection.

(b) If $Y \cap H = \varnothing$, then $Y \subset \mathbb{P}^n - H$. And $\mathbb{P}^n - H$ is affine, thus Y is an affine variety. Then by 1.3.1.(e), Y is only one point. But Y has dimensiongeq1, which makes a contradiction.

Solution 1.3.8. Write $X = \mathbb{P}^n - (H_i \cap H_j)$. Then if $f \in \mathscr{O}(X)$, we have $f = \frac{g}{h}$ for some homogeneous polynomial with same degree, and h has only zeros in $H_i \cap H_j$, i.e. $\dim Z(h) \leq \dim H_i \cap H_j \leq n - 2$. But $Z(h)$ must be a hypersurface and has dimension $n - 1$, which makes a contradiction.

Solution 1.3.9. Clearly $S(X) = k[x, y]$, $S(Y) = k[x, y, z]/(xy - z^2) \cong k[x, y] \oplus k[x, y]$. And $[S(Y) : S(X)] = 2$, thus $S(X) \ncong S(Y)$.

Solution 1.3.10 (Subvarieties)**.** If $X' = X \cap U$ and $Y' = Y \cap V$ for some open sets U and V, since $\varphi : X \to Y$ is a morphism, we know that for all regular function $f : Y \to k$, $f \circ \varphi$ is regular. If $g : Y' \to k$ is regular, since V is open, we know that $g = \frac{f}{h}$ for some regular function on Y and h on Y has only zeros on $Y - Y'$. Then since $\varphi(X') \subset Y'$, we know that h has no zeros on $\varphi(X')$, i.e. $g \circ \varphi|_{X'}$ is regular. Hence $\varphi|_{X'}$ is a morphism.

Solution 1.3.11. This problem is local, hence we may assume X is affine. Since the subvarieties containing P correspond to prime ideals of $A(X)$ contained in $\mathfrak{m}_P$, and those prime ideals correspond to prime ideals of the ring $\mathscr{O}_P$, so clearly.

Solution 1.3.12. Firstly we may assume that X is affine. By 3.2.(c) the proof is finished. For general case, we may assume $X \in \mathbb{P}^n$, and denote $p \in X_i = X \cap U_i$. Then we have $\dim \mathscr{O}_p(X_i) = \dim X_i = \dim X$. Clearly, $\mathscr{O}_p(X) \subset \mathscr{O}_p(X_i)$. Inversely, if $f \in \mathscr{O}_p(X_i)$, f is regular on a neighbourhood $p \in U \subset X_i$. But U in X is also an open set, so $f \in \mathscr{O}_p(X)$, i.e. $\mathscr{O}_p(X) = \mathscr{O}_p(X_i)$. Thus $\dim \mathscr{O}_p = \dim X$.

Solution 1.3.13 (The Local Ring of a Subvariety)**.** Obviously $\mathscr{O}_{Y,X}$ is a ring, then we only need to prove that $\mathfrak{m}_Y = \{$all function in $\mathscr{O}_{Y,X}$ which vanishes on $Y\}$ is the unique maximal ideal in $\mathscr{O}_{Y,X}$, i.e. $\mathfrak{m}_Y$ equals to the Jacobian of $\mathscr{O}_{Y,X}$. If $f \in \mathfrak{m}_Y$, and $g \in \mathscr{O}_{Y,X}$, we know that $1 + fg \neq 0$ on Y. So $\frac{1}{1+fg}$ is well-defined on Y, hence it's regular on an open set U which contains Y. Thus $(1 + fg)$ has inverse on $\mathscr{O}_{Y,X}$.

Moreover, $\dim X = \dim k[x]/I(Y) + \text{height } I(Y) = \dim k[Y] + \text{height } I(Y) = \dim Y + \dim \mathscr{O}_{Y,X}$.

Solution 1.3.14 (Projection from a Point)**.** (a) Under some coordinates changing, we may assume that $P = (1, 0, \ldots, 0)$ and $\mathbb{P}^n = (x_0 = 0) \subset \mathbb{P}^{n+1}$, then φ is $(x_0, x_1, \ldots, x_n) \mapsto (0, x_1, \ldots, x_n)$, hence a morphism.

(b) $\varphi : (t^3, t^2u, tu^2, u^3) \mapsto (t^3, t^2u, 0, u^3)$, i.e. on $\mathbb{P}^2$, it is (t^3, t^2u, u^3) which has equation $y^3 = x^2z$.

Solution 1.3.15 (Products of Affine Varieties)**.** (a) If $X \times Y = Z_1 \cup Z_2$ for two closed subsets Z_1 and Z_2, we may define $X_i = \{x \in X, \{x\} \times Y \subset U_i\}$ for $i = 1, 2$. For any point $x \in X - X_1$, $\{x\} \times Y$ is closed in $X \times Y$, hence there exists a open subset containing $\{x\} \times Y$ and disjoint to Z_1. So there exists an open subset in X containing x and disjoint to X_1, i.e. X_1 is closed. And similarly we know X_2 is also closed. Moreover, for any $x \in X$, since $Y \cong \{x\}Y \subset Z_1 \cup Z_2$, by the irreducibility of Y we know $\{x\}Y$ must be contained in at least one of Z_1 and Z_2, so $x \in X_1$ or $x \in X_2$. So $X = X_1 \cup X_2$. Since X is irreducible, we may assume $X = X_1$, i.e. $X \times Y = U_1$, hence irreducible.

(b) We can define $\varphi : A(X) \otimes_k A(Y) \to A(X \times Y)$, $(f \otimes g) \mapsto fg$. Then we need to prove this φ is an isomorphism. For the surjectivity, since the coordinate functions $x_1, \dots, x_m, y_1, \dots, y_n \in A(X \times Y)$ all have preimages, then every function in $A(X\times Y)$ is a polynomial of these coordinates, hence it must have a preimage in $A(X) \otimes_k A(Y)$. For the injectivity, if $0 \neq \sum_{i=1}^k f_i \otimes g_i \in \ker \varphi$ with a minimal k, we know $f_1(x_0) \neq 0$ for some $x_0 \in X$. since $\sum_{i=1}^k f_i(x_0) g_i(y) = 0$ for any $y \in Y$, we know $g_1(y) = -f_1(x_0)^{-1} \sum_{i=2}^k f_i(x_0) \otimes g_i(y)$. So this contradicts with the minimality of k, hence φ is injective.

(c) (i) If $f : X \to k$ is regular, we know $f \circ \rho_X : X \times Y \to k$ is just polynomial of X, hence regular. So $\rho_X : X \times Y \to X$ is a morphism. And so does $\rho_X : X \times Y \to Y$.

(ii) If $\varphi_X : Z \to X$, $\varphi_Y : Z \to Y$. By product, we have a unique morphism $\varphi_X \times \varphi_Y : Z \to X \times Y$.

(d) If $X_0 \subsetneq \dots \subsetneq X_n$ and $Y_0 \subsetneq \dots \subsetneq Y_m$ is chains in X and Y, then we have $X_0 \times Y_0 \subsetneq \dots \subsetneq X_n \times Y_0 \subsetneq X_n \times Y_1 \subsetneq \dots \subsetneq X_n \times Y_m$ is a chain in $X \times Y$. Hence $\dim X \times Y \geq \dim X + \dim Y$. Conversely, we clearly have $\dim X \times Y \leq \dim X + \dim Y$.

Solution 1.3.16 (Products of Quasi-Projective Varieties)**.** (a) We just need to prove that $\mathbb{P}^n \times \mathbb{P}^m$ is a projective variety. Take $\varphi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$, $((x_0, \dots, x_n), (y_0, \dots, y_m)) \mapsto (x_0y_0, x_0y_1, \dots, x_0y_m, x_1y_0, \dots, x_ny_m)$. Then clearly φ is isomorphic between $\mathbb{P}^n \times \mathbb{P}^m$ and $\varphi(\mathbb{P}^n \times \mathbb{P}^m)$. And $\varphi(\mathbb{P}^n \times \mathbb{P}^m)$ is define in $\mathbb{P}^{nm+n+m}$ by $z_{ij}z_{kl} = z_{il}z_{kj}$ for every i, j, k, l, where z_{ij} is the term corresponding to the term x_iy_j. Thus $\mathbb{P}^n \times \mathbb{P}^m$ is a projective variety.

(b) Similar with (a).

(c) If $\varphi_X : Z \to X$ and $\varphi_Y : Z \to Y$, we can define $(\varphi_X, \varphi_Y) : Z \to X \times Y$, $z \mapsto (\varphi_X(Z), \varphi_Y(z))$, where we treat the $X \times Y$ as a variety in $\mathbb{P}^{nm+n+m}$. Then we have $Z \to X \times Y$.

Solution 1.3.17 (Normal Varieties)**.** (a) Since every conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$, it is smooth, hence normal.

(b) Since Jacobian matrix $(y, x, -w, -z)$ has rank 1 on all points of Q_1, then Q_1 is smooth, hence normal.

At the cone point of Q_2, since local ring is $k[x, y, z]/(z^2 - xy)$ and xy is square-free, it is integrally closed, hence normal. At other points, similarly by Jacobian we know that is smooth, hence normal.

(c) At the cusp, it is obviously singular, hence not normal.

(d) Y is normal $\Leftrightarrow$ every $\mathscr{O}_p$ is integrally closed $\Leftrightarrow$ $\mathscr{O}(Y)$ is integrally closed, where the second arrow is implied by Atiyah 5.12.

(e) Take A as th integral closure of $k[Y]$ in $k(Y)$. Then we just need to find a $\bar{Y}$ to satisfy $k[\bar{Y}] = A$, i.e. $k[\bar{Y}]$ is finitely generated over k without zero-divisors. Then we just need to prove that $k[\bar{Y}]/k[Y]$ is finitely generated.

By Noether normalization, there exists $B \subset k[Y]$ such that $B \supset k[T_1, \dots, T_r]$ and $k[Y]$ is integral over B. Then $k(T_1, \dots, T_r) \subset k(Y)$ but

$$k(T_1, \dots, T_r) \supset B \subset k[Y] \subset A \subset k(Y).$$

Thus A is the integral closure of B in $k(Y)$, and $k(Y)$ is a finite field extension of $k(T_1, \dots, T_r)$. Then B is integral closed, thus A/B is finitely generated, hence $A/k[Y]$.

Solution 1.3.18 (Projective Normal Varieties)**.** (a) If Y is projectively normal, then $S[Y]$ is integrally closed. So every localization of $S[Y]$ is integrally closed, i.e. Y is normal.

(b) Since Jacobian matrix is

$$\begin{bmatrix} w & -z & -y & x \\ 2xz & -3y^2 & x^2 & 0 \end{bmatrix},$$

it has rank n everywhere, so Y is normal. But $S(Y) = S/(xw - yz, x^2z - y^3)$ is not integral closed.

(c) We have $\varphi : \mathbb{P}^1 \to Y$, $(t, u) \mapsto (t^4, t^3u, tu^3, u^4)$ and $\psi : Y \to \mathbb{P}^1$, $(x, y, z, w) \to (x, y)$. Then $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$, hence isomorphic.

Solution 1.3.19 (Automorphisms of $\mathbb{A}^n$)**.** If φ is isomorphism, there exists a ψ such that $\psi\circ\varphi = \mathrm{id}$ and $\varphi\circ\psi = \mathrm{id}$. If $x_i = x_i(f_1, \ldots, f_n)$ are the equations of ψ, we have $(\frac{\partial f_i}{\partial x_j}) \cdot (\frac{\partial x_i}{\partial f_j}) = I$, i.e. we get the determinant $J \cdot J_{x/f} = 1$ for two polynomial J and $J_{x/f}$. Then J =non-zero constant.

Solution 1.3.20. (a) If f is regular on $Y - P$, then $f = \frac{g}{h}$ for two polynomials and $h|_{Y,p} \neq 0$. Since $\dim Y \geq 2$, the zeros of h is empty or has at least dimension 1 on Y, hence h is no zeros, i.e. h =nonzero constant. So $\frac{g}{h}$ can define on the whole Y, i.e. f extends to a regular function on Y.

(b) If $\dim Y = 1$, we just take $f = \frac{1}{x-p}$.

Solution 1.3.21 (Group Varieties)**.** (a) For any regular function $\mathbb{A}^1 \to k$, we have $f \circ \mu : \mathbb{A}^2 \to k$ is also a regular function obviously. Hence it is a group variety.

(b) For any regular function $\mathbb{A}^1 - \{0\} \to k$, we have $f \circ \mu : (\mathbb{A}^1 - \{0\})^2 \to k$ is also a regular function.

(c) For any $f \in \mathrm{Hom}(X, G)$, we can define a morphism $-f : X \to G$, $x \mapsto -f(x)$. And for any $f, g \in \mathrm{Hom}(X, G)$, we can define $f + g : X \to G$, $x \mapsto f(x) + g(x)$. Then clearly $-f$ and $f + g$ are both in $\mathrm{Hom}(X, G)$, hence $\mathrm{Hom}(X, G)$ has a group structure from the structure on G.

(d) Define $\varphi : \mathrm{Hom}(X, \mathbb{G}_\mathrm{a}) \to \mathscr{O}(X)$, $(f : X \to \mathbb{G}_\mathrm{a}) \mapsto f$. φ is clearly a bijection, and for every $f, g \in \mathrm{Hom}(X, \mathbb{G}_\mathrm{a})$, we have $f + g : X \to \mathbb{G}_\mathrm{a}$, $x \mapsto f(x) + g(x)$, so $\varphi(f + g) = \varphi(f) + \varphi(g)$, hence an isomorphism.

(e) Define $\varphi : \mathrm{Hom}(X, \mathbb{G}_\mathrm{m}) \to \mathscr{O}(X)$, $(f : X \to \mathbb{G}_\mathrm{m}) \mapsto f$. Then for every $f \in \mathrm{Hom}(X, \mathbb{G}_\mathrm{m})$, we have $f \neq 0$, i.e. $\frac{1}{f} \in \mathscr{O}(X)$, hence $\mathrm{Im}\varphi = \mathscr{O}^\times(X)$. For every $f, g \in \mathrm{Hom}(X, \mathbb{G}_\mathrm{m})$, we have $fg : X \to \mathbb{G}_\mathrm{m}$, $x \mapsto f(x)g(x)$, then $\varphi(fg) = \varphi(f) varphi(g)$, hence an isomorphism.

1.4 Rational Maps

Solution 1.4.1. Define h on $U \cup V$ as

$$h(x) = \begin{cases} f(x), & x \in U \\ g(x), & x \in V \end{cases}$$

Since $f|_{U\cap V} = g|_{U\cap V}$, h is well-defined. Then we need to prove that h is regular. Thus if $f = \frac{f_1}{f_2}$, $g = \frac{g_1}{g_2}$, for some $f_1, f_2 \in k[U]$ and $g_1, g_2 \in k[V]$. On the $U \cap V \backslash (f_2 g_2 = 0)$, we have $\frac{f_1}{f_2} = \frac{f_1 g_2}{f_2 g_2} = \frac{f_2 g_1}{f_2 g_2} = \frac{g_1}{g_2}$. Then h is regular on $U \backslash V$, $V \backslash U$ and $U \cap V$, hence it is regular.

Solution 1.4.2. A map is regular iff it is regular in a neighbourhood of every point, so by 1.4.1. and Zorn's lemma, there exists a largest subset U of X such that f is regular on U.

Solution 1.4.3. (a) Clearly f is defined on the open subset $x_0 \neq 0$, which is isomorphic to $\mathbb{A}^1$. And f is the projection of $\mathbb{P}^2 \to \mathbb{A}^1$ of its second coordinate.

(b) We may define $\varphi : \mathbb{P}^2 \to \mathbb{P}^1$, $(x_0, x_1, x_2) \mapsto (x_0, x1)$, which is defined on the open set $\mathbb{P}^2 - \{(0, 0, 1)\}$. So φ is just the projection of first two coordinates.

Solution 1.4.4. (a) By 1.3.1.(c), every conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$, hence birational equivalent to $\mathbb{P}^1$.

(b) We can define $\varphi : \mathbb{A}^1 \to (y^2 = x^3)$, $t \mapsto (t^2, t^3)$, and $\psi : (y^2 = x^3) \to \mathbb{A}^1$, $(x, y) \mapsto \frac{x}{y}$. Then $(y^2 = x^3)$ is birational equivalent to $\mathbb{A}^1$, and $\mathbb{A}^1$ is birational equivalent to $\mathbb{P}^1$.

(c) Clearly, $\varphi : Y \to \mathbb{P}^1$, $(x, y, z) \mapsto (x, y)$, and $\psi : \mathbb{P}^1 \to Y$, $(x_0, x_1) \mapsto (x_1^2 x_0 - x_1^3, x_1^3 - x_0^2 x_1, x_0^3)$. Then Y is birational equivalent to $\mathbb{P}^1$.

Solution 1.4.5. Since $\varphi : Q \to \mathbb{P}^2$, $(x, y, z, w) \mapsto (x, y, z)$, and $\psi : \mathbb{P}^2 \to Q$, $(x_0, x_1, x_2) \mapsto (x_0 x_2, x_1 x_2, x_2^2, x_0 x_1)$, hence birational equivalent. But not isomorphic because $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$.

Solution 1.4.6 (Plane Cremona Transformations)**.** (a) Define $U = (xyz \neq 0)$. Since $\varphi^2 : U \to U$, $(x, y, z) \mapsto (x^2 yz, xy^2 z, xyz^2) = (x, y, z)$, hence φ is birational.

(b) Just take $U = V = (xyz \neq 0)$.

(c) On $U \to V$, it is just $(x, y, z) \mapsto (\frac{1}{x}, \frac{1}{y}, \frac{1}{z})$.

Solution 1.4.7. We may assume that X, Y are affine since it is a local problem, and $P = Q = 0$. Then $A(X)_{\mathfrak{m}_0} \cong A(X)_{\mathfrak{m}_0}$ induced $k(X) \cong k(Y)$, hence a birational equivalence between X and Y. Under the map of images $\frac{f_i}{g_i}$ of x_i in $k(Y)$. Since x_i is not invertible in $A(X)_{\mathfrak{m}_0}$, we have $g_i(x_i) \neq 0$, $f_i(x_i) = 0$, then $(\frac{f_1}{g_1}(0), \ldots, \frac{f_n}{g_n}(0)) = 0$, i.e. $P \mapsto Q$. Then the others is under corollary 4.5.

Solution 1.4.8. (a) Firstly, k is algebraic closed, then $|k|$ is infinite, i.e. $|\mathbb{A}^n| = |k^n| = |k|$. Then $|\mathbb{P}^n| = |\mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \ldots \cup \mathbb{A}^0| = (n-1)|k| + 1 = |k|$. Then for general X, if $X \subset \mathbb{P}^n$ for some n, we have $|X| \leq |\mathbb{P}^n| = |k|$. And for some open affine $U \subset X$ which $\dim U \geq 1$, so by normalisation there exists a finite surjection $U \to \mathbb{A}^{\dim U}$, so $|U| \geq |\mathbb{A}^{\dim U}| \geq |k|$. Then $|k| \leq |U| \leq |X| \leq |\mathbb{P}^n| \leq |k|$, hence $|X| = |k|$.

(b) Any two curves over k has same cardinality, then they are homeomorphic under cofinite topology.

Solution 1.4.9. Since X has dimension r, we may assume $\frac{x_1}{x_0}, \ldots, \frac{x_r}{x_0} \in k(X)$ is algebraic independent. And we have some $f \in k(X)$ such that $(\frac{x_1}{x_0}, \ldots, \frac{x_r}{x_0}, f)$ generates $k(X)$, and $f = x_0^{-1} \sum_{i=r+1}^{n} a_i x_i$. Define $H_1 = (\sum a_i x_i = 0)$ and $H_2 = (x_{n+1} = 0)$, and $P = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is the $r+2$-th term. By construction, f is taken to $\frac{x_{r+1}}{x_0}$ under transformation $H_1 \to H_2$, then we may assume $P \notin X$, and $\frac{x_1}{x_0}, \ldots, \frac{x_r}{x_0}$ is the transcendence basis of $k(X)$, and $\frac{x_{r+1}}{x_0} = f$.

Define $H_2 = (x_{r+2} = 0)$ and $\pi : X \to X'$ is the projection about P and H_2. Then by this construction, $k(X) \cong k(X')$, hence birational.

Solution 1.4.10. Easily we know $\bar{Y} = Z(y_2^2 - x_1 y_1^2, y_2^3 - x_2 y_1^3) \subset \mathbb{A}^2 \times \mathbb{P}^1$. Supposing $x1x_2 \neq 0$, $x_1 y_2 = x_2 y_1$ and $x_1^3 = x_2^2$, we have $y_2 = \frac{x_2 y_1}{x_1}$, $y_2^2 = x_1 y_1^2$ and $y_2^3 = x_2 y_1^3$. So $\bar{Y} \supset \phi^{-1}(Y - \{0\})$.

If $(0, 0, y_1, y_2) \in \bar{Y}$, then $y_2 = 0$, i.e. $(0, 0, 1, 0) \in \bar{Y}$.

1.5 Nonsingular Varieties

Solution 1.5.1. (a) Tacnode. (b) Node. (c) Cusp. (d) Triple Point.

Solution 1.5.2. (a) Pinch Point. (b) Conical double point. (c) Double line.

Solution 1.5.3 (Multiplicities). (a) Clearly, $\mu_P(Y) = 1 \Leftrightarrow f(P) = 0$ and $\nabla_f P \neq 0 \Leftrightarrow P$ is nonsingular.

(b) $\mu_{(0,0)}(a) = \mu_{(0,0)}(b) = \mu_{(0,0)}(c) = 2$, $\mu_{(0,0)}(d) = 3$.

Solution 1.5.4 (Intersection Multiplicity). (a) We may assume that $P = (0, 0)$. Firstly, take a neighbourhood U of P such that $(Y \cap Z) \cap U$ is only point T. Then we may take $I_P \subset A(U)$, by Nullstellensatz, $I_P^r \subset (f, g)$ for some $r > 0$. So in $\mathscr{O}_P$, we have $\mathfrak{m}_P^r \subset (f, g)$. Thus we just need to show $\mathscr{O}_P/\mathfrak{m}_P^r$ has finite length. Since $\mathscr{O}_P/\mathfrak{m}_P^r \supset \mathfrak{m}_P/\mathfrak{m}_P^r \supset \ldots \supset \mathfrak{m}_P^{r-1}/\mathfrak{m}_P^r \supset 1$, and $(\mathfrak{m}_P^i/\mathfrak{m}_P^r)/(\mathfrak{m}_P^{i+1}/\mathfrak{m}_P^r) \cong \mathfrak{m}_P^i/\mathfrak{m}_P^{i+1}$ has finite dimension, then $\mathfrak{O}_p/\mathfrak{m}_P^r$ has finite length.

Secondly, denote the length of $\mathscr{O}_P/(f, g) = l$, $m = \mu_P(Y)$ and $n = \mu_Q(Y)$. We may assume $m \leq n$. Since there exists a chain of length mn in $k[x, y]/(f, g)$, this chain induces a chain in $\mathscr{O}_P/(f, g) = k[x, y]_{(x,y)}/(f, g)$. Thus $l \geq mn$.

(b) If L is not in the tangent cone of Y at P (only finite lines in the tangent cone), we may assume $P = (0, 0)$, and $L = (y = 0)$. Then $f = f_m$+higher terms and $f_m = x^m + y \cdot$ (polynomial of degree $m - 1$). So $\mathscr{O}_P(y, f) = k[x, y]_{(x,y)}/(y, x^m + y(\ldots)) = k[x, y]_{(x,y)}/(y, x^m) \cong k[x]/x^m$ has length m, so $(L \cdot Y)_P = m = \mu_P(Y)$.

(c) Under some coordinate changing, we may assume $L \cap Y$ does not happen in $x_2 = 0$ in $\mathbb{P}^2$. So this problem will be considered in $\mathbb{A}^2$, and $L \cap Y$ does not in infinity. We may assume that $L = (y = 0)$, and $Y = (f = 0)$ for some $f \in k[x, y]$. And $Y \cap L$ does not in infinity means that the highest order of f is $f_d = x^d + y \cdot ((d-1)$-dimensional terms). Then points in $Y \cap L$ corresponds to the zero of $f_d(x, 0)$, i.e. the multiplicity of $P = (\alpha, 0)$ is the length of $\mathscr{O}_P(y, f) \cong k[x, y]_{(x-a,y)}/(y, f) \cong k[x, y]_{(x-a)}/(f_d(x, 0))$. Thus this module has length which equals to the multiplicity $\mathfrak{m}_a$ of a at $f_d(x, 0)$. Hence $\sum_p (Y, L)_p = \sum_a \mathfrak{m}_a = d$.

Solution 1.5.5. If $\operatorname{char}(k) = 0$ or $\operatorname{char}(k) \nmid d$, we just need to take $f = x_0^d + x_1^d + x_2^d$. If $\operatorname{char}(k)|d$, we just need to $f = x_0^{d-1}x_1 + x_1^{d-1}x_2 + x_2^{d-1}x_0$.

Solution 1.5.6 (Blowing Up Curve Singularities)**.** (a) If Y is given by $f = y^2 - x^3 + x^4 + y^4$, for coordinates x, y, t, u in $\mathbb{A}^2 \times \mathbb{P}^1$ and $xt - yu = 0$, we may consider $U = (t = 1)$. Then $\phi^{-1}(Y) \cap U$ is given by $f = 0$ and $x = yu$, i.e.

$$\begin{cases} x = yu \\ y^2(1 + u^4y^2 + y^2 - u^3y) = 0 \end{cases}$$

Denote $W = \phi^{-1}(Y) \cap U$. Then $\overline{W \cap \phi^{-1}(Y - \{0\})}$ satisfies $x = yu$ and $g = 1 + u^4y^2 + y^2 - u^3y = 0$, i.e. $I(\tilde{Y} \cap U) = (x - yu, g)$. Since $\tilde{Y} \cap U \cap E = \varnothing$, we have ϕ on $\tilde{Y} - (\tilde{Y} \cap E)$ is isomorphic. Thus $\tilde{Y} \cap U$ is nonsingular.

Denote $U' = (u = 1)$, then similarly we have $I(\tilde{Y} \cap U') = (h, y - xt)$ for $h = t^2 + x^2 + t^4x^2 - x$. Since $\tilde{Y} \cap U' \cap E = (0, 0, 0, 1)$, out of this point the $\tilde{Y}$ is nonsingular. And at this point, its Jacobian is

$$\begin{bmatrix} 2x(1 + t^4) - 1 & 0 & 2t + 4t^3x^2 \\ -t & 1 & -x \end{bmatrix}\Big|_{x=y=t=0} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has rank 2, hence nonsingular.

If Y is given by $f = xy - (x^6 + y^6)$, we may take $U = (t = 1)$. Then $\phi^{-1}(Y) \cap U$ is given by $x = uy$ and $uy^2 - y^6(1 + u^6)$. Similarly we have that $\tilde{Y} \cap U$ is given by

$$\begin{cases} x = yu \\ u - y^4(1 + u^6) = 0 \end{cases}$$

Then $\tilde{Y} \cap U \cap E = (0, 0, 1, 0)$. And its Jacobian is

$$\begin{bmatrix} 0 & -4y^3(1 + u^6) & 1 - 6x^4u^5 \\ 1 & -u & -y \end{bmatrix}\Big|_{x=y=u=0} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

has rank 2. Thus $\tilde{Y} \cap U$ is nonsingular.

Take $U' = (u = 1)$. Similarly $\phi^{-1}(Y) \cap U'$ is given by $tx = y$ and $tx^2 - x^6(1 + t^6)$ and $\tilde{Y} \cap U'$ is given by $y = x$ and $tx^2 - x^6(1 + t^6)$. Then $\tilde{Y} \cap U' \cap E = (0, 0, 0, 1)$. And its Jacobian $\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$ has rank 2. Thus $\tilde{Y} \cap U'$ is also nonsingular.

(b) We may assume $P = (0, 0)$ and Y is given by $f = f_2$+higher terms, where f_2 is homogeneous polynomial in degree 2. Under some coordinate changing, we may assume that $f_2 = xy$. So similarly with (a) we have $\tilde{Y} \cap U \cap E = (0, 0, 1, 0)$ and $\tilde{Y} \cap U' \cap E = (0, 0, 0, 1)$, and both two points are nonsingular.

(c) If Y is given by $f = x^2 - (x^4 + y^4)$, we know in $U = (t = 1)$, $\phi^{-1}(Y) \cap U$ is given by $x = yu$ and $y^2u^2 - y^4(1 + u^4)$, hence $\tilde{Y} \cap U$ is given by $x - yu = 0$ and $g = u^2 - y^2(1 + u^4) = 0$. Then $\tilde{Y} \cap U \cap E = (0, 0, 1, 0)$ and the Jacobian of this point is

$$\begin{bmatrix} 0 & -2y(1 + u^4) & 2u - 4y^2u^3 \\ 1 & -u & -y \end{bmatrix}\Big|_{x=y=u=0} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which has rank 1, hence node.

(d) On $U = (t = 1)$, $\phi^{-1}(Y) \cap U$ is given by $x = yu$ and $x(x^2 - u^3) = 0$, i.e. gives a cusp. Then one more blow-up gives a smooth point.

Solution 1.5.7. (a) At $P = (0, 0, 0)$, all partials are zero, so P is singular. If $Q = (a, b, c) \neq P$, we may assume $c \neq 0$, and denote $g(x, y) = f(x, y, 1)$. Since Y is nonsingular, g_x, g_y will not be simultaneously zero. Sense $f_x(a, b, c) = c^{d-1}f_x(\frac{a}{c}, \frac{b}{c}, 1)$, f_x, f_y will not be simultaneously zero, hence nonsingular.

(b) Take x, y, z, s, t, u as the coordinates in $\mathbb{A}^3 \times \mathbb{P}^2$. Denote $U = (s = 1)$. Then $\tilde{X} \cap U$ is given by $y = tx$ and $z = ux$ and $f(x, tx, ux) = 0$. Since $f(x, tx, ux) = u^d f(1, t, x)$. Thus $\tilde{X} \cap U$ is given by $y = tx$, $z = ux$ and $f(1, t, u) = 0$. Denote $g(t, u) = f(1, t, u)$. Since f is nonsingular, g is nonsingular too. Thus $\tilde{X}$ is nonsingular.

(c) Since $\tilde{X} \cap U$ is isomorphic to $Z(f(1, t, u)) \subset \mathbb{A}^3$, we know that $\tilde{X} \cap U \cap E$ is isomorphic to $Z(g(t, u)) \subset \mathbb{A}^3$. Similar to $(t = 1)$ and $(u = 1)$, we have that $\phi^{-1}(0) \cong \tilde{X} \cap E \cong Y$.

Solution 1.5.8. Since f is homogeneous, rank $(J)|_P$ is independent of the coordinate of P. So we may assume that $x_0 = 1$ and denote $g_i(x_1, \ldots, x_n) = f_i(1, x_1, \ldots, x_n)$. Then $(\partial x_i f_j)|_P$ is $(\partial x_i g_j)|_P$ adding $(\partial x_0 f_i)|_P$ at the top. By Euler's lemma, these two matrices have the same rank. Then by some condition in affine case, the exercise has proved.

Solution 1.5.9. If $f = gh$, then by 1.3.7., there exists a point P such that $g(P) = h(P) = 0$. Then

$$\partial_x f(P) = g(P) \cdot \partial_x h(P) + h(P) \cdot \partial_x g(P) = 0$$
$$\partial_y f(P) = g(P) \cdot \partial_y h(P) + h(P) \cdot \partial_y g(P) = 0 \qquad \partial_z f(P) = g(P) \cdot \partial_z h(P) + h(P) \cdot \partial_z g(P) = 0$$

which clearly makes a contradiction.

Solution 1.5.10. (a) By 5.2.A, we know $\dim T_P(X) = \dim \mathfrak{m}/\mathfrak{m}^2 \geq \dim X$. And the equality holds iff $\mathscr{O}_P$ is regular.

(b) Since we have local ring morphism $\mathscr{O}_{\varphi(P)} \to \mathscr{O}_P$, we have $T_P(\varphi) : T_P(X) \to T_{\varphi(P)}(Y)$.

(c) Since $\varphi^* : \mathscr{O}_{(0)} \to \mathscr{O}_{(0,0)}$ makes $\mathfrak{m}_{(0)} \mapsto \mathfrak{m}^2_{(0,0)}$. And $\mathfrak{m}_{(0)} = (x)$, we have $x \mapsto (x = y^2) \in \mathfrak{m}^2_{(0,0)}$, i.e. a zero map.

Solution 1.5.11 (The Elliptic Quartic Curve in $\mathbb{P}^3$)**.** Denote $f_1 = x^2 - xz - yw$, $f_2 = yz - xw - zw$ and $g = y^2z - x^3 + xz^2$, $\pi : \mathbb{P}^3 \to \mathbb{P}^2$ as $(x, y, z, w) \mapsto (x, y, z)$. First, since $g = (x + z)f_1 + yf_2$, we know $f_1 = f_2 = 0$ implies $g = 0$, i.e. $\pi(Y - P) = W$. Second, if $g(a, b, c) = 0$ and $Q = (a, b, c) \neq (1, 0, -1) = P'$, we have $b \neq 0$ or $a + c \neq 0$.

In the first case, define $\psi(x, y, z) = (x, y, z, \frac{x^2 - xz}{y})$. In the second case, define $\psi(x, y, z) = (x, y, z, \frac{yz}{x+z})$. Thus we construct a map $\psi : (W - P') \to Y - P$ as the inverse of $\varphi : Y - P \to W - P'$. So by 1.5.9. we know that W is irreducible, hence $W - P'$ is, then $Y - P$, hence Y is.

Solution 1.5.12 (Quadric Hypersurfaces)**.** (a) Since every quadric form has form $x^T A x$ for some symmetric matrix A, and k is algebraic closed, so A is similar to $\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$ of some $I_{r \times r}$.

(b) If $f = gh$ is reducible, g and h must be linear form. If g and h are proportional, $gh = x_0^2$. And if g and h are nor proportional, $gh = x_0^2 + x_1^2$.

(c) Since for any $P \in \mathrm{Sing}Q$, we have that every partial derivatives are zero, i.e. $x_1 = \ldots = x_r = 0$. Thus $\mathrm{Sing}Q$ is variety for dimension $n - r - 1$.

(d) Define $\mathbb{P}^r = (x_{r+1} = \ldots = x_n = 0)$, and $Q' = Q \cap \mathbb{P}^r$. If P is a point on line L of $R \in Q'$, and $R' \in Z = \mathrm{Sing}Q$. Then we may define $P = (aR_0, \ldots, aR_r, bR'_{r+1}, \ldots, bR'_n)$. Then $f(R) = 0$ implies $f(P) = 0$. Conversely, if $P = (P_i) \in Q$, we know that $R = (P_0, \ldots, P_r, 0, \ldots, 0)$ has $f(R) = 0$, i.e. $R \in Q'$, and $(0, \ldots, 0, P_{r+1}, \ldots, P_n) \in \mathrm{Sing}Q$.

Solution 1.5.13. We may assume X is affine, and denote the integral closure $A(X) \in K(X)$ as A, and $A = (f_1, \ldots, f_m)$. If $f_i = \frac{g_i}{h_i}$, f_i is defined exactly on $D(h_i)$. Then $\mathscr{O}_X$ is integrally closed$\Leftrightarrow f_i \in \mathscr{O}_X \Leftrightarrow h_i(X) \neq 0$ for all i. Then X is normal$\Leftrightarrow x \in \bigcap D(h_i)$, i.e. open.

Solution 1.5.14 (Analytically Isomorphic Singularities)**.** (a) This isomorphic induces $\mathfrak{m}_P^r \mapsto \mathfrak{m}_Q^r$, so $\mathfrak{m}_P^r/\mathfrak{m}_P^{r+1} \mapsto \mathfrak{m}_Q^s/\mathfrak{m}_Q^{s+1}$ implies $r = s$, i.e. $\mu_P(Y) = \mu_Q(Z)$.

(b) Similar process, we have $f_{r+1} = h_t g_{s+1} + g_s h_{t+1}$, and so on.

(c) If f has an ordinary 2-fold point on P, hence we may assume $f = f_1 f_2$. Then we have an automorphism φ of $k[[x, y]]$ such that $f_1 \mapsto x$ and $f_2 \mapsto y$. Hence φ induces an isomorphism $\varphi : k[[x, y]]/(xy) \to k[[x, y]]/(f)$. Thus every ordinary 2-fold point is analytically isomorphic to $P = 0$ on $g = xy$.

If f has an ordinary 3-fold point on P, hence we may assume $f = f_1 f_2 f_3$. Then we have a unique automorphism φ of $k[[x, y]]$ such that $f_1 \mapsto x$, $f_2 \mapsto ay$ and $f_3 \mapsto x + y$ for some $a \in k$. Hence φ induces an isomorphism $\varphi : k[[x, y]]/(xy(x + y)) \to k[[x, y]]/(f)$. Thus every ordinary 2-fold point is analytically isomorphic to $P = 0$ on $g = xy(x + y)$.

Denote $g_a = xy(x + y)(x + ay)$ for some $a \in k$. If $\varphi : k[[x, y]]/(g_a) \cong k[[x, y]]/(g_b)$, we know that $\varphi : (x, y) \mapsto (x, y)$ and $(g_a) \mapsto (g_b)$, i.e. $x \mapsto x$, $y \mapsto u_1 y$, $x + y \mapsto u_2(x + y)$ and $x + ay \mapsto u_3(x + by)$. Then clearly $u_1 = u_2 = u_3 = 1$, so $a = b$. So, if $a \neq b$, g_a is not isomorphic to g_b.

(d) If $y^2 - x^r$ and $y^2 - x^s$ for some $r \leq s$ are analytically isomorphic at $P = 0$, we have automorphism φ of $k[[x, y]]$ as $y^2 - x^r \mapsto u \cdot (y^2 - x^s) = y^2 +$(higher terms of y)$- x^s +$(higher terms with only x), where $u = 1+$higher terms. Denote l_x as a linear term of $\varphi(x)$, and l_y as a linear term of $\varphi(y)$. So $\varphi(y)^2 = y^2 + \varphi(x)^r +$ (terms of degree> 2). Then we have $l_y^2 = y^2$, i.e. $l_y = y$. Thus we may assume $l_x = ax + by$, and clearly $a \neq 0$. So $r = s$, by comparing terms of x in above equation.

Then for any double points on $f(x, y)$, if $f = f_2 g$ for some $g \in (x, y)^3$, and $f_2 = l_1 l_2$ with $l_1 \neq l_2$, we may as in above, map $(f_2) \mapsto (x^2 - y^2)$. If $f_2 = l^2$, we may assume $l = x$ under some coordinate changing. Since $g \in (x, y)^3$, we can find $r \geq 3$ and $h \in \mathfrak{m}$, such that $g = -(ax+y)^r + h(x^2 - (ax+y)^r)$ for some $a \in k$. Then we have an automorphism φ of $k[[x, y]]$ as $x \mapsto x$ and $y \mapsto ax+y$. So $(1+f) \circ \varphi(x^2 - y^r) = (1+f)(x^2 - (ax+y)^r) = x^2 + g$. Hence $(x^2 - y^r) \mapsto (x^2 + g)$, i.e. f is analytically isomorphic to $x^2 - y^r$.

Solution 1.5.15 (Families of Plane Curves)**.** (a) Clearly we have a correspondence

$$\begin{aligned}
&(\text{points of } \mathbb{P}^N) \longleftrightarrow (\text{algebraic sets in } \mathbb{P}^2)\\
&(a_{rst})_{r+s+t=d} \longrightarrow \sum_{r,s,t} a_{rst} x^r y^s z^t = 0\\
&(a_{rst})_{r+s+t=d} \longleftarrow \text{algebraic set} \subset \mathbb{P}^2 \text{ defined by } f = \sum_{r,s,t} a_{rst} x^r y^s z^t = 0
\end{aligned}$$

(b) The 1-1 correspondence is in (a). Then by elimination theorem, for $\frac{\partial f}{\partial x_0}, \dots \frac{\partial f}{\partial x_n}$), there are polynoials in coefficients of f which are zero when f is singular or reducible.

1.6 Nonsingular Curves

Solution 1.6.1. (a) Since every non-singular curve is isomorphic to some abstract nonsingular curve, hence isomorphic to an open set of some projective curve. And since Y is not isomorphic to $\mathbb{P}^1$, we know Y must be isomorphic to some open subset of $\mathbb{A}^1$.

(b) We may assume $Y \cong \mathbb{A}^1 - \{a_1, \dots, a_n\}$ by (a), then Y is isomorphic to $y \cdot \prod(x - a_i) = 1$ in $\mathbb{A}^2$, hence affine.

(c) Since $A(Y) \cong k[x]_{(x-a_1,\dots,x-a_n)}$, it is clearly a UFD with all primes $(x - a_i)$.

Solution 1.6.2 (An Elliptic Curve)**.** (a) If $P = (x, y) \in Y$ is a singular point, we have $y^2 = x^3 - x$, $2y = 0$ and $-3x^2 + 1 = 0$, which have no zeros, i.e. Y is nonsingular. Thus $y^2 - x^3 + x$ is irrducible, hence $A(Y) = k[x, y]/(y^2 - x^3 + x)$ is a domain. Moreover, since Y is nonsingular, hence normal, i.e. $A(Y)$ is integral closed.

(b) $k[x]$ is obviously a polynomial ring. Since $y^2 \in k[x]$, then y is in the closure of $k[x]$, i.e. $A \subset$the closure of $k[x]$. Conversely, since A is integral closed, then equals.

(c) If $f(x, y) \in A$, it must have the form $y \cdot g(x) + h(x)$ for some $g, h \in k[x]$. So $N(f) = h^2 - y^2 g^2 = h^2 - g^2(x^3 - x) \in k[x]$. And clearly, $N(1) = 1 \cdot 1 = 1$, $N(a \cdot b) = N(a) \cdot N(b)$.

(d) If $a \in A$ is a unite, there exists a $b \in A$ such that $ab = 1$. Then $N(a)N(b) = 1$ in $k[x]$, i.e. $N(a) \in k^\times$. If $a = yg + h$, then $h^2 - g^2(x^3 - x) \in k^\times$, i.e. $g = 0$ and $h \in k^\times$. So $a = h \in k^\times$. Since we clearly notice that no element has norm with degree 1, x and y are clearly irreducible in A. Finally, $y^2 = y \cdot y = x(x-1)(x+1)$ has two ways of factorization, i.e. A is not a UFD.

(e) Since A is non-trivial and not a UFD, by 6.1.(c), A is not isomorphic to $A(X)$ for any rational curve X, hence Y is not a UFD.

Solution 1.6.3. (a) $\mathbb{A}^2 - \{(0,0)\} \to \mathbb{P}^1$, $(x,y) \mapsto (x,y)$.

(b) $\mathbb{P}^1 - \{(1,0)\} \to \mathbb{A}^1$, $(x,y) \mapsto \frac{x}{y}$.

Solution 1.6.4. f induces a map $Y \to \mathbb{A}^1$ as $x \mapsto f(x)$. Then by 6.8. we have $\varphi : Y \to \mathbb{P}^1$. If Y is irreducible, since f is non-constant, $\mathrm{Im}\varphi = \mathbb{P}^1$ and φ is dominant, we know that φ induces a $\varphi^* : k(Y) \to k(\mathbb{P}^1)$. If $P \in \mathbb{P}^1$, $\varphi^{-1}(P)$ is closed. Since φ is non-constant, $\varphi^{-1}(P)$ are just finite points. If Y is reducible, we may assume $Y = Y_1 \cup \ldots \cup Y_n$. So for every Y_i, $\varphi^{-1}(P) \cap Y_i$ is finite, so $\varphi^{-1}(P)$ is finite.

Solution 1.6.5. Since $\bar{X}$ is a curve, if $x \in \bar{X} - X$, by 6.8, for any $X \to X$ can induce a $X \cup \{x\} \to X$, which makes a contradiction. Hence $X = \bar{X}$.

Solution 1.6.6 (Automorphisms of $\mathbb{P}^1$)**.** (a) If $ad - bc \neq 0$, then $x \mapsto \frac{ax+b}{cx+d}$ has inverse $x \mapsto (ad-bc)^{-1}\frac{dx-b}{-cx+a}$, hence an automorphism.

(b) By 6.12, clearly $\mathrm{Aut}\mathbb{P}^1 = \mathrm{Aut}k(x)$.

(c) If $\varphi \in \mathrm{Aut}k(x)$, we may assume $\varphi(x) = \frac{g(x)}{h(x)}$ for some $(g,h) = 1$. Then $\psi = \varphi^{-1}$ is rational, i.e. if $x \neq y$, $\varphi(x) \neq \varphi(y)$. But if f or g has degree> 1, $\frac{g}{h} = a$ has more than 1 solution, which makes a contradiction. So f and g are linear transformation. So $\mathrm{PGL}(1) \cong \mathrm{Aut}k(x) \cong \mathrm{Aut}\mathbb{P}^1$.

Solution 1.6.7. Denote $\varphi : \mathbb{A}^1 - \{P_1, \ldots, P_s\} \cong \mathbb{A}^1 - \{Q_1, \ldots, Q_t\}$. Then φ can be extended as isomorphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$, which gives a 1-1 correspondence between $\{P_1, \ldots, P_s\}$ and $\{Q_1, \ldots, Q_t\}$.

Since any set with at most 3 points in $\mathbb{P}^1$ can be mapped to any other set with same size under $\mathrm{Aut}(\mathbb{P}^1)$. so the converse if true when $r \leq 3$, but cannot hold in case $r \geq 4$.

1.7 Intersections in Projective Space

Solution 1.7.1. (a) Clearly $P_{d-\text{uple}}(x) = P_n(xd)$, and $P_n(x) = \binom{x+n}{n}$. So $P_{d-\text{uple}} = \binom{xd+n}{n} = \frac{(xd)^n}{n!}$+lower-terms, hence $\deg(d-\text{uple}) = d^n$.

(b) Since $P_{r\times s}(x) = P_r(X) \times P_s(x)$. So $P_{r\times s}(x) = \binom{x+r}{r} \cdot \binom{x+s}{s} = \frac{x^r}{r!} \cdot \frac{x^s}{s!}$+lower-terms, hence deg(Segre embedding) $= \frac{r!\cdot s!}{(r+s)!} = \binom{r+s}{r}$.

Solution 1.7.2. (a) Clearly $P_{\mathbb{P}^n}(0) = 1$, then $p_a(\mathbb{P}^n) = 0$.

(b) If Y is a plane curve in degree d, then $P_Y(x) = \binom{x+2}{2} - \binom{x-d+2}{2} = \frac{d}{2}(2x-d+3)$. Then $p_a(Y) = \frac{1}{2}(d-1)(d-2)$.

(c) If H is any hypersurface of degree d, then $P_H(x) = \binom{x+n}{n} - \binom{x-d+n}{n}$. So $p_a(H) = \binom{d-1}{n}$.

(d) Clearly $P_Y(x) = \binom{x+3}{3} - \binom{x-a+3}{3} - \binom{x-b+3}{3} + \binom{x-a-b+3}{3}$. So $p_a(Y) = \frac{1}{2}ab(a+b-4)+1$.

(e) Since $P_{Y\times Z}(x) = P_Y(x) \cdot P_Z(x)$, then $p_a(Y \times Z) = (-1)^{r+s}(P_Y(0)\cdot P_Z(0) - 1) = p_a(Y)p_a(Z) + (-1)^s p_a(Y) + (-1)^r p_1(Z)$.

Solution 1.7.3 (The Dual Curve)**.** Clearly if $P$is nonsingular and Y is defined by f. So we may denote $P = (x_0, y_0, z_0)$, and the coordinates of tangent line in $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})|_{(x_0,y_0,z_0)}$, which is well-defined. Thus $P \mapsto T_P(Y)$ is a morphism.

Solution 1.7.4. If a line meets Y in d points, this line is not a tangent line, and does not pass through any singular point. Since Y has only finite points, all lines pass through and form a closed set of $\mathbb{P}^{2*}$. So by 1.7.3. all tangent lines contain in a closed set of $\mathbb{P}^{2*}$. Thus out of these two closed sets is an open set U.

Solution 1.7.5. (a) If Y has a point with multiplicity$\geq d$, we may assume it is $P=(0,0)$. Then if f defines Y, f has all terms of degree d, i.e. a product of linear factors, which is contradict with that Y is irreducible of $d>1$.

(b) In $\mathbb{A}^2$ we may assume that f defining Y has the form $f=g+h$ with g homogeneous of degree $d-1$ and h homogeneous of degree d. Then denoting $t=\frac{y}{x}$, we have $f=0\Leftrightarrow y=-\frac{g(t,1)}{h(t,1)}$ and $x=yt$, which is the inverse rational map of $Y\to\mathbb{A}^1$. Hence Y is rational.

Solution 1.7.6 (Linear Varieties)**.** ($\Leftarrow$) Trivial. ($\Rightarrow$) Since Y has pure dimension r, by 7.6., Y is irreducible. And since the Hilbert polynomial of divisor variety has leading term as linear degree, if Y has degree 1, for hyperplane H not containing Y, we have $\deg(Y\cap H)=1$, i.e. linear.

Solution 1.7.7. (a) Denote the hyperplane of infinity in $\mathbb{P}^n$ as H. Then we may assume that H does not contain P and Y. So we can define $X\to\operatorname{Cone}(Y)$ as $PQ-$(points in infinity) $\mapsto$affine line on $\operatorname{Cone}(Y)$, which has $Q\mapsto Q$ and $P\mapsto$the cone point. Thus this map is birational, hence X is a variety with dimension$=\dim(\operatorname{Cone}(Y))=r+1$.

(b) If $\dim Y=0$, the $\deg Y=d$ means Y has d points, i.e. X has $d-1$ lines. If $\dim Y=r>0$, H is hyperplane containing P but not containing Y, by 7.7. and 7.6., $\deg X\cap H=\deg X$. But $\deg X\cap H\leq\deg Y\cap H<\deg Y<d$.

Solution 1.7.8. 1.7.7. means Y is contained in a variety H with degree 1, dimension r. So 1.7.6. means that H is a linear variety, which is isomorphic to $\mathbb{P}^{r+1}$. So Y is isomorphic to quadric hypersurface in $\mathbb{P}^{r+1}$.

1.8 What is Algebraic Geometry?

The reason of my death.

2 Schemes

2.1 Sheaves

Solution 2.1.1. Denote the constant presheaf as $\mathscr{P}$. Then for any $x \in X$, we just need to prove that $\mathscr{P}_x = \mathscr{A}_x$. Trivially we have $\mathscr{P}_x = A$. Otherwise, for any open set U containing x, we may denote the connected component of x in U as V. Then $V \subset U$ is open, i.e. $\mathscr{A}(V) = A$. So $\mathscr{A}_x = \varinjlim_{x \in U} \mathscr{A}(U) = A$. i.e. $\mathscr{A}_x = \mathscr{P}_x$ for any $x \in X$. Thus $\mathscr{A} = \mathscr{P}^+$.

Solution 2.1.2. (a) Since the sheaf of kernal is just the presheaf of kernal, we clearly have $(\ker\varphi)_P = \varinjlim_{P \in U}(\ker\varphi)(U) = \ker(\varinjlim_{P \in U}(\varphi)(U)) = \ker(\varphi_P)$, where the second equality is from the fact that $(\ker\varphi)(U) = \ker(\varphi(U))$ for all open set U.

Denote $\mathscr{P}$ as the image of φ as presheaf and $\mathscr{C}$ as the cokernal of φ as presheaf, i.e. $\mathscr{P}^+ = \mathrm{Im}\varphi$, $\mathscr{C}^+ = \mathrm{coker}\varphi$. Then $(\mathrm{Im}\varphi)_P = \mathscr{P}_P = (\ker(\mathscr{G} \to \mathrm{coker}\varphi))_P = \ker(\mathscr{G}_P \to (\mathrm{coker}\varphi)_P) = \ker(\mathscr{G}_P \to \mathscr{C}_P) = \ker(\mathscr{G}_P \to \mathrm{coker}(\varphi_P)) = \mathrm{Im}(\varphi_P)$.

(b) φ is injective$\Leftrightarrow \ker\varphi = 0 \Leftrightarrow \ker(\varphi_P) = 0$ for all $P \in X \Leftrightarrow \varphi_P$ is injective. And same for the surjective case.

(c) For any i, $\ker\varphi^i = \mathrm{Im}\varphi^{i-1} \Leftrightarrow (\ker\varphi^i)_P = (\mathrm{Im}\varphi^{i-1})_P$ for all $P \in X \Leftrightarrow \ker\varphi^i_P = \mathrm{Im}\varphi^i_P$ for all $P \in X$. So the exactness of sequence of sheaves is equivalent to the exactness of sequence of every stalks.

Solution 2.1.3. (a) $(\Leftarrow)$ If φ is surjective, by 2.1.2., we know that φ_P is surjective for every $P \in X$. So for every $P \in U$, s_P have a preimage (V, t) such that $(\varphi(t))_P = s_P$. Shrinking V if necessary, we have some sets $(V_i, t_i)_{i \in I}$ such that $\varphi(t_i) = s_{V_i}$.

$(\Rightarrow)$ We clearly have that φ_P is surjective for any $P \in X$. Then by 2.1.2., we have done.

(b) Take $X = \mathbb{C} - \{0\}$, $\mathscr{F} = \mathscr{G}$ are the sheaves of holomorphic functions on X and $\varphi(f) = \exp(f)$ for every section f. Then if open set $U \subset X$ is not simply connected, we know that $\varphi(U)$ is not surjective.

Solution 2.1.4. (a) By 2.1.2., we know that $(\mathrm{Im}\varphi^+)_P = \mathrm{Im}(\varphi^+_P) = \mathrm{Im}(\varphi_P) = (\mathrm{Im}\varphi)_P = 0$ for all $P \in X$, where the last equality if from the fact that φ is injective. So by 2.1.2., φ^+ is injective.

(b) Since $\mathrm{Im}\varphi$ is a subsheaf of $\mathscr{G}$, we have a morphism $g : \mathrm{Im}\varphi^+ \to \mathscr{G}$ by sheafification. And $g_P : \mathrm{Im}\varphi^+_P \cong \mathrm{Im}\varphi_P \to \mathscr{G}_P$ is injective. So g is injective, i.e. $\mathrm{Im}\varphi^+$ is a subsheaf of $\mathscr{G}$ by first part of 2.1.6.(b) below.

Solution 2.1.5. By proposition 1.1., the morphism of sheaves is an isomorphism iff it is an isomorphism on every stalks. So it is equivalent to the injection and surjection on every stalks, i.e. the injection and surjection of the morphism by 2.1.2.

Solution 2.1.6. (a) The natural map is $\mathscr{F}(U) \to \mathscr{F}(U)/\mathscr{F}'(U) \to (\mathscr{F}/\mathscr{F}')(U)$. And the surjection and exactness can be confirmed on stalks.

(b) Since $\varphi : \mathscr{F}' \to \mathscr{F}$ is an injection, if $\varphi(s) = 0 \in \mathscr{F}(U)$ for some $s \in \mathscr{F}'(U)$, we know for every $p \in U$, $\varphi(s)_P = 0$, and by injection on stalks we know that $s_P = 0$, i.e. $s = 0$. So $\mathscr{F}'$ is a subsheaf of $\mathscr{F}$.

On every $P \in X$ we know $0 \to \mathscr{F}'_P \to \mathscr{F}_P \to \mathscr{F}''_P \to 0$ is exact by 1.1.1.(c). So $\mathscr{F}''_P \cong \mathscr{F}_P/\mathscr{F}'_P = (\mathscr{F}/\mathscr{F}')_P$, that is, $\mathscr{F}'' \cong \mathscr{F}/\mathscr{F}'$, i.e. $\mathscr{F}''$ is a quotient sheaf of $\mathscr{F}$.

Solution 2.1.7. (a) Since we have an exact sequence $0 \to \ker\varphi \to \mathscr{F} \to \mathrm{coIm}\varphi \to 0$, and the category is a Abelian category, i.e. $\mathrm{coIm}\varphi \cong \mathrm{Im}\varphi$, we have $\mathrm{Im}\varphi \cong \mathscr{F}/\ker\varphi$ by 2.1.6.(b).

(b) Since we have an exact sequence $0 \to \mathrm{Im}\varphi \to \mathscr{G} \to \mathrm{coker}\varphi \to 0$, we have $\mathrm{coker}\varphi \cong \mathscr{G}/\mathrm{Im}\varphi$ by 2.1.6.(b).

Solution 2.1.8. Denote $f : \mathscr{F}' \to \mathscr{F}$ and $g : \mathscr{F} \to \mathscr{F}''$. Then the injection of f is in 2.1.6.(b), so we just need to prove the exactness at $\mathscr{F}(U)$.

For any $P \in U$, we have $(g_U \circ f_U)_P = \varinjlim_{P \in V \subset U}(g_U \circ f_U)_V = \varinjlim_{P \in V \subset U}(g_V \circ f_V) = (g \circ f)_P = 0$. Moreover, if we have some $s \in \mathscr{F}(U)$ such that $f(s) = 0$ in $\mathscr{F}''_U$, we can decompose $U = \bigcup_{i \in I} V_i$ and $s|_{V_i}$ has a preimage t_i

in $\mathscr{F}'_{V_i}$. And on $V_i \cap V_j$ for any $i, j \in I$, $f(t_i|_{V_i\cap V_j}) = f(t_j|_{V_i\cap V_j}) \in \mathscr{F}(V_i \cap V_j) = s|_{V_i\cap V_j}$. So by injection of f, we know that $t_i|_{V_i\cap V_j} = t_j|_{V_i\cap V_j}$, i.e. we may glue all t_i to get a $t \in \mathscr{F}'(U)$ which maps to s.

Solution 2.1.9 (Direct Sum)**.** It is trivially a sheaf. For any sheaf morphism $f : \mathscr{H} \to \mathscr{F}$ and $g : \to \mathscr{G}$, we can define $f \oplus g$ as $(f \oplus g)(s) = f(s) \oplus g(s) \in (\mathscr{F} \oplus \mathscr{G})(U)$ for any $s \in \mathscr{H}(U)$, which is a morphism too. So $\mathscr{F} \oplus \mathscr{G}$ we defined is the direct product of $\mathscr{F}$ and $\mathscr{G}$ in the category of sheaves of abelian groups. Since in abelian category, the direct product of finite objects is just the direct sum and vice versa, $\mathscr{F} \oplus \mathscr{G}$ is the direct sum of $\mathscr{F}$ and $\mathscr{G}$ too.

Solution 2.1.10 (Direct Limit)**.** Denote $\mathscr{F} = \varinjlim \mathscr{F}_i$ as presheaf, and $\mathscr{F}^+$ is the direct limit as sheaf. Then for any collection of $\mathscr{F}_i \to \mathscr{G}$ compatible with the maps of direct system, on every open set $U \subset X$ we have $\mathscr{F}_i(U) \to \mathscr{G}(U)$ compatible with the maps of direct system, hence we have a morphism $\varinjlim \mathscr{F}_i(U) \to \mathscr{G}(U)$, i.e. we get a presheaf morphism $\mathscr{F} \to \mathscr{G}$. By sheafification, this morphism factor as $\mathscr{F} \to \mathscr{F}^+ \to \mathscr{G}$. And we may compose $\mathscr{F}_i \to \mathscr{F} \to \mathscr{F}^+$, and we have $\mathscr{F}_i \to \mathscr{F}^+ \to \mathscr{G}$.

Solution 2.1.11. Denote this presheaf as $\mathscr{F}$. Then we need to prove that is a sheaf, i.e. to cheek the condition of glueing.

If we have $U = \bigcup_{k\in I} U_k$, then by noetherian we may assume that I is finite and $U = \bigcup_{k=1}^n U_k$. And if we have a set of sections $s_k \in \mathscr{F}(U_k)$ such that $s_k|_{U_k\cap U_l} = s_l|_{U_k\cap U_l}$ for all k, l, for s_k and s_l for $k, l = 1, \ldots, n$, we may find some i_{kl} such that $s_k^{i_{kl}} \in \mathscr{F}_{i_{kl}}(U_k)$ and $s_l^{i_{kl}} \in \mathscr{F}_{i_{kl}}(U_l)$ as the preimage of s_k and s_l such that $s_k^{i_{kl}}|_{U_k\cap U_l} = s_l^{i_{kl}}|_{U_k\cap U_l}$. Then take a $i > \max_{k,l}\{i_{kl}\}$, we have $s_k^i|_{U_k\cap U_l} = s_l^i|_{U_k\cap U_l}$. Hence those s_k^i can be glued up in $\mathscr{F}_i$ to get a s^i. Then just take the image s of s^i in $\mathscr{F}(U)$. And by the previous process, the s is unique since $\mathscr{F}_i$ is a sheaf.

Solution 2.1.12 (Inverse Limit)**.** We may denote $\mathscr{F} = \varprojlim \mathscr{F}_i$ as presheaf. If we have $U = \bigcup_{k\in I} U_k$ and a set of sections $s_k \in \mathscr{F}(U_k)$ such that $s_k|_{U_k\cap U_l} = s_l|_{U_k\cap U_l}$ for all k, l, we have $s_k^i|_{U_k\cap U_l} = s_l^i|_{U_k\cap U_l}$ for all k, l, where s_k^i is the image of s_k in $\mathscr{F}_i$ and same to s_l^i. So $\{s_k^i\}$ can be glued up to a unique $s^i \in \mathscr{F}_i(U)$, which has a direct limit $s \in \mathscr{F}(U)$, i.e. $\mathscr{F}$ is a sheaf.

If we have $\mathscr{G} \to \mathscr{F}_i$ satisfying the condition of inverse limit, for any open set U we have $\mathscr{G}(U) \to \mathscr{F}_i(U)$ satisfying that condition, so we have a morphism $\mathscr{G}(U) \to \varprojlim \mathscr{F}_i(U) = \mathscr{F}(U)$, i.e. we have defined a $\mathscr{G} \to \mathscr{F}$, which means $\mathscr{F}$ satisfies the universal property we need.

Solution 2.1.13 (Espace Étalé of a Presheaf)**.** For any $s \in \mathscr{F}^+(U)$, by strongest topology we have $\bar{S}$ is continuous. Then we just need to prove s is continuous. If V is open in Spé $(\mathscr{F})$ and $P \in s^{-1}(V)$, then clearly $P \in U$. Take a neighbourhood $U' \subset U$ of P, such that there exists a $t \in \mathscr{F}(U')$ with $s|_{U'} = t$. Then $s|_{U'}^{-1}(V) = t^{-1}(V)$ is an open neighbourhood of P, which is contained in $s^{-1}(V)$. So every point in $s^{-1}V$ has an open neighbourhood contained in the preimage, i.e. s is continuous.

If $s : U \to$ Spé $(\mathscr{F})$ is continuous, $V \subset X$ open and $t \in \mathscr{F}(V)$, we have $s(x) = t(x)$ for every $x \in t^{-1}(s(U))$, i.e. there exists an open subset $W \subset t^{-1}(s(U))$ such that $s|_W = t|_W$. Since t is continuous by strongest topology, $t^{-1}(s(U))$ is open in U for any $t \in \mathscr{F}(U)$, we know $s(U)$ is open in Spé $(\mathscr{F})$. So if $x \in U$, $s(x) = t_x$ in $\mathscr{F}$, and the continuity of s makes $s^{-1}(t(W))$ open. So for any open $W' \subset W$, we have $t|_{W'} = s|_{W'}$. So s locally gives a section of $\mathscr{F}$, i.e. a section of $\mathscr{F}^+$.

Solution 2.1.14 (Support)**.** If $P \in U$ such that $s_P = 0$, we know by definition there exists a subset $p \in V \subset U$ such that $s|_V = 0$. So $U -$ Supp (s) is open, Supp (s) is closed.

Solution 2.1.15 (Sheaf $\mathscr{H}om$)**.** It is clearly a presheaf. If we have $U = \bigcup_{k\in I} U_k$ and a set of sections $f_k \in \mathscr{H}om(\mathscr{F}, \mathscr{G})(U_k)$ such that $f_k|_{U_k\cap U_l} = f_l|_{U_k\cap U_l}$ for all k, l, for $s \in \mathscr{F}(U)$, we have $f_k(s|_{U_k})|_{U_k\cap U_l} = (f_k|_{U_k\cap U_l})(s|_{U_k\cap U_l}) = (f_l|_{U_k\cap U_l})(s|_{U_k\cap U_l}) = f_l(s|_{U_l})|_{U_k\cap U_l}$. So $f_k(s|_{U_k})$ can be uniquely glued up and get a $t = f(s)$, i.e. $\{f_k\}$ can be uniquely glued up. Hence $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is a sheaf.

Solution 2.1.16 (Flasque Sheaves)**.** (a) Since every subset of irreducible topological space is connected, we know that for every open set U, $\mathscr{F}(U) = A$, and the restriction map is always identity, hence surjective. So constant sheaf on irreducible topological space is flasque.

(b) If $\mathscr{F}'$ is flasque, we take a section $s'' \in \mathscr{F}''(U)$. Since $\mathscr{F} \to \mathscr{F}''$ is surjective, we may take $U = \bigcup_{i\in I} U_i$ and $s_i \in \mathscr{F}_{U_i}$ whose image in $\mathscr{F}''$ is s''_{U_i}. So on $U_i \cap U_j$, $s_{ij} = s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j}$ has the image 0 in $\mathscr{F}''$, hence has a preimage in $\mathscr{F}'(U_i \cap U_j)$, namely t_{ij}.

Let $\mathcal{S} = \{(t_i)_{i\in J} | J \subset I, t_i \in \mathscr{F}'(U_i)$ such that $t_{ij} = t_i|_{U_i\cap U_j} - t_j|_{U_i\cap U_j}$ for all $i, j \in J\}$. By Zorn's lemma, $\mathcal{S}$ must have a maximal element, namely $(t_i)_{i\in J}$. If $J \subsetneq I$, we may take $i_0 \in I - J$. For any $i, j \in J$, we have $t_i|_{U_i\cap U_j\cap U_{i_0}} - t_j|_{U_i\cap U_j\cap U_{i_0}} = t_{ij}|_{U_i\cap U_j\cap U_{i_0}} = t_{i_0 j}|_{U_i\cap U_j\cap U_{i_0}} - t_{i_0 i}|_{U_i\cap U_j\cap U_{i_0}}$. Since $(\bigcup_{i\in J} U_i) \cap U_{i_0} = \bigcup_{i\in J} (U_i \cap U_{i_0})$ and $t_i|_{U_i\cap U_{i_0}} + t_{i_0 i} \in \mathscr{F}'(U_i \cap U_{i_0})$ satisfying the glueing condition, they can be glued as $t_{i_0} \in \mathscr{F}'(U_{i_0})$. So $(t_i)_{i\in J\cup\{i_0\}} \in \mathcal{S}$, which contradict with the maximality of J. So we must have $I = J$, i.e. we find $(t_i)_{i\in I}$ such that $t_{ij} = t_i|_{U_i\cap U_j} - t_j|_{U_i\cap U_j}$. Then we have $(s_i - \varphi(t_i))|_{U_i\cap U_j} = (s_j - \varphi(t_j))|_{U_i\cap U_j}$, where $\varphi : \mathscr{F}' \to \mathscr{F}$. So we can glue all $(s_i - \varphi(t_i))$ up as $s \in \mathscr{F}(U)$, which has the image $s'' \in \mathscr{F}''(U)$, i.e. $\mathscr{F}(U) \to \mathscr{F}''(U)$ is surjective. So $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ is exact.

(c) If $\mathscr{F}'$ and $\mathscr{F}$ are flasque, since we have the following commutative diagram for open sets U, V such that $V \subset U$:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}'(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{F}''(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}'(V) \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{F}''(V) \arrow[r] & 0
\end{tikzcd}$$

we know that $\mathscr{F}''(U) \to \mathscr{F}''(V)$ is surjective by the exactness of first two vertical arrow.

(d) For every open sets $V \subset U \subset Y$, we have $f^{-1}(V) \subset f^{-1}(U) \subset X$. Since $\mathscr{F}(f^{-1}(U)) \to \mathscr{F}(f^{-1}(V))$ is surjective, $f_*\mathscr{F}(U) \to f_*\mathscr{F}(V)$ is surjective, hence $f_*\mathscr{F}$ is flasque.

(e) For every open sets $V \subset U$, and $s \in \mathscr{G}(V)$, we can define a $t \in \mathscr{G}(U)$ such that $s(P) = t(P)$ for $P \in V$ and $s(P) = 0$ for $P \notin V$. So the restriction map is surjective, i.e. $\mathscr{G}$ is flasque.

And we can define a map $\varphi : \mathscr{F} \to \mathscr{G}$ as $\varphi(s) = t$ for every $s \in \mathscr{F}(U)$, where t satisfies $t(P) = s_P \in \mathscr{F}_P$. Since if a section is zero on every stalks, this section must be zero, we know that φ is injective.

Solution 2.1.17 (Skyscraper Sheaves)**.** If $Q \in \overline{\{P\}}$ and $Q \in U$ for some open set U, we clearly have $Q \in U$, so $i_P(A)_Q = \varinjlim_{Q\in U} i_P(A)(U) = \varinjlim_{Q\in U} A = A$. Otherwise, if $Q \notin \overline{\{P\}}$, then there exists a very small open neighbourhood V of Q such that $P \notin V$, hence $i_P(A)_Q = \varinjlim_{Q\in U} i_P(A)(U) = \varinjlim_{Q\in U\subset V} i_P(A)(U) = 0$.

If $P \in U$ for some open set U, we have $i_*\mathscr{A}(U) = \mathscr{A}(i^{-1}(U)) = \mathscr{A}(\overline{\{P\}}) = A$. And if $P \notin U$ for some open set U, we have $i_*\mathscr{A}(U) = \mathscr{A}(i^{-1}(U)) = \mathscr{A}(\varnothing) = 0$. Hence $i_*\mathscr{A} \cong i_P(A)$.

Solution 2.1.18 (Adjoint Property of f^{-1})**.** Since $f^{-1}f_*\mathscr{F}$ is the sheaf associated to the presheaf $U \mapsto \varinjlim_{U\subset f^{-1}(V)} \mathscr{F}(f^{-1}(V))$, which clearly has a morphism to $\mathscr{F}$ as presheaf. Then by sheafification we have a sheaf morphism $f^{-1}f_*\mathscr{F} \to \mathscr{F}$. And for the other one, for any open subset $U \subset Y$, we have $f(f^{-1}(U)) \subset U$. So we have $\mathscr{G}(U) \to \varinjlim_{f(f^{-1}(U))\subset V} \mathscr{G}(V) \to f^{-1}\mathscr{G}(f^{-1}(U)) = f_*f^{-1}\mathscr{G}(U)$, i.e. we have define a morphism $\mathscr{G} \to f_*f^{-1}\mathscr{G}$.

We can define $\alpha : \mathrm{Hom}(\mathscr{G}, f_*\mathscr{F}) \to \mathrm{Hom}(f^{-1}\mathscr{G}, \mathscr{F})$ as $(\phi : \mathscr{G} \to f_*\mathscr{F}) \to (f^{-1}\mathscr{G} \xrightarrow{f^{-1}\phi} f^{-1}f_*\mathscr{F} \to \mathscr{F})$. And similarly a morphism $\beta : \mathrm{Hom}(f^{-1}\mathscr{G}, \mathscr{F}) \to \mathrm{Hom}(\mathscr{G}, f_*\mathscr{F})$ as $(\phi : f^{-1}\mathscr{G} \to \mathscr{F}) \to (g \to f_*f^{-1}\mathscr{G} \xrightarrow{f_*\phi} f_*\mathscr{F})$. Then clearly $\alpha \circ \beta = \mathrm{id}$ and $\beta \circ \alpha = \mathrm{id}$ by counting all stalks for $\beta(\alpha(\phi))$ or $\alpha(\beta(\phi))$.

Solution 2.1.19 (Extending a Sheaf by Zero)**.** (a) If $P \in Z$, we have $(i_*\mathscr{F})_P = \varinjlim_{P\in V} i_*\mathscr{F}(V) = \varinjlim_{P\in V\cap Z} \mathscr{F}(V \cap Z) = \mathscr{F}_P$. If $P \notin Z$, there exists a open subset $P \in W \subset X$ such that $W \cap Z = \varnothing$. So $(i_*\mathscr{F})_P = \varinjlim_{P\in V} i_*\mathscr{F}(V) = \varinjlim_{P\in V\subset W} i_*\mathscr{F}(V) = 0$.

(b) If $P \in U$, we have $(j_!\mathscr{F})_P = \varinjlim_{P\in V} j_!\mathscr{F}(V) = \varinjlim_{P\in V\subset U} \mathscr{F}(V) = \mathscr{F}_P$. If $P \notin U$, we have $(j_!\mathscr{F})_P = \varinjlim_{P\in V} j_!\mathscr{F}(V) = 0$ since there must exist a $P \in W \subset X$ such that $W \not\subset U$. Since $j_!\mathscr{F}$ is a sheaf, it must be unique to satisfy those conditions on stalks. And for any $V \subset U$, we have $(j_!\mathscr{F})|_U(V) = j_!\mathscr{F}(V) \cong \mathscr{F}(V)$, i.e. $j_!\mathscr{F})|_U \cong \mathscr{F}$ as sheaves on U.

(c) Since $0 \to j_!(\mathscr{F}|_U)_P \to \mathscr{F}_P \to i_*(\mathscr{F}|_Z)_P \to 0$ is $0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0 \to 0$ if $P \in U$, or $0 \to 0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0$ if $P \in Z$ by (a) and (b), they are both exact. So by 2.1.2.(c), the sequence is exact.

Solution 2.1.20 (Subsheaf with Supports)**.** (a) For any open set U with a covering $U = \bigcup_{i\in I} U_i$, and sections $s_i \in \Gamma_{Z\cap U_i}(U_i, \mathscr{F}|_{U_i})$ satisfying the compatibility condition, we may treat s_i as a section in $\mathscr{F}|(U_i)$ whose support is in Z. Then obviously all s_i satisfy the compatibility condition, i.e. s_i can be glued up as a unique $s \in \mathscr{F}(U)$. Since for any $P \in U$, $s_P = (s_i)_P$ for some i, then the support of s is just the union of all supports of s_i, so is contained in Z, i.e. $s \in \Gamma_{Z\cap U}(U, \mathscr{F}|_U)$. Hence $\mathscr{H}^0_Z(\mathscr{F})$ is a sheaf.

(b) It is clearly exact by counting the stalks like 2.1.19. And furthermore, since $j_*(\mathscr{F}|_U)(V) = \mathscr{F}|_U(U\cap V) = \mathscr{F}(U\cap V)$. Then on every open set V, $\mathscr{F}(V) \to j_*(\mathscr{F}|_U)(V)$ is just $\mathscr{F}(V) \to \mathscr{F}(V\cap U)$, which is surjective since $\mathscr{F}$ is flasque.

Solution 2.1.21 (Some Examples of Sheaves on Varieties)**.** (a) For any open set U with a covering $U = \bigcup_{i\in I} U_i$, and sections $f_i \in \mathscr{I}_Y(U_i)$ satisfying the compatibility condition, we know that every f_i is a regular function vanishing at all points of $Y\cap U_i$. Then there exists a unique $f \in \mathscr{O}_X(U)$ as the glueing of all f_i, hence it vanishes on every points of $Y \cap U$, i.e. $\mathscr{I}_Y$ is a sheaf.

(b) By construction, we clearly have a injective $\mathscr{I}_Y \to \mathscr{O}_X$. And for any open set $U \subset X$, $\operatorname{coker}(\mathscr{I}_Y(U) \to \mathscr{O}_X(U)) = \{f \text{ is a regular function on } Y\} = i_*(\mathscr{O}_Y)(U)$. So by 2.1.7.(b), we have $\mathscr{O}_X/\mathscr{I}_Y = {}_*(\mathscr{O}_Y)$.

(c) The global section of this exact sequence is $0 \to 0 \to k \to k\oplus k$. So the last arrow is clearly not surjective.

(d) For any open set U we clearly have a morphism $\mathscr{O}(U) \to \mathscr{K}(U)$ as $f \mapsto \bar{f}$, where $\bar{f}$ is the extension of f on X. If $\bar{f} = 0$, $f = \bar{f}|_U = 0$, hence injective. Moreover, on every point $P \in X$, $(\mathscr{K}/\mathscr{O})_P = \mathscr{K}_P/\mathscr{O}_P = I_P = (i_P(I_P))_P = (\sum_{P\in X} i_P(I_P))_P$, hence $\mathscr{K}/\mathscr{O} \cong \sum_{P\in X} i_P(I_P)$.

(e) Only need to show that $\Gamma(X, \mathscr{K}) \to \Gamma(X, \sum_{P\in X} i_P(I_P))$ is surjective, i.e. $K \to \bigoplus_{P\in X} I_P$ is surjective. So we just need show that for every $f \in K$ for some $P \in X$, then for every $Q \in X$, $Q \neq P$ we have some $g \in K$ such that $g \in \mathscr{O}_Q$ and $g - f \in \mathscr{P}$. If f is constant, trivial. If not, we may assume $f = x^p \cdot \frac{\alpha(x)}{\beta(x)}$, where $\alpha = \sum a_i x^i$ and $\beta = \sum b_i x^i$ for some $a_0, b_0 \neq 0$. Since PGL_1 is transitive, we may assume $P = 0$. Then if $p \geq 0$, we just take $g = 1$. If $p < 0$, we define $g = x^p \cdot \sum_{i=0}^{-p} c_i$, where $c_0 = \frac{a_0}{b_0}$, and $c_i = x^i b_0^{-1}(a_i - \sum_{j=0}^{i-1} c_j b_{i-j})$. Then $g \in \mathscr{O}_Q$ for any $Q \neq P$, and $f - g = x^p \beta(x)^{-1}(\alpha(x) - \beta(x)\sum_{i=0}^{-p} c_i) \in \mathscr{O}_P$.

Solution 2.1.22 (Glueing Sheaves)**.** For any open set $U \subset X$, we just define $\mathscr{F}(U) = \{(s_i) \in \mathscr{F}(U\cap U_i) \mid s_i|_{U\cap U_i\cap U_j} = s_j|_{U\cap U_i\cap U_j}\}$. Then $\mathscr{F}$ is clearly a sheaf on X.

2.2 Schemes

Solution 2.2.1. Since $(A_f)_g = A_{fg}$, and $\{D(f)\}$ form a basis of open set on $\operatorname{Spec} A$, we just need to prove that $\mathscr{O}(D(f)) = A_f$. We construct a $\varphi : A_f \to \mathscr{O}(D(f))$, $\frac{a}{f^k} \mapsto (D(f) \to \coprod_{\mathfrak{p}\in \operatorname{Spec} A} A_{\mathfrak{p}}, \mathfrak{p} \mapsto \frac{a}{f^k}$, then we just need to prove φ is a bijection.

If $\varphi(\frac{a}{f^k} = 0$, for any $\mathfrak{p} \in D(f)$, we have $\frac{a}{f^k} = 0$ in $A_{\mathfrak{p}}$, i.e. $\exists t \notin \mathfrak{p}$ such that $ta = 0$. So $\operatorname{Ann}(a) \not\subset \mathfrak{p}$ for all $\mathfrak{p}$, i.e. $V(\operatorname{Ann}(a)) \subset V((f))$. Then $f \in \sqrt{\operatorname{Ann}(a)}$. So $\frac{a}{f^k} = 0$ in A_f, hence φ is injective.

If $s \in \mathscr{O}(D(f))$, then we have an open covering of $D(f)$ as $\{U_i\}_{i\in I}$ and for any U_i there exist $a_i, f_i \in A$ such that $f_i \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{a_i}{f_i}$ for any $\mathfrak{q} \in U_i$. We may assume $U_i = D(g_i)$ for some $g_i \in A$ since we have a basis of open set $\{D(f)\}$, and this covering is finite covering since $\operatorname{Spec} A$ is noetherian. Then for any $\mathfrak{q} \in D(g_i)$ we have $f_i \notin \mathfrak{q}$, i.e. $D(g_i) \subset D(f_i)$, hence $\sqrt{(g_i)} \subset \sqrt{(f_i)}$. So $g_i^{k_i} = h_i f_i$ for some $h_i \in A$. Then $s_q = \frac{h_i a_i}{g_i^{k_i}}$. Since $D(g_i) = D_{g_i^{k_i}}$, we may replace g_i by $g_i^{k_i}$ and a_i by $h_i a_i$, i.e. we have an open covering $\{D(g_i)\}_{i\in I}$ for $D(f)$, and $s(q) = \frac{a_i}{g_i}$ for every $\mathfrak{q} \in D(g_i)$.

For any $\mathfrak{q} \in D(g_i) \cap D(g_j)$, we have $\frac{a_i}{g_i} = \frac{a_j}{g_j} \in A_\mathfrak{q}$. So for any $\mathfrak{q} \in D(g_ig_j)$, there exists a $t \notin \mathfrak{q}$ such that $t(a_ig_j - a_jg_i) = 0$, i.e. Ann $(a_ig_j - a_jg_i) \not\subseteq \mathfrak{q}$. So $V(\text{Ann}\,(a_ig_j - a_jg_i)) \subset V((g_ig_j))$, i.e. $g_ig_j \in \sqrt{\text{Ann}\,(a_ig_j - a_jg_i)}$. Since I is finite, we may take a k sufficiently large, such that $(g_ig_j)^k(a_ig_j - a_jg_i) = 0$. Then we may replace our $\frac{a_i}{g_i}$ by $\frac{a_ig_i^k}{g_i^{k+1}}$, i.e. we have an open covering $\{D(g_i)\}_{i\in I}$ for $D(f)$, and $s(q) = \frac{a_i}{g_i}$ for every $\mathfrak{q} \in D(g_i)$, and $a_ig_j = a_jg_i$ for every pairs i, j.

Since $D(f) = \bigcup_i D(g_i)$, we have $f^k = \sum b_ig_i$ for some $b_i \in A$. So for any j, we have $a_jf^k = a_j \sum b_ig_i = \sum b_ia_jg_i = \sum b_ia_ig_j = g_j \sum b_ia_i$, i.e. $\frac{a_j}{g_j} = \frac{\sum b_ia_i}{f^k}$. So φ is surjective.

Solution 2.2.2. Since X is a scheme, then it has an open covering $X = \bigcup V_i$, such that $(V_i, \mathscr{O}_X|_{V_i}) \cong (\text{Spec}\, A_i, \mathscr{O}_{\text{Spec}\, A_i})$ is an affine scheme. If U is some V_i, trivial. If not, then $\{U \cap V_i\}_{i\in I}$ form an open covering of U. And since U is open in X, then $U \cap V_i$ is open in V_i. Then $U \cap V_i = \bigcup_j \text{Spec}\,(A_i)_{f_{ij}}$ for some $f_{ij} \in A_i$ since $\{D(f)\}_{f\in A_i}$ forms a basis of open set of Spec A_i. And on every Spec $(A_i)_{f_{ij}}$, $\mathscr{O}_X|_U|_{\text{Spec}\,(A_i)_{f_{ij}}} = \mathscr{O}_{\text{Spec}\, A_i}|_{\text{Spec}\,(A_i)_{f_{ij}}} \cong \mathscr{O}_{\text{Spec}\,(A_i)_{f_{ij}}}$ by 2.2.1. So $(U, \mathscr{O}_X|_U)$ is a scheme.

Solution 2.2.3 (Reduced Schemes)**.** (a) ($\Rightarrow$) If $s \in \mathscr{O}_P$ is nilpotent, i.e. $s^n = 0$ for some n, there exists a neighbourhood U of p and $t \in \mathscr{O}(U)$ such that $t_p = s$ and $t^n = 0$. Then $t = 0$ because $\mathscr{O}(U)$ is reduced. So $s = t_P = 0$, hence $\mathscr{O}_P$ is reduced.

($\Leftarrow$) If $s \in \mathscr{O}(U)$ is nilpotent, i.e. $s^n = 0$ for some n. Then $s_P^n = 0$ for every $P \in U$, hence $s_P = 0$ for every $P \in U$, i.e. $s = 0$. So $\mathscr{O}(U)$ is reduced.

(b) If $X = \text{Spec}\, A$ is affine, we just take the Spec A_{red} as the reduced schemes of X, since $\text{Spec}\, A \cong \text{Spec}\, A_{\text{red}}$ as topological space, and $(A_f)_{\text{red}} = (A_{\text{red}})_{\bar{f}}$, where $\bar{f}$ is the image of f in A_{red}. For general case, if $X = \bigcup \text{Spec}\, A_i$, then clearly $\mathscr{O}_{X_{\text{red}}}(\text{Spec}\, A_i) = (A_i)_{\text{red}}$.

On every affine piece, the canonical morphism $A \to A_{\text{red}}$ induces a morphism $\text{Spec}\, A \to \text{Spec}\, A_{\text{red}}$, then we just glue them up to get a morphism $X \to X_{\text{red}}$. Since on every affine piece, $\text{Spec}\, A \to \text{Spec}\, A_{\text{red}}$ is a homeomorphism on topological space, hence so is $X \to X_{\text{red}}$.

(c) Since X is reduced, then for any open set $U \in Y$, $\mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U))$ induces a morphism $(\mathscr{O}_Y(U))_{\text{red}} \to \mathscr{O}_X(f^{-1}(U))$, hence we have a presheaf morphism $(U \mapsto (\mathscr{O}_Y(U))_{\text{red}}) \to f_*\mathscr{O}_X$. So by sheafification we have a morphism $\mathscr{O}_{Y_{\text{red}}} \to f_*\mathscr{O}_X$, i.e. a scheme morphism $X \to Y_{\text{red}}$. By construction, $X \to Y$ obviously factors through $X \to Y_{\text{red}}$.

Solution 2.2.4. We can define a $\beta : \text{Hom}(A, \mathscr{O}_X(X)) \to \text{Hom}(X, \text{Spec}\, A)$ as follows, for any $\phi : A \to \mathscr{O}_X(X)$, we may assume $X = \bigcup U_i = \bigcup \text{Spec}\, A_i$ is an affine covering, then the ring morphism $A \xrightarrow{\phi} \mathscr{O}_X(X) \to \mathscr{O}_X(U_i)$ induces a $f_i : U_i \to \text{Spec}\, A_i$. So for any $U_i \cap U_j$, we may cover it by some affine pieces U_{ijk}, where $U_{ijk} = \text{Spec}\, A_{ijk}$. Then $f_i|_{U_{ijk}} : \text{Spec}\, A_{ijk} \to \text{Spec}\, A$ and $f_j|_{U_{ijk}} : \text{Spec}\, A_{ijk} \to \text{Spec}\, A$ are clearly the same, i.e. f_i and f_j are the same on $U_i \cap U_j$. Hence $\{f_i\}$ can be glued up to get a $f : X \to \text{Spec}\, A$, i.e. β is well-defined. Clearly β is the inverse of α, i.e. α is bijection.

Solution 2.2.5. Clearly all the prime ideals of $\mathbb{Z}$ are (0) and (p) for all prime numbers p, and the unique minimal ideal of $\mathbb{Z}$ is (0). For every ideal $(n) \subset \mathbb{Z}$, $\mathscr{O}_\mathbb{Z}(D(n)) = \{\frac{a}{b} \mid p \nmid b \text{ for all } p|n\}$. And on every point of Spec $\mathbb{Z}$, $\mathbb{O}_{\mathbb{Z},(0)} = \mathbb{Q}$, and $\mathbb{O}_{\mathbb{Z},(p)} = \mathbb{Z}_p$.

Since $\mathbb{Z}$ is the initial object of the category of rings, then by 2.2.4., Spec $\mathbb{Z}$ is the final object of the category of schemes.

Solution 2.2.6. Since zero ring has no prime ideal, the spectrum of the zero ring is $\varnothing$. Since the zero ring is the final object of the category of rings, the spectrum of the zero ring is the initial object of the category of schemes.

Solution 2.2.7. If we have a morphism $\text{Spec}\, K \to X$, clearly the unique point of Spec K maps to a point $x \in X$. And we have a morphism $\mathscr{O}_x \to K$. Since K is a field, the maximal ideal $\mathfrak{m}_x$ of $\mathscr{O}_x$ maps to zero in K, i.e. it induces a morphism $k(x) = \mathscr{O}_x/\mathfrak{m}_x \to K$.

Conversely, if we fix a point $x \in X$ and a morphism $k(x) \to K$, we can define a sheaf morphism f : Spec $K \to X$ as the point of Spec $K \mapsto x$, and $\mathscr{O}_X \to f_* \mathscr{O}_{\text{Spec } K}$ as

$$\begin{cases} \mathscr{O}_X(U) \to 0 & \text{if } x \notin U \\ \mathscr{O}_X(U) \to \mathscr{O}_x \to k(x) \to K & \text{if } x \in U \end{cases}$$

Solution 2.2.8. Since $k[\varepsilon]/(\varepsilon^2)$ has only one prime (ε), Spec $k[\varepsilon]/(\varepsilon^2)$ has only one point. If we have a morphism Spec $k[\varepsilon]/(\varepsilon^2) \to X$, the point of Spec $k[\varepsilon]/(\varepsilon^2)$ has a image x. And it induces a local ring morphism $\phi : \mathscr{O}_x \to k[\varepsilon]/(\varepsilon^2)$. So $\phi(\mathfrak{m}_x) \subset (\varepsilon)$. Since $\varepsilon^2 = 0$, we can define a $\psi : \mathfrak{m}_x \to k$ as $a \mapsto$
$frac\phi(a)\varepsilon$, and it induces a morphism $T_x = \mathfrak{m}_x/\mathfrak{m}_x^2 \to k$. This morphism is from vector space to the constant field, so it is induced by a element of the vector space, and the morphism is the inner product.

Conversely, if we fix an $x \in X$ and an element of T_x, it induces the ψ above. Then we can define ϕ : $\mathscr{O}_x \to k[\varepsilon]/(\varepsilon^2)$ as $a + b \to a + \psi(b)\varepsilon$, where $a \in k$ and $b \in \mathfrak{m}_x$. So we can define a scheme morphism f : Spec $k[\varepsilon]/(\varepsilon^2) \to X$ as the point of Spec $k[\varepsilon]/(\varepsilon^2) \to x$. And $\mathscr{O}_X \to \mathscr{O}_{k[\varepsilon]/(\varepsilon^2)}$ is

$$\begin{cases} \mathscr{O}_X(U) \to 0 & \text{if } x \notin U \\ \mathscr{O}_X(U) \to \mathscr{O}_x \xrightarrow{\phi} k[\varepsilon]/(\varepsilon^2) & \text{if } x \in U \end{cases}$$

Solution 2.2.9. If $U =$ Spec A is an affine piece of X and $U \cap Z \neq \varnothing$, $U \cap Z$ must be an irreducible closed subset of U, so has the form $V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$. Since $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$, we may denote ζ as the point corresponding to $\mathfrak{p}$. So the closure of Z in U is $U \cap Z$, but $U \cap Z$ is a nonempty open subset in the irreducible space Z, hence dense in it, i.e. $Z = \overline{\{\zeta\}}$. If $Z = \overline{\{\zeta'\}}$ for another point ζ', we have $\zeta' \in U \cap Z$. If $\mathfrak{p}'$ is the prime ideal corresponding to ζ', we have $V(\mathfrak{p}') = \overline{\{\mathfrak{p}'\}} = U \cap Z = V(\mathfrak{p})$, which means $\mathfrak{p} = \mathfrak{p}'$, i.e. ζ is unique.

Solution 2.2.10. Since $\mathbb{R}[x]$ is a PID, its prime ideals are: i. the unique minimal prime ideal (0), whose residue field is $\mathbb{R}(x)$; ii. $(x + a)$ for some $a \in \mathbb{R}$, whose residue field is $\mathbb{R}$; iii. $(x^2 + bx + c)$ for some $b, c \in \mathbb{R}$ such that $b^2 < 4c$, whose residue field is $\mathbb{C}$. So Space (Spec $\mathbb{R}[x]) \cong \mathbb{R} \cup \mathbb{H} \cup \{0\}$ as a set.

Solution 2.2.11. Since $\mathbb{F}_p[x]$ is a PID, its prime ideals are: i. the unique minimal prime ideal (0), whose residue field is $\mathbb{F}_p(x)$; ii. (f) for some irreducible polynomial in $\mathbb{F}_p(x)$, whose residue field is $\mathbb{F}_{p^d}$ if $\deg f = d$. Since $x^{p^n} - x$ is the product of all prime polynomial with degree $d|n$, so the number of all the prime polynomial with degree d is $\sum_{d|n} \mu(d) p^{n/d}$, i.e. $\sharp\{\text{points with residue field } \mathbb{F}_{p^d}\} = \sum_{d|n} \mu(d) p^{n/d}$.

Solution 2.2.12 (Glueing Lemma)**.** We may glue the underlying space as $X = \coprod X_i / \sim$, where the equivalent relation is $x_i \in X_i \sim x_j \in X_j \Leftrightarrow \varphi_{ij}(x_i) = x_j$. Then we clearly have maps $\psi_i : X_i \to X$. So we can define the sheaf on $\mathbb{X}$ as glueing all $\psi_{i*}\mathscr{O}_{X_i}$ on $\mathbb{X}$, hence $(X, \mathscr{O}_X$ is a scheme, and the four conclusion is naturally gotten.

Solution 2.2.13. (a) ($\Leftarrow$) If X is noetherian and U is a open set in X with an open covering $U = \bigcup U_i$. If U does not have a finite subcovering, then we can construct a ascending chain $\{V_i\}$ in U as: $V_1 = U_{i_1}$ for some U_{i_1}, and $V_{n+1} = V_n \cup U_{i_n}$ for some $U_{i_n} \not\subseteq V_n$. Since U does not have a finite subcovering, this construction is well-defined, i.e. we get a ascending chain $V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_n \subsetneq \ldots$, which is contradict with the fact that X is noetherian.

($\Rightarrow$) If $U_1 \subset U_2 \subset \ldots \subset U_n \subset \ldots$ is an ascending chain in X, then $U = \bigcup_i U_i$ has an open covering $\bigcup_i U_i$. Since it is quasi-compact, U has a finite covering, which will involve U_n as the maximal index, hence $U_n = U_{n+1} = \ldots$.

(b) If $\{U_i\}$ covers $X = \text{Spec} A$, we may assume $X - U_i = V(\mathfrak{a}_i)$ closed, then $\sum_i \mathfrak{a}_i = A$, i.e. we have $1 = \sum_i^n f_i g_i$ for some $f_i \in A$ and $g_i \in \mathfrak{a}_i$ for some $\mathfrak{a}_i$. So $X = \bigcup_{i=1}^n U_i$ for those U_i, hence quasi-compact.

For example of non-noetherian but quasi-compact spectrum, we just take $A = k[x_1, x_2, \ldots]$, then we have an ascending chain of closed subsets $V((x_1)) \subsetneq V((x_1, x_2)) \subsetneq \ldots \subsetneq V((x_1, x_2, \ldots, x_n)) \subsetneq \ldots$.

(c) For any descending chain $Z_1 \supset Z_2 \supset \ldots \supset Z_n \supset \ldots$ in Spec A, we have a corresponding ascending chain of ideals in A as $I_1 \subset I_2 \subset \ldots \subset I_n \subset \ldots$, which will be stable since A is noetherian. So the descending chain $\{Z_i\}$ will be stable, hence Space (Spec A) is noetherian.

(d) Take $A = k[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$. So Space (Spec A) is a singleton, hence noetherian. But A itself is not noetherian.

Solution 2.2.14. (a) ($\Leftarrow$) If Proj $S = \varnothing$, then every graded prime ideal in S containing S_+. For any prime ideal $\mathfrak{p}$, we have a graded prime ideal $\mathfrak{q} = \bigoplus_{d=0}^{\infty} \mathfrak{p} \cap S_d$, then $\mathfrak{p} \supset \mathfrak{q} \supset S_+$. Hence S_+ is contained in the nilpotent radical, i.e. every element of S_+ is nilpotent.

($\Rightarrow$) If every element of S_+ is nilpotent, every prime ideal in S contain S_+, hence Proj $S = \varnothing$.

(b) For any $\mathfrak{p} \in U$, there exists some $s \in S_+$ such that $\varphi(s) \notin \mathfrak{p}$. So $D(\varphi(s)) \subset U$ is a neighbourhood of $\mathfrak{p}$. Hence U is open. Moreover, for every $\mathfrak{p} \in U$, we may define $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p}) \not\supseteq S_+$, hence well-defined. And the morphism on sheaves $f_\sharp$ is given by $S_{(\varphi^{-1}\mathfrak{p})} \to T_{(\mathfrak{p})}$.

(c) For any $\mathfrak{p} \not\supseteq T_+$, there exists some $t \in T_k$, such that $t \notin \mathfrak{p}$. So $t^n \notin \mathfrak{p}$ for some n such that $nk \geq d_0$. Then there exists some $s \in S_+$ such that $s = \varphi(t^n)$, hence $\imath \not\supseteq \varphi(S_+)$, i.e. $U =$ Proj T.

For any $\mathfrak{p} \in$ Proj S, we denote $\mathfrak{q} = \sqrt{\varphi(\mathfrak{p})}$. If $a \in \varphi^{-1}\mathfrak{q}$, we have $\varphi(a^n) \in \varphi(\mathfrak{p})$, i.e. $\varphi(a^n) = \sum b_i \varphi(s_i)$ for some $b_i \in T$ and $s_i \in \mathfrak{p}$. Since $(\sum b_i \varphi(s_i))^m$ is a polynomial in $\varphi(s_i)$ with coefficients in $T_{\geq d_0}$, $\varphi(a^{nm}) \in \varphi(\mathfrak{p})$, i.e. $a \in \mathfrak{p}$. Thus $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, i.e. f is surjective. Conversely, if $f(\mathfrak{p}) = f(\mathfrak{q})$ for some $\mathfrak{p}, \mathfrak{q} \in$ Proj T, we have $\varphi^{-1}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{q})$. For $t \in \mathfrak{p} \cap T_+$, there exists a $s \in S$ such that $\varphi(s) = t^k$ for some sufficiently large k. So $t^k = \varphi(s) \in \mathfrak{q}$, i.e $t \in \mathfrak{q}$. Hence $\mathfrak{p} \subset \mathfrak{q}$, and vice versa, which means injection.

Since all $D_+(s)$ cover Proj S, then for sufficiently large i, $f^{-1}D_+(s^i) = D_+(t) \subset$ Proj T for some t. Then we need to prove that $S_{(s^i)} \to T_{(t)}$ is an isomorphism. Changing s^i into s, if $\frac{f}{s^n} \mapsto 0$, we have $0 = t^m \varphi(f) = \varphi(s^m f)$ for some large m, i.e. $s^m f \in \ker \varphi$. Since $S_d \cong T_d$ for sufficiently large d, some large powers of $s^m f$ maps to 0, i.e. $\frac{f}{s^n} = 0$, hence $S_{(s)} \to T_{(t)}$ is injective. If $\frac{g}{t^n} \in T_{(t)}$, we know that $\frac{t^{d_0} g}{t^{n+d_0}}$ has a preimage in $S_{(s)}$, hence surjective.

(d) Since $V = \{$maximal ideals of S which do not contain $S_+\}$, we know that $t(V) = \{$irreducible components of $V\} = \{$prime ideals of S which do not contain $S_+\}$, i.e. $t(V) =$ Proj S. And clearly $t(V)$ and Proj S have same topology. Moreover, for every U open in Proj S, denote the preimage of U in V as U', $\alpha_* \mathscr{O}_V(U) = \mathscr{O}_V(U') = \{$regular functions which have no poles in $U'\} = \{$regular functions which have no poles in $U\} = \mathscr{O}_{\text{Proj } S}(U)$, hence $(t(V), \alpha_* \mathscr{O}_V) \cong (\text{Proj } S, \mathscr{O}_{\text{Proj } S})$.

Solution 2.2.15. (a) P is not a closed point$\Leftrightarrow$the corresponding irreducible closed subset Z of P is not a point$\Leftrightarrow$the residue field of Z is has transcendental degree≥ 1. So P is a closed points iff the residue field of P is just k.

(b) Since we have the morphism of sheaves $\mathscr{O}_Y \to f_* \mathscr{O}_X$, it induces a morphism of residue fields $k(f(P)) \hookrightarrow k(P)$. Since X and Y are both fields over k, $k(f(P))$ and $k(P)$ are both field extension of k. But $k(P) = k$, so we have $k \hookrightarrow k(f(P)) \hookrightarrow k$, i.e. $k(f(P)) = k$.

(c) Denote this morphism by υ. For any $\varphi : V \to W$, we can define $\upsilon(\varphi)$ as φ on closed points, and for any nonclosed points which corresponding to irreducible closed subset Y of V, we can define $\upsilon(\varphi)(Y) = \overline{\varphi(Y)}$. So since $\upsilon(\varphi)$ is an extension of φ, the injection is obvious. For surjection, if $\psi : t(V) \to t(W)$ is a morphism of schemes, we have $\upsilon^{-1}(\psi) = \psi|_V$. Since ψ maps closed points to closed points, this is well-defined. If $\psi(P) = Q$ for closed points P and Q in $t(V)$ and $t(W)$, we may assume $P = \mathfrak{p} \subset A$ in some affine piece $U =$ Spec A and $Q = \mathfrak{q} \subset B$ in some affine piece $V =$ Spec $B \subset f^{-1}(U)$. Then $P \mapsto Q$ is just Spec $B \to$ Spec A on stalks, hence $\upsilon^{-1}(\psi)$ is regular, i.e. $\upsilon^{-1}(\psi) \in \mathrm{Hom}_{\mathfrak{Var}}(V, W)$.

Solution 2.2.16. (a) If $x \in X_f \cap U$, we know $f_x \notin \mathfrak{m}_x$, i.e. $\bar{f}_x \notin \mathfrak{m}_x$, hence $x \in D(\bar{f})$. If $x \in D(\bar{f}) \subset U$, we have $\bar{f}_x \notin \mathfrak{m}_x$, i.e. $f_x \notin \mathfrak{m}$, hence $x \in X_f$.

(b) Since X is quasi-compact, we may assume $X = \bigcup U_i$ for finite i, and $U_i =$ Spec A_i is affine. For every i, since $a|_{X_f} = 0$, we know that $a_i = a|_{U_i}$ restricts on $(U_i)_{f_i}$ is zero, where f_i is the restriction of f on U_i. So there exists a n_i such that $f_i^{n_i} a_i = 0$. If we denote $n = \max\{n_i\}$ since the index set is finite, we have $f_i^n a_i = 0$, i.e. $(f^n a)|_{U_i} = 0$ for every i. Hence we have $f^n a = 0$ on whole X.

(c) On every U_i, $b|_{U_i \cap X_f} = b|_{D(f_i)}$, there exists some n_i such that $f_i^{n_i} b|_{D(f_i)}$ can be extended into U_i. So we may take an $n = \max\{n_i\}$ so that $f_i^n b|_{D(f_i)}$ can be extended into U_i as some b_i. On every i, j, we may assume $U_i \cap U_j = \bigcup_{k \in I_{ij}} U_{ijk}$ for some finite index set I_{ij}. On every U_{ijk}, since $b_i|_{U_{ijk}} - b_j|_{U_{ijk}} = 0$, there exists some m_{ijk} such that $f_{U_{ijk}}^{n_{ijk}}(b_i|_{U_{ijk}} - b_j|_{U_{ijk}}) = 0$. Then since the index ijk is finite, we can take some $m = \max\{m_{ijk}\}$ such that $f^m(b_i|_{U_{ij}} - b_j|_{U_{ij}}) = 0$. So $f^{n+m} b$ can be extended to some global section.

(d) Since $A = \mathscr{O}_X(X) \to \mathscr{O}_X(X_f)$ maps f to zero, it induces a morphism $A_f \to \mathscr{O}_X(X_f)$. But (b) means injection and (c) means surjection. So we have $A_f \cong \mathscr{O}_X(X_f)$.

Solution 2.2.17 (A Criterion for Affineness)**.** (a) Denote the inverse of $f|_{f^{-1}(U_i)}$ as g_i. Since two maps $f|_{f^{-1}(U_i)}|_{f^{-1}(U_i \cap U_j)}$ and $f|_{f^{-1}(U_j)}|_{f^{-1}(U_i \cap U_j)}$ are just $f|_{f^{-1}(U_i \cap U_j)}$, and they are obviously both isomorphism, there exists g_{ij} are the inverse of $f|_{f^{-1}(U_i \cap U_j)}$. Clearly $g_{ij} = g_i|_{U_i \cap U_j} = g_j|_{U_i \cap U_j}$, which means all g_i can be glued together into a $g : Y \to X$ as the inverse of f. So f is isomorphism.

(b) The identity $A \to \mathscr{O}(X)$ induces a $\varphi : X \to \operatorname{Spec} A$. Since $\{f_i\}$ generate the unit ideal, $\{D(f_i)\}$ forms an open covering of $\operatorname{Spec} A$. Since clearly $\varphi^{-1}(D(f_i)) = X_f$, by 2.2.16.(d) we have $\mathscr{O}_X(X_{f_i}) \cong A_{f_i}$. Since φ is isomorphic on $\varphi^{-1}(D(f_i))$, by (a), φ is an isomorphism.

Solution 2.2.18. (a) f is nilpotent$\Leftrightarrow$there exists an n such that $f^n = 0 \Leftrightarrow D(f) = D(f^n) = D(0) = \varnothing$.

(b) ($\Leftarrow$) If φ is injective, for any prime $\mathfrak{q} \in B$, the $f^\sharp_{\varphi^{-1}(\mathfrak{q})}$ is just $A_{\varphi^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$, which is injective. Hence $f^\sharp$ is injective.

($\Rightarrow$) Just take the global section of $f^\sharp$. Then we can get $\varphi = f^\sharp : A \to B$ is injective.

Furthermore, for any $U \subset X$, we have $U \subset V(\bigcap_{\mathfrak{p} \in U} \mathfrak{p})$, i.e. $\bar{U} \subset V(\bigcap_{\mathfrak{p} \in U} \mathfrak{p})$. If $\bar{U} = V(\mathfrak{a})$ for some $\mathfrak{a} \subset A$, for any $\mathfrak{p} \in U$, we have $\mathfrak{a} \subset \mathfrak{p}$, i.e. $\mathfrak{a} \subset \bigcup_{\mathfrak{p} \in U} \mathfrak{p}$. So we get $\bar{U} = V(\bigcap_{\mathfrak{p} \in U} \mathfrak{p})$. Thus,

$$\overline{f(Y)} = V(\bigcap_{\mathfrak{p} \in f(Y)} \mathfrak{p}) = V(\bigcap_{\mathfrak{q} \in Y} \varphi^{-1}(\mathfrak{q})) = V(\varphi^{-1}(\bigcap_{\mathfrak{q} \in Y} \varphi^{-1}\mathfrak{q})) = V(\varphi^{-1}(\sqrt{(0)})) = V(\ker\varphi) = V(0) = X$$

(c) If φ is surjective, $B \cong A/\ker\varphi$ induces $\operatorname{Spec} B \cong \operatorname{Spec} A/\ker\varphi = V(\ker\varphi) \subset \operatorname{Spec} A$ is closed. Furthermore, we have $f^\sharp_{\mathfrak{p}} = A_{\mathfrak{p}} \to B \otimes_A A_{\mathfrak{P}}$ is surjective, hence $f^\sharp$ is surjective.

(d) If $f^\sharp$ is surjective, for any $b \in B \in \mathscr{O}_Y(Y)$, there exists an open covering $X = \bigcup U_i$ such that $b|_{U_i}$ has preimage $\frac{a_i}{f^{N_i}}$ in $\mathscr{O}_X(U_i)$. Since we may cover each U_i by affine pieces and X can be covered by such affine pieces, we may assume the covering $X = \bigcup U_i$ is affine covering for $U_i = \operatorname{Spec} A_{f_i} = D(f_i)$ for some $f_i \in A$. Since X is quasi-compact, we may assume this covering is finite as $X = \bigcup_{i=1}^n U_i$, and $N = \max\{N_i\}$. Hence we have some $g_i \in A$ such that $\sum_{i=1}^n f_i^N g_i = 1$ since $\{D(f_i^N)\}$ covers X. So $b = \sum \varphi(f_i)^N \varphi(g_i) b = \varphi(\sum g_i f_i^{N-N_i} a_i)$.

Solution 2.2.19. (i$\Rightarrow$ii) If $X = \operatorname{Spec} A$ is disconnected, there exists two disjoint open subsets U_1, U_2. So there exist $e_1, e_2 \in \mathscr{O}_X(X) = A$ such that $e_1|_{U_1} = 1$, $e_2|_{U_1} = 0$ in $\mathscr{O}_X(U_1)$ and $e_1|_{U_2} = 0$, $e_2|_{U_2} = 1$ in $\mathscr{O}_X(U_2)$ since $\mathscr{O}_X$ is a sheaf. So exactly $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1 + e_2 = 1$.

(ii$\Rightarrow$iii) Just define $A_1 = e_1 \cdot A$ and $A_2 = e_2 \cdot A$. We have $A = A_1 \times A_2$.

(iii$\Rightarrow$i) Since prime ideals of $A_1 \times A_2$ have two kinds: $\mathfrak{p} \times A_2$ for some prime $\mathfrak{p} \subset A_1$, and $A_1 \times \mathfrak{q}$ for some prime $\mathfrak{q} \subset A_2$. Clearly we have $\operatorname{Spec} A = \operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2$.

2.3 First Properties of Schemes

Solution 2.3.1. ($\Rightarrow$) If $Y = \bigcup V_i = \bigcup \operatorname{Spec} B_i$ is an affine covering of Y, and $f^{-1}(\operatorname{Spec} B_i) = \bigcup \operatorname{Spec} A_{ij}$ for some finitely generated B_i-algebra A_{ij}. Since $V \cup V_i$ is open in V_i, there exists an affine covering $V \cup V_i = \bigcup \operatorname{Spec}(B_i)_{f_{ik}}$ for some $f_{ik} \in B_i$. If we denote $\varphi : B_i \to A_{ij}$, then $(A_{ij})_{\varphi(f_{ik})}$ is a finitely generated $(B_i)_{f_{ik}}$-algebra.

So we may assume that $\operatorname{Spec} B = \bigcup \operatorname{Spec} B_i$ with $f^{-1}(\operatorname{Spec} B_i) = \bigcup \operatorname{Spec} A_{ij}$ for some finitely generated B_i algebra A_{ij}. For any $\mathfrak{p} \in \operatorname{Spec} B$, we have $\mathfrak{p} \in \operatorname{Spec} B_i$ for some i. Since $\operatorname{Spec} B_i$ is open in $\operatorname{Spec} B$, there exists some $f_\mathfrak{p} \in B$ such that $\mathfrak{p} \in \operatorname{Spec} B_{f_\mathfrak{p}} \subset \operatorname{Spec} B_i$. So we have $\operatorname{Spec} (B_i)_{f_\mathfrak{p}} \cong \operatorname{Spec} B_{f_\mathfrak{p}}$. Since $f^{-1}(\operatorname{Spec} (B_i)_{f_\mathfrak{p}}) = \bigcup \operatorname{Spec} (A_{ij})_{\varphi(f_\mathfrak{p})}$, we know $(A_{ij})_{\varphi(f_\mathfrak{p})}$ is a finitely generated $B_{f_\mathfrak{p}}$-algebra, hence a finitely generated B-algebra.

($\Leftarrow$) Obvious.

Solution 2.3.2. ($\Rightarrow$) We may assume $Y = \bigcup V_i = \bigcup \operatorname{Spec} B_i$ is an affine open covering, and $f^{-1}(V_i)$ is quasi-compact. If $V \subset Y$ is an affine piece, $V \cap V_i = \bigcup D(g_{ij})$ for some $g_{ij} \in V_i$. Then V can be covering by these $D(f_{ij})$. Since V is affine, i.e. quasi-compact, we may choose finitely many $D(g_{ij})$ covering V. If for every V_i, we have $f^{-1}(V_i) = \bigcup \operatorname{Spec} A_{ik}$ for finitely many A_{ik}. So $f^{-1}(D(g_{ij})) = \bigcup \operatorname{Spec} (A_{ik})_{\varphi(g_{ij})}$, where $\varphi : B_i \to A_{ik}$, i.e. $f^{-1}(D(g_{ij}))$ is quasi-compact. So $f^{-1}(V) = \bigcup f^{-1}(D(g_{ij}))$ is quasi-compact.

($\Leftarrow$) Obvious.

Solution 2.3.3. (a) ($\Rightarrow$) We only need to show f is quasi-compact. If V is an affine piece in Y, then $f^{-1}(V)$ can be covered by finite affine piece in X. Since every affine piece is quasi-compact, $f^{-1}(V)$ is quasi-compact.

($\Leftarrow$) Obvious.

(b) For any affine piece $V = \operatorname{Spec} B$, f is of finite type $\Leftrightarrow$ f is locally of finite type and quasi-compact $\Leftrightarrow$ $f^{-1}(V)$ is quasi-compact and can be covered by $\operatorname{Spec} A_i$ for finitely generated B-algebra A_i $\Leftrightarrow$ $f^{-1}(V)$ can be covered by finitely many $\operatorname{Spec} A_i$ with same property.

(c) Since f is of finite type, $f^{-1}(V) = \bigcup U_i = \bigcup \operatorname{Spec} A_i$ for finitely many finitely generated B-algebra A_i. If $U = \operatorname{Spec} A \subset f^{-1}(V)$, for any point $\mathfrak{p} \in U \cap U_i$, there exists same $f' \in A$ such that $\mathfrak{p} \in \operatorname{Spec} A_{f'} \subset U \cap U_i$. Since $\operatorname{Spec} A_{f'}$ is open in U_i, there exists some $f_i \in A_i$ such that $\mathfrak{p} \in \operatorname{Spec} (A_i)_{f_i} \subset \operatorname{Spec} A_{f'}$. This induces a morphism $A_i \to A_{f'} \to (A_i)_{f_i}$. So the image of f_i in $A_{f'}$ has a preimage in A, namely f. So we have $\operatorname{Spec} A_f \cong \operatorname{Spec} (A_i)_{f_i}$. Clearly $(A_i)_{f_i}$ is a finitely generated A_i-algebra, hence a finitely generated B-algebra with basis $\{\frac{a_{i1}}{1}, \dots, \frac{a_{i,n_i}}{1}, \frac{1}{f_i}\}$.

So, we may assume that $U = \bigcup \operatorname{Spec} A_{f_i}$ for some f_i with A_{f_i} are all finitely generated B-algebra. Since U is quasi-compact, we may assume this covering is finite. For any $a \in A$, we have $\frac{a}{1} = \frac{s_i}{f_i^k}$ for some $s_i \in B[\frac{a_{i1}}{1}, \dots, \frac{a_{i,n_i}}{1}, \frac{1}{f_i}]$ and a fixed sufficiently large k. Thus there exists another sufficiently large l such that $f_i^l(f_i^k a - s_i)$. We may change $k + l$ as k and $f_i^l s_i$ as s_i and have $f_i^k a = s_i$. Since $U = \bigcup D(f_i^k)$, there exists some $g_i \in A$ such that $\sum f_i^k g_i = 1$. So $a = \sum g_i s_i \in B[S]$ with $S = \{a_{ij}, f_i \text{ for all } i, j\}$.

Solution 2.3.4. ($\Rightarrow$) We may assume $Y = \bigcup V_i = \bigcup \operatorname{Spec} B_i$, $f^{-1}(V_i) = U_i = \operatorname{Spec} A_i$ with some finitely generated B_i-module A_i. If we denote $\varphi_i : B_i \to A_i$, for every $f \in B_i$, clearly $f^{-1}(D(f)) = D((\varphi(f))) \subset \operatorname{Spec} A_i$. And more, $(A_i)_{\varphi(f)}$ is a finitely generated $(B_i)_f$-module.

So for any affine piece $V = \operatorname{Spec} B \subset Y$, similar with 2.3.3., we know that $V \cap V_i$ can be covered by basis open sets of both V and V_i, which preimage is basis open sets of U_i and finite over their images. So we may assume $V = \bigcup \operatorname{Spec} B_i$ with $f^{-1}(V_i) = U_i = \operatorname{Spec} A_i$ for some finite B_i-module A_i with basis $\{\frac{a_{i1}}{f_i^k}, \dots, \frac{a_{i,n_i}}{f_i^k}\}$. By 2.2.17., we know $f^{-1}(V) = U$ is affine. Moreover, we may assume $V = \bigcup V_i$ is finite, since V is quasi-compact. So if $V_i = \operatorname{Spec} B_{f_i}$ for some $f_i \in B$, $\{f_1, \dots, f_n\}$ generates B, then the preimages of them generates A. For any $a \in A$, we have $\frac{a}{1} = \frac{\sum s_j a_{ij}}{f_i^k}$. So there exists an l such that $f_i^l(f_i^k a - \sum s_j a_{ij}) = 0$. So we may replace $k + l$ to k and $f_i^l a_{ij}$ to a_{ij}. And we have $f_i^k a = \sum s_j a_{ij}$. Since $D(f_i^k)$ covers U, we have some $g_i \in U$ such that $\sum f_i^k g_i = 1$, hence $a = \sum_i \sum_j g_i s_j a_{ij}$, i.e. A is finite over B.

Solution 2.3.5. (a) Since this question is local, we may assume $Y = \operatorname{Spec} B$ is affine. Hence $X = f^{-1}(Y) = \operatorname{Spec} A$ is affine and A is finite generated B-module. For any $y \in Y$, we have $f^{-1}y = \operatorname{Spec} A \otimes_B \kappa(y)$, which is a finitely generated $\kappa(y)$-module, i.e. a vector space of finite rank. So $f^{-1}(y)$ is finite.

(b) We may assume $Y = \bigcup V_i = \bigcup \operatorname{Spec} B_i$ and $f^{-1}(V_i) = U_i = \operatorname{Spec} A_i$ for some finitely generated B_i-module A_i. For any closed set $Z \subset X$, we know that $Z \cap U_i$ is closed in U_i. If $f(Z \cap U_i)$ is also closed in V_i, $V_i - f(Z \cap U_i)$ is open in Y, i.e. $Y - f(Z)$ is open in Y, hence $f(Z)$ is closed. Hence we reduce this problem into affine case.

So we may assume $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$. Then any closed subset $Z \subset X$ has the form $Z = V(\mathfrak{b})$ for some ideal $\mathfrak{b} \subset A$. If we denote the morphism $\varphi : A \to B$ induced by $X \to Y$, $\mathfrak{a} = \varphi^{-1}(\mathfrak{b}) \subset B$ is an ideal, and $W = V(\mathfrak{a}) \subset Y$. So we only need to prove $f(Z) = W$. Obviously we have $f(Z) \subset W$. Conversely, for any point $\mathfrak{p} \in W$, i.e. a prime ideal $\mathfrak{p} \subset B$ with $\mathfrak{b} \subset \mathfrak{p}$. By going-up theorem, it can be lifted to a prime $\mathfrak{q} \subset A$ such that $\mathfrak{a} \subset \mathfrak{q}$, which means $f(Z) \supset W$, hence $f(Z) = W$.

(c) The ring morphism $k[t] \to k[t, \frac{1}{t}] \oplus k[t, \frac{1}{t-1}]$ induces a morphism of schemes: $\operatorname{Spec} k[t, \frac{1}{t}] \oplus k[t, \frac{1}{t-1}] \to \operatorname{Spec} k[t]$, which is surjective, finite-type and quasi-finite but not finite.

Solution 2.3.6. If ξ is in some affine piece $U = \operatorname{Spec} A$, it corresponds to a minimal prime ideal $\mathfrak{p} \subset A$. Since X is an integral scheme, A must be an integral domain, hence $\mathfrak{p} = (0)$. So $\mathscr{O}_\xi = (\mathscr{O}|_U)_\xi = A_{(0)} = \operatorname{Frac}(A)$ is a field. And if $X = \operatorname{Spec} A$, we have had $\mathscr{O}_\xi = \operatorname{Frac}(A)$.

Solution 2.3.7. Since X, Y are integral, i.e. irreducible, we can denote the generic points of X and Y are η_X and η_Y. Since f is dominant, we have $f(\eta_X) = \eta_Y$, because $\overline{f(\eta_X)} = \overline{f(\overline{\{\eta_X\}})} = \overline{f(X)} = Y = \overline{\eta_Y}$, and the generic point of Y is unique. So we have an injective morphism on stalk $f^\sharp_{\eta_Y} : \mathscr{O}_{\eta_Y} \to \mathscr{O}_{\eta_X}$. Since X, Y are integral, $\mathscr{O}_{\eta_X}$ and $\mathscr{O}_{\eta_Y}$ are integral, i.e. $f^\sharp_{\eta_Y}$ induces a morphism $f' : k(Y) \to k(X)$ with $\ker f' = 0$. So $k(X)/k(Y)$ is a field extension. We may assume $\eta_Y \in V \subset Y$ for some affine piece $V = \operatorname{Spec} B$, and $\eta_X \in U \subset f^{-1}(V)$ for some affine piece $U = \operatorname{Spec} A$, then clearly $k(X) = \operatorname{Frac}(A)$ and $k(Y) = \operatorname{Frac}(B)$, and η_X and η_Y are corresponding to (0) in A and B. Since f is generically finite, there are only finitely many prime ideal in A lying over $(0) \subset B$. So if $k(X)/k(Y)$ is transcendental, i.e. A/B is transcendental extension, by going-up theorem there exists infinite prime ideals lying over $(0) \subset B$, which makes a contradiction. So $k(X)/k(Y)$ is algebraic extension. And since A is a finitely generated B-algebra, $k(X)/k(Y)$ is finite.

In the affine case, i.e. $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. We may denote $\{a_i\}_i$ is a set of generators of A as a B-algebra. Then for any a_i, it satisfies a polynomial with B-coefficients with leading term $b_i a_i^{n_i}$. So we may assume $b = \prod b_i$, then in $A_b \to B_b$, the generator a_i satisfies a monic polynomial, hence A_b is integral over B_b, hence A_b is a finitely generated B_b-module, i.e. on $U = D(b)$, X is finite over Y. In the general case, since X and Y are integral, i.e. irreducible, any affine piece is densed in them, so we've done.

Solution 2.3.8 (Normalization)**.** For affine pieces $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ in X, we need to glue $\tilde{U} = \operatorname{Spec} \tilde{A}$ and $\tilde{V} = \operatorname{Spec} \tilde{B}$ together. For every $\mathfrak{p} \in U \cap V$, we can find a open set $W \subset U \cap V$ with $\mathfrak{p} \in W$ and W is principal open subset of both U and V as we do in 2.3.3. We may assume $W = \operatorname{Spec} A_f = \operatorname{Spec} B_g$. Denoting $\varphi : \tilde{U} \to U$ and $\psi : \tilde{V} \to V$ as canonical morphisms, we have $\varphi^{-1}(W) = \operatorname{Spec} \tilde{A}_f$ and $\psi^{-1}(W) = \operatorname{Spec} \tilde{B}_g$. Since $\tilde{A}_f$ and $\tilde{B}_g$ are both normalized, so by uniqueness we have $\tilde{A}_f \cong \tilde{B}_g$, i.e. $\varphi^{-1}(W) \cong \psi^{-1}(W)$, which means we can glue $\tilde{U}$ and $\tilde{V}$, or other affine piece together with a canonical map $\phi : \tilde{X} \to X$.

If we have a normal integral scheme Z with $f : Z \to X$, for every affine piece $U = \operatorname{Spec} A \subset X$, we have $f : f^{-1}(U) \to U$. Since Z is normal, we clearly know that $f^{-1}(U)$ is normal. So f induces a ring morphism $A \to \mathscr{O}_Z(f^{-1}(U))$, which can be extended to $\tilde{A} \to \mathscr{O}_Z(f^{-1}(U))$. So we have a morphism $\tilde{f} : f^{-1}(U) \to \tilde{U}$. Then glueing them together, we get a morphism $\tilde{f}Z \to \tilde{X}$.

Finally, by construction, the morphism ϕ is clearly affine, so this problem is local. We may assume $X = \operatorname{Spec} A$ and $\tilde{X} = \operatorname{Spec} \tilde{A}$. Then if X is of finite type over some field k, i.e. A is a finitely generated k-algebra, we know that $\tilde{A}$ is finitely generated A-module by Theorem 3.9A. in Chapter I. So ϕ is finite.

Solution 2.3.9 (The Topological Space of a Product)**.** (a) Clearly $\mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k = \operatorname{Spec}(k[x] \otimes_k k[y]) = \operatorname{Spec} k[x, y] = \mathbb{A}^2_k$. And, $(x - y) \in \operatorname{Spec} k[x, y]$ is not contained in Space $\mathbb{A}^1_k \times$ Space $\mathbb{A}^1_k$.

(b) Obviously $k(x) \otimes_k k(y) = \{\frac{a(x,y)}{f(x)g(y)} \mid a(x, y) \in k[x, y], f(x) \in k[x], g(y) \in k[y]\}$. So the primes of $k(x) \otimes_k k(y)$ are all height 1 primes of $k[x, y]$ which are not in forms of just $f(x)$ or $g(y)$. Hence $\operatorname{Spec} k(x) \times_{\operatorname{Spec} k} \operatorname{Spec} k(y)$ has infinitely many points.

Solution 2.3.10 (Fibres of a Morphism)**.** (a) Clearly the morphism Space $X_y \to f^{-1}(y)$ is induced by the projection $\pi : X \times_Y \operatorname{Spec} k(y) \to X$. Then we need to prove that π is a homeomorphism on the underlying space.

Since the problem is local for Y, we may assume $Y = \operatorname{Spec} B$. For any open subset $U \subset X$, we have $\pi^{-1}(U) = U \times_Y \operatorname{Spec} k(y) = U_y$, so we only need to prove that $U_y \cong f^{-1}(y) \cap U$. So we can also assume $X = \operatorname{Spec} A$ affine. Denoting $\mathfrak{p}$ is the prime ideal of B corresponding to y, $\varphi : B \to A$ is the ring homomorphism corresponding to the scheme homomorphism $f : X \to Y$, we have $X \times_Y \operatorname{Spec} k(y) = \operatorname{Spec} A \otimes_B B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = \operatorname{Spec} A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Since the prime ideals of $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ are corresponding to primes ideals of $A_{\mathfrak{p}}$ containing $\mathfrak{p}A_{\mathfrak{p}}$, which is corresponding to prime ideals $\mathfrak{q} \subset A$ such that $\mathfrak{q} \cap \varphi(B - \mathfrak{p}) = \varnothing$ and $\varphi(\mathfrak{p}) \subset \mathfrak{q}$, which is corresponding to prime ideals $\mathfrak{q} \subset A$ with $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, we have $X \times_Y \operatorname{Spec} k(y) \cong f^{-1}(y)$.

(b) Clearly we have $X_y = \operatorname{Spec} (k[s,t]/(s-t^2)) \otimes_{k[s]} (k[s]/(s-a)) = \operatorname{Spec} k[s,t]/(s-t^2, s-a)$. If $a \neq 0$, $k[s,t]/(s-t^2, s-a) \cong k \oplus k$, so X_y has only two points and each point has residue field k. If $a = 0$, $k[s,t]/(s-t^2, s-a) \cong k[t]/(t^2)$, so $\operatorname{Spec} k[t]/(t^2)$ is a non-reduced one-point scheme. Since η is corresponding to $(0) \subset Y$, so $X_\eta = \operatorname{Spec} (k[s,t]/(s-t^2) \otimes_{k[s]} k(s) = \operatorname{Spec} k(s)[t]/(s-t^2)$. Since $k(s)[t]/(s-t^2)$, X_η is a singleton. Moreover, $k(s)[t]/(s-t^2)$ is an extension of degree 2 over $k(\eta)$.

Solution 2.3.11 (Closed Subschemes)**.** (b) If $\varphi : Y \to X = \operatorname{Spec} A$ is a closed immersion, we firstly prove that Y is an affine scheme. For any point $P \in Y$, we can find an neighbourhood $U \subset X$ of P such that $U \cap Y = \operatorname{Spec} A'$ is an affine subset of Y. Take $f \in A$ such that $D(f) \subset U$, then $D(f) \cap Y$ is an affine subset of Y. Denote g is the image of f under $A \to \mathscr{O}(U) \to \mathscr{O}_Y(U \cap Y) = A'$. So $D(f) \cap Y = D(g) \cong \operatorname{Spec} A'_g$. Since we can cover $X - Y$ by some principal open set, combining with above $D(f)$, we can covering X by $\{D(f_i\}_{i \in I}$ such that $D(f_i) \cap Y$ is affine subset of Y. Since $\operatorname{Spec} A$ is quasi-compact, we may assume the above covering is finite for $i = 1, \ldots, n$, i.e. all f_i generate the A. If we denote the image of f_i under $\mathscr{O}_X(X) \to \mathscr{O}_Y(Y)$ as $\bar{f}_i$, all $\bar{f}_i$ generate the $\mathscr{O}_Y(Y)$. Moreover, $D(\bar{f}_i) = D(f_i) \cap Y$ is affine, we know Y is affine by 2.2.17.(b).

So we may assume $Y = \operatorname{Spec} B$. Then the closed immersiom $Y \to X$ is induced by ring homomorphism $\phi : A \to B$. Denote $\mathfrak{a} = \ker \phi$, we have a commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[r, "\varphi"] \arrow[d, "\theta"] & \operatorname{Spec} A \\
\operatorname{Spec} A/\mathfrak{a} \arrow[ur] &
\end{tikzcd}$$

where θ is induced by injection $A/\mathfrak{a} \to B$. Since $\varphi(\operatorname{Spec} B) = \overline{\varphi(\operatorname{Spec} B)} = V(\bigcap_{\mathfrak{p} \in \varphi(\operatorname{Spec} B)} \mathfrak{p}) = V(\bigcap_{\mathfrak{q} \in \operatorname{Spec} B} \phi^{-1}(\mathfrak{q})) = V(\phi^{-1}(\bigcap \mathfrak{q})) = V(\phi^{-1}(\sqrt{(0)})) = V(\sqrt{\mathfrak{a}}) = V(\mathfrak{a})$. So θ is a homeomorphism on the underlying topological space. Moreover, for any $f \in A$, we have a homomorphism $(A/\mathfrak{a})_{\bar{f}} \to B_{\phi(f)}$ is injective, because $A/\mathfrak{a} \to B$ is injective, i.e. $\theta^\sharp$ is injective. Since θ is a closed immersion, we have $\theta^\sharp$ is an isomorphism. So $Y = \operatorname{Spec} A/\mathfrak{a}$.

(a) For any affine piece $U = \operatorname{Spec} A \subset X$, then $f^{-1}(U) \to U$ is a closed immersion, so $f^{-1}(U) = \operatorname{Spec} A/I$ for some ideal $I \subset A$. For any $g : X' \to X$, take an affine piece $U' = \operatorname{Spec} A' \subset g^{-1}(U)$, then $f'^{\,-1}(U') = \operatorname{Spec} A' \otimes_A A/I = \operatorname{Spec} A'/IA'$, so $f'^{\,-1}(U') \to U'$ is a closed immersion. Then $Y' \to X'$ is just the glueing-together of all $f'^{-1}(U') \to U'$, so is a closed immersion.

(c) If $X = \operatorname{Spec} A$ is affine, then $Y = \operatorname{Spec} A/I$ for some radical ideal I. If $Y' \to X$ is a closed immersion such that $Y \cong Y'$ on the underlying space, then $Y' = \operatorname{Spec} A/I'$, for some $V(I') = V(I)$. So $I = \sqrt{I} = \sqrt{I'}$. So we have a canonical ring homomorphism $A \to A/I' \to A/I$, which induces the scheme homomorphism $Y \to Y' \to X$. For general case, we just consider the affine covering $X = \bigcup U$, then $U \cap Y \to U$ may factor through $U \cap Y'$. Then we may glue together all $U \cap Y \to U \cap Y' \to U$ to get a $Y \to Y' \to X$.

(d) **This question may need to add a condition that f is quasi-compact and quasi-separated, and the proof will use the theory of sheaves of modules.** Denote $\mathscr{I} = \ker f^\sharp$. Since f is quasi-compact and quasi-separated, we know that $f_*\mathscr{O}_Z$ is quasi-coherent by preposition 5.8.(c), so $\mathscr{I}$ is a quasi-coherent $\mathscr{O}_X$-module. Then we can define $(Y, \mathscr{O}_Y) = (\operatorname{Supp}(\mathscr{O}_Z/\mathscr{I}), (\mathscr{O}_Z/\mathscr{I})|_{\operatorname{Supp}(\mathscr{O}_Z/\mathscr{I})})$. So $i : Y \to X$ is a closed immersion. Clearly $f(Z) \subset \operatorname{Supp}(f_*\mathscr{O}_Z) \subset \overline{f(Z)}$, since Y is closed, we have $Y = \overline{f(Z)}$. Then we can define $g : Z \to Y$ on the underlying space as induced by f. And $f^\sharp : \mathscr{O}_X \to f_*\mathscr{O}_Z$ induces $\mathscr{O}_X/\mathscr{I} \to f_*\mathscr{O}_Z = i_*g_*\mathscr{O}_Z$, which induces $\mathscr{O}_Y \to g_*\mathscr{O}_Z$. Hence $g : Z \to Y$ is a scheme homomorphism, and clearly $f = ig$.

If $i' : Y' \to X$ is a closed immersion, $g' : Z \to Y'$ with $f = i'g'$, we have $f(Z) \subset i'(Y')$, i.e. $\overline{f(X)} \subset i'(Y')$. So we have a continuous map $j : Y \to Y'$ with $i = i'j$ on the underlying topological space. Denoting $\mathscr{I}'$ as the ideal sheaf of Y', we clearly know $\mathscr{I}' = \ker i'^\sharp$. Since $f = i'g'$, $\mathscr{I}'$ is contained in $\ker f^\sharp$, i.e. $\mathscr{I}'$ is a subsheaf of $\mathscr{I}$. So we have $\mathscr{O}_X/\mathscr{I}' \to \mathscr{O}_X/\mathscr{I}$, i.e. $i'_*\mathscr{O}_{Y'} \to i_*\mathscr{O}_Y = i'_*j_*\mathscr{O}_Y$. Hence we have $\mathscr{O}_{Y'} \to j_*\mathscr{O}_Y$. So $j : Y \to Y'$ is a morphism of schemes, i.e. $Y \to X$ will factor through Y'.

By our construction, if Z is a reduced scheme, Y is clearly the reduced induced structure on $\overline{f(Z)}$.

Solution 2.3.12 (Closed Subschemes of Proj S)**.** (a) Since φ preserves the degrees, we know $\varphi(S_+) = T_+$. Then $U = \{\mathfrak{p} \in \operatorname{Proj} T \mid \mathfrak{p} \not\supseteq T_+\} = \operatorname{Proj} T$. Since $T \cong S/\ker\varphi$, we have a one-to-one corresponding between homogeneous prime ideals of S containing $\ker\varphi$ and homogeneous prime ideals of T, so $f : \operatorname{Proj} T \to V(\ker\varphi)$ is a homeomorphism. For any $a \in S$, we have $\mathscr{O}_{\operatorname{Proj} S}(D_+(a)) = S_a \to T_{\varphi(a)} = \mathscr{O}_{\operatorname{Proj} T}(f^{-1}(D_+(a))$ is sujective, so $f^\sharp$ is surjective.

(b) If $I' = \bigoplus_{d\geq d_0} I_d$, then $(S/I)_d \cong (S/I')_d$ for all $d \geq d_0$. Then the morphism $f : \operatorname{Proj}(S/I) \to \operatorname{Proj}(S/I')$ induced by $S/I' \to S/I$ is an isomorphism by 2.2.14.(c).

Solution 2.3.13 (Properties of Morphisms of Finite Type)**.** (a) If $f : Z \to X$ is a closed immersion. For any affine piece $U = \operatorname{Spec} A$ of X, $f^{-1}(U) = Z \cap U \to U$ is also a closed immersion, hence $Z \cap U = \operatorname{Spec} A/I$ for some ideal I of A. Since A/I is a finitely generated A-algebra, we know f is of finite type.

(b) If $i : Y \to X$ is an open immersion. For any affine piece $U = \operatorname{Spec} A$ of X, $U \cap Y$ is open in U, hence can be $U \cap Y = \bigcup_i D(f_i) = \bigcup_i \operatorname{Spec} A_{f_i}$ for some $f_i \in A$. Since A_{f_i} is a finitely generated A-algebra, we know f is locally of finite type. So f is of finite type since f is also quasi-compact.

(c) If $f : X \to Y$ and $g : Y \to Z$ is of finite type. So for any affine piece $U = \operatorname{Spec} A$ of Z, there are finitely many $V_i \subset Y$ such that $g^{-1}(U) = \bigcup V_i$ and $V_i = \operatorname{Spec} B_i$ for some finitely generated A-algebra B_i. Then there are finitely many $W^i_j \subset X$ such that $f^{-1}(V_i) = \bigcup W^i_j$ and $W^i_j = \operatorname{Spec} C^i_j$ for some finitely generated B_i algebra C^i_j. So all C^i_j are finitely generated A-algebra, and the index set of i, j is finite. Hence $g \circ f$ is of finite type.

(d) If $f : X \to Y$ is of finite type, we need to show that $X' = X \times_Y Y' \to Y'$ is of finite type.

(i) If $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Y' = \operatorname{Spec} B'$, we have A is finitely generated B-algebra, hence $A \otimes_B B'$ is finitely generated B'-algebra, i.e. $X' \to Y'$ is of finite type.

(ii) If $Y = \operatorname{Spec} B$ and Y' are affine, there exists a finite open affine covering $X = \bigcup U_i$ such that $U_i = \operatorname{Spec} A_i$ for some finitely generated B-algebra A_i. Hence X' has an finite open affine covering $X' = \bigcup \operatorname{Spec} A_i \otimes_B B'$, where $A_i \otimes_B B'$ is finitely generated B'-algebra, i.e. $X' \to Y'$ is of finite type.

(iii) If $Y = \operatorname{Spec} B$ is affine, and $Y' = \bigcup Y'_i$ is an open affine covering of Y' such that $Y'_i = \operatorname{Spec} B'_i$ for some B'_i, we have $X \times_Y Y'_i$ is finite type over Y'_i by (ii). Sine $X \times_Y Y'_i$ is the preimage of Y'_i, we know that $X' \times_X Y$ is finite type over X'.

(iv) Denote the morphism of $Y' \to Y$ as g. If $Y = \bigcup Y_i$ is an open affine covering of Y such that $Y_i = \operatorname{Spec} B_i$ for some ring B_i, $f^{-1}(Y_i) \times_{Y_i} g^{-1}(Y_i)$ is finite type over $g^{-1}(U_i)$ by (iii). So $X \times_Y g^{-1}(U_i) \to g^{-1}(U_i)$ is of finite type. So $X' \to Y'$ is the glueing-together of all above morphism, hence of finite type.

(e) Since $X \times_S Y \to Y$ is the base change of $X \to S$, hence of finite type. Since $X \times_S Y \to S$ is the compound of $X \times_S Y \to Y$ and $Y \to S$, hence of finite type.

(f) For any affine piece $\operatorname{Spec} C \subset Z$, any affine piece $\operatorname{Spec} B \subset g^{-1}(\operatorname{Spec} C) \subset Y$, and any affine piece $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B) \subset X$, we have $\operatorname{Spec} A \subset (g \circ f)^{-1}(\operatorname{Spec} C)$, and A is a finitely generated C-algebra with the morphism $C \to B \to A$. If $\{a_i\}_{i=1}^n$ generates A as C-algebra, we have a surjective morphism $C[x_1, \ldots, x_n] \to A$ as $x_i \mapsto a_i$. Since this morphism is factored through $C[x_1, \ldots, x_n] \to B[x_1, \ldots, x_n] \to A$, we know $B[x_1, \ldots, x_n] \to A$ is surjective. So A is a finitely generated B-algebra, i.e. $g \circ f$ is locally of finite type. Since g is of finite type, hence quasi-compact, and f is quasi-compact too, we know that $g \circ f$ is also quasi-compact. Hence $g \circ f$ is of finite type.

(g) Since Y is noetherian, we may assume $Y = \bigcup Y_i$ for some $Y_i = \operatorname{Spec} B_i$ is a finite open affine covering of Y. So $f^{-1}(Y_i)$ can be covered by finitely many $X_{ij} = \operatorname{Spec} A_{ij}$ with some finitely generated B_i-algebra A_{ij}. Since each B_i is noetherian, $\{\operatorname{Spec} A_{ij}\}$ is a finite covering of X with A_{ij} noetherian, hence X is noetherian.

Solution 2.3.14. This problem is local, so we may assume that $X = \operatorname{Spec} A$ is affine. Since X is of finite type over a field k, we know that A is a finitely generated k-algebra, hence a Jacobson ring. Denote $V = \operatorname{Spm} A \subset X$ as the set of all maximal ideals of A, i.e. the closed points set of X. If some open subset $U \subset X$ satisfies $U \cap V = \varnothing$, as we may assume $U = D(f)$ as a principal open subset, we have $f \in \mathfrak{m}$ for all maximal ideal of A. Then f is nilpotent since A is Jacobson, i.e. $U = D(f) = \varnothing$. Hence V is dense in X.

For the counterexample, we just need to take a integral local ring A, then $\operatorname{Spec} A = \{(0), \mathfrak{m}\}$. Then taking an $f \in \mathfrak{m}\backslash\{0\}$, we know $\mathfrak{m} \notin D(f)$, hence not dense.

Solution 2.3.15. (a) (i$\Rightarrow$ii) and (iii$\Rightarrow$i) is trivial. (ii$\Rightarrow$iii): We only need to show that if K/k is purely inseperable, then $\pi : X_K \to X$ is a homeomorphism. If X_K is not irreducible, we may have two affine piece V_1 and V_2 disjoint, then U_K defined as the direct sum of those affine pieces is also affine, and clearly U_K is not integral. Hence we only need to consider U_K and $U = \pi(U_K)$, so this problem is local, i.e. we may assume $X = \operatorname{Spec} A$ is affine. And since $\operatorname{Spec} A$ is homeomorphic to $\operatorname{Spec} A_{\text{red}}$, we may assume that A is reduced. Firstly we assume that K/k is generated by a single element, i.e. $K \cong k[T]/(T^q - c)$ for some $c \in k$ and q is a power of $\operatorname{char}(k)$. Then for any prime ideal $\mathfrak{p} \subset A$, and prime ideal $\mathfrak{q} \subset A_K$ lying above $\mathfrak{p}$. Then $\mathfrak{p}A_K \subset \mathfrak{q}$. Conversely, if $a^q \in \mathfrak{p}$ for all $a \in \mathfrak{q}$, we have $\mathfrak{q} \subset \sqrt{\mathfrak{p}A_K}$. So $\mathfrak{q} = \sqrt{\mathfrak{p}A_K}$, i.e. π is injective. The surjection is trivial. Moreover, for any ideal $I \subset A_K$ and $J = I \cap A$, we have $I^q \subset J$, i.e. $\pi(V(I)) = V(J)$. So π is closed, hence homeomorphism. Secondly for K/k is a purely seperable finite extension, then K/k is made up of a series of simple extension, then trivial. Finally for general case, we will prove that X_K is homeomorphic to some $X_{K'}$ for K'/k is finite. For any closed subset $Z_K \subset X_K$, we may assume $Z_K = V(I_K)$ for some redical ideal $I_K \subset A_K$. Since X_K is a variety, we know that A_K is noetherian, i.e. I_K is generated by $f_1, \dots, f_r$. So there exists a finite extension K' of k, such that $f_1, \dots, f_r \in A_{K'}$ and generate an ideal $I_{K'}$. So $I_{K'}$ induces a closed subset $Z_{K'}$ of $A_{K'}$ and clearly $Z_K = Z_{K'} \times_{K'} K$, i.e. the closed subset of X_K is homeomorphic to a closed subset of some $X_{K'}$, hence homeomorphic to a closed subset of X. So X_K is homeomorphic to X.

(b) (i$\Rightarrow$ii) and (iii$\Rightarrow$i) is trivial. (ii$\Rightarrow$iii): We only need to prove that if K/k is seperable, and X is reduced, we have X_K is reduced. This problem is clearly local, we may assume that $X = \operatorname{Spec} A$ is affine. For all minimal prime ideals $\mathfrak{p}_i$ of A (since A is finitely generated k-algebra, there exist only finitely many $\mathfrak{p}_i$), we have $A \to \prod_i A/\mathfrak{p}_i$ is injective, then $A_K \to \prod_i (A/\mathfrak{p}_i)_K$ is injective since $\times_k K$ is flat. So we may assume that A is integral. Since A is a subring of $\operatorname{Frac}(A)$, we only need to show that for all separable field extension F/k, we have $F_K = F \otimes_k K$ is reduced. Since every element F_K is contained in some $F_{K'}$ for some finite separable extension K'/k, so we only need to show that F_k is reduced for finite separable extension K/k. Since $K = k[T]/f$ for some separable polynomial $f \in k[T]$, we have $F_K = F[t]/f$, and $f \in F[T]$ is also separable. So F_K is reduced.

(c) Take $X = \operatorname{Spec} (\mathbb{R}/(t^2+1)) \cong \operatorname{Spec} \mathbb{C}$, then X is integral. But $X_{\mathbb{C}} = \operatorname{Spec} (\mathbb{C}/(t^2+1)) \cong \operatorname{Spec} \mathbb{C} \coprod \operatorname{Spec} \mathbb{C}$ is neither irreducible nor reduced.

Solution 2.3.16 (Noetherian Induction)**.** Denote $\mathcal{S} = \{V \mid V$ is a closed subset of X which do not hold the property $\mathscr{P}\}$. If X does not hold $\mathscr{P}$, $\mathcal{S}$ is not empty. Since X is noetherian, $\mathcal{S}$ has a minimal element, namely Y. Since every proper closed subset of Y is not in $\mathscr{S}$, they all hold the property $\mathscr{P}$, then Y also holds the $\mathscr{P}$, which makes a contradiction.

Solution 2.3.17 (Zariski Spaces)**.** (d) If $x \in X$ is the generic point, and $x \notin U \subset X$ for some proper open subset U, then $X = \overline{\{x\}} \subset X - U$, which makes a contradiction.

(a) Only need to show that if X is a notherian scheme, every irreducible subscheme Y of X has a unique generic point. If $U = \operatorname{Spec} A$ is an affine piece of X with $U \cap Y \neq \varnothing$, $U \cap Y \subset U$ is an irreducible closed subset, i.e. $U \cap Y = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$. Clearly $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ in U. Denote $y \in Y$ corresponding to $\mathfrak{p}$. Then $U \cap Y \subset \overline{\{y\}}$ in Y. Since $U \cap Y$ is open in Y, and Y is irreducible, we know that $U \cap Y$ is dense in Y. Hence $\overline{\{y\}} \supset \overline{U \cap Y} = Y$, i.e. $\overline{\{y\}} = Y$. If y and y' are both generic point of Y, by (d) we know that $y, y' \in U \cap Y$, and there exist two corresponding prime ideal $\mathfrak{p}$ and $\mathfrak{p}'$ in A. So $V(\mathfrak{p}') = U \cap Y = V(\mathfrak{p})$, i.e. $\sqrt{\mathfrak{p}'} = \sqrt{\mathfrak{p}}$, hence $y = y'$.

(b) If Y is a minimal nonempty closed subset, for every $y \in Y$, we know $\overline{\{y\}} \subset Y$ is closed. Since Y is minimal, we have $\overline{\{y\}} = Y$, i.e. y is the generic point of Y. By the uniqueness of generic point, Y has only one point.

(c) For any two points $x, y \in X$. If $x \in X/\overline{\{y\}}$ or $y \in X/\overline{\{x\}}$, we've done. So, if $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$, we know that $\overline{\{x\}} = \overline{\{y\}} := Z$, i.e. x and y are both generic points of Z. By uniqueness we know that $x = y$. So X satisfies the axiom T_0.

(e) If $x \in X$ is a minimal point, then if $y \in \overline{\{x\}}$, we have $y = x$ by minimality. Hence $\overline{\{x\}}$ is a singleton. Then by (b) we know that x is a closed point. If $x \in X$ is a maximal point, we know $x \subset Y$ for some irreducible component of X. Denote y as the generic point of Z. Clearly $x \in \overline{\{y\}}$, then $x = y$ by maximality of x. So x is a generic point of some irreducible component. If $Z \subset X$ is a closed subset, $z \in Z$, and x is a specialization of z, clearly we have $x \in \overline{\{z\}} \subset Z$.

(f) Clearly $t(X)$ is noetherian since every closed subset of $t(X)$ is induced by some closed subset of X. For any nonempty closed irreducible subset $V \subset t(X)$, we have a closed irreducible subset $Z \subset X$ such that $V = t(Z)$. Then Z itself is a point of $t(Z)$, and clearly Z is the generic point of $t(Z)$, hence $t(X)$ is a Zariski space. Furthermore, if X itself is a Zariski space, then every closed irreducible subset of X is one-to-one corresponding to its generic point, i.e. this is a bijection $t(X) \to X$. And the continuity is obvious.

Solution 2.3.18 (Constructible Sets)**.** (a) ($\Leftarrow$) Conditions (1) and (3) imply every closed subset of X belongs to $\mathfrak{F}$. Condition (2) implies locally closed subset of X belongs to $\mathfrak{F}$. Conditions (2) and (3) imply finite union of elements of $\mathfrak{F}$ is in $\mathfrak{F}$, so any finite union of distinct locally closed subset of X belongs to $\mathfrak{F}$.

($\Rightarrow$) Denote $\mathfrak{G}$ as the set of all subset of X which can be written as a finite disjoint union of locally closed subset, then $\mathfrak{G} \subset \mathfrak{F}$. Clearly every open subset of X belongs to $\mathfrak{G}$, i.e. $\mathfrak{G}$ satisfies condition (1). If $\coprod_i (Z_i \cap U_i), \coprod_j (Z'_j \cap U'_j) \in \mathfrak{G}$, we have $(\coprod_i (Z_i \cap U_i)) \cap (\coprod_j (Z'_j \cap U'_j) = \coprod_{i,j}((Z_i \cap Z'_j) \cap (U_i \cap U'_j)) \in \mathfrak{G}$, hence $\mathfrak{G}$ satisfies condition (2). For condition (3), if U is an open subset of X and Z is a closed subset of Z, then $X - (U \cap Z) = (X - U) \cup (X - Z) = (X - U) \coprod (U \cap (X - Z)) \in \mathfrak{G}$. For general case, $X - (\coprod_i (Z_i \cap U_i)) = \bigcap (X - (Z_i \cap U_i))$. Since $X - Z_i \cap U_i) \in \mathfrak{G}$, by condition (2) we know that $X - (\coprod_i (Z_i \cap U_i)) \in \mathfrak{G}$, hence $\mathfrak{G}$ satisfies condition (3). Then by minimality of $\mathfrak{F}$, we have $\mathfrak{G} = \mathfrak{F}$, i.e. every element of $\mathfrak{F}$ is a disjoint union of locally closed subset of X.

(b) ($\Leftarrow$) If $\eta \in V$ with $V \in \mathfrak{F}$, obviously $\bar{V} \supset \overline{\{\eta\}} = X$, i.e. $\bar{V} = X$, hence dence.

($\Rightarrow$) If $V = \coprod_i (Z_i \cap U_i) \in \mathfrak{F}$ is dense in X, we have $\bigcup Z_i \supset \bar{V} = X$, hence $\bigcup Z_i = X$. Then $\eta \in Z_i$ for some i. Since U_i are all open, by 2.3.17.(d) we know that $\eta \in U_i$. So $\eta \in Z_i \cap U_i \subset V$.

Furthermore, since every nonempty open subset contains the generic point by 2.3.17.(d), the ($\Leftarrow$) is trivial. Conversely, since $\bigcup Z_i = X$ and X is irreducible, there must exist an i such that $Z_i = X$. So $Z_i \cap U_i = U_i$, i.e. $U_i \subset V$.

(c) ($\Rightarrow$) is trivial. ($\Leftarrow$) Obviously we only need to consider the cast $S = Z \cap U$. If $x \in S$, we have $\overline{\{x\}} \subset S$ since S is stable under specialization. For $\bar{S}$, denote all generic points of its irreducible components as $\{\eta_1, \dots, \eta_n\}$. Since $\overline{\{\eta_i\}} \subset \bar{S}$, we have $\overline{\{\eta_i\}} \subset \overline{Z_j \cap U_j}$ for some j. Then $\eta_i \in U_j$ by 2.3.17.(d). So $\overline{\{\eta_i\}} \subset S$, hence $\bar{S} \subset S$, i.e. S is closed. Furthermore, if T is open, it is obviously constructed and stable under generalization. Conversely, if T is constructed and stable under generalization, $X - T$ is constructed and stable under specialization, so $X - T$ is closed, i.e. T is open.

(d) Clearly if f is continuous, $f^{-1}(Z_i)$ is closed and $f^{-1}(U_i)$ is open. So $f^{-1}(V) = f^{-1}(\coprod_i (Z_i \cap U_i)) = \coprod_i (f^{-1}(Z_i) \cap f^{-1}(U_i))$ is constructible of X.

Solution 2.3.19. (a) Since X is noetherian, $X = \bigcup \operatorname{Spec} A_i$ for finitely many i and $\operatorname{Spec} A_i$ are all irreducible.

So if every $f(\text{Spec } A_i)$ is constructible, $f(X)$ is constructible, hence we reduce the question to the case that $X = \text{Spec } A$ is affine and irreducible. Since Y is also noetherian, $Y = \bigcup \text{Spec } B_i$ for finitely many i and all $\text{Spec } B_i$ are all irreducible. So if every $f(X) \cap \text{Spec } B_i$ is constructible in $\text{Spec } B_i$, we know that $f(X) \cap \text{Spec } B_i$ is constructible in Y, hence $f(X)$ is constructible in Y. So we reduce the question to the case that $Y = \text{Spec } B$ is also affine and irreducible. By 2.2.3.(b), we clearly can assume that X and Y are reduced, i.e. we may assume X and Y are integral. Finally, we may denote $\overline{f(X)} = Z \subset Y$, then we have $Z = \text{Spec } B/I$ for some reduced I by 2.3.11.(b). So $f(X) = f(X) \cap Z$, the constructibility are the same for those two forms, so we may assume that $Z = Y$, i.e. f is dominant.

(b) Firstly we prove this algebraic result by using induction on the number r of generators of B as A-algebra. If $r = 1$, we have $B \cong A[x]$ or $A[x]$ quotient something. If $B \cong A[x]$, every $b \in B$ has the form $b = b_n x^n + \ldots + b_0$, then we just define $a = b_n$. Then for $\varphi : A \to K$ with $\varphi(a) \neq 0$, the polynomial $\varphi(a_n)x^n + \ldots + \varphi(a_0)$ in K has only finitely many roots in K since K is algebraic closed. Taking an $\alpha \in K$ is not a root of this polynomial, and define $\varphi'(x) = \alpha$, we clearly have $\varphi'(b) \neq 0$. For the second condition, we may assume x is a generator of B as an A-algebra, then it is algebraic over A with the minimal polynomial $a_m x^m + \ldots + a_0 = 0$. Since b is also algebraic over A, we may assume $b_n b^n + \ldots + b_0 = 0$. And we can formally write $b_n + \ldots + b_0 b^{-n} = 0$. Then we can define $a = a_m b_0$. For any $\phi : A \to K$ such that $\phi(a) \neq 0$, then ϕ can be extended to a morphism $\varprojlim_k A[a^{-k}] \to K$. Since x is integral over $A[a^{-1}]$, we know B must be contained in some $\varprojlim_k A[a^{-k}]$, hence we have a morphism $\phi' : B \to K$. And moreover, we know b^{-1} is also integral over $A[a^{-1}]$, it is also contained in $\varprojlim_k A[a^{-k}]$. Since $b \in B$, we know that b has valuation 0 in the valuation ring $\varprojlim_k A[a^{-k}]$, i.e. an invertible element. So $\phi'(b) \neq 0$.

For general r, we know that $B \cong A[x_1, \ldots, x_r]/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$. We clearly have a morphism $\psi : A[x_1, \ldots, x_{r-1}] \hookrightarrow A[x_1, \ldots, x_r]$ and $\psi^{-1}(\mathfrak{p}) \subset A[x_1, \ldots, x_{r-1}]$ is a prime ideal. Then we have $A \to A[x_1, \ldots, x_{r-1}]/\psi^{-1}(\mathfrak{p}) \to B$ are ring extension, so $\varphi : A \to K$ can be extended to $\varphi'' : A[x_1, \ldots, x_{r-1}]/\psi^{-1}(\mathfrak{p}) \to K$ satisfies the condition, which can be extended to $\varphi' : B \to K$ by induction. Since x is integral over $A[a^{-1}]$,

Back the the question, we may assume $X = \text{Spec } A$, $Y = \text{Spec } B$ for noetherian integral domain A, B and f is dominant. Then $f : X \to Y$ is induced by a $\varphi : B \to A$. Then for $1 \in A$ we have an $b \in B$ as the algebraic result above. Then we just need to prove that $D(b) \subset f(X)$. For every $\mathfrak{p} \in D(b)$, i.e. $b \notin \mathfrak{p}$, then $\varphi : B \to \overline{\text{Frac}(B/\mathfrak{p})}$ is a morphism such that $\varphi(b) \neq 0$. Then we know that φ can be extended to a φ' with $\varphi'(1) \neq 0$. Then $\mathfrak{q} = \ker \varphi' \subset A$ is a prime ideal. Clearly $\mathfrak{q} \cap B = \mathfrak{p}$, which means $f(\mathfrak{q}) = \mathfrak{p}$. So $D(b) \subset f(X)$.

(c) For $V(b) = \text{Spec } B/(b)$, we have a morphism $\tilde{\varphi} : B/(b) \to A/(b)A$, which induces a $\tilde{f} : \text{Spec } A/(b)A \to \text{Spec } B/(b)$. Since φ is injective, $\tilde{\varphi}$ is injective, i.e. $\tilde{f}$ is dominant. Decompose (b) as $(b) = \cap \mathfrak{p}_i$ for some primary ideal $\mathfrak{p}_i$. Then $V(b) = \bigcup V(\sqrt{\mathfrak{p}_i})$. Since we have $f_i : \text{Spec } A/\sqrt{\mathfrak{p}_i}A \to \text{Spec } B/\sqrt{\mathfrak{p}_i}$, by noetherian induction we know that $\text{Im} f_i$ is constructible in $V(\mathfrak{p}_i)$. So $\text{Im}\tilde{f} = \bigcup \text{Im} f_i$ is constructible in $\text{Spec } B/(b)$, i.e. $f(X) \cap V(b) = \text{Im}\tilde{f}$ is constructible in $V(b)$. Hence $f(X) = D(b) \coprod (f(X) \cap V(b))$ is constructible.

(d) Consider $X = \mathbb{A}^1_{\mathbb{C}}$ and $Y = \mathbb{P}^2_{\mathbb{C}}$, and $f : X \to Y$ is $f(x) = (x : 1 : 0)$. Then $\text{Im} f$ is neither closed nor open.

Solution 2.3.20 (Dimension). (a) Clearly, since X is integral scheme, for any $P \in X$ and affine neighbourhood $U = \text{Spec } A$ of P, if P corresponds to $\mathfrak{p} \subset A$, we have $\{Z \subset X \mid Z \text{ closed irreducible with } P \in Z\} \cong \{Z \subset U \mid Z \text{ closed irreducible with } P \in Z\} \cong I \subset A \mid I \text{ prime with } I \subset \mathfrak{p}\} \cong \{I \subset A_{\mathfrak{p}} = \mathscr{O}_P \mid I \text{ prime}\}$. Hence (a) is obvious since we assume more that P is a closed point.

(b) For closed P, clearly $\text{Frac}\mathscr{O}_P = K(X)$. So $\text{tr.d.}K(X)/k = \text{tr.d.Frac}\mathscr{O}_P/k = \dim \mathscr{O}_P = \dim X$.

(c) By the one-to-one corresponding in (a), trivial.

(e) Since $K(X) = K(U)$ and (b), trivial.

(d) For the case that $X = \text{Spec } A$ and $Y = \text{Spec } B$ are both affine and Y is irreducible, we have $B = A/\mathfrak{p}$ for some prime $\mathfrak{p} \subset A$. So $\dim Y + \text{codim}(Y, X) = \text{height } \mathfrak{p} + \dim A/\mathfrak{p} = \dim A = \dim X$. If X and Y are not necessarily affine but Y is still irreducible, we may pick an affine subset $X' \subset X$, and $Y' = X' \cap Y$ is affine. Then clearly $\dim X' = \dim X$, $\dim Y' = \dim Y$ and $\text{codim}(Y, X) = \text{codim}(Y', X')$ by (e) or just counting the dimension of stalk, so $\dim X = \dim X' = \dim Y' + \operatorname{codim}(Y', X') = \dim Y + \operatorname{codim}(Y, X)$. For general case, if Y is not irreducible, we may pick an irreducible component with maximal dimension Z of Y, then $\dim Y = \dim Z$ and $\operatorname{codim}(Y, X) = \operatorname{codim}(Z, X)$, then obvious.

(f) By (e) we only need to consider the affine case, i.e. $X = \operatorname{Spec} A$. Then $\dim X = \text{tr.d.Frac}(A)/k = \text{tr.d.Frac}(A) \otimes_k k'/k' = \dim X'$.

Solution 2.3.21. (a) If $\mathfrak{m}_R = (u)$ for some $u \in R$, then $I = (ut - 1) \subset R[t]$ is a maximal ideal. Since $R[t]/I \cong \operatorname{Frac}(R)$, which has dimension 1. And since X clearly has dimension 2, so $\dim \mathscr{O}_P < \dim X$.

(d) Just take $Y = \{P\}$, then $\dim Y + \operatorname{codim}(Y, X) = 0 + 1 < 2 = \dim X$.

(e) Define $U = \operatorname{Spec}\,(R[t])_{\mathfrak{m}_R[t]}$, then $\dim U = 1 \neq 2 = \dim X$.

Solution 2.3.22 (Dimension of the Fibres of a Morphism)**.** (a) This problem is local, so we may assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, $f : X \to Y$ induces a ring homomorphism $\varphi : B \to A$. Since $Y' \subset Y$ closed and irreducible, we may assume $Y' = \operatorname{Spec} B/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset B$, then it's generic point η' is the point in Y corresponding to $\mathfrak{p}$. Since $f^{-1}(Y') = \{\mathfrak{q} \subset A \mid \mathfrak{q} \text{ prime and } \varphi^{-1}(\mathfrak{q}) \subset \mathfrak{p}\}$. So every closed irreducible subset $Z \subset X$ with $\mathfrak{p} \in f(Z)$ must has the form $\operatorname{Spec} A/\mathfrak{q}$ with some $\mathfrak{q}$ satisfying $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. For every point $P \in Y'$ corresponding to $\mathfrak{p}'$, i.e. $\mathfrak{p}' \subset \mathfrak{p}$, then there exists a point in $Q \in Z \cap f^{-1}(Y')$ corresponding to $\mathfrak{q}' \subset \mathfrak{q}$ is the preimage of P. Hence this induces a surjective morphism $\mathscr{O}_{P,Y} \to \mathscr{O}_{Q,X}$, hence $\dim \mathscr{O}_{P,Y} \geq \dim \mathscr{O}_{Q,X}$. Then by 2.3.20.(c), $\operatorname{codim}(Y', Y) = \inf\{\dim \mathscr{O}_{P,Y} \mid P \in Y'\} \geq \inf\{\dim \mathscr{O}_{Q,X} \mid Q \in Z\} = \operatorname{codim}(Z, X)$.

(b) Denote the irreducible component of X_y as Z. Take $Y' = \overline{\{y\}}$, then by (a) we have $\operatorname{codim}(Z, X) \leq \operatorname{codim}(Y', Y)$. By 2.3.20.(d), we have $\operatorname{codim}(Z, X) = \dim X - \dim Z$ and $\operatorname{codim}(Y', Y) = \dim Y - \dim Y'$, we have $\dim X \geq \dim X - \dim Y = e$.

(c) This problem is local for Y, so we may assume $Y = \operatorname{Spec} B$ is affine. If $X = \operatorname{Spec} A$ is also affine, we know that A is a finitely generated B-algebra. So we can pick $t_1, \ldots, t_n \in A$ as a transcendental base of $K(X)/K(Y)$. Define $X_1 = \operatorname{Spec} B[t_1, \ldots, t_e]$, we clearly know that $X_1 \cong \mathbb{A}^e_Y$, and $g : X \to X_1$ is generically finite. Then by 3.7. we know there exists an open dense $U' \subset X'$, such that $g^{-1}(U') \cong U'$, then $U = g^{-1}(U') \subset X$ is open and dense. And obviously, $h : X' \to Y$ clearly have dimension e on every fiber X'_y. Hence $U \subset X$ is the open set we need. For general integral scheme X, since every affine piece $V \subset X$ is open and dense, and for V we have a $U \subset V$ open dense with the property we need, we know that $U \subset X$ is also open and dense.

(d) (1) By (b) this is trivial. (2) By (c), there exists a open set U such that $E_h \subset X - U$ for $h > e$, hence E_h cannot be dense. (3) For $h > e$, we have $E_h \subset X - U$, then we may define $X' = X - U$ with the reduced induced closed subscheme structure, then clearly X' is an integral scheme. Then $f : X \to Y$ induces a $f' : X' \to Y$, and E_h is clearly the set of all point $x \in X'$ such that $\dim X'_{f(x)} \geq h$. Then by induction of dimension of X, E_h is closed in X', hence closed in X.

(e) Clearly $C_h = f(E_h - E_{h-1}) = f(E_h) - f(E_{h-1})$, so we only need to show each $f(E_h)$ is constructible. By 2.3.19. this is obviously true by closedness of E_h, hence we've done.

Solution 2.3.23. Clearly $t(V) \times_{\operatorname{Spec} k} t(W)$ is a integral scheme of finite type over k, i.e. $t(V) \times_{\operatorname{Spec} k} t(W) = t(X)$ for some variety X. Since every closed point of $t(X)$ will project to closed points on $t(V)$ and $t(W)$, i.e. every point of X corresponds to a point in V and a point in W, which gives us a bijection $X \cong V \times W$. And the continuity is obvious.

2.4 Separated and Proper Morphisms

Solution 2.4.1. If $f : X \to Y$ is finite, by definition f is affine, hence seperated. Finiteness clearly means finitely type. And since finiteness is stable under base change by 2.3.13.(d), and by 2.3.5.(b) we know that finiteness means closed, so f is universally closed.

Solution 2.4.2. Firstly consider the morphism of underlying space. Since f and g are S-morphisms, we can define a $h = f \times g : X \to Y \times_S Y$, then $h = \Delta \circ f$ on U. Since Y is seperated over S, $\Delta(Y)$ is closed in $Y \times_S Y$, hence $h^{-1}(\Delta(Y))$ is closed is X. Since $U \subset h^{-1}(\Delta(Y))$ and U is dense, we know that $h^{-1}(\Delta(Y)) = X$, i.e. $f = g$ on whole X. Then we just need to consider the sheaf structure. For $x \in X$, we have a $y = f(x) = g(x) \in Y$, we have $f_y^\sharp, g_y^\sharp : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$. We may assume that y is in an affine piece Spec B corresponding to a prime ideal $\mathfrak{p}$ and Spec $A \subset f^{-1}(\text{Spec } B)$ with $x \in$ Spec A. Hence we have two morphisms $\varphi, \psi : B \to A$ corresponding to $f, g :$ Spec $A \to$ Spec B. For any $b \in B$ we can define $a = \varphi(b) - \psi(b) \in A$. Clearly we have $U \cap \text{Spec } A \subset V(a)$ closed in Spec A and $U \cap$ Spec A is dense in Spec A, we know that $V(a) =$ Spec A. Since X is reduced we have $a = 0$, i.e. $f = g$ on Spec A. Since x is arbitrary, $f = g$ on whole X.

For counterexamples. (a) Define $X = Y = \text{Spec } k[x, y]/(x^2, xy)$. Then X and Y are homeomorphic to $\mathbb{A}_k^1$ but have a nilpotent at (0). Then we may define the $f : X \to Y$ be the identity, and $g : X \to Y$ by the morphism corresponding to $k[x, y]/(x^2, xy) \to k[x, y]/(x^2, xy)$ with $x \mapsto 0$, $y \mapsto y$. Then $f = g$ on $X - \{(0)\}$ but not on (0). (b) Define X and Y as the affine line over k with P doubled as in Example 2.3.6., and $f : X \to Y$ as the identity, $g : X \to Y$ as the switching of the doubled point. Then $f = g$ out of the doubled point, but not on whole X.

Solution 2.4.3. We may assume that $S =$ Spec A. Since X is seperated over S, we know that $\Delta : X \to X \times_S X$ is a closed immersion. If we denote two projections from $X \times_S X$ to X as π_1 and π_2, we have $U \cap V = \Delta^{-1}(\pi_1^{-1}(U) \cap \pi_2^{-1}(V))$. So we have a closed immersion $U \cap V \to \pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times_S V$. Since $U \times_S V$ is clearly affine, $U \cap V$ is also affine.

For counterexample, define X to be the affine surface with a doubled point P, U and V are two copies of $\mathbb{A}^2$, then $U \cap V = \mathbb{A}^2 - P$ is not affine.

Solution 2.4.4. Since $Z \to S$ is proper and $Y \to S$ is separated, we know that $Z \to Y$ is proper, hence $f(Z)$ is closed. Moreover, since $f(Z) \to Y$ is a closed immersion, hence of finite type by 2.3.11.(a), and $Y \to Z$ is of finite type, we know that $f(Z)$ is of finite type over Z. Since $\Delta(f(Z)) = \Delta(Y) \cap (f(Z) \times_S f(Z))$, and $\Delta(Y)$ is closed in $Y \times_S Y$, we know that $\Delta(f(Z))$ is closed in $f(Z) \times_S f(Z)$, i.e. $f(Z) \to S$ is separated. Finally, since properness and surjection is stable under base change, we only need to show $f(Z) \to S$ is closed. Denote the morphism $X \to S$ and $Y \to S$ as p and q. Then for any $W \subset f(Z)$, we know that $W = f(f^{-1}(W))$, so $q(W) = q(f(f^{-1}(W))) = p(f^{-1}(W))$. Since p is proper, we know that $p(f^{-1}(W))$ is closed, hence $f(Z) \to S$ is closed.

Solution 2.4.5. (a) If R a valuation ring of K has center x, then $\mathfrak{m}_R \cap \mathscr{O}_{X,x} = \mathfrak{m}_x$. So we have a commutative diagram

$$\begin{tikzcd} \text{Spec } K \arrow[r] \arrow[d] & X \arrow[d] \\ \text{Spec } R \arrow[r] \arrow[ur] & \text{Spec } k \end{tikzcd}$$

Since x is unique, the morphism Spec $R \to X$ is unique, hence X is seperated over k.

(b) If R is a valuation ring of K, and X is proper over k, then the following commutative diagram

$$\begin{tikzcd} \text{Spec } K \arrow[r] \arrow[d] & X \arrow[d] \\ \text{Spec } R \arrow[r] & \text{Spec } k \end{tikzcd}$$

induces a morphism Spec $R \to X$. Then obviously the point in X corresponding to the point $\mathfrak{m}_R$ in Spec R is the center of R.

(c) If S is a valuation ring with fraction field L with a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} L \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} S \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

we may assume the image of Spec L in X is the generic point x of X, or we may take a closed subset $Z \subset X$ with generic point x and change X to Z. Then Spec $L \to X$ induces a field extension $K \to L$. Hence valuation on L induce a valuation ring on K with a valuation ring $R = \{r \in K \mid v(r) \geq 0\} \subset K$. So we have the following commutative ring

$$\begin{tikzcd} \operatorname{Spec} L \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} S \arrow[r] & \operatorname{Spec} R \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

In the case of (a), if we have two morphisms Spec $S \to X$, then the different two images of $\mathfrak{m}_S$ is the different two images of $\mathfrak{m}_R$ in the morphism Spec $R \to X$, which makes a contradiction. Hence X is seperated over k. In the case of (b), by assumption we have a unique morphism Spec $R \to X$, i.e. Spec $S \to$ Spec $R \to X$. Hence X is proper over k.

(d) If $a \in \mathscr{O}_X(X)$ with $a \notin k$, since k is algebraically closed, a is transcendental over k, so $k[a^{-1}]_{(a^{-1})}$ is a local ring contained in K. Hence there exists a valuation ring $R \subset K$ dominates it, i.e. $a^{-1} \in \mathfrak{m}_R$. By morphism Spec $R \to X$, we have a morphism $\mathscr{O}_X(X) \to R$, hence the image of a in R has valuation≥ 0. Thus $v(1) = v(a/a) = v(a) + v(a^{-1}) > 0$, which makes a contradiction.

Solution 2.4.6. We may assume $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ for some finitely generated k-algebra A and B. Then we have a k-algebra morphism $\varphi : B \to A$. Denote $K = \operatorname{Frac}(A)$. And there exists a valuation ring R with fraction field K containing Imφ. Then we have a commutative ring

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] & Y \end{tikzcd}$$

Since $X \to Y$ is proper, we have a morphism Spec $R \to X$, i.e. a ring morphism $A \to R$. By 4.11.A, the integral closure of Imφ in K is the intersection of all valuation ring in K containing Imφ, so A is contained in this integral closure, i.e. every element of A is integral over B. Since f is of finite type, f is finite.

Solution 2.4.7 (Scheme over Spec $\mathbb{R}$)**.** (a) First consider the case that $X = \operatorname{Spec} A$ is affine. Since X has a semilinear involution σ, we have a ring morphism $\tau : A \to A$ commutating with the conjugation on $\mathbb{C}$. Then $B = \{a \in A \mid \tau(a) = a\}$ is clearly a subring of A over $\mathbb{R}$ of finite type, since A is a finitely generated $\mathbb{C}$-algebra. Then we can just define $X_0 = \operatorname{Spec} B$, then $X_0 \times_{\mathbb{R}} \mathbb{C} = X$.

For general case, for every $x \in X$, we have an affine piece Spec A containing x and $\sigma(x)$. Since X is seperated over $\mathbb{C}$, Spec $A \cap \sigma(\operatorname{Spec} A)$ is affine and contains x and $\sigma(x)$, hence there exists a ring A_x such that $x, \sigma(x) \in \operatorname{Spec} A_x$ and $\sigma(\operatorname{Spec} A_x) = \operatorname{Spec} A_x$. Then by affine case we know that there exists a finitely generated $\mathbb{R}$-algebra B_x can be base changed to A_x. Since $\{A_x\}$ is an affine covering of X, we only need to glue all Spec B_x together to get a X_0. For any A_x and A_y, as before Spec $A_x \cap$ Spec A_y is affine, namely Spec A. And there exists a ring B, which can be base changed to A. Hence we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_x \arrow[d] & \operatorname{Spec} A \arrow[l] \arrow[r] \arrow[d] & \operatorname{Spec} A_y \arrow[d] \\ \operatorname{Spec} B_x & \operatorname{Spec} B \arrow[l] \arrow[r] & \operatorname{Spec} B_y \end{tikzcd}$$

Hence all B_x can be glued together to be a X_0, which satisfies $X_0 \times_{\mathbb{R}} \mathbb{C} = X$.

(b) If X_0 is affine, then X is obviously affine. Conversely, if $X = \operatorname{Spec} A$ is affine, then by construction above, $X_0 = \operatorname{Spec} B$ for the ring $B = \{a \in A \mid \sigma(a) = a\}$, hence affine.

(c) Clearly if we have a f_0, we clearly have a $f = f \times_{\mathbb{R}} \mathbb{C}$ such that $f \circ \sigma_X = \sigma_Y \circ f$. Conversely, in the case that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, we have a ring homomorphism $\varphi : B \to A$. Since $f \circ \sigma_X = \sigma_Y \circ f$, we know φ induces a morphism $\varphi : B^\tau \to A^\tau$, where τ is the complex conjugation on X or Y. Then by the construction above we know a morphism $f_0 : \operatorname{Spec} A^\tau \to \operatorname{Spec} B^\tau$, which can be base changed to f. In general case, for Y we can take affine piece $\operatorname{Spec} B$ stable under σ as above, and for any $x \in f^{-1}(\operatorname{Spec} B)$, we know that $f(\sigma(x)) = \sigma(f(x)) \in \operatorname{Spec} B$, i.e. $f^{-1}(\operatorname{Spec} B)$ is stable under σ, so we can take affine piece $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$ stable under σ as above. So we have the f_0 as in the affine case. Hence we may glue all f_0 together to get a f_0, which can be base changed to f as we want.

(d) If $X = \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[t]$, we know that $X_0 = \operatorname{Spec} (\mathbb{C}[t])^\tau = \operatorname{Spec} \mathbb{R}[t] = \mathbb{A}^1_{\mathbb{R}}$.

(e) If $X = \mathbb{P}^1_{\mathbb{C}}$ and σ has a fixed closed point x, we may define $U = X - \{x\}$, which is affine and isomorphic to $\operatorname{Spec} \mathbb{C}[t]$. Then $\sigma : U \to U$ induces a morphism $\tau : \mathbb{C}[t] \to \mathbb{C}[t]$, and τ is an involution commutating with complex conjugation. Hence we may assume that $\tau(at + b) = \bar{a}t + \bar{b}$, i.e. $U_0 = \operatorname{Spec} \mathbb{R}[t]$ by construction in (a). Hence $X_0 = \mathbb{P}^1_{\mathbb{R}}$.

If *sigma* does not have any fixed closed point, for any closed point $x \in X$, we may define $U = X - \{x, \sigma(x)\}$, which is isomorphic to $\operatorname{Spec} \mathbb{C}[t, t^{-1}]$. Then $\sigma : U \to U$ induces an involution $\tau : \mathbb{C}[t, t^{-1}] \to \mathbb{C}[t, t^{-1}]$ commutating with complex conjugation. Since $\tau(\tau(t)) = t$, it has two cases: $\tau(t) = -t$ or $\tau(t) = at^{-1}$ for some $a \in \mathbb{C}$. (1) If $\tau(t) = -t$, τ will fixed $\mathbb{C}$, which is contradict with the assumption that σ does not have any fixed closed point. (2) If $\tau(t) = at^{-1}$, then $\tau(a) = \tau(t\tau(t)) = \tau(t)t = a$, hence $a \in \mathbb{R}$. If $a \geq 0$, then $\tau(t - \sqrt{a}) = at^{-1} - \sqrt{a} = -\sqrt{a}t^{-1}(t - \sqrt{a})$, i.e. σ can fix the prime ideal $(t - \sqrt{a})$, which makes a contradiction, hence $a < 0$. Under the coordinate changing $t \mapsto a^{-1/2}t$, we may assume that $a = -1$, i.e. $\tau(-t) = 1/t$. Since we have an isomorphism $\mathbb{C}[\frac{X}{Z}, \frac{Y}{Z}]/(1 + \frac{XY}{Z^2})$ via $-t \mapsto \frac{Y}{Z}$ and $t^{-1} \mapsto \frac{X}{Z}$, and the corresponding $\mathbb{C}[-t] \cong \mathbb{C}[\frac{Y}{X}, \frac{Z}{X}]/(\frac{Y}{X} + (\frac{Z}{X})^2)$ and $\mathbb{C}[t^{-1}] \cong \mathbb{C}[\frac{X}{Y}, \frac{Z}{Y}]/(\frac{X}{Y} + (\frac{Z}{Y})^2)$, we know that τ switch X and Y. Since $\mathbb{P}^1_{\mathbb{C}} \cong \operatorname{Proj} \mathbb{C}[X, Y, Z]/(XY + Z^2)$, we may change the coordinates $\frac{X+Y}{2} \mapsto U$ and $\frac{i(Y-X)}{2} \mapsto V$ and have $\mathbb{P}^1_{\mathbb{C}} \cong \operatorname{Proj} \mathbb{C}[U, V, Z]/(U^2 + V^2 + Z^2)$. So by the construction above, $X_0 = \operatorname{Proj} \mathbb{C}[U, V, Z]/(U^2 + V^2 + Z^2)$.

Solution 2.4.8. (d) If $f : X \to Y$ and $g : Z \to W$ have $\mathscr{P}$, by base change, $X \times A \to Y \times A$ and $Y \times A \to Y \times B$ have $\mathscr{P}$. So the composition $X \times A \to Y \times B$ has $\mathscr{P}$, which is just $f \times g$.

(e) Since $X \to X \times_Z Y$ is the base change of $Y \to Y \times_Z Y$ by $f \times \mathrm{id}$, and Y is separated, i.e. $Y \to Y \times Y$ is closed immersion, $X \to X \times_Z Y$ has $\mathscr{P}$. Since $X \times_Z Y \to Y$ is the base change of $g \circ f$, so $X \to X \times_Z Y \to Y$ has $\mathscr{P}$.

(f) Since $X_{\mathrm{red}} \to X$ is a closed immersion, we know $X_{\mathrm{red}} \to X \to Y$ has $\mathscr{P}$. Since $X_{\mathrm{red}} \to Y$ can factor through $X_{\mathrm{red}} \to Y_{\mathrm{red}}$ by 2.2.3.(c), and $Y_{\mathrm{red}} \to Y$ is separated, we know $X_{\mathrm{red}} \to Y_{\mathrm{red}}$ has $\mathscr{P}$.

Solution 2.4.9. If $f : X \to Y$ and $g : Y \to Z$ are projective, we may assume they have factorizations $X \to \mathbb{P}^n_Y \to Y$ and $Y \to \mathbb{P}^m_Z \to Z$. Since $(\mathbb{P}^m_Z \times_Z \mathbb{P}^n_Z) \times_{\mathbb{P}^n_Z} Y \cong \mathbb{P}^m_Z \times_Z Y \cong \mathbb{P}^m_Y$, hence $g \circ f$ factorize as $X \to \mathbb{P}^m_Y \to \mathbb{P}^m_Z \times_Z \mathbb{P}^n_Z \to \mathbb{P}^n_Z \to Z$. Since $X \to \mathbb{P}^m_Z \times_Z \mathbb{P}^n_Z$ is clearly a closed immersion, we only need to prove that there exists a closed immersion $\mathbb{P}^m_Z \times_Z \mathbb{P}^n_Z \to \mathbb{P}^{mn+m+n}_Z$, namely the Segre immersion.

For the case $Z = \operatorname{Spec} \mathbb{Z}$, we know that $\mathbb{P}^m_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[x_0, \ldots, x_m]$, $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[y_0, \ldots, y_n]$, $\mathbb{P}^{mn+m+n}_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[z_{00}, \ldots, z_{mn}]$. On open piece $D_+(x_i) \times D_+(y_j) \to D_+(z_{ij})$ we have the closed immersion induced by

$\mathbb{Z}[z_{00}, \ldots, z_{mn}]_{(z_{ij})} \to \mathbb{Z}[x_0, \ldots, x_m]_{(x_i)} \otimes_{\mathbb{Z}} \mathbb{Z}[y_0, \ldots, y_n]_{(y_j)}$, $\frac{z_{kl}}{z_{ij}} \to \frac{x_k}{x_i} \otimes \frac{y_l}{y_j}$. So glueing them together we have a closed immersion $\mathbb{P}^m_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}} \to \mathbb{P}^{mn+m+n}_{\mathbb{Z}}$. For general Z, we only need to prove that projective property is stable under base change.

If $f : X \to Y$ is projective, it may be factorized as $X \to \mathbb{P}^n_Y \to Y$. Then for any $Y' \to Y$ and $X' = X \times_Y Y'$, we have $f' : X' \to Y'$ can be factorized as $X' \to \mathbb{P}^n_Y \times_Y Y' \to Y'$. Since $\mathbb{P}^n_Y \times_Y Y' = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y \times_Y Y' = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y' = \mathbb{P}^n_{Y'}$, f' is projective, i.e. the property of projective is stable under base change.

For (a) in 2.4.8., closed immersion $X \to Y$ is clearly projective via factorizing $\mathbb{P}^0_Y$. And we've done the (b) and (c). So by 2.4.8. projective morphisms have properties (d)-(f).

Solution 2.4.10 (Chow's Lemma)**.** (a) Since S is noetherian, X is also noetherian, hence X have irreducible components $X_1, \ldots, X_n$. For every X_i, we denote $V_i = X_i - \cup_{j \neq i} X_j$. Then V_i is open dense in X_i, and open in X. Since $f : X \to S$ is proper, the induced morphism $f_i : X_i \to S$ is proper. If we have $g_i : X'_i \to X_i$ have the property we need, and $U_i \subset X_i$ open dense is isomorphic to $g_i^{-1}(U_i)$, we just only need to define $X' = \coprod X'_i$ and we have a morphism $g : X' \to X$ such that g and fg is projective. Take $U = \cup(U_i \cap V_i)$. Then U is dense in X and clearly $U \cong g^{-1}(U)$. So we only need to consider the case that X is irreducible.

(b) S has a finite affine covering $S = \cup \operatorname{Spec} B_i$, and each $f^{-1}(\operatorname{Spec} B_i)$ has a finite affine covering $f^{-1}(\operatorname{Spec} B_i) = \cup \operatorname{Spec} A_{ij}$ for some finitely generated B_i-algebra A_{ij}. If $a_1, \ldots, a_n$ is a set of generators of A_{ij}, we have a surjective morphism $B_i[x_1, \ldots, x_n] \to A_{ij}$ as $x_k \to a_k$, which induces a closed immersion $\operatorname{Spec} A_{ij} \to \operatorname{Spec} B_i[x_1, \ldots, x_n]$. So we have an immersion $\operatorname{Spec} A_{ij} \to \mathbb{P}^n_{B_i}$. Since we have an open immersion $\operatorname{Spec} B_i \to S$, which induces an immersion $\mathbb{P}^n_{B_i} \to \mathbb{P}^n_S$, i.e. we have an immersion $\operatorname{Spec} A_{ij} \to \mathbb{P}^n_S$. So it can be factorized as $\operatorname{Spec} A_{ij} \to P_{ij} \to \mathbb{P}^n_S$ for some open immersion $\operatorname{Spec} A_{ij} \to P_{ij}$. And we only need to change notations $\operatorname{Spec} A_{ij}$ to U_i and P_{ij} to P_i.

(c) Since X and P are both proper over S, $X \times_S P$ is proper over S, so X' closed in $X \times_S P$ is also proper over S. Then by corollary 4.8.(e), X' is proper over P. So we only need to show that $X' \to P$ is an immersion. Define W_i as the preimage of U_i of the projection $P \to P_i$. Then we have $h^{-1}(W_i) = X' \cap (X \times_S W_i)$. Since $X' \cap (X \times_S W_i)$ form an open covering of X', we only need to show that $X' \cap (X \times_S W_i) \to W_i$ is a closed immersion. Consider $W_i \to W_i \to X$, the graph map $W_i \to X \times_S W_i$ is a closed immersion. Since the image of $U \to P$ is contained in every W_i, we have morphisms $U \to W_i$. Similarly we have a morphism $U \to X \times_S W_i$ induced by $U \to X \times_S P$, and this morphism factorize W_i. Since $X' \cap (X \times_S W_i)$ is the scheme-theoretic image of $U \to X \times_S W_i$, the morphism $X' \cap (X \times_S W_i) \to X \times_S W_i$ factorizes through the closed immersion $W_i \to X \times_S W_i$. So $X' \cap (X \times_S W_i) \to W_i$ are closed immersions.

(d) Clearly $X' \cap (U \times_S P)$ is the scheme-theoretic image of $U \to U \times_S P$ and this is closed since it is a graph morphism, so $g^{-1}(U) = X' \cap (U \times_S P)$ is clearly U itself.

Solution 2.4.11. (a) If L/K has transcendental degree n, then we may find $x_1, \ldots, x_n \in L$ such that $L/K(x_1, \ldots, x_n)$ is finite. Then $\mathscr{O}[x_1, \ldots, x_n]_{(\mathfrak{m}[x_1, \ldots, x_n])}$ is a noetherian local domain with quotient field $K(x_1, \ldots, x_n)$, hence we just need to consider the case that L/K is finite. Consider a system of parameters $\{x_1, \ldots, x_n\}$ of $\mathfrak{m}$ i.e. $\sqrt{(x_1, \ldots, x_n)} = \mathfrak{m}$ and $x_1 \notin \sqrt{(x_2, \ldots, x_n)}$. Then in the ring $\mathscr{O}' = \mathscr{O}[x_2/x_1, \ldots, x_n/x_1]$, $\mathfrak{m}\mathscr{O}' = (x_1)$ is not the unit ideal. Hence for any minimal prime ideal $\mathfrak{p}$ lying over (x_1), then by Krull's principal ideal theorem we have height $\mathfrak{p} = 1$, i.e. $\dim \mathscr{O}'_{\mathfrak{p}} = 1$. Hence by Krull-Akizuki theorem, $\dim \tilde{\mathscr{O}}'_{\mathfrak{p}}$ is noetherian of dimension 1. So if R is any localization of $\tilde{\mathscr{O}}'_{\mathfrak{p}}$ at one of its maximal ideals, R is a discrete valuation ring in L dominating $\mathscr{O}$ since R is principal.

(b) For any valuation ring $\mathscr{O}$ and its fraction field K with the commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} \mathscr{O} \arrow[r] & Y \end{tikzcd}$$

We have a field extension L/K with a discrete valuation ring $R \subset L$ dominating $\mathscr{O}$ satisfying the commutative diagram

$$\begin{tikzcd} \operatorname{Spec} L \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} \mathscr{O} \arrow[r] & Y \end{tikzcd}$$

Then every morphism $\operatorname{Spec} \mathscr{O} \to X$ corresponds to a morphism $\operatorname{Spec} R \to X$ by composing $\operatorname{Spec} R \to \operatorname{Spec} \mathscr{O}$. Conversely, every morphism $\operatorname{Spec} R \to X$ corresponds to a morphism $\operatorname{Spec} \mathscr{O} \to X$ set-theoretically. On the sheaf, if we denote $x \in X$ as the image of $(0) \subset R$, we have $\mathscr{O}_x \to R_{(0)} = L$ factors through K, i.e. it induces a morphism $\mathscr{O}_x \to \mathscr{O}_{(0)}$. And if we denote $x \in X$ as the image of $\mathfrak{m}_R \subset R$, since R dominate $\mathscr{O}$, the morphism $\mathscr{O}_x \to R_{\mathfrak{m}_R}$ factors through $\mathscr{O}_{\mathfrak{m}}$, i.e. we have a morphism of schemes $\operatorname{Spec} \mathscr{O} \to X$. Hence the morphism from $\operatorname{Spec} R$ to X is one-to-one corresponds to the morphism from $\operatorname{Spec} \mathscr{O}$ to X, so the criteria of separable or proper is just need to consider the case of discrete valuation rings.

Solution 2.4.12 (Examples of Valuation Rings)**.** (a) Consider the smooth projective curve C_K, we have the following commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & C_K \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

Then we denote the image of the maximal ideal $\mathfrak{m}$ of R in C_X as x. If x is the generic point, we have a morphism $K = \mathscr{O}_{X,x} \to R$, which makes a contradiction with that R is not a field. Hence x is a closed point, and we have a morphism $\mathscr{O}_{X,x} \to R$, which means R dominate $\mathscr{O}_{X,x}$. By theorem 6.1.A in chapter I, we know that $\mathscr{O}_{X,x}$ is a maximal element in the set of local ring in K, hence $\mathscr{O}_{X,x} \cong R$, i.e. R is a discrete valuation ring.

(b.1) Take an affine piece $U = \operatorname{Spec} A$ containing x_1. If x_1 corresponds to a prime ideal $\mathfrak{p} \subset A$, we know that $K = \operatorname{Frac}(A)$ and $\mathscr{O}_{X,x_1} = A_{\mathfrak{p}}$. Since Y has codimension 1, we know that height $\mathfrak{p} = 1$, so $\dim A_{\mathfrak{p}} = 1$. Since X is nonsingular, A is normal, hence $A_{\mathfrak{p}}$ is normal. And since $A_{\mathfrak{p}}$ is clearly noetherian, $A_{\mathfrak{p}}$ is a discrete valuation ring. Obviously $A_{\mathfrak{p}}$ has center x_1.

(b.2) We may assume that X' is smooth or we just need to blow up it several times to make it smooth. Then by (b.1) R is a discrete valuation ring. Since f induces an injection $\mathscr{O}_{X,x_0} \to R$, we clearly have R dominate $\mathscr{O}_{X,x_0}$, i.e. R has center x_0.

(b.3) By definition R is a valuation ring and dominate R_0. Since R_0 dominate $\mathscr{O}_{X,x_0}$, clearly R dominate $\mathscr{O}_{X,x_0}$, i.e. has center x_0.

2.5 Sheaves of Modules

Solution 2.5.1. (a) For every open subset $U \subset X$, we can define $\phi : \mathscr{E}(U) \to \check{\check{\mathscr{E}}}(U) = \operatorname{Hom}(\mathscr{H}om(\mathscr{E}, \mathscr{O}_X)|_U, \mathscr{O}_X|_U)$ as $s \mapsto \{\rho_{s,V}\}_V$, where V runs through all open subset of U, and $\rho_{s,V} : \operatorname{Hom}(\mathscr{E}, \mathscr{O}_X)(V) \to \mathscr{O}_X(V)$ as $t \mapsto t_V(s|_V)$. Since $\mathscr{E}$ is locally free of finite rank, we have $\mathscr{H}om(\mathscr{E}, \mathscr{O}_X)_P = \operatorname{Hom}(\mathscr{E}_P, \mathscr{O}_{X,P})$ for every point $P \in X$. Hence on stalks, the ϕ induces $\mathscr{E}_P \to \operatorname{Hom}(\operatorname{Hom}(\mathscr{E}_P, \mathscr{O}_{X,P}), \mathscr{O}_{X,P})$ as $s_P \mapsto \rho_{s,P}$ for some $\rho_{s,P}(t_P) = t_P(s_P)$. Since $\mathscr{E}_P$ is free of finite rank, the morphism ϕ_P is clearly an isomorphism, hence ϕ is an isomorphism.

(b) For every open subset $U \subset X$, we can define $\phi : \check{\mathscr{E}}(U) \otimes \mathscr{F}(U) \to \operatorname{Hom}(\mathscr{E}|_U, \mathscr{F}|_U)$ as $\varepsilon \otimes s \mapsto \theta_{\varepsilon,f,U}$, where $\theta_{\varepsilon,f,U}(e) = \varepsilon(e) \cdot f|_V$ for any $e \in \mathscr{E}(V)$ for some open subset V of U. So on every $P \in X$, we have $\phi_P : \check{\mathscr{E}}_P \otimes \mathscr{F}_P \to \operatorname{Hom}(\mathscr{E}_P, \mathscr{F}_P)$ as $\phi_P(\varepsilon \otimes f) = \theta_{\varepsilon,f,P}$ where $\theta_{\varepsilon,f,P}(e) = \varepsilon(e) \cdot f$. It is clearly an isomorphism, hence ϕ is an isomorphism.

(c) We may define a $\phi : \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) \to \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F}))$ as on every $V \subset U \subset X$ both open, $\phi(\psi)(s)(t) = \psi_V(t \otimes s|_V)$ for $s \in \mathscr{F}(U)$ and $t \in \mathscr{E}(V)$. Firstly, if $\phi(\psi) = 0$, for every $U \subset X$ open, $s \in \mathscr{F}(U)$

and $t \in \mathscr{G}(U)$, we have $\psi(t \times s) = 0$, i.e. ψ is a zero map. Hence ϕ is injective. Conversely, for any morphism $\theta : \mathscr{F} \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{G})$, on open subset $U \subset X$, we can define a $\psi : \mathscr{E}(U) \otimes \mathscr{F}(U) \to \mathscr{G}(U)$ as $\psi(t \otimes s) = \theta(s)(t)$ for any $s \in \mathscr{F}(U)$ and $t \in \mathscr{G}(U)$. Hence ψ is a morphism of sheaves as the preimage of θ, so ϕ is also surjective.

(d) (Projection Formula) $f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{E} \cong \mathscr{H}om_{\mathscr{O}_Y}(\check{\mathscr{E}}, f_*\mathscr{F}) \cong f_*\mathscr{H}om_{\mathscr{O}_X}(f^*(\check{\mathscr{E}}), \mathscr{F}) \cong f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{E})$.

Solution 2.5.2. (a) For every $\mathscr{O}_X$-module $\mathscr{F}$, we may define $M = \mathscr{F}(X)$ and $L = \mathscr{F}(U)$ where $U = \{(0)\}$ open in X, then we clearly have the morphism ρ. Conversely, for every M, L, ρ, we may define a $\mathscr{O}_X$-module $\mathscr{F}$ as $\mathscr{F}(X) = M$ and $\mathscr{F}(U) = L$, and the restriction map $\mathscr{F}(X) \to \mathscr{F}(U)$ is the map $M \otimes 1 \to L$.

(b) If $\mathscr{F}$ is quasi-coherent, i.e. $\mathscr{F} = \tilde{M}$, then $L = M_{\mathfrak{m}} = M \otimes_R K \xrightarrow{\rho} L$. So ρ is an isomorphism. Conversely, if ρ is an isomorphism, i.e. $L = M_{\mathfrak{m}}$. So $\mathscr{F} = \widetilde{\mathscr{F}(X)} = \tilde{M}$, i.e. quasi-coherent.

Solution 2.5.3. For any $\phi : M \to \mathscr{F}(X)$, we may define $\psi : \tilde{M} \to \mathscr{F}$ as $M_f \to \mathscr{F}(D(f))$, $\frac{m}{f^n} \mapsto \frac{\phi(m)|_{D(f)}}{f^n}$ on every principal open subset $D(f)$ for some $f \in A$. Conversely, for any $\psi : \tilde{M} \to \mathscr{F}$, we only need to define $\phi = \psi(X)$.

Solution 2.5.4. ($\Rightarrow$) For every $x \in X$, there exists an affine open neighborhood $U = \operatorname{Spec} A$ of x. If we denote $\mathscr{F}(U) = M$, we know $\mathscr{F}|_U = \tilde{M}$ since $\mathscr{F}$ is quasi-coherent. Since M is an A-module, it must be a cokernal of a morphism between free A-modules. Hence on U, $\mathscr{F}|_U$ is a cokernal of a morphism between free $\mathscr{O}_X|_U$-modules. Moreover, if $\mathscr{F}$ is coherent, i.e. M is finitely generated A-module, we can take two free A-modules above finitely generated, i.e. $\mathscr{F}$ is a cokernal of a morphism between free $\mathscr{O}_X|_U$-modules of finite rank.

($\Leftarrow$) For every $x \in X$ we have a neighborhood $U \subset X$ such that $\mathscr{F}|_U$ is the cokernal of a morphism between free $\mathscr{O}_X|_U$-modules. Shrinking U to make $U = \operatorname{Spec} A$ is affine, we know on U, free $\mathscr{O}_X|_U$-module is of the form $\tilde{A^n}$ for any cardinality n, i.e. quasi-coherent, so the cokernal is quasi-coherent. So $\mathscr{F}|_U = \tilde{M}$ for some A-module M. Hence $\mathscr{F}$ is quasi-coherent. Moreover, if $\mathscr{F}|_U$ is the cokernal of a morphism between free $\mathscr{O}_X|_U$-modules of finite rank, i.e. cokernal of a morphism between two coherent sheaves, it is clearly coherent.

Solution 2.5.5. (a) Just take $X = \operatorname{Spec} k[s,t]$ and $Y = \operatorname{Spec} k[s]$. Then $f_*\mathscr{O}_X$ is not coherent, since $k[s,t]$ is not finitely generated over $k[s]$.

(b) If $f : Z \to X$ is a closed immersion, for some affine piece $U = \operatorname{Spec} A \subset X$, we have $U \cap Z$ is closed in U, hence homeomorphic to some $\operatorname{Spec} A/I$ by corollary 5.10. On U, since $A/I = \mathscr{O}_X(U \cap Z) \to \mathscr{O}_Z(U \cap Z) = B$ is surjective, we have $A \to A/I \to B$ is surjective. And since B is clearly a finite A-module, f is finite.

(c) Since $f : X \to Y$ is finite, for any affine piece $U = \operatorname{Spec} A$ in Y, $f^{-1}(U) = \operatorname{Spec} B$ for some finite A-module B. Since $\mathscr{F}$ is coherent, we know that $\mathscr{F}|_{f^{-1}U} = \tilde{M}$ for some finitely generated B-module M. So by proposition 5.2., $f_*\mathscr{F}|_U = \widetilde{{}_AM}$, where ${}_AM$ means to treat M as an A-module, which is clearly finitely generated. So $f_*\mathscr{F}$ is coherent.

Solution 2.5.6 (Support)**.** (a) $\operatorname{Supp} m = \{x \in \operatorname{Spec} A \mid m_x \neq 0\} = \{\mathfrak{p} \subset A \text{ prime} \mid m_{\mathfrak{p}} \neq 0\} = V(\operatorname{Ann} m)$.

(b) Clearly $\operatorname{Supp} \mathscr{F} = \bigcup_{m \in M} V(\operatorname{Ann} m)$. Since M is finitely generated, we may take a set of generators $\{m_1, \dots, m_n\}$, hence $\bigcup_{m \in M} V(\operatorname{Ann} m) = \bigcup_{i=1}^n V(\operatorname{Ann} m_i) = V(\bigcap_{i=1}^n \operatorname{Ann} m_i) = V(\operatorname{Ann} M)$.

(c) If X is a noetherian scheme, we have an affine finite covering $X = \bigcup \operatorname{Spec} A_i$ for some ring A_i, and $\mathscr{F}$ on every affine piece $\operatorname{Spec} A_i$ has the form $\tilde{M}_i$ for some A_i-module M_i. Since $\operatorname{Supp} \mathscr{F} \cap \operatorname{Spec} A_i = V(\operatorname{Ann} M_i)$ is closed, $\operatorname{Supp} \mathscr{F} = \bigcup V(\operatorname{Ann} M_i)$ is also closed.

(d) By 2.1.20. we have an exact sequence $0 \to \mathscr{H}_Z^0(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U)$, where $U = X - Z$ and $j : U \to X$ is the open immersion. Then by proposition 5.8., $j_*(\mathscr{F}|_U)$ is quasi-coherent, so $\mathscr{H}_Z^0(\mathscr{F})$ is quasi-coherent by proposition 5.7. Since A is noetherian, we know $\Gamma_Z(\mathscr{F}) = \{m \in M \mid \operatorname{Supp} m \subset V(\mathfrak{a})\} = \{m \in M \mid \mathfrak{a}^n m = 0 \text{ for some } n\} = \Gamma_{\mathfrak{a}}(M)$, so $\mathscr{H}_Z^0(\mathscr{F}) = \widetilde{\Gamma_{\mathfrak{a}}(M)}$.

(e) Similarly with (d), we have the exact sequence $0 \to \mathscr{H}_Z^0(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U)$. By proposition 5.8., $j_*(\mathscr{F}|_U)$ is quasi-coherent (resp. coherent), then $\mathscr{H}_Z^0(\mathscr{F})$ is quasi-coherent (resp. coherent).

Solution 2.5.7. (a) For any affine neighbourhood $U = \operatorname{Spec} A$ of x, we may assume x corresponds to prime ideal $\mathfrak{p} \subset A$, $\mathscr{F}|_U = \tilde{M}$ for some finitely generated A-module M, and $M_{\mathfrak{p}} \cong A_{\mathfrak{p}}^{\oplus n}$ for some $n \in \mathbb{Z}$. If M is generated by $\{m_1, \dots, m_k\}$, we may assume that $\frac{m_i}{1}$ corresponds to $(\frac{a_{i1}}{s_{i1}}, \dots, \frac{a_{in}}{s_{in}}) \in A_{\mathfrak{p}}^n$. So we may define $s = \prod_{i,j} s_{ij}$, hence $m_i|_{D(s)} = \sum_j \frac{a_{ij}}{s_{ij}} e_j$, where $e_j \in M_{\mathfrak{p}}$ corresponding to $(0, \dots, 0, 1_j, 0, \dots, 0) \in A_{\mathfrak{p}}^{\oplus n}$. So we have a surjection $\phi_s : A_s^{\oplus n} \to M_s$. Moreover, since A is noetherian, M is finite presentation, i.e. $\ker \phi_s = \operatorname{Im}(\theta)$ for some $\theta : A_s^{\oplus m} \to A_s^{\oplus n}$ and some m. Since θ can be written as a matrix $(a_{ij})_{i,j}$, and $\ker \phi_s$ will vanish in $A_{\mathfrak{p}}$, there exist some $t_{ij} \in A - \mathfrak{p}$ such that $a_{ij} t_{ij} = 0$. So we just need to define $t = s \cdot \prod_{i,j} t_{ij}$, then $D(t) \subset D(s)$, and $\ker \phi_s$ vanish in $D(t)$, i.e. $\mathfrak{F}|_{D(t)} \cong \mathscr{O}_X|_{D(t)}^n$.

(b) ($\Rightarrow$) Trivial. ($\Leftarrow$) By (a), trivial.

(c) ($\Rightarrow$) Just define $\mathscr{G} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$. So we have a morphism $\mathscr{F} \otimes \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X) \to \mathscr{O}_X$, and it is clearly isomorphism on stalks since $\mathscr{F}$ is locally free of rank 1, i.e. this morphism is an isomorphism.

($\Leftarrow$) Since $\mathscr{F} \otimes \mathscr{G} \cong \mathscr{O}_X$, for every $P \in X$ we have $\mathscr{F}_P \otimes \mathscr{G}_P \cong \mathscr{O}_{X,P}$. Since $\mathscr{F}_P$ and $\mathscr{G}_P$ are both $\mathscr{O}_{X,P}$-modules, and $\mathscr{O}_{X,P}$ is a local ring, we know $\mathscr{F}_P \cong \mathscr{O}_{X,P}$. Then by (b), $\mathscr{F}$ is locally free of rank 1.

Solution 2.5.8. (a) For every n, and $x \in X$ such that $\varphi(x) = k < n$, we can choose an affine neighbourhood $U = \operatorname{Spec} A$ of x. So we may assume $\mathscr{F}|_U = \tilde{M}$ for some finitely generated A-module M, and M is generated by $\{m_1, \dots, m_r\}$. If prime ideal $\mathfrak{p} \subset A$ corresponds to x, we have $\mathscr{F}_x \otimes k(x) \cong M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$. So we can take a set of elements $\{u_1, \dots, u_k\}$ in M such that they form a basis of $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ as $k(x)$ vector space. Then Nakayama lemma implies $\{u_1, \dots, u_k\}$ generate $M_{\mathfrak{p}}$ as $A_{\mathfrak{p}}$-module, so there exist $\frac{a_{ij}}{f_{ij}} \in M_{\mathfrak{p}}$ such that $m_j = \sum_i \frac{a_{ij}}{f_{ij}} u_i$. Define $f = \prod_{i,j} f_{ij}$. Then $\mathfrak{p} \in D(f)$, and for any $\mathfrak{q} \in D(f)$, $m_j \in M_{\mathfrak{q}}$, then $\{m_1, \dots, m_r\}$ generate $M_{\mathfrak{q}}$ as an $A_{\mathfrak{q}}$-module. Hence $\{u_1, \dots, u_k\}$ generate $M_{\mathfrak{q}}$ as an $A_{\mathfrak{q}}$-module, i.e. $\varphi(\mathfrak{q}) \le k < n$. Thus $\{x \in X \mid \varphi(x) < n\}$ is open, i.e. φ is upper semi-continuous.

(b) For any $n \in \mathbb{N}$, we can define $U_n = \varphi^{-1}(n) \subset X$. For any $x \in U_n$, by 5.7.(a), there exists an open neighbourhood U of x such that $\mathscr{F}|_U \cong \mathscr{O}_X|_U^n$ is free. Hence for every $y \in U$, we have $\varphi(y) = n$. So U_n is an open set. Since $U_i \cap U_j = \varnothing$ for $i \ne j$, and $\bigcup U_i = X$ by definition, and since X is connected, we know there exists some $n \in \mathbb{N}$ such that $U_n = X$, and $U_i = \varnothing$ for all $i \ne n$, i.e. $\varphi \equiv n$ is a constant function.

(c) For any $x \in X$ and any affine neighbourhood $U = \operatorname{Spec} A$ of x, we may assume $\mathscr{F}|_U = \tilde{M}$ for some finitely generated A-module M. Take a set $\{m_1, \dots, m_n\}$ of M forming a basis of $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ as an $k(x)$ vector space. Then by Nakayama lemma, $\{m_1, \dots, m_n\}$ generate $M_{\mathfrak{p}}$ as $A_{\mathfrak{p}}$-module, hence they generate $M_{\mathfrak{q}}$ as $A_{\mathfrak{q}}$-module for all $\mathfrak{q} \subset \mathfrak{p}$. Since φ is constant, the images of $m_1, \dots, m_n$ in $M_{\mathfrak{q}}/\mathfrak{q}M_{\mathfrak{q}}$ are linearly independent. So if $\sum a_i m_i = 0$ in $M_{\mathfrak{p}}$, we have $a_i = 0$ in all $A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}$ for all $\mathfrak{q} \subset \mathfrak{p}$. Since X is reduced, we know $a_i = 0$ in $A_{\mathfrak{p}}$, hence m_i are linearly independent over $A_{\mathfrak{p}}$, i.e. $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module. Then by 5.7.(b), $\mathscr{F}$ is locally free.

Solution 2.5.9. (a) Since S_1 generates S, if $S_1 = \{s_i\}_i$, $\{D_+(s_i)\}_i$ is an affine covering of X. For all $m \in M$, it must be contained in some M_d, hene m has degree 0 in $M(d)_{s_i} = \widetilde{M(n)})(D_+(s_i))$ for all s_i, i.e. a section in $D_+(s_i)$. Since $D_+(s_i) \cap D_+(s_j) = D_+(s_i s_j)$, we know the sections defined by m above agree on all intersections, hence they can be glued together to be a section of $\Gamma_*(\tilde{M})$, i.e. we have a morphism $\alpha : M \to \Gamma_*(\tilde{M})$. Moreover, for any $m \in M_d$, $s \in S_{d'}$, we have $s \cdot \alpha(m)$ is the image of $m \times s$ in $\Gamma(X, \widetilde{M(d)} \otimes \mathscr{O}_X(d')) = \Gamma(X, M(\widetilde{d + d'}))$, i.e. α preserves the grade.

(b) First consider the case $\tilde{M} = 0$ and we need to prove that $M_d = 0$ for $d \gg 0$. By proposition 7.4. in Chapter I, we have a filtration of M as $0 = M_0 \subset \dots M_n = M$, with $0 \to M_{i-1} \to (S/\mathfrak{p}_i)(l_i) \to 0$ for all i, where p_i are homogeneous prime ideals in S and $l_i \in \mathbb{Z}$. Since $\tilde{M} = 0$, by induction of exact sequences $0 \to \tilde{M_{i-1}} \to \tilde{M_i} \to S\widetilde{/\mathfrak{p}_i(l_i)} \to 0$, we know that $\tilde{M_i} = 0$ and $\widetilde{S/\mathfrak{p}_i} = 0$ for all i. So $\mathfrak{p}_i \subset S_+$ for all i, i.e. $(S/\mathfrak{p}_i)(l_i)$ has non-zero elements in only one degree. Since the filtration is finite, we know that M has non-zero elements in finitely many some degrees, i.e. $M_d = 0$ for all $d \gg 0$. For general M, we may denote $M' = \Gamma_*(\tilde{M})$ and have $M' \subset M$. Then M/M' satisfies the case above, i.e. $(M/M')_d = 0$ for all $d \gg 0$. Then $M_d = M'_d$ for all $d \gg 0$.

(c) By (b), if M is finitely generated, M is equivalent to $\Gamma_*(\tilde{M})$. And by proposition 5.15., we know that if $\mathscr{F}$ is quasi-coherent, $\Gamma_*(\tilde{\mathscr{F}}) \cong \mathscr{F}$. So we only need to show that for any quasi-finitely generated graded S-module M, $\tilde{M}$ is coherent, and conversely for any coherent sheaf $\mathscr{F}$, $\Gamma_*(\mathscr{F})$ is quasi-finitely generated.

Firstly, for any quasi-finitely generated graded S-module M, there exists a finitely generated graded S-module M' such that for some d, $M_{\geq d} \cong M'_{\geq d}$. So for any $s_i \in S_1$, since $\frac{m}{s_i^n} = \frac{ms_i^d}{s_i^{n+d}}$, $M_{(s_i)} \cong M'_{(s_i)}$, hence finitely generated. Since $\{D_+(s_i)\}_i$ forms an affine covering of X, and $\tilde{M}|_{D_+(s_i)}$ are all coherent, we clearly have $\tilde{M}$ is coherent.

Conversely, for every coherent sheaf $\mathscr{F}$, by theorem 5.17., $\mathscr{F}(n)$ is generated by a finite number of global sections for $n \gg 0$. So we can define M as the submodule of $\Gamma_*(\mathscr{F})$ generated by these sections. Then we have an injection $\tilde{M} \to \mathscr{F}$. Since we obviously have the isomorphism $\widetilde{M(n)} \cong \mathscr{F}(n)$, $\tilde{M} \cong \mathscr{F}$, and hence $M_d \cong \widetilde{M(d)}(X) \cong \mathscr{F}(d)(X) = \Gamma_*(\mathscr{F})_d$ for $d \gg 0$, i.e. $\Gamma_*(\mathscr{F})$ is quasi-finitely generated.

Solution 2.5.10. (a) For any $s \in \bar{I}$, we have $s = \sum s_d$ for some homogeneous elements s_d with degree d. Since $x_i^n s = \sum x_i^n s_d \in I$, $x_i^n s_d$ are homogeneous elements, and I is a homogeneous ideal, we have $x_i^n s_d \in I$ for each i and d, i.e. $s_d \in \bar{I}$ for all d, hence $\bar{I}$ is a homogeneous ideal.

(b) Clearly for any homogeneous ideal I we have $I_{x_i} \cong \bar{I}_{x_i}$ for all x_i by definition. So I_1 and I_2 define the same closed subscheme $\Leftrightarrow \tilde{I}_1 = \tilde{I}_2 \Leftrightarrow I_{1,x_i} = \tilde{I}_1(D_+(x_i)) = \tilde{I}_2(D_+(x_i)) = I_{2,x_i} \Leftrightarrow \bar{I}_1 = \bar{I}_2$.

(c) Denote $I = \Gamma_*(\mathscr{I}_Y)$. Since $\bar{I}$ is homogeneous, for any homogeneous $s \in \bar{I}$ with degree d, i.e. $x_i^n s \in I$ for all x_i and some n. Since $x_i^n s$ has degree $n+d$, i.e. $x_i^n s \in \Gamma(\mathscr{I}_Y(n+d))$. So $s = x_i^{-n} x_i^n s \in \Gamma(U_i, \mathscr{I}_Y(n+d) \otimes \mathscr{O}_X(-n)) = \Gamma(U_i, \mathscr{I}_Y(d))$ for all i. Since $\{U_i\}$ forms an open covering of X, there exists an $s \in \Gamma(\mathscr{I}_Y(d))$, i.e. $I = \bar{I}$.

(d) By (a), (b) and (c), trivial.

Solution 2.5.11. For any homogeneous element $f \in S$ and $g \in T$ with same degree greater than 0, we have an isomorphism $S_f \otimes_A T_g \cong (S \times_A T)_{f \otimes g}$ as $(\frac{s}{f^n}, \frac{t}{g^m}) \mapsto \frac{f^m g^{n+m} s \otimes f^{n+m} g^n t}{(f \otimes g)^{n+m}}$ and $(\frac{s}{f^n}, \frac{t}{g^n}) \leftarrow \frac{s \otimes t}{(f \otimes g)^n}$. Hence we have an isomorphism $D_+(f) \times_A D_+(g) \cong D_+(f \otimes g)$. Moreover, for any $f, f' \in S$, $g, g' \in T$ homogeneous for $\deg f = \deg g$ and $\deg f' = \deg g'$, we have a cumbersome but trivial commutative diagram for restriction

$$\begin{tikzcd}
D_+(f) \times_A D_+(g) \arrow[r, "\cong"] & D_+(f \otimes g) \\
D_+(ff') \times_A D_+(gg') \arrow[r, "\cong"] \arrow[u] \arrow[d] & D_+(ff' \otimes gg') \arrow[u] \arrow[d] \\
D_+(f') \times_A D_+(g') \arrow[r, "\cong"] & D_+(f' \otimes g')
\end{tikzcd}$$

So, since $D_+(f)$ for all $f \in S$ homogeneous with degree greater than 0 form an open covering for S, and same for T, we have Proj $(S \times_A T) \cong X \times_A Y$ by construction of fibre product.

To prove $\mathscr{O}(1) \cong p_1^*(\mathscr{O}_X(1)) \otimes p_2^*(\mathscr{O}_Y(1))$, we just need to check on all affine piece $D_+(f \otimes g)$. But actually $\mathscr{O}(1)(D_+(f \otimes g)) = (S \times_A T)_{f \otimes g}$, and $(p_1^*(\mathscr{O}_X(1)) \otimes p_2^*(\mathscr{O}_Y(1)))(D_+(f \otimes g)) = (p_1^*(\mathscr{O}_X(1)))(D_+(f \otimes g)) \otimes (p_2^*(\mathscr{O}_Y(1)))(D_+(f \otimes g)) = \mathscr{O}_X(1)(D_+(f)) \otimes \mathscr{O}_Y(1)(D_+(g)) = S_f \otimes_A T_g$, then trivial.

Solution 2.5.12. (a) We may assume $i_n : X \to \mathbb{P}_Y^n$ and $i_m : X \to \mathbb{P}_Y^m$ are two closed immersions and $\mathscr{L} \cong i_n^* \mathscr{O}_{\mathbb{P}_Y^n}(1)$, $\mathscr{M} \cong i_m^* \mathscr{O}_{\mathbb{P}_Y^m}(1)$. Then consider $i : X \to \mathbb{P}_Y^n \times_Y \mathbb{P}_Y^m \to \mathbb{P}_Y^N$, where $N = nm + n + m$, the first arrow is just $i_n \times i_m$, and the second arrow is the Segre embedding. Then $\mathscr{L} \otimes \mathscr{M} = i_n^*(\mathscr{O}_{\mathbb{P}_Y^n}(1)) \otimes i_m^*(\mathscr{O}_{\mathbb{P}_Y^m}(1)) = i^*(\mathscr{O}_{\mathbb{P}_Y^N}(1))$, hence very ample.

(b) We may assume $i_n : X \to \mathbb{P}_Y^n$ and $i_m : Y \to \mathbb{P}_Z^m$ are two closed immersions and $\mathscr{L} \cong i_n^* \mathscr{O}_{\mathbb{P}_Y^n}(1)$, $\mathscr{M} \cong i_m^* \mathscr{O}_{\mathbb{P}_Z^m}(1)$. Then consider $i : X \to \mathbb{P}_Y^n \cong \mathbb{P}_{\mathbb{Z}}^n \times Y \to \mathbb{P}_{\mathbb{Z}}^n \times \mathbb{P}_Z^m \cong \mathbb{P}_{\mathbb{Z}}^n \times \mathbb{P}_{\mathbb{Z}}^m \times Z \cong \mathbb{P}_{\mathbb{Z}}^N \times Z \cong \mathbb{P}_Z^N$, where $N = nm + n + m$, and the three arrows are i_n, $\mathrm{id} \times i_m$ and the Segre embedding$\times$id. Then $\mathscr{L} \otimes f^* \mathscr{M} = i_n^*(\mathscr{O}_{\mathbb{P}_Y^n}(1)) \otimes f^* i_m^*(\mathscr{O}_{\mathbb{P}_Z^m}(1)) = i^*(\mathscr{O}_{\mathbb{P}_Z^N}(1))$, hence very ample.

Solution 2.5.13. Clearly, $S^{(d)}$ is generated by S_d as an S_0-algebra. So $\{D_+(f)\}_{f \in S_d}$ forms an open covering on Proj $S^{(d)}$ and also X. So we just need to check on each affine piece, i.e. to prove the isomorphism $\varphi : S^{(d)}_{(f)} \cong S_{(f)}$, which is just $\frac{s}{f^n} \mapsto \frac{s}{f^n}$, so we've done. Moreover, on every affine piece $D_+(f)$, $\mathscr{O}(1)(D_+(f)) = S^{(d)}(1)_{(f)} = S(d)_{(f)} = \mathscr{O}_X(d)(D_+(f))$. So just glue together all isomorphism, we have $\mathscr{O}(1) \cong \varphi^*\mathscr{O}_X(d)$.

Solution 2.5.14. (a) X is reduced since it's normal, so X is irreducible since it's connected. So S is a domain. Defining $\mathscr{I} = \bigoplus_{n\geq 0} \mathscr{O}_X(n)$ as hint, then we have $\mathscr{I}_\mathfrak{p} = \{\frac{s}{f} \in S_\mathfrak{p} \mid \deg s \geq \deg f\}$ for all $\mathfrak{p} \in X$. Since $S_\mathfrak{p}$ is integrally closed, $\mathscr{I}_\mathfrak{p}$ is integral over $S_\mathfrak{p}$, i.e. $\mathscr{I}$ is a sheaf of integrally closed domain. Then $S' = \Gamma(X, \mathscr{I})$ is integrally closed, which clearly contains S. Moreover, S' is obviously the minimal one which contains S and is integrally closed as we've seen in the proof of theorem 5.19., i.e. S' is the integral closure of S.

(b) By definition we clearly know that $\tilde{S} \cong \tilde{S'}$, then by 2.5.9.(b) we know that $S_d = S'_d$ for $d \gg 0$.

(c) By (b), we know for $d \gg 0$, we have $S_{nd} = S'_{nd}$ for all n since $S_0 = S'_0 = k$, i.e. $S^{(d)} \cong S'^{(d)}$. Then for any x integral over $S^{(d)}$, i.e. x satisfies $x^n + s_{n-1}x^{n-1} + \ldots + s_0 = 0$ for some $s_i \in S^{(d)}$, we know $x \in S'$ and clearly has degree being divisible by d. So $x \in S'^{(d)} = S^{(d)}$. So $S^{(d)}$ is integrally closed.

(d) ($\Rightarrow$): If X is projectively normal, S is integrally closed, i.e. $S = S'$. Then by (a), we have $S_n = \Gamma(X, \mathscr{O}_X(n))$. Since $A[x_0, \ldots, x_r] \to A[x_0, \ldots, x_r]/I = S$ is clearly surjective and perverse the degree, we have $\Gamma(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(n)) \to \Gamma(X, \mathscr{O}_X(n))$ is surjective.

($\Leftarrow$): Just need to prove $S = S'$. If not, the projection $A[x_0, \ldots, x_r] \to S \to S'$ is clearly not surjective, i.e. $\Gamma(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(n)) \to \Gamma(X, \mathscr{O}_X(n))$ is not surjective, which makes a contradiction.

Solution 2.5.15 (Extension of Coherent Sheaves)**.** (a) If $X = \operatorname{Spec} A$ is an noetherian affine scheme, $\mathscr{F}$ is a quasi-coherent sheaf on X. We may assume $\mathscr{F} = \tilde{M}$ for some A-module M. Since X is noetherian, A is also noetherian, so $M = \bigcup M_\alpha$ for all finitely generated A-module M_α. Then on any principal open subset $D(f)$ for some $f \in A$, clearly we have $M_f = \bigcup M_{\alpha,f}$. And for any open subset $U \subset X$, we just take a direct limit from principal open subset to U to get that $\mathscr{F} = \bigcup \mathscr{F}_\alpha$, where $\mathscr{F}_\alpha = \tilde{M}_\alpha$.

(b) By proposition 5.8., we know that $i_*\mathscr{F}$ is quasi-coherent. Then by (a), we have $i_*\mathscr{F} = \bigcup \mathscr{F}_\alpha$ for all coherent subsheaves $\mathscr{F}_\alpha = \tilde{M}_\alpha$ for $i_*\mathscr{F}$. Since A is noetherian, $\tilde{M}_\alpha$ has maximal element, namely M, so $i_*\mathscr{F} = \mathscr{F}' := \tilde{M}$, i.e. coherent.

(c) Clearly $\rho^{-1}(i_*\mathscr{F})$ is quasi-coherent, and clearly $\rho^{-1}(i_*\mathscr{F})|_U \cong \mathscr{F}$. Then as the same method of (b), we may take $\rho^{-1}(i_*\mathscr{F}) = \bigcup \mathscr{F}_\alpha$ and take a maximal element of coherent sheaf $\mathscr{F}'$ of the union, we know $\mathscr{F}' = \rho^{-1}(i_*\mathscr{F})$ is the extension of $\mathscr{F}$.

(d) If $X = \bigcup V_i$ is an affine open covering, which we can assume this covering is finite since X is noetherian, by induction we just need to prove that $\mathscr{F}$ on U can be extended to a coherent sheaf on $U \cap V_1$. Since $\mathscr{F}_{U\cap V_1}$ is a coherent sheaf on $U \cap V_1$, it can be extended to be a sheaf $\mathscr{G}$ on V_1. Since $\mathscr{F}$ and $\mathscr{G}$ is compatible on $U \cap V_1$, they can be glued together to be a coherent sheaf $\mathscr{F}'$ on $U \cup V_1$. By induction we can get a coherent sheaf $\mathscr{F}'$ on $U \cup (\bigcup V_i) = X$.

(e) Obviously $\mathscr{F} \supset \bigcup \mathscr{F}_\alpha$ for all coherent $\mathscr{F}_\alpha \subset \mathscr{F}$. Conversely, for any open subset $U \subset X$ and $s \in \mathscr{F}(U)$, s can generate a coherent sheaf $\mathscr{G}$ on U. Then by (d), $\mathscr{G}$ can be extended to be a coherent subsheaf $\mathscr{F}'$ of $\mathscr{F}$ such that $s \in \mathscr{F}'(U) = \mathscr{G}(U)$. Hence $\mathscr{F} \subset \bigcup \mathscr{F}_\alpha$, i.e. equal.

Solution 2.5.16 (Tensor Operations on Sheaves)**.** (a) Since $\mathscr{F}$ is locally free of rank n, for any point $x \in X$, there exists a neighbourhood U of x such that $\mathscr{F}|_U = \mathscr{O}_X|_U^n$ with basis $e_1, \ldots, e_n$. Then $T^r(\mathscr{F})|_U$ is free with basis $\{e_{i_1} \otimes \ldots \otimes e_{i_r} \mid 1 \leq i_1, \ldots, i_r \leq n\}$, $S^r(\mathscr{F})|_U$ is free with basis $\{e_{i_1} \otimes \ldots \otimes e_{i_r} \mid 1 \leq i_1 \leq \ldots \leq i_r \leq n\}$ and finally $\Lambda^r(\mathscr{F})|_U$ is free with basis $\{e_{i_1} \otimes \ldots \otimes e_{i_r} \mid 0 \leq i_1 < \ldots < i_r \leq n\}$. So $T^r(\mathscr{F})$, $S^r(\mathscr{F})$ and $\Lambda^r(\mathscr{F})$ are locally free with rank n^r, $\binom{n+r-1}{n-1}$ and $\binom{n}{r}$.

(b) For any point $x \in X$, and a neighbourhood U of x such that $\mathscr{F}|_U = \mathscr{O}_X|_U^n$ with basis $e_1, \ldots, e_n$. Then we clearly have a morphism

$$\Lambda^r(\mathscr{F})|_U \otimes \Lambda^{n-r}(\mathscr{F})|_U \to \Lambda^n(\mathscr{F})|_U$$

$$(e_{i_1} \otimes \ldots \otimes e_{i_r}) \otimes (e_{j_1} \otimes \ldots \otimes e_{j_{n-r}}) \mapsto (-1)^{\sigma(i_1,\ldots,i_r,j_1,\ldots,j_{n-r})} e_1 \otimes \ldots \otimes e_n$$

where the σ is the sign of permutation. It is clearly a perfect pairing.

(c) Similarly, take open subset U such that $\mathscr{F}'|_U, \mathscr{F}|_U, \mathscr{F}''|_U$ are free, then $\mathscr{F}|_U = \mathscr{F}'|_U \oplus \mathscr{F}''|_U$. So we have $S^r\mathscr{F}|_U = \bigoplus_{p=0}^r (S^p\mathscr{F}'|_U \otimes S^{r-p}\mathscr{F}''|_U)$. So we just define $F^i = \bigoplus_{p=i}^r (S^p\mathscr{F}'|_U \otimes S^{r-p}\mathscr{F}''|_U)$ to get such filtration.

(d) Similarly with (c), $\mathscr{F}|_U = \mathscr{F}'|_U \oplus \mathscr{F}''|_U$ are all free and $n = n' + n''$. Then $\Lambda^r\mathscr{F}|_U = \bigoplus_{p=0}^r (\Lambda^p\mathscr{F}'|_U \otimes \Lambda^{r-p}\mathscr{F}''|_U)$. But if $r > n$, we have $\Lambda^r\mathscr{F}|_U = 0$ and same for $\mathscr{F}'$ and $\mathscr{F}''$, so $\Lambda^n\mathscr{F}|_U = \Lambda^{n'}\mathscr{F}'|_U \otimes \Lambda^{n''}\mathscr{F}''|_U$ because the rest terms are all zero.

(e) For T^r, clearly f^{-1} can commute with T^r, then we do this by induction. If $r = 1$, trivial. If $f^*(T^{r-1}(\mathscr{F})) = T^{r-1}(f^*(\mathscr{F}))$, we have $T^r(f^*(\mathscr{F})) = f^*(\mathscr{F})\otimes_{\mathscr{O}_X} T^{r-1}(f^*(\mathscr{F})) = f^{-1}(\mathscr{F})\otimes_{f^{-1}\mathscr{O}_Y}\mathscr{O}_X\otimes_{\mathscr{O}_X} f^*(T^{r-1}(\mathscr{F})) = f^{-1}(\mathscr{F})\otimes_{f^{-1}\mathscr{O}_Y} f^{-1}(T^{r-1}(\mathscr{F}))\otimes_{f^{-1}\mathscr{O}_Y}\mathscr{O}_X = f^{-1}(T^n(\mathscr{F}))\otimes_{f^{-1}\mathscr{O}_Y}\mathscr{O}_X = f^*(T^r(\mathscr{F}))$. For S^r or Λ^r, we may denote $\mathscr{I}$ as the ideal of $x\otimes y - y\otimes x$ or $x\otimes x$ as in definition, we have $S^r(f^*(\mathscr{F})) = T^r(f^*(\mathscr{F}))/f^*\mathscr{I} = f^*(T^r(\mathscr{F}))/f^*\mathscr{I} = f^*(T^r(\mathscr{F})/\mathscr{I}) = f^*(S^r(\mathscr{F}))$, and the same for Λ^r.

Solution 2.5.17 (Affine Morphisms)**.** (a) If $f : X \to Y$ is affine, and $Y = \bigcup V_i$ is an affine open covering for some $V_i = \text{Spec } B_i$, we have $U_i = f^{-1}(V_i) = \text{Spec } A_i$ are all affine. Then for any affine subset $V \subset Y$, we can take an affine covering $V \cap V_i = \bigcup D(f_{ij})$ for some $f_{ij} \in B_i$, and $f^{-1}(D(f_{ij})) = \text{Spec } (A_i)_{f_{ij}}$ are all affine. So we now may assume that $V = \text{Spec } B$ can be covered by some affine $V_i = \text{Spec } B_i$ and $f^{-1}(V_i) = U_i = \text{Spec } A_i$ is affine. Moreover, we can cover V_i by some $D(f_{ij})$ for some $f_{ij} \in B$, and denote f'_{ij} as the image of restriction map $\mathscr{O}_Y(V) \to \mathscr{O}_Y(V_i)$ of f_{ij}. Then clearly $D(f_{ij}) = D(f'_{ij})$, and $f^{-1}(D(f'_{ij}) = f^{-1}\text{Spec } (B_i)_{f'_{ij}} = \text{Spec } (A_i)_{f'_{ij}}$. So we now can assume that $V = \text{Spec } B$ can be covered by some principal affine subset $V_i = D(f_i) = \text{Spec } B_{f_i}$ for some $f_i \in B$, and $f^{-1}(V_i) = \text{Spec } A_{f_i}$. Finally, since $V = \text{Spec } B$, we can assume this covering is finite. Then denote $U = f^{-1}(V)$, and g_i as the image of f_i under $B \to \mathscr{O}_X(U)$, so $\{g_i\}$ generates $\mathscr{O}_X(U)$. Furthermore, we know that all $D(g_i)$ cover U, and $U_{g_i} = f^{-1}(D(f_i))$ are all affine. Then by 2.2.17., we know that U is affine.

(b) If $f : X \to Y$ is affine, we may take an affine open covering $Y = \bigcup V_i$ and $f^{-1}(V_i) = U_i$ are all affine. Then since affine schemes are quasi-compact, i.e. V_i and U_i are all quasi-compact, so f is quasi-compact. Moreover, since $f : U_i \to V_i$ are separated by theorem 4.1., and the diagonal morphism $X \to X \times_Y X$ factors through $X \to \bigcup U_i \to \bigcup U_i \times_{V_i} U_i \to X \times_Y X$, we know that $X \to X \times_Y X$ is a closed immersion, i.e. f is separated. And finally, if f is finite, then by definition it must be affine.

(c) If $Y = \text{Spec } B$ is affine, we just define $X = \text{Spec } A$ for $A = \mathscr{A}(Y)$. Then for principal affine subset $D(g)$ for $g \in B$, we have $f^{-1}(D(g)) = \text{Spec } A_g = \text{Spec } (\mathscr{A}(D(g)))$. And for general open subset $V \subset Y$, V is a direct limit of principal open subset, and similar about $f^{-1}(V)$ and $\mathscr{A}(V)$. So $X = \textbf{Spec } \mathscr{A}$ as we want. And the uniqueness of X is trivial by the property we want.

For general Y, we can define $X_V = \text{Spec } \mathscr{A}(V)$ for any affine $V = \text{Spec } B \subset Y$, and $\varphi_V : X_V \to V$ is the corresponding morphism. Then we just need to glue all X_V together. For $V = \text{Spec } B$ and $V' = \text{Spec } B'$ affine in Y, by we have $V \cap V' = \bigcup W_i$ for some $W_i = D(f_i) = D(f'_i)$ principal in both two affine piece for some $f_i \in B$ and $f'_i \in B'$. Then by axiom of sheaves, $\mathscr{A}(V) \to \mathscr{A}(W_i)$ is a morphism of rings, then we have a morphism of affine schemes $X_{W_i} \to X_V$. And since W_i is principal in V, $\mathscr{A}(W_i)$ is a localization of $\mathscr{A}(V)$. Therefore, $\varphi_V^{-1}(W_i) \cong X_{W_i} \cong \varphi_{V'}^{-1}(W_i)$, i.e. X_V and $X_{V'}$ are compatible on the part of $V \cap V'$. So we can glue X_V all together to get a X. Finally, the uniqueness of X is from the uniqueness of glueing lemma and the uniqueness of the affine case.

(d) On any affine piece $V \subset Y$, we have $f_*\mathscr{O}_X(V) = \mathscr{O}_X(f^{-1}(V)) = \mathscr{A}(V)$. Since all affine subsets form a basis of open subsets, we have $f_*\mathscr{O}_X \cong \mathscr{A}$. Conversely, for any affine piece $U \subset V \subset Y$, we have $\mathscr{O}_X(f^{-1}(V)) = f_*\mathscr{O}_X(V) = \mathscr{A}(V)$, and the restriction maps are clearly compatible. Moreover, $\mathscr{A}(V) = A := \mathscr{O}_X(f^{-1}(V))$ is a B-algebra for $B = \mathscr{O}_Y(V)$, so clearly a B-module, i.e. $\mathscr{A}$ is quasi-coherent. Then by existence and uniqueness of $\textbf{Spec } \mathscr{A}$, we have $X \cong \textbf{Spec } \mathscr{A}$.

(e) For any quasi-coherent $\mathscr{A}$-module $\mathscr{M}$, we can define a quasi-coherent $\mathscr{O}_X$-module $\tilde{\mathscr{M}}$ like $\tilde{\mathscr{M}}(f^{-1}(V)) = \mathscr{A}(V)$ for all affine piece $V \subset Y$. Furthermore, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we clearly know that $\widetilde{f_*\mathscr{F}}$ is the same with $\mathscr{F}$ on every affine piece, so $\widetilde{f_*\mathscr{F}} \cong \mathscr{F}$. Conversely, for any quasi-coherent $\mathscr{A}$-module

$\mathscr{M}$, $f_*\tilde{\mathscr{M}}$ is the same with $\mathscr{M}$ at every affine piece too, so $f_*\tilde{\mathscr{M}} \cong \mathscr{M}$. Hence f_* and $\tilde{}$ are inverse to each other, i.e. those two categories are equivalent.

Solution 2.5.18 (Vector Bundles)**.** (a) Just need to confirm the transition function. For any affine $U_i = \operatorname{Spec} A_i$ where $\mathscr{E}$ is free, by definition we have $f^{-1}(U_i) = \operatorname{Spec} A_i[x_1, \ldots, x_n]$. For any affine $U_i = \operatorname{Spec} A_i$, $U_j = \operatorname{Spec} A_j$ in Y where $\mathscr{E}$ is free, and some affine $V = \operatorname{Spec} B \subset U_i \cap U_j$, we have two canonical morphisms $\rho_i : B \to A_i$ and $\rho_j : B \to A_j$ from restriction maps. So the transition function is just $\psi = \psi_j \circ \psi_i^{-1} = \operatorname{Spec}(\rho_j^{-1}\rho_j\rho_i^{-1}\rho_i) = \mathrm{id}$, hence linear. So $(X, f, \{U\}, \{\psi\})$ is a vector bundle of rank n over Y.

(b) This problem is local, so we may assume $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} \mathbb{A}^n_A$ and define a $\mathscr{O}_Y$-module structure of $\mathscr{S}(X/Y)$. So now, a section $s : Y \to X$ induces a A-algebra homomorphism $\theta : A[x_1, \ldots, x_n] \to A$, which is depended on a n-tuple $(e_1, \ldots, e_n)$ for $\theta(x_i) = e_i$. Hence $\mathscr{S}(X/Y) \cong A^n$, it clearly has an A-module structure. So $\mathscr{S}(X/Y)$ is a locally free $\mathscr{O}_Y$-module for rank n.

(c) For any open set V where $\mathscr{E}$ is free on. For every $s \in \Gamma(V, \check{\mathscr{E}}) = \operatorname{Hom}(\mathscr{E}|_V, \mathscr{O}_Y|_V)$, it induces a morphism $s_{\text{sym}} : S(\mathscr{E}|_V) \to \mathscr{O}_Y|_V$. So s_{sym} induces a morphism **Spec** s : **Spec** $\mathscr{O}_Y|_V \to$ **Spec** $S(\mathscr{E}|_V)$. Since $V \cong \mathscr{O}_Y|_V$, the morphism **Spec** $s : V \to$ **Spec** $S(\mathscr{E}|_V) \hookrightarrow X$ is a section of $\mathscr{S}(X/Y)$. Conversely, if we have a section $\sigma : V \to X$ with image in **Spec** $S(\mathscr{E}|_V)$, it induces a morphism of global section $S(\mathscr{E}|_V) \to \mathscr{O}_Y|_V$, which is exactly a morphism $s|_\sigma : \mathscr{E}|_V \to S(\mathscr{E}|_V) \to \mathscr{O}_Y|_V$. So we have $\mathscr{S}(X/Y) \cong \check{\mathscr{E}}$.

(d) For every locally free sheaf $\mathscr{E}$ of rank n, by (a), we have a geometric vector bundle **Spec** $S(\mathscr{E})$. Conversely, if we have a vector bundle $f : X \to Y$, by (b), we have a locally free sheaf of sections $(\mathscr{S}(X/Y))\check{}$ of rank n. So by (c) and 2.5.1.(a), we know this corresponding is one-to-one.

2.6 Divisors

Solution 2.6.1. Clearly $X \times \mathbb{P}^n$ is integral and separated. And since we can cover $X \times \mathbb{P}^n$ by n pieces of $X \times \mathbb{A}^n$, hence regular. Moreover, we have a exact sequence

$$\mathbb{Z} \xrightarrow{i} \operatorname{Cl} X \times \mathbb{P}^n \xrightarrow{j} \operatorname{Cl} X \times \mathbb{A}^n \to 0$$

where $i(1) = Z$, where $Z = \pi_2^{-1}([0, \ldots, 0, 1])$, for $\pi_2 : X \times \mathbb{P}^n \to \mathbb{P}^n$. Firstly, if $n \in \ker i$ for some $n > 0$, i.e. $nZ \sim 0$. Since the function field of $X \times \mathbb{P}^n$ is $K(t_1, \ldots, t_n)$, where K is the function field of X, then we have a $f = f(t_1, \ldots, t_n) \in K(t_1, \ldots, t_n)$ such that $v_Z(f) = n$ and $v_Y(f) = 0$ for any other prime divisor Y. We may assume $f = t_1^{r_1} \ldots t_n^{r_n} \cdot \frac{g}{h}$ for some $g, h \in K[t_1, \ldots, t_n]$ prime to each other and have no factors of $t_1, \ldots, t_n$. If there exists an i satisfies $n_i \neq 0$ and g, h have degree 0 about t_i, then clearly $v_Y(f) = -n_i$ for $Y = \pi_2^{-1}([0, \ldots, 0, 1, 0, \ldots, 0])$ for 1 in the i-th tuple. So g or h has non-zero degree, hence has a non-zero irreducible factor in $K[t_1, \ldots, t_n]$ corresponding to a $Y = \pi_2^{-1}(t)$ for some $t \in \mathbb{P}^n$, so $v_Y(f) \neq 0$, which makes a contradiction. So $\ker i = 0$.

Secondly, we may consider the morphism $\operatorname{Cl} X \to \operatorname{Cl} X \times \mathbb{P}^1$, $Y \mapsto \pi_1^{-1}(Y)$, where Y is a prime divisor of X, and $\pi_1 : X \times \mathbb{P}^n \to X$. So the component $\operatorname{Cl} X \to \operatorname{Cl} X \times \mathbb{P}^n \to \operatorname{Cl} X \times \mathbb{A}^n \cong \operatorname{Cl} X$, we have $Y \mapsto Y \times \mathbb{P}^n \mapsto Y \times \mathbb{A}^n \mapsto Y$, hence the exact sequence above is split, so $\operatorname{Cl} X \times \mathbb{P}^n \cong (\operatorname{Cl} X) \times \mathbb{Z}$.

Solution 2.6.2 (Varieties in Projective Space)**.** (a) Since $V \cap X$ is noetherian, hence has finitely many irreducible components, i.e. $\sum n_i Y_i$ is a finite sum. And by 1.1.8., we know that Y_i all have codimension 1 in X, so $\sum n_i Y_i$ is a Weil divisor of X. So we have a morphism $\operatorname{Div} \mathbb{P}^n \to \operatorname{Div} X$ as linearly extension of $V \mapsto \sum n_i Y_i$ since $\operatorname{Div} \mathbb{P}^n$ is generated freely.

(b) We may assume D corresponds to $f \in k(\mathbb{P}^n)$. Since we have the immersion $X \hookrightarrow \mathbb{P}^n$, it induces an endomorphism $i : k(\mathbb{P}^n) \to k(X)$. Denoting $\bar{f}$ as the image of f by above morphism, we have $D.X = (\bar{f})$. Hence we have a morphism $\operatorname{Cl} \mathbb{P}^n \to \operatorname{Cl} X$.

(c) Recall the definition $i(X, V; Y) = \mu_{\mathfrak{p}}(S/(I_V + I_X))$, where $\mathfrak{p}$ is the prime ideal corresponding to Y. If we take a prime decomposition of $I_V + I_X = \mathfrak{p}_1^{r_1} \dots \mathfrak{p}_n^{r_n}$, we have a filtration of $S/(I_V + I_X)$ as $0 = S/S \subset S/\mathfrak{p}_1 \subset \dots \subset S/\mathfrak{p}_1^{r_1} \dots \mathfrak{p}_n^{r_n - 1} \subset S/\mathfrak{p}_1^{r_1} \dots \mathfrak{p}_n^{r_n} = S/(I_V + I_X)$. So by the uniqueness of proposition 7.4. in chapter I, we know that $\mu_{\mathfrak{p}_i}(S/(I_V + I_X)) = n_i$ for $\mathfrak{p} = \mathfrak{p}_i$, and $\mu_{\mathfrak{p}}(S/(I_V + I_X)) = 0$ for any other $\mathfrak{p}$. For any U_i with $Y \not\subset U_i$, we know that $v_Y(\bar{f}) = v_{\mathfrak{p}}(I_V + I_X)$, since $v_{\mathfrak{p}}$ is the extension of v_Y to the function field of $\mathbb{P}^n$, hence equals to $i(V, X; Y)$. Furthermore, by theorem 7.7. in chapter I, we have $\sum i(V, X; Y) \deg Y = \deg V \cdot \deg X$, we have $\deg(V.X) = \deg V \cdot \deg X$. And we may extend this equality linearly and have $\deg(D.X) = \deg D \cdot \deg X$.

(d) Since D is principal, there exists some $g \in k(X)$ such that $D = (g)$. Since we have an endomorphism $i : k(\mathbb{P}^n) \to k(X)$, we may pick a $f \in i^{-1}(g)$. Then by (b), we have $(f).X = (g) = D$. Hence $\deg D = 0$. And the rest is obvious.

Solution 2.6.3 (Cones)**.** (a) Define $U_i = V \cap D_+(x_i)$ for $i = 0, \dots, n$. Then $U_i \subset \mathbb{A}^n$ is an affine variety. We may assume U_i is generated by $I = (f_1, \dots, f_d)$ for some homogeneous $f = f(x_0, \dots, 1, \dots, x_n) \in k[x_0, \dots, \hat{x}_i, \dots, x_n]$. Then $C(U_i)$ is generated by $I = (\bar{f}_1, \dots, \bar{f}_d)$ for $\bar{f} = \bar{f}(x_0, \dots, x_n) = x_i^{\deg f} \cdot f(\frac{x_0}{x_i}, \dots, 1, \dots, \frac{x_n}{x_i})$. So $\overline{C(U_i)}$ is generated by $I = (F_1, \dots, F_d)$ for $F = F(x_0, \dots, x_n, x_{n+1}) = \bar{f}(x_0, \dots, x_n)$, which just ignore the last coordinate. So clearly $\pi^{-1}(U_i) = \overline{C(U_i)} - P = U_i \times \mathbb{A}^1$. Moreover, since the morphism π^* is given by $\sum n_i Y_i \mapsto \sum n_i \pi^{-1}(Y_i)$. Firstly, if $D \in \text{Div } V$, and $\pi^*(D) = (f)$ for some $f \in k(\bar{X} - P) = k(\bar{X}) = k(V)(t)$. Clearly $\pi^*(D)$ only involve prime ideals in the form $\pi^{-1}(Y)$ for some prime divisor Y of X, we know $f \in k(V)$, i.e. $\pi^*(D)$ is principal, hence π^* is injective. Secondly, we just need to prove that any prime divisor Z in $\bar{X} - P$ in the form $\pi(Z) = V$ is linearly equivalent to some divisor $\sum n_i \pi^{-1}(Y_i)$. Since Z corresponds to a prime ideal $\mathfrak{p} \in k(\bar{X} - P) = k(V)[t]$. Since $\mathfrak{p}$ is principal, we may find a generator f, then $(f) - Z$ has the form $\sum n_i \pi^{-1}(Y_i)$ for some prime divisors Y_i in X, hence π^* is surjective. So we have $\text{Cl }(V) \cong \text{Cl }(\bar{X} - P) \cong \text{Cl }(\bar{X})$.

(b) Define $H_\infty = (x_{n+1} = 0)$, then clearly $V = \bar{X} \cap H_\infty$. So if $g = a_{n+1} x_{n+1} + \sum_{i=0}^n a_i x_i$, and $H = (g)$, we have $V \not\subset H$. So if $V = (f_1, \dots, f_d)$, we have $V.H = (f_1, \dots, f_d, g)$, and $\pi^{-1}(V.H) = (f_1, \dots, f_d, g')$, where $g' = \sum_{i=0}^n a_i x_i$. So if we set $f = \frac{g}{x_{n+1}} \in k(\mathbb{P}^{n+1})$, and $\bar{f}$ is the image of f under the morphism $k(\mathbb{P}^{n+1}) \to k(\bar{X})$, hence $V + (\bar{f}) = \pi^{-1}(V.H)$. Furthermore, by proposition 6.5., we have $\mathbb{Z} \to \text{Cl } \bar{X} \to \text{Cl } X \to 0$, and the first arrow is just $1 \mapsto 1 \cdot V$. Since $\deg V \neq 0$, the first morphism is injective, hence we have $0 \to \mathbb{Z} \to \text{Cl } \bar{X} \to \text{Cl } X \to 0$. Then by (a), we have $0 \to \mathbb{Z} \to \text{Cl } \bar{X} \to \text{Cl } X \to 0$, and the first morphism is just $1 \mapsto V.H$ and the H is determined by the isomorphism $\text{Cl } V \cong \text{Cl } \bar{X}$ as above.

(c) ($\Rightarrow$): (1) Clearly the unique factorization domain is integrally closed, hence V is projectively normal. (2) By proposition 6.2., we have $\text{Cl } X = 0$ since $S(V)$ is a unique factorization domain, hence by (b), we have $\mathbb{Z} \cong \text{Cl } V$.

($\Leftarrow$): Similarly with 1.3.18., since $S(V)$ is integrally closed, for every prime $\mathfrak{p}$ (may not homogeneous), $S(V)_{\mathfrak{p}}$ is integrally closed, hence $X = \text{Spec } S(V)$ is normal. So by proposition 6.2., $S(V)$ is a unique factorization domain.

(d) Consider the maximal ideal $(x_0, \dots, x_n) \subset k[x_0, \dots, x_n]$, it induces a maximal ideal $\mathfrak{m} = (x_0, \dots, x_n) S(V) \subset S(V)$, so $\mathscr{O}_P = S(V)_{\mathfrak{m}}$. Since $X = \text{Spec } S(V)$, and $\text{Spec } \mathscr{O}_P = \text{Spec } S(V)_{\mathfrak{m}}$, we have a morphism $\beta : \text{Spec } \mathscr{O}_P \to X$ induced by the morphism $\alpha : S(V) \to S(V)_{\mathfrak{m}}$. Then β induces a $\text{Div } X \to \text{Div Spec } \mathscr{O}_P$ as $\sum n_i Y_i \mapsto \sum n_i \beta^{-1}(Y_i)$, where $\beta^{-1}(Y_i) = Z(\alpha(I(Y_i)))$. So we have a morphism $\beta^* : \text{Cl } X \to \text{Cl Spec } \mathscr{O}_P$, then we need to prove this is isomorphic. Firstly, if $\beta^{-1}(Y_i) = Z(\alpha(I(Y))) = 0$ for some prime divisor $Y \subset X$, i.e. Y does not contain 0, since Y is corresponding to a prime ideal $\mathfrak{p} \subset S(V)$ for height 1. Since $\mathfrak{p} \not\subset \mathfrak{m}$, $\mathfrak{p} = (f)$ is principal for some $f \in S(V)$, hence $Y = (f)$, i.e. $Y \sim 0$. So if $\beta^*(\sum n_i Y_i) = (g)$ for some $g \in S(V)_{\mathfrak{m}}$, we may assume that all Y_i containing the vertex. So $\sum n_i Y_i = (\beta(g))$, i.e. $\sum n_i Y_i \sim 0$, hence β^* is injective. Secondly, Since every prime ideal $\mathfrak{p} \subset S(V)_{\mathfrak{m}}$ corresponds to a prime ideal $\mathfrak{p}' \subset S(V)$ contained in $\mathfrak{m}$, so β^* is clearly surjective. So $\text{Cl } X \cong \text{Cl Spec } \mathscr{O}_{\mathfrak{p}}$.

Solution 2.6.4. Denote $K = \text{Frac}A$, $B = k[x_1, \dots, x_n]$, and $L = \text{Frac}B$. For any element $\frac{g+zh}{g'+zh'} \in K$ for some $g, g', h, h' \in B$, then $\frac{g+zh}{g'+zh'} = \frac{g+zh}{g'+zh'} \cdot \frac{g'-zh'}{g'-zh'} = \frac{(gg'-fhh')+(g'h-gh')z}{g'^2-fh'^2}$. So $K = L[z]/(z^2 - f)$, i.e. K/L is an extension of degree 2, hence Galois. So for any $\alpha = g + zh \in K$ for some $g, h \in L$, its minimal polynomial is just

$X^2 - 2gX + (g^2 - fh^2)$. So α is integral over B iff $2g, g^2 - fh^2 \in B$, i.e. $g, h \in B$ since f is square-free. So A is the integral closure of B in K, hence integrally closed.

Solution 2.6.5 (Quadric Hypersurfaces). (a) By 1.5.12., when $r \geq 2$, $f = x_0^2 + \ldots + x_r^2$ is irreducible, hence square-free. So by 2.6.4., $A = k[x_0, \ldots, x_n]/(f)$ is integrally closed, so $X = \operatorname{Spec} A$ is normal.

(b) Just take a linear change $x_0 \mapsto \frac{x_0+x_1}{2}$ and $x_1 \mapsto \frac{x_0-x_1}{2\sqrt{-1}}$, then f will be changed to $f = x_2^2 + \ldots + x_n^2 - x_0x_1$. (1) The case $r = 2$ is just example 6.5.2., so $\operatorname{Cl} X \cong \mathbb{Z}/2\mathbb{Z}$. (2) If $r = 3$, similarly with example 6.5.2., we may denote $\bar{x}_1 \in A$ as the image of x_1 from $k[x_0, \ldots, x_n]$. Then we may define $Z = V(x_1)$ closed in X, hence we have an exact sequence $\mathbb{Z} \to \operatorname{Cl} X \to \operatorname{Cl}(X - Z) \to 0$. Since $X - Z = \operatorname{Spec} A_{\bar{x}_1}$ for $A_{\bar{x}_1} \cong k[x_2, \ldots, x_n](x_1)$, which is a unique factorization domain. Hence $\operatorname{Cl}(X - Z) = 0$ by proposition 6.2., so we have a surjection $\mathbb{Z} \to \operatorname{Cl} X$. So $\operatorname{Cl} X$ only has three choices: $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$ for some n, and 0. Moreover, if we also take a linear change $x_2 \mapsto \frac{x_2+x_3}{2}$ and $x_3 \mapsto \frac{x_2-x_3}{2\sqrt{-1}}$, then $f = x_0x_1 + x_2x_3$. So X is the cone of $V = \operatorname{Proj} A$. By example 6.6.1., $\operatorname{Cl} V = \mathbb{Z} \oplus \mathbb{Z}$. And by 2.6.3.(b), we have an exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \operatorname{Cl} X \to 0$, so $\operatorname{Cl} X = \mathbb{Z}$. (3) Same with (2), we have a surjection $\mathbb{Z} \to \operatorname{Cl} X$ as $1 \mapsto 1 \cdot Z$. So we just need to prove that the corresponding ideal $(\bar{x}_1)$ of Z is prime. Since $A/(\bar{x}_1) \cong k[x_0, x_2, \ldots, x_n]/(g)$ where $g = x_2^2 + \ldots + x_n^2$, and we already have g is irreducible in $k[x_0, x_2, \ldots, x_n]$ since $r \geq 4$ by 1.5.12. So $A/(\bar{x}_1)$ is a domain, hence $(\bar{x}_1)$ is a prime ideal, i.e. $\operatorname{Cl} X = 0$.

(c) (1) By 2.6.3.(b), we have an exact sequence $0 \to \mathbb{Z} \to \operatorname{Cl} Q \to \mathbb{Z}/2\mathbb{Z}$, where the first arrow is $1 \mapsto Q.H$. Tensoring with $\mathbb{Q}$, we have $\mathbb{Q} \to \operatorname{Cl} Q \otimes \mathbb{Q} \to 0$, we know that $\operatorname{Cl} Q \cong \mathbb{Z}$ or $\mathbb{Z} \oplus T$ with some torsion T. Tensoring with $\mathbb{Z}/p\mathbb{Z}$ for some prime $p \neq 2$, we have $\mathbb{Z}/p\mathbb{Z} \to \operatorname{Cl} Q \otimes \mathbb{Z}/p\mathbb{Z} \to 0$, so $T \cong (\mathbb{Z}/2\mathbb{Z})^n$ if exists. Tensoring with $\mathbb{Z}/2\mathbb{Z}$, we have $\mathbb{Z}/2\mathbb{Z} \to \operatorname{Cl} Q \otimes \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$, so $T \cong \mathbb{Z}/2\mathbb{Z}$ if exists. In example 6.5.2., we know that $Y = (x_1 = x_2 = 0)$ or $Y' = (x_0 = x_2 = 0)$ is the generator of the group $\operatorname{Cl} X = \mathbb{Z}/2\mathbb{Z}$. Since the preimage of Y or Y' in $\operatorname{Cl} Q$ is $R = [1 : 0 : 0]$ or $R' = [0 : 1 : 0]$. Since $Q.H = R + R'$, if T exists, the preimage of generator of $\mathbb{Z}/2\mathbb{Z}$ must be a torsion element, which makes a contradiction. So we know T does not exists, so we have $\operatorname{Cl} Q = \mathbb{Z}$ with generator $R \sim R'$. (2) The case $r = 3$ is just the example 6.6.1., we have $\operatorname{Cl} Q \cong \mathbb{Z} \oplus \mathbb{Z}$. (3) Since $\operatorname{Cl} X = 0$, we have $\operatorname{Cl} Q \cong \mathbb{Z}$ by 2.6.3.(b).

(d) Similarly with (a), we have $S(Q)$ is integrally closed. And moreover by (c) we have $\operatorname{Cl} Q \cong \mathbb{Z}$, we know that $S(Q)$ is a unique factorization domain by 2.6.3.(c). Since Y is a irreducible subvariety or codimension 1 in Q, it corresponds to a prime ideal $\mathfrak{p}$ in $S(Q)$ with height 1, hence principal. So $\mathfrak{p} = (\bar{f})$ for some $\bar{f} \in S(Q)$, it has a preimage $f \in k[x_0, \ldots, x_n]$. Then we just need to define $V = Z(f)$, so $V \cap Q = Y$ is a complete intersection.

Solution 2.6.6. (a) ($\Rightarrow$) If P, Q, R are collinear, i.e. there exists a line L with $L \cap X = \{P, Q, R\}$ by Bezout theorem. Since $L \sim (z = 0)$, i.e. $P + Q + R = 3P_0 = 0$.

($\Leftarrow$) If $P + Q + R = 0$, we may denote L as the line of P and Q. So $L \cap X = \{P, Q, T\}$ for some T by Bezout theorem. Since $P + Q + T = 0$, we have $R = -P - Q = T$, i.e. R is on the line L.

(b) Denote the tangent line of P as L, so L intersect X at P with multiplicity ≥ 2. Then we may assume that L intersect X as $\{P, P, T\}$ for some $T \in X$, i.e. $P + P + T = 0$. Since $P + P = 0$, we have $T = P_0$, which means L passes through P_0. Conversely, if L passes through P_0, we have $L \cap X = \{P, P, P_0\}$, i.e. $P + P = P + P + P_0 = 0$.

(c) If $P + P + P = 0$, then by (a), there exists a line L' intersect X with $3P$, then L' is the tangent line, i.e. L. So P is an inflection point. Conversely, if L intersect X with $3P$, we have $P + P + P = 0$ by (a).

(d) Clearly $P_0 \in X(\mathbb{Q})$. Moreover, if $R = P + Q$ with $P, Q \in X(\mathbb{Q})$, we have a line L intersect X with $P, Q, -R$. The intersection forms a set of functions, so by Vieda's theorem, we know that the coordinates of R are in $\mathbb{Q}$ since the coordinates of P and Q are in $\mathbb{Q}$. Hence $X(\mathbb{Q})$ is a group. Furthermore, by Mordell's theorem, $X(\mathbb{Q})$ is a finitely generated abelian group.

Solution 2.6.7. Similarly with example example 6.11.4., we have a one-to-one correspondence from closed points in $X - Z$ to Cartier divisor classes of degree 0, i.e. elements in $\operatorname{CaCl}^0(X)$. Then Since we have an isomorphism $\mathbb{G}_m \to X - Z$ as $t \mapsto (4t - 4t^2, 4t + 4t^2, (1-t)^3)$, and $\frac{y-x}{y+x} \leftarrow (x, y, z)$. So we have $\operatorname{CaCl}^0(X) \cong \mathbb{G}_m$.

Solution 2.6.8. (a) Clearly we have a map f^* : Pic $Y \mapsto$ Pic X. For the group structure, if $\mathscr{L}, \mathscr{M} \in$ Pic Y, for every point $P \in Y$, there exists an affine neighbourhood $V = \operatorname{Spec} B$ such that $\mathscr{L} = \tilde{M}$ and $\mathscr{M} = \tilde{N}$ on V. So for any $U = \operatorname{Spec} A \subset f^{-1}(V)$, we have $f^*(\mathscr{L} \otimes \mathscr{M})|_U = f^*(\widetilde{M \otimes_B N})|_U = M \otimes_B \widetilde{N \otimes_B} A \cong (M \otimes_B A) \widetilde{\otimes_A} (N \otimes_B A) = f^*(\mathscr{L}) \otimes f^*(\mathscr{M})$. Hence we have a group homomorphism f^* : Pic $Y \to$ Pic X.

(b) For any $D = (U_i, f_i) \in \Gamma(Y, \mathscr{K}^*/\mathscr{O}^*)$, we can define $f^{\mathrm{CaCl}\,*}(D) = (f^{-1}(U_i), f^*(f_i))$, hence a Cartier divisor on X. So we have a morphism $f^{\mathrm{CaCl}\,*}$: CaCl $Y \to$ CaCl X. On one hand, for any prime divisor $V \subset Y$, if $U_i \cap V \neq \varnothing$, then clearly $f^{\mathrm{Cl}\,*}(\nu_V(f_i)V) = \nu_V(f_i) f^{\mathrm{Cl}\,*}(V) = \nu_V(f_i) \cdot \sum_{f(U)=V} \nu_U(t)U$, where $t \in K(Y)$ such that $\nu_V(t) = 1$. But $\nu_V(f_i) \cdot \nu_U(t) = \nu_V(f_i)$, hence the morphism $f^{\mathrm{Cl}\,*}$ corresponds to $f^{\mathrm{CaCl}\,*}$. On the other hand, the corresponding between $f^{\mathrm{CaCl}\,*}$ and $f^{\mathrm{Pic}\,*}$ is trivial by definition. Hence $f^{\mathrm{Cl}\,*}$, $f^{\mathrm{CaCl}\,*}$ and $f^{\mathrm{Pic}\,*}$ is the same, we may denote it as f^*.

(c) For any prime Weil divisor $V \subset \mathbb{P}^n$, then clearly for any local parameter f_i of Y_i on U_i, we have $\bar{f}_i = f_i|_{U_i \cap X}$ is the $\bar{f}_i$ in 2.6.2., so we have $f^{\mathrm{Cl}\,*}(V) = \sum \nu_{Y_i}(\bar{f}_i) Y_i = V.X$. Hence the morphism $f^{\mathrm{Cl}\,*}$ corresponds to the f^* defined in 2.6.2., so by (b), these all f^* are the same.

Solution 2.6.9 (Singular Curves)**.** (a) We already have a morphism π^* : Pic $X \to$ Pic $\tilde{X}$ as in 2.6.8. Denote the morphism $\theta : \Gamma(X, \mathscr{K}^*) \to \Gamma(X, \mathscr{K}^*/\mathscr{O}_X^*)$, and $\tilde{\theta} : \Gamma(\tilde{X}, \mathscr{K}^*) \to \Gamma(\tilde{X}, \mathscr{K}^*/\mathscr{O}_{\tilde{X}}^*)$, then Pic $X \cong$ CaCl $X = \operatorname{coker}(\theta)$ and Pic $\tilde{X} \cong \operatorname{coker}(\tilde{\theta})$, $\ker(\theta) = \Gamma(X, \mathscr{O}_X^*)$ and $\ker(\tilde{\theta}) = \Gamma(X, \mathscr{O}_{\tilde{X}}^*)$. As the hint, we consider the exact sequence $0 \to \pi_*\mathscr{O}_{\tilde{X}}^* \to \mathscr{K}^*/\mathscr{O}_X^* \to \mathscr{K}^*/\pi_*\mathscr{O}_{\tilde{X}}^* \to 0$, then since $H^1(\pi_*\mathscr{O}_{\tilde{X}}^*) = 0$, we have an exact sequence $0 \to \Gamma(X, \pi_*\mathscr{O}_{\tilde{X}}^*) \to \Gamma(X, \mathscr{K}^*/\mathscr{O}_X^*) \to \Gamma(X, \mathscr{K}^*/\pi_*\mathscr{O}_{\tilde{X}}^*) \to 0$. Then by snack lemma, we have an exact sequence $\Gamma(X, \mathscr{O}_X^*) \to \Gamma(X, \mathscr{O}_{\tilde{X}}^*) \to \Gamma(X, \pi_*\mathscr{O}_{\tilde{X}}^*/\mathscr{O}_X^*) \to$ Pic $X \to$ Pic $\tilde{X} \to 0$. So $\ker \pi^* = \operatorname{Im}(\Gamma(X, \mathscr{O}_{\tilde{X}}^*) \to \Gamma(X, \pi_*\mathscr{O}_{\tilde{X}}^*/\mathscr{O}_X^*)) = \Gamma(X, \pi_*\mathscr{O}_{\tilde{X}}^*/\mathscr{O}_X^*) = \bigoplus_{P \in X} \tilde{\mathscr{O}}_P^*/\mathscr{O}_P^*$. Hence we have the exact sequence $0 \to \bigoplus_{P \in X} \tilde{\mathscr{O}}_P^*/\mathscr{O}_P^* \to$ Pic $X \to$ Pic $\tilde{X} \to 0$.

(b) Clearly whenever X is cuspidal or nodal cubic curve, the normalization of X is just the blow up at the singular point, i.e. $\mathbb{P}^1$. Since clearly Pic $\mathbb{P}^1 = \mathbb{Z}$ by corollary 6.17., we have $0 \to \bigoplus_{P \in X} \tilde{\mathscr{O}}_P^*/\mathscr{O}_P^* \to$ Pic $X \to \mathbb{Z} \to 0$. If $P \in X$ is regular point, we have $\mathscr{O}_P^* \cong \tilde{\mathscr{O}}_P^*$, so the $\bigoplus$ is non-vanish on those singular points. If X is the plane cuspidal cubic curve $y^2 = x^3$, the unique singular point is just $Z = (0, 0)$, then $\tilde{\mathscr{O}}_Z^*/\mathscr{O}_Z^* \cong k = \mathbb{G}_{\mathrm{a}}$. If X is the nodal cubic curve $y^2 z = x^3 + x^2 z$, the unique singular point is just $Z = (0, 0, 1)$, then $\tilde{\mathscr{O}}_Z^*/\mathscr{O}_Z^* \cong k^* = \mathbb{G}_{\mathrm{m}}$.

Solution 2.6.10 (The Grothendieck Group $K(X)$)**.** (a) If $X = \mathbb{A}_k^1 = \operatorname{Spec} k[x]$, $\mathscr{F} = \tilde{M}$ for some finitely generated module M. Since $k[x]$ is a principal ideal domain, there exists an exact sequence $0 \to (k[x])^n \to (k[x])^m \to M \to 0$, i.e. $0 \to \mathscr{O}_X^n \to \mathscr{O}_X^m \to \mathscr{F} \to 0$. Hence in $K(X)$, we have $\gamma(\mathscr{F}) = (m - n)\gamma(\mathscr{O}_X)$. Conversely, we clearly have $\gamma(\mathscr{O}_X^n) = n\gamma(\mathscr{O}_X)$. So we have an isomorphism $K(X) \cong \mathbb{Z}$.

(b) For any exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, we have $0 \to \mathscr{F}'_\xi \to \mathscr{F}_\xi \to \mathscr{F}''_\xi \to 0$, hence $\dim_K \mathscr{F}_\xi = \dim_K \mathscr{F}'_{xi} + \dim_K \mathscr{F}''_\xi$, hence the morphism $K(X) \to \mathbb{Z}$ is well-defined. Moreover, for any $n \in \mathbb{Z}$, we have $\dim_K (\mathscr{O}_X^n)_\xi = n$, so surjective.

(c) For any coherent sheaf $\mathscr{F}$ on $X - Y$, then $\mathscr{F}$ can be extended to be a coherent sheaf $\mathscr{F}'$ on X by 2.5.15, so $K(X) \to K(X - Y)$ is surjective. For the exactness at the middle, for any $\mathscr{F} \in \ker(K(X) \to K(X - Y))$, on every affine piece $U = \operatorname{Spec} A \subset X$, we have $Y \cap U = \operatorname{Spec} A/I$ for some ideal $I \subset A$, $i_* i^* \mathscr{F}|_U = \widetilde{M/IM}$, where $\mathscr{F}|_U = \tilde{M}$ for some A-module M. So $\eta : \mathscr{F} \to i_* i^* \mathscr{F}$ is surjective. Then we may define $\mathscr{F}_0 = \mathscr{F}$, and $\mathscr{F}_i = \ker(\mathscr{F}_{i-1} \to i_* i^* \mathscr{F}_{i-1})$ for $i > 0$, so $\mathscr{F}_i|_U = I^i \tilde{M}$. Since M is finitely generated, $I^i M = 0$ for sufficiently large i, so the filtration $\mathscr{F}_0 \supset \mathscr{F}_1 \supset \ldots \supset \mathscr{F}_n$ is finite. And by definition, $\mathscr{F}_i/\mathscr{F}_{i+1}$ is a coherent sheaf extended from a coherent sheaf on Y, so $\gamma(\mathscr{F}) = \sum \gamma(\mathscr{F}_i/\mathscr{F}_{i+1}) \in \operatorname{Im}(K(Y) \to K(X))$, i.e. $\ker(K(X) \to K(X - Y)) \subset \operatorname{Im}(K(Y) \to K(X))$. The converse is obvious, so $K(Y) \to K(X) \to K(X - Y) \to 0$ is exact.

Solution 2.6.11 (The Grothendieck Group of a Nonsingular Curve)**.** (a) Define $\mathscr{F}_P$ as the skyscraper sheaf of $\operatorname{coker}(\mathscr{I}_{D,P} \to \mathscr{O}_{X,P}) = \mathscr{O}_{X,P}/\mathfrak{m}_P^n$ at P, where n is the coefficient of P in D. Then we have an exact sequence $0 \to \mathscr{I}_D \to \mathscr{O}_X \to \bigoplus_{P \in D} \mathscr{F}_P \to 0$ since all $P \in D$ are closed points. So $\mathscr{O}_D \cong \bigoplus \mathscr{F}_P$, i.e. $\gamma(\mathscr{O}_D) = \sum \gamma(\mathscr{F}_P)$. Since we have a natural $\mathscr{O}_{X,P}$-modules exact sequence $0 \to \mathfrak{m}^{i-1}/\mathfrak{m}^i \to \mathscr{O}_{X,P}/\mathfrak{m}^{i-1} \to \mathscr{O}_{X,P}/\mathfrak{m}^i \to 0$, and since $\mathfrak{m}^{i-1}/\mathfrak{m}^i \cong \mathfrak{m}/\mathfrak{m}^2 \cong k$, we have $\gamma(\mathscr{F}_P) = n\gamma(k(P))$. So we have $\gamma(\mathscr{O}_D) = \psi(D)$. If $D \sim D'$ are both effective divisors, we have $\psi(D) = \gamma(\mathscr{O}_D) = \gamma(\mathscr{O}_X) - \gamma(\mathscr{I}_D) = \gamma(\mathscr{O}_X) - \gamma(\mathscr{L}(-D)) = \gamma(\mathscr{O}_X) - \gamma(\mathscr{L}(-D')) = \psi(D')$. For general divisor D, we can choose two effective divisor D_+ and D_- such that $D = D_+ - D_-$, so $\psi(D) = \psi(D_+) - \psi(D_-)$ is independent of linear equivalence, hence we have a morphism $\psi : \mathrm{Cl}\, X \to K(X)$.

(b) We may embed X into $\mathbb{P}^n$, and define Y as the closure of X in $\mathbb{P}^n$. So $\mathscr{F}$ can be extended to be a coherent sheaf $\mathscr{F}'$ on Y, so there exists a free module $\mathscr{E}$ with surjection $\mathscr{E} \to \mathscr{F}'$ by corollary 5.18. Restrict on X, we have a free module $\mathscr{E}_0 = \mathscr{E}|_X$ with surjection $\mathscr{E}_0 \to \mathscr{F}$. Denote the kernal of this surjection as $\mathscr{E}_1$. Since for every closed point $P \in X$, the module $\mathscr{E}_{1,P}$ is a submodule of $\mathscr{E}_{0,P} = \mathscr{O}_{X,P}^n$. Then since $\mathscr{O}_{X,P}$ is a reduced regular local ring of dimension one, hence principal ideal domain, we have $\mathscr{E}_{0,P}$ is a free module. So by 2.5.7., $\mathscr{E}_0$ is locally free of finite rank.

If we also have another resolution $0 \to \mathscr{E}_1' \to \mathscr{E}_0' \to 0$, we may denote $\mathscr{G} = \ker(\mathscr{E}_0 \oplus \mathscr{E}_0' \to \mathscr{F}$. Then by nine lemma we have two exact sequences $0 \to \mathscr{E}_1 \to \mathscr{G} \to \mathscr{E}_0' \to 0$ and $0 \to \mathscr{E}_1' \to \mathscr{G} \to \mathscr{E}_0 \to 0$. Then clearly $\operatorname{rank} \mathscr{G} = r_0 + r_1' = r_0' + r_1$. So $\det \mathscr{F} = (\wedge^{r_0}\mathscr{E}_0) \otimes (\wedge^{r_1}\mathscr{E}_1)^{-1} = (\wedge^{r_0}\mathscr{E}_0) \otimes (\wedge^{r_1}\mathscr{E}_1)^{-1} \otimes (\wedge^{r_0'}\mathscr{E}_0')^{-1} \otimes (\wedge^{r_0'}\mathscr{E}_0') = (\wedge^{r_0}\mathscr{E}_0) \otimes (\wedge^{\operatorname{rank}\mathscr{G}}\mathscr{G})^{-1} \otimes (\wedge^{r_0}\mathscr{E}_0) = (\wedge^{r_0}\mathscr{E}_0) \otimes (\wedge^{r_0}\mathscr{E}_0)^{-1} \otimes (\wedge^{r_1'}\mathscr{E}_1')^{-1} \otimes (\wedge^{r_0'}\mathscr{E}_0') = (\wedge^{r_0'}\mathscr{E}_0') \otimes (\wedge^{r_1'}\mathscr{E}_1')^{-1}$. Clearly, if we have $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, we have $\det \mathscr{F} = \det \mathscr{F}' \otimes \det \mathscr{F}''$, so we have a morphism $\det : K(X) \to \operatorname{Pic} X$.

If D is effective, we have an exact sequence $0 \to \mathscr{I}_D \to \mathscr{O}_X \to \mathscr{O}_D \to 0$. Since $\mathscr{O}_X$ and $\mathscr{I}_D$ is locally free of rank 1, then $\det(\psi(D)) = \mathscr{O}_X \otimes \mathscr{I}_D^{-1} = \mathscr{I}_D^{-1} = \mathscr{L}(-D)^{-1} = \mathscr{L}(D)$. Then for general divisor D, we have $D = D_+ - D_-$ for two effective divisors D_+ and D_-, we have $\det(\psi(D)) = \det(\psi(D_+)) \otimes (\det(\psi(D_-)))^{-1} = \mathscr{L}(D_+) \otimes (\mathscr{L}(D_-)) = \mathscr{L}(D)$.

(c) Take an affine open covering $\{U_i\}$ of X, $U_i = \operatorname{Spec} A_i$. Then we may assume $\mathscr{F}|_{U_i} = \tilde{M}_i$ for some A_i-module M_i. Since the generic point η is contained in every U_i, and corresponds to the ideal (0) in every A_i, so $\mathscr{F}_\eta = \operatorname{Frac}(A_i) \otimes_{A_i} M_i$. Denote $\{e_1, \ldots, e_r\}$ be a set of basis of $\mathscr{F}_\eta$ as $K(X)$-module, then we may assume $e_j = \frac{m_{ij}}{a_i}$ for all A_i and e_j. So we need to construct a Cartier divisor about these a_i. Denote $Z = \bigcap V(a_i)$. If $Z = \varnothing$, we define $V_i = U_i - V(a_i)$. If not, since Z is a closed subset of X, it is just finite set of closed points since X is a curve. Then we may divide Z as $Z = \coprod Z_i$, such that $Z_i \subset A_i$. Then we may define $V_i = U_i - (V(a_i) - Z_i)$. Hence for any $i \neq j$, $V(a_i) \cap V_i \cap V_j = \varnothing$, so a_i is invertible in $\mathscr{O}_X(V_i \cap V_j)$. Then we can define a Cartier define $D' = (V_i, a_i)$. So $\mathscr{F}_\eta$ is generated by $\frac{m_{ij}}{a_i} \in \Gamma(V_i, \mathscr{L}(D') \otimes \mathscr{F})$, and this induces an injection $\mathscr{O}_X^r \to \mathscr{L}(D') \otimes \mathscr{F}$. Hence if we define $D = -D'$, then we have an injection $\mathscr{L}(D)^r \to \mathscr{F}$, so we may denote $\mathscr{T}$ as the cokernal of this injection. Then we only need to prove that $\mathscr{T}$ is a torsion sheaf. But consider $0 \to (\mathscr{L}(D)^r)_\eta \to \mathscr{F}_\eta \to \mathscr{T}_\eta \to 0$, the first two terms are both vector spaces of rank r over $K(X)$, hence $\mathscr{T}_\eta = 0$, then $\mathscr{T}$ is a torsion sheaf.

To prove $\gamma(\mathscr{F}) - r\gamma(\mathscr{O}_X) \in \operatorname{Im}\psi$, since $\gamma(\mathscr{F}) - r\gamma(\mathscr{O}_X) = r(\gamma(\mathscr{L}(D)) - \gamma(\mathscr{O}_X)) + \gamma(\mathscr{T})$, we just need to prove that $\gamma(\mathscr{L}(D)) - \gamma(\mathscr{O}_X)$ and $\gamma(\mathscr{T})$ are both in the image of ψ. Firstly, if D is effective, we clearly have $\psi(\mathscr{O}_D) = \psi(\mathscr{O}_X) - \psi(\mathscr{L}(D))$ by (a), then we've done. For general D, we may denote $D = D_+ - D_-$. Then $\psi(D) = \gamma(\mathscr{O}_{D_+}) - \gamma(\mathscr{O}_{D_-}) = \gamma(\mathscr{L}(D_-)) - \gamma(\mathscr{L}(D_+)) = \gamma(\mathscr{L}(D_-) \otimes (\mathscr{L}(D_-))^{-1}) - \gamma(\mathscr{L}(D_+) \otimes (\mathscr{L}(D_-))^{-1}) = \gamma(\mathscr{O}_X) - \gamma(\mathscr{L}(D))$. Secondly, by 2.5.6., we know $\operatorname{Supp} \mathscr{T}$ is a closed subset of X, hence a finite set of closed points. So $\mathscr{T} = \oplus \mathscr{T}^P$ for finitely some skyscraper sheaves $\mathscr{T}^P$. For any P, we may choose an affine subset $U = \operatorname{Spec} A$ containing P, then $\mathfrak{p}$ is the corresponding prime ideal of P, and $\mathscr{T}^P = \tilde{M}$ for some A-module M. Then by 2.5.6.(b), we have an N such that $\mathfrak{p}^N M = 0$. Since we have an exact sequence $0 \to \mathfrak{p}^{i+1}M \to \mathfrak{p}^i M \to \mathfrak{p}^i M/\mathfrak{p}^{i+1}M \to 0$, and $\mathfrak{p}^i M/\mathfrak{p}^{i+1}M \cong (A/\mathfrak{p})^{\oplus n_i}$ for some n_i. So $\gamma(\mathscr{T}^P) = \sum n_i \gamma(k(P)) \in \operatorname{Im}\psi$, hence $\gamma(\mathscr{T}) = \sum \gamma(\mathscr{T}^P) \in \operatorname{Im}\psi$. Thus, $\gamma(\mathscr{F}) - r\gamma(\mathscr{O}_X) \in \operatorname{Im}\psi$.

(d) Define a morphism $\phi : \mathbb{Z} \to K(X)$ as $n \mapsto n\gamma(\mathscr{O}_X)$. Since we have $\operatorname{rank} : K(X) \to \mathbb{Z}$, $\det : K(X) \to \operatorname{Pic} X$, $\psi : \operatorname{Pic} X \to K(X)$ and ϕ such that $\operatorname{rank} \circ \psi = 0$, $\det \circ \psi = \mathrm{id}_{\operatorname{Pic} X}$ and $\operatorname{rank} \circ \phi = \mathrm{id}_{\mathbb{Z}}$, we clearly have $K(X) = \operatorname{Pic} X \oplus \mathbb{Z}$.

Solution 2.6.12. By 2.6.11., we have $\det \mathscr{F} \in \operatorname{Pic} X$, then we can define $\deg \mathscr{F}$ as the degree of the Weil divisor corresponding to $\det \mathscr{F}$. So the properties (1) and (3) is immediately following from (a) and (b) in 2.6.11. For (2), by 2.6.11.(d), $\mathscr{F} = \bigoplus \mathscr{F}^P$, so $\deg \mathscr{F} = \sum \deg \mathscr{F}^P$, where $\mathscr{F}^P$ is the skyscraper sheaf of $\mathscr{F}_P$ at P. For any P, we can pick $U = \operatorname{Spec} A$ containing P, $\mathfrak{p}$ corresponds to P, and $\mathscr{F}^P = \tilde{M}$ on U, then

$\deg \mathscr{F}^P = \text{length } (M_\mathfrak{p}) = \text{length } \mathscr{F}_P$, hence $\deg \mathscr{F} = \sum \text{length } \mathscr{F}_P$. For uniqueness, if $\mathscr{F}$ has rank zero, (2) insures the uniqueness. If $\mathscr{F}$ has rank one, (1) implies it. For higher rank, the problem can be reduced to rank one case by the induction from (3).

2.7 Projective Morphisms

Solution 2.7.1. By 1.1.2., we just need to check on stalks. Since $\mathscr{L}$ and $\mathscr{M}$ are both invertible, hence rank 1, for every point $P \in X$ we can take an affine neighbourhood $U = \operatorname{Spec} A$ of P such that $\mathscr{L}|_U \cong \mathscr{O}_X|_U$ and $\mathscr{M}|_U \cong \mathscr{O}_X|_U$. So this original question is equivalent to prove that if we have a surjection of $A_\mathfrak{p}$-module as $\varphi : A_\mathfrak{p} \to A_\mathfrak{p}$, then φ is isomorphic. Since φ is a morphism of $A_\mathfrak{p}$-modules, it just depends on $\varphi(1)$, i.e. $\varphi(a) = a \cdot \varphi(1)$. Since φ is surjective, 1 has a preimage, named ι. Then $\varphi^{-1}(b) = b \cdot \iota \in A_\mathfrak{p}$, hence φ is an isomorphism.

Solution 2.7.2. Since $\{s_i\}$ and $\{t_j\}$ generate the same linear space, we may assume that $s_i = \sum a_{ij} t_j$. Then on $\mathscr{O}_{\mathbb{P}^m}(1)$, we define $u_i = \sum a_{ij} x_j$, hence $\varphi^*(u_i) = \sum a_{ij} \varphi^*(x_i) = \sum a_{ij} t_j = s_i$. So if we define $L = Z(u_0, \ldots, u_n)$, by the uniqueness in theorem 7.1., we know that $\rho \circ \varphi = \psi$ with the projection $\rho : \mathbb{P}^m - L \to \mathbb{P}^n$, $(u_0, \ldots, u_n, v_0, \ldots, v_{m-n-1}) \mapsto (u_0, \ldots, u_n)$.

Solution 2.7.3. (a) By theorem 7.1.(2), the morphism φ is equivalent to an invertible sheaf $\mathscr{L}$ on $\mathbb{P}^n$ with global sections $s_0, \ldots, s_m$ generate $\mathscr{L}$. Since every invertible sheaf on $\mathbb{P}^n$ is isomorphic to some $\mathscr{O}(d)$, then we may assume $\mathscr{L} \cong \mathscr{O}(d)$ and $s_i \in k[x_0, \ldots, x_n]_{(d)}$. In the case $m < n$, if $d \geq 1$, then $V_+(s_0) \cap \ldots \cap V_+(s_m)$ has dimension > 0, hence non-empty, which means there exists some point P such that $s_0, \ldots, s_m$ cannot generate P, hence contradict. So $d \leq 0$ in this case, then there are only trivial global section, hence $\operatorname{Im}\varphi$ is just a point. In the case $m \geq n$, if φ is surjective, then $\dim(\operatorname{Im}\varphi) = m \geq n$, but $\dim(\operatorname{Im}\varphi) \leq n$, hence $\dim(\operatorname{Im}\varphi) = n$. If not, there exists some $P \in \mathbb{P}^m$ not in the image of φ, so we have a morphism $\varphi' : \mathbb{P}^n \to \mathbb{P}^m - P \to \mathbb{P}^{m-1}$, where the second arrow is the projection $\pi : \mathbb{P}^m - P \to \mathbb{P}^{m-1}$. Then by induction, we have $\operatorname{Im}\varphi'$ is a singleton or $\dim(\operatorname{Im}\varphi') = n$. If $\operatorname{Im}\varphi' = \{Q\}$ is a singleton, then $\operatorname{Im}\varphi \subset \pi^{-1}(Q) \cong \mathbb{A}^1$. Since clearly $\operatorname{Im}\varphi$ is closed and connected, $\operatorname{Im}\varphi$ is a point in $\mathbb{A}$. Else if $\dim \operatorname{Im}\varphi' = n$, we have $\dim \operatorname{Im}\varphi \geq n$. Since $\dim \operatorname{Im}\varphi \leq n$, we have $\dim \operatorname{Im}\varphi = n$.

(b) As we discuss in (a), we know that $\mathscr{L} \cong \mathscr{O}_d$ for some $d \geq 1$. Then we may assume $s_i = x_0^{e_{i,0}} \ldots x_m^{e_{i,n}}$ for $\sum_j e_{i,j} = d$. So for d-uple embedding $\rho_d : \mathbb{P}^n \to \mathbb{P}^N$ for $N = \binom{n+d}{n} - 1$, we have $\rho_{d,*}\mathscr{L} \subset \mathscr{O}_{\mathbb{P}^N,1}$ as a subsheaf, with $s_i \mapsto u_i = u_{e_{i,0},\ldots,e_{i,n}}$. Then defining $L = V_+(u_0, \ldots, u_m)$, we have a projection $\mathbb{P}^N - L \to \mathbb{P}^m$ with $u_i \mapsto y_i$, hence φ factors through this projection.

Solution 2.7.4. (a) If X has an invertible ample sheaf $\mathscr{L}$, which means has an invertible very ample sheaf $\mathscr{L}^n$, hence induces a closed immersion $i : \mathscr{X} \to \mathbb{P}^m_A$ for some m and $\mathscr{L}^n \cong i^*\mathscr{O}(1)$. Then i is proper, hence separated. Since $\mathbb{P}^m_A \to \operatorname{Spec} A$ is separated, $X \to \operatorname{Spec} A$ is also separated.

(b) We may denote two copies of affine line as $U_0 \cong U_1 \cong \operatorname{Spec} k[x]$, and $U_0 \cap U_1 = V \cong \operatorname{Spec} k[x, x^{-1}]$. Then we already have that $\operatorname{Cl} U_0 = \operatorname{Cl} U_1 = \operatorname{Cl} V = 0$. So for any invertible sheaf $\mathscr{L}$ on X, we have $\mathscr{L}_0 = \mathscr{L}|_{U_0} \cong \mathscr{O}_{U_0}$, and $\mathscr{L}_1 = \mathscr{L}|_{U_1} \cong \mathscr{O}_{U_1}$, and $\mathscr{M} = \mathscr{L}|_V = \mathscr{O}_V$, with $\mathscr{L}_0|_V \cong \mathscr{M} \cong \mathscr{L}_1$. So the morphism $\mathscr{L}_0|_V \to \mathscr{L}_1|_V$ is determined by an automorphism of $k[x, x^{-1}]$. Since $\operatorname{Aut}(k[x, x^{-1}]) = \{ax^n \mid a \in k,\ n \in \mathbb{Z}\}$, every $\mathscr{L}$ actually corresponds to a Cartier divisor $D_{ax^n} = \{(U_0, 1), (U_1, ax^n)\}$. If $D_{ax^n} \sim D_{bx^m}$, we clearly have $n = m$, and vice versa. So $\operatorname{Pic} X \cong \mathbb{Z}$.

For Cartier divisor $D_n = \{(U_0, 1), (U_1, x^n)\}$ with $n > 0$, and $\mathscr{L}_n$ is the corresponding invertible sheaf. Then $s \in \Gamma(X, \mathscr{L}_n)$ means, $s_0 = s|_{U_0} \in k[x]$ and $s_1 = s|_{U_1} \in x^{-n}k[x]$ agree on V. Since $x^{-n}k[x]$ have a homogeneous component of non-negative degree, we know that the origin of U_1 cannot be generated by global sections. For $n < 0$, $D_n = \{(U_0, 1), (U_1, x^n)\} \sim \{(U_0, x^{-n}), (U_1, 1)\}$, similarly we know that U_0 cannot be generated by global sections. Clearly we have $\mathscr{L}_n \otimes \mathscr{L}_m \cong \mathscr{L}_{n+m}$, i.e. $\mathscr{L}_n^m = \mathscr{L}_{nm}$, so $\mathscr{L}_n$ is not ample since $\mathscr{L}_{nm}$ is not very ample for any m. For $n = 0$ case, we have $\mathscr{L}_0 \cong \mathscr{O}_X$, hence not ample.

Solution 2.7.5. (a) Clearly $\mathscr{M}^n$ is generated by global sections. Then for any coherent sheaf $\mathscr{F}$, $\mathscr{F}\otimes\mathscr{L}^n$ is generated by global sections for large n, hence $\mathscr{F}\otimes(\mathscr{L}\otimes\mathscr{M})^n = (\mathscr{F}\otimes\mathscr{L}^n)\otimes(\mathscr{M}^n)$ is generated by global sections, hence $\mathscr{L}\otimes\mathscr{M}$ is ample.

(b) Since $\mathscr{M}$ is at least coherent, $\mathscr{M}\otimes\mathscr{L}^n$ is generated by global sections for sufficient large n. For general coherent sheaf $\mathscr{F}$, $\mathscr{F}\otimes\mathscr{L}^m$ is generated by global sections for some n, hence $\mathscr{F}\otimes(\mathscr{M}\otimes\mathscr{L}^{n+1})^m = (\mathscr{F}\otimes\mathscr{L}^m)\otimes(\mathscr{M}\otimes\mathscr{L}^n)^m$ is generated by global sections, i.e. $\mathscr{M}\otimes\mathscr{L}^p$ is ample for $p \geq n+1$.

(c) For any coherent sheaf $\mathscr{F}$, $\mathscr{F}\otimes\mathscr{L}^n$ is generated by global sections for $n > n_1$, and $\mathscr{O}_X\otimes\mathscr{L}^m$ is generated by global sections for $n > n_2$. So for $n > \max\{n_1, n_2\}$, $\mathscr{F}\otimes(\mathscr{L}\otimes\mathscr{M})^n = (\mathscr{F}\otimes\mathscr{L}^n)\otimes(\mathscr{O}_X\otimes\mathscr{M}^n)$ is generated by global sections, hence $\mathscr{L}\otimes\mathscr{M}$ is ample.

(d) Since we have a closed immersion $\iota : X \to \mathbb{P}^n$ and a morphism $\varphi : X \to \mathbb{P}^m$ such that $\mathscr{L} = \iota^*\mathscr{O}_{\mathbb{P}^n}$ and $\mathscr{M} = \varphi^*\mathscr{O}_{\mathbb{P}^m}$. Since we have a closed immersion $f : X \to X\times\mathbb{P}^m \to \mathbb{P}^n\times\mathbb{P}^m \to \mathbb{P}^N$, where the last arrow is the Segre embedding, and the second arrow is a closed immersion because it is the base change of ι, we know $\mathscr{L}\otimes\mathscr{M} = f^*\mathscr{O}_{\mathbb{P}^N}(1)$ is very ample.

(e) Since $\mathscr{L}^m$ is very ample for some m, and $\mathscr{L}^d$ is generated by global sections for $d > d_0$, then we just take $n_0 = m + d_0$ and by (d) we know that $\mathscr{L}^n$ is very ample for $n > n_0$.

Solution 2.7.6 (The Riemann-Roch Theorem). (a) Since $\iota : X \to \mathbb{P}^n$ is a closed embedding, we may take $S = k[x_0, \dots, x_n]/I(X)$ and have $X = \operatorname{Proj} S$. Since $\mathscr{L}$ is very ample w.r.t. ι, we have $\mathscr{L} = \iota^*\mathscr{O}_{\mathbb{P}^n} = \widetilde{S(1)}$. By 2.5.9. we have $\Gamma(X, \mathscr{L}^n) = \Gamma(X, \widetilde{S(n)}) \cong S(n)$ for sufficiently large n. So $\dim|nD| = \dim\Gamma(X,\mathscr{L}^n) - 1 = \dim S_n - 1 = P_X(n) - 1$.

(b) If $r|n$, we know that $nD = 0$, then $|nD|$ is trivial, hence $\dim|nD| = 0$. In the case $r \nmid n$, if $nD \sim E$ for some effective divisor E, then $0 = \deg r(nD) = \deg rE > 0$, which makes a contradiction. So $\dim|nD| = -1$.

Solution 2.7.7 (Some Rational Surfaces). (a) By definition, the embedding $\iota : \mathbb{P}^2 \to \mathbb{P}^5$ induced by $|D|$ satisfies $\iota^*(t_{ijk}) = x^iy^jz^k$ for $i+j+k = 2$. Hence ι induces a morphism of graded rings: $k[t_{200}, \dots, t_{110}] \to k[x, y, z]$ as $t_{ijk} \mapsto x^iy^jz^k$, which is just the definition of 2-uple embedding, hence $\operatorname{Im}\iota$ is the Veronese surface.

(b) For any two points $P = (a, b, c)$ and $P' = (a', b', c')$ in $\mathbb{P}^2$, if $a = a' = 0$, since y^2, z^2 and $xy - yz$ can generate $\mathscr{O}(2)$ on $\mathbb{P}^1 = (a = 0)$, hence V separates P and P'. If $a = 0$ and $a' = 1$, x^2 separated P and P'. If $a = a' = 1$ and $(b', c') \neq (-b, -c)$, we know that y^2 and z^2 can separates them. If $a = a' = 1$, $b' = -b$ and $c' = -c$, $xy - yz$ can separates them. For $P = (a, b, c)$, if $c = 1$ and $a \neq 0$, the section $x^2 - a^2$ and $xz - yz - a + b$ can separate the tangent space of P. If $c = 1$ and $a = b = 0$, $xy - yz$ and x^2 can separate the tangent space of P. The case of P on $D(a)$ or $D(b)$ are same with this, hence this corresponding morphism is a closed embedding.

(c) We may assume $P = (0, 0, 1)$, then $\mathfrak{d} =< x^2, y^2, xy, xz, yz >$. Then the morphism maps U to an open subset U' of $V(zw - xt, yw - zt)$. Since U is open dense in $\tilde{X}$, we know that $\bar{U}'$ is the image of $\tilde{X}$, which is corresponding to the divisor $V(x, y, z) + V(x, z, w) + V(x, w, t)$ in $\mathbb{P}^4$, hence degree 3. Every line thought P has the form $\alpha a + \beta b = 0$, so it corresponds to the line $V(\alpha x + \beta z, \alpha y + \beta z, \alpha t + \beta w)$. Hence if $\alpha : \beta \neq \alpha' : \beta'$, two different line in $\tilde{X}$ do not meet.

Solution 2.7.8. This is just the easy version of proposition 7.12. about $\mathrm{id} : X \to X$.

Solution 2.7.9. (a) We can define a morphism $\varphi : \operatorname{Pic} X\times\mathbb{Z} \to \operatorname{Pic}\mathbb{P}(\mathscr{E})$, $(\mathscr{L}, n) \mapsto \pi^*\mathscr{L}\otimes\mathscr{O}(n)$. For injectivity, if $\pi^*\mathscr{L}\otimes\mathscr{O}(n) \cong \mathscr{O}_{\mathbb{P}(\mathscr{E})}$, we have $\mathscr{O}_X = \pi_*\mathscr{O}_{\mathbb{P}(\mathscr{E})} = \pi_*(\pi^*\mathscr{L}\otimes\mathscr{O}(n)) \cong \mathscr{L}\otimes\pi_*\mathscr{O}(n)$. Then by proposition 7.11., we know that $\pi_*\mathscr{O}(n) \cong S^n(\mathscr{E})$, hence there we need it to be invertible and get $n = 0$, and $\mathscr{L} \cong \mathscr{O}_X$. For surjectivity, we may choose a finite affine covering $\{U_i\}$ of X such that $\mathscr{E}$ is trivial on each $U_i = \operatorname{Spec} A_i$. Then all $V_i = \mathbb{P}^{r-1}_{A_i} \cong U_i\times\mathbb{P}^{r-1}$ cover $\mathbb{P}(\mathscr{E})$. So we have $\operatorname{Pic} V_i = \operatorname{Pic} U_i\times\mathbb{Z}$. For any $\mathscr{M} \in \operatorname{Pic}\mathbb{P}(\mathscr{E})$, we have $\mathscr{M}_i = \mathscr{M}|_{V_i} \in \operatorname{Pic} V_i$, hence there exists $\mathscr{L}_i \in \operatorname{Pic} U_i$ and n_i such that $\mathscr{M}_i \cong \pi_i^*\mathscr{L}_i\otimes\mathscr{O}_i(n_i)$. Denote $U_{ij} = U_i\cap U_j$ and $V_{ij} = V_i\cap V_j$. Clearly $\mathscr{M}_i|_{V_{ij}} = \mathscr{M}_j|_{V_{ij}}$, hence we have a transition isomorphism $\pi_i^*\mathscr{L}_i\otimes\mathscr{O}_i(n_i)|_{V_{ij}} \cong \pi_j^*\mathscr{L}_j\otimes\mathscr{O}_j(n_j)|_{V_{ij}}$, hence $\mathscr{L}_i\otimes\pi_*\mathscr{O}_i(n_i)|_{V_{ij}} \cong \mathscr{L}_j\otimes\pi_*\mathscr{O}_j(n_j)|_{V_{ij}}$. By proposition 7.11. we have $n_i = n_j = n$. Since $\mathscr{O}_i|_{V_{ij}} \cong \mathscr{O}_j|_{V_{ij}}$, we have $\mathscr{L}_i|_{U_{ij}} \cong \mathscr{L}_j|_{U_{ij}}$. So all $\mathscr{L}_i$ can be glued together to be an $\mathscr{L}$, i.e. $\mathscr{M} \cong \mathscr{L}\otimes\mathscr{O}(n)$.

(b) If we have an $f : \mathbb{P}(\mathscr{E}) \cong \mathbb{P}(\mathscr{E}')$, there exists an $\mathscr{L} \in \operatorname{Pic} X$ such that $f^*\mathscr{O}'(1) \cong \mathscr{O}(1) \otimes \pi^*\mathscr{L}$. So $\mathscr{E}' \cong \pi'_*(\mathscr{O}'(1)) = \pi_*(\mathscr{O}(1) \otimes \pi^*\mathscr{L}) = \mathscr{E} \otimes \mathscr{L}$. Conversely, if $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L}$, we have a surjection $\pi^*\mathscr{E}' \cong \pi^*\mathscr{E} \otimes \pi^*\mathscr{L} \to \mathscr{O}(1) \otimes \pi^*\mathscr{L}$. Then by theorem 7.12., we have a morphism $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E}')$.

Solution 2.7.10 ($\mathbb{P}^n$-Bundles Over a Scheme)**.** (a) We may define a projective n-space bundle over X as a scheme P with a morphism $\pi : P \to X$, such that there exists an open covering $\{U_i\}$ of X such that $\pi^{-1}(U_i) \cong U_i \times \mathbb{P}^n$, and for any $V = \operatorname{Spec} A \subset U_i \cap U_j$, the transition morphism $(U_i \times \mathbb{P}^n)|_{V\times\mathbb{P}^n} \to (U_j \times \mathbb{P}^n)|_{V\times\mathbb{P}^n}$ is an A-linear automorphism. (**Is this really a question?**)

(b) Since $\mathscr{E}$ is locally free, we have an affine open covering $\{U_i = \operatorname{Spec} A_i\}$ of X such that $\mathscr{E}|_{U_i} \cong \mathscr{O}_{U_i}^{n+1}$. Then $\pi^{-1}U_i = \mathbb{P}^n_{A_i}$. For any $V = \operatorname{Spec} B \subset U_i \cap U_j$, then $\mathbb{P}^n_{A_i} \to \mathbb{P}^n_B$ is induced by $B[x_0, \dots, x_n] \to A_i[x_0, \dots, x_n] \to A_i[x_0, \dots, x_n 0$, where the first arrow is just $x_k \mapsto x_k$ and morphism on coefficient $B \to A_i$, and the second arrow is linear. Hence the transition morphism is linear, so $\mathbb{P}(\mathscr{E})$ is a $\mathbb{P}^n$-bundle.

(c) If P is a projective bundle with projection $\pi : P \to X$, we have an affine open cover $\{U_i\}$ of X such that $\pi^{-1}(U_i) = U_i \times \mathbb{P}^n$. So we can define $\mathscr{L}_i = \mathscr{O}_{U_i\times\mathbb{P}^n}(1)$. For any $V = \operatorname{Spec} B \subset U_i \cap U_j$, since the transition morphism $(U_i \times \mathbb{P}^n)|_{V\times\mathbb{P}^n} \to (U_j \times \mathbb{P}^n)|_{V\times\mathbb{P}^n}$ is linear, it induces the isomorphism $\mathscr{L}_i|_{V\times\mathbb{P}^n} \to \mathscr{L}_j|_{V\times\mathbb{P}^n}$, hence all $\mathscr{L}_i$ can be glued together to be an $\mathscr{L}$ on P. Since all $\mathscr{L}_i$ is invertible, $\mathscr{L}$ is invertible too. So we can define a locally free sheaf $\mathscr{E} = \pi_*\mathscr{L}$. Then by definition we clearly have $\mathbb{P}(\mathscr{E}) \cong P$.

(d) By definition, we know that $\mathbb{P}(\mathscr{E}_{\mathbb{P}(\mathscr{F})}) \cong \mathbb{P}(\mathscr{F})$. And by 2.7.9.(b), $\mathscr{E}_{\mathbb{P}(\mathscr{F})} \cong \mathscr{F} \otimes \mathscr{L}$ for some invertible sheaf $\mathscr{L}$. So we have an equivalence between the category of projective bundles and the category of equivalent classes locally free sheaves.

Solution 2.7.11. (a) For any affine piece $U \subset X$, by 2.5.13. we already have $\operatorname{Proj}(\bigoplus_{n=1}^{\infty} \mathscr{I}(U)^{nd}) \cong \operatorname{Proj}(\bigoplus_{n=1}^{\infty} \mathscr{I}(U)^n)$. And clearly those isomorphism can be glued together by 2.5.13., so we have an isomorphism $\mathbf{Proj} \bigoplus_n \mathscr{I}^n \cong \mathbf{Proj} \bigoplus_n \mathscr{I}^{nd}$.

(b) Clearly, $\bigoplus_n (\mathscr{I} \cdot \mathscr{J})^n = (\bigoplus_n \mathscr{I}^n) * \mathscr{J}$, hence by lemma 7.9., we have $\mathbf{Proj} \bigoplus_n (\mathscr{I} \cdot \mathscr{J})^n \cong \mathbf{Proj} \bigoplus_n \mathscr{I}^n$.

(c) Actually, we will prove that for any blow-up $f : \tilde{X} = \mathrm{Bl}_{\mathscr{I}} X \to X$, the open set $U = X - Z(\mathscr{I})$ is the maximal open subset of X such that $f^{-1}(U) \cong U$, where $Z(\mathscr{I})$ is the closed subset corresponding to $\mathscr{I}$. For any affine piece $V = \operatorname{Spec} B$ with $V \cap Z(\mathscr{I}) \neq \varnothing$, $\mathscr{I}(V) \subsetneq \mathscr{O}(V) = B$ is a proper ideal, namely I. Then $\pi^{-1}(V) = \mathbf{Proj} \bigoplus_{n=0}^{\infty} I^n$, hence not isomorphic to V. So U is the maximal open subset of X such that $f^{-1}(U) \cong U$. So in theorem 7.17., if $Z = \mathrm{Bl}_{\mathscr{I}}(X)$ and $U \subset X$ is the maximal open subset such that $f^{-1}(U) \cong U$, we must have $\operatorname{Supp} I = X - U$.

Solution 2.7.12. If $P \in \tilde{Y} \cap \tilde{Z} \subset \tilde{X}$, we can take an affine piece $U = \operatorname{Spec} A$ containing $\pi(P)$. Then we may denote I_Y, I_Z as the ideals corresponding to $Y \cap U$ and $Z \cap U$. So $\pi^{-1}(U) = \operatorname{Proj} \bigoplus (I_Y + I_Z)^n$. Since $\pi^{-1}(Y \cap U) = \operatorname{Proj} \bigoplus ((I_Y + I_Z)/I_Y)^n$ and $\pi^{-1}(Z \cap U) = \operatorname{Proj} \bigoplus ((I_Y + I_Z)/I_Z)^n$, and the embedding $\pi^{-1}(Y \cap U) \to \pi^{-1}(U)$ is induced by quotient $\varphi_Y : \bigoplus (I_Y + I_Z)^n \to \bigoplus ((I_Y + I_Z)/I_Y)^n$ and so does $\pi^{-1}(Z \cap U) \to \pi^{-1}(U)$, we know that P corresponds to a prime ideal in $\bigoplus (I_Y + I_Z)^n$ and containing $\ker \varphi_Y = \bigoplus I_Y^n$ and $\ker \varphi_Z = \bigoplus I_Z^n$, which makes a contradiction because $\bigoplus I_Y^n + \bigoplus I_Z^n = \bigoplus ((I_Y + I_Z)/I_Y)^n$.

Solution 2.7.13 (A Complete Nonprojective Variety)**.** (a) Since $\mathbb{A}^1 \to \operatorname{Spec} k$ is clearly proper, $\pi|_{C\times(\mathbb{P}^1-\{0\})} : C \times (\mathbb{P}^1 - \{0\}) \to C$ is proper because it is the base change of $\mathbb{A}^1 \to \operatorname{Spec} k$. And so does $\pi|_{C\times(\mathbb{P}^1-\{\infty\})}$, hence π is proper.

(b) By 2.6.9.(a), we have an exact sequence $0 \to \bigoplus_{P\in C\times\mathbb{A}^1} \tilde{\mathscr{O}}_P/\mathscr{O}_P \to \operatorname{Pic}(C\times\mathbb{A}^1) \to \operatorname{Pic}(\mathbb{P}^1\times\mathbb{A}^1) \to 0$. Since we only need to consider the singular locus $P = P_0 \times \mathbb{A}^1$ term of the direct sum, and $\tilde{\mathscr{O}}_P = k[t,z]_{(t)}$ and $\mathscr{O}_P = k[t^2, t^3, z]_{(t^2,t^3)}$, so we have $0 \to \mathbb{G}_{\mathrm{m}} \to \operatorname{Pic}(C\times\mathbb{A}^1) \to \mathbb{Z} \to 0$. Since we clearly have a morphism $\mathbb{Z} \to \operatorname{Pic}(C\times\mathbb{A}^1)$ as $1 \mapsto f_*\mathscr{O}(1)$, where $f : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{A}^1 \to C \times \mathbb{A}^1$, this exact sequence is split, hence $\operatorname{Pic}(C\times\mathbb{A}^1) \cong \mathbb{G}_{\mathrm{m}} \times \mathbb{Z}$. For $C \times (\mathbb{A}^1 - \{0\})$, we also have $0 \to \bigoplus_{P\in C\times(\mathbb{A}^1-\{0\})} \tilde{\mathscr{O}}_P/\mathscr{O}_P \to \operatorname{Pic}(C \times (\mathbb{A}^1 - \{0\})) \to \operatorname{Pic}(\mathbb{P}^1 \times (\mathbb{A}^1 - \{0\})) \to 0$. And we only need to consider $P = P_0 \times (\mathbb{A}^1 - \{0\})$, and $\tilde{\mathscr{O}}_P/\mathscr{O}_P \cong k[t, z, z^{-1}]_{(t)}/k[t^2, t^3, z, z^{-1}]_{(t^2,t^3)} \cong \mathbb{G}_{\mathrm{m}}$. Moreover,

$\mathrm{Pic}\,(\mathbb{P}^1 \times (\mathbb{A}^1 - \{0\})) \cong \mathbb{Z} \times \mathbb{Z}$ by using proposition 6.7., and the exact sequence is split by similar reason, so $\mathrm{Pic}\,(C \times (\mathbb{P}^1 \times (\mathbb{A}^1 - \{0\})) \cong \mathbb{G}_\mathrm{m} \times \mathbb{Z} \times \mathbb{Z}$.

(c) We clearly have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{G}_\mathrm{m} \arrow[r] \arrow[d, "i"] & \mathrm{Pic}\,(C \times \mathbb{A}^1) \arrow[r] \arrow[d, "j"] & \mathbb{Z} \arrow[r] \arrow[d, "k"] & 0 \\
0 \arrow[r] & \mathbb{G}_\mathrm{m} \arrow[r] & \mathrm{Pic}\,(C \times (\mathbb{A}^1 - \{0\})) \arrow[r] & \mathbb{Z} \times \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Since clearly i is the identity, and $k : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $n \mapsto (0, n)$, we know that $j : \mathrm{Pic}\,(C \times \mathbb{A}^1) \to \mathrm{Pic}\,(C \times (\mathbb{A}^1 - \{0\}))$ is just $(t, n) \mapsto (t, 0, n)$. For the other one, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{G}_\mathrm{m} \arrow[r] \arrow[d, "i"] & \mathrm{Pic}\,(C \times (\mathbb{A}^1 - \{0\})) \arrow[r] \arrow[d, "j"] & \mathbb{Z} \times \mathbb{Z} \arrow[r] \arrow[d, "k"] & 0 \\
0 \arrow[r] & \mathbb{G}_\mathrm{m} \arrow[r] & \mathrm{Pic}\,(C \times (\mathbb{A}^1 - \{0\})) \arrow[r] & \mathbb{Z} \times \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

And similarly i is the identity, and k is just $(d, n) \mapsto (d + n, n)$, so $j : (t, d, n) \mapsto (t, d + n, n)$.

(d) By (b) and (c) we have the following diagram

$$\begin{tikzcd}
 & \mathrm{Pic}\,(C \times (\mathbb{P}^1 - \{0\})) \arrow[r] & \mathrm{Pic}\,(C \times (\mathbb{P}^1 - \{0\} - \{\infty\})) \arrow[dd] \\
\mathrm{Pic}\,X \arrow[ur] \arrow[dr] & & \\
 & \mathrm{Pic}\,(C \times (\mathbb{P}^1 - \{\infty\})) \arrow[r] & \mathrm{Pic}\,(C \times (\mathbb{P}^1 - \{0\} - \{\infty\}))
\end{tikzcd}$$

Then if any element in $\mathrm{Pic}\,X$ with image (t, n) and (t', n') in $\mathrm{Pic}\,(C \times (\mathbb{P}^1 - \{0\}))$ or $\mathrm{Pic}\,(C \times (\mathbb{P}^1 - \{\infty\}))$, we have $(t, 0, n) = (t', n', n')$, hence $t = t'$ and $n = n' = 0$. So the image of $\mathrm{Pic}\,X$ in $\mathrm{Pic}\,(C \times \mathbb{A}^1)$ is just $\{(t, n) \mid t \in \mathbb{G}_\mathrm{m}\}$. Since $\mathrm{Pic}\,(C \times \mathbb{A}^1) \to \mathrm{Pic}\,(C \times \{0\})$ is just identity, the image of the restriction map $\mathrm{Pic}\,X \to \mathrm{Pic}\,(C \times \{0\})$ consists entirely of divisors of degree 0 on C. If X is projective over k, there exists a closed immersion $X \to \mathbb{P}^n$. Then we have a morphism $\mathrm{Pic}\,\mathbb{P}^n \to \mathrm{Pic}\,X \to \mathrm{Pic}\,(C \times \{0\})$. Since $\mathrm{Pic}\,\mathbb{P}^n \to \mathrm{Pic}\,(C \times \{0\})$ is just $n \mapsto (1, n)$, hence not in the image of $\mathrm{Pic}\,X$ in $\mathrm{Pic}\,(C \times \{0\})$, which makes a contradiction.

Solution 2.7.14. (a) Take $X = \mathbb{P}^1$, and $\mathscr{E} = \mathscr{O}(-1)$. Then if $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ is very ample relative to X, i.e. $\mathscr{E}$ is the pullback of $\mathscr{O}_{\mathbb{P}^n}(1)$ on X, which is contradict with the non-negativity.

(b) Since $\mathscr{L}$ is ample, $\mathscr{L}^n$ is very ample on X for sufficiently large n. By theorem 7.10., $\mathscr{O}_P(1) \otimes \pi^*\mathscr{L}^m$ is very ample on P relative to X for sufficiently large m. Hence by 2.5.12.(b), $\mathscr{O}_P(1) \otimes \pi^*\mathscr{L}^{m+n}$ is very ample relative to Y.

2.8 Differentials

Solution 2.8.1. (a) By proposition 8.4. we already have $\mathrm{coker}\delta = \Omega_{k(B)/k}$. For injectivity of δ, it equivalent to show that $\delta' : \mathrm{Der}_k(B, k)\mathrm{Hom}_k(\Omega_{B/k} \otimes k, k) \to \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is surjective. By theorem 8.25., we know that $k(B)$ is already contained in A, so for any $b \in B$, b can uniquely write as $b = \lambda + c$ for $\lambda \in k(B)$ and $c \in \mathfrak{m}$. For any $h \in \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$, we can define $db = h(\bar{c})$, hence δ' is surjective, i.e. δ is injective.

(b) Similarly, $k(B) \subset B$. ($\Rightarrow$) By (a) we have $\dim_{k(B)} \Omega_{B/k} \otimes k(B) = \dim B + \dim \Omega_{k(B)/k} = \dim B + \mathrm{tr.d.}k(B)/k$. Then by lemma 8.9., we only need to show that $\dim_K \Omega_{B/k} \otimes K = \dim B + \mathrm{tr.d.}k(B)/k$. By theorem 8.2., we have $\Omega_{B/k} \otimes_B K = \Omega_{K/k}$. Since k is perfect, so K is separably generated, i.e. $\dim_K \Omega_{K/k} = \text{tr.d.}K/k$, then $\dim_K \Omega_{B/k} \otimes K = \text{tr.d.}K/k$. For any prime ideal $\mathfrak{p} \subset A$, we have $\text{Frac}A = \text{Frac}A_{\mathfrak{p}}$, and height $\mathfrak{p} = \dim A_{\mathfrak{p}}$. So $\text{tr.d.}K/k = \dim A = \text{height } \mathfrak{p} + \dim A/\mathfrak{p} = \dim B + \dim A/\mathfrak{p} = \dim B + \text{tr.d.}k(B)$. So $\dim_{k(B)} \Omega_{B/k} \otimes k(B) = \dim_K \Omega_{B/k} \otimes K = \dim B + \text{tr.d.}k(B)/k$, then $\Omega_{B/k}$ is free of rank $\dim B + \text{tr.d.}k(B)/k$.

($\Leftarrow$) By theorem 8.6, we have $\dim \Omega_{k(B)/k} = \text{tr.d.}k(B)/k$. So by (a), we have $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim B$, hence regular.

(c) For any affine neighbourhood $U = \text{Spec}\, A$ of x and x corresponds to a prime ideal $\mathfrak{p} \subset A$. By (b), $\mathscr{O}_{X,x}$ is regular iff $\Omega_{A_{\mathfrak{p}}/k} = \Omega_{A/k} = (\Omega_{X/k})_x$ is free of rank $\dim A_{\mathfrak{p}} + \text{tr.d.}k(A_{\mathfrak{p}})/k = \dim A = \dim X$.

(d) By corollary 8.16., there exists an open dense set V such that $\mathscr{O}_{X,x}$ is regular if $x \in V$, hence $U \supseteq V$ is dense. For openness, if $x \in U$, since $\Omega_{X/k}$ is locally free by (c), there exists an open neighbourhood W of x such that $\Omega_{X/k}|_W$ is free of rank $n = \dim X$. So every for every $y \in W$, $\mathscr{O}_{X,y}$ is free of rank n, hence $w \in U$, i.e. U is open.

Solution 2.8.2. Define $B = \{(x, s) \mid s_x \in \mathfrak{m}_x\mathscr{E}_x\} \subset X \times V$ with projection $\pi_X : B \to X$ and $\pi_V : B \to V$. For any $x \in X$, then $\pi_X^{-1}(x) = \ker(V \otimes_k k(x) \to \mathscr{E}_x \otimes_{\mathscr{O}_x} k(x) \cong \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x$. Since $\mathscr{E}$ is generated by global section, this map is surjective. And since $\dim\ker = \dim V - \text{rank}\,\mathscr{E}$, we have $\dim B = \dim X + \dim V - \text{rank}\,\mathscr{E}$. Since $r > n$, we have $\dim B < \dim V$, i.e. π_V is not surjective. So there exists $s \in V$ such that for any $x \in X$, we have $s_x \notin \mathfrak{m}_x\mathscr{E}_x$. Fixed such an s, we have an injection $\mathscr{O}_X \to \mathscr{E}$ as $\times s$. Then by 2.5.7.(b), we know that the cokernal is locally free of rank (rank $\mathscr{E} - 1$).

Solution 2.8.3 (Product Schemes)**.** (a) By proposition 8.10., we have $\Omega_{(X\times_S Y)/Y} \cong p_1^*\Omega_{X/S}$ and $\Omega_{(X\times_S Y)/X} \cong p_2^*\Omega_{Y/S}$. Then by proposition 8.11., we have $\Omega_{(X\times_S Y)/Y} \to \Omega_{(X\times_S Y)/S} \to \Omega_{(X\times_S Y)/X} \to 0$, and $\Omega_{(X\times_S Y)/X} \to \Omega_{(X\times_S Y)/S} \to \Omega_{(X\times_S Y)/Y} \to 0$, i.e.

$$p_1^*\Omega_{X/S} \to \Omega_{(X\times_S Y)/S} \to p_2^*\Omega_{Y/S} \to 0$$

$$p_2^*\Omega_{Y/S} \to \Omega_{(X\times_S Y)/S} \to p_1^*\Omega_{X/S} \to 0$$

On every $\text{Spec}\, A \subset X$, $\text{Spec}\, B \subset Y$ and $\text{Spec}\, C \subset S$ with $\text{Spec}\, A$ and $\text{Spec}\, B$ contained in the preimage of $\text{Spec}\, C$, we have $\Omega_{(A\times_C B)/A} \to \Omega_{B/C} \otimes_B (B \otimes_C A) \to \Omega_{(A\times_C B)/C} \to \Omega_{(A\times_C B)/A}$, with $\mathrm{d}(1 \otimes b) \mapsto \mathrm{d}b \otimes (1 \otimes 1) \mapsto \mathrm{d}(1 \otimes b) \mapsto \mathrm{d}(1 \otimes b)$. So the above two sequences are split on every affine pieces, hence $\Omega_{(X\times_S Y)/S} = p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$.

(b) Denote the dimension of X and Y as n and m. Then clearly by 2.5.16., $\omega_{X\times Y} = \Lambda^{mn}\Omega_{(X\times Y)/S} = \Lambda^{mn}(p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}) = p_1^*\Lambda^n\Omega_{X/S} \otimes p_2^*\Lambda^m\Omega_{Y/S} = \omega_{X/S} \otimes \omega_{Y/S}$.

(c) By 1.7.2.(b), we have $p_a(Y) = 1$, then $p_a(Y \times Y) = -1$ by 1.7.2.(e). By example 8.20.3., we have $\omega_Y \cong \mathscr{O}_Y$. Then $\omega_{Y\times Y} \cong p_1^*\mathscr{O}_Y \otimes p_2^*\mathscr{O}_Y \cong \mathscr{O}_{Y\times Y}$. Since Y is proper over k, i.e. $Y \times Y$ is proper over Y, hence proper over k. Then by 2.4.5.(d), we have $\Gamma(Y \times Y, \mathscr{O}_{Y\times Y}) \cong k$, i.e. $p_g = 1$.

Solution 2.8.4 (Complete Intersections in $\mathbb{P}^n$)**.** (a) ($\Rightarrow$) If $I_Y = (f_1, \ldots, f_r)$, we only need to define $H_i = Z(f_i)$ and get $Y = H_1 \cap \ldots \cap H_r$.

($\Leftarrow$) If $Y = H_1 \cap \ldots \cap H_r$, and each H_i corresponds to a prime ideal $I_i = (f_i) \subset S$. Since f_{i+1} is not a zero divisor in $S/(f_1, \ldots, f_i)$, $(f_1, \ldots, f_r)$ is a regular sequence of and $(f_1, \ldots, f_r) \subset I_Y$. Since $S/(f_1, \ldots, f_r)$ has degree $\sum \deg H_i$, there exists an ideal J such that $(f_1, \ldots, f_r) = I \cap J$ for $\text{codim}J > 2$. By unmixedness theorem, the primary components of $(f_1, \ldots, f_r)$ has codimension ≤ 1, hence $J = \varnothing$. So $I_Y = (f_1, \ldots, f_r)$.

(b) If Y is normal, we know that $\text{Sing}Y$ has codimension ≥ 2, hence $\text{Sing}\mathit{Cone}(Y)$ has codimension ≥ 2. So by proposition 8.23.(b), we know that $S(\text{Cone}(Y))$ is integrally closed, i.e. $S(Y) = S(\text{Cone}(Y))$ is integrally closed, hence Y is projectively normal.

(c) Since Y is projectively normal, we have $S(Y) = S/I_Y$, i.e. the projection $S \to S_Y$ is surjective. Hence $\Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(l)) \to \Gamma(Y, \mathscr{O}_Y(l))$ is just the l-grade of the projection above, i.e. surjective. Taking $l = 0$, we have a surjection $k \to \Gamma(Y, \mathscr{O}_Y)$, i.e. $\dim \Gamma(Y, \mathscr{O}_Y) \leq 1$, hence the number of connected component ≤ 1, i.e. connected.

(d) For any hyperplane H, there exists a nonsingular hypersurface H_1 in $|dH|$ such that H_1 has degree d_1. Since $\mathbb{P}^n|_{H_1} \cong \mathbb{P}^{n-1}$, then we repeat this process in $\mathbb{P}^{n-1}, \ldots, \mathbb{P}^{n-r+1}$, to get a subscheme $Y = H_1 \cap \ldots \cap H_r$ such that Y is nonsingular with $\deg H_i = d_i$.

(e) Clearly, $\omega_{H_1} = \mathscr{O}_{\mathbb{P}^n}(-n-1) \otimes \mathscr{O}_{H_1}(d_1) = \mathscr{O}_{H_1}(d_1-n-1)$. Then $\omega_{H_1\cap H_2} = \mathscr{O}_{H_1}(d_1-n-1) \otimes \mathscr{O}_{H_1\cap H_2}(H_1.H_2) = \mathscr{O}_{H_1}(d_1-n-1) \otimes (\mathscr{O}_{H_1}(d_2)|_{H_1\cap H_2}) \cong \mathscr{O}_{H_1\cap H_2}(d_1+d_2-n-1)$. Repeating this method, we have $\omega_Y = \omega_{H_1\cap \ldots \cap H_r} = \mathscr{O}_Y(\sum d_i - n - 1)$.

(f) By (c), we have an exact sequence $0 \to \mathscr{I}_Y \to \mathscr{O}_{\mathbb{P}^n} \to \mathscr{O}_Y \to 0$. Since $\deg Y > d - n - 1$, which means there are no section of degree $d - n - 1$ vanishing on Y, hence $\dim_k \Gamma(Y, \mathscr{I}_Y(d - n - 1)) = 0$. So by this and (e), we have $p_g(Y) = \dim_k \Gamma(Y, \omega_Y) = \dim_k \Gamma(Y, \mathscr{O}_Y(d - n - 1)) = \dim_k \Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d - n - 1)) = \binom{d-1}{n}$. By 1.7.2., we've already know that $p_a(Y) = \binom{d-1}{n}$.

(g) Denote $Y = H \cap H'$, where $\deg H = d$, $\deg H' = e$ and H, H' is generated by f and g. Then we have an exact sequence $0 \to (\mathscr{O}_{\mathbb{P}^3}(-4))|_Y \to (\mathscr{O}_{\mathbb{P}^3}(d-4) \oplus \mathscr{O}_{\mathbb{P}^3}(e-4))|_Y \to (\mathscr{O}_{\mathbb{P}^3}(d+e-4))|_Y \to \mathscr{O}_Y(d+e-4) \to 0$, where the second arrow is $s \mapsto (fs, gs)$, and the third arrow is $(s, t) \mapsto (gs - ft)$. Hence $p_g(Y) = \dim_k \Gamma(Y, \mathscr{O}_Y(d + e - n - 1)) = \binom{d+e-1}{3} + \binom{-1}{3} - \binom{d-1}{3} - \binom{e-1}{3} + 1 = \frac{1}{2}de(d + e - 4) + 1$, hence equal to the arithmetic genus by 1.7.2.

Solution 2.8.5 (Blowing up a Nonsingular Subvariety)**.** (a) By proposition 6.5., we have an exact sequence $\mathbb{Z} \to \operatorname{Pic} \tilde{X} \to \operatorname{Pic} (\tilde{X} - Y') \to 0$. Since we have $\tilde{X} - Y' \cong X - Y$, and Y has codimension ≥ 2, we know that $\operatorname{Pic} (\tilde{X} - Y') \cong \operatorname{Pic} (X - Y) \cong \operatorname{Pic} X$. Moreover, the morphism $\mathbb{Z} \to \operatorname{Pic} \tilde{X}$ is just $1 \mapsto Y'$. And by theorem 8.24., we know that $\mathscr{O}_{\tilde{X}}(nY')|_{Y'} \cong \mathscr{O}_Y(n)$. So this morphism is injective, i.e. we have an exact sequence $0 \to \mathbb{Z} \to \operatorname{Pic} \tilde{X} \to \operatorname{Pic} X \to 0$. Since we have a morphism $\pi^* : \operatorname{Pic} X \to \operatorname{Pic} \tilde{X}$, this sequence is split, i.e. $\operatorname{Pic} \tilde{X} \cong \operatorname{Pic} X \oplus \mathbb{Z}$.

(b) By (a) we may assume $\omega_{\tilde{X}} \cong f^*\mathscr{M} \otimes \mathscr{L}(qY')$ for some invertible sheaf $\mathscr{M}$ on X and $q \in \mathbb{Z}$. Denote $U = X - Y$, we have $\pi^{-1}(U) \cong U$. Since $\omega_{\tilde{X}}|_U \cong \omega_U \cong \omega_X|_U$, and as in (a), $\operatorname{Pic} X \cong \operatorname{Pic} U$, we know that $\omega_X \cong \omega_{\tilde{X}}|_U = \mathscr{M}$, hence $\omega_{\tilde{X}} \cong f^*\omega_X \otimes \mathscr{L}(qY')$. To determine q, by theorem 8.24., we have $\omega_{Y'} \cong \omega_{\tilde{X}} \otimes \mathscr{L}(Y') \otimes \mathscr{O}_{Y'} = f^*\omega_X \otimes \mathscr{L}((q+1)Y') \otimes \mathscr{O}_{Y'}$. By proposition 6.18., we have $\mathscr{L}((q+1)Y') = \mathscr{I}_{Y'}^{-q-1} \cong \mathscr{O}_{Y'}(-q-1)$, hence $\omega_{Y'} \cong f^*\omega_X \otimes \mathscr{O}_{Y'}(-q-1)$. Fix a closed point $y \in Y$, and $Z = \pi^{-1}(y) = \{y\} \times_Y Y'$, then by 2.8.3.(b), we have $\omega_Z = \pi_1^*\omega_y \otimes \pi_2^*\omega_{Y'} = \mathscr{O}_Z \otimes \pi_2^*\omega_{Y'} = \omega_Z(-q-1)$. By theorem 8.24., Z is a projective space of dimension $r - 1$, i.e. $\omega_Z = \mathscr{O}_Z^{-r}$, hence $q = r - 1$, i.e. $\omega_{\tilde{X}} \cong f^*\omega_X \otimes \mathscr{L}((r-1)Y')$.

Solution 2.8.6 (The Infinitesimal Lifting Property)**.** (a) Denote the morphism $I \to B \to B'$ as ψ and φ. Since $\varphi \circ \theta = f - f = 0$, we have $\operatorname{Im}\varphi \subset I$. Since g and g' are both ring homomorphism, $\theta(1) = g(1) - g'(1) = 1 - 1 = 0$. Hence for every $a \in k$, we have $\theta(a) = \theta(a) \cdot \theta(1) = 0$. Moreover, for any $a, b \in A$, $\theta(ab) = g(ab) - g'(ab) = g(a)g(b) - g'(a) - g'(b) - g(a)g(b) - g'(a)g(b) + g'(a)g(b) - g'(a)g'(b) = g'(a)\theta(b) + g(b)\theta(a)$. So $\theta \in \operatorname{Hom}_A(\Omega_{A/k}, I)$. Conversely, if $\theta \in \operatorname{Hom}_A(\Omega_{A/k}, I)$, we have a morphism $\psi \circ \theta \circ d : A \to I \to B$. Then clearly $\varphi \circ \psi \circ \theta \circ d = 0$ by exactness, we know that $g + \theta$ is another k-linear homomorphism lifting f. Moreover, since $I^2 = 0$, we have $\theta(a)\theta(b) = 0$ for any $a, b \in A$, then $g(ab) + \theta(ab) = g(a)g(b) + g(a)\theta(b) + g(b)\theta(a) + \theta(a)\theta(b) = (g(a) + \theta(a))(g(b) + \theta(b))$, hence $g + \theta$ is a ring homomorphism.

(b) For any x_i, we fix a $b_i \in B'$ as the lifting of $f(\bar{x}_i)$. Then the morphism $h : P \to B'$ is defined as $x_i \to b_i$. If $a \in J \subset P$, we know that the image of a in B is zero, hence $h(a) \in I$. Moreover, if $a \in J^2$, we have $h(a) \in I^2 = 0$. Hence h induces a morphism $\bar{h} : J/J^2 \to I$.

(c) Take theorem 8.17. on $\operatorname{Spec} P$ and $\operatorname{Spec} A$, we have an exact sequence $0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0$. Since A is nonsingular, $\Omega_{A/k}$ is locally free, hence $\operatorname{Ext}_A^i(\Omega_{A/k}, I) = 0$ for all $i > 0$. So we have an exact sequence $0 \to \operatorname{Hom}_A(\Omega_{A/k}, I) \to \operatorname{Hom}_A(\Omega_{P/k}, I) \to \operatorname{Hom}_A(J/J^2, I) \to 0$. Then we have $\theta : \Omega_{P/k} \to I$ as the lifting of $\bar{h}$. So we can define $\theta' : P \to \Omega_{P/k} \to I \to B'$ as a k-homomorphism. Define $h' = h - \theta$, then for any $a \in J$ we have $h'(a) = 0$, i.e. h' induces a morphism $g : A \to B'$.

Solution 2.8.7. This problem is just an algebraic question as: If A' is a ring, $I \subset A'$ is an ideal with $I^2 = 0$, $A'/I \cong A$, and A-module M is isomorphic to I, then we need to prove that $A' = A \oplus M$ as abelian groups, with multiplication $(a,m)\cdot(a',m') = (aa', am' + a'm)$. By 2.8.6., the identity morphism on A can be lifted to a morphism $A \to A'$. Then the exact sequence $0 \to I \to A' \to A \to 0$ is split as an exact sequence of abelian groups, i.e. $A' = A \oplus M$ as abelian groups. Moreover, for any $a \in A$, we clearly know that $(a,0)\cdot(a',m') = (aa', am')$ by the A-module structure on A. And for any $m \in I$, we have $(0,m)\cdot(a',m') = (0, a'm + mm') = (0, a'm)$ since $mm' \in I^2 = 0$. Hence A' is the trivial extension.

Solution 2.8.8. If X' is a nonsingular variety birational to X with morphism $X \to X'$. We can take V as the largest open subset of X representing this rational morphism, and $f : V \to X'$ is the representing morphism. By proposition 8.11., we have $f^*\Omega_{X'} \to \Omega_V$, which induces morphisms $f^*\omega^n_{X'} \to \omega^n_V$ and $f^*\Omega^q_{X'} \to \Omega^q_V$. By corollary 4.5. in chapter 1, there exists an open subset $U \subset V$ such that $f^{-1}(U) \cong U$, so $\Omega_V|_U \cong \Omega_{X'}|_{f^{-1}(U)}$. Hence we have the following two commutative diagrams

$$\begin{tikzcd}
\Gamma(X', \omega^n_{X'}) \arrow[r] \arrow[d] & \Gamma(V, \omega^n_V) \arrow[d] \\
\Gamma(f^{-1}(U), \omega^n_{f^{-1}(U)}) \arrow[r, "\cong"] & \Gamma(U, \omega^n_U)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(X', \Omega^q_{X'}) \arrow[r] \arrow[d] & \Gamma(V, \Omega^q_V) \arrow[d] \\
\Gamma(f^{-1}(U), \Omega^q_{f^{-1}(U)}) \arrow[r, "\cong"] & \Gamma(U, \Omega^q_U)
\end{tikzcd}$$

Since $f(U)$ is open dense in X', and global section of $\omega^n_{X'}$ or $\Omega^q_{X'}$ will not vanish on $f(U)$, hence we have injections $\Gamma(X', \omega^n_{X'}) \to \Gamma(V, \omega^n_V)$ and $\Gamma(X', \Omega^q_{X'}) \to \Gamma(V, \Omega^q_V)$. As in the proof of theorem 8.19., $X - V$ has codimension > 1. For every point in X, take an affine neighbourhood U of that point such that ω^n_X or Ω^q_X is free on U. By proposition 6.3., we have $\Gamma(U, \mathscr{O}_U) \cong \Gamma(U \cap V, \mathscr{O}_{U\cap V})$ since $U - V$ has codimension > 1 in U, hence we have $\Gamma(X, \omega^n_X) \cong \Gamma(V, \omega^n_V)$ and $\Gamma(X, \Omega^q_X) \cong \Gamma(V, \Omega^q_V)$. So we have injections $\Gamma(X', \omega^n_{X'}) \to \Gamma(X, \omega^n_X)$ and $\Gamma(X', \Omega^q_{X'}) \to \Gamma(X, \Omega^q_X)$, i.e. $P_n(X') \le P_n(X)$ and $h^{q,0}(X') \le h^{q,0}(X)$. Then by symmetry we have $P_n(X') = P_n(X)$ and $h^{q,0}(X') = h^{q,0}(X)$.

2.9 Formal Schemes

Solution 2.9.1. (a) By theorem 8.17., we have an injection $\mathscr{I}/\mathscr{I}^2 \to \Omega_{X/k} \otimes \mathscr{O}_Y$. Since $X = \mathbb{P}^n$, we have an injection $\Omega_{X/k} \to \mathscr{O}_X(-1)^{n+1}$ by theorem 8.13., which induces an injection $\Omega_{X/k} \otimes \mathscr{O}_Y \to \mathscr{O}_Y(-1)^{n+1}$ since they are both locally free. Hence we have an injection $\mathscr{I}/\mathscr{I}^2 \to \mathscr{O}_Y(-1)^{n+1}$.

(b) For $r = 1$, we already have $\Gamma(Y, \mathscr{I}/\mathscr{I}^2) \hookrightarrow \Gamma(Y, \mathscr{O}_Y(-1)^{n+1}) = 0$, hence $\Gamma(Y, \mathscr{I}/\mathscr{I}^2) = 0$. For $r > 1$, we have $S^r(\mathscr{I}/\mathscr{I}^2) \cong \mathscr{I}^r/\mathscr{I}^{r+1}$, then $\Gamma(Y, \mathscr{I}^r/\mathscr{I}^{r+1}) = S^r(\Gamma(Y, \mathscr{I}/\mathscr{I}^2)) = 0$.

(c) For $r = 1$, $\Gamma(Y, \mathscr{O}_X/\mathscr{I}) = \Gamma(Y, \mathscr{O}_Y) = k$, since Y has positive dimension. For greater, by induction we may assume $\Gamma(Y, \mathscr{O}_X/\mathscr{I}^r) = k$. Since we have an exact sequence $0 \to \mathscr{I}^r/\mathscr{I}^{r+1} \to \mathscr{O}_X/\mathscr{I}^{r+1} \to \mathscr{O}_X/\mathscr{I}^r \to 0$, hence $\Gamma(Y, \mathscr{O}_X/\mathscr{I}^{r+1}) \hookrightarrow \Gamma(Y, \mathscr{O}_X/\mathscr{I}^r)$ by (b). Since they are both k-algebra, and we've already had a non-zero morphism $k = \Gamma(Y, \mathscr{O}_Y) \to \Gamma(Y, \mathscr{O}_X/\mathscr{I}^{r+1})$, we know that $\Gamma(Y, \mathscr{O}_X/\mathscr{I}^{r+1}) = k$.

(d) Clearly $\Gamma(\hat{X}, \mathscr{O}_{\hat{X}}) = \varprojlim_r \Gamma(Y, \mathscr{O}_X/\mathscr{I}^r) = k$.

Solution 2.9.2. We may replace Z as the scheme-theoretic image of f, so we may assume that f is dominant. Then we have an injection $\mathscr{O}_Z \to f_*\mathscr{O}_X$. Hence it induces an injection $\mathscr{O}_{\hat{Z}} \to f_*\mathscr{O}_{\hat{X}}$. Then $\Gamma(\hat{Z}, \mathscr{O}_{\hat{Z}}) \hookrightarrow \Gamma(\hat{X}, \mathscr{O}_{\hat{X}}) = k$ by 2.9.1., where $\hat{Z}$ is the completion of Z along P. Since $\Gamma(\hat{Z}, \mathscr{O}_{\hat{Z}}) \cong \hat{\mathscr{O}}_{Z,P}$ is the completion of $\mathscr{O}_{Z,P}$ along its maximal ideal. So if Z has positive dimension, we have tr.d.$\mathscr{O}_{Z,P}/k > 0$, hence contradict. So Z has zero degree. Since X is connected, we know that Z is just a single point, i.e. $f(X) = P$.

Solution 2.9.3. (a) We may assume $\mathfrak{X}$ is the completion of $X = \operatorname{Spec} A$ along $\tilde{I}$, and $\mathfrak{F}' = \hat{\mathscr{F}}'$, and $\mathscr{F}' = \tilde{M}$. Then $\mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}' = \widetilde{M/I^nM} = \tilde{M}_n$. For any affine open subset $\mathfrak{U} \subset \mathfrak{X}$, it is induced from $U = \operatorname{Spec} A_f \subset X$. For any $s \in \Gamma(\mathfrak{U}, \mathfrak{F}'')$, for any $x \in \mathfrak{U}$ there exists a open neighbourhood $D(fg) \cap \mathfrak{X}$ of x such that $s|_{D(fg)\cap\mathfrak{X}}$ lifts to section $t \in (\mathfrak{F}/\mathfrak{I}^n\mathfrak{F}')(D(fg) \cap \mathfrak{X})$. Since we can cover $\mathfrak{U}$ with finite open sets $D(fg_i) \cap \mathfrak{X}$ such that for each i,

$s|_{D(fg_i)\cap\mathfrak{X}}$ lifts to $t_i \in (\mathfrak{F}/\mathfrak{I}^n\mathfrak{F}')(D(fg)\cap\mathfrak{X})$. Since $D(fg)\cap D(fg_i) = D(fgg_i)$, then $t, t_i \in (\mathfrak{F}/\mathfrak{I}^n\mathfrak{F}')(D(fgg_i)\cap\mathfrak{X})$ both lift s, hence $t - t_i \in (\mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}')(D(fgg_i)\cap\mathfrak{X})$. Then there exists $n > 0$ such that $g^n(t - t_i)$ extends to a section $u_i \in (\mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}')(D(fg_i)\cap\mathfrak{X})$. So we may pick an n sufficiently large such that $t_i' = g^n t_i + u_i$ is a lifting of $g^n s$ on $D(fg_i)$ for all i, and furthermore t_i' and $g^n t$ agree on $D(fgg_i)$. Since we have t_i' and t_j' on $D(fg_ig_j)$ of $(\mathfrak{F}/\mathfrak{I}^n\mathfrak{F}')$, and they are both lifting of $g^n s$, so $t_i' - t_j' \in (\mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}')(D(fg_ig_j)\cap\mathfrak{X})$. Since $t_i' = t_j'$ on $D(fgg_ig_j)\cap\mathfrak{X}$, we have $g^m(t_i' - t_j') = 0$ for some m. Hence we can glue all $g^m t_i'$ together for sufficient large m to get a section $t'' \in \Gamma(\mathfrak{U}, (\mathfrak{F}/\mathfrak{I}^n\mathfrak{F}'))$, which lifts $g^m s$. So there exists some $n > 0$, $g^n s$ can lift to a section in $\Gamma(\mathfrak{U}, (\mathfrak{F}/\mathfrak{I}^n\mathfrak{F}'))$.

Since we can cover $\mathfrak{U}$ by finite many $D(ff_i)\cap\mathfrak{X}$ such that $s|_{D(ff_i)\cap\mathfrak{X}}$ lifts to a section of $(\mathfrak{F}/\mathfrak{I}^n\mathfrak{F}')$ over $D(ff_i)\cap\mathfrak{X}$ for all i. Then there exists an n such that $t_i \in \Gamma(\mathfrak{U}, (\mathfrak{F}/\mathfrak{I}^n\mathfrak{F}'))$ is a lifting of $f_i^n s$. Since we have $f = \sum a_i f_i^n$ for some $a_i \in A$, we may define $t = \sum a_i t_i \in \Gamma(\mathfrak{U}, (\mathfrak{F}/\mathfrak{I}^n\mathfrak{F}'))$ whose image in $\Gamma(\mathfrak{U}, \mathfrak{F}'')$ is s, hence surjective.

(b) By proposition 9.6., we know that $\mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}' \to \mathfrak{F}'/\mathfrak{I}^m\mathfrak{F}'$ is surjective for $n \geq m$, hence $\{\mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}'\}$ satisfies (ML) condition and $\mathfrak{F}' = \varprojlim \mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}'$. Moreover, $\varprojlim \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}' = \varprojlim \mathfrak{F}\cdot\mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}' = \mathfrak{F}\cdot\mathfrak{F}' = \mathfrak{F}$. Then by proposition 9.1., we have an exact sequence $0 \to \varprojlim \Gamma(\mathfrak{U}, \mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}') \to \varprojlim \Gamma(\mathfrak{U}, \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}') \to \varprojlim \Gamma(\mathfrak{U}, \mathfrak{F}'') \to 0$, hence $0 \to \Gamma(\mathfrak{U}, \mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}) \to \Gamma(\mathfrak{U}, \mathfrak{F}'') \to 0$. Since $\mathfrak{X}$ is affine, we have $0 \to \Gamma(\mathfrak{X}, \mathfrak{F}') \to \Gamma(\mathfrak{X}, \mathfrak{F}) \to \Gamma(\mathfrak{X}, \mathfrak{F}'') \to 0$.

Solution 2.9.4. Since this problem is local, we may assume $\mathfrak{X} = \hat{X}$ is affine, where $X = \operatorname{Spec} A$ and $\hat{X}$ is the completion of X along $\tilde{I}$. (**Here we may need to add a condition that A is I-adic complete.**) Hence we have $0 \to \Gamma(\mathfrak{X}, \mathfrak{F}') \to \Gamma(\mathfrak{X}, \mathfrak{F}) \to \Gamma(\mathfrak{X}, \mathfrak{F}'') \to 0$ since $\mathfrak{F}'$ is coherent. We may assume that $\mathfrak{F}' = M'^{\triangle}$ and $\mathfrak{F}'' = M''^{\triangle}$, hence $0 \to \hat{M}' \to N \to \hat{M}'' \to 0$, where $N = \Gamma(\mathfrak{X}, \mathfrak{F})$. Since A is I-adic complete, we know that N is an A-module, $\hat{M}' = M'$, $\hat{M}'' = M''$, and the above exact sequence is an A-module sequence. Since on every affine piece $D(f)\cap\mathfrak{X}$ we have the same thing, and clearly have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'_f \arrow[r] & N' \arrow[r] & M''_f \arrow[r] & 0
\end{tikzcd}$$

We must have $N' = N_f$. Hence $\Gamma(\mathfrak{U}, \mathfrak{F}) = \Gamma(\mathfrak{U}, N^{\triangle})$ for every principal open set $\mathfrak{U}$. And principal open set forms a topological basis on $\mathfrak{X}$, we have $\mathfrak{F} = N^{\triangle}$, i.e. coherent.

Here we will explain why we assume A is I-adic complete. We know that every N, the extension of $\hat{M}'$ and $\hat{M}''$ is corresponding to an element of $\operatorname{Ext}^1_{\hat{A}}(\hat{M}'', \hat{M}')$. And since $\hat{A}$ is a flat A-module, we have $\operatorname{Ext}^1_{\hat{A}}(\hat{M}'', \hat{M}') \cong \operatorname{Ext}^1_A(M'', M')\hat{}$. If A is not I-adic complete, we know that the A-module homomorphism $\operatorname{Ext}^1_A(M'', M') \to \operatorname{Ext}^1_A(M'', M')\hat{}$ is not surjective, hence we can pick an extension N of $\hat{M}'$ and $\hat{M}''$, which is not induced by some A-module M, then $\mathfrak{F}$ is not coherent.

Solution 2.9.5. We may assume $\mathfrak{F}$ can be generated by global sections $\{s_i\}_i$. Define $\mathfrak{U}_i = \{x \in \mathfrak{X} \mid s_{i,x} \notin \mathfrak{m}_x\}$. Then $\mathfrak{X} = \bigcup U_i$. Since $\mathfrak{X}$ is noetherian, we can take a finite sub-covering $\{U_i\}_{i=1}^n$ such that $\mathfrak{X} = \bigcup_{i=1}^n U_i$, i.e. $\mathfrak{F}$ can be generated by $s_1, \ldots, s_n$.

Solution 2.9.6. (a) Denote the morphism $\Gamma(Y_m, \mathscr{O}_{Y_m}) \to \Gamma(Y_n, \mathscr{O}_{Y_n})$ as φ_{mn}, and the morphism $\Gamma(Y_m, \mathscr{O}^*_{Y_m}) \to \Gamma(Y_n, \mathscr{O}^*_{Y_n})$ as φ^*_{mn}. Since $\{\Gamma(Y_n, \mathscr{O}_{Y_n})\}$ satisfies (ML) condition, for every n, there exists an n_0 such that for all $m \geq n_0$, we have $\operatorname{Im}\varphi_{mn} = \operatorname{Im}\varphi_{n_0n}$. For every $s \in \Gamma(Y_m, \mathscr{O}^*_{Y_m})$, there exists an $t \in \Gamma(Y_m, \mathscr{O}^*_{Y_m})$ such that $st = 1$. Since $\operatorname{Im}\varphi_{mn} = \operatorname{Im}\varphi_{n_0n}$, there exists a' and b' in $\Gamma(Y_{n_0}, \mathscr{O}_{Y_{n_0}})$ such that $\varphi_{mn}(a) = \varphi_{n_0n}(a')$ and $\varphi_{mn}(b) = \varphi_{n_0n}(b')$. So $\varphi_{n_0n}(a'b') = 1$. Since clearly $\ker\varphi_{n_0n}$ is nilpotent, we can write $a'b' = 1 + \epsilon$ for some nilpotent element ϵ. Since $1+\epsilon$ is invertible, a' is also invertible, i.e. $a' \in \Gamma(Y_{n_0}, \mathscr{O}^*_{Y_{n_0}})$, which means $\operatorname{Im}\varphi^*_{mn} = \operatorname{Im}\varphi^*_{n_0n}$. So $\{\Gamma(Y_n, \mathscr{O}^*_{Y_n})\}$ satisfies (ML) condition.

(b) Since $\mathfrak{F}$ is coherent, for any point in $\mathfrak{X}$, and for every affine neighbourhood $\hat{X}$ of $\mathfrak{X}$ containing this point, where $X = \operatorname{Spec} A$ and $\hat{X}$ is the completion of X along $\tilde{I}$ for some $I \subset A$, such that $\mathfrak{F}|_{\hat{X}} = \varprojlim \mathscr{F}_n$ for some coherent sheaf $\mathscr{F}_n = \tilde{M}_n$, where M_n are A/I^n-module. Denote $\mathfrak{F}(\hat{X}) = \varprojlim M_n = M$. Then $(\mathfrak{F}/\mathfrak{I}^n\mathfrak{F})(\hat{X}) = M/\tilde{I}M = M_n$. Since $\mathfrak{F}/\mathfrak{I}^n\mathfrak{F} \cong \mathscr{O}_{Y_n}$, we know that $M_n = (\mathfrak{F}/\mathfrak{I}^n\mathfrak{F})(\hat{X}) = A/I^n$. Hence for any point in $\mathfrak{X}$, there exists a neighbourhood $\mathfrak{U}$ such that $\mathfrak{F}|_\mathfrak{U} \cong \mathscr{O}_\mathfrak{U}$. Since the point is ambiguous, $\mathfrak{F} \cong \mathscr{O}_\mathfrak{X}$. So for $\phi : \operatorname{Pic} \mathfrak{X} \to \varprojlim \operatorname{Pic} Y_n$, and $\mathfrak{F} \in \ker \phi$, we know that $\mathfrak{F}/\mathfrak{I}^n\mathfrak{F} \cong \mathscr{O}_{Y_n}$, hence $\mathfrak{F} \cong \mathscr{O}_\mathfrak{X}$. So ϕ is injective.

(c) Since $\mathscr{L}_n$ are invertible sheaf, we can define $\mathscr{M}_n = \mathscr{H}om_{Y_n}(\mathscr{L}_n, \mathscr{O}_{Y_n})$, by 2.5.7. we have $\mathscr{L}_n \otimes \mathscr{M}_n \cong \mathscr{O}_{Y_n}$. Clearly, $\mathscr{M}_{n+1} \otimes \mathscr{O}_n = \mathscr{H}om_{Y_{n+1}}(\mathscr{L}_{n+1}, \mathscr{O}_{Y_{n+1}}) \otimes \mathscr{O}_n = \mathscr{H}om_{Y_n}(\mathscr{L}_n, \mathscr{O}_{Y_n}) = \mathscr{M}_n$, i.e. $\{\mathscr{M}_n\}$ forms an invertible system. So we may define $\mathfrak{M} = \varprojlim \mathscr{M}_n$, hence $\mathfrak{M} \otimes \mathfrak{L} = \varprojlim \mathscr{M}_n \otimes \mathscr{L}_n = \varprojlim \mathscr{O}_{Y_n} = \mathscr{O}_\mathfrak{X}$. Clearly 2.5.7.(c) is correct for formal schemes, then $\mathscr{L}$ is locally free of rank 1. So obviously, $\operatorname{Pic} \mathfrak{X} \to \varprojlim \operatorname{Pic} Y_n$ is surjective.

(d) If $\mathfrak{X}$ is affine, we may assume $\mathfrak{X} = \hat{X}$ for some completion of $X = \operatorname{Spec} A$ along $\tilde{I}$. Then $\Gamma(Y_n, \mathscr{O}_{Y_n}) = A/I^n$, hence $\varphi_{mn} : A/I^m \to A/I^n$ is surjective. So $\{\Gamma(Y_n, \mathscr{O}_{Y_n})\}$ satisfies (ML) condition. If Y_n is projective over k, we know that $\Gamma(Y_n, \mathscr{O}_{Y_n}) = k$, and φ_{mn} is just identity on k. So $\{\Gamma(Y_n, \mathscr{O}_{Y_n})\}$ satisfies (ML) condition.

3 Cohomology

3.1 Derived Functors

Nothing.

3.2 Cohomology of Sheaves

Solution 3.2.1. (a) By 2.1.17., we have $0 \to \mathbb{Z}_U \to \mathbb{Z}_X \to i_P\mathbb{Z}_P \oplus i_Q\mathbb{Z}_Q \to 0$. So we have a long exact sequence $0 \to \Gamma(X, \mathbb{Z}_U) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H^1(X, \mathbb{Z}_U) \to \dots$. Since $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is not surjective, we have $H^1(X, \mathbb{Z}_U) \neq 0$.

(b) We can assume this "suitably general position" means $H_i = (x_i)$. Then we can take the Godement resolution of $\mathbb{Z}_Y$ as

$$0 \to \mathbb{Z}_Y \to \bigoplus_i \mathbb{Z}_{H_i} \to \bigoplus_{i<j} \mathbb{Z}_{H_i \cap H_j} \to \dots \to \mathbb{Z}_{H_0 \cap \dots \cap H_n} \to 0$$

So this is a flasque resolution. Then we can have $H^{n-1}(Y, \mathbb{Z}_Y) = \mathbb{Z}$ and $H^n(Y, \mathbb{Z}_Y) = 0$ when $n > 1$. Moreover, we also have $H^{n-1}(X, \mathbb{Z}_X) = H^n(X, \mathbb{Z}_X) = 0$ for $n > 1$ in next section. Since we have an exact sequence $0 \to \mathbb{Z}_U \to \mathbb{Z}_X \to \mathbb{Z}_Y \to 0$, we have $0 \to \mathbb{Z} \to H^n(X, \mathbb{Z}_U) \to 0$, i.e. $H^n(X, \mathbb{Z}_U) = \mathbb{Z}$.

Solution 3.2.2. Since every open subset of X is connected, the constant sheaf $\mathscr{K}$ is clearly flasque. For $\mathscr{K}/\mathscr{O}$, we know this is just $\bigoplus_{P \in X} i_P(I_P)$, i.e. the direct sum of skyscraper sheaves, hence flasque. So $0 \to \mathscr{O} \to \mathscr{K} \to \mathscr{K}/\mathscr{O} \to 0$ is a flasque resolution. By 2.1.21.(e), we have $H^1(X, \mathscr{O}) = 0$. For $n > 1$, we have the long exact sequence $\dots \to 0 \to H^n(X, \mathscr{O}) \to 0 \to \dots$, i.e. $H^n(X, \mathscr{O}) = 0$.

Solution 3.2.3 (Cohomology with Supports)**.** (a) For any short exact sequence of sheaves $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, we denote $\varphi : \Gamma_Y(X, \mathscr{F}') \to \Gamma_Y(X, \mathscr{F})$ and $\psi : \Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{F}'')$. Since φ and ψ is restricted from the morphism $\varphi' : \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F})$ and $\psi' : \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'')$, we have $\psi \circ \varphi = 0$, and φ is injective. For any $s \in \ker \psi$, there exists $t \in \Gamma(X, \mathscr{F}')$ such that $s = \varphi'(t)$. Checking the stalk, we have $t_P = 0$ for $P \in X - Y$, since $s_P = 0$. So $t \in \Gamma_Y(X, \mathscr{F}')$, i.e. we have the exact sequence $0 \to \Gamma_Y(X, \mathscr{F}') \to \Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{F}'')$.

(b) The left exactness is from (a). For any $s \in \Gamma_Y(X, \mathscr{F}'') \subset \Gamma(X, \mathscr{F}'')$, there exists a $t \in \Gamma(X, \mathscr{F})$ such that $s = \psi'(t)$. Checking the stalk, we have $t_P \mapsto s_P = 0$ for $P \in X - Y$. Hence from the exact sequence at stalk $0 \to \mathscr{F}'_P \to \mathscr{F}_P \to \mathscr{F}''_P \to 0$, there exists $r_P \in \mathscr{F}'_P$ such that $\varphi'_P(r_P) = t_P$. The morphism $r_P \mapsto t_P$ is induced from some $r_i \in \Gamma(U_i, \mathscr{F}')$ such that $r_i \mapsto t|_{U_i}$ for some neighbourhood U_i of P. So we have an open covering $\{U_i\}$ of $X - Y$. Since $r_i|_{U_i \cap U_j} - r_j|_{U_i \cap U_j} \mapsto t|_{U_i \cap U_j} - t|_{U_i \cap U_j} = 0$, we have $r_i|_{U_i \cap U_j} = r_j|_{U_i \cap U_j}$ because $\Gamma(U_i \cap U_j, \cdot)$ is left exact. So all r_i can be glued together to be an $r \in \Gamma(U, \mathscr{F}')$. Since $\mathscr{F}'$ is flasque, there exists an $r' \in \Gamma(X, \mathscr{F}')$ such that $r'|_U = r$. Then $\psi'(t - \varphi'(r')) = s$, and $(t - \varphi'(r'))|_P = t|_P - \varphi'_P(r'|_p) = 0$. So $t - \varphi'(r') \in \Gamma_Y(X, \mathscr{F})$ is a preimage of s, i.e. ψ is surjective.

(c) By lemma 2.4., there exists injective sheaf $\mathscr{I}$, such that $\mathscr{F} \hookrightarrow \mathscr{I}$, and we may define $\mathscr{G} = \mathscr{I}/\mathscr{F}$. Since $\mathscr{F}$ and $\mathscr{I}$ are both flasque, by 2.1.16.(c) we know that $\mathscr{G}$ is flasque. By (b), we know that $H^1_Y(X, \mathscr{F}) = H^1_Y(X, \mathscr{G}) = 0$. And since $H^n_Y(X, \mathscr{I}) = 0$ for all n, we have $H^n(X, \mathscr{F}) \cong H^{n-1}(X, \mathscr{G})$. So by induction, $H^n(X, \mathscr{F}) = 0$.

(d) By 2.1.20.(b), since $\mathscr{F}$ is flasque, we have $0 \to \mathscr{H}^0_Y(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_{X-Y}) \to 0$. So we have $0 \to \Gamma_Y(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}) \to \Gamma(X - Y, \mathscr{F})$. Since $\mathscr{F}$ is flasque, the morphism $\Gamma(X, \mathscr{F}) \to \Gamma(X - Y, \mathscr{F})$ is surjective, then we've done.

(e) For any injective sheaf $\mathscr{I}$, by lemma 2.4. and (d), we have $0 \to \Gamma_Y(X, \mathscr{I}) \to \Gamma(X, \mathscr{I}) \to \Gamma(U, \mathscr{I}|_U) \to 0$. Take an injective resolution $\mathscr{I}^\cdot$ of $\mathscr{F}$, since $\cdot|_U$ preserves injection, $\mathscr{I}^\cdot|_U$ is an injective resolution of $\mathscr{F}|_U$. So we have the following commutative diagram with exact rows

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \Gamma_Y(X,\mathscr{I}^k) \arrow[r] \arrow[d] & \Gamma(X,\mathscr{I}^k) \arrow[r] \arrow[d] & \Gamma(U,\mathscr{I}^{k+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma_Y(X,\mathscr{I}^{k+1}) \arrow[r] \arrow[d] & \Gamma(X,\mathscr{I}^k) \arrow[r] \arrow[d] & \Gamma(U,\mathscr{I}^{k+1}) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

So the long exact sequence is by using snack lemma of the above diagram.

(f) We just need show that we have an isomorphism of functors $\Gamma_Y(X,\cdot) \to \Gamma_Y(V,\cdot|_V)$. For any U open in X, we can treat $\mathscr{F}(U)$ as a set of continuous morphisms $U \to \text{Spé}\ (\mathscr{F})$ by 2.1.13. So any $s \in \Gamma_Y(X,\mathscr{F})$ means a continuous morphism $s : X \to \text{Spé}\ (\mathscr{F})$ with $s(P) = 0 \subset \mathscr{F}_P$ for all $P \in X - Y$, hence $\Gamma_Y(X,\mathscr{F}) \to \Gamma_Y(V,\mathscr{F}|_V)$ is injective. Moreover, for any $s \in \Gamma_Y(V,\mathscr{F}|_V)$, then we can define a morphism $t : X \to \text{Spé}\ (\mathscr{F})$ as $t(P) = s(P)$ if $P \in Y$, and $t(P) = 0 \subset \mathscr{F}_P$ if $P \in X - Y$. We can take an open covering $\{U_i\}$ of X such that for each i, we have either $U_i \subset V$ or $U_i \cap Y = \varnothing$. So if $U_i \subset V$, we know that $t(U_i) = s(U_i)$, hence continuous, if $U_i \cap Y = \varnothing$, $t(U_i) = 0$, hence continuous. So t is a continuous morphism, hence $t \in \Gamma_Y(X,\mathscr{F})$. So $\Gamma_Y(X,\cdot) \cong \Gamma_Y(V,\cdot|_V)$. For every sheaf $\mathscr{F}$ with injective resolution $\mathscr{I}^\cdot$, we have an injective resolution $\mathscr{I}^\cdot|_V$ of $\mathscr{F}|_V$. Hence the isomorphism above induces the isomorphism of cohomology group $H^i_Y(X,\mathscr{F}) \cong H^i_Y(V,\mathscr{F}|_V)$.

Solution 3.2.4 (Mayer-Vietoris Sequence)**.** For every injective sheaf $\mathscr{I}$, we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma_{Y_1\cap Y_2}(X,\mathscr{I}) \arrow[r] \arrow[d] & \Gamma_{Y_1}(X,\mathscr{I}) \oplus \Gamma_{Y_2}(X,\mathscr{I}) \arrow[r] \arrow[d] & \Gamma_{Y_1\cup Y_2}(X,\mathscr{I}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X,\mathscr{I}) \arrow[r] \arrow[d] & \Gamma(X,\mathscr{I}) \oplus \Gamma(X,\mathscr{I}) \arrow[r] \arrow[d] & \Gamma(X,\mathscr{I}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X - Y_1\cap Y_2,\mathscr{I}) \arrow[r] \arrow[d] & \Gamma(X - Y_1,\mathscr{I}) \oplus \Gamma(X - Y_2,\mathscr{I}) \arrow[r] \arrow[d] & \Gamma(X - Y_1\cap Y_2,\mathscr{I}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The last two rows are clearly exact, and three columns are exact since $\mathscr{I}$ is flasque, so by nine lemma we know the first row is exact. Hence for injective resolution $\mathscr{I}^\cdot$ of $\mathscr{F}$, we have an exact sequence of complexes $0 \to \Gamma_{Y_1\cap Y_2}(X,\mathscr{I}^\cdot) \to \Gamma_{Y_1}(X,\mathscr{I}^\cdot) \oplus \Gamma_{Y_2}(X,\mathscr{I}^\cdot) \to \Gamma_{Y_1\cup Y_2}(X,\mathscr{I}^\cdot) \to 0$. Taking the long exact sequence of this, we have $\ldots \to H^i_{Y_1\cap Y_2}(X,\mathscr{F}) \to H^i_{Y_1}(X,\mathscr{F}) \oplus H^i_{Y_2}(X,\mathscr{F}) \to H^i_{Y_1\cup Y_2}(X,\mathscr{F}) \to H^{i+1}_{Y_1\cap Y_2}(X,\mathscr{F}) \to \ldots$.

Solution 3.2.5. We first show that $\Gamma_P(X,\mathscr{F}) \cong \Gamma_P(X_P,\mathscr{F}_P)$. Since $\Gamma(X_P,\mathscr{F}_P) = \varinjlim_{P\in U} \Gamma(U,\mathscr{F}|_U) = \mathscr{F}_P$. So we have a morphism $\Gamma(X,\mathscr{F}) \to \mathscr{F}_P \cong \Gamma(X_P,\mathscr{F}_P)$, hence it induces a morphism $\Gamma_P(X,\mathscr{F}) \to \Gamma_P(X_P,\mathscr{F}_P)$. For injectivity, if $s,t \in \Gamma_P(X,\mathscr{F})$ with $s_P = t_P$, since for every $Q \in X$ with $Q \neq P$, we have $s_Q = t_Q = 0$, hence $s = t$.

For surjectivity, for any $s_P \in \Gamma_P(X_P, \mathscr{F}_P) \cong \mathscr{F}_P$, there exists a neighbourhood U of P and a $s \in \Gamma(U, \mathscr{F})$ such that $s_P = s|_P$. We may shrink U such that $s_Q = 0$ for any $Q \in U$ such that $P \notin \overline{\{Q\}}$, then we have a section $t \in \Gamma_P(X, \mathscr{F})$ as the extension of s with 0 outside, hence $t|_P = s_P$, i.e. $\Gamma_P(X, \mathscr{F}) \to \Gamma_P(X_P, \mathscr{F}_P)$ is surjective. So $\Gamma_P(X, \mathscr{F}) \cong \Gamma_P(X_P, \mathscr{F}_P)$. For higher cohomology, if we have some sheaf $\mathscr{F}$ with injective resolution $\mathscr{I}^{\cdot}$, there exists an injective resolution of $\mathscr{F}|_{X_P}$ as $\mathscr{I}^{\cdot}|_{X_P}$ like we've done in 3.2.3.(f), hence $H^i_P(X, \mathscr{F}) = H^i_P(X_P, \mathscr{F}_P)$.

Solution 3.2.6. Followed by the hint, we firstly prove that $\mathscr{I}$ is injective iff for any U open in X, and for any sheaf $\mathscr{R} \subset \mathbb{Z}_U$, the morphism $\mathscr{R} \to \mathscr{I}$ can be extended to be a morphism $\mathbb{Z}_U \to \mathscr{I}$. The only if part is trvial. Conversely, for any sheaves $\mathscr{F} \subset \mathscr{G}$ with morphism $\phi : \mathscr{F} \to \mathscr{I}$, we can define $\mathcal{S} = \{\mathscr{H} \mid \mathscr{F} \subset \mathscr{H} \subset \mathscr{G}$ such that *phi* can be extended to a morphism $\mathscr{H} \to \mathscr{I}\}$. Then $\mathcal{S}$ is a ordered set. By Zorn's lemma, there exists a maximal element of $\mathcal{S}$ as $\mathscr{H}$. If $\mathscr{H} \subsetneq \mathscr{G}$, there exists an open set $U \subset X$, and $s \in \mathscr{G}(U) - \mathscr{H}(U)$. Then we can define $\mathbb{Z}_U$ as a subsheaf of $\mathscr{G}$ generated by s, and $\mathscr{R} = \mathscr{Z}_U \cap \mathscr{F}$. So the morphism $\mathscr{R} \to \mathscr{I}$ can be extended to $\mathbb{Z}_U \to \mathscr{I}$, which contradict with the maximality of $\mathscr{H}$, hence $\mathscr{H} = \mathscr{G}$, i.e. $\mathscr{I}$ is injective.

Secondly we prove that for any finitely generated sheaf $\mathscr{R}$ with morphism $\varphi : \mathscr{R} \to \varinjlim \mathscr{I}_\alpha$, this morphism must factor through some $\mathscr{I}_\alpha$. We may assume $\mathscr{R}$ is generated by some $s_i \in \mathscr{R}(U_i)$. Then there exists $t_i \in \mathscr{I}_{\alpha_i}(U_i)$ representing the image of s_i. Since A is a direct system, there exists a α greater than all α_i, so we may choose all $t_i \in \mathscr{I}_\alpha(U_i)$. Hence φ must factor through $\mathscr{I}_\alpha$.

Finally, any subsheaf $\mathscr{R}$ of $\mathbb{Z}_U$ must be finitely generated, since any morphism $\varphi : \mathscr{R} \to \varinjlim \mathscr{I}_\alpha$ must factor through $\mathscr{I}_\alpha$, then $\mathscr{R} \to \mathscr{I}_\alpha$ can be extended to $\mathbb{Z}_U \to \mathscr{I}_\alpha$, hence φ can be extended to $\mathbb{Z}_U \to \varinjlim \mathscr{I}_\alpha$. So $\varinjlim \mathscr{I}_\alpha$ is injective.

Solution 3.2.7. (a) By definition, the constant sheaf $\mathcal{Z}$ is the sheaf $\mathcal{Z}(U) = \{s : U \to \mathbb{Z} \mid s \text{ locally constant}\}$. Then we may fix a $P \in \mathbb{S}^1$, we can define $Y = \mathbb{S}^1 - \{P\}$, and $\mathscr{Z}$ as $\mathscr{Z}(U) = \mathbb{Z}(U \cap Y)$. Then $\mathscr{Z}$ is a sheaf, with $\mathscr{Z}_Q \cong \mathbb{Z}$ for all $Q \neq P$, and $\mathscr{Z}_P \cong \mathbb{Z} \oplus \mathbb{Z}$. Hence we the injection $\mathcal{Z} \to \mathscr{Z}$ has cokernal $i_P(\mathbb{Z})$, i.e. $0 \to \mathcal{Z} \to \mathscr{Z} \to i_P(\mathbb{Z}) \to 0$. So we have $0 \to H^0(\mathbb{S}^1, \mathcal{Z}) \to H^0(\mathbb{S}^1, \mathscr{Z}) \to H^0(\mathbb{S}^1, i_P(\mathbb{Z})) \to H^1(\mathbb{S}^1, \mathcal{Z}) \to H^1(\mathbb{S}^1, \mathscr{Z}) \to \ldots$. Since we clearly have $H^k(\mathbb{S}^1, \mathscr{Z}) \cong H^k(I, \mathcal{Z}^I)$, where $I = (0, 1)$ and $\mathcal{Z}^I$ is the constant sheaf on I of $\mathbb{Z}$, and I is contractible, we have $H^1(I, \mathcal{Z}^I) = H^1(\mathrm{pt}, \mathcal{Z}^{\mathrm{pt}}) = 0$, so we have $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to H^1(\mathbb{S}^1, \mathcal{Z}) \to 0$. Since the second arrow is isomorphic, we have $H^1(\mathbb{S}^1, \mathcal{Z}) = \mathbb{Z}$.

(b) Define $\mathscr{D}$ as $\mathscr{D}(U) = \{s : U \to \mathbb{R}\}$. $\mathscr{D}$ is clearly flasque, hence $H^1(\mathbb{S}^1, \mathscr{D}) = 0$, i.e. $0 \to H^0(\mathbb{S}^1, \mathscr{R}) \to H^0(\mathbb{S}^1, \mathscr{D}) \to H^0(\mathbb{S}^1, \mathscr{D}/\mathscr{R}) \to H^1(\mathbb{S}^1, \mathscr{R}) \to 0$. So we just need to prove that $\Gamma(\mathbb{S}^1, \mathscr{D}) \to \Gamma(\mathbb{S}^1, \mathscr{D}/\mathscr{R})$ is surjective. Since every element in $\Gamma(\mathbb{S}^1, \mathscr{D}/\mathscr{R})$ can be described as $s = \sum_{i=1}^\infty (s_i, U_i)$ for some covering $\mathbb{S}^1 = \bigcup U_i$ with $s_i \in \mathscr{D}(U_i)$, and for any $U_i \cap U_j \neq \varnothing$, we have $s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j} \in \mathscr{R}(U_i \cap U_j)$, where the sum is finite sum because $\mathbb{S}^1$ is compact. For $s = \sum_i (s_i, U_i)$, we may shrink U_i and assume $(\overline{U_i \cap U_{i+1}}) \cap (\overline{U_{i+1} \cap U_{i+2}}) = \varnothing$. So we can define $r_i = s_{i+1}|_{U_i \cap U_{i+1}} - s_i|_{U_i \cap U_{i+1}} \in \mathscr{R}(U_i \cap U_{i+1})$. Then there exists a continuous function r_i' such that $r_i'|_{U_i \cap U_{i+1}} = r_i|_{U_i \cap U_{i+1}}$, and $r_i'|_{U_{i-1} \cap U_i} = 0$. Then we can define a $t \in \Gamma(\mathbb{S}^1, \mathscr{D})$ such that $t(P) = s_i(P) + r_i'(P)$ if $P \in U_i$. So t is a preimage of s, i.e. $\Gamma(\mathbb{S}^1, \mathscr{D}) \to \Gamma(\mathbb{S}^1, \mathscr{D}/\mathscr{R})$ is surjective.

3.3 Cohomology of Noetherian Affine Scheme

Solution 3.3.1. ($\Rightarrow$) By definition, trivial. ($\Leftarrow$) Define $\mathscr{N}$ as the nilpotent sheaf of X, then $\mathscr{N}^d = 0$ for sufficiently large d since X is noetherian. So for any coherent sheaf $\mathscr{F}$ on X, we define $\mathscr{G}_d = \mathscr{N}^d \cdot \mathscr{F} / \mathscr{N}^{d+1} \cdot \mathscr{F}$. Since X and X_{red} have the same underlying topological space, and $\mathscr{O}_{X_{\mathrm{red}}} = \mathscr{O}_X / \mathscr{N}$, so $\mathscr{G}_d$ is a coherent $\mathscr{O}_{X_{\mathrm{red}}}$-module when $d \geq 0$. By theorem 3.5., we have $H^n(X, \mathscr{G}_d) = H^n(X_{\mathrm{red}}, \mathscr{G}_d) = 0$. By definition we clearly have $0 \to \mathscr{N}^{d+1}\mathscr{F} \to \mathscr{N}^d\mathscr{F} \to \mathscr{G}_d \to 0$, hence $H^1(X, \mathscr{N}^{d+1}\mathscr{F}) \to H^1(X, \mathscr{N}^d\mathscr{F})$ is surjective. Since $H^1(X, \mathscr{N}^d\mathscr{F}) = 0$ for sufficiently large d, hence by induction, we know that $H^1(X, \mathscr{F}) = 0$. So by theorem 3.7., X is affine.

Solution 3.3.2. ($\Rightarrow$) Since every irreducible component is closed in X, hence affine by 2.3.11.(b). ($\Leftarrow$) For any sheaf of ideal $\mathscr{I}$ on X corresponding to closed subset Z, then for any irreducible component $Y \subset X$, we have an exact sequence $0 \to \mathscr{I}_{Y\cup Z} \to \mathscr{I}_Z \to \mathscr{F} \to 0$, where $\mathscr{F} = i_*\mathscr{I}_{Y\cap Z}$, and $i : Y \to X$ is the closed embedding. Since $H^1(X, \mathscr{F}) = H^1(Y, \mathscr{I}_{Y\cap Z}) = 0$, where the second equality is because Y is affine. So we have surjective morphism $H^1(X, \mathscr{I}_{Y\cup Z}) \twoheadrightarrow H^1(X, \mathscr{I})$. Then we can have a series of surjections $H^1(X, \mathscr{I}_{Z\cup Y_1\cup\ldots\cup Y_n}) \twoheadrightarrow \ldots \twoheadrightarrow H^1(X, \mathscr{I}_{Z\cup Y_1}) \twoheadrightarrow H^1(X, \mathscr{I})$. Since X is reduced, we have $H^1(X, \mathscr{I}_X) = 0$. But we clearly have $Z\cup Y_1 \cup \ldots \cup Y_n = X$, we know that $0 = H^1(X, \mathscr{I}_X) \twoheadrightarrow H^1(X, \mathscr{I})$, hence $H^1(X, \mathscr{I}) = 0$. So X is affine.

Solution 3.3.3. (a) Suppose we have exact sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$. If $m \in \ker(\Gamma_\mathfrak{a}(M') \to \Gamma_\mathfrak{a}(M))$, m is clearly in $\ker(M' \to M)$, hence $m = 0$. If $m \in \ker(\Gamma_\mathfrak{a}(M) \to \Gamma_\mathfrak{a}(M''))$, similarly $m \in \ker(M \to M'')$, hence there exists $m' \in M'$ such that $f(m') = m$. Since for some large n, we have $\mathfrak{a}^n m = 0$, hence $f(\mathfrak{a}^n m') = \mathfrak{a}^n m = 0$. Since f is injective, we have $\mathfrak{a}^n m' = 0$, i.e. $m' \in \Gamma_\mathfrak{a}(M')$. So we have $0 \to \Gamma_\mathfrak{a}(M') \to \Gamma_\mathfrak{a}(M) \to \Gamma_\mathfrak{a}(M'')$.

(b) For any M with injective resolution $0 \to M \to I^\cdot$, we have a flasque resolution $0 \to \tilde{M} \to \tilde{I}^\cdot$ of $\tilde{M}$. So we only need to prove that $\Gamma_\mathfrak{a}(M) = \Gamma_Y(X, \tilde{M})$. For any $m \in \Gamma_\mathfrak{a}(M)$, there exists a big n such that $\mathfrak{a}^n m = 0$. Then for any $\mathfrak{p} \notin Y$, i.e. there exists some $a \in \mathfrak{a}$ but $a \notin \mathfrak{p}$. Then a^n is invertible in $M_\mathfrak{p}$, hence $m = \frac{a^n m}{a^n} = 0$ in $M_\mathfrak{p}$, i.e. $m \in \Gamma_Y(X, \tilde{M})$. Conversely, if $s \in \Gamma_Y(X, \tilde{M})$, then Supp $(s) \subset V(\mathfrak{a})$. By 2.5.6.(a), we know that Supp $(s) = V(\text{Ann }(s))$, so $\mathfrak{a} \subset \sqrt{\text{Ann }(s)}$. We may denote $\mathfrak{a} = (f_1, \ldots, f_n)$, there exists k_i such that $f_i^{k_i} \in \text{Ann }(s)$. So we may fix an $N > \sum k_i$, we know that $\mathfrak{a}^N \subset \text{Ann }(s)$, hence $\mathfrak{a}^N s = 0$, i.e. $s \in \Gamma_\mathfrak{a}(M)$.

(c) Clearly $\Gamma_\mathfrak{a}(H^i_\mathfrak{a}(M)) \subset H^i_\mathfrak{a}(M)$. Conversely, fix an injective resolution $0 \to M \to I^\cdot$ of M. Then we have $0 \to \Gamma_\mathfrak{a}(M) \to \Gamma_\mathfrak{a}(I^\cdot)$. So $H^i_\mathfrak{a}(M) = \ker d^{i+1}/\text{Im} d^i$, hence every element can be represented as some $m \in \ker d^{i+1}$. So $H^i_\mathfrak{a}(M) \subset \Gamma_\mathfrak{a}(H^i_\mathfrak{a}(M))$.

Solution 3.3.4 (Cohomological Interpretation of Depth)**.** (a) If $\text{depth}_\mathfrak{a} M \geq 1$, there exists at least an $x \in \mathfrak{a}$ is not a zero divisor of M. If $m \in \Gamma_\mathfrak{a}(M)$, then $\mathfrak{a}^n m = 0$ for sufficiently large n, hence $x^n m = 0$. So we only have $m = 0$, i.e. $\Gamma_\mathfrak{a}(M) = 0$. Conversely, if $\Gamma_\mathfrak{a}(M) = 0$ and M is finitely generated, for any $m \in M$, there exists an $x \in \mathfrak{a}$ such that $x^n m \neq 0$ for any $n \geq 0$. So $\mathfrak{a} \not\subseteq \text{Ann }(m)$, then $\mathfrak{a} \not\subseteq \bigcup \text{Ann }(m)$. Hence $\mathfrak{a} \not\subseteq \text{Ann }(M)$, i.e. $\text{depth}_\mathfrak{a}(M) \neq 0$.

(b) We prove this by induction. The case $n = 0$ is just (a). Suppose $\text{depth}_\mathfrak{a}(M) \geq n$ is equivalent to $H^i_\mathfrak{a}(M) = 0$ for all $i < n$. (i⇒ii) If M is an A-module such that $\text{depth}_\mathfrak{a}(M) \geq n + 1$, there exists some $x \in \mathfrak{a}$ such that $\text{depth}_\mathfrak{a}(M/xM) \geq n$. Since we have an exact sequence $0 \to M \xrightarrow{\times x} M \to M/xM \to 0$, we have $\ldots \to H^{n-1}_\mathfrak{a}(M/xM) \to H^n_\mathfrak{a}(M) \xrightarrow{\times x} H^n_\mathfrak{a}(M) \to \ldots$. By induction, we have $H^n_\mathfrak{a}(M) \xrightarrow{\times x} H^n_\mathfrak{a}(M)$ is injective. Hence by 3.3.3.(c), we have $H^n_\mathfrak{a}(M) = 0$. (ii⇒i) Since $H^i_\mathfrak{a}(M) = 0$ for $i < n + 1$, by same exact sequence we have $H^i_\mathfrak{a}(M/xM) = 0$ for $i < n$. Hence $\text{depth}_\mathfrak{a}(M/xM) \geq n$, i.e. $\text{depth}_\mathfrak{a}(M) \geq n + 1$.

Solution 3.3.5. Since this problem is local, we may assume $U = \text{Spec } A$ is affine, and P is corresponding to $\mathfrak{p} \subset A$. In this question, we will prove there two conditions are equivalent to (iii) $\text{depth}_\mathfrak{p} A \geq 2$.

(i⇒iii) Since $\mathscr{O}_P = A_\mathfrak{p}$, if $(\frac{x_1}{y_1}, \frac{x_2}{y_2})$ is a regular sequence of $A_\mathfrak{p}$, then (x_1, x_2) is a regular sequence of A and $x_1, x_2 \notin \mathfrak{p}$.

(iii⇒i) If (x_1, x_2) is a regular sequence of A with $x_1, x_2 \notin \mathfrak{p}$, then $(\frac{x_1}{1}, \frac{x_2}{1})$ is a regular sequence of $A_\mathfrak{p}$.

(ii⇒iii) By 3.2.3.(e) we have $0 \to H^0_P(X, \mathscr{O}_X) \to H^0(U, \mathscr{O}_X) \to H^0(U - P, \mathscr{O}_X) \to H^1_P(X, \mathscr{O}_X) \to H^1(U, \mathscr{O}_X) \to \ldots$. Since U is affine, we have $H^1(U, \mathscr{O}_X) = 0$. If any section of $\mathscr{O}_X$ over $U - P$ can be extended uniquely to a section over U, we have $H^0(U, \mathscr{O}_X) \cong H^0(U - P, \mathscr{O}_X)$. So we have $H^0_P(X, \mathscr{O}_X) = H^1_P(X, \mathscr{O}_X) = 0$, i.e. $H^0_\mathfrak{p}(A) = H^1_\mathfrak{p}(A) = 0$. Then by 3.3.4.(b), we know that $\text{depth}_\mathfrak{p} A \geq 2$.

(iii⇒ii) If $\text{depth}_\mathfrak{p} A \geq 2$, we know that $H^0_P(X, \mathscr{O}_X) = H^1_P(X, \mathscr{O}_X) = 0$, hence $H^0(U, \mathscr{O}_X) \cong H^0(U - P, \mathscr{O}_X)$ for same reason.

Solution 3.3.6. (a) For every injective morphism $\mathscr{F} \to \mathscr{F}'$ of quasi-coherent sheaves on X, and morphism $\mathscr{F} \to \mathscr{G}$, the morphism $\mathscr{F} \to f_{i*}(\tilde{I}_i)$ corresponds to $\mathscr{F}|_{U_i} \to \tilde{I}_i$. Since $\tilde{I}_i$ is injective, we have a morphism $\mathscr{F}'|_{U_i} \to \tilde{I}_i$, which corresponds to $\mathscr{F}' \to f_{i*}(\tilde{I}_i)$. So the morphism $\mathscr{F} \to \mathscr{G}$ can be extended to a morphism $\mathscr{F}' \to \mathscr{G}$, i.e. $\mathscr{G}$ is injective.

(b) If $\mathscr{F}$ and $\mathscr{G}$ are two quasi-coherent sheaves on U with an injection $\mathscr{F} \to \mathscr{G}$, then $i_*\mathscr{F} \to i_*\mathscr{G}$ is also injective, and those are both quasi-coherent, where $i : U \to X$ is the embedding map. Since $\mathscr{F} \to i^{-1}\mathscr{I}$ induces a map $i_*\mathscr{F} \to \mathscr{I}$. So it induces a map $i_*\mathscr{G} \to \mathscr{I}$, which is corresponding to the morphism $\mathscr{G} \to i^{-1}\mathscr{I}$. Hence $\mathscr{I}|_U$ is injective in $\mathfrak{Qco}(U)$. For any open subset V in X, we want to prove that $\mathscr{I}(X) \to \mathscr{I}(V)$ is surjective. Suppose X has a finie affine open covering $\bigcup_i U_i$ for $U_i = \operatorname{Spec} A_i$. For any $s \in \mathscr{I}(V)$, it induces an $s_i \in \mathscr{I}(V \cap U_i)$. Since $\mathscr{I}|_{U_i}$ is injective in $\mathfrak{Qco}(U_i)$, I_i is injective in $\mathfrak{Mod}(A_i)$, where $\mathscr{I}|_{U_i} = \tilde{I}_i$. Then $\mathscr{I}|_{U_i}$ is injective in $\mathfrak{Ab}(U_i)$, so $\mathscr{I}(U_i) \to \mathscr{I}(V \cap U_i)$ is surjective, i.e. s_i has a preimage t_i. Since t_i's are clearly compatible, there exists a $t \in \mathscr{I}(X)$, which is the preimage of s. Hence $\mathscr{I}$ is surjective.

(c) By (b), any injective resolution in $\mathfrak{Qco}(X)$ is a flasque resolution, hence trivial.

Solution 3.3.7. (a) Suppose $\mathfrak{a} = (f_1, \ldots, f_r)$, then $U = \bigcup D(f_i)$. For any element $s \in \operatorname{Hom}_A(\mathfrak{a}^n, M)$, we have $\frac{s(f_i^n)}{f_i^n} \in M_{f_i} = \Gamma(D(f_i), \tilde{M})$. Since $\frac{s(f_i^n)}{f_i^n}$ and $\frac{s(f_j^n)}{f_j^n}$ are same in $M_{f_if_j}$, the map $\operatorname{Hom}_A(\mathfrak{a}^n, M) \to \prod_i \Gamma(D(f_i), \tilde{M})$, $s \mapsto (\frac{s(f_1^n)}{f_1^n}, \ldots, \frac{s(f_r^n)}{f_r^n})$ induces a morphism $\operatorname{Hom}_A(\mathfrak{a}^n, M) \to \Gamma(U, \tilde{M})$. So we can take a direct limit and get a morphism $\phi : \varinjlim \operatorname{Hom}_A(\mathfrak{a}^n, M) \to \Gamma(U, \tilde{M})$. If $\phi(s) = 0$ for some $s = (s_n) \in \varinjlim \operatorname{Hom}_A(\mathfrak{a}^n, M)$, for sufficiently large n we have $(\frac{s_n(f_1^n)}{f_1^n}, \ldots, \frac{s_n(f_r^n)}{f_r^n}) = 0$. Then $f_i^{k_i} s_n(f_i^n) = 0$ for some k_i. Denote $k = \max\{k_i\}$. Then so for $N > r(k+n)$, we have $s_k(f_i^k) = 0$ for all i, hence $s_N(\mathfrak{a}) = 0$. So ϕ is injective. For any $t \in \Gamma(U, \tilde{M})$, we may denote the restriction of t in $D(f_i)$ as $\frac{t_i}{f_i^{k_i}}$. Denote $k = \max\{k_i\}$, we can replace $\frac{f_i^{k-k_i} t_i}{f_i^k}$, hence t can be represent by $(\frac{t_1}{f_1^k}, \ldots, \frac{t_r}{f_r^k})$. Since clearly $\frac{t_i}{f_i^k} = \frac{t_j}{f_j^k}$ in $M_{f_if_j}$, there exists l_{ij} such that $(f_if_j)^{l_{ij}}(t_if_j^k - t_jf_i^k) = 0$. Denoting $l = \max\{l_{ij}\}$, we may replace $\frac{f_i^l t_i}{f_i^{k+l}}$ by $\frac{t_i}{f_i^n}$. Then t can be represented by $(\frac{t_1}{f_1^k}, \ldots, \frac{t_r}{f_r^k})$ such that $t_if_j^n = t_jf_i^n$. Since if $N > rn$, $\mathfrak{a}^n$ is generated by f_i^n, then we can define $s : \mathfrak{a}^N \to M$ as $s(f_i^n) = t_i$, hence ϕ is surjective.

(b) For any open subset U in X, we can take $X - U = V(\mathfrak{a})$. By (a) we just need to prove $\varinjlim \operatorname{Hom}_A(A^n, I) \to \varinjlim \operatorname{Hom}_A(\mathfrak{a}^n, I)$ is surjective. Since I is surjective, any morphism $s : \mathfrak{a}^n \to I$ can be extended to a morphism $t : A^n \to I$, hence the morphism $\operatorname{Hom}_A(A^n, I) \to \operatorname{Hom}_A(\mathfrak{a}^n, I)$ is surjective. Take a direct limit and we know that the restriction map is surjective, hence $\tilde{I}$ is flasque.

Solution 3.3.8. If $I \to I_{x_0}$ is surjective, the element $\frac{1}{x_0} \in I_{x_0}$ must have a preimage $m \in I$, hence there exists some n such that $x_0^n(x_0m - 1) = 0$. Then $0 = x_{n+1}x_0^n(x_0m - 1) = x_{n+1}x_0^{n+1}m - x_{n+1}x_0^n$. Since $x_{n+1}x_0^{n+1} = 0$, we have $x_{n+1}x_0^n = 0$, which makes a contradiction.

3.4 Čech Cohomology

Solution 3.4.1. Take an affine covering $\{V_i\}$ of Y. Since Y is separated, $V_{i_0,\ldots,i_n}$ is also affine. And since f is affine, $\{U_i = f^{-1}(V_i)\}$ is an affine covering of X. So if $\mathscr{F}|_{V_{i_0,\ldots,i_n}} = \tilde{M}$, we have $\mathscr{F}(V_{i_0,\ldots,i_n}) = M$ and $f_*\mathscr{F}(U_{i_0,\ldots,i_n}) = M$. Hence the Čech complex of $\mathscr{F}$ and $f_*\mathscr{F}$ are the same, i.e. $\check{H}^i(X, \mathscr{F}) \cong \check{H}^i(Y, f_*\mathscr{F})$. Then by theorem 4.5., $H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$.

Solution 3.4.2. (a) Denote function fields of X and Y as K_X and K_Y. Since f is finite, we know that K_X is a finite generated K_Y-module with generators $e_1, \ldots, e_r$. We may assume $e_j \in \mathscr{O}_{X,\eta_X}$ can be represented by $s_j \in \Gamma(U_j, \mathscr{O}_X)$. If $i_j : U_j \to X$ is the canonical embedding, we can define $\mathscr{M} = \bigoplus i_{j,*}(s_j\mathscr{O}_{U_j})$. Hence we have a canonical morphism $\alpha : \mathscr{O}_Y^r \to f_*\mathscr{M}$, and $(f_*\mathscr{M})_{\eta_Y} = K_Y[e_1, \ldots, e_r] = K_X$.

(b) Just take $\mathscr{H}om(\cdot, \mathscr{F})$ to α to get a morphism $\beta : \mathscr{H}om(f_*\mathscr{M}, \mathscr{F}) \to \mathscr{H}om(\mathscr{O}_Y^r, \mathscr{F})$ and it is isomorphic at the generic point of Y. Since clearly $\mathscr{H}om(\mathscr{O}_Y^r, \mathscr{F}) \cong \mathscr{F}^r$, and by 2.5.17.(e) we know that $\mathscr{H}om(f_*\mathscr{M}, \mathscr{F})$ is a quasi-coherent $f_*\mathscr{O}_X$-module, hence $\mathscr{H}om(f_*\mathscr{M}, \mathscr{F}) \cong f_*\mathscr{G}$ for some coherent sheaf $\mathscr{G}$ on Y, then β is actually $\beta : f_*\mathscr{G} \to \mathscr{F}^r$.

(c) By 3.3.1. we know that X, Y are affine iff X_{red}, Y_{red} are affine. And clearly if f is finite surjective, f_{red} is also finite and surjective. So we may reduce the problem to the reduced case. If Y has irreducible components

$Y_1, \dots, Y_n$, then $f_i : f^{-1}(Y_i) \to Y_i$ is also finite and surjective. Then by 3.3.2. we just need to prove every Y_i is affine, hence we may reduce the problem to the case that Y is irreducible. Since $f : X \to Y$ is surjective, there exists an irreducible component of X containing the preimage of η_Y, hence we just replace X to this irreducible component, and by 3.3.2. we may assume additively that X is irreducible. Hence we reduce the problem to the case that X and Y are both integral.

For any coherent sheaf $\mathscr{F}$ on Y, by (b) there exists such a $\beta : f_*\mathscr{G} \to \mathscr{F}$ with $(\ker\beta)_{\eta_Y} = 0$ and $(\mathrm{coker}\beta)_{\eta_Y} = 0$. Since we have two exact sequences $0 \to \ker\beta \to f_*\mathscr{G} \to \mathrm{Im}\beta \to 0$ and $0 \to \mathrm{Im}\beta \to \mathscr{F} \to \mathrm{coker}\beta \to 0$. By 3.4.1. we have $H^i(Y, f_*\mathscr{G}) \cong H^i(X, \mathscr{G}) = 0$, hence $H^i(Y, \mathrm{Im}\beta) \cong H^{i+1}(Y, \ker\beta)$. Moreover, since $(\ker\beta)_{\eta_Y} = 0$, the support of $\ker\beta$ is contained in a closec subset $Z \subset X$. Denote the embedding $j : Z \to X$, we have $\ker\beta = j_*j^*\ker\beta$, and $j^*\ker\beta$ is coherent, hence $H^i(Y, \ker\beta) \cong H^i(X, j^*\ker\beta) = 0$. And similarly $H^i(Y, \mathrm{coker}\beta) = 0$ for all i. Then $H^i(Y, \mathrm{Im}\beta) = 0$ for all i, and $H^i(Y, \mathscr{F}) = 0$ for all i. Hence by theorem 3.7. we know that Y is affine.

Solution 3.4.3. Take $U_x = \mathrm{Spec}\, k[x, y, x^{-1}]$ and $U_y = \mathrm{Spec}\, k[x, y, y^{-1}]$. Then U_x and U_y cover U, and $U_x \cap U_y = U_{xy} = \mathrm{Spec}\, k[x, y, x^{-1}, y^{-1}]$. Then we have the Čech complex $0 \to k[x, y, x^{-1}] \oplus k[x, y, y^{-1}] \xrightarrow{d} k[x, y, x^{-1}, y^{-1}] \to 0$, where $d(f, g) = f - g$. Then $H^1(U, \mathscr{O}_U) \cong k[x, y, x^{-1}, y^{-1}]/\mathrm{Im} d = \{\sum a_{ij} x^i y^j \mid i, j < 0\}$, i.e. $H^1(U, \mathscr{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^i y^j \mid i, j < 0\}$.

Solution 3.4.4. (a) By the condition, the restriction map $\mathscr{F}(U_{\lambda(j)}) \to \mathscr{F}(V_j)$ induces a map $\lambda_n : C^n(\mathfrak{U}, \mathscr{F}) \to C^n(\mathfrak{V}, \mathscr{F})$. Since all λ_n are commutative with d in the complexes $C^\cdot(\mathfrak{U}, \mathscr{F})$ and $C^\cdot(\mathfrak{V}, \mathscr{F})$, they induces morphisms $\lambda^i : \check{H}^i(\mathfrak{U}, \mathscr{F}) \to \check{H}^i(\mathfrak{V}, \mathscr{F})$.

(b) We may define $\lambda_n : \mathscr{C}^n(\mathfrak{U}, \mathscr{F}) \to \mathscr{C}^n(\mathfrak{V}, \mathscr{F})$ in the same way. If we have an injective resolution $0 \to \mathscr{F} \to \mathscr{I}^\cdot$, the morphism $\mathscr{C}^\cdot(\mathfrak{U}, \mathscr{F}) \to \mathscr{I}^\cdot$ and $\mathscr{C}^\cdot(\mathfrak{U}, \mathscr{F}) \to \mathscr{C}^\cdot(\mathfrak{V}, \mathscr{F}) \to \mathscr{I}^\cdot$ are both induced from the identity of $\mathscr{F}$, hence homotopic. So the following diagram is commutative

$$\begin{tikzcd} \check{H}^i(\mathfrak{U}, \mathscr{F}) \arrow[r, "\lambda^i"] \arrow[dr] & \check{H}^i(\mathfrak{V}, \mathscr{F}) \arrow[d] \\ & H^i(X, \mathscr{F}) \end{tikzcd}$$

hence compatible.

(c) Following the hint, we define $D^\cdot(\mathfrak{U})$ as the cokernal of $C^\cdot(\mathscr{U}, \mathscr{F}) \to C^\cdot(\mathscr{U}, \mathscr{G})$. Then by proposition 4.3. we have $\check{H}^{i+1}(\mathfrak{U}, \mathscr{F}) \cong H^i(D^\cdot(\mathfrak{U}))$. Since we have the natural map $D^\cdot(\mathfrak{U}) \to C^\cdot(\mathfrak{U}, \mathscr{R})$, it induces the map $H^i(D^\cdot(\mathfrak{U})) \to \check{H}^i(\mathfrak{U}, \mathscr{R})$. By induction, we have $H^i(X, \mathscr{R}) \cong \varinjlim \check{H}^i(\mathfrak{U}, \mathscr{R})$. So we only need to prove $\varinjlim H^i(D^\cdot(\mathfrak{U})) = \varinjlim \check{H}^i(\mathfrak{U}, \mathscr{R})$.

Since for any $s \in \Gamma(U, \mathscr{R})$ for some open subset U, there exists an open covering $\{U_i\}$ of U such that $s|_{U_i}$ has a preimage $t_i \in \Gamma(U_i, \mathscr{G})$. So any element of $C^i(\mathfrak{U}, \mathscr{R})$, there exists a refinement $\mathfrak{V}$ with a preimage in $C^i(\mathfrak{V}, \mathscr{G})$, hence $\varinjlim C^i(\mathfrak{U}, \mathscr{G}) \to \varinjlim C^i(\mathfrak{U}, \mathscr{R})$ is surjective, i.e. we have an exact sequence $0 \to \varinjlim C^\cdot(\mathfrak{U}, \mathscr{F}) \to \varinjlim C^\cdot(\mathfrak{U}, \mathscr{G}) \to \varinjlim C^\cdot(\mathfrak{U}, \mathscr{R}) \to 0$. But since direct limit is an exact functor, we have an exact sequence $0 \to \varinjlim C^\cdot(\mathfrak{U}, \mathscr{F}) \to \varinjlim C^\cdot(\mathfrak{U}, \mathscr{G}) \to \varinjlim D^\cdot(\mathfrak{U}) \to 0$, i.e. $\varinjlim D^\cdot(\mathfrak{U}) \cong \varinjlim C^\cdot(\mathfrak{U}, \mathscr{R})$. Then we have $\varinjlim \check{H}^{i+1}(\mathfrak{U}, \mathscr{F}) \cong \varinjlim D^i(\mathfrak{U}) \cong \varinjlim \check{H}^i(\mathfrak{U}, \mathscr{R}) \cong H^i(X, \mathscr{R}) \cong H^{i+1}(X, \mathscr{F})$. Then we just take $i = 0$.

Solution 3.4.5. For any open covering $\mathfrak{U} = \{U_i\}_i$, we can define $\mathrm{Pic}\,(\mathfrak{U})$ to be the subgroup of $\mathrm{Pic}\,(X)$ consisting of the isomorphic classes of all $\mathscr{L}$ satisfying $\mathscr{L}|_{U_i} \cong \mathscr{O}_{U_i}$. Then clearly we have $\mathrm{Pic}\,(X) = \varinjlim \mathrm{Pic}\,(\mathfrak{U})$. So if we can prove $\check{H}^1(\mathfrak{U}, \mathscr{O}_X^*) \cong \mathrm{Pic}\,(\mathfrak{U})$, by 3.4.4. we will get $H^1(X, \mathscr{O}_X^*) \cong \mathrm{Pic}\,(X)$.

Now let's prove $\check{H}^1(\mathfrak{U}, \mathscr{O}_X^*) \cong \mathrm{Pic}\,(\mathfrak{U})$. Denote $Z^1(\mathfrak{U}, \mathscr{O}_X^*) = \ker(C^1(\mathfrak{U}, \mathscr{O}_X^*) \to C^2(\mathfrak{U}, \mathscr{O}_X^*)) = \{(s_{ij}) \mid s_{ij}|_{U_{ijk}} s_{jk}|_{U_{ijk}} = s_{ik}|_{U_{ijk}}\}$ and $B^1(\mathfrak{U}, \mathscr{O}_X^*) = \mathrm{Im}(C^0(\mathfrak{U}, \mathscr{O}_X^*) \to C^1(\mathfrak{U}, \mathscr{O}_X^*)) = \{(s_{ij}) \mid s_{ij} = t_i^{-1}|_{U_{ij}} t_j|_{U_{ij}}$ for some $(t_i)\}$. Then by definition, we have $\check{H}(\mathfrak{U}, \mathscr{O}_X^*) \cong Z^1(\mathfrak{U}, \mathscr{O}_X^*)/B^1(\mathfrak{U}, \mathscr{O}_X^*)$. For any $(s_{ij}) \in Z^1(\mathfrak{U}, \mathscr{O}_X^*)$, we can denote $\phi_{ij} : \mathscr{O}_{U_i}|_{U_{ij}} \to \mathscr{O}_{U_j}|_{U_{ij}}$ as $\times s_{ij}$. So if $(s_{ij}) \in Z^1(\mathfrak{U}, \mathscr{O}_X^*)$, all $\mathscr{O}_{U_{ij}}$ can be glued together through $\{\phi_{ij}\}$ to get an $\mathscr{L} \in \mathrm{Pic}\,(\mathfrak{U})$. Hence we define a morphism $\Phi : Z^1(\mathfrak{U}, \mathscr{O}_X^*) \to \text{Pic}\,(\mathfrak{U})$. For any $\mathscr{L} \in \text{Pic}\,(\mathfrak{U})$, we have a set of $\phi_i : \mathscr{L}|_{U_i} \to \mathscr{O}_{U_i}$. Then $\phi_j\phi_i^{-1}$ is an automorphism of $\mathscr{O}_{U_{ij}}$, which corresponds to an $s_{ij} \in \mathscr{O}_X^*(U_{ij})$. Then clearly $(s_{ij}) \in Z^1(\mathfrak{U}, \mathscr{O}_X^*)$, i.e. Φ is surjective. If $(s_{ij}) \in \ker\Phi$, i.e. we have a morphism $\psi : \mathscr{O}_X \cong \mathscr{L}$, since we have $\mathscr{L}_{U_i} \cong \mathscr{O}_{U_i}$ by construction, compositing with ψ, we get an automorphism on $\mathscr{O}_{U_i}$, which corresponds to an element $t_i \in \mathscr{O}_X^*(U_i)$. Then clearly we have $s_{ij} = t_j|_{U_{ij}} t_i|_{U_{ij}}^{-1}$, i.e. $\ker\Phi = B^1(\mathfrak{U}, \mathscr{O}_X^*)$. So we have $\check{H}^1(\mathfrak{U}, \mathscr{O}_X^*) \cong \text{Pic}\,(\mathfrak{U})$.

Solution 3.4.6. On every point $P \in X$, we can denote $A = \mathscr{O}_{X,P}$ and have $\mathscr{I}_P = I$, $\mathscr{O}_{X,P}^* = A^*$, $\mathscr{O}_{X_0,P}^* = (A/I)^*$. Since $I^2 = 0$, we clearly have an exact sequence $0 \to I \to A^* \to (A/I)^* \to 0$. So we have $0 \to \mathscr{I} \to \mathscr{O}_X^* \to \mathscr{O}_{X_0}^* \to 0$. Taking the long exact sequence and using 3.4.5. we have $\ldots \to H^1(X, \mathscr{I}) \to \text{Pic}\, X \to \text{Pic}\, X_0 \to H^2(X, \mathscr{I}) \to \ldots$.

Solution 3.4.7. Since X does not contain $(1,0,0)$, we clearly know that $U \cap V = X$. And clearly $U = \text{Spec}\, k[x,y]_{f(x,1,y)}$, $V = \text{Spec}\, k[z,w]_{f(z,w,1)}$ and $U \cap V = \text{Spec}\, k[s,t,t^{-1}]_{f(s,t,1)}$, the map $d : \Gamma(U, \mathscr{O}_X) \oplus \Gamma(V, \mathscr{O}_X) \to \Gamma(U \cap V, \mathscr{O}_X)$ is $(g(x,y), h(z,w)) \mapsto g(st^{-1}, t^{-1}) - h(s,t)$.

If $(g,h) \in \ker d$, there exists some $e(s,t) \in k[s,t,t^{-1}]$ such that $g - h = fe$. Since $(1,0,0) \notin X$, we may assume $f(x,y,1) = \sum_{0 \le i,j \le d} a_{ij} x^i y^i$ with $a_{d0} = 1$. Then we may split $e(s,t)$ as $e = e_0 + e_1 + e_2$, where e_0 is the sum of all terms with $i \le -d - j$, e_1 is the sum of all terms with $j \ge 0$, and the rest is e_2. So $e_0 e \in \text{Im}(k[x,y]/(f(x,1,y)) \to k[s,t,t^{-1}]/(f(s,t,1)))$, and $e_1 \in \text{Im}(k[z,w]/(f(z,w,1)) \to k[s,t,t^{-1}]/f(s,t,1)))$. So we must have $e_2 = 0$, $g = e_0 e + C$, $h = e_1 e + C$, hence $\ker d = \{(g,h) = (C,C) \mid C \in k\}$, and $\dim H^0(X, \mathscr{O}_X) = 1$.

If $e \in \text{coker} d$, we may assume e can be represented by $\sum_{i \ge 0, j \in \mathbb{Z}} a_{ij} x^i y^j \in k[x,y,y^{-1}]$. Since every term $x^i y^i$ with $j \ge 0$ has preimage $(0, x^i y^j)$, and every term $x^i y^j$ with $-j \ge i$ has preimage $(x^i y^{-j-i}, 0)$, we may assume e is represented by $\sum_{0 < -j < i} a_{ij} x^i y^j$. Since $(1,0,0) \notin X$, we may assume $f(x,y,z) = x^d + f'(x,y,z)$. Then $x^d = -f'(x,y,1)$ in $k[x,y]/(f(x,y,1))$, hence e can be uniquely represented by $\sum_{0 < -j < i < d} a_{ij} x^i y^j$, and every $\sum_{0 < -j < i < d} a_{ij} x^i y^j$ is in the cokernal. Hence $\dim H^1(X, \mathscr{O}_X) = \frac{1}{2}(d-1)(d-2)$.

Solution 3.4.8 (Cohomological Dimension)**.** (a) For any quasi-coherent sheaf $\mathscr{G}$, by 2.5.15. we know that $\mathscr{G} = \varinjlim \mathscr{F}_\alpha$ for some coherent sheaves $\mathscr{F}_\alpha$. If $H^n(X, \mathscr{F}) = 0$ for all coherent sheaf $\mathscr{F}$, by proposition 2.9., we know that $H^n(X, \mathscr{G}) = \varinjlim H^n(X, \mathscr{F}_\alpha) = 0$. Hence we can only consider coherent sheaves on X.

(b) By corollary 5.18. in chapter II, there exists a locally free sheaf $\mathscr{E}$ with exact sequence $0 \to \mathscr{G} \to \mathscr{E} \to \mathscr{F} \to 0$. Then if when $i > n$ we have $H^i(X, \mathscr{E}) = 0$, there exists isomorphism $H^i(X, \mathscr{F}) \cong H^{i+1}(X, \mathscr{G})$. So we may take induction from $i > \dim X$ to smaller, then we have $H^i(X, \mathscr{F}) = 0$. Hence we can only consider locally free sheaves.

(c) If X can be covered by $r+1$ open affine subsets, then there are no indices in $C^i(\mathfrak{U}, \mathscr{F})$ if $i > r$. Hence clearly $\text{cd}(X) \le r$.

(d) If X is a quasi-projective scheme of dimension r over a field k, there exists an embedding $X \to \mathbb{P}_k^n$. Then we can define $U_i = X \cap \{x_i \neq 0\}$, then $\{U_i\}$ is an affine covering of X. Then by (c) we have $\text{cd}(X) \le r = \dim X$.

(e) By definition Y is the intersection of hypersurfaces $H_1, \ldots, H_r$. By proposition 2.5. in chapter II, we know that $X - H_i$ is affine. Hence $\{X - H_i\}$ is an affine covering of $X - Y$. Then by (c) we know that $\text{cd}(X - Y) \le r - 1$.

Solution 3.4.9. If Y is a set-theoretic complete intersection in X, in the same way with 3.4.8.(e) we have $\text{cd}(X - Y) \le 1$. Then we just need to prove that $H^2(X - Y, \mathscr{O}_X) \neq 0$ to get a contradiction. Since X is affine, we have $H^i(X, \mathscr{O}_X) = 0$ for all $i > 0$. Then by 3.2.3., we have $H^2(X - Y, \mathscr{O}_X) \cong H_Y^2(X, \mathscr{O})$. By 3.2.4., we have $\ldots \to H_{Y_1}^3(X, \mathscr{O}_X) \oplus H_{Y_2}^3(X, \mathscr{O}_X) \to H_Y^3(X, \mathscr{O}_X) \to H_P^4(X, \mathscr{O}_X) \to H_{Y_1}^4(X, \mathscr{O}_X) \oplus H_{Y_2}^4(X, \mathscr{O}_X) \to \ldots$. Since $X - Y_1 = \text{Spec}\, k[x_1, x_2, x_3, x_4, x_1^{-1}, x_2^{-1}]$ is affine, by 3.2.3. we have $H_{Y_1}^i(X, \mathscr{O}_X) \cong H^i(X - Y_1, \mathscr{O}_X) = 0$ for all $i > 0$, and same for $X - Y_2$. So we have $H_Y^3(X, \mathscr{O}_X) \cong H_P^4(X, \mathscr{O}_X)$. By 3.2.3. we also have $H_P^4(X, \mathscr{O}_X) \cong H^3(X - P, \mathscr{O}_X)$. So we need to prove that $H^3(X - P, \mathscr{O}_X) \neq 0$.

Clearly we have $X - P = \bigcup_{i=1}^4 U_i$, where $U_i = (x_i \neq 0) = \text{Spec}\, k[x_1, x_2, x_3, x_4, x_i^{-1}]$. By 3.4.8.(d) we have $C^4(\mathfrak{U}, \mathscr{O}_X) = 0$, hence $H^3(X - P, \mathscr{O}_X) = \text{coker}(C^2(\mathfrak{U}, \mathscr{O}_X) \to C^3(\mathfrak{U}, \mathscr{O}_X))$. Since this morphism is just

$k[x_1, x_2, x_3, x_4, x_2^{-1}, x_3^{-1}, x_4^{-1}] \oplus \ldots \oplus k[x_1, x_2, x_3, x_4, x_1^{-1}, x_2^{-1}, x_3^{-1}] \to k[x_1, x_2, x_3, x_4, x_1^{-1}, x_2^{-1}, x_3^{-1}, x_4^{-1}]$, the cokernal is spanned by all $x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$ with all $i_j < 0$. Hence $H^3(X - P, \mathscr{O}_X) \neq 0$. So $H^2(X - Y, \mathscr{O}_X) \cong H^3(X - P, \mathscr{O}_X) \neq 0$.

If $\bar{Y}$ is a set-theoretic complete intersection, we may restrict those hypersurfaces to X. Then Y is a set-theoretic complete intersection, hence contradict.

Solution 3.4.10. Take an affine covering $\mathfrak{U}$ of X. For any infinitesimal extension $(X', \mathscr{I})$, we have an exact sequence $0 \to \mathscr{I} \to \mathscr{O}_{X'} \to \mathscr{O}_X \to 0$. Then by 2.8.7. this sequence is split on every affine piece U_i, which is given by a lifting $\alpha_i : \mathscr{O}_X|_{U_i} \to \mathscr{O}_{X'}|_{U_i}$. So on every U_{ij}, there exist two lifting $\alpha_i|_{U_{ij}}$ and $\alpha_j|_{U_{ij}}$. Hence there exists $\beta_{ij} \in \mathrm{Hom}_{\mathscr{O}(U_{ij})}(\Omega_{X/k}(U_{ij}), \mathscr{I}(U_{ij})) \cong (\mathscr{F} \otimes \mathscr{T})(U_{ij})$ such that $\alpha_i|_{U_{ij}} - \alpha_j|_{U_{ij}} = \beta_{ij}$. Since on each U_{ijk}, we have $\beta_{ij}|_{U_{ijk}} + \beta_{jk}|_{U_{ijk}} + \beta_{ki}|_{U_{ijk}} = \alpha_i|_{U_{ijk}} - \alpha_j|_{U_{ijk}} + \alpha_j|_{U_{ijk}} - \alpha_k|_{U_{ijk}} + \alpha_k|_{U_{ijk}} - \alpha_i|_{U_{ijk}} = 0$. Hence $(\beta_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F} \otimes \mathscr{T})$. For any other lifting $\alpha_i' : \mathscr{O}_X(U_i) \to \mathscr{O}_{X'}(U_i)$, we can get $(\beta_{ij}') \in Z^1(\mathfrak{U}, \mathscr{F} \otimes \mathscr{T})$. By 2.8.6.(a), there exists $\gamma_i \in \mathrm{Hom}_{\mathscr{O}_X(U_i)}(\Omega_{X/k}(U_i), \mathscr{I}(U_i))$ such that $\alpha - \alpha' = \gamma$. So $\beta_{ij} - \beta_{ij}' = \gamma_i|_{U_{ij}} - \gamma_j|_{U_{ij}}$, hence (β_{ij}) and (β_{ij}') are in the same class of $H^1(\mathfrak{U}, \mathscr{F} \otimes \mathscr{T})$. Thus we have a map {isomorphism classes of infinitesimal extension} $\to H^1(X, \mathscr{F} \otimes \mathscr{T})$, $(X', \mathscr{I}) \mapsto (\beta_{ij})$.

Conversely, on each U_i we have trivial lifting $\mathscr{O}_X|_{U_i} \to \mathscr{O}_X|_{U_i} \otimes \mathscr{F}|_{U_i}$. Then for any $(\beta_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F} \otimes \mathscr{T})$, we may glue all $\mathscr{O}_X|_{U_i} \otimes \mathscr{F}|_{U_i}$ together through β_{ij} to get a sheaf $\mathscr{O}_{X'}$. Hence we have the inverse $H^1(X, \mathscr{F} \otimes \mathscr{T}) \to$ {isomorphism classes of infinitesimal extension}, i.e. the map is bijective.

Solution 3.4.11. Here we need to use spectral sequence. Define a presheaf $\mathscr{H}^q(\mathscr{F})$ as $\mathscr{H}^q(\mathscr{F})(U) = H^q(U, \mathscr{F}_U)$. Consider the bicomplex $(K^{pq}) = (C^p(\mathfrak{U}, \mathscr{I}^q))$, where $\mathscr{I}^{\cdot}$ is an injective resolution of $\mathscr{F}$. Then $H_I^p H_{II}^q(K^{\cdot\cdot}) = \check{H}^p(\mathfrak{U}, \mathscr{H}^q(\mathscr{F}))$. Since $H_I^q(K^{\cdot,p}) = \check{H}^q(\mathfrak{U}, \mathscr{I}^p)$, and $\mathscr{I}^p$ are all injective, we have

$$H_{II}^p H_I^q(K^{\cdot\cdot}) = \begin{cases} 0 & \text{if } q \geq 1 \\ H^p(X, \mathscr{F}) & \text{if } q = 0 \end{cases}$$

Hence the second spectral sequence of $K^{\cdot\cdot}$ degenerates and $H^n(X, \mathscr{F}) = H^n(K^{\cdot\cdot})$. So the spectral sequence of $K^{\cdot\cdot}$ is $\check{H}^P(\mathfrak{U}, \mathscr{H}^p(\mathscr{F})) \Rightarrow H^{p+q}(X, \mathscr{F})$. So if all $H^q(U_{i_0 \ldots i_n}, \mathscr{F}) = 0$, this spectral sequence is degenerated. So $H^p(X, \mathscr{F}) = \check{H}^p(\mathfrak{U}, \mathscr{H}^0(\mathscr{F})) = \check{H}^p(\mathfrak{U}, \mathscr{F})$.

3.5 The Cohomology of Projective Space

Solution 3.5.1. By the short exact sequence we have $0 \to H^0(X, \mathscr{F}') \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{F}'') \to H^1(X, \mathscr{F}') \to \ldots$. Since $\mathscr{F}', \mathscr{F}, \mathscr{F}''$ are all coherent, and X has finite dimension, this long exact sequence will stop by zeros. Hence $\sum_i (-1)^i (\dim_k H^i(X, \mathscr{F}') - \dim_k H^i(X, \mathscr{F}) + \dim_k H^i(X, \mathscr{F}''))$ is a finite sum and equals to zero, i.e. $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$.

Solution 3.5.2. (a) Firstly we need to find some $x \in \mathscr{O}_X(1)$ such that $\phi_x : \mathscr{F}(-1) \xrightarrow{\times x} \mathscr{F}$ is injective. If ϕ_x is not injective, there exists at least one point $P \in X$ such that $\phi_{x,P}$ is not injective. Assume $P \in U = \operatorname{Spec} A$, $\mathscr{F}|_U = \tilde{M}$ and P corresponds to a prime $\mathfrak{p}$. Then $M_{\mathfrak{p}} \xrightarrow{x} M_{\mathfrak{p}}$ is not injective, i.e. $\mathfrak{p}$ is an associated prime of M, and $x_P \in \mathfrak{p}$. Since X is noetherian, the associated points on X are finite, we only need to take a hyperplane H do not pass through all associated points, and this hyperplane corresponds to x. This is trivial in linear algebra. Then we have some $x \in \mathscr{O}_X(1)$ such that $\mathscr{F}(-1) \xrightarrow{\times x} \mathscr{F}$ is injective.

So, if we denote the cokernal of this morphism as $\mathscr{G}$, we have $\chi(\mathscr{G}(m)) = \chi(\mathscr{F}(m)) - \chi(\mathscr{F}(m-1))$. Moreover, clearly $\operatorname{Supp}(\mathscr{G}) = \operatorname{Supp}(\mathscr{F}) \cup V(x)$, i.e. $\dim \operatorname{Supp}(\mathscr{G}) = \dim \operatorname{Supp}(\mathscr{F}) - 1$. Hence by induction, $\Delta(\chi(\mathscr{F}(m)))$ is a polynomial, we have $\chi(\mathscr{F}(m))$ is a polynomial, namely the Hilbert polynomial $P(m)$.

(b) Since $H^i(X, \mathscr{F}(m)) = 0$ for $i > 0$ and $m \gg 0$, we have $P(m) = \chi(\mathscr{F}(m)) = \dim H^0(X, \mathscr{F}(n)) = \dim M_n = P_M(m)$ for $m \gg 0$.

Solution 3.5.3 (Arithmetic Genus)**.** (a) Since X is integral, we clearly know that X is a projective variety. Then by theorem 3.4.(a) in chapter I, we have $H^0(X, \mathscr{O}_X) = k$. Hence $p_a(X) = \sum_{i=0}^{r-1}(-1)^i \dim_k H^{r-i}(X, \mathscr{O}_X)$.

(b) By 3.5.2.(b) we have $\chi(\mathscr{O}_X) = P_X(0)$, i.e. $(-1)^r(\chi(\mathscr{O}_X) - 1) = (-1)^r(P_X(0) - 1)$ are equivalent.

(c) If X and Y are both nonsingular projective curve over k and birational equivalent, we only need to show $X \cong Y$, hence trivially $p_a(X) = p_a(Y)$. We only need to define the morphism $X \to Y$ and $Y \to X$. Define the function field of X and Y as K. For any point $P \in X$, since P is nonsingular, $\mathscr{O}_P$ is a valuation ring with fractional field K. Hence we have a commutative diagram

$$\begin{tikzcd} & \operatorname{Spec} K \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & \operatorname{Spec} \mathscr{O}_P \arrow[r] \arrow[ur, "\exists !"] & \operatorname{Spec} k \end{tikzcd}$$

Since Y is proper over Spec k, by valuation criterian there exists a morphism $X \to \operatorname{Spec} \mathscr{O}_P \to Y$. Hence we have a set-theoretic map $X \to Y$. By the proof of theorem 6.9. in chapter I, the above map is a morphism from abstract curve X to Y. Hence we have a morphism $X \to Y$. Similarly we have $Y \to X$ as the inverse of $X \to Y$, i.e. $X \cong Y$.

Solution 3.5.4. (a) For any sheaf $\mathscr{F}$, we may define $P(\gamma(\mathscr{F})) = P(\mathscr{F}) \in \mathbb{Q}(z)$ by 3.5.2.(a). By 3.5.1. we know that for any $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ we have $P(\gamma(\mathscr{F})) = P(\gamma(\mathscr{F}')) + P(\gamma(\mathscr{F}''))$. Then by definition of Grothendieck group, we have a additive homomorphism $P : K(X) \to \mathbb{Q}[z]$.

(b) By hint, we firstly prove (1⇒2). If $K(X)$ is a free abelian group generated by $\{\gamma(\mathscr{O}_{L_i})\}$. Since there exists a linear embedding $j : \mathbb{P}^i_k \to \mathbb{P}^r_k$ such that $\mathscr{O}_{L_i} = j_*\mathscr{O}_{\mathbb{P}^i_k}$. So $P(\gamma(\mathscr{O}_{L_i})) = P(\gamma(\mathscr{O}_{\mathbb{P}^i_k})) = \binom{i+z}{i}$. So if $\alpha = \sum a_i\gamma(\mathscr{O}_{L_i})$ such that $P(\alpha) = 0$, we have $\sum a_i\binom{i+z}{i} = 0$, i.e. $a_i = 0$ for all i. Hence P is injective.

Then we prove (1) and (2) simultaneously by induction on r. The case $r = 0$ is trivial. We may assume the case $r - 1$. We may assume $L_0 \subset L_1 \subset \ldots \subset L_r$. By 2.6.2. we have $K(\mathbb{P}^{r-1}) \to K(\mathbb{P}^r \to K(\mathbb{P}^r - \mathbb{P}^{r-1}) \to 0$. We may take $L_{r-1} \cong \mathbb{P}^{r-1}$. Then the morphism $P_{r-1} : K(\mathbb{P}^{r-1}) \to \mathbb{Q}[z]$ factors through $K(\mathbb{P}^{r-1}) \to K(\mathbb{P}^r) \to \mathbb{Q}(z)$, hence $K(\mathbb{P}^{r-1}) \to k(\mathbb{P}^r)$ is injective. Moreover, we have $K(\mathbb{P}^r - \mathbb{P}^{r-1}) = K(\mathbb{A}^r) = \mathscr{O}_{\mathbb{A}^r} \cdot \mathbb{Z}$, i.e. we have $0 \to \mathbb{Z}^{r-1} \to K(\mathbb{P}^r) \to \mathbb{Z} \to 0$ by induction. Since $\operatorname{Ext}^1(\mathbb{Z}, \mathbb{Z}^{r-1}) = 0$, we know $K(\mathbb{P}^r)$ must be the unique extension of $\mathbb{Z}$ by $\mathbb{Z}^{r-1}$, i.e. $\mathbb{Z}^r$.

Solution 3.5.5. (a and c) We will use induction on the codimension of Y. If Y has codimension 0, i.e. $Y = X$. Hence trivially $H^0(X, \mathscr{O}_X(n)) \to H^0(Y, \mathscr{O}_Y(n))$ is surjective. For higher codimension d of Y, by 2.8.4. we know $Y = H_1 \cup \ldots \cup H_s$ for some hypersurfaces $H_1, \ldots, H_s$. Then we may define $Z = H_1 \cup \ldots \cup H_{s-1}$. By induction we have $H^0(X, \mathscr{O}_X(n)) \to H^0(Z, \mathscr{O}_Z(n))$ is surjective and $H^i(Z, \mathscr{O}_Z(n)) = 0$ for $0 < i < \dim q$. Denote $i : Y \to Z$ as the canonical closed embedding. Then if $d = \deg H_s$, we have an exact sequence $0 \to \mathscr{O}_Z(n-d) \to \mathscr{O}_Z(n) \to i_*\mathscr{O}_Y(n) \to 0$. Since $H^1(Z, \mathscr{O}_Z(n-d)) = 0$ by theorem 5.2., we have $H^0(Z, \mathscr{O}_Z(n)) \to H^0(Y, \mathscr{O}_Y(n))$ is surjective, hence $H^0(X, \mathscr{O}_X(n)) \to H^0(Z, \mathscr{O}_Z(n)) \to H^0(Y, \mathscr{O}_Y(n))$ is surjective. Moreover, for any $0 < i < \dim Y = \dim Z - 1$, we have $\ldots \to H^i(Z, \mathscr{O}_Z(n-d)) \to H^i(Z, \mathscr{O}_Z(n)) \to H^i(Y, \mathscr{O}_Y(n)) \to H^{i+1}(Z, \mathscr{O}_Z(n-d)) \to \ldots$. By induction hypothesis $H^i(Z, \mathscr{O}_Z(n-d)) = H^{i+1}(Z, \mathscr{O}_Z(n-d)) = 0$, then $H^i(Y, \mathscr{O}_Y(n)) = H^i(Z, \mathscr{O}_Z(n)) = 0$.

(b) Since $k = H^0(X, \mathscr{O}_X) \to H^0(Y, \mathscr{O}_Y)$ is surjective, and both cohomology groups are k-modules, we clearly have $H^0(Y, \mathscr{O}_Y) = k$. Since k has no idempotent, Y is connected.

(d) By theorem 5.2. we have $H^i(Y, \mathscr{O}_Y) = 0$ for any $i > \dim Y$. Then $\chi(\mathscr{O}_Y) = \dim_k H^0(Y, \mathscr{O}_Y) + (-1)^q \dim_k H^q(Y, \mathscr{O}_Y) = 1 + (-1)^q H^q(Y, \mathscr{O}_Y)$. Then $p_a(Y) = (-1)^q(\chi(\mathscr{O}_Y) - 1) = \dim_k H^q(Y, \mathscr{O}_Y)$.

Solution 3.5.6 (Curves on a Nonsingular Quadric Surface)**.** (a) (1) We denote $i : Q \to X$ and $j : \mathbb{P}^1 \to Q$ as the canonical embeddings. If $a = b$, there exists an exact sequence $0 \to \mathscr{O}_X(a-2) \to \mathscr{O}_X(a) \to i_*\mathscr{O}_Q(a) \to 0$. Since $H^1(X, \mathscr{O}_X(a)) = H^2(X, \mathscr{O}_X(a-2)) = 0$, we have $H^1(Q, \mathscr{O}_Q(a)) = 0$. If $a = b + 1$, we have an exact sequence $0 \to \mathscr{O}_Q(a, a+1) \to \mathscr{O}_Q(a+1) \to j_*\mathscr{O}_{\mathbb{P}^1}(a+1) \to 0$. Since we've already have $H^1(Q, \mathscr{O}_Q(a+1)) = 0$, and

$H^0(Q, \mathscr{O}_Q(a+1)) \to H^0(Y, \mathscr{O}_{\mathbb{P}^1}(a+1))$ is surjective because this is just the restriction. Hence $H^1(Q, \mathscr{O}_Q(a, a+1)) = 0$. And the case $(a+1, a)$ is the same.

(2) If $a \leq b$, we may write $a = b - n$ for some $n \geq 0$. Then we have $0 \to \mathscr{O}_Q(b-n, b) \to \mathscr{O}_Q(b) \to j_*\mathscr{O}_{\mathbb{P}^1}(b)^n \to 0$. Hence similarly, $H^1(Q, \mathscr{O}_Q(b)) = 0$ and $H^0(Q, \mathscr{O}_Q(b)) \to \mathscr{O}_{\mathbb{P}^1}(b)^n)) = 0$ is surjective. Then $H^1(Q, \mathscr{O}_Q(a, b)) = 0$ if $a, b < 0$.

(3) Denote $n = -a \geq 2$. Then we have $0 \to \mathscr{O}_Q(-n, 0) \to \mathscr{O}_Q \to j_*\mathscr{O}^n_{\mathbb{P}^1} \to 0$. Hence the long exact sequence is $0 \to 0 \to k \to k^n \to H^1(Q, \mathscr{O}_Q(a, 0)) \to 0$, i.e. $H^1(Q, \mathscr{O}_Q(a, 0)) = k^{n-1} \neq 0$.

Actually we can use Künneth formula $H^n(X \times Y, \mathscr{F} \boxtimes \mathscr{G}) = \bigoplus_{p+q=n} H^p(X, \mathscr{F}) \otimes H^q(Y, \mathscr{G})$ to do this exercise.

(b) (1) Clearly $\mathscr{I}_Y = \mathscr{O}_Q(-a, -b)$. Then $H^0(Q, \mathscr{O}_Q(-a, -b)) = H^1(Q, \mathscr{O}_Q(-a, -b)) = 0$. Moreover, by the exact sequence $0 \to \mathscr{I}_Y \to \mathscr{O}_Q \to i_*\mathscr{O}_Y \to 0$, where $i : Y \to Q$ is the canonical embedding. Then the long exact sequence is $0 \to 0 \to k \to H^0(Y, \mathscr{O}_Y) \to 0$. Then $H^0(Y, \mathscr{O}_Y) = k$, i.e. Y is connected.

(2) Since $\mathscr{O}_Q(a, b)$ corresponds to a closed embedding $Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^a \times \mathbb{P}^b \to \mathbb{P}^N$. Then by Bertini's theorem and the bracket in the theorem 8.18. in chapter II, i.e. the remark 7.9.1., there exists a hyperplane $H \subset \mathbb{P}^N$ such that $H \not\subseteq Q$, and $Y = H \cap Q$ is irreducible smooth. Since Q has dimension 2, Y is a curve.

(3) Clearly Q is normal. Then by 2.8.4.(b) we know that Q is projective normal. Then by 2.5.14.(d), $\Gamma(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(n)) \to \Gamma(Q, \mathscr{O}_Q(n))$ is surjective for every n. Since Y is normal, by 2.5.14.(d) $\Gamma(Q, \mathscr{O}_Q(n)) \to \Gamma(Y, \mathscr{O}_Y(n))$ is surjective for every n iff Y is projective normal. Since $0 \to \mathscr{I}_Y(n) \to \mathscr{O}_Q(n) \to \mathscr{O}_Y(n) \to 0$, we know Y is projective normal iff $H^1(Q, \mathscr{I}_Y(n)) = 0$. Since $\mathscr{I}_Y(n) = \mathscr{O}_Q(-a, -b) \otimes \mathscr{O}_Q(n) = \mathscr{O}_Q(n-a, n-b)$. Then $H^1(Q, \mathscr{I}_Y(n)) = 0$ iff $|(n-a) - (n-b)| \leq 1$, i.e. $|a - b| \leq 1$.

(c) Since Y is in the form (a, b), we have $Y = Y_1 \coprod Y_2$ for $Y_1 = a$ copies of $\mathbb{P}^1$ and $Y_2 = b$ copies of $\mathbb{P}^1$. Then since $\mathscr{I}_{Y_2}$ is clearly flat, we have an exact sequence $0 \to \mathscr{I}_{Y_1} \otimes \mathscr{I}_{Y_2} \to \mathscr{I}_{Y_2} \to \mathscr{O}_{Y_1} \otimes \mathscr{I}_{Y_2} \to 0$, i.e. $0 \to \mathscr{O}_Q(-a, -b) \to \mathscr{O}_Q(0, -b) \to \mathscr{O}_{Y_1} \otimes \mathscr{O}_Q(0, -b) \to 0$. By (a) we have $H^0(Q, \mathscr{O}_Q(-a, -b)) = H^1(Q, \mathscr{O}_Q(-a, -b)) = 0$. Then we only need to calculate $H^2(Q, \mathscr{O}_Q(-a, -b)$. As in (a.3), we have $0 \to \mathscr{O}_Q(0, -b) \to \mathscr{O}_Q \to j_*\mathscr{O}^b_{\mathbb{P}^1}) \to 0$. Hence $0 \to 0 \to k \to k^n \to H^1(Q, \mathscr{O}_Q(0, -b) \to 0 \to 0 \to H^2(Q, \mathscr{O}_Q(0, -b) \to 0 \to \ldots$. Hence $H^2(Q, \mathscr{O}_Q(0, -b)) = 0$. By (a), we have $H^1(Q, \mathscr{O}_Q(0, -b)) = k^{b-1}$, and by similar method of (a), we have $H^1(Q, \mathscr{O}_Q(0, -b) \otimes \mathscr{O}_Y) = k^{a(b-1)}$. Then for exact sequence $0 \to \mathscr{O}_Q(-a, -b) \to \mathscr{O}_Q(0, -b) \to \mathscr{O}_{Y_1} \otimes \mathscr{O}_Q(0, -b) \to 0$, we have $0 = H^1(Q, \mathscr{O}_Q(-a, -b) \to k^{b-1} \to k^{a(b-1)} \to H^2(Q, \mathscr{O}_Q(-a, -b)) \to 0 \to \ldots$. Hence $p_a(Y) = \chi(\mathscr{O}_Q(-a, -b)) = \dim_k H^2(Q, \mathscr{O}_Q(-a, -b)) = a(b-1) - (b-1) = ab - a - b + 1$.

Solution 3.5.7. (a) For any coherent sheaf $\mathscr{F}$ on Y, we have $H^i(Y, \mathscr{F} \otimes (i^*\mathscr{L})^n) = H^i(X, i_*(\mathscr{F} \otimes (i^*\mathscr{L})^n)) = H^i(X, i_*\mathscr{F} \otimes \mathscr{L}^n)) = 0$ for $i > 0$ and $n \gg 0$ since $\mathscr{L}$ is ample. Hence $i^*\mathscr{L}$ is ample.

(b) Clearly $X_{\text{red}} \to X$ is a closed embedding, we only need to prove ($\Leftarrow$). For any coherent $\mathscr{F}$ on X, if we denote $\mathscr{N}$ is the nilpotent sheaf of $\mathscr{O}_X$, i.e. $\mathscr{N}^r = 0$ for some r, we have a filtration $\mathscr{F} \supsetneq \mathscr{N}\mathscr{F} \supsetneq \ldots \supsetneq \mathscr{N}^r\mathscr{F} = 0$. And clearly we have $\mathscr{N}^j\mathscr{F}/\mathscr{N}^{j+1}\mathscr{F}$ is simultaneously a coherent $\mathscr{O}_{X_{\text{red}}}$-module and a coherent $\mathscr{O}_X$-module. Hence $H^i(X, (\mathscr{N}^j\mathscr{F}/\mathscr{N}^{j+1}\mathscr{F}) \otimes \mathscr{L}^n) = H^i(X_{\text{red}}, (\mathscr{N}^j\mathscr{F}/\mathscr{N}^{j+1}\mathscr{F}) \otimes \mathscr{L}^n_{\text{red}}) = 0$ for $n \gg 0$ and $i > 0$ since $\mathscr{L}_{\text{red}}$ is ample. By the filtration we have $0 \to \mathscr{N}^{j+1}\mathscr{F} \to \mathscr{N}^j\mathscr{F} \to \mathscr{N}^j\mathscr{F}/\mathscr{N}^{j+1}\mathscr{F} \to 0$, hence $H^i(X, \mathscr{F} \otimes \mathscr{L}^n) = H^i(X, \mathscr{N}\mathscr{F} \otimes \mathscr{L}^n) = \ldots = H^i(X, 0 \otimes \mathscr{L}^n) = 0$. Hence $\mathscr{L}$ is ample.

(c) For any irreducible component X_i of X, the canonical embedding $X_i \to X$ is closed, hence by (a) we only need to prove ($\Leftarrow$). By (b) we may assume X is reduced. Write $X = \bigcup_i^m X_i$ for some irreducible components X_i. Then we prove it by induction on m. Denote the ideal sheaf of X_1 by $\mathscr{I}$. Then for any coherent sheaf $\mathscr{F}$ on X, we have $0 \to \mathscr{I}\mathscr{F} \to \mathscr{F} \to \mathscr{F}/\mathscr{I}\mathscr{F} \to 0$. Clearly $\operatorname{Supp}(\mathscr{I}\mathscr{F}) \subset X_2 \cup \ldots \cup X_m$ and $\operatorname{Supp}(\mathscr{F}/\mathscr{I}\mathscr{F}) \subset X_1$. By induction hypothesis we know $H^i(X, \mathscr{I}\mathscr{F} \otimes \mathscr{L}^n) = 0$ and $H^i(X, \mathscr{F}/\mathscr{I}\mathscr{F} \otimes \mathscr{F}^n) = 0$ for $i > 0$ and $n \gg 0$. Then clearly $H^i(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for $i > 0$ and $n \gg 0$.

(d) ($\Rightarrow$) For any coherent $\mathscr{F}$ on X, we have $f_*\mathscr{F}$ is coherent on Y. Then by 3.4.1. we have $H^i(X, \mathscr{F} \otimes (f^*\mathscr{L})^n) = H^i(Y, f_*(\mathscr{F} \otimes (f^*\mathscr{L})^n)) = H^i(Y, f_*\mathscr{F} \otimes \mathscr{L}^n) = 0$ for $i > 0$ and $n \gg 0$. Hence $f_*\mathscr{L}$ is ample.

($\Leftarrow$) By (b) and (c) we can reduced to the case that X and Y are both integral. Then we will prove this by noetherian induction. By 3.4.2.(b), there exists a coherent sheaf $\mathscr{G}$ on X such that $u : f_*\mathscr{G} \to \mathscr{F}^{\oplus m}$ is an isomorphism at the generic point of Y. Then Supp $(\ker(u)) \subsetneq$ Supp $(\mathscr{F})$ and Supp $(\operatorname{coker}(u)) \subsetneq$ Supp $(\mathscr{F})$. So by induction we have $H^i(Y, \ker(u) \otimes \mathscr{L}^n) = 0$ and $H^i(Y, \operatorname{coker}(u) \otimes \mathscr{L}^n) = 0$ for $i > 0$ and $n \gg 0$. Hence $H^i(Y, \mathscr{F} \otimes \mathscr{L}^n) = H^i(Y, f_*\mathscr{G} \otimes \mathscr{L}^n) = H^i(X, \mathscr{G} \otimes (f^*\mathscr{L})^n) = 0$ for $i > 0$ and $n \gg 0$. Hence $\mathscr{L}$ is ample on Y.

Solution 3.5.8. (a) (Is this question an interrogative sentence?)

(b) By theorem 6.2A. in chapter I we know that normality implies nonsingularity. And the completeness is clear. Hence by (a), $\tilde{X}$ is projective. For any very ample invertible sheaf $\mathscr{L}$ on $\tilde{X}$, there exists a closed embedding $i : \tilde{X} \hookrightarrow \mathbb{P}^m$ for some m such that $\mathscr{L} = i^*\mathscr{O}(1)$. Since the preimage in $\tilde{X}$ of singular points in X is just finite point, by Bertini's theorem, there exists a hyperplane $H \subset \mathbb{P}^m$ such that $D = i^*H = \sum P_i$ is an effective divisor on $\tilde{X}$ and $f(P_i)$ are all nonsingular points on X. Then we denote $D_0 = \sum f(P_i)$, and $\mathscr{L}_0$ is the invertible sheaf on X corresponding to D_0. Hence $\mathscr{L} = f^*\mathscr{L}_0$. Then by 3.5.7.(d), we know $\mathscr{L}_0$ is ample on X, hence $\mathscr{L}_0^m$ is very ample on X for some $m \gg 0$. So X is projective.

(c) If $X = \bigcup_{i=1}^n X_i$ for some irreducible components X_i, we denote $Y = \bigcup_{i=2}^n X_i$, and $X = X_1 \cup Y$. Since we have an exact sequence $0 \to \mathscr{O}_X^* \to \mathscr{O}_{X_1}^* \oplus \mathscr{O}_Y \to \mathscr{O}_{X_1 \cap Y}^* \to 0$. Since $X_1 \cap Y$ is just points, hence $H^1(X_1 \cap Y, \mathscr{O}_{X_1 \cap Y}^*) =$ Pic $(X_1 \cap Y) = 0$. So Pic $X \to$ Pic $X \oplus$Pic Y is surjective. By induction on the number of irreducible components of X, we know Pic $X \to \bigoplus$ Pic X_i is surjective. By (b), X_i are all projective, hence we have invertible sheaves $\mathscr{L}_i$ on X_i such that $\mathscr{L}_i$'s are all very ample. Then by 3.5.7.(c), we know that $\mathscr{L} = \mathscr{L}_1 \boxtimes \ldots \boxtimes \mathscr{L}_n$ is ample on X. So $\mathscr{L}^m$ is very ample on X for some $m \gg 0$, i.e. X is projective.

(d) Take the nilpotent sheaf $\mathscr{N}$ of $\mathscr{O}_X$. Then $\mathscr{N}^r = 0$ for some r. We take $i > \frac{r}{2}$ and $\mathscr{I} = \mathscr{N}^i$. By 3.4.6. we have an exact sequence $0 \to \mathscr{I} \to \mathscr{O}_X^* \to \mathscr{O}_{X_0}^* \to 0$, where $X_0 = (X, \mathscr{O}_X/\mathscr{I})$. Then since X has dimension 1, we know $H^2(X, \mathscr{I}) = 0$, hence Pic $X \to$ Pic X_0 is surjective. Since $\mathscr{O}_{X_0}/\mathscr{N}^{[\frac{r}{2}]} = \mathscr{O}_{X_{\text{red}}}$, hence we can use the induction of r and know Pic $X \to$ Pic $X_0 \to \ldots \to$ Pic X_{red} is surjective. Then X is projective by 3.5.7.(b).

Solution 3.5.9 (A Nonprojective Scheme)**.** Suppose $k = 0$, we know that $\delta(\mathscr{O}(1)) = (-\frac{x_0}{x_1^2}) \cdot (-\frac{x_1}{x_2^2}) \cdot (-\frac{x_2}{x_0^2}) = -x_0^{-1}x_1^{-1}x_2^{-1} \in H^2(X, \omega)$. Hence the morphism δ is injective. So The morphism $H^1(X, \omega) \to$ Pic X' is surjective. Since $H^1(X, \mathscr{O}(-3)) = 0$, we have Pic $X' = 0$. So X' is not projective.

Solution 3.5.10. For the exact sequence $\mathscr{F}^1 \to \ldots \mathscr{F}^r$, we have r exact sequences: $0 \to \ker^i \to \mathscr{F}^i \to \operatorname{Im}^i \to 0$. Then by Serre' theorem, there exists n_i such that for all $n > n_i$, we have $0 \to \Gamma(X, \ker^i(n)) \to \Gamma(X, \mathscr{F}^i(n)) \to \Gamma(X, \operatorname{Im}^i(n)) \to 0$. So take an $N = \max\{n_i\}$. Then for any $n > N$, we have $\Gamma(X, \mathscr{F}^1(n)) \to \ldots \to \Gamma(X, \mathscr{F}^r(n))$.

3.6 Ext Groups and Sheaves

Solution 3.6.1. We denote the set $E(\mathscr{F}'', \mathscr{F}') = \{$all extension of $\mathscr{F}''$ by $\mathscr{F}'$ up to isomorphism$\}$. Then we need to prove the morphism $\Phi : E(\mathscr{F}'', \mathscr{F}') \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ is a bijection set-theoretically.

We need to define a map $\Psi : \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}') \to E(\mathscr{F}'', \mathscr{F}')$. Fixing an injective resolution $0 \to \mathscr{F}' \to \mathscr{I}^\cdot$, we know every element $e \in \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ has a representing element $e' : \mathscr{F}'' \to Z^1(\mathscr{I}^\cdot) \in \operatorname{Hom}(\mathscr{F}'', Z^1(\mathscr{I}^\cdot))$. So we may define $\Psi(e) = \mathscr{F}$ as the pulling-back of $0 \to \mathscr{F}'' \to \mathscr{I}^0 \to Z^1(\mathscr{I}^\cdot) \to 0$ by e', i.e. $\mathscr{F} = \mathscr{I}^0 \oplus_{e'} \mathscr{F}'$ consisting of all section (s, t) of $\mathscr{I}^0 \oplus \mathscr{F}'$ which satisfying $d^0(s) = e'(t)$. The morphism $\mathscr{F}'' \to \mathscr{I}^0 \oplus_{e'} \mathscr{F}'$ is the direct sum of $\mathscr{F}'' \to \mathscr{I}^0$ and the zero morphism. And the morphism $\mathscr{I}^0 \oplus_{e'} \mathscr{F}' \to \mathscr{F}'$ is the projection. Suppose $e'' : \mathscr{F}'' \to Z^1(\mathscr{I}^\cdot)$ represents e too. There exists a morphism $f : \mathscr{F}'' \to \mathscr{I}^0$ such that $e' - e'' = d^0 f$. Then we have an isomorphism $\mathscr{I}^0 \oplus_{e'} \mathscr{F}'' \to \mathscr{I}^0 \oplus_{e''} \mathscr{F}''$ as $(s, t) \mapsto (s - f(t), t)$ for every section (s, t). Hence the Ψ is well-defined.

Easy to see that Φ and Ψ are inverse to each other. So $E(\mathscr{F}'', \mathscr{F}') \cong \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$.

Solution 3.6.2. (a) Suppose there exists such a $\mathscr{P}$. For any open subset $U \subset X$ and any closed point $x \in U$, we know $\mathscr{O}_X \to i_x(k(x))$ is surjective, where $i_x(k(x))$ is the skyscraper sheaf of the local field $k(x)$. Then $\mathscr{P} \to i_x(k(x)) \to 0$ is surjective. For any $V \subsetneq U$ open and containing x (the existence of such V is based on the fact that k is infinite), we have another surjection $j_!(\mathscr{O}_X|_V) \to i_x(k(x))$, where $j : V \to X$ is the embedding. So by the projectivity of $\mathscr{P}$, we have a morphism $\mathscr{P} \to j_!(\mathscr{O}_X|_V)$, and the following commutative diagram

$$\begin{tikzcd} \mathscr{P} \arrow[r] \arrow[d] & \mathscr{O}_X \arrow[r] \arrow[d] & 0 \\ j_!(\mathscr{O}_X|_V) \arrow[r] & i_x(k(x)) \arrow[r] & 0 \end{tikzcd}$$

Hence the morphism $\mathscr{P}(U) \to i_x(k(x))(U) = k(x)$ must factor through zero. So every section in $\mathscr{P}(U)$ has the zero stalk on x. By the ambiguity of x, $\mathscr{P}(U) = 0$. Hence $\mathscr{P} = 0$, which makes a contradiction.

(b) Suppose there exists a such $\mathscr{P}$. In the $\mathfrak{Coh}(X)$ case, there exists some n such that $\Gamma(X, \mathscr{P}(n)) \to \Gamma(X, \mathscr{O}_X(n))$ is surjective by theorem 5.2. So $\Gamma(X, \mathscr{P}(n))$ is not empty. In the $\mathfrak{Qco}(X)$ case, we can pick an affine piece $U = \operatorname{Spec} A$ of X. Then $\mathscr{P}|_U \to \mathscr{O}_X|_U \to 0$ is induced by an A-module injection $A \to M$. So clearly the image M' of this injection is a finitely generated A-module, which induces a surjection $\tilde{M}' \to \mathscr{O}_X|_U$. By 2.5.15. there exists a coherent subsheaf $\mathscr{P}'$ for $\mathscr{P}$, such that there exists a surjection $\mathscr{P}' \to \mathscr{O}_X$ and $\mathscr{P}'|_U = \tilde{M}'$. Then by theorem 5.2. for $n \gg 0$ we have $\Gamma(X, \mathscr{P}'(n)) \to \Gamma(X, \mathscr{O}_X(n))$ is surjective. So $\Gamma(X, \mathscr{P}'(n))$ is not empty, i.e. $\Gamma(X, \mathscr{P}(n)$ is not empty.

Since $i_x(k(x)) \otimes \mathscr{O}_X(-n-1) \cong i_x(k(x))$, we have a surjection $\mathscr{O}_X(-n-1) \to i_x(k(x)) \to 0$. Hence we have a morphism $\mathscr{P} \to \mathscr{O}_X(-n-1)$, which induces a twisting $\mathscr{P}(n) \to \mathscr{O}_X(-1)$. But $\Gamma(X, \mathscr{O}_X(-1))$ is empty, which makes a contradiction.

Solution 3.6.3. (b) By theorem 6.3. we may assume $X = \operatorname{Spec} A$ is affine. Then $\mathscr{F} = \tilde{M}$ for some finitely generated A-module M and $\mathscr{G} = \tilde{N}$ for some A-module N. Since M is finitely generated, there exists a finite free resolution $A^{n_\cdot} \to M \to 0$. Then $\operatorname{Ext}^i(M, N) = H^i(\operatorname{Hom}(A^{n_\cdot}, N)) = H^i(N^{n_\cdot})$ is an A-module. Hence by theorem 6.7. $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) = (\operatorname{Ext}^i(M, N))^\sim$ is quasi-coherent.

(a) As in (b), since N is finitely generated, $\operatorname{Ext}^i(M, N) = H^i(N^{n_\cdot})$ is also finitely generated. Hence $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is coherent.

Solution 3.6.4. By theorem 1.3. we just need to show that $(\mathscr{E}xt^i(\cdot, \mathscr{G})$ is a coeffaceable functor. Since $\mathfrak{Coh}(X)$ has enough locally frees, for any $\mathscr{F} \in \mathfrak{Coh}(X)$, there exists a locally free sheaf $\mathscr{L}$ such that $\mathscr{L} \to \mathscr{F} \to 0$. So we only need to show for any locally free $\mathscr{L}$ and $\mathscr{G} \in \mathfrak{Mod}(X)$, we have $\mathscr{E}xt^i(\mathscr{L}, \mathscr{G}) = 0$ for $i > 0$.

For every affine piece $U = \operatorname{Spec} A$ such that $\mathscr{L}|_U = \mathscr{O}_U^n$, we have $\mathscr{E}xt^i(\mathscr{L}, \mathscr{G})|_U = \mathscr{E}xt^i(\mathscr{O}_U^n, \mathscr{G}|_U)$ by theorem 6.7. Take an injective resolution $0 \to \mathscr{G} \to \mathscr{I}^\cdot$. We have $\mathscr{E}xt^i(\mathscr{L}, \mathscr{G})|_U = H^i(\mathscr{H}om(\mathscr{O}_U^n, \mathscr{I}^\cdot|_U) = H^i(\mathscr{I}^\cdot|_U)^n = 0$. Hence $\mathscr{E}xt^i(\mathscr{L}, \mathscr{G}) = 0$ in the global.

Solution 3.6.5. (a) ($\Rightarrow$) We've done in 3.6.4.

($\Leftarrow$) If $\mathscr{E}xt^1(\mathscr{F}, \mathscr{G}) = 0$, we have $\operatorname{Ext}^1(\mathscr{F}_x, \mathscr{G}_x) = \mathscr{E}xt^1(\mathscr{F}, \mathscr{G})_x = 0$ for every $x \in X$. Then by ambiguity of $\mathscr{G}$, we know that $\mathscr{F}_x$ is projective. Since it is also finitely generated, $\mathscr{F}_x$ is free. Then by 2.5.7., $\mathscr{F}$ is locally free.

(b) ($\Rightarrow$) By theorem 6.5. and (a), trivial.

($\Leftarrow$) Since we may take a locally free resolution $0 \to \mathscr{L}^n \ldots \to \mathscr{L}^0 \to \mathscr{F} \to 0$, denoting $\mathscr{H} = \ker(\mathscr{L}^0 \to \mathscr{F})$, we have a locally free resolution $0 \to \mathscr{L}^n \ldots \to \mathscr{L}^1 \to \mathscr{H} \to 0$ for $\mathscr{H}$, hence $\operatorname{hd}\mathscr{H} \leq n-1$. Since we have $0 \to \mathscr{H} \to \mathscr{L}_0 \to \mathscr{F} \to 0$, we know that $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) = \mathscr{E}xt^{i-1}(\mathscr{H}, \mathscr{G})$. Hence by induction we know $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) = 0$ for all $i > n$.

(c) Take the locally free resolution $0 \to \mathscr{L}^n \ldots \to \mathscr{L}^0 \to \mathscr{F} \to 0$ of $\mathscr{F}$. Then we have $0 \to \mathscr{L}_x^\cdot \to \mathscr{F}_x \to 0$. Hence $\operatorname{hd}\mathscr{F} \geq \sup_x \operatorname{pd}_{\mathscr{O}_x} \mathscr{F}_x$. Conversely, if $\operatorname{hd}\mathscr{F} > \sup_x \operatorname{pd}_{\mathscr{O}_x} \mathscr{F}_x$ strictly, by theorem 6.10. we know $\operatorname{Ext}^i(\mathscr{F}_x, N) = 0$ for every $x \in X$, $i \geq \operatorname{hd}\mathscr{F}$ and A-module N. So for every $\mathscr{G}$, we know $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) = 0$ for $i \geq \operatorname{hd}\mathscr{F}$, which makes a contradiction with (b).

Solution 3.6.6. (a) ($\Rightarrow$) Trivial by definition.

($\Leftarrow$) Firstly, we know $\mathrm{Ext}^i(M, A^n) = 0$ for all n by induction on n via the exact sequence $0 \to A^{n-1} \to A^n \to A \to 0$. Then for any finitely generated A-module N, there exists $0 \to K \to A^n \to N \to 0$ for some n and finitely generated K. Hence we have $\mathrm{Ext}^i(M, N) = \mathrm{Ext}^{i+1}(M, K)$. Then by decreasing induction and proposition 6.11., we know $\mathrm{Ext}^i(M, N) = 0$ for all $i > 0$. Since M is finitely generated, there exists L and m such that $0 \to L \to A^m \to M \to 0$. Then $\mathrm{Ext}^i(M, L) = 0$ for all i. Hence $\mathrm{Hom}(M, A^m) \to \mathrm{Hom}(M, M)$ is surjective. So the identity $M \to M$ factors through $M \to A^m \to M$, hence M is a summand of A^m, i.e. projective.

(b) ($\Rightarrow$) By proposition 6.10., trivial.

($\Leftarrow$) We will prove it by induction on n. If $n = 0$, this is just (a). For generous n, since M is finitely generated, there exists an L and m such that $0 \to L \to A^m \to M \to 0$. Hence $\mathrm{Ext}^{i-1}(N, A) = \mathrm{Ext}^i(M, A) = 0$ for all $i > n$. By induction $\mathrm{pd} N \le n - 1$. Hence $\mathrm{pd} M \le \mathrm{pd} N + 1 = n$.

Solution 3.6.7. Take a free resolution of M as $A^{n_\cdot} \to M \to 0$. Then we have a locally free resolution of $\tilde{M}$ as $\mathscr{O}_X^{n_\cdot} \to \tilde{M} \to 0$. Then $\mathrm{Ext}^i_X(\tilde{M}, \tilde{N}) = H^i(\mathrm{Hom}_X(\mathscr{O}_X^{n_\cdot}, \tilde{N})) = H^i(\mathrm{Hom}_A(A^{n_\cdot}, N)) = \mathrm{Ext}^i_A(M, N)$, and $\mathscr{E}xt^i_X(\tilde{M}, \tilde{N})) = H^i(\mathscr{H}om_X(\mathscr{O}_X^{n_\cdot}, \tilde{N})) = H^i(\mathrm{Hom}_X(A^{n_\cdot}, N)^\sim) = (H^i(\mathrm{Hom}_X(A^{n_\cdot}, N))^\sim = \mathrm{Ext}^i_A(M, N)^\sim$.

Solution 3.6.8. (a) For every open subset $U \subset X$, we denote $Z = X - U$. Firstly we consider the case that Z is irreducible. In this case, Z is a Weil divisor of X, which corresponds to a Cartier divisor $D = \{(U_i, f_i)\}$. Then consider the invertible sheaf $\mathscr{L} = \mathscr{L}(D)$, which satisfies $\mathscr{L}|_{U_i} \cong \mathscr{O}_{U_i}$, $f_i^{-1} \leftarrow 1$. Then we consider the global section $s \in \Gamma(X, \mathscr{L})$ glued by all f_i^{-1} on U_i. So $X_s = X - Z = U$. For general case, if $Z = Z_1 \cup \ldots \cup Z_n$, there exists $\mathscr{L}_i$ and global sections s_i such that $X_{s_i} = X - Z_i$. Then consider $\mathscr{L} = \bigotimes \mathscr{L}_i$ and the global section $s = \bigotimes s_i$. So we have $X_s = \bigcup X_{s_i} = U$.

(b) Since $\mathscr{F}$ is coherent, we may cover X by some $U_i = \mathrm{Spec}\, A_i$ such that $\mathscr{F}|_{U_i} = \tilde{M}_i$ for some finitely generated A_i-modules, i.e. $\mathscr{F}|_{U_i}$ is generated by finitely many section $m_{ij} \in M_i$. By (a), there exist $s_{ij} \in \Gamma(U_i, \mathscr{L}_{ij})$ for some $\mathscr{L}_{ij}$ such that U_i is covered by $X_{s_{ij}}$ and $m_{ij} \in \Gamma(X_{s_{ij}}, \mathscr{F})$. Then by theorem 5.14. in chapter II, $s_{ij}^{n_{ij}} m_{ij} \in \Gamma(X, \mathscr{L}_{ij}^{n_{ij}} \otimes \mathscr{F})$, which determine a morphism $\mathscr{O}_X \to \mathscr{L}_{ij}^{n_{ij}} \otimes \mathscr{F}$. Tensoring with $\mathscr{L}_{ij}^{n_{ij}}$ and direct summing up all i, j, we have a morphism $\bigoplus_{i,j} \mathscr{L}_{ij}^{n_{ij}} \to \mathscr{F}$. Since m_{ij} generates $\mathscr{F}$ locally, this morphism is surjective.

Solution 3.6.9. (a) By 3.6.8. the existence of locally free resolution of $\mathscr{F}$ is obvious. By theorem 6.11A. and the regularity of X, we know $\mathrm{pd}\mathscr{F}_x \le \dim \mathscr{O}_{X,x} \le \dim X$ for all $x \in X$. And by 3.6.5. we know $\mathrm{hd}\mathscr{F} \le \sup_x \mathrm{pd}\mathscr{F}_x \le \dim X < \infty$, hence there exists a finite locally free resolution of $\mathscr{F}$.

(b) **STEP 1.** We need to show if there exists two surjections $\mathscr{F}' \to \mathscr{F}$ and $\mathscr{F}'' \to \mathscr{F}$, there exists a locally free sheaf $\mathscr{E}$ and a commutative diagram

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d] & \mathscr{F}' \arrow[d] \\ \mathscr{F}'' \arrow[r] & \mathscr{F} \end{tikzcd}$$

with every arrows surjective.

Denote the kernal of $\mathscr{F}' \oplus \mathscr{F}'' \to \mathscr{F}$ as $\mathscr{G}$. Since $\mathscr{F}' \to \mathscr{F}$ and $\mathscr{F}'' \to \mathscr{F}$ are both surjective, we know the morphism $\mathscr{G} \to \mathscr{F}' \oplus \mathscr{F}'' \to \mathscr{F}'$ and $\mathscr{G} \to \mathscr{F}' \oplus \mathscr{F}'' \to \mathscr{F}''$ are both surjective. Then by 3.6.8., there exists a locally free sheaf $\mathscr{E}$ with a surjection $\mathscr{E} \to \mathscr{G}$, hence the $\mathscr{E}$ is the one we need.

STEP 2. We need to show that if we have exact sequences $0 \to \mathscr{G}' \to \mathscr{E}' \to \mathscr{F}' \to 0$ and $0 \to \mathscr{G}'' \to \mathscr{E}'' \to \mathscr{F}'' \to 0$, and two surjections $\mathscr{F}' \to \mathscr{F}$ and $\mathscr{F}'' \to \mathscr{F}$, there exists an exact sequence $0 \to \mathscr{G} \to \mathscr{E} \to \mathscr{F} \to 0$

with a commutative diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathscr{G}' \arrow[r] \arrow[u] & \mathscr{E}' \arrow[r] \arrow[u] & \mathscr{F}' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathscr{G} \arrow[r] \arrow[u] \arrow[d] & \mathscr{E} \arrow[r] \arrow[u] \arrow[d] & \mathscr{F} \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{G}'' \arrow[r] \arrow[d] & \mathscr{E}'' \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with every rows and strips exact.

By Step 1, there exists a locally free sheaf $\mathscr{E}_1$ satisfying the diagram

$$\begin{tikzcd}
\mathscr{E}_1 \arrow[r] \arrow[d] & \mathscr{F} \arrow[d] \\
\mathscr{E}' \arrow[r] & \mathscr{F}'
\end{tikzcd}$$

with all arrows surjective. Then there exists a locally free sheaf $\mathscr{E}_2$ satisfying the diagram

$$\begin{tikzcd}
\mathscr{E}_2 \arrow[r] \arrow[d] & \mathscr{E}_1 \arrow[r] & \mathscr{F} \arrow[d] \\
\mathscr{E}'' \arrow[rr] & & \mathscr{F}''
\end{tikzcd}$$

with all arrows surjective. So this $\mathscr{E}_2$ satisfies the following diagram

$$\begin{tikzcd}
0 & 0 & \\
\mathscr{E}' \arrow[r] \arrow[u] & \mathscr{F}' \arrow[r] \arrow[u] & 0 \\
\mathscr{E}_2 \arrow[r, "c"] \arrow[u, "a"] \arrow[d, "b"] & \mathscr{F} \arrow[r] \arrow[u] \arrow[d] & 0 \\
\mathscr{E}'' \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[r] \arrow[d] & 0 \\
0 & 0 &
\end{tikzcd}$$

By 3.6.8. we have two locally free sheaves $\mathscr{E}_3$ and $\mathscr{E}_4$ such that $\mathscr{E}_3 \xrightarrow{d} \mathscr{G}' \to 0$ and $\mathscr{E}_4 \xrightarrow{e} \mathscr{G}'' \to 0$. So we may define $\mathscr{E} = \mathscr{E}_2 \oplus \mathscr{E}_3 \oplus \mathscr{E}_4$ and $\mathscr{G} = \ker(c) \oplus \mathscr{E}_3 \oplus \mathscr{E}_4$ and

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathscr{G}' \arrow[u] \arrow[r, "g"] & \mathscr{E}' \arrow[u] \arrow[r] & \mathscr{F}' \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \ker(c) \oplus \mathscr{E}_3 \oplus \mathscr{E}_4 \arrow[u, "a|_{\ker(c)}\oplus 0 \oplus e"] \arrow[d, "b|_{\ker(c)}\oplus d \oplus 0"] \arrow[r] & \mathscr{E}_2 \oplus \mathscr{E}_3 \oplus \mathscr{E}_4 \arrow[u, "a\oplus 0 \oplus ge"] \arrow[d, "b\oplus hd \oplus 0"] \arrow[r, "c\oplus 0\oplus 0"] & \mathscr{F} \arrow[u] \arrow[d] \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{G}'' \arrow[d] \arrow[r, "h"] & \mathscr{E}'' \arrow[d] \arrow[r] & \mathscr{F}'' \arrow[d] \arrow[r] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Hence, if we have two locally free resolutions $\mathscr{E}'^{\cdot} \to \mathscr{F} \to 0$ and $\mathscr{E}''^{\cdot} \to \mathscr{F} \to 0$, there exists a locally free resolution $\mathscr{E}^{\cdot} \to \mathscr{F} \to 0$ with surjections $\mathscr{E}^{\cdot} \to \mathscr{E}'^{\cdot}$ and $\mathscr{E}^{\cdot} \to \mathscr{E}''^{\cdot}$ with all things commutative.

STEP 3. We need to show the independence of the choice of the resolution of $\mathscr{F}$ for δ.

By Step 2 we only need to consider the case that two choices $\mathscr{E}^{\cdot} \to \mathscr{F} \to 0$ and $\mathscr{E}'^{\cdot} \to \mathscr{F} \to 0$ have the surjections $\mathscr{E}^{\cdot} \to \mathscr{E}'^{\cdot}$. We denote the kernal of this surjections as $\mathscr{G}^{\cdot}$. Then $\sum_i (-1)^i [\mathscr{E}^i] = \sum_i (-1)^i ([\mathscr{E}'^i] + [\mathscr{G}^i]) = \sum_i (-1)^i [\mathscr{E}'^i] + \sum_i (-1)^i [\mathscr{G}^i] = \sum_i (-1)^i [\mathscr{E}'^i]$. Hence the definition of δ is well-defined.

STEP 4. We need to show that δ is actually a group morphism, i.e. for any exact sequence $0 \to \mathscr{F}'' \to \mathscr{F} \to \mathscr{F}' \to 0$, we have $\delta([\mathscr{F}]) = \delta([\mathscr{F}']) + \delta([\mathscr{F}''])$.

Since we can find two locally free sheaf $\mathscr{E}'^0$ and $\mathscr{E}''^0$ with surjections $\mathscr{E}'^0 \to \mathscr{F}' \to 0$ and $\mathscr{E}''^0 \xrightarrow{c} \mathscr{F}'' \to 0$. By Step 1, there exists a locally free sheaf $\mathscr{E}$ and diagram

$$\begin{tikzcd}
\mathscr{E} \arrow[r, "a"] \arrow[d, "b"] & \mathscr{E}'^0 \arrow[d] \\
\mathscr{F} \arrow[r] & \mathscr{F}'
\end{tikzcd}$$

with all arrows surjective. Then we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(a) \oplus \mathscr{E}''^0 \arrow[r] \arrow[d, "b|_{\ker(a)}\oplus c"] & \mathscr{E} \oplus \mathscr{E}''^0 \arrow[r, "a\oplus 0"] \arrow[d, "b\oplus dc"] & \mathscr{E}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}'' \arrow[r, "d"] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

So by this progress, we construct three locally free resolution $\mathscr{E}^{\cdot} \to \mathscr{F} \to 0$, $\mathscr{E}'^{\cdot} \to \mathscr{F}' \to 0$ and $\mathscr{E}''^{\cdot} \to \mathscr{F}'' \to 0$

with commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \mathscr{E}''^1 \arrow[r] \arrow[d] & \mathscr{E}''^0 \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \mathscr{E}^1 \arrow[r] \arrow[d] & \mathscr{E}^0 \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \mathscr{E}'^1 \arrow[r] \arrow[d] & \mathscr{E}'^0 \arrow[r] \arrow[d] & \mathscr{F}' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

So clearly $\delta([\mathscr{F}]) = \delta([\mathscr{F}']) + \delta([\mathscr{F}''])$, i.e. δ is a group morphism.

STEP 5. δ and ε are clearly inverse to each other. Hence $K(X) \cong K_1(X)$.

Solution 3.6.10 (Duality for a Finite Flat Morphism)**.** (a) For any affine piece $U = \operatorname{Spec} A$ of Y, we may assume $\mathscr{G}|_U = \tilde{N}$, and $f^{-1}(U) = \operatorname{Spec} B \subset X$. Then $\mathscr{H}om_Y(f_*\mathscr{O}_X, \mathscr{G})|_U = (\operatorname{Hom}_A(B, N))^\sim$. Since B is a finite A-module, we know that $\operatorname{Hom}_A(B, N)$ is a B-module. Hence $\mathscr{H}om_Y(f_*\mathscr{O}_X, \mathscr{G})$ is a quasi-coherent $f_*\mathscr{O}_X$-module.

(b) Firstly we define a morphism $\alpha : f_*\mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G}) \to \mathscr{H}om_Y(f_*\mathscr{F}, f_*f^!\mathscr{G})$ as for any $\phi \in \operatorname{Hom}_{f^{-1}(U)}(\mathscr{F}|_{f^{-1}(U)}, f^!\mathscr{G}|_{f^{-1}(U)})$ maps to $\psi \in \operatorname{Hom}_U(f_*\mathscr{F}|_U, f_*f^!\mathscr{G}|_U)$ as $\psi_W = \phi_{f^{-1}(W)}$ for any open subset $W \subset U$. Then by (a) we have an isomorphism $f_*f^!\mathscr{G} \cong \mathscr{H}om_Y(f_*\mathscr{O}_X, \mathscr{G})$, which induces a morphism $\beta : \mathscr{H}om_Y(f_*\mathscr{F}, f_*f^!\mathscr{G}) \to \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{H}om_Y(f_*\mathscr{O}_X, \mathscr{G}))$. Finally, since $f_*\mathscr{O}_X$ is an $\mathscr{O}_Y$-algebra, we have a morphism $\mathscr{H}om_Y(f_*\mathscr{O}_X, \mathscr{G}) \to \mathscr{H}om_Y(\mathscr{O}_Y, \mathscr{G}) \cong \mathscr{G}$, hence we have a $\gamma : \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{H}om_Y(f_*\mathscr{O}_X, \mathscr{G})) \to \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{G})$. So we just need to define $\delta = \gamma \circ \beta \circ \alpha : f_*\mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G}) \to \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{G})$. On every affine piece $U = \operatorname{Spec} A \subset Y$, we assume $f^{-1}(U) = \operatorname{Spec} B \subset Y$, $\mathscr{F} = \tilde{M}$ and $\mathscr{G} = \tilde{N}$ for some A-algebra B, B-module M and A-module N. The δ on U is just $\operatorname{Hom}_B(M, \operatorname{Hom}_A(B, N)) \to \operatorname{Hom}_A(M \otimes_A B, N)$ as $\phi \mapsto (m \otimes 1 \mapsto \phi(m)(1))$, which is an isomorphism trivially. Hence δ is isomorphic.

(c) As we've done in (b), we have a natural translation $\mathscr{H}om_X(\mathscr{F}, \cdot) \to \mathscr{H}om_Y(f_*\mathscr{F}, f_*\cdot)$. Composing with $\Gamma(\cdot)$, we have $\operatorname{Hom}_X(\mathscr{F}, \cdot) \to \operatorname{Hom}_Y(f_*\mathscr{F}, f_*\cdot)$. Since $\operatorname{Ext}^i_X(\mathscr{F}, \cdot)$ and $\operatorname{Ext}^i_Y(f_*\mathscr{F}, f_*\cdot)$ are both δ-functors, this makes a morphism of δ-functors, $\operatorname{Ext}^i_X(\mathscr{F}, \cdot) \to \operatorname{Ext}^i_Y(f_*\mathscr{F}, f_*\cdot)$. Hence we have a morphism $\operatorname{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \operatorname{Ext}^i_Y(f_*\mathscr{F}, f_*f^!\mathscr{G})$. Moreover, we have a morphism $f_*f^!\mathscr{G} \to \mathscr{G}$ as in (b), by the functorality of Ext we have a morphism $\phi^i : \operatorname{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \operatorname{Ext}^i_Y(f_*\mathscr{F}, f_*f^!\mathscr{G}) \to \operatorname{Ext}^i_Y(f_*\mathscr{F}, \mathscr{G})$.

(d) We'll prove this by induction on the homological dimension of $\mathscr{F}$. If $\operatorname{hd}\mathscr{F} = 0$, i.e. $\mathscr{F}$ is locally free, by proposition 6.2. we only need to consider the case that $\mathscr{F} = \mathscr{O}_X$. Then $\operatorname{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) = H^i(X, f^!\mathscr{G})$. Since f is affine, 3.4.1. implies $H^i(X, f^!\mathscr{G}) \cong H^i(Y, f_*f^!\mathscr{G}) \cong H^i(Y, (f_*\mathscr{O}_X)^\vee \otimes \mathscr{G})$. And by theorem 6.3., we know $H^i(Y, (f_*\mathscr{O}_X)^\vee \otimes \mathscr{G}) \cong \operatorname{Ext}^i_Y(f_*\mathscr{O}_X, \mathscr{G})$. For general case, for any $\mathscr{F}$, there exists a locally free sheaf $\mathscr{E}$ and a coherent sheaf $\mathscr{H}$ such that $\operatorname{hd}\mathscr{H} = \operatorname{hd}\mathscr{F} - 1$ satisfying $0 \to \mathscr{H} \to \mathscr{E} \to \mathscr{F} \to 0$. Since f is affine, we have $0 \to f_*\mathscr{H} \to f_*\mathscr{E} \to f_*\mathscr{F} \to 0$. Hence for every $i > 0$, we have

$$\begin{tikzcd}
\operatorname{Ext}^i(\mathscr{H}, f^!\mathscr{G}) \arrow[r] \arrow[d, "\cong"] & \operatorname{Ext}^i(\mathscr{E}, f^!\mathscr{G}) \arrow[r] \arrow[d, "\cong"] & \operatorname{Ext}^i(\mathscr{F}, f^!\mathscr{G}) \arrow[r] \arrow[d, "\phi_i"] & \operatorname{Ext}^{i+1}(\mathscr{H}, f^!\mathscr{G}) \arrow[r] \arrow[d, "\cong"] & \operatorname{Ext}^{i+1}(\mathscr{E}, f^!\mathscr{G}) \arrow[d, "\cong"] \\
\operatorname{Ext}^i(f_*\mathscr{H}, \mathscr{G}) \arrow[r] & \operatorname{Ext}^i(f_*\mathscr{E}, \mathscr{G}) \arrow[r] & \operatorname{Ext}^i(f_*\mathscr{F}, \mathscr{G}) \arrow[r] & \operatorname{Ext}^{i+1}(f_*\mathscr{H}, \mathscr{G}) \arrow[r] & \operatorname{Ext}^{i+1}(f_*\mathscr{E}, \mathscr{G})
\end{tikzcd}$$

So by five-lemma, we know $\phi_i : \operatorname{Ext}^i(\mathscr{F}, f^!\mathscr{G}) \cong \operatorname{Ext}^i(f_*\mathscr{F}, \mathscr{G})$.

3.7 The Serre Duality Theorem

Solution 3.7.1. Since X is projective and $\mathscr{L}$ is ample, we know there exists an n such that $\mathscr{L}^n$ is very ample. Hence if $i : X \to \mathbb{P}^N_k$ is the canonical embedding, we have $\dim_k \Gamma(X, \mathscr{L}^n) = \dim_k \Gamma(X, i^*\mathscr{O}(1)) = \dim X + 1$. Hence if $\Gamma(X, \mathscr{L}^{-1})$ is not empty, there exists some nonzero $s \in \Gamma(X, \mathscr{L}^{-n})$. So the morphism $\Gamma(X, \mathscr{L}^n) \xrightarrow{\times s} \Gamma(X, \mathscr{O}_X)$ has image for more than dimension 1 over k. But $\dim_k \Gamma(X, \mathscr{O}_X) = 1$, which is contradict.

Solution 3.7.2. (a) For every coherent sheaf $\mathscr{F}$ on Y, clearly we have a morphism $\mathrm{Hom}_Y(\mathscr{F}, \omega_Y^\circ) \cong \mathrm{Ext}^n_Y(\mathscr{O}_Y, \mathscr{F})'$. Then for every coherent sheaf $\mathscr{G}$ on X, we know $f_*\mathscr{G}$ is coherent since f is finite. Then by 3.6.10., $\mathrm{Hom}_X(\mathscr{G}, f^!\omega_Y^\circ) = \mathrm{Hom}_Y(f_*\mathscr{G}, \omega_Y^\circ) = \mathrm{Ext}^n_Y(\mathscr{O}_Y, f_*\mathscr{G})' = \mathrm{Ext}^n_X(\mathscr{O}_X, \mathscr{G})'$. Hence $f^!\omega_Y^\circ$ is a dualizing sheaf of X. So by proposition 7.2., we have $\omega_X^\circ \cong f^!\omega_Y^\circ$.

(b) By corollary 7.12., we know that $\omega_X \cong \omega_X^\circ$ and $\omega_Y \cong \omega_Y^\circ$. Hence $\omega_X \cong f^!\omega_Y$. So we have a natural morphism $t : f_*\omega_X \cong f_*f^!\omega_Y \to \omega_Y$.

Solution 3.7.3. By 2.5.16.(c), we have a filtration $\wedge^r(\mathscr{O}_X(-1)^{n+1}) = \mathscr{F}^0 \supset \dots \mathscr{F}^r \supset \mathscr{F}^{r+1} = 0$, which satisfies $\mathscr{F}^p/\mathscr{F}^{p+1} \cong \Omega^p_X \otimes \wedge^{r-p}\mathscr{O}_X$. Clearly for $r \neq 0, 1$, we have $\wedge^r\mathscr{O}_X \cong 0$, and for $r = 0, 1$, we have $\wedge^r\mathscr{O}_X \cong \mathscr{O}$. So $\mathscr{F}^0 = \dots = \mathscr{F}^{r-1}$, hence the filtration is actually is just $\wedge^r(\mathscr{O}_X(-1)^{n+1}) \supset \mathscr{F}^r \supset \mathscr{F}^{r+1} = 0$ such that $\mathscr{F}^r \cong \mathscr{F}^r/\mathscr{F}^{r+1} \cong \Omega^r_X \otimes \wedge^0\mathscr{O}_X \cong \Omega^r_X$, and $\wedge^r(\mathscr{O}_X(-1)^{n+1})/\Omega^r = \mathscr{F}^0/\mathscr{F}^r \cong \Omega^{r-1}_X \otimes \wedge^1\mathscr{O}_X \cong \Omega^{r-1}_X$. Moreover, since $\mathscr{O}_X(-1)$ is a line bundle, we have $\wedge^r(\mathscr{O}_X(-1)^{n+1}) \cong \mathscr{O}_X(-r)^{\oplus N}$ for $N = \binom{r}{n+1}$. So we have an exact sequence $0 \to \Omega^r_X \to \mathscr{O}_X(-r)^{\oplus N} \to \Omega^{r-1}_X \to 0$.

Since $H^i(X, \mathscr{O}_X(-r)) = 0$ for all $i < n$ or $r < n+1$, we have $H^i(X, \Omega^r_X) \cong H^{i-1}(X, \Omega^{r-1}_X)$ for $1 \le i$ when $r < n+1$, or $1 \le i < n$ when $r \ge n+1$. Since $H^0(X, \Omega^0_X) \cong H^0(X, \mathscr{O}_X) \cong k$, we have $H^i(X, \Omega^i_X) \cong k$ for all $0 \le i \le n$. Moreover, for $i < n$, we have $H^i(X, \Omega^n_X) \cong H^i(X, \mathscr{O}_X(-n-1)) \cong 0$, so when $i < r$ and $0 \le r \le n$ we already have $H^i(X, \Omega^r_X) = 0$. Then by corollary 7.13., we have $H^i(X, \Omega^r_X) = H^{n-i}(X, \Omega^{n-r}_X)' = 0$ for all $i > r$ and $0 \le r \le n$. So we have $H^i(X, \Omega^r_X) = 0$ for all $i \neq r$.

Solution 3.7.4 (The Cohomology Class of a Subvariety)**.** (a) In this case, we know that $p = n$, and $H^0(X, \mathscr{O}_X) \cong H^0(P, \mathscr{O}_P) \cong k$. The morphism $H^0(X, \mathscr{O}_X) \to H^0(P, \mathscr{O}_P)$ is just the restriction $s \mapsto s_P$, which is just the identity on k. Hence it corresponds to the element 1 in $H^n(X, \omega_X) \cong k$, i.e. $t_X(\eta(P)) = 1$.

(b) By Bertini's theorem, there exists a hyperplane H of dimension $n-p$, such that $Z = H \cap Y$ completely. Then we restrict the morphism $H^{n-p}(X, \Omega^{n-p}_X) \to H^{n-p}(Y, \Omega^{n-p}_Y)$ on the H as $H^{n-p}(H, \Omega^{n-p}_H) \to H^{n-p}(Z, \Omega^{n-p}_Z)$. Since Z is just $\deg(Y)$ points. Then by (a) we know this morphism is just $s \mapsto (s_{P_1}, \dots, s_{P_{\deg(Y)}})$, i.e. the $\deg(Y)$-copies of identities of k. Since the morphism $H^{n-p}(Z, \Omega^{n-p}_Z) \cong k^{\deg(Y)} \to k$ is just a sum, we know the composition $k \cong H^{n-p}(H, \Omega^{n-p}_H) \to k$ is $\times\deg(Y)$. Hence it corresponds to the element $\deg(Y) \cdot 1$ in $H^p(H, \mathscr{O}^p_H) \cong H^p(X, \mathscr{O}_X)$, where we need the fact that the morphism $k \cong H^p(H, \mathscr{O}^p_H) \cong H^p(X, \mathscr{O}_X) \cong k$ is just identity from the proof in 3.7.3.

(c) We just need to prove that $d\log$ is a morphism of sheaves of abelian groups. Since on Ω_X, we know $d(fg) = f\,dg + g\,df$. So $d\log(fg) = f^{-1}g^{-1}d(fg) = f^{-1}g^{-1}(f\,dg + g\,df) = f^{-1}df + g^{-1}dg = d\log(f) + d\log(g)$.

(d) Denote $\deg\mathscr{Y} = d$. Then we may denote the Cartier divisor corresponding to Y as $\{(U_i, f_i)\}$ for some affine covering U_i and $f_i \in \Gamma(U_i, \mathscr{O}_X)$ with degree d. Hence the cocycle on $H^1(X, \mathscr{O}^*_X)$ corresponding to $\mathscr{L}_Y$ is $f_i/f_j \in U_{ij} = U_i \cap U_j$. Then $c(\mathscr{L}_Y) = (f_i^{-1}df_i - f_j^{-1}df_j, U_{ij})$. So via the morphism $H^1(X, \Omega_X) \to H^0(X, \mathscr{O}_X)$ in 3.7.3., the element $c(\mathscr{L}_Y)$ maps to $\sum f_i x_k \frac{\partial f_i}{\partial x_k} = d$. Hence $c(\mathscr{L}_Y) = d = \eta(Y)$, where the second equality is from (b).

3.8 Higher Direct Images of Sheaves

Solution 3.8.1. Fix an injective resolution $0 \to \mathscr{F} \to \mathscr{I}^\cdot$ in $\mathfrak{Mod}(X)$. Then $f_*\mathscr{I}^i$ are all injective. Moreover, since $\mathrm{R}^if_*(\mathscr{F}) = 0$, we know $0 \to f_*\mathscr{F} \to f_*\mathscr{I}^\cdot$ is exact, hence an injective resolution of $f_*\mathscr{F}$ in $\mathfrak{Mod}(Y)$. Since $\Gamma(X, \mathscr{I}^i) \cong \Gamma(Y, f_*\mathscr{I}^i)$, we know $H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$.

Solution 3.8.2. For any affine V in Y, we know $f^{-1}(V)$ is also affine, hence $H^i(f^{-1}(V),\mathscr{F})=0$ for all $i>0$. Since $\mathrm{R}^i f_*\mathscr{F}$ is the sheaf associated to the presheaf $V\mapsto H^i(f^{-1}(V),\mathscr{F})$, which is the zero sheaf. So $\mathrm{R}^i f_*\mathscr{F}=0$ for all $i>0$.

Solution 3.8.3. Fix an injective resolution $0\to\mathscr{F}\to\mathscr{I}^\cdot$ of $\mathscr{F}$. Then since $f^*\mathscr{E}$ is locally free, by proposition 6.7. we know $0\to\mathscr{F}\otimes f^*\mathscr{E}\to\mathscr{I}^\cdot\otimes f^*\mathscr{E}$ is an injective resolution of $\mathscr{F}\otimes f^*\mathscr{E}$. By 2.5.1.(d), we know $f_*(\mathscr{I}^\cdot\otimes f^*\mathscr{E})\cong f_*\mathscr{I}^\cdot\otimes\mathscr{E}$. Since tensoring a locally free sheaf is exact, we have $\mathrm{R}^i f_*(\mathscr{F}\otimes f^*\mathscr{E})=H^i(f_*(\mathscr{I}^\cdot\otimes f^*\mathscr{E}))=H^i(f_*\mathscr{I}^\cdot\otimes\mathscr{E})=H^i(f_*\mathscr{I}^\cdot)\otimes\mathscr{E}=\mathrm{R}^i f_*\mathscr{F}\otimes\mathscr{E}$.

Solution 3.8.4. (a) Since $\mathscr{E}$ is locally free, we may take an affine covering of $X=\bigcup U_j=\bigcup\operatorname{Spec}A_j$ such that $\mathscr{E}|_{U_j}$ are free. Then $H^i(\pi^{-1}(U_j),\mathscr{O}(l))=H^i(\mathbb{P}^n_{A_i},\mathscr{O}(l))=0$ in the case $0<i<n$, $l\in\mathbb{Z}$ or the case $i=n$, $l>-n-1$. Then by theorem 8.1., we know $\mathrm{R}^i f_*\mathscr{O}(l)=0$ in these cases.

(b) By theorem 7.11.(b) in chapter II, we have a surjection $\pi^*\mathscr{E}\to\mathscr{O}(1)$. So tensoring by $\mathscr{O}(-1)$, we have a surjection $\pi^*\mathscr{E}(-1)\to\mathscr{O}_X$, whose kernal we denote as $\mathscr{F}$. So for any affine piece $U=\operatorname{Spec}A$ on Y such that $\mathscr{E}$ is free on U, we have $0\to\mathscr{F}|_{\mathbb{P}^n_A}\to\pi^*\mathscr{E}(-1)|_{\mathbb{P}^n_A}\to\mathscr{O}|_{\mathbb{P}^n_A}\to0$. Then we know $\mathscr{F}|_{\mathbb{P}^n_A}\cong\Omega_{X/Y}|_{\mathbb{P}^n_A}$. So we have an exact sequence $0\to\Omega_{X/Y}\to\pi^*\mathscr{E}(-1)\to\mathscr{O}_X\to0$.

By 2.5.16.(e), we know $\omega_{X/Y}=\wedge^n\Omega_{X/Y}\cong(\pi^*\wedge^{n+1}\mathscr{E})(-n-1)$. Moreover, since we have $\omega_{X/Y}|_{\mathbb{P}^n_A}\cong\mathscr{O}_{\mathbb{P}^n_A}(-n-1)$, then $(\mathrm{R}^n\pi_*\omega_{X/Y})|_U=\mathrm{R}^n\pi_*(\omega_{X/Y}|_U)=H^n(\mathbb{P}^n_A,\omega_{\mathbb{P}^n_A/A})^\sim=\tilde{A}=\operatorname{Spec}A$. Hence $\mathrm{R}^n\pi_*\omega_{X/Y}\cong\mathscr{O}_Y$.

(c) When $l>-n-1$, by (a) and the fact that $\pi_*\mathscr{O}(-l-n-1)=0$, trivial. When $l=-n-1$, we may consider the Koszul complex $0\to\pi^*(\wedge^{n+1}\mathscr{E})(-n-1)\to\dots\to\pi^*(\wedge^2\mathscr{E})(-2)\to\pi^*(\mathscr{E})(-1)\to\mathscr{O}_X\to0$. Denote $L^{i,q}=\mathrm{R}^q\pi_*(\pi^*\wedge^{-i}\mathscr{E})(i)\cong\wedge^{-i}\mathscr{E}\otimes\mathrm{R}^q\pi_*(\mathscr{O}_X(i))$. Then we have a spectral sequence $E_2^{p,q}=H^p(L^{\cdot,q})$. So clearly, $E_2^{0,0}=\pi_*\mathscr{O}_X\cong\mathscr{O}_Y$, $E^{-n-1,n}=\wedge^{n+1}\mathscr{E}\otimes\mathrm{R}^n\pi_*(\mathscr{O}_X(-n-1))$, and $E_2^{p,q}=0$ in other cases. Hence, $d_{n+1}^{0,0}:\mathscr{O}_Y\to\wedge^{n+1}\mathscr{E}\otimes\mathrm{R}^n\pi_*(\mathscr{O}_X(-n-1))$ is an isomorphism, i.e. $\mathrm{R}^n\pi_*(\mathscr{O}(-n-1))\cong\wedge^{n+1}\mathscr{E}^\vee$. When $l<-n-1$, we may consider the map $\pi^*(S^{-l-n-1}\mathscr{E})\otimes\mathscr{O}(l)\cong\mathscr{O}(-n-1)$. Then by projection formula, $S^{-l-n-1}(\mathscr{E})\otimes\mathrm{R}^n\pi_*\mathscr{O}(l)\cong\mathrm{R}^n\pi_*\mathscr{O}(-n-1)\cong\wedge^{n+1}\mathscr{E}^\vee$. So $\mathrm{R}^n\pi_*\mathscr{O}(l)\cong S^{-l-n-1}(\mathscr{E})\otimes\wedge^{n+1}\mathscr{E}^\vee\cong\pi_*(\mathscr{O}(-l-n-1))\otimes\wedge^{n+1}\mathscr{E}^\vee$.

(d) Firstly, $p_a(Y)=\sum_{i=0}^{n-1}(-1)^i h^{n-i}(\mathscr{O}_Y)$. So $p_a(X)=\sum_{i=0}^{2n-1}(-1)^i h^{2n-i}(\mathscr{O}_X)=\sum_{i=0}^{n-1}(-1)^{i+n}h^{n-i}(\mathscr{O}_X)=(-1)^n p_a(X)$. Secondly, if ω_X has global section s, on each projective piece U of X, we know $s|_U\in\Gamma(U,\omega_X|_U)=\Gamma(U,\omega_U)$. Since $\Gamma(U,\omega_U)=0$, this makes a contradiction. Hence $\Gamma(X,\omega_X)=0$, i.e. $p_g(X)=\dim\Gamma(X,\omega_X)=0$.

(e) **Again. Is this a question?**

3.9 Flat Morphisms

Solution 3.9.1. Since the morphism $f|_U$ is also of finite type, we just need to prove that $f(X)$ is open. By 2.3.18. we've already know that $f(X)$ is a constructible set. For any $y\in f(X)$ with preimage $x\in X$, and any y' such that $y\in\overline{\{y'\}}$, we know that y' corresponds to a prime ideal in $\mathscr{O}_{Y,y}$. So since the morphism $\mathscr{O}_{Y,y}\to\mathscr{O}_{X,x}$ is flat, by going-down theorem of flatness version, there exists a prime ideal in $\mathscr{O}_{X,x}$ corresponding to y', i.e. there exists a $x'\in X$ such that $x\in\overline{\{x'\}}$ and $f(x')=y'$. Hence $y'\in f(X)$, i.e. $f(X)$ is open.

Solution 3.9.2. In the $w=1$ plane, the curve is $(x,y,z)=(t^3,t^2,t)$. Consider the curve family X_a as $(x,y,z)=(t^3,t^2,at)$. Then $X_a=\operatorname{Spec}k[a,x,y,z]/I$, where $I=(y^3-x^2,z^2-a^2y,z^3-a^3x,zy-ax,zx-ay^2)$. Hence if $a=0$, we have $I=(y^3-x^2,z^2,zy,zx)$, i.e. the fibre at $a=0$ supports on the cubic curve $y^2=x^3$ in $\mathbb{A}^2$ with nilpotent point z. The embedding point is just $(0,0)$, and at the prime ideal (x,y), z is a nilpotent.

Solution 3.9.3. (a) Since finite morphism is quasi-finite, the fibre is just finite points and hence zero-dimension. Then by 3.10.9. we know that the morphism is flat since all fibre are in the same dimension.

(b) Take the point $P=(x,y)$ as the intersecting point of X. Then by theorem 9.1A. we know that $\mathscr{O}_{X,P}$ is a free $\mathscr{O}_{Y,f(P)}$-module. Since $\mathscr{O}_{X,P}/\mathfrak{m}_{Y,f(P)}\mathscr{O}_{X,P}\cong k$, we know $\mathscr{O}_{X,P}$ is a free $\mathscr{O}_{Y,f(P)}$-module of rank 1, i.e. $\phi:\mathscr{O}_{Y,f(P)}\cong\mathscr{O}_{X,P}$ as an $\mathscr{O}_{Y,f(P)}$-module. So there exists a decomposition of z as $z=f\cdot\phi(1)$ for some $f\in\mathscr{O}_{Y,f(P)}$, which makes a contradiction.

(c) Clearly, $(k[x,y,z,w]/(z^2,zw,w^2,xz-yw))_{\text{red}} = k[x,y,z,w]/\sqrt{(z^2,zw,w^2,xz-yw))} = k[x,y,z,w]/(z,w) = k[x,y]$, i.e. $X_{\text{red}} \cong Y$. Since the Jacobian matrix is

$$J = \begin{bmatrix} 0 & 0 & 0 & z \\ 0 & 0 & 0 & -w \\ 2z & w & 0 & x \\ 0 & z & 2w & -y \end{bmatrix}$$

When $z, w = 0$, J has rank 1 or 0, but there does not exists nilpotent in local ring, hence no embedding points at the locus $z, w = 0$. When z and w are not simultaneously zero, J has rank 3, so non-singular, i.e. no embedding points. Moreover, the morphism $f : X \to X_{\text{red}}$ has changing fibre dimension, that is, $f^{-1}(0,0)$ has dimension 0, $f^{-1}(x,0)$ and $f^{-1}(0,y)$ has dimension 1, and $f^{-1}(x,y)$ has dimension 2 with $x, y \neq 0$. Hence by 3.10.9., f is not flat.

Solution 3.9.4 (Open Nature of Flatness)**.** We may assume $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, and denote U as the subset of U of all x at which f is flat. For any $x \in U$, i.e. $B_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$, where $\mathfrak{q}$ corresponds to x and $\mathfrak{p}$ is the image of $\mathfrak{q}$, there exists a $g \in B - \mathfrak{q}$ such that for all prime $\mathfrak{q}' \subset B$ such that $\mathfrak{q} \subset \mathfrak{q}'$ and $g \notin \mathfrak{q}'$, we have $B_{\mathfrak{q}'}$ is flat over $A_{\mathfrak{p}}$ by commutative algebra. Hence $U \cap \overline{\{x\}}$ contains a non-zero open subset of $\overline{\{x\}}$, hence U is constructible. Moreover, for any $\mathfrak{q}$ such that $B_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$, for any $\mathfrak{q}' \supset \mathfrak{q}$, $B_{\mathfrak{q}'}$ is flat over $A_{\mathfrak{p}'}$. Hence U contains all generalizations of its points. Then by 2.3.18.(c), U is open.

Solution 3.9.5 (Very Flat Families)**.** (a) Consider the flat family $\{X_t\}$ that $X_t = \{(1:0:0), (0:1:0), (1:1:t)\} \subset \mathbb{P}^2$. Notice that only when $t = 0$ the three points in X_t are colinear. When $t \neq 0$, $I(X_t) = (xz - txy, yz - txy, z^2 - t^2xy)$. And denote the closure of $Z((xz - txy, yz - txy, z^2 - t^2xy)) \subset \mathbb{A}^4$, we know that $I(Y) = (xz - txy, yz - txy, z^2 - t^2xy, x^2y - xy^2)$. So Y is a closure of a flat family over a smooth curve, hence Y is flat too, and when $t \neq 0$, $Y_t = C(X_t)$. When $t = 0$, X_0 supports on $z = 0$, hence $C(X_0) \neq Y_0$. So $C(X_t)$ is not a flat family.

(b) (**Seems Hartshorne's book has so many exercises which are not exactly exercises?**)

(c) Since for all $d \geq 0$, the $\dim_{k(t)}(S_t/I_t)_d$ is a constant, i.e. the Hilbert polynomial P_t are the same. Then by theorem 9.9., $\{X_t\}$ is a flat family. For $\{C(X_t)\}$, if $I = I(X) \subset k[x_0, \ldots, x_n]$, then $I(C(X)) = I \subset k[x_0, \ldots, x_n, x_{n+1}]$. So for any t we have $\dim_{k(t)}(k(t)[x_0, \ldots, x_{n+1}]/I_t)_d = \dim_{k(t)}(S_t/I_t)_d + \ldots + \dim_{k(t)}(S_t/I_t)_0$, i.e. the Hilbert polynomial of $\overline{C(X_t)}$ are all the same. Hence by the same reason we know $\overline{C(X)} \to \mathbb{P}^{n+1}$ is flat, so $C(X) \to \mathbb{P}^{n+1}$ is flat.

(d) Since X_t are all projectively normal varieties in $\mathbb{P}^n_k$, we know that $\dim(S_t/I_t)_d = H(d)$ for $d \geq$the Hilbert regularity, where $H(t)$ is the common Hilbert polynomial since $\{X_t\}$ is a flat family by theorem 9.11. Consider the integrally closed ring S_t/I_t, it is generated by the degree 1 elements $x_0, \ldots, x_n$, hence the Hilbert regularity is less than $\deg X_t - n + 1 \leq 0$. So $\dim(S_t/I_t)_d = H(d)$ for all d, i.e. $\{X_t\}$ is a very flat family.

Solution 3.9.6. ($\Rightarrow$) Obviously. ($\Leftarrow$) Treating Y' as a closed subscheme of $\mathbb{P}^n$, then $I(Y') = \langle I(Y), L \rangle$, where L is the linear function who generates H. So if Y' is a complete intersection in $\mathbb{P}^{n-1}$, in the ring $k[x_0, \ldots, x_n]$, we have length $I(Y')/(L) = \operatorname{codim}_{\mathbb{P}^{n-1}}(Y') = \operatorname{codim}_{\mathbb{P}^n}(Y)$. And since length $I(Y')/(L) =$ length $I(Y)$, we have length $I(Y) = \operatorname{codim}_{\mathbb{P}^n}(Y)$, i.e. Y is a complete intersection in $\mathbb{P}^n$.

Solution 3.9.7. If $X = \operatorname{Spec} A$ is affine, we may assume $Y = Z(I)$ for some ideal I. Since $X \otimes_k D = \operatorname{Spec} A[t]/(t^2)$, the infinitesimal deformation Y' corresponds to an ideal $I' \subset A[t]/(t^2)$. Since the generic fibre is Y, we know that the projection of I' in A is just I. And since Y' is flat over D, by theorem 9.1.(a) we know that the kernel of the morphism $A \to (A[t]/(t^2))/I' \xrightarrow{\times t} (A[t]/(t^2))/I'$ is contained in I'. Conversely, if an ideal $I' \subset A[t]/(t^2)$ satisfies the above two conditions, it corresponds to an infinitesimal deformation Y'.

For any $\phi \in \operatorname{Hom}_A(I/I^2, A/I)$, we define $I' = I \cup \{a + bt | b = \phi(a)\} \subset A[t]/(t^2)$. Then by the above two conditions, I' is an ideal in $A[t]/(t^2)$. Conversely, if we have such an I', since $I' \to I$ is surjective, for any $a \in I$, there exists $a + bt \in I'$ for some b, then we may define $b = \phi(a)$. Then by the second condition above, this is well-defined. Then ϕ is clearly a morphism from I/I^2 to A/I. Hence we have a corresponding between the infinitesimal extension of Y and the element of $\mathrm{Hom}_A(I/I^2, A/I)$.

For any ring homomorphism $\psi : A \to B$ with two ideal $I \subset A$, $J \subset B$ such that $\psi^{-1}J \subset I$, we clearly have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(I/I^2, A/I) \arrow[r, "\cong"] \arrow[d] & \text{the infinitesimal extension of Spec } A/I \arrow[d] \\
\mathrm{Hom}_B(J/J^2, B/J) \arrow[r, "\cong"] & \text{the infinitesimal extension of Spec } B/J
\end{tikzcd}$$

So the general case is just glueing.

Solution 3.9.8. Clearly we have an exact sequence $0 \to k \xrightarrow{\times t} D \to k \to 0$. Since A' is flat over D, and $A' \otimes_D k \cong A$, we have an exact sequence $0 \to A \xrightarrow{\times t} A' \to A \to 0$. Moreover, defining $P' = P \otimes_k D$, we have an exact sequence $0 \to P \to P' \to P \to 0$. Then we can define a morphism $f : P' \to A'$ with kernel J'. And we have the following exact sequence

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J \arrow[r] \arrow[d] & J' \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d, "f"] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Hence the equivalent classes of A' is just the choice of the ideal J' in P' quotient to the choice of f.

Fix a J'. For any $x \in J$ for the right J, we can lift it to $x + yt \in J'$ for some $y \in P$, where we may lift it to another $x + y't \in J'$ for some $y' \in P$ but $y - y' \in J$. So we may define a morphism $\phi : x \to \bar{y} \in A$, where the lifting of x is $x + yt \in J'$. And $\phi \in \mathrm{Hom}_P(J, A)$. Conversely, for any $\phi \in \mathrm{Hom}_P(J, A)$, we may define $J' = \{x + yt \mid x \in J, y \in P \text{ such that } \bar{y} = \phi(x) \in A\}$ as an ideal in P'. And $K \cap J' = J$, hence we have $0 \to J \to J' \to J \to 0$ and J' defines an A'. So $\{\text{choices of } K\} \cong \mathrm{Hom}_P(J, A)$.

Moreover, by 2.8.6.(a) we know that $\{\text{choices of } f\} \cong \mathrm{Hom}_P(\Omega_{P/k}, A) \cong \mathrm{Hom}_A(\Omega_{P/k} \otimes A, A)$. And since $\mathrm{Hom}_A(J/J^2, A) \cong \mathrm{Hom}_P(J, A)$, we have $T^1(A) \cong \{\text{choices of } A'\}$.

Solution 3.9.9. By 3.9.8. we only need to show the surjectivity of the morphism $\Theta : \mathrm{Hom}_A(\Omega_{P/k} \otimes A, A) \to \mathrm{Hom}_A(J/J^2, A)$, where $P = k[x, y, z, w]$, $J = (x, y) \cap (z, w) = (xz, xw, yz, yw)$. Since J/J^2 has generators xz, xw, yz, yw as an A-module, any morphism $\psi : J/J^2 \to A$ is determined by the quaternity $(a, b, c, d) = (\psi(\bar{xz}), \psi(\bar{xw}), \psi(\bar{yz}), \psi(\bar{yw}))$.

Since $\Omega_{P/k}$ has basis $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z, \mathrm{d}w$, we may consider the morphism $\phi_x : \Omega_{P/k} \otimes A \to A$ such that $\phi_x(\mathrm{d}x) = 1$ and $\phi_x(\mathrm{d}y) = \phi_x(\mathrm{d}z) = \phi_x(\mathrm{d}w) = 0$. Then the morphism $\mu : J/J^2 \to \Omega_{P/k} \otimes A$, where $f \mapsto \mathrm{d}f \otimes 1$, maps to $(z, w, 0, 0) \in \mathrm{Hom}_A(J/J^2, A)$ via Θ. Similarly, $\Theta(\phi_y) = (0, 0, z, w)$, $\Theta(\phi_z) = (x, 0, y, 0)$ and $\Theta(\phi_w) = (0, x, 0, y)$.

For any $(a, b, c, d) \in \mathrm{Hom}_A(J/J^2, A)$, we have $ay = \psi(xyz) = x\psi(yz) = cx$. Similarly, $aw = bz$, $by = dx$, $cw = dz$. So there exists constant $\alpha, \beta, \gamma, \delta$ such that $(a, b, c, d) = \alpha(z, w, 0, 0) + \beta(0, 0, z, w) + \gamma(x, 0, y, 0) +$

$\delta(0,x,0,y) = \Theta(\alpha\phi_x + \beta\phi_y + \gamma\phi_z + \delta\phi_w)$ by noticing that $xz = xw = yz = yw = 0$ in A. Hence Θ is surjective.

Solution 3.9.10. (a) By Example 9.13.2., any infinitesimal deformation corresponds to an element in $H^1(\mathbb{P}^1, \mathscr{T}_{\mathbb{P}^1})$. Since $\mathscr{T}_{\mathbb{P}^1} = \mathscr{O}(2)$, and $H^1(\mathbb{P}^1, \mathscr{O}(2)) = 0$, $\mathbb{P}^1$ is rigid.

(b) Consider $Y = Z((a-1)x^2 + (b-1)y^2 + z^2) \subset \mathbb{A}^2 \otimes \mathbb{P}^2$. The projection $\pi : Y \to \mathbb{A}^2$ is smooth on $S = D(a-1, b-1) = \operatorname{Spec} k[a,b,(a-1)^{-1},(b-1)^{-1}]$. Hence we may define $X = \pi^{-1}(S)$ and $f = \pi_X$. Firstly, $X_0 = f^{-1}(0,0) = Z(x^2+y^2-z^2) \cong \mathbb{P}^1$. Secondly, the generic fibre X_η is the quadratic curve $(a-1)x^2+(b-1)y^2+z^2 = 0$ in $\mathbb{P}^2_{k(a,b)}$, which has no rational points on $k(a,b)$. So this flat family is not locally trivial everywhere.

(c) Since f is flat, every fibre has the same Hilbert polynomial, i.e. every fibre is $\mathbb{P}^1$. For an affine $U \subset T$ containing t, we consider an effective divisor associated to $\omega_{X/T}^{-1}$ which is supported on two distinct points in each fibre. Then $D \to U$ is clearly flat. So we may consider $T' = D$ and the family $f' : X' \to T'$. Moreover, for any $x \in T'$, clearly $x \in X$, hence there exists a $x' \in X \times_T T' = X'$, which define a section $s : T' \to X'$, $s(x) = x'$. Then we will prove that if a flat morphism $f' : X' \to T'$ has fibres $\mathbb{P}^1$ and a section $s : T' \to X'$, this family is trivial.

Denote $Y' \subset X'$ as the schema-theoretic image of s, then Y' is flat over T'. And Y' intersect every fibre on just one point, it is a Cartier divisor on X'. Denote $\mathscr{L}$ as the associated invertible sheaf on X'. For any $t' \in T'$, $H^0(X'_{t'}, \mathscr{L}_{t'})$ is a vector space of dimension 2, and $H^1(X'_{t'}, \mathscr{L}_{t'}) = 0$. Then consider the morphism $\phi^i(s) : \mathrm{R}^i f'_* \mathscr{L} \otimes k(t') \to H^i(X'_{t'}, \mathscr{L}_{t'})$, by theorem 12.11., $H^1(X'_{t'}, \mathscr{L}_{t'}) = 0$, ϕ^1 is surjective, hence isomorphism, so $\mathrm{R}^i f'_* \mathscr{L} = 0$. Moreover, the zero sheaf is free, hence ϕ^0 is surjective, hence isomorphism. Since $f'_*\mathscr{L}$ is locally free of rank 2 on T', we may denote it as $\mathscr{E}$. Then the nature morphism $f'^*\mathscr{E} \to \mathscr{L}$ induces the morphism $X' \to \mathbb{P}(\mathscr{E})$, which is isomorphic on each fibre, hence isomorphic globally.

Solution 3.9.11. By corollary 3.11. in chapter IV, any curve is birational to a plane curve. So we just need to consider the case of plane curve. Than $p_a(C) = \frac{1}{2}(d-1)(d-2) - \sum_P \delta_P$ by 4.1.8. Hence $p_a(C) \leq \frac{1}{2}(d-1)(d-2)$. Moreover, $p_a(C) = p_g(C)$, hence non-negative.

3.10 Smooth Morphisms

Solution 3.10.1. The Jacobian matrix of this curve is $\begin{bmatrix} 0 \\ 2y \end{bmatrix}$, which is rank 1 everywhere since $y \neq 0$. So every local ring of X is regular local ring.

If $X \to \operatorname{Spec}(k)$ is smooth, we may consider the smooth base change $X_{\bar{k}} \to \bar{k}$, which is defined by $y^2 = x^p - t = (x - t^{1/p})^p$ over $\bar{k}$. Hence $X_{\bar{k}}$ is not regular at $(t^{1/p}, 0)$, which makes a contradict. So X is not smooth.

Solution 3.10.2. Denote the relative dimension of the morphism $X_y \to \operatorname{Spec}(k(y))$ as n. Then $\Omega_{X_y/k(y)}$ is a locally free sheaf of rank n on X_y. So, $\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = \dim_{k(x)}(\Omega_{X_y/k(y)} \otimes k(x)) = n$. Since $\Omega_{X/Y} \otimes k(x) = (\Omega_{X/Y})_x / \mathfrak{m}_x (\Omega_{X/Y})_x$, by Nakayama's lemma, there exists $s_1, \ldots, s_n \in \Omega_{X/Y}(U_x)$ for some open neighbourhood of x such that the images of s_i's in $\Omega_{X/Y} \otimes k(x) = (\Omega_{X/Y})_x / \mathfrak{m}_x(\Omega_{X/Y})_x$ form a $k(x)$-basis and they generate $\Omega_{X/Y}(U_x)$. So for every $x' \in U_x$, we have $\dim_{k(x')}(\Omega_{X/Y} \otimes k(x')) \leq n$. Conversely, by theorem 8.6 in chapter II we have $\dim_{k(x')}(\Omega_{X/Y} \otimes k(x')) \geq n$, hence equals, i.e. $\Omega_{X/Y}$ is a locally free sheaf of rank n on U_x. Denoting $U = \bigcup_{x \in X_y} U_x$, we know that $\Omega_{X/Y}$ is locally free of rank n on U.

Moreover, since f is proper, the base change $X_y \to \operatorname{Spec}(k(y))$ is proper. So X_y is quasi-compact, hence there exists finitely many $x_1, \ldots, x_m \in X_y$, such that $X_y \subset \bigcup_{i=1}^m U_{x_i}$. Since f is flat, we have $f(U_{x_i})$ is open and containing y. Define $V = \bigcup_{i=1}^m f(U_{x_i})$, then this is a open neighbourhood of y, which preimage is smooth over itself.

Solution 3.10.3. (ii)$\Leftrightarrow$(iii) is a consequence of theorem 8.6. in chapter II. And (i)$\Leftrightarrow$(ii) is just the definition.

Solution 3.10.4. ($\Rightarrow$) By 3.10.3., étale implies unramify and flatness, hence $k(x)/k(y)$ is separable, and $\mathfrak{m}_y\mathscr{O}_x = \mathscr{O}_y$. Since $\mathscr{O}_y \to \mathscr{O}_x$ is flat, we have an exact sequence $0 \to \mathfrak{m}^{r+1} \otimes_{\mathscr{O}_y} \mathscr{O}_x \to \mathfrak{m}^r \otimes_{\mathscr{O}_y} \mathscr{O}_x \to \mathfrak{m}^r/\mathfrak{m}^{r+1} \otimes_{\mathscr{O}_y} \mathscr{O}_x \to 0$.

Since $\mathfrak{m}^r/\mathfrak{m}^{r+1}\otimes_{\mathscr{O}_y}\mathscr{O}_x = \mathfrak{m}^r\mathscr{O}_x/\mathfrak{m}^{r+1}\mathscr{O}_x$, we have $\operatorname{Gr}\mathscr{O}_x \cong \operatorname{Gr}\mathscr{O}_y\otimes_{\mathscr{O}_y}\mathscr{O}_x$. Hence $\hat{\mathscr{O}}_x \cong \hat{\mathscr{O}}_y\otimes_{\mathscr{O}_y}\mathscr{O}_x$. So $\hat{\mathscr{O}}_y\otimes_{k(y)}k(x) = \hat{\mathscr{O}}_y\otimes_k k\otimes_{\mathscr{O}_y}\mathscr{O}_x = \hat{\mathscr{O}}_x \cong \hat{\mathscr{O}}_y\otimes_{\mathscr{O}_y}\mathscr{O}_x$.

($\Leftarrow$) Unramify is clear. For flatness, if $M \to N$ is an injective morphism of $\mathscr{O}_y$-modules, we want to show that $M\otimes_{\mathscr{O}_y}\mathscr{O}_x \to N\otimes_{\mathscr{O}_y}\mathscr{O}_x$ is injective. Since $\mathscr{O}_y \to \hat{\mathscr{O}}_y$ is flat, $\mathscr{O}_y \to \mathscr{O}_x\otimes_{\mathscr{O}_y}\hat{\mathscr{O}}_y$ is also flat. So $M\otimes_{\mathscr{O}_y}\mathscr{O}_x\otimes_{\mathscr{O}_y}\hat{\mathscr{O}}_y \to N\otimes_{\mathscr{O}_y}\mathscr{O}_x\otimes_{\mathscr{O}_y}\hat{\mathscr{O}}_y$ is injective. Since $\hat{\mathscr{O}}_y$ is faithfully flat, $M\otimes_{\mathscr{O}_y}\mathscr{O}_x \to N\otimes_{\mathscr{O}_y}\mathscr{O}_x$ is injective.

Solution 3.10.5. For any $x \in X$, there exists an étale morphism $f : U \to X$ such that $f^*\mathscr{F}$ is free $\mathscr{O}_U$-module. So $(f^*\mathscr{F})_{x'}$ is a free $\mathscr{O}_{x'}$-module. Denote $\mathscr{F}_x$ as an $\mathscr{O}_x$-module M. Then $M\otimes_{\mathscr{O}_x}\mathscr{O}_{x'} = (f^*\mathscr{F})_{x'} = \mathscr{O}_{x'}^n = \mathscr{O}_x^n\otimes_{\mathscr{O}_x}\mathscr{O}_{x'}$ for some n. Since $\mathscr{O}_x$ and $\mathscr{O}_{x'}$ are local, then flatness implies faithful flatness. So $M \cong \mathscr{O}_x^n$. Then by 2.5.7., $\mathscr{F}$ is locally free.

Solution 3.10.6. Consider $X = \operatorname{Spec} k[t,s]/(t^2-(s^2-1)^2)$. Then the morphism $k[x,y]/(y^2-x^2(x+1)) \to k[t,s]/(t^2-1, t(t^2-1))$, where $x \mapsto s^2-1$ and $y \mapsto st$, induces the morphism $f : X \to Y$. For unramify, since $f^*\Omega_{Y/k} = \Omega_{X/k}$, we have $\Omega_{X/Y} = 0$. For flatness, the fibre at node is $X_0 = \operatorname{Spec} k[x,y]/(x,y)\otimes_{k[x,y]/(y^2-x^2(x+1))} k[t,s]/(t^2-1, t(t^2-1)) = \operatorname{Spec} k[s,t]/(t^2, s^2-1, st) = \operatorname{Spec} k[s,t]/(t, s^2-1) = \operatorname{Spec} k[s]/(s^2-1)$. The fibre at any other point is clearly isomorphic to X_0, hence f is flat.

Solution 3.10.7. (a) Suppose $f(x,y,z) = \sum_{i+j+k=3,i,j,k\geq 0} a_{ijk}x^iy^jz^k \in \mathfrak{d}$. Since $(1,0,0),(0,1,0),(0,0,1) \in V(f)$, we have $a_{300} = a_{030} = a_{003} = 0$. Then because $(1,1,0) \in V(f)$, we have $a_{210} = a_{120}$, and similarly $a_{021} = a_{012}$ and $a_{102} = a_{201}$. Moreover, $(1,1,1) \in V(f)$, so $a_{111} = 0$. Hence f has the form $f = axy(x+y)+byz(y+z)+czx(z+x)$. So $\mathfrak{d}$ is generated by 3 cubic terms, hence dimension 2. And there exists a morphism $\mathbb{P}^2 - \{P_i\} \to \mathbb{P}^2$ as $(x,y,z) \mapsto (xy(x+y), yz(y+z), zx(z+x))$.

On affine piece $D_+(x)$, the morphism above is $(x,y) \mapsto (\frac{x(x+y)}{y+1}, \frac{x(x+1)}{y(y+1)}$. So in algebraic language, we have a morphism $g' : k(s,t) \to k(x,y)$, $s \mapsto \frac{x(x+y)}{y+1}$, $t \mapsto \frac{x(x+1)}{y(y+1)}$. Since $y\cdot g'(t)+g'(s) = x$, we have $0 = g'(s)+g'(s) = g'(s)+\frac{x(x+y)}{y+1} = \frac{g'(s)(y+1)+y^2g'(t)^2+yg'(ts)+y^2g'(t)+yg'(ts)+g'(s)^2+yg'(s)}{y+1}$, i.e. $y^2g'(t)(g'(t)+1)+g'(s)(g'(s)+1) = 0$. So we have a minimal polynomial $y^2 = -\frac{g'(s)(g'(s)+1)}{g'(t)(g'(t)+1)}$, hence inseparable of degree 2 on this piece. Similarly we can do the same thing on the piece $D_+(y)$ and $D_+(z)$.

(b) For $f = axy(x+y)+byz(y+z)+czx(z+x) \in \mathfrak{d}$, we have $0 = \frac{\partial f}{\partial x} = ay^2+bz^2$, $0 = \frac{\partial f}{\partial y} = ax^2+cz^2$ and $0 = \frac{\partial f}{\partial z} = bx^2+cy^2$. So f corresponds to a singular point $(\sqrt{c}, \sqrt{b}, \sqrt{a})$, this is a 1-1 correspondence between $\mathfrak{d}$ and $\mathbb{P}^2$.

Solution 3.10.8 (A Linear System with Moving Singularities Contained in the Base Locus (Any Characteristic))**.** Simple calculate, the cone of C in $\mathbb{A}^3$ is $(x-1+\frac{z}{t})^2+y^2 = (1-\frac{z}{t})^2$. Y_t is $(tx-tw+z)^2+(ty)^2-(tw-z)^2 = 0$ in $\mathbb{P}^3$. Hence $\{Y_t\}$ forms a linear system of dimension 1 with a moving singularity at P. And the base locus of $\{Y_t\}$ is $\{(x-w)^2+y^2 = w^2\} \cup \{x = y = 0\}$, i.e. the conic C plus the z-axis.

Solution 3.10.9. This problem is local, so we may assume $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ for two regular ring A and B with a ring homomorphism $A \to B$. For any $y \in Y$, it may correspond to a prime $\mathfrak{p} \subset A$, the fibre $X_y = \operatorname{Spec} B\otimes k(y) = \operatorname{Spec} B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$. So if X is Cohen-Macaulay, $B_\mathfrak{p}$ is also CM. And we have $\dim B_\mathfrak{p} = \dim A_\mathfrak{p} + \dim B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$. So $\operatorname{depth}(\mathfrak{p}B_\mathfrak{p}, B_\mathfrak{p}) = \dim A_\mathfrak{p}$. Since $A_\mathfrak{p}$ is regular, we may take a regular sequence $x_1,\ldots,x_d$ of $\mathfrak{p}$, where $d = \dim A_\mathfrak{p}$. Take the Koszul complex $K = K(x_1,\ldots,x_d)$. Then the Koszul complex of the generators of $\mathfrak{p}B_\mathfrak{p}$ is $K\otimes B_\mathfrak{p}$, which has no homology except for H^d. Hence $\operatorname{Tor}_1^{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}, B_\mathfrak{p}) = H^{d-1}(K\otimes B_\mathfrak{p}) = 0$, i.e. $B_\mathfrak{p}$ is flat over $A_\mathfrak{p}$. So B is flat over A.

3.11 The Theorem on Formal Functions

Solution 3.11.1. The sheaf $\mathrm{R}^{n-1} f_* \mathscr{O}_U$ is the one associated to the presheaf $V \mapsto H^{n-1}(f^{-1}(V), \mathscr{O}_U|_{f^{-1}(V)})$. Hence we may take the Cech complex of U for the open covering $U_i = \operatorname{Spec} k[x_1, \dots, x_n, x_i^{-1}]$:

$$\dots \to \bigoplus_{i=1}^{n} k[x_1, \dots, x_n, x_1^{-1}, \dots, \hat{x_i^{-1}}, \dots, x_n^{-1}] \to k[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}] \to 0 \to \dots$$

So the $n-1$-cohomology group is the linear combinations of monomials of negative degree, which is non-empty. Hence $\mathrm{R}^{n-1} f_* \mathscr{O}_U \neq 0$.

Solution 3.11.2. Consider the Stein factorization of f: $X \xrightarrow{f'} Y' \xrightarrow{g} Y$. For any $y' \in Y$, $f^{-1}(g(y'))$ is discrete and hence $f'^{-1}(y') \subset f^{-1}(g(y'))$ is discrete. But $f'^{-1}(y')$ is non-empty and connected. So $f'^{-1}(y')$ is just one point, i.e. f' is 1-1 and onto. Moreover, since f' is projective, i.e. proper, there exists a homeomorphism between the underlying spaces. Combining with $f'^{\sharp} : \mathscr{O}_{Y'} \to f'_* \mathscr{O}_X$ is an isomorphism, we know that f' is an isomorphism of schemes. So $f = g$ is finite.

Solution 3.11.3. Consider the Stein factorization of f as $X \xrightarrow{f'} X' \xrightarrow{g} \mathbb{P}^n_k$. Then since the fibres of f' are all closed and connected, we only need to prove that for all $D \in \mathfrak{d}$ and the associated closed subscheme Z, $f'(Z)$ is connected. By definition of f, any $D \in \mathfrak{d}$ is of the form $D = f^* H$ for some Cartier divisor H in the complete linear system of the invertible sheaf $\mathscr{O}(1)$ on $\mathbb{P}^n$, where H is clearly ample. Since $f^* H = f'^* g^* H$, and g is finite, so $g^* H$ is ample. Since X' is normal and of dimension ≥ 2, so by 3.7.9., $g^* H$ is connected, i.e. $f'(Z)$ is connected.

Solution 3.11.4 (Principle of Connectedness)**.** The problem is stable under base change, so we may assume that T is smooth. Since f is flat and projective, we have $f_* \mathscr{O}_X$ is torsion free. Since T smooth, we know $f_* \mathscr{O}_X$ is locally free. Then since f has connected fibre over U, $f_* \mathscr{O}_X|_U$ has rank 1, hence $f_* \mathscr{O}_X$ has rank 1. So $f_* \mathscr{O}_X$ is the sheaf of sections of a line bundle on T. Moreover, since $f_* \mathscr{O}_X(T) = \mathscr{O}_X(X)$, $f_* \mathscr{O}_X$ has a nowhere vanishing section corresponding to the section 1 on $\mathscr{O}_X$, so $f_* \mathscr{O}_X$ is the trivial line bundle, i.e. $f_* \mathscr{O}_X \cong \mathscr{O}_T$. Then by corollary 11.3., the fibre of f is connected for every closed point of T.

Solution 3.11.5. By 2.9.6., we have $\operatorname{Pic} \mathfrak{X} = \varprojlim \operatorname{Pic} X_n$. Then we only need to show that $\varprojlim \operatorname{Pic} X_n \cong \operatorname{Pic} Y$. Denote $\mathscr{I}$ as the ideal sheaf of Y, hence generated by some polynomial $f \in k[x_0, \dots, x_N]$. Then by 3.4.6., considering X_n with the sheaf $\mathscr{I}^{n-1}$, we have an exact sequence $\dots \to H^1(X_n, \mathscr{I}^{n-1}) \to \operatorname{Pic} X_n \to \operatorname{Pic} Y \to H^2(X_n, \mathscr{I}^{n-1}) \to \dots$. Since X_n has the same underlying topological space with X_{n-1} and Y, which is of dimension ≥ 3, and $H^i(X_n, \mathscr{I}^{n-1}) \cong H^i(X_{n-1}, \mathscr{I}^{n-1})$. Since $X_{n-1} = \operatorname{Proj} k[x_0, \dots, x_N]/(f^{n-1})$, $\mathscr{I}^{n-1} = \mathscr{O}_{X_{n-1}}$, by 3.5.5. we have $H^i(X_{n-1}, \mathscr{O}^{n-1}) = 0$ for $i = 1, 2$. So we have $\operatorname{Pic} X_n \cong \operatorname{Pic} Y$ for all n. Then $\varprojlim \operatorname{Pic} X_n \cong \operatorname{Pic} Y$.

Solution 3.11.6. (a) We have an exact sequence $\dots \to H^{N-1}(X - Y, \check{\mathscr{F}}) \to H^N_Y(X, \check{\mathscr{F}}) \to H^N(X, \check{\mathscr{F}}) \to H^N(X - Y, \check{\mathscr{F}}) \to$. Since Y is closed in X, we have $H^{N-1}(X - Y, \check{\mathscr{F}}) = H^N(X - Y, \check{\mathscr{F}}) = 0$, i.e. we have $H^N_Y(X, \check{\mathscr{F}}) \cong H^N(X, \check{\mathscr{F}})$. Since $\check{\mathscr{F}}$ is locally free, by Serre duality we have $H^0(X, \mathscr{F}) \cong H^N(X, \check{\mathscr{F}})'$. Then we only need to prove the formal duality $H^0(\hat{X}, \hat{\mathscr{F}}) = H^N_Y(X, \check{\mathscr{F}})'$.

We've already had $H^0(\hat{X}, \hat{\mathscr{F}}) = \varprojlim H^0(X_m, \mathscr{F}_m)$. But $H^0(X_m, \mathscr{F}_m) = H^0(X, \mathscr{F}_m) \cong \operatorname{Ext}^N(\mathscr{F}_m, \omega_{X/k})' \cong \operatorname{Ext}^N(\mathscr{O}_X/\mathscr{I}^m_Y, \hat{\mathscr{F}})'$, we have $H^0(\hat{X}, \hat{\mathscr{F}}) = \varprojlim \operatorname{Ext}^N(\mathscr{O}_X/\mathscr{I}^m_Y, \hat{\mathscr{F}})' = (\varinjlim \operatorname{Ext}^N(\mathscr{O}_X/\mathscr{I}^m_Y, \hat{\mathscr{F}}))' = H^N_Y(X, \check{\mathscr{F}})'$.

(b) (i⇒ii) For any n, we have an exact sequence $0 \to \mathscr{I}^k \mathfrak{F}(n)/\mathscr{I}^{k+1}\mathfrak{F}(n) \to \mathfrak{F}(n)/\mathscr{I}^{k+1}\mathfrak{F}(n) \to \mathfrak{F}(n)/\mathscr{I}^k \mathfrak{F}(n) \to 0$, hence we have $H^0(X_{k+1}, \mathscr{F}_{k+1}(n)) \to H^0(X_k, \mathscr{F}_k(n)) \to H^1(X_1, (\mathscr{I}^k \mathfrak{F}/\mathscr{I}^{k+1}\mathfrak{F})(n))$. Since $X_1 = Y$ is projective over k, there exists an n_0 such that for any $n \geq n_0$, we have $H^1(Y, (\mathscr{I}^k \mathfrak{F}/\mathscr{I}^{k+1}\mathfrak{F})(n)) = 0$. So when $n \geq n_0$, the morphism $H^0(X_{k+1}, \mathscr{F}_{k+1}(n)) \to H^0(X_k, \mathscr{F}_k(n))$ is surjective. Hence if we take a projective limit, we have a surjective morphism $H^0(\hat{X}, \mathfrak{F}(n)) \to H^0(X_1, \mathscr{F}_1(n))$. Adding n_0 if necessary, we assume that $\mathscr{F}_1(n)$ is generated by global section, then we may pick $s_1, \dots, s_m \in H^0(\hat{X}, \mathfrak{F}(n))$ such that their images in $H^0(X_1, \mathscr{F}_1(n))$

can generate the sheaf. Then for any $y \in Y$, we have a morphism $\mathscr{O}^m_{\hat{X},y} \to \mathfrak{F}(n)_y$ as $e_i \mapsto s_{i,y}$. Since the morphism $\mathscr{O}^m_{Y,y} \to \mathscr{F}_1(n)_y$ induced by this morphism is clearly surjective, then by Nakayama lemma we know this morphism is surjective. Hence $\mathfrak{F}(n)$ is generated by global sections $s_1, \ldots, s_m$.

(ii⇒i) Since $\mathfrak{n}$ is generated by global section, there exist finitely some q_i and a surjective morphism $\bigoplus \mathscr{O}_{\hat{X}}(q_i) \to \mathfrak{F}(n)$. And clearly the kernel of this morphism $\mathfrak{K}$ also satisfies the Serre theorem, i.e. there exists m such that $\mathfrak{K}(m)$ is generated by finitely many global sections. Then there also exist finitely some p_i and an exact sequence $\bigoplus \mathscr{O}_{\hat{X}}(p_i) \to \bigoplus \mathscr{O}_{\hat{X}}(q_i + m) \to \mathfrak{F}(n + m) \to 0$. Tensoring with $\mathscr{O}_{\hat{X}}(-n - m)$ we have an exact sequence $\hat{E}_1 \to \hat{E}_0 \to \mathfrak{F} \to 0$, where $\hat{E}_1$ and $\hat{E}_0$ have the form $\bigoplus \mathscr{O}_X(q_i)$. Then by (a) we have $H^0(X, \mathscr{H}om_X(\mathscr{E}_1, \mathscr{E}_0) \cong H^0(\hat{X}, \mathscr{H}om_{\hat{X}}(\hat{\mathscr{E}}_1, \hat{\mathscr{E}}_0)$, hence $\mathfrak{F}$ correspond to some $\mathscr{F}$, i.e. $\mathfrak{F} = \hat{\mathscr{F}}$.

(c) By 3.11.5. we have $\operatorname{Pic} Y \cong \operatorname{Pic} \hat{X}$. And by 2.6.2.(d) we have an injective morphism $\operatorname{Pic} X \to \operatorname{Pic} \hat{X}$. By (b), any $\mathfrak{F} \in \operatorname{Pic} \hat{X}$ is algebraizable, i.e. $\mathfrak{F} = \hat{\mathscr{F}}$ for some $\mathscr{F} \in \operatorname{Pic} X$, hence we have a morphism $\operatorname{Pic} \hat{X} \to \operatorname{Pic} X$. Then we have isomorphism $\operatorname{Pic} X \cong \operatorname{Pic} \hat{X}$.

Solution 3.11.7. (a) Similarly with 3.11.5., we have $\operatorname{Pic} \hat{X} = \varprojlim \operatorname{Pic} X_n$. For any X_n, we have $H^1(X_n, \mathscr{I}^{n-1}) \to \operatorname{Pic} X_n \to \operatorname{Pic} X_1 \to H^2(X_n, \mathscr{I}^{n-1})$. Since X_n has dimension 1, we have $H^2(X_n, \mathscr{I}^{n-1}) = 0$. Hence in limit, we have $\operatorname{Pic} \hat{X} = \varprojlim \operatorname{Pic} X_n \to \operatorname{Pic} Y$ is surjective. Moreover, since $H^1(X_n, \mathscr{I}^{n-1}) = H^1(X_{n-1}, \mathscr{I}^{n-1}) = k[x_0, x_1, x_2]/(f^{n-1})$. Take projective limit, we have $\ker(\varprojlim \operatorname{Pic} X_n \to \operatorname{Pic} Y) = \varinjlim k[x_0, x_1, x_2]/(f^{n-1})$ =the completion of $k[x_0, x_1, x_2]$ of (f), hence infinite dimensional.

(b) Denote $A = k[x_0, x_1, x_2]$ and $\hat{A}$ the completion of A of (f). Then take $1 \in A$, it maps to 1 in $H^1(X_n, \mathscr{I}^{n-1})$, via the morphism $H^1(X_n, \mathscr{I}^{n-1}) \to H^1(X_n, \mathscr{O}^*_{X_n}) = \operatorname{Pic} X_n$, 1 maps to an invertible sheaf $\mathscr{L}_n$. Then take a projective limit of $\{\mathscr{L}_n\}$ and get a sheaf $\mathfrak{L}$ on $\hat{X}$. Since all $\mathscr{L}_n$ have inverse $\mathscr{M}_n$, we have $\mathfrak{L} \otimes \mathfrak{M} = \mathscr{O}_{\hat{X}}$, where $\mathfrak{M} = \varprojlim \mathscr{M}_n$. Hence $\mathfrak{L}$ is invertible.

(c) Just take $\mathfrak{F} = \mathfrak{L}$ as in (b). If $\mathfrak{F}(n)$ can be generated by global section for all $n \geq n_0$ for some n_0, by 3.11.6. we have $\mathfrak{F}$ is algebraizable, which makes a contradiction.

Solution 3.11.8. Since $H^i(X_y, \mathscr{F} \otimes \mathscr{O}_y/\mathfrak{m}_y) = 0$, and $\mathscr{F}$ is flat, we have $0 \to \mathscr{F}_y \otimes \mathfrak{m}_y^k/\mathfrak{m}_y^{k+1} \to \mathscr{F}_y \otimes \mathscr{O}_y/\mathfrak{m}_y^{k+1} \to \mathscr{F}_y \otimes \mathscr{O}_y/\mathfrak{m}_y^k \to 0$. So by induction, we have $H^i(X_y, \mathscr{F} \otimes \mathscr{O}_y/\mathfrak{m}_y^k) = 0$ since $\mathfrak{m}_y^k/\mathfrak{m}_y^{k+1}$ is a direct sum of copies of $\mathscr{O}_y/\mathfrak{m}_y$. So by formal function theorem we have $(\mathrm{R}^i f_*(\mathscr{F}))_y\hat{} = 0$, hence there exists a neighbour of y such that $\mathrm{R}^i f_*(\mathscr{F})$ is 0 on it.

3.12 The Semicontinuity Theorem

Solution 3.12.1. We may assume that Y is affine. Since Y is of finite type over k, we may assume $Y = \operatorname{Spec} k[x_1, \ldots, x_n]/I$, where I is an ideal of $k[x_1, \ldots, x_n]$ generated by $f_1, \ldots, f_m$. Then clearly $\phi(y) = \dim_k(\mathfrak{m}_y/\mathfrak{m}_y^2) = n + 1 - \operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(y)\right)$. Then if $\phi(y) = n + 1 - t$, we may assume $\det(\frac{\partial f_i}{\partial x_j}(y))_{1 \leq i,j \leq t} \neq 0$. Then the open subset $S_t = \{y' \mid \det(\frac{\partial f_i}{\partial x_j}(y'))_{1 \leq i,j \leq t} \neq 0\}$ is a neighbourhood of y and for any $y' \in S_t$, we have $\phi(y') \leq n + 1 - t$, hence ϕ is upper semicontinuous.

Solution 3.12.2. For any hypersurface Y of degree d, we may assume $Y = \operatorname{Proj} k[x_0, \ldots, x_n]/(f)$ for some polynomial f of degree d. Then $h^r(Y, \mathscr{O}_Y) = \dim_k S(Y)^r = \dim_k(k[x_0, \ldots, x_n])^r - \dim_k(k[x_0, \ldots, x_n])^{r-d} = \binom{n+r}{r} - \binom{n+r-d}{r-d}$ is a constant.

Solution 3.12.3. According to (https://mathoverflow.net/questions/90260/trouble-with-semicontinuity) the book has a typo here.

Clearly X_a is parametrized with $(t^4, t^3u, at^2u^2, tu^3, u^4)$, then $\mathscr{I} \subset \mathscr{O}_{\mathbb{P}^4 \otimes \mathbb{A}^1}$ is generated by $(a^2x_0x_4 - x_2^2, a^2x_1x_3 - x_2^2, x_0x_2 - ax_1^2, x_2x_4 - ax_3^2, ax_0x_3 - x_1x_2, ax_1x_4 - x_2x_3)$. For all $t \neq 0$, we easily know that $h^0(X_t, \mathscr{O}_{X_t}) = 1$ and $h^1(X_t, \mathscr{O}_{X_t}) = h^1(\mathscr{P}^4, \mathscr{I}_t) = h^2(\mathscr{P}^4, \mathscr{I}_t) = 0$. When $t = 0$, we have $\mathscr{I}_0$ is generated by $(x_0x_2, x_1x_2, x_2^2, x_2x_3, x_2x_4)$.

Hence $H^0(X_0, \mathscr{O}_{X_0}) = k[x_2]/(x_2^2)$, i.e. $h^0(X_0, \mathscr{O}_{X_0}) = 2$. Since $H^1(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}) = 0$, we have $h^1(\mathscr{P}^4, \mathscr{I}_0) = 1$. Moreover, by flatness, the arithmetic genus is constant, hence $h^1(X_0, \mathscr{O}_{X_0}) = h^0(X_0, \mathscr{O}_{X_0}) - \chi(\mathscr{O}_{X_0}) = 2-1 = 1$. And similarly $h^2(\mathbb{P}^4, \mathscr{I}_0) = 1$.

Solution 3.12.4. Consider $\mathscr{F} = \mathscr{L} \otimes \mathscr{M}^{-1}$. We have $\mathscr{F}_y = \mathscr{O}_{X_y}$ for each $y \in Y$. Then all we need to do is define $\mathscr{N} = f_*\mathscr{F}$, and prove that $f^*\mathscr{N} = \mathscr{F}$. On every fibre X_y, we have $\mathscr{F}_y = \mathscr{O}_{X_y}$, hence $h^0(X_y, \mathscr{F}_y) = h^0(X_y, \mathscr{O}_{X_y}) = 1$, a constant. Then by corollary 12.9., the sheaf $\mathscr{N} = f_*\mathscr{F}$ is locally free of rank 1. Since we've already had a morphism $f^*\mathscr{N} = f^*f_*\mathscr{F} \to \mathscr{F}$, it's a surjective morphism between two invertible sheaves, hence isomorphic.

Solution 3.12.5. Define a morphism $\Phi : \operatorname{Pic} Y \times \mathbb{Z} \to \operatorname{Pic} X$ as $(\mathscr{F}, m) \mapsto \pi^*\mathscr{F} \otimes \mathscr{O}_X(m)$. First we will prove Φ is injective. By proposition 7.11. in chapter II, we have $\pi_*\mathscr{O}_X \cong \mathscr{O}_Y$. Then if $\Phi(\mathscr{F}, m) = \mathscr{O}_X$ for some $\mathscr{F}$ and m, we have $\mathscr{O}_Y \cong \pi_*(\pi^*\mathscr{F} \otimes \mathscr{O}_X(m)) \cong (\pi_*\mathscr{O}_X(m)) \otimes \mathscr{F}$. Since $\mathscr{F}$ is invertible, we have $\pi_*\mathscr{O}_X(m) \cong \mathscr{F}^{-1}$. So by proposition 7.11. in chapter II, we have $m > 0$. But $\pi_*\mathscr{O}_X(m) \cong S^m(\mathscr{E})$, which has the rank $\binom{m+r-1}{r-1} > 1$, where r is the rank of $\mathscr{E}$. Hence it makes a contradiction.

For surjection, for any $\mathscr{M} \in \operatorname{Pic} X$, we restrict it to X_y for any $y \in Y$ and know that $\mathscr{M}_y$ is an invertible sheaf on $\mathbb{P}^n$, hence $\mathscr{M}_y \cong \mathscr{O}_{\mathbb{P}^r_k}(m_y)$ for some m_y. Since X is flat over Y, all $\mathscr{M}_y$ have the same Hilbert polynomial, i.e. m_y are all the same. We may define $m = m_y$ for some y. Then we define $\mathscr{F} = \pi_*(\mathscr{M} \otimes \mathscr{O}_X(-m))$. Since for any y, we have $(\mathscr{M} \otimes \mathscr{O}_X(-m))_y \cong \mathscr{O}_{X_y}$. So by 3.12.4., we have $\mathscr{M} \otimes \mathscr{O}_X(-m))_y \cong \mathscr{O}_{X_y} \cong \pi^*\mathscr{F}$ for some invertible sheaf $\mathscr{F}$ on Y, i.e. $\mathscr{M}$ has a preimage $(\mathscr{F}, m)$.

Solution 3.12.6. (a) We denote the projections as $p : X \times T \to X$ and $q : X \times Y \to Y$. Then q is projective. Define $\mathscr{M} = (p^*\mathscr{L}_t) \otimes \mathscr{L}^{-1}$. So $\mathscr{M}_t \cong \mathscr{O}_{X \times \{t\}} \cong \mathscr{O}$. Since $H^1(X, \mathscr{O}_X) = 0$, we have $q_*\mathscr{M} \otimes k(t) \to H^1((X \times T)_t, \mathscr{M}_t) = H^1(X, \mathscr{O}_X) = 0$ is clearly surjective. Then by theorem 12.11., $q_*\mathscr{M}$ is locally free of rank 1 on a neighbourhood U of t. Shrinking U and we may assume that $q_*\mathscr{M}$ is free of rank 1 on U, i.e. $q_*\mathscr{M}|_U \cong \mathscr{O}_U$. So we can pick some $s \in \mathscr{M}(X \times U)$ corresponding to $1 \in \mathscr{O}_U(U)$. Shrinking U again as we may assume that q_*s does not vanish on any point of U. Hence for any $t' \in U$, we have $\mathscr{M}|_{X \times \{t'\}} \cong \mathscr{O}_{X \times \{t'\}} \cong \mathscr{O}_X$. Hence every $\mathscr{L}_t$ for $t \in U$ is isomorphic. Since T is connected and noetherian, any two $\mathscr{L}_t$ and $\mathscr{L}_{t'}$ are isomorphic.

(b) We define $\Phi : \operatorname{Pic} X \times \operatorname{Pic} T \to \operatorname{Pic} X \times T$ as $(\mathscr{F}, \mathscr{G}) \mapsto \mathscr{F} \boxtimes \mathscr{G}$, and $\Psi : \operatorname{Pic} X \times T \to \operatorname{Pic} X \times \operatorname{Pic} T$ as $\mathscr{L} \mapsto (\mathscr{L}_t, q_*\mathscr{L})$ for some closed point t. By (a) we know that the $\mathscr{L}_t$ is well-defined, and $q_*\mathscr{L}$ is locally free of rank 1. So Φ and Ψ are both well-defined, and clearly they are invertible with each other. Hence $\operatorname{Pic} X \times \operatorname{Pic} T \cong \operatorname{Pic} X \times T$.

4 Curves

4.1 Riemann-Roch Theorem

Solution 4.1.1. Take another point $Q \neq P$, and define $D = 2P - Q$. Then $\deg D = 1$. We can pick an n such that $\deg(nD) = n > \max\{2g-2, g, 1\}$. By Riemann-Roch we have $l(nD) = n + 1 - g$. So $nD - (f) \sim D'$ for some $f \in K(X)$ and effective divisor D'. So we have $(f) \sim D' - 2nP + nQ$, so f has a pole at P and regular everywhere else since D' is effective and of degree n.

Solution 4.1.2. Take another point $Q \neq P_1, \dots, P_r$ and define $D = 2P_1 + \ldots + 2P_r - (2r-1)Q$. Then $\deg D = 1$. We can pick an n such that $\deg(nD) = n > \max\{2g-2, g, 1\}$ like in 4.1.1. By Riemann-Roch we have $l(nD) = n + 1 - g \geq 1$. So $nD - (f) \sim D'$ for some $f \in K(X)$ and effective divisor D'. So $(f) \sim D' - 2n(P_1 + \ldots + P_r) + n(2r-1)Q$. So f has poles at each P since D' is effective and of degree n.

Solution 4.1.3. By remark 4.10.2.(e) in chapter II, we may embed X in a complete variety $\bar{X}$ as an open subset. Then $\bar{X} \backslash X = \{P_1, \dots, P_r\}$. Then by 4.1.2., there exists some $f \in K(\bar{X})$ such that f has poles at all P_i and no poles everywhere else. Since f gives a finite morphism from $\bar{X}$ to $\mathbb{P}^1$, hence affine, we have $X = f^{-1}(\mathbb{A}^1)$ is affine.

Solution 4.1.4. By 3.3.1. and 3.3.2., we may assume X is integral. Then since X is not proper, the normalization $\tilde{X}$ is not proper either. So by 4.1.3. we know that $\tilde{X}$ is affine. Then by 3.4.2., since $\tilde{X} \to X$ is finite, we know X is affine.

Solution 4.1.5. By Riemann-Roch, we have $\dim|D| = l(D) - 1 = \deg D - g + l(K - D)$. Since D is effective, we have $l(K-D) \leq l(K)$. Moreover, we have $l(K) - l(K-K) = 2g - 2 + 1 - g = g$. So $\dim|D| \leq \deg D - g + g = \deg D$. If the equality holds, we have $l(K - D) = l(K)$, i.e. $D = 0$ or $g = 0$.

Solution 4.1.6. For any $P \in X$, we may define $D = gP$. Then $\deg D = g$, and $l(D) \geq \deg D + 1 - g = 1$. Then there exists some $f \in K(X)$ which has a pole at P with order g and regular everywhere else. So $f : X \to \mathbb{P}^1$ with $D \mapsto \infty$, and $\deg f = \deg D = g$.

Solution 4.1.7. (a) If $g = 2$, we have $\deg K = 2g - 2 = 2$ and $\dim|K| = l(K) - 1 = g - 1 = 1$. For any $P \in X$, we have $\dim|P| - \dim|K - P| = 1 + 1 - g = 0$. So if $\dim|P| = 1$, there exists a rational morphism $f : X \to \mathbb{P}^1$, which is contradict with the fact that $g > 1$. Then $\dim|P| = 0$, hence $\dim|K - P| = 0$. Hence $\dim|K - P| = \dim|K| - 1$, i.e. P is not a base point. Moreover, since $\deg K = 2$, we have a morphism of degree 2 from X to $\mathbb{P}^1$, i.e. X is hyperelliptic.

(b) If X is a curve on quadric surface corresponding to the divisor of degree $(g+1, 2)$, we consider the morphism $f : X \to \mathbb{P}^1$, where $f = \pi_2|_X$ is the second projection $Q = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Then by theorem 6.8. in chapter II, this morphism is finite and non-constant. So by theorem 6.9. in chapter II, $\deg f^*(P) = \deg f \cdot \deg P$, i.e. $\deg f = 2$.

Solution 4.1.8. (a) Since X and $\tilde{X}$ are both projective, we have $H^0(X, \mathscr{O}_X) = k$ and $H^0(X, f_*\mathscr{O}_{\tilde{X}}) = H^0(\tilde{X}, \mathscr{O}_{\tilde{X}}) = k$. And since $\sum \tilde{\mathscr{O}}_P/\mathscr{O}_P$ is a direct sum of flasque sheaf, we have $H^1(X, \sum \tilde{\mathscr{O}}_P/\mathscr{O}_P) = 0$. So we have an exact sequence $0 \to H^0(X, \sum \tilde{\mathscr{O}}_P/\mathscr{O}_P) \to H^1(X, \mathscr{O}_X) \to H^1(\tilde{X}, \mathscr{O}_{\tilde{X}}) \to 0$. So by 3.5.3., $p_a(X) = p_a(\tilde{X}) + \sum_{P \in X} \dim_k \tilde{\mathscr{O}}_P/\mathscr{O}_P = p_a(\tilde{X}) + \sum \delta_P$.

(b) Since p_a and δ_P are all non-negative, if $p_a(X) = 0$, we have $\delta_P = 0$ for all P. So every local rings are normal, then the curve X is nonsingular. Since $p_a(X) = 0$, by example 1.3.5., X is just isomorphic to $\mathbb{P}^1$.

(c) For arbitrary curve X, we may embed it into $\mathbb{P}^3$. Consider the projection $\pi : \mathbb{P}^3 \to \mathbb{P}^2$, we denote $\pi' : \tilde{X} \to \pi(X)$. So $\delta_P = \text{length } \tilde{\mathscr{O}}_P/\mathscr{O}_P = \text{length } \pi'(\tilde{\mathscr{O}})_{\pi(P)}/\mathscr{O}_{\pi P} = \delta_{\pi(P)}$. So we may assume that X is a plane curve. So we may assume we have an affine neighbourhood of the node or the original cusp P as the zero of

$\phi(x,y)$ in $\mathbb{A}^2$, where ϕ has the form $\phi = g_2(x,y)$+higher terms, and $g_2(x,y)$ the the degree 2 term. So we may formally decompose ϕ as two terms ψ_1 and ψ_2 in degree 1. Then $\tilde{\mathscr{O}}_P = \mathscr{O}_P(\psi_1) = \mathscr{O}_P(\psi_2)$. So there does not exists other k-module A such that $\mathscr{O}_P \subset A \subset \tilde{\mathscr{O}}_P$, i.e. length $\tilde{\mathscr{O}}_P/\mathscr{O}_P = 1$.

Solution 4.1.9 (Riemann-Roch for Singular Curves)**.** (a) Denote $f : X_{\text{reg}} \to X$ as the canonical morphism. Since we have an exact sequence $0 \to \mathscr{L}(D) \to f_*\mathscr{L}_{\text{reg}}(D) \to \sum_{P\in X}(f^*\mathscr{L}(D))_P/\mathscr{L}(D)_P \to 0$. And since $\mathscr{L}(D)$ is locally free, in the local ring we have $\mathscr{L}(D)_P = \mathscr{O}_{X,P}$ and $(f_*\mathscr{L}_{\text{reg}}(D))_P = \tilde{\mathscr{O}}_{X,P}$. So $H^0(X, (f_*\mathscr{L}_{\text{reg}}(D))_P/\mathscr{L}(D)_P) = \delta_P$ as in 4.1.8. Hence $\chi(\mathscr{L}(D)) = \chi(f_*\mathscr{L}_{\text{reg}}(D)) - \sum_{P\in X}\delta_P = \deg D + 1 - p_a(X_{\text{reg}}) - \sum_{P\in X}\delta_P = \deg D + 1 - p_a(X)$.

(b) For any Cartier divisor D and very ample divisor L, there exists some $n > 0$ such that $D + nL$ is generated by global section. Then by 2.7.5.(d), $M = D + nL + L = D + (n+1)L$ is also very ample. Then $D = M - (m+1)L$ is the difference of two very ample divisors.

(c) By (b) we only need to consider the case that $\mathscr{L}$ is very ample. Then there exists some closed embedding $f : X \to \mathbb{P}^n$ such that $\mathscr{L} = f^*\mathscr{O}(1)$. By Bertini's theorem, we may choose a hyperplane $H \subset \mathbb{P}^n$ such that $H \cap X \subset X_{\text{reg}}$ does not contain any singular point of X, hence we may define $D = H \cap X$, and $\mathscr{L} = \mathscr{L}(D)$.

(d) By proposition 8.23. in chapter II, we know that X is CM. Hence by theorem 7.6. in chapter III, we have $H^1(X, \mathscr{L}(D)) \cong \operatorname{Ext}^0_X(\mathscr{L}(D), \omega^0_X)' = \operatorname{Ext}^1_X(\mathscr{O}_X, \mathscr{L}(-D) \otimes \omega_X)' \cong H^0(X, \omega_X \otimes \mathscr{L}(-D))$. So $l(D) - l(K-D) = \chi(\mathscr{L}_D) = \deg D + 1 - p_a(X)$.

Solution 4.1.10. By 4.1.9.(d), we have $\deg K = p_a - 1 = 0$. Then if D is a divisor of degree 0, we have $D + P_0$ corresponding to an invertible sheaf $\mathscr{L}(D')$ for some divisor D' supported on X_{reg}. And on X_{reg}, the divisor $D' - P_0$ has degree 0 corresponding to some $P - P_0$ on X_{reg} by example 1.3.7. Hence $D \sim P - P_0$ for some $P \in X_{\text{reg}}$. Conversely, for any $P \in X_{\text{reg}}$, there exists some invertible sheaf $\mathscr{L}$ corresponding to the divisor $P - P_0$ clearly. Hence we have a bijection $X_{\text{reg}} \to \operatorname{Pic}^0_X$.

4.2 Hurwitz's Theorem

Solution 4.2.1. We use induction on n. If we've proved that for any $i < n$, $\mathbb{P}^i$ is simply connected, if $f : X \to \mathbb{P}^n$ is an étale covering, for any hyperplane $H \subset \mathbb{P}^n$, f^*H is ample. By corollary 7.9. in chapter III, f^*H is connected. Then since H is simply connected, $f : f^*H \to H$ is an étale covering, hence trivial. Then $f|_H$ is an isomorphism, i.e. $\deg f = 1$. So f is an isomorphism globally.

Solution 4.2.2 (Classification of Curves of Genus 2)**.** (a) Since $g_{\mathbb{P}^1} = 0$, by Hurwitz's formula, we have $2g_X - 2 = 2 \times (0-2) + \deg R$. So $\deg R = 6$. For any ramified point P, since $e_P \leq \deg f = 2$ and $p > 2$, we have $e_P = 2$. So R is the sum of 6 points and each ramified point has ramification index 2.

(b) Denote the integral ring of K as O_K. For any prime $(x - \alpha) \in k[x]$, it will be decomposed into two primes in O_K if and only if $\alpha = \alpha_i$ for some $i = 1, \ldots, 6$. And the infinity valuation of $k[x]$ does not split in O_K. So the morphism $f : X \to \mathbb{P}^1$ has 6 ramification points $\alpha_1, \ldots, \alpha_6$, and the ramification index of each point is 2. Hence the ramification divisor $R = \sum \alpha_i$, $\deg R = 2$. So by Hurwitz's formula, we have $g_X = 2$. Moreover, suppose f is given by a divisor D of degree 2. We have $|D| - |K_X - D| = \deg D + 1 - g_X = 1$. So $|K_X - D| = 0$. And since $|K_X - D| - |D| = \deg(K_X - D) + 1 - g_X$, we have $\deg(K_X - D) = 0$, i.e. $K_X - D$ is trivial because $\operatorname{Pic}^0\mathbb{P}^1$ is trivial. So $K_X = D$, f is given by just K_X.

(c) If one of three points is the infinity, we may assume $P_1 = x_1$ and $P_2 = x_2$ in $\mathbb{A}^1$, and $P_3 = \infty$, then the linear transformation $f(x) = \frac{x-x_1}{x_2-x_1} : \mathbb{A}^1 \to \mathbb{A}^1$ maps P_1, P_2, P_3 to $0, 1, \infty$. If no one of three points is the infinity, we may assume $P_1 = x_1, P_2 = x_2, P_3 = x_3 \in \mathbb{A}^1$. Then the fractional transformation $f(x) = \frac{x-x_1}{x-x_3} \cdot \frac{x_2-x_3}{x_2-x_1}$ maps P_1, P_2, P_3 to $0, 1, \infty$.

(d) **Every chapter has some exercises which are not exactly questions.**

(e) By (a), (b) and (d), trivial.

Solution 4.2.3 (Plane Curves)**.** (a) ($\Leftarrow$) (1) If $P \in L$, we may assume $P = (0,0) \in \mathbb{A}^2$, and $L = (y = 0)$, $T_P = (x = 0)$. If $X \cap \mathbb{A}^2$ is generated by a polynomial f, then for every $Q = (a,b) \in X$, $T_Q = (f'_x(Q)(x-a) + f'_y(Q)(y-b) = 0)$, which intersects L at $(a + \frac{f'_y(Q)}{f'_x(Q)}b, 0)$. So if t is a local parameter at $0 \in \mathbb{A}^1$, we have $\phi^*(t) = \frac{f'_y}{f'_x}y + x$. On the y-axis, we have $f'_y(0) = 0$ and $\phi(0) = 0$. So x vanishes at 0 of order ≥ 2, i.e. ϕ is ramified at 0.

($\Leftarrow$) (2) If $P \in L$, all things are the same with (1). So we may assume $P = (0,0) \in \mathbb{A}^2$, $T_P = (x = 0)$, and L is the line at infinity. Then similarly, the $\phi : X \to \mathbb{A}^1$ is just $Q \mapsto -\frac{f'_y(Q)}{f'_x(Q)}$. Since P is an inflection point, we may assume X is generated by f in $\mathbb{A}^2$. Then if f restricted to $x = 0$ has degree ≥ 3 in y, we have $f'_y \in \mathfrak{m}_P^2$. Hence for any local parameter t at P, $\phi^*(t) \in \mathfrak{m}_P^2$, i.e. ϕ is ramified at P.

($\Rightarrow$) If $P \in L$, we've done. If else, we similarly may assume $P = (0,0) \in \mathbb{A}^2$, $T_P = (x = 0)$, and L is the line at infinity. And $\phi : X \to \mathbb{A}^1$ is $Q \mapsto -\frac{f'_y(Q)}{f'_x(Q)}$. For any local parameter t at P, $\phi^*(t) \in \mathfrak{m}_P^2$ implies $f'_y \in \mathfrak{m}_P^2$ locally. Hence P is an inflection point of X.

Finally, by Hurwitz's formula, we know that ramified points are finite, hence inflection points are also finite.

(b) We may assume $L = (z = 0)$ on $\mathbb{P}^2$ is a multiple tangent as we want. If for some P such that L is the tangent at P, we may assume $P = [a : b : 0]$. Then $t = \frac{bx-ay}{ax+by}$ is a local coordinate on X near p. Since L is not an inflection line near P, near P, $\frac{z}{x} = t^2 f$ at $\mathscr{O}_{X,P}$ for some invertible $f \in \mathscr{O}_{X,P}$. Since $\hat{\mathscr{O}}_{X,P} \cong k[[t]]$, we may assume $f(t) = f_0 + f_1 t + f_2 t^2 + \ldots$. So near P, there exists an open neighbourhood such that t is a coordinate and f is invertible. So the tangent line at any t_0 is $(z - t_0^2 f(t_0)x) - (2t_0 f(t_0) + t_0^2 f'(t_0))(y - t_0 x) = 0$, i.e. $t_0^2(f(t_0) - t_0 f'(t_0))x + t_0(-2f(t_0) - t_0 f'(t_0))y + z = 0$. So the tangent of L in the dual curve corresponding to P is $[0 : -2f_1 : 0]$. Hence L is an ordinary r-fold point on X^*.

(c) We may assume that $O = (0,0,1)$, $P = (0,1,1)$ and L is a line at infinity which does not contain O. Then $\psi : (x,y) \mapsto [x : y]$. So on $D(y)$, we may assume $\psi : U \to D(y)$ as $(x,y) \mapsto \frac{x}{y}$ for some neighbourhood U of P. So $\psi(P) = 0$. If ψ is ramified at P, we have $\psi^*(t) = \frac{x}{y} \in \mathfrak{m}_P^2$, where t is a local parameter at 0. Since $y \neq 0$, we have $x \in \mathfrak{m}_P^2$, i.e. the line $x = 0$ is tangent to X at P. By Hurwitz's formula, we have $(d-1)(d-2) - 2 = -2d + \deg R$, i.e. $\deg R = d(d-1)$. Since 0 is not an inflection point or on the tangent line, R is reduced. So the number of tangent lines is $\deg R = d(d-1)$. And the ramification index in each P is just 2.

(d) For any point $O \in X$ not on any inflections or multiple tangents, we define $\psi : X \to \mathbb{P}^1$ as the projection from P. So $\deg \psi = d-1$. By Hurwitz's theorem $(d-1)(d-2) - 2 = -2d + 2 + \deg(R)$, i.e. $\deg R = (d+1)(d-2)$. So the number of tangent lines is $\deg R = (d+1)(d-2)$.

(e) Since $\phi^{-1}(P) = \{Q \in X \mid P \in T_Q(X)\}$, if P is not an inflection or on a multiple tangent, by (c) we have $\sharp\phi^{-1}(P) = d(d-1)$, i.e. $\deg \phi = d(d-1)$. So by Hurwitz's formula, we have $\deg R = 3d^2 - 5d$. So by ignoring the ramification of type (1) in part (a), we know that the number of inflection points is $3d^2 - 6d$. And the last is obvious.

(f) Since the map $\phi : X \to X^*$ is finite and birational, and X is normal, we have X is the normalization of X^*. So $p_a(X^*) = \frac{1}{2}(d(d-1)-1)(d(d-1)-2)$. Since $p_a(X^*) = p_a(X) + \sharp\text{inflections} + \sharp\text{bitangents}$, $p_a(X) = \frac{1}{2}(d-1)(d-2)$, and $\sharp$inflections$= 3d^2 - 6d$ by (e), we have $\sharp$bitangents$= \frac{1}{2}d(d-2)(d-3)(d+3)$.

(g) By (e), $\sharp$inflections$= 3 \times 3 \times (3-2) = 9$. And since degree is just 3, they are all ordinary. Choose a coordinate system such that $P = (0,0,1)$ and $Q = (0,1,0)$ are inflection points and the tangent lines are $(y = 0)$ and $(z = 0)$. So the cubic curve is just $yz(ax + by + cz) + dx^3 = 0$ for some a,b,c,d. So the line passing through P, Q intersect X at $(0, -c, b)$, which is an inflection point.

(h) By (f), $\sharp$bitangents$= \frac{1}{2} \times 4 \times 2 \times 1 \times 7 = 28$.

Solution 4.2.4 (A Funny Curve in Characteristic p)**.** Since $f_x = z^3$, $f_y = x^3$, $f_z = y^3$, and the common zero $(0,0,0) \notin \mathbb{P}^2$, X is smooth. Moreover, since the Hessian matrix is just 0 at all points, every point is an inflection point. For any $P = (a,b,c)$, the tangent line at P is $c^3(x-a) + a^3(y-b) + b^3(z-c) = 0$, i.e. $c^3 x + a^3 y + b^3 z = 0$. So the map $X \to X^*$ is just the Frobenious morphism, hence isomorphic and purely inseparable.

Solution 4.2.5 (Automorphisms of a Curve of Genus ≥ 2)**.** (a) For any ramified point $P \in X$ with $e_P = r$, we may denote $f^{-1}(f(P)) = \{P_1, \ldots, P_s\}$. They form an orbit of G on X. So P_i's have conjugate stabilizers. So $s =$the index of the stabilizer$= \frac{|G|}{r} = \frac{n}{r}$. By Hurwitz's formula, we have $2g(X)-2 = n(2g(Y)-2)+\sum_{i=1}^{s} \frac{n}{r_i}(r_i-1)$, i.e. $\frac{2g-2}{n} = 2g(Y) - 2 + \sum_{i=1}^{s}(1 - \frac{1}{r_i})$.

(b) If $g(Y) \geq 1$ and $\sum_{i=1}^{s}(1 - \frac{1}{r_i}) = 0$, we have $g(Y) \geq 2$, hence by (a), $n \geq g(X) - 1$. If $g(Y) \geq 1$ and $\sum_{i=1}^{s}(1 - \frac{1}{r_i}) > 0$, we have $\sum_{i=1}^{s}(1 - \frac{1}{r_i}) \geq \frac{1}{2}$, so $2g(Y) - 2 + \sum_{i=1}^{s}(1 - \frac{1}{r_i}) \geq \frac{1}{2}$, i.e. $n \geq 4(g(X) - 1)$. If $g(Y) = 0$, then by (a) we have $\sum_{i=1}^{s}(1 - \frac{1}{r_i}) = \frac{2g(X)-2}{n} + 2 \geq 2$. So if $s \geq 5$, we have $\sum_{i=1}^{s}(1 - \frac{1}{r_i}) \geq s \cdot (1 - \frac{1}{2}) \geq 2 + \frac{1}{2}$. If $s = 4$, the minimal case is $(r_1, r_2, r_3, r_4) = (2, 2, 2, 3)$, hence $\sum_{i=1}^{s}(1 - \frac{1}{r_i}) \geq 2 + \frac{1}{6}$. If $s = 3$, the minimal case is $(r_1, r_2, r_3) = (2, 3, 7)$, hence $\sum_{i=1}^{s}(1 - \frac{1}{r_i}) \geq 2 + \frac{1}{42}$. So $\sum_{i=1}^{s}(1 - \frac{1}{r_i}) \geq 2 + \frac{1}{42}$ at all, and $n \geq 84(g(X) - 1)$.

Solution 4.2.6 (f_* for Divisors)**.** (a) We may assume X and Y are both affine. For any effective divisor D, $\mathscr{L}(-D)$ is quasi-coherent. So consider the exact sequence $0 \to \mathscr{L}(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0$. By theorem 8.1. in chapter III, we have $\mathrm{R}^1 f_*\mathscr{L}(-D) = 0$, so $0 \to f_*\mathscr{L}(-D) \to f_*\mathscr{O}_X \to f_*\mathscr{O}_D \to 0$. Then by theorem 6.11.(b) in chapter II, we have $\det(f_*\mathscr{L}(-D)) \cong \det(f_*\mathscr{O}_X) \otimes (\det(f_*\mathscr{O}_D))^{-1}$. Since $f_*\mathscr{O}_D \cong \bigoplus_{i=1}^{n} \mathscr{O}_{f_*D}$, i.e. $\det(f_*\mathscr{O}_D) \cong \bigotimes \det(\mathscr{O}_{f_*D}) = \mathscr{L}(f_*D)$. So $\det(f_*\mathscr{O}_D)^{-1} \cong \mathscr{L}(-f_*D)$, and $\det(f_*\mathscr{L}(-D)) \cong \det(f_*\mathscr{O}_X) \otimes \mathscr{L}(-f_*D)$. For arbitrary D, we may write $D = D_1 - D_2$ for two effective divisors D_1 and D_2 and consider the exact sequence $0 \to \mathscr{L}(D) \to \mathscr{L}(-D_2) \to \mathscr{O}_{D_1} \to 0$. And similarly we have $\det(f_*\mathscr{L}(D)) = \det(f_*\mathscr{L}(-D_2)) \otimes \mathscr{L}(f_*D_1) = \det(f_*\mathscr{O}_X) \otimes \mathscr{L}(-f_*D_2)) \otimes \mathscr{L}(f_*D_1) = \det(f_*\mathscr{O}_X) \otimes \mathscr{L}(f_*D)$.

(b) Since $\mathscr{L}(D)$ only depends on its linear equivalent class, by (a), so does f_*D. If $\deg f = n$, the pullback of a point is a divisor of degree n, so $f_*f^* = n$.

(c) By 3.7.2., we have $f^!\Omega_Y \cong \Omega_X$. By 3.6.10., we have $f_*\Omega_X = f_*\mathrm{Hom}_X(\mathscr{O}_X, f^!\Omega_Y) = \mathrm{Hom}_Y(f_*\mathscr{O}_X, \Omega_Y) = (f_*\mathscr{O}_X)^{-1} \otimes \Omega_Y$. Since both side are locally free of rank n, we have $\det f_*\Omega_X \cong \det((f_*\mathscr{O}_X)^{-1} \otimes \Omega_Y) = (\det f_*\mathscr{O}_X)^{-1} \otimes \Omega_Y^{\otimes n}$.

(d) Since $K_X \sim f^*K_Y + R$, we have $f_*K_X \sim f_*f^*K_Y + f_*R = nK_Y + B$. So $\mathscr{L}(-B) \cong \Omega_Y^{\otimes n} \otimes \mathscr{L}(f_*K_X)^{-1}$. By (a) we have $\mathscr{L}(f_*K_X)^{-1} \cong \det f_*\mathscr{O}_X \otimes (\det f_*\Omega_X)^{-1}$, we have $\mathscr{L}(-B) \cong (\det f_*\mathscr{O}_X)^2$.

Solution 4.2.7 (Étale Covers of Degree 2)**.** (a) For every $P \in Y$, $(f_*\mathscr{O}_X)_P$ is a rank 2 free module over $\mathscr{O}_{Y,P}$. So $\mathscr{L}_P$ is a rank 1 free module, hence $\mathscr{L}$ is invertible. So by the exact sequence $0 \to \mathscr{O}_Y \to f_*\mathscr{O}_X \to \mathscr{L} \to 0$, we have $\mathscr{L} \cong \det \mathscr{L} \cong \det f_*\mathscr{O}_X \otimes (\det \mathscr{O}_Y)^{-1} \cong \det f_*\mathscr{O}_X$. Then $\mathscr{L}^2 = (\det f_*\mathscr{O}_X) = \mathscr{L}(-B)$. Since f is étale, hence unramified, i.e. $B = 0$, so $\mathscr{L}^2 \cong \mathscr{O}_Y$.

(b) Clearly the canonical morphism $f : X \to Y$ is finite, so x is integral, separated, of finite type over k, and of dimension 1. So X is a curve. Moreover, since X is obviously normal, it is smooth. The function field is clearly an extension of degree 2, so by 3.10.3., f is étale.

(c) For exact sequence $0 \to \mathscr{O}_Y \to f_*\mathscr{O}_X \to \mathscr{L} \to 0$, we have a section $f_*\mathscr{O}_X \to \mathscr{O}_Y$ as $\sigma \mapsto \frac{\sigma+\tau\sigma}{2}$. So the exact sequence is split, and $f_*\mathscr{O}_X \cong \mathscr{O}_Y \oplus \mathscr{L}$. So by 2.5.17., $X \cong$ **Spec** $(\mathscr{O}_Y \oplus \mathscr{L})$.

4.3 Embeddings in Projective Space

Solution 4.3.1. ($\Leftarrow$) By corollary 3.2., trivial. ($\Rightarrow$) If D is very ample, then $l(D) = l(D - P - Q) \geq 2$ for any two points P, Q. Since $g = 2$, we have $\dim |D| \neq 1$ so $l(D) \neq 2$, i.e. $l(D) > 2$. So, if $\deg D \leq 1$, then by 4.1.5., $l(D) \leq degD + 1 \leq 2$, which contradicts. If $\deg D = 2$, $l(D) = l(K - D) + 1 < l(K) + 1 = 2$, which contradicts. If $\deg D = 3$, we have $l(K - D) = 0$, so $l(D) = 2$, which contradicts. If $\deg D = 4$, by corollary 3.2., D has no base point. So $|D|$ gives a morphism to $\mathbb{P}^2$ and X is a plane curve. So $g = \frac{1}{2}(4 - 1)(4 - 2) = 3 \neq 2$, which contradicts. Hence by all above, $\deg D \geq 5$.

Solution 4.3.2. (a) Define $D = X.L$ for some line L. By $g(X) = 3$, we have $l(K) = 3$, $\deg K = 4$. By Bézout theorem, $\deg D = 4$. Since $\dim |L| = 2$, we have $l(D) = 3$. So $l(K - D) = l(D) + g(X) - \deg D - 1 = 1$, i.e. $\deg K - D) = 0$. So by 4.1.5., $K = D$.

(b) Suppose $D = P + Q$ for two points P, Q. Since K is very ample and gives an embedding to $\mathbb{P}^2$, we have $\dim|K| = 2$. So define L as the line PQ. We have $K = X.L$ by (a). Thus we may assume $K = P + Q + R + S$ for some two points R, S. So $\dim|D| = \dim|K - P - Q| = \dim|K| - 2 = 0$.

(c) If $\deg D = 2$, by (b), $\dim|D|$ cannot be 1. So there cannot exist a close morphism $X \to \mathbb{P}^1$ of degree 2, i.e. not hyperelliptic.

Solution 4.3.3. We may assume $X = \bigcup Hi$ for some hypersurfaces. By 2.8.4.(e), we have $\mathscr{L}(K) \cong \mathscr{O}_X(n)$ for some $n > 0$. So $|K|$ induces an n-tuple embedding, hence K is very ample.

If $g(X) = 2$, we have $\deg K = 2g - 2 = 2$, so K is not very ample by 4.3.1. Hence X is not a complete intersection.

Solution 4.3.4. (a) Since $\mathbb{P}^1$ is projectively normal, and by 2.5.14., the d-uple embedding is projectively normal, the image, i.e. the rational normal curve of degree d in $\mathbb{P}^d$ is projectively normal. Moreover, clearly, the kernel is generated by $x_i x_j = x_k x_l$ for all 4-tuple (i, j, k, l) such that $i + j = k + l$.

(b) For any hyperplane $H \subset \mathbb{P}^n$, we define $D = X.H$. Then $\dim|D| = n$ and $\deg D = d$. So $l(D) = n + 1 \leq \deg D + 1 = d + 1 \leq n + 1$. So $d = n$, and by 4.1.5., $g(X) = 0$. So $X \cong \mathbb{P}^1$. Since $X \not\subseteq \mathbb{P}^{n-1}$, the morphism $\Gamma(\mathbb{P}^n, \mathscr{O}(1)) \to \Gamma(X, \mathscr{L}(D))$ is injective. So D corresponds to a $(n + 1)$-dimension subspace $V \in \Gamma(\mathbb{P}^1, \mathscr{O}(n))$. Since $\dim_k \Gamma(\mathbb{P}^1, \mathscr{O}(n)) = n + 1$, they are equal. Hence $X \subset \mathbb{P}^n$ is induced by $|D|$, hence it is a rational normal curve.

(c) Take n small enough such that $X \subset \mathbb{P}^n$ and $X \not\subseteq \mathbb{P}^{n-1}$ for any $\mathbb{P}^{n-1}$. As in (b), we have $n = 2$ and X is a rational normal curve of degree 2, i.e. a conic in some $\mathbb{P}^2$.

(d) If X is not a plane cubic curve, by (b), $X \subset \mathbb{P}^3$ and it is a rational normal curve of degre 3, hence a twisted cubic.

Solution 4.3.5. (a) If $\phi(X)$ is nonsingular, by 2.5.5. we have $\dim H^0(\phi(X), \mathscr{O}_{\phi(X)}(1)) \leq \dim H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)) = 3$. By since X is not a plane curve, we have $H^0(\mathbb{P}^3, \mathscr{I}_X(1)) = 0$. So $H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) \to H^0(X, \mathscr{O}_X(1))$ is injective, i.e. $\dim H^0(X, \mathscr{O}_X(1)) \geq 4$, which contradicts.

(b) Since X is the normalization of $\phi(X)$, by 4.1.8. we have $g(X) \leq g(\phi(X)) = \frac{1}{2}(d-1)(d-2)$. Since $\phi(X)$ is not nonsingular, the inequality above is strict.

(c) If X_0 has no nilpotent element, it is a nodal plane curve, which has more bigger genus than X_t with $t \neq 0$, which contradicts that flat family has same genus.

Solution 4.3.6. (a) Suppose $X \subset \mathbb{P}^n$ for smallest n. If $n \geq 4$, by 4.3.4.(b), we have X is a rational normal curve. If $n = 2$, then $g(X) = \frac{1}{2}(4-1)(4-2) = 3$. If $n = 3$, by 4.3.5.(b) we have $g < 3$. If $g = 2$, by 4.3.1., any divisor of degree 3 is not very ample, so X cannot be embedded in $\mathbb{P}^3$, which contradicts. If $g = 0$, X is clearly a rational quartic curve.

(b) If $g = 1$, we may consider the exact sequence $0 \to \mathscr{I}_X(2) \to \mathscr{O}_{\mathbb{P}^3}(2) \to \mathscr{O}_X(2) \to 0$. Since $\dim H^0(\mathbb{P}^3, \mathscr{O}_{\mathscr{P}^3}(2)) = 1$, and $\dim H^0(X, \mathscr{O}_X(2)) = l(2K) = 8 + 1 - 1 = 8$ by Riemann-Roch. So $\dim H^0(\mathbb{P}^3, \mathscr{I}_X(2)) \geq 2$. So X is a complete intersection of more than 2 irreducible surface. Since the intersection of 2 quadric curves has degree 4 by Bézout theorem, this intersection is all of X.

Solution 4.3.7. Clearly $xy + x^4 + y^4 = 0$ has only one node $(0, 0)$. If it can by represented as a projection of a nonsingular curve X in $\mathbb{P}^3$, X must have degree 4 and genus 2. But by 4.3.6., it is impossible.

Solution 4.3.8. (a) Obviously, $(1, 0, 0)$ is on every tangent line.

(b) We may assume $X \subset \mathbb{P}^n$ is a closed embedding. Denote P as the strange point of X. If $O \notin X$ is a point, ϕ is the projection about O, $\phi(P)$ is clearly on every tangent line of $\phi(X)$. So we may assume P is in $\mathbb{P}^3$. Then by theorem 3.9. X is $\mathbb{P}^1$.

Solution 4.3.9. Since the tangent variety has dimension ≤ 2, and the dimension of multisecant lines ≤ 1, the hyperplanes factor through any tangent line or multisecant line has dimension ≤ 2 in $(\mathbb{P}^3)'$, hence there exists an open subset U of $(\mathbb{P}^3)'$ and every hyperplane H in U does not factor through any tangent line and multisecant lines, i.e. $X.H$ consists of d distinct points and no three of them are collinear.

Solution 4.3.10. Consider $(\mathbb{P}^n)^{\times n}$, and a hyperplane $H \subset (\mathbb{P}^n)^{\times n}$ consisting of all points $(x_1, \ldots, x_n)$ such that $x_1, \ldots, x_n$ are collinear. Hence H is closed. Consider $X^{\times n} \subset (\mathbb{P}^n)^{\times n}$. Since X is not contained in any $\mathbb{P}^{n-1}$, so $X^{\times n} \not\subseteq H$. Since $(\mathbb{P}^n)^{\times n} - H$ is open in $(\mathbb{P}^n)^{\times n}$, we have $X^{\times n} \cap ((\mathbb{P}^n)^{\times n} - H)$ is open in $X^{\times n}$, and it is non-empty. So for all $\{P_1, \ldots, P_n\} \subset X$ such that $(P_1, \ldots, P_n) \in X^{\times n} \cap ((\mathbb{P}^n)^{\times n} - H)$, $P_1, \ldots, P_n$ are not collinear.

Solution 4.3.11. (a) Similarly with proposition 3.4., O induces a closed immersion $X \to \mathbb{P}^{n-1}$ iff O is not on any secant line or tangent space of X. Moreover, since **Sec** is locally like $(X \times X - \Delta) \times \mathbb{P}^1$, which has dimension $2r + 1$, and the tangent bundle has dimension $2r$, there exists point $O \subset \mathbb{P}^n$ such that O is not on any secant line or tangent space of X since $n > 2r + 1$.

(b) Denote the Veronese embedding as ρ. For any point $R \in \mathbb{P}^5$ on the secant line $\overline{\rho(P)\rho(Q)}$ for some two points $P, Q \in \mathbb{P}^2$. Consider $Y = \rho(\overline{PQ})$, the image of secant line in $\mathbb{P}^2$. R is on the plane spanned by Y. So every line on this plane passing through R is also a secant line to Y, i.e. to X. Moreover, any point on a secant line of X lies on a family of secant lines of dimension 1 of X. So **Sec** has dimension $\leq 2 + 2 + 1 - 1 = 4$. But conversely, the secant line of the four point $\rho(1:0:0), \rho(0:1:0), \rho(0:0:1), \rho(1:1:0)$ are linearly independent, i.e. $\dim \mathbf{Sec} \geq 4$. Hence $\dim \mathbf{Sec} = 4$.

Solution 4.3.12. 1. Case $r = 0$ and $d = 2, 3, 4, 5$. The curve $x^d + y^d + z^d = 0$ over $\mathbb{C}$ is smooth.

2. Case $r = 1$ and $d = 3, 4, 5$. The curve $xyz^{d-2} + x^d + y^d = 0$ over $\mathbb{C}$ has only one node $(0:0:1)$.

3. Case $d = 4, r = 3$ or $d = 5, r = 6$. Consider rational normal curve X of degree d in $\mathbb{P}^d$. We may project it into $\mathbb{P}^2$ as Y. Then $\deg Y = d$. And $g(Y) = \frac{1}{2}(d-1)(d-2)$. Since X is isomorphic to $\mathbb{P}^1$, i.e. $g(X) = 0$, by 4.1.8. Y has $\frac{1}{2}(d-1)(d-2)$ nodes.

4. Case $d = 4, r = 2$. The curve $x^2yz + x^2z^2 + xy^3 + xyz^2 - y^2z^2 = 0$ over $\mathbb{C}$ has only two nodes $(0:0:1)$ and $(1:0:0)$.

5. Case $d = 5, r = 2$. The curve $-x^4y + x^4z - x^3y^2 - x^3yz + x^3z^2 - x^2y^3 - x^2y^2z + x^2yz^2 + x^2z^3 - xy^4 - xyz^3 - y^5 - y^4z + y^3z^2 + y^2z^3 = 0$ over $\mathbb{C}$ has only two nodes $(0:0:1)$ and $(0, 1, -1)$.

6. Case $d = 5, r = 3$. The curve $x^5 + x^4z + x^3y^2 + x^3yz - x^3z^2 - x^2y^2z - x^2yz^2 + x^2z^3 - xy^4 + xy^2z^2 - xyz^3 - y^5 - y^3z^2 + y^2z^3 = 0$ over $\mathbb{C}$ has only three nodes $(0:0:1)$, $(\frac{1-\sqrt{-3}}{2} : 1 : 1)$ and $(-\frac{1-\sqrt{-3}}{2} : 1 : 1)$.

7. Case $d = 5, r = 4$. The curve $x^2y^2z - x^2yz^2 + x^2z^3 + xy^2z^2 - xyz^3 + y^4z + y^3z^2 + y^2z^3 = 0$ over $\mathbb{C}$ has only four nodes $(0:0:1)$, $(1:0:0)$, $(1:\sqrt{-1}:0)$ and $(-1:\sqrt{-1}:0)$.

8. Case $d = 5, r = 5$. The curve $x^4y - x^4z + x^3y^2 + x^3yz + x^2y^2z - x^2yz^2 + x^2z^3 + xy^4 + xy^2z^2 - xyz^3 + y^4z + y^2z^3 = 0$ over $\mathbb{C}$ has only five nodes $(0:0:1)$, $(1:-\sqrt{-1-\sqrt{-1}}:-1)$, $(1:\sqrt{-1-\sqrt{-1}}:-1)$, $(1:-\sqrt{-1+\sqrt{-1}}:-1)$ and $(1:\sqrt{-1+\sqrt{-1}}:-1)$.

Mathematica is Power, France is Bacon.

4.4 Elliptic Curves

Solution 4.4.1. We may assume $X \subset \mathbb{P}^2$ has the equation $y^2 = x(x-1)(x-\lambda)$ and P is the infinity point by proposition 4.6. So we may define a morphism $\phi : k[x, y, t] \to R$ as $x \mapsto x$, $y \mapsto y$ and $t \mapsto 1 \in H^0(X, \mathscr{O}_X(P))$. For any $s \in H^0(X, \mathscr{O}_X(nP))$ for some n, s has only one pole at P, i.e. there exists an $f \in k(\mathbb{A}^2)$ such that $f|_X = s$. So ϕ is surjective. Conversely, $(y^2 - x(x-t^2)(x-\lambda t^2)) \subset \ker(\phi)$. And by Riemann-Roch, $\dim H^0(X, \mathscr{O}_X(nP)) = n$, which is equal to $\dim R_{(n)}$. So ϕ is an isomorphism.

Solution 4.4.2. We need to prove a lemma from Castelnuovo, for any coherent sheaf $\mathscr{F}$ on X such that $H^1(X,\mathscr{F}) = H^1(X,\mathscr{F}\otimes\mathscr{L}(-D)) = 0$, we have $H^0(X,\mathscr{F}\otimes\mathscr{L}(nD))\otimes H^0(X,\mathscr{L}(D)) \to H^0(X,\mathscr{F}\otimes\mathscr{L}((n+1)D))$ is surjective. If $\operatorname{Supp}\mathscr{F}$ has dimension 0, there exists an $s\in\mathscr{L}(D)$ such that $s(x)\neq 0$ for all $x\in\operatorname{Supp}\mathscr{F}$. So the morphism $H^0(X,\mathscr{F})\otimes_k (s\cdot k)\to H^0(X,\mathscr{F}\otimes\mathscr{L}(D))$ is an isomorphism. So $H^0(X,\mathscr{F})\otimes H^0(X,\mathscr{L}(D))\to H^0(X,\mathscr{F}\otimes\mathscr{L}(D))$ is surjective. If $\operatorname{Supp}\mathscr{F}$ has dimension 1, we may consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathscr{F}\otimes\mathscr{L}(-D))\otimes H^0(X,\mathscr{L}(D)) \arrow[r] \arrow[d, "\alpha"] & H^0(X,\mathscr{F})\otimes H^0(X,\mathscr{L}(D)) \arrow[r] \arrow[d, "\beta"] & H^0(X,\mathscr{G})\otimes H^0(X,\mathscr{L}(D)) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & H^0(X,\mathscr{F}) \arrow[r] \arrow[ur, "\psi"] & H^0(X,\mathscr{F}\otimes\mathscr{L}(D)) \arrow[r] & H^0(X,\mathscr{G}\otimes\mathscr{L}(D)) \arrow[r] & 0
\end{tikzcd}$$

where $\mathscr{G}$ is the cokernel of $\mathscr{F}\otimes\mathscr{L}(-D)\to\mathscr{F}$ as $a\mapsto a\otimes s$ for some $s\in H^0(X,\mathscr{L}(D))$ such that $s|_x$ is not a zero divisor of $\mathscr{F}_x$ for every $x\in X$. So we have a surjective morphism $\operatorname{coker}\beta\to\operatorname{coker}\gamma$. Since $\operatorname{Supp}\mathscr{G}$ has dimension 1, we've already have $\operatorname{coker}\gamma = 0$. Moreover, we can define a morphism $\psi : H^0(X,\mathscr{F})\to H^0(X,\mathscr{F})\otimes H^0(X,\mathscr{L}(D))$ as $a\mapsto a\otimes s$. This morphism is commutative with the diagram, hence the morphism $\operatorname{coker}\alpha\to\operatorname{coker}\beta$ is zero morphism. So, $\operatorname{coker}\beta = 0$, i.e. $H^0(X,\mathscr{F})\otimes H^0(X,\mathscr{L}(D))\to H^0(X,\mathscr{F}\otimes\mathscr{L}(D))$ is surjective. For higher n, we just need to replace $\mathscr{F}$ to $\mathscr{F}\otimes\mathscr{L}((n-1)D)$ and prove by induction.

Denote the embedding $X\to\mathbb{P}^n$ by ϕ. Take an effective divisor E of degree $d-2$ supported on $\operatorname{Supp} i_*\mathscr{O}_D$, and consider the exact sequence $0\to\mathscr{L}(-E)\to\mathscr{O}_X\to i_*\mathscr{O}_E\to 0$. So we have $0\to\mathscr{L}(D-E)\to\mathscr{L}(D)\to i_*\mathscr{O}_E\to 0$. Since $\deg D-E = 2$, by Serre duality we have $H^1(\mathscr{L}(D-E)) = 0$. Hence we have a commutative diagram

$$\begin{tikzcd}
H^0(X,\mathscr{L}(D-E))\otimes H^0(X,\mathscr{L}(nD)) \arrow[r] \arrow[d, "f"] & H^0(X,\mathscr{L}(D))\otimes H^0(X,\mathscr{L}(nD)) \arrow[r] \arrow[d, "g"] & H^0(X,i_*\mathscr{O}_E)\otimes H^0(X,\mathscr{L}(nD)) \arrow[r] \arrow[d, "h"] & 0 \\
H^0(X,\mathscr{L}(D-E+nD)) \arrow[r] & H^0(X,\mathscr{L}(D+nD)) \arrow[r] & H^0(X,i_*\mathscr{O}_E\otimes\mathscr{L}(nD)) &
\end{tikzcd}$$

So by five lemma, we have an exact sequence $\operatorname{coker}f\to\operatorname{coker}g\to\operatorname{coker}h$. Since nD has no base point for $n\geq 1$. By Castelnuovo's lemma we have $\operatorname{coker}h = 0$. Moreover, since $D-E$ has degree 2, $D-E$ has no base point. So again by Castelnuovo's lemma, we have $\operatorname{coker}f = 0$. Hence $\operatorname{coker}g = 0$, i.e. $H^0(X,\mathscr{L}(D))\otimes H^0(X,\mathscr{L}(nD))\to H^0(X,\mathscr{L}((n+1)D)$ is surjective.

Since $|D|$ is complete, we know $H^0(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(1))\to H^0(X,\mathscr{O}_X(1))$ is surjective. For $n>0$, since we have

$$\begin{tikzcd}
H^0(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(1))\otimes H^0(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(k)) \arrow[r] \arrow[d] & H^0(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(k+1)) \arrow[d] \\
H^0(X,\mathscr{O}_X(1))\otimes H^0(X,\mathscr{O}_X(k)) \arrow[r, "g"] & H^0(X,\mathscr{O}_X(k+1))
\end{tikzcd}$$

Since g is surjective as above, and the left arrow is surjective by induction, we have $H^0(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(k))\to H^0(X,\mathscr{O}_X(k))$ is surjective for all k. So by 2.5.14.(d), X is projectively normal.

Solution 4.4.3. Denote $f = y^2 - x(x-1)(x-\lambda)$. So we may write $R = k[x,y]/(f)$ as the ring of functions on X which are regular out of P_0. Then $K(X) = \operatorname{Frac}(R)$. So for every $g\in K(X)$, it must has the form $g = a(x)+b(x)y$ for some $a(x), b(x)\in k(x)$. So if $\phi\in\operatorname{Aut}(X,P_0)$, we may assume $\phi(x,y) = (a(x)+b(x)y, c(x)+d(x)y)$. For any $P = (x,y)\in X$, we have $\phi(P)+\phi(-P) = 0$, i.e. $0 = \phi(x,y)+\phi(x,-y) = (a(x)+b(x)y, c(x)+d(x)y)+(a(x)-b(x)y, c(x)-d(x)y)$. So $(a(x)+b(x)y, c(x)+d(x)y) = (a(x)-b(x)y, -c(x)+d(x)y)$, hence $b(x) = c(x) = 0$ for all x, i.e. $\phi(x,y) = (a(x), d(x)y)$. Since $\phi(x,\infty) = (a(x), d(x)\infty) = (x,\infty) = P_0$ for all x, we have $d(x)$ is a constant d, i.e. $\phi(x,y) = (a(x), dy)$. Since $(a(x), dy)\in X$, we clearly know that $a(x)$ has degree 1, i.e. $a(x)$ has the form $ax+b$. So $\phi\in\operatorname{Aut}(X,P_0)$ has the form $\phi(x,y) = (ax+b, dy)$.

Solution 4.4.4. Take the automorphism $y \mapsto y - \frac{a_1x+a_3}{2}$, the equation $f = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6$ transform to $f = y^2 - x^3 - (a_2 + \frac{a_1^2}{4})x^2 - (a_4 + \frac{a_1a_3}{2})x - (a_6 + \frac{a_3^2}{4}) = y^2 - (x-a)(x-b)(x-c)$ for some a, b, c. Then take the automorphism $x \mapsto \frac{x-a}{x-b}$, $f = (b-a)^3y^2 - x(x-1)(x-\lambda)$ for some $\lambda = \frac{a-c}{a-b}$. Then $j = 2^8\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2} = \frac{(a^2+b^2+c^2-ab-bc-ca)^3}{((a-b)(b-c)(c-a))^2}$. So by Veda's theorem and Mathematica, we have

$$j = \frac{1728\left((a_1^2+4a_2)^2 - 24(a_1a_3+2a_4)\right)^3}{\left((a_1^2+4a_2)^2 - 24(a_1a_3+2a_4)\right)^3 - \left(-36\left(a_1^2+4a_2\right)(a_1a_3+2a_4) + \left(a_1^2+4a_2\right)^3 + 216\left(a_3^2+4a_6\right)\right)^2}$$

hence a rational function with coefficient in $\mathbb{Q}$. For the existence of j, we consider the following case

1. Case $\mathrm{char}(k) = 2$ and $j = 0$. $y^2 + y = x^3$.
2. Case $\mathrm{char}(k) = 2$ and $j \neq 0$. $y^2 + xy = x^3 + x^2 + j^{-1}$.
3. Case $\mathrm{char}(k) = 3$ and $j = 0$. $y^2 = x^3 + x$.
4. Case $\mathrm{char}(k) = 3$ and $j \neq 0$. $y^2 = x^3 + x^2 - j^{-1}$.
5. Case $\mathrm{char}(k) \neq 2, 3$ and $j = 0$. $y^2 = x^3 + 1$.
6. Case $\mathrm{char}(k) \neq 2, 3$ and $j = 1728$. $y^2 = x^3 + x$.
7. Case $\mathrm{char}(k) \neq 2, 3$ and $j \neq 0, 1728$. $y^2 = x^3 + 2kx + 2k$, where $k = \frac{j}{1728-j}$.

Solution 4.4.5. (a) We may define π and π' by the linear system $|2P_0|$ and $|2f(P_0)|$. So clearly there exists a morphism $g : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 2 satisfying $\pi \circ f = g \circ \pi'$.

(b) We may assume X has the function $y^2 = x^3 + ax + bx + c$. Since $f(P_0) = P_0$, we just define π and π' as the projection of the x-coordinate in $\mathbb{A}^2$. So g is $(x : z) \mapsto (x^2 : z^2)$.

(c) Suppose X has the form $y^2 = x(x-1)(x-\lambda)$. Clearly g has two branch point 0 and 1. And by Hurwitz's formula, π and π' both have 4 branch points $0, 1, \lambda$ and ∞. So g is branched over 0 and ∞. Define $D = (0,0) + (1,0) + (\lambda,0) + P_0$ as a divisor of X. Then $R_{\pi\circ f} = f^*D$. Since $R_{g\circ\pi'} = D + \pi'^*((0)+(\infty)) = D + (2\cdot(0,0) + 2\cdot P_0)$, and $g\circ\pi' = \pi\circ f$, we know $f^*D = 3\cdot(0,0) + (1,0) + (\lambda,0) + 3\cdot P_0$. So $g^{-1}(\{1,\lambda\}) = \{0, 1, \lambda, \infty\}$ consists of the four branch points of π'. Easily calculate, we have $(0, 1, \lambda, \infty) \sim (-1, 1, \sqrt{\lambda}, -\sqrt{\lambda})$. So $\frac{256(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2} = \frac{16(\lambda^2-10\lambda+1)^3}{\lambda(\lambda-1)^4}$.

(d) By (c), we have $256\lambda^{11} - 1808\lambda^{10} + 5504\lambda^9 - 21696\lambda^8 - 5760\lambda^7 + 47008\lambda^6 - 5760\lambda^5 - 21696\lambda^4 + 5504\lambda^3 - 1808\lambda^2 + 256\lambda = 0$, i.e. $\lambda = -1, 0, 1, 3-2\sqrt{2}, 3+2\sqrt{2}, \frac{1-3\sqrt{-7}}{32}, \frac{1+3\sqrt{-7}}{32}, \frac{1-3\sqrt{-7}}{2}, \frac{1+3\sqrt{-7}}{2}$. So for nonsingular curve, the corresponding $j = 1728, 8000, -3375$.

Solution 4.4.6. (a) Similarly, we may take a line L which is not a tangent line of X. Then we can define $\phi : X \to L$ as $P \mapsto T_p(X) \cup L$. Then consider the morphism $\psi : X \to L$ as the projection from some $O \notin L$, by Hurwitz's formula, $2g-2 = -2d + \deg R_\psi$, i.e. $\deg R_\psi = 2g-2+2d = d^2-d-2r$. So $\deg X^* = \sharp\phi^{-1}(P) = d^2-d-2r$. Then by Hurwitz's formula, $\deg R_\phi = 2g-2+2(d^2-d-2r) = 3d^2-5d-6r$. Ignoring d-ramified points, we have $\sharp$inflection points$= 3d^2 - 6d - 6r$.

(b) If $n = 3$, we can take some $O \in \mathbb{P}^3$ not in any hyperosculating hyperplane, and $\psi : X \to \mathbb{P}^2$ is the projection from O. Denote the image plane curve as $\bar{X}$. Then $\bar{X}$ has degree d, r nodes and genus $g = \frac{1}{2}(d-1)(d-2) - r$. In $\mathbb{P}^{3*}$, the hyperplane corresponding to O intersects X^* at $\deg X^*$ points. And every point corresponding to a hyperplane in $\mathbb{P}^3$ which is an osculating hyperplane for some point on X. So $\deg X^* = \sharp$hyperosculating points of $\bar{X} = \sharp$inflection$= 6(g-1) + 3d$. So by Hurwitz's formula, $\sharp$hyperosculating points of $X = 2g-2+2\deg X^* - (2g-2+2d) = 12(g-1) + 4d$.

If $n > 3$, we may prove it by induction. We may take some $O \in \mathbb{P}^n$ not in and hyperosculating hyperplane, and the image projection from O of X, $\bar{X}$, is smooth. And similarly, $\deg X^* = \sharp$hyperosculating points of $\bar{X} = n(n-1)(g-1) + nd$. So by Hurwitz's formula, $\sharp$hyperosculating points of $X = 2g-2+2\deg X^* - ((n-2)(n-1)(g-1) + (n-1)d) = n(n+1)(g-1) + (n-1)d$.

(c) Clearly, the infinity point P_0 is a hyperosculating point. So for any hyperosculating hyperplane X in $\mathbb{P}^{d-1}$ intersecting X with P, we have $H.X = dP$. Since all hyperplanes in $\mathbb{P}^{d-1}$ are linearly equivalent, we have

$d(P-P_0)=0$, i.e. P is a d-torsion point. Conversely, for any d-torsion point, dP is linearly equivalent to dP_0, i.e. it is a divisor obtained by intersection X with some hyperplane in $\mathbb{P}^{d-1}$. Hence P is a hyperosculating point.

Solution 4.4.7 (The Dual of a Morphism)**.** (a) By theorem 4.11., Pic is just Jacobian. And clearly the induced homomorphism f^* is the morphism between Jacobians Jac$X' \to$ Jac$X$, which is just $(X', P_0') \to (X, P_0)$ since X and X' are elliptic curves.

(b) Since f^* is just $\hat{f}$, by $(gf)^* = f^* \cdot g^*$, trivial.

(c) Separable Case. If f is separable, since f has degree n, we have $\sharp \ker f = n$, i.e. every element in $\ker f$ has order dividing n. So $\ker f \subset \ker n_X$. By Galois theory, we have $n_X^* K(E) \subset f^* K(E) \subset K(E)$. Then we may fine a map g satisfying $f^* g^* K(E) = n_X^* K(E)$, i.e. $g \circ f = n_X$, So clearly $g = \hat{f}$.

Purely Inseparable Case. We may assume f is just the Frob$_p$, where $p = \mathrm{char}(k)$. Then $\deg f = p$. Since p_X is clearly not separable, by Galois theory, p_X can be decomposed into separable and purely inseparable part, i.e. $p_X = g \circ \mathrm{Frob}_p^e$. Hence $\hat{f} = g \circ \mathrm{Frob}_p^{e-1}$.

(d) Fix an invertible sheaf $\mathscr{L}$ on X'. Define Pic $^0_\sigma$ as in hint. Then clearly Pic $^0_\sigma \cong$ Pic $(X'/X)^0$ as $f \mapsto \mathscr{L}_f = f'^* \mathscr{L}$, where $f' = f \times \mathrm{id} : X \times X' \to X' \times X'$, $\mathscr{L} = \mathscr{L}(\Delta_{X'}) \otimes \pi_2^* \mathscr{L}(-P_0') \in$ Pic $^0(X'/X')$ is the sheaf in the definition of Jacobian, and $\pi_2 : X' \times X' \to X'$ is the second projection. So we have $\mathscr{L}_{f+g} \cong \mathscr{L}_f \otimes \mathscr{L}_g$. And for any $x \in X$, we have $\Gamma_g^* \mathscr{L}_f = \mathscr{L}_{f,(x,g(x))} = \mathscr{L}_{f(x),g(x)}$. If $f(x) = g(x)$, we have $\mathscr{L}_{f(x),g(x)} = \mathscr{O}_{X',f(x)}$. And if $f(x) \neq g(x)$, we have $\mathscr{L}_{f(x),g(x)} = 0$. So $\Gamma_g^*(\mathscr{L}_f) = \Gamma_f^* \mathscr{L}_g)$. Since for any $\mathscr{F} \in$ Pic X', $\mathscr{F}' = p_2^* \mathscr{F} \in$ Pic $^0_\sigma$, there exists some $h : X \to X'$ such that $\mathscr{L}_h = \mathscr{F}'$. And we have $\Gamma_{f+g}^* \mathscr{L}_h = \Gamma_h^* \mathscr{L}_{f+g} = \Gamma_h^* \mathscr{L}_f \otimes \Gamma_h^* \mathscr{L}_g = \Gamma_f^* \mathscr{L}_h \otimes \Gamma_g^* \mathscr{L}_h$. Hence $(f+g)^* \mathscr{F} \cong f^* \mathscr{F} \otimes g^* \mathscr{F}$, i.e. $(f+g)\hat{} \cong \hat{f} + \hat{g}$.

(e) Since $n_X = 1_X + \ldots + 1_X$. And $1_X = \mathrm{id}_X$ has the dual id_X. So by (d), in the case $X' = X$, $\hat{n}_X = \hat{1}_X + \ldots + \hat{1}_X = 1_X + \ldots + 1_X = n_X$. So $n_X = \hat{n}_X$. By (c), since $n_X \circ n_X = n_X^2$, we have $\deg n_X = n^2$.

(f) By (c), we have $(\deg f)^2 = \deg \hat{f} \circ \deg f = \deg f \circ \deg \hat{f} = (\deg \hat{f})^2$. So $\deg f = \deg \hat{f}$.

Solution 4.4.8. For any étale covering $X \to E$, since étale means unramified, we have $g(X) = g(E)$, i.e. X is an elliptic curve. For any étale covering $f : E' \to E$, by 4.4.7., there exists a dual morphism $g : E \to E'$ such that $g \circ f = n_X$, where $n = \deg f$. So $\pi_1(E) = \varprojlim_n \mathrm{Gal}(n_E)$. Then consider the following three cases:

1. Case $\mathrm{char}(k) = 0$. Clearly n_E is an étale morphism for all n. For any $P \in \ker n_E$, we define $\tau_P : E \to E$ as $Q \mapsto Q + P$. Then $n_E \circ \tau_P = n_E$, hence $\tau_P \in \mathrm{Gal}(n_E)$. Conversely, for any $f \in \mathrm{Gal}(n_E)$, we have $\deg f = \deg n_X / \deg n_X = 1$. So f has the form τ_P for some $P \in X$. Then by $n_E \circ \tau_P = n_E$ we have $P \in \ker n_E$. Hence $\mathrm{Gal}(n_E) = \ker(n_E) = (\mathbb{Z}/n\mathbb{Z})^2$. So $\pi_1(E) = \varprojlim_n \mathrm{Gal}(n_E) = \varprojlim_n (\mathbb{Z}/n\mathbb{Z})^2 = \prod_{l \text{ prime}} \mathbb{Z}_l \times \mathbb{Z}_l$.

2. Case $\mathrm{char}(K) = p$ and the Hasse invariant is 0. In this case, if $(n,p) = 1$, n_X is étale. And if $p | n$, n_X is ramified. So similarly with case 1, we have $\mathrm{Gal}(n_E) = \ker(n_E) = (\mathbb{Z}/n\mathbb{Z})^2$ for $(n,p) = 1$. So $\pi_1(E) = \varprojlim_{p \nmid n} \mathrm{Gal}(n_E) = \varprojlim_{p \nmid n} (\mathbb{Z}/n\mathbb{Z})^2 = \prod_{l \neq p} \mathbb{Z}_l \times \mathbb{Z}_l$.

3. Case $\mathrm{char}(K) = p$ and the Hasse invariant is 1. For any étale covering $f : E' \to E$, we have a decomposition $f = g \circ \hat{\mathrm{Frob}}^r$ for some g such that $p \nmid n = \deg g$. So we may consider $n_E \circ \hat{\mathrm{Frob}}^r = \hat{g} \circ \hat{E}$, and similarly with case 1, the we can only take the term $(n_E \circ \hat{\mathrm{Frob}}^r)$ in the inverse limit. And since $\mathrm{Gal}(n_E \circ \hat{\mathrm{Frob}}^r) = (\mathbb{Z}/n\mathbb{Z})^2 \times (\mathbb{Z}/p^r\mathbb{Z})$ by 4.4.15., we have $\pi_1(E) = \varprojlim_{n,r} (\mathbb{Z}/n\mathbb{Z})^2 \times (\mathbb{Z}/p^r\mathbb{Z}) = \mathbb{Z}_p \times \prod_{l \neq p} (\mathbb{Z}_l \times \mathbb{Z}_l)$.

Solution 4.4.9. (a) If $f : X \to X'$ and $g : X' \to X''$ are finite, clearly $g \circ f$ is finite. Moreover, we have a finite dual morphism $\hat{f} : X' \to X$. Hence isogeny is an equivalent relation.

(b) Fix an X. If $f : X \to X'$ is an isogeny, clearly $\ker f$ is a finite subgroup of X. Conversely, if $g : X \to X''$ is another isogeny with kernel G, we have $K(X') = K(X)^G = K(X'')$. Moreover, X' and X'' are both smooth, they are isomorphic, i.e. the isogeny class of X is a subset of the set of finite subgroup of X, hence countable.

Solution 4.4.10. For any $\mathscr{M} \in$ Pic $(X \times X)$, we define $\mathscr{L}_1 = \mathscr{M}|_{X \times \{P_0\}}$, $\mathscr{L}_2 = \mathscr{M}|_{\{P_0\} \times X}$, and $\mathscr{F} = \mathscr{M} \otimes (p_1^* \mathscr{L}_1 \otimes p_2^* \mathscr{L}_2)^{-1}$. Then clearly $\mathscr{F}|_{X \times \{P_0\}}$ and $\mathscr{F}|_{\{P_0\} \times X}$ is trivial. Since X is smooth over k, we have $X \times X \to X$ is a flat morphism. Then $\mathscr{F}|_{X\times\{P\}}$ has the same degree with $\mathscr{F}|_{X\times\{P_0\}}$ for all $P \in X$, i.e. has degree 0. Hence we may define a morphism $\Phi : \mathrm{Pic}\,(X\times X) \to R$ as $\phi(\mathscr{M})(P) = \psi(\mathscr{F}|_{X\times\{P\}})$ for any $P \in X$, where $\psi : \mathrm{Pic}^{\,0} X \cong X$. Clearly Φ is surjective. For any $\mathscr{M} \in \ker\Phi$, we know $\mathscr{F}|_{X\times\{P\}}$ is trivial for all $P \in X$, so $\mathscr{F} = p_2^*\mathscr{L}$ for some $\mathscr{L} \in \mathrm{Pic}\,X$. Hence $\mathscr{M} = p_1^*\mathscr{L}_1 \otimes p_2^*(\mathscr{L}_2 \otimes \mathscr{L})$, which means $\ker\Phi = p_1^*\mathrm{Pic}\,X \oplus p_2^*\mathrm{Pic}\,X$. So we have an exact sequence $0 \to p_1^*\mathrm{Pic}\,X \oplus p_2^*\mathrm{Pic}\,X \to \mathrm{Pic}\,(X\times X) \to R \to 0$.

Solution 4.4.11. (a) Denote the area of period parallelogram of lattice L as A_L. Then clearly $\deg f = \frac{A_{\alpha L}}{A_L} = |\alpha|^2$.

(b) Since f is induced by the morphism $\mathbb{C} \to \mathbb{C}$ as $\times\alpha$, and N_X is induced by $\times N$, where $N = \deg f = |\alpha|^2 = \alpha\bar{\alpha}$, we know the morphism $\times\bar{\alpha}$ induces the dual morphism $\hat{f} : X \to X$.

(c) Since $\tau \in \mathbb{Q}(\sqrt{-d})$ and is integral over $\mathbb{Z}$, τ^2 is a linear combination of τ and 1 with integral coefficients, so $\mathbb{Z}[\tau] = \mathbb{Z} \oplus \mathbb{Z}\tau = L_\tau$, and $\mathbb{Z}[\tau] \subset R$. Conversely, for any $f \in R$, there exists some $\alpha \in \mathbb{C}$ corresponding to f such that $\alpha L_\tau \subset L_\tau$. Then $\alpha \cdot 1 \in L_\tau$, i.e. $\alpha \in \mathbb{Z}[\tau]$. So $R = \mathbb{Z}[\tau]$.

Solution 4.4.12. (a) If X has automorphisms leaving P_0 fixed other than ± 1, $\mathrm{End}(X, P_0)$ must bigger than $\mathbb{Z}$. Hence X has complex multiplication, so $\tau \in \mathbb{Q}(\sqrt{-d})$ for some square-free $d > 0$. If f is an automorphism of X leaving P_0 fixed corresponding to some $\alpha \in \mathbb{C}$, we have $\deg f = 1$. By 4.4.11.(a), we have $|\alpha| = 1$. So $\alpha = \zeta_n$ for some n, where $\zeta_n \neq 1$ satisfies $\zeta_n^n = 1$. Moreover, since $\mathbb{Q}(\alpha) \subset \mathbb{Q}(\tau)$ has extension ≤ 2 over $\mathbb{Q}$, and $\alpha \neq \pm 1$, we have $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$. So $n = 4$ or 6, i.e. $\alpha = i$ or ω, in which case, $\tau = i$ or ω.

(b) In this case, X has complex multiplication too, since 2 is not a square. We may assume $\tau \in \mathbb{Q}(\sqrt{d})$ for some square-free $d < 0$. Then we have two cases:

1. $d \equiv 1 \mod 4$. Then the integral ring O of $\mathbb{Q}(\sqrt{d})$ is $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$. So $\alpha = (a + \frac{b}{2}) + \frac{b}{2}\sqrt{d}$ for some integers a, b. Then $(a + \frac{b}{2})^2 - \frac{b^2 d}{4} = 2$. We only have an integer solution $d = -7$, $a = 0$ and $b = \pm 2$. So in this case, we clearly have $\tau = \frac{-1+\sqrt{-7}}{2}$.

2. $d \equiv 2, 3 \mod 4$. Then the integral ring is $\mathbb{Z}[\sqrt{d}]$. So $\alpha = a + b\sqrt{d}$ for some integers a, b, and $a^2 - b^2 d = 2$. So we have $(a, b, d) = (\pm 1, \pm 1, -1), (0, \pm 1, -2)$. In each case, we have $\tau = i$ or $\tau = \sqrt{-2}$.

Solution 4.4.13. If E is a supersingular elliptic curve over $\mathbb{F}_{13}$, we may assume it has the form $E_\lambda : y^2 = x(x-1)(x-\lambda)$ for some $\lambda \in \bar{\mathbb{F}}_{13}$. Then by corollary 4.22., E is supersingular iff $h_{13}(\lambda) = \sum_{i=0}^{6}\binom{6}{i}^2\lambda^i = \lambda^6 + 10\lambda^5 + 4\lambda^4 + 10\lambda^3 + 4\lambda^2 + 10\lambda + 1 = 0$. So $h_{13}(\lambda)$ has at most 6 roots. If λ is a root, $\lambda' = \lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}$ or $\frac{\lambda-1}{\lambda}$, λ' must be a root of h_{13} since $j_\lambda = j_{\lambda'}$ and by theorem 4.1., $E_\lambda \cong E_{\lambda'}$. Since if $j_\lambda = 0$, we have $h_{13}(\lambda) = \lambda^4 + 11\lambda^3 + 1\lambda^2 + 3)(\lambda^2 - \lambda + 1) - 2 \neq 0$, and if $j_\lambda = 1728$, we have $h_{13}(\lambda) = (7\lambda^3 + 9\lambda^2 + 5)(2\lambda^3 - 3\lambda^2 - 3\lambda + 2) + (\lambda^2 + 12\lambda + 4) = \lambda^2 + 12\lambda + 4 \neq 0$. Hence every root of h_{13} is not corresponding to $j = 0$ or 1728. So the 6 roots of h_{13} correspond to one j, i.e. we only have one supersingular elliptic curve over $\mathbb{F}_{13}$. By Mathematica, if $h_{13}(\lambda) = 0$, we have $j = 5 \mod 13$. And in this case, E has the form $y^2 = x^3 + 4x + 7$.

Solution 4.4.14. By proposition 4.21., $X_{(p)}$ is supersingular iff $(xyz)^{p-1}$ has coefficient 0 in $(x^3 + y^3 - z^3)^{p-1}$. So if $3 \nmid p-1$, there cannot exist the term $(xyz)^{p-1}$. If $3 | p-1$, clearly the coefficient is $\frac{(p-1)!}{(((p-1)/3)!)^3}$, which cannot be zero since $(p-1)!$ and $((p-1)/3)!)$ don't have the prime factor p. So $X_{(p)}$ is supersingular iff $p \equiv 2 \mod 3$. By Dirichlet's density theorem, the set $\mathfrak{P}$ has density $\frac{1}{2}$.

Solution 4.4.15. By proposition 2.1. and example 2.1.5, $\hat{F}' : X \to X_p$ is separable $\Leftrightarrow$ we have $0 \to \hat{F}'^*\Omega_{X_p} \to \Omega_X$ $\Leftrightarrow$ $0 \to H^0(X_p, \Omega_{X_p}) \to H^0(X, \Omega_X)$ $\Leftrightarrow$ by Serre's duality, the morphism $F'^* : H^1(X, \mathscr{O}_X) \to H^1(X_p, \mathscr{O}_{X_p})$ is surjective $\Leftrightarrow$ the Hasse invariant of X is 1.

Since F' is purely inseparable, we have $\deg_s(\hat{F}') = \deg_s(p_X)$. So $\sharp\ker p_X = \deg_s(\hat{F}')$. Since $\deg F' = p$ and $\deg\hat{F}' = p$, we have $\deg_s(\hat{F}') = 1$ or p because p is a prime. So by above, if the Hasse invariant is 1, $\hat{F}'$ is separable, $\deg_s(\hat{F}') = p$, i.e. $\sharp\ker p_X = p$, hence $\ker p_X = \mathbb{Z}/p$. Or if the Hasse invariant is 0, $\hat{F}'$ is inseparable, $\deg_s(\hat{F}') = 1$, i.e. $\sharp\ker p_X = 1$, hence $\ker p_X = 0$.

Solution 4.4.16. (a) Clearly the Frobenious morphism on $\mathrm{Spec}\,\mathbb{F}_q$ is the isomorphism, since $\mathbb{F}_q \to \mathbb{F}_q$, $x \mapsto x^q$ is an isomorphism. So the base change $X \to \mathbb{F}_q$ is an isomorphism $X_q \to X$. If X has homogeneous equation $f(x,y,z) = 0$, we have $X = \operatorname{Proj} A$ for $A = k[x,y,z]/(f)$. Then $X_p = \operatorname{Proj} A_p$ for $A_p = A \otimes_{F'} k = k[x,y,z]/(f_p)$, where f_p is the polynomial which change every coordinate of f to the qth-power. So $f_p(x^p, y^p, z^p) = (f(x,y,z))^p$. Hence the morphism $X \to X$ is the qth-power map on the coordinates of points of X embedding in $\mathbb{P}^2$.

(b) Suppose X has the form $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. Consider the invariant differential $\omega = \frac{\mathrm{d}x}{2y+a_1x+a_3}$ of X. We have $(1-F')^*\omega = \omega + F'^*\omega$. Since $F'^*\omega = \frac{\mathrm{d}(x^q}{2y^q+a_1x^q+a_3} = \frac{qx^{q-1}\mathrm{d}x}{2y^q+a_1x^q+a_3} = 0$. So $(1-F')^*\omega = \omega$. Since Ω_X is defined by ω, we have $0 \to (1-F')^*\Omega_X \to \Omega_X$. Hence $(1-F')$ is separable by proposition 2.1. Clearly $\ker(1-F') = \{(x,y,z) \in X \mid (x,y,z) = (x^p, y^p, z^p\} = X(\mathbb{F}_q)$.

(c) Define $a = 1 + \deg F' - \deg(1-F')$. Then clearly $a_X = 1_X + (\deg F')_X - (\deg(1-F'))_X = 1_X + F'\hat{F}' - (1_X - F')(1_X - \hat{F}') = 1_X + F'\hat{F}' - 1_X - F'\hat{F}' + (F' + \hat{F}') = F' + \hat{F}'$. Moreover, by (b), $\sharp X(\mathbb{F}_q) = \deg_s(1_X - F') = \deg(1_X - F') = -a + 1 + \deg F' = q - a + 1$.

(d) By (c), $(m + nF')(m + n\hat{F}') = m^2 + n^2F'\hat{F}' + mn(F' + \hat{F}') = m^2 + n^2q + amn \geq 0$. We have $|a| \leq \frac{m^2+n^2q}{|mn|} \leq \frac{2\sqrt{m^2n^2q}}{|mn|} = 2\sqrt{q}$.

(e) If $q = p$, in the same method with (b), we know $(m + nF')$ is separable iff $p \nmid m$. So the Hasse invariant is $0 \Leftrightarrow \hat{F}' = a - F'$ is inseparable $\Leftrightarrow p|a$. When $p \geq 5$, we have $p|a$ and $|a| \leq 2\sqrt{q}$. So $a = 0$, i.e. $N = q + 1$.

Solution 4.4.17. (a) If X has the Weierstrass equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2) \in X$, we have: (1) if $x_1 = x_2$ and $y_1 + y_2 + a_1x_2 + a_3 = 0$, then $P_1 + P_2 = O$; (2) if $x_1 = x_2$ and $y_1 + y_2 + a_1x_2 + a_3 \neq 0$, then $(x_3, y_3) = (\lambda^2 + a_1\lambda - a_2 - x_1 - x_2, -(\lambda + a_1)x_3 - \nu - a_3)$, where $\lambda = \frac{3x_1^2+2a_2x_1+a_4-a_1y_1}{2y_1+a_1x_1+a_3}$ and $\nu = \frac{-x_1^3+a_4x_1+2a_6-a_3y_1}{2y_1+a_1x_1+a_3}$; (3) if $x_1 \neq x_2$, then $(x_3, y_3) = (\lambda^2 + a_1\lambda - a_2 - x_1 - x_2, -(\lambda + a_1)x_3 - \nu - a_3)$, where $\lambda = \frac{y_2-y_1}{x_2-x_1}$ and $\nu = \frac{y_1x_2-y_2x_1}{x_2-x_1}$.

So in this case, $P + Q = (\frac{-a^3+ab+b^2}{a^2}, -\frac{b^3}{a^3} + \frac{b}{a} - a + b - 1)$. And $P = (0,0)$, $2P = (1,0)$, $3P = (-1,-1)$, $4P = (2,-3)$, $5P = (\frac{1}{4}, -\frac{5}{8})$, $6P = (6,14)$, $7P = (-\frac{5}{9}, \frac{8}{27})$, $8P = (\frac{21}{25}, -\frac{69}{125})$, $9P = (-\frac{20}{49}, -\frac{435}{343})$, $10P = (\frac{161}{16}, -\frac{2065}{64})$.

(b) For this curve, we have $\Delta = 37$. So for any $p \neq 37$, $\Delta \neq 0$, so the reduction X_p is smooth, i.e. good reduction.

Solution 4.4.18. By Mathematica, we at least has these points: $(-3,\pm2)$, $(-2,\pm4)$, $(-1,\pm4)$, $(1,\pm2)$, $(2,\pm2)$, $(3,\pm4)$, $(5,\pm10)$, $(9,\pm26)$, $(13,\pm46)$, $(31,\pm172)$, $(41,\pm262)$, $(67,\pm548)$, $(302,\pm5248)$. They are all generated by P and Q.

Solution 4.4.19. Suppose X is defined as a Weierstrass equation $f(x,y,z) = 0$. Then we can define $\bar{X}$ as a subscheme of $\mathbb{P}^2_Z$ as the equation $f(x,y,z) = 0$. So the generic fibre is $X/\mathbb{Q}$, and for any closed point $(p) \in \operatorname{Spec}\mathbb{Z}$, the fibre is X_p. Define $T = D(2\Delta) \subset \operatorname{Spec}\mathbb{Z}$ and $\bar{X}_0 = \pi^{-1}(T) \subset \bar{X}$. Then the generic fibre of $\bar{X}_0$ is also X, and $\bar{X}_0$ is smooth over T. Moreover, the addition morphism $\mu : X \times X \to X$ and the negation morphism $\iota : X \to X$ can be extended to the morphisms $\mu : \Delta(\bar{X}_0) \to \bar{X}_0$ and $\iota : \bar{X}_0 \to \bar{X}_0$. So $n_X : \bar{X}_0 \to \bar{X}_0$ and defined over T for all $n \in \mathbb{Z}$. For any $(p) \in T$, consider $(n_X)_p : X_{(p)} \to X_{(p)}$. And for any prime ℓ, by theorem 6.8. in chapter II, $(\ell_X)_p : X_{(p)} \to X_{(p)}$ is either constant or flat. Since it is not constant for all p, we have ℓ_X is flat. And since for any $n = \ell_1 \ldots \ell_r$, we have $n_X = \ell_{1,X} \cdot \ldots \ell_{r,X}$ is flat. And by theorem 4.17., n_X is also finite. So $\ker n_X$ is flat over T. Finally, for all $(p) \in T$, $X_{(p)}$ is nonsingular. So we have an exact sequence $0 \to \ker\pi_p \to X(\mathbb{Q}) \to X_{(p)}(\mathbb{F}_p) \to 0$, where π_p is the reduction at prime p. So if $(n,p) = 1$, we have an injection $X(\mathbb{Q})[n] \to X_{(p)}(\mathbb{F}_p)$.

Solution 4.4.20. (a) If X is defined by $y^2 = x(x-1)(x-\lambda) = x^3 - (\lambda+1)x^2 + \lambda x$ for some λ, we have $j_X = 256\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$. So by 4.4.16., X_p is defined by $y^2 = x^3 - (\lambda+1)^{1/p}x^2 + \lambda^{1/p}x$, we have $j_{X_p}\frac{256\lambda^{-2/p}\left((-\lambda-1)^{2/p}-3\lambda^{\frac{1}{p}}\right)^3}{(-\lambda-1)^{2/p}-4\lambda^{\frac{1}{p}}}$. Hence $j_{X_p}^p \equiv 256\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2} = j_X \mod p$. Hence $j(X_p) = j(X)^{1/p}$ in $\bar{F}_p$.

(b) ($\Leftarrow$) Just take $\pi = F'$, then trivial. ($\Rightarrow$) We have p_X is inseparable, and $\deg p_X = p^2$. If $p_X = \pi\hat{\pi}$, so $\deg\pi = \deg\hat{\pi} = p$. And at least one of π and $\hat{\pi}$ is purely inseparable. We may assume π is purely inseparable.

Then by theorem 2.5. π is equivalent to a morphism $\pi : Y \to X$ with $K(Y) = K(X)^{1/p}$. Since $\pi \in R$, we have $K(X) = K(Y) = K(X)^{1/p}$, hence $X \cong X_p$ since $K(X_p) = K(X)^{1/p}$.

Since $\pi : X \to X$ is purely inseparable, there must exist some unit $\phi \in R$ such that $\pi = \phi F'$. The Hasse invariant is $0 \Leftrightarrow \hat{F}'$ is purely inseparable $\Leftrightarrow \hat{\pi} = \hat{F}'\hat{\phi}$ is purely inseparable of degree $p \Leftrightarrow$ there exists some unit $\psi \in R$ such that $\hat{\pi} = \psi F'$, i.e. $\hat{\pi} = \psi\phi^{-1}\pi$ for some unit $\psi\phi^{-1} \in R$.

(c) If Hasse$(X) = 0$, we know $\hat{F}'$ is purely inseparable. Then $p_X = F'\hat{F}' : X \to X$ is purely inseparable of degree p^2. So $X_{p^2} \cong X$. Hence $j_X = j_{X_{p^2}}^{p^2} = j_X^{p^2}$, i.e. $j_X \in \mathbb{F}_{p^2}$.

(d) For any $f \in R$, we may assume $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$ for three homogeneous f_i with coefficient in k. So $(f_i(x,y,z))^p = f_i(x^p,y^p,z^p)$ since they are all in k with char$(k) = p$. Hence $f \circ F = F \circ f$. So if Hasse $\neq 0$, we have $\lambda_F \neq 0$ by definition. Then for any $f \in R$, if $\lambda_f \notin \mathbb{F}_p$, we have $\lambda_f \cdot F^*(x) \neq F^*(\lambda_f)F^*(x) = F^*(\lambda x)$, which contradicts that they actually commutate with each other, hence $\lambda_f \in \mathbb{F}_p$. So clearly $\phi : R \to \mathbb{F}_p$ is surjective. Define $\mathfrak{p} = \ker\phi$, we have $R/\mathfrak{p} \cong \mathbb{F}_p$.

Solution 4.4.21. Denote d as the discriminant of O. Then $(1,\omega)$ is a base of O as a $\mathbb{Z}$-module, where $\omega = \frac{d+\sqrt{d}}{2}$. So if we define f as the smallest positive integer such that $f\omega \in R$. For any $a + b\omega \in R$, we have $f|b$, hence $R = \mathbb{Z} + \mathbb{Z}(f\omega) = \mathbb{Z} + f \cdot O$.

Solution 4.4.22. Denote $\phi : X \to \mathbb{A}^1_{\mathbb{C}}$ the family and $s_0 : \mathbb{A}^1_{\mathbb{C}} \to X$ the section. Then for any closed $t \in \mathbb{A}^1_{\mathbb{C}}$, we define $s_0(t)$ as the zero of group structure of X_t. So consider the set $S = \overline{\bigcup_{t\in\mathbb{A}^1_{\mathbb{C}}} \ker 2_{X_t}}$ and $\psi = \phi|_S : S \to \mathbb{A}^1_{\mathbb{C}}$. Then for any $t \in \mathbb{A}^1_{\mathbb{C}}$, S_t has only four closed points. So by theorem 9.9. in chapter II, $\psi : S \to \mathbb{A}^1_{\mathbb{C}}$ is a flat family. And ψ is obviously unramified, hence ψ is étale. But since $\mathbb{A}^1_{\mathbb{C}}$ is simply connected, S must be four copies of $\mathbb{A}^1_{\mathbb{C}}$, one of the copy is just s_0. And we may denote the rest three as s_1, s_2, s_3. So for any $t \in \mathbb{A}^1_{\mathbb{C}}$, we may choose a coordinate of X_t such that $s_0(t) = \infty$, $s_1(t) = 0$, $s_2(t) = 1$, and $s_3(t) = \lambda$. Then $t \mapsto s_3(t) \mapsto \lambda$ is a morphism $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}} - \{0,1\}$. Since the morphism like this must be constant morphism, i.e. λ are the same for all X_t. Hence all X_t are isomorphic to each other, i.e. the family is trivial.

4.5 The Canonical Embedding

Solution 4.5.1. By 4.3.3. and proposition 5.2., trivial.

Solution 4.5.2. Hyperelliptic Case. If X is hyperelliptic, there exists a unique morphism $f : X \to \mathbb{P}^1$ of degree 2. By Hurwitz's formula, it has $2g+2$ ramified points. For any $\phi \in \mathrm{Aut}(X)$, we have $f \circ \phi : X \to \mathbb{P}^1$ of degree 2, hence $f \circ \phi$ and f differs by some automorphism of $\mathbb{P}^1$. So ϕ maps ramified points to ramified points. If ϕ fixed all $2g+2$ ramified points, by Lefschetz's theorem, it is just identity. So $|\mathrm{Aut}(X)| \leq |$the group of permutations of ramified points$| < \infty$.

Non-Hyperelliptic Case. If X is not hyperelliptic, we may consider the canonical embedding $f : X \to \mathbb{P}^{g-1}$. Then by 4.4.6., there are $g(g-1)^2 + dg$ hyperosculation points of X in $\mathbb{P}^{g-1}$. In this case, we have $g \geq 3$, hence $g(g-1)^2 + dg \geq 2g + 2$. And obviously, any $\phi \in \mathrm{Aut}(X)$ maps hyperosculation points to hyperosculation points by definition. And if ϕ fixes all hyperosculation points, by Lefschetz's theorem we know $\phi = \mathrm{id}_X$. So $|\mathrm{Aut}(X)| \leq |$the group of permutations of hyperosculation points$| < \infty$.

Solution 4.5.3 (Moduli of Curves of Genus 4)**.** Hyperelliptic Case. By example 5.5.5., the hyperelliptic curves of genus 4 form an irreducible family of dimension $2 \times 4 - 1 = 7$.

Non-Hyperelliptic Case. By example 5.2.2., any nonhyperelliptic curve of genus 4 is the complete intersection of a quadric hypersurface and a cubic hypersurface in $\mathbb{P}^3$, and vice versa. Since the moduli of quadric hypersurface in $\mathbb{P}^3$ has dimension 9, and for any quadric hypersurface Q defined by f, we have $0 \to H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) \xrightarrow{\times f} H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(3)) \to H^0(Q, \mathscr{O}_Q(3)) \to 0$. So $\dim H^0(Q, \mathscr{O}_Q(3)) = \dim H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(3)) - \dim H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) - 1 = 20 - 4 - 1 = 15$. So the moduli of nonhyperelliptic curve of genus 4 has dimension $15 + 9$ quotient by PGL(3), which has dimension 15, hence has dimension 9.

Unique Trigonal Case. By example 5.5.2., any curve of genus with only one g_3^1 is the complete intersection of a quadric cone and a cubic hypersurface in $\mathbb{P}^3$, and vice versa. Since any quadric cone in $\mathbb{P}^3$ corresponds to a symmetric 4×4 matrix of rank 3 up to scalar, we know the dimension of moduli of quadric cone has dimension 8. Then similarly with the non-hyperelliptic case, the moduli of curves of genus 4 with unique trigonal has dimension $15 + 8 - 15 = 8$.

Solution 4.5.4. (a) ($\Rightarrow$) We may assume X is in $\mathbb{P}^3$. If X is a nonhyperelliptic curve with two g_3^1, they gives two morphism ϕ_1 and ϕ_2 from X to $\mathbb{P}^1$ of degree 3. Then we define $\phi = \phi_1 \times \phi_2$, and $Y = \mathrm{Im}\phi$. So $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$ has form (a, b) for some a, b by example 6.6.1. in chapter II. Denote $\deg \phi = d$, we have $da = 3$ and $db = 3$ since they are from g_3^1's. If $a = b = 1$, clearly two g_3^1 are the same, which makes a contradiction. So $a = b = 3$ and $d = 1$, hence $X \cong Y$. Fix a point $P_0 \in X$, denote π as the projection from P_0 to $\mathbb{P}^2$. If $\pi(P) = \pi(Q)$ for two P, Q, the trisecant PQP_0 must lie in X. But on X, there exist only two lines through P_0, so $\pi(X)$ has only two singularities. If there exists some quatersecant T of X, since X is the complete intersection of a quadric surface and a cubic surface, T intersect the quadric surface at 4 points, which contradicts with Bézout's theorem. So X does not have any quatersecant. So $\pi(X)$ has only nodes. In this case $\pi(X)$ is a plane curve of genus 4 with two nodes, clearly it has degree 5.

($\Leftarrow$) If X is a plane quintic curve with two nodes, its normalization $\tilde{X}$ must have genus 4. Since X has degree 5, any line passing through one of nodes of X meets X in 4 points by Bézout's theorem. So we get two g_3^1's corresponding to two nodes of X. If $\tilde{X}$ is hyperelliptic, it has a g_2^1 and also a g_3^1. So we have two morphism ϕ_2 and ϕ_3 from X to $\mathbb{P}^1$ with degree 2 and 3. Then if $\phi = \phi_2 \times \phi_3$ has degree d and image Y corresponding to form (a, b) in $\mathbb{P}^1 \times \mathbb{P}^1$. So $da = 2$ and $db = 3$, i.e. $d = 1$. So $\tilde{X}$ is isomorphic to the curve $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, which has genus 6 and makes a contradiction. Hence X is nonhyperelliptic of genus 4 with two g_3^1's.

(b) By example 5.5.2., X is the complete intersection of an irreducible quadric cone C and a cubic surface F. Fix a point $P_0 \in X$ and denote π as the projection from P_0. If P_0 lies in a trisecant T of X, by Bézout's theorem, $T \subset C$. Since for P_0, there exists only one line passing through P_0 and lying in C. So $\pi(X)$ has only one singularity. If there exists a quatersecant T' of X, similarly with (a) it makes a contradiction. So X has only trisecants. Moreover, since $4 = \frac{1}{2}(d-1)(d-2) - \delta$ with $\delta \leq 2$, we have $d = 5$ and $r = 2$, i.e. $\pi(X)$ is a quintic plane curve with only one tacnode.

If we have a plane curve X with degree $d < 6$ and only r nodes of genus 4, we have $4 = \frac{1}{2}(d-1)(d-2) - r$, so the only solution is $d = 5$ and $r = 2$. By (a), the normalization of X is a curve of genus 4 with two g_3^1's, which makes a contradiction.

Solution 4.5.5 (Curves of Genus 5)**.** (a) Clearly the moduli of quadric hypersurface in $\mathbb{P}^4$ has dimension 14. If we fix a quadric hypersurface H_1 and H_2 which intersect completely at a surface S, if a third hypersurfaces H_3 intersect completely with S, we have $S \not\subseteq H_3$. So, the moduli of this kinds of H_3 has dimension $14 - \binom{4}{2} = 8$. Hence the dimension of the moduli of this kinds of curves of genus 5 is $14 + 14 + 8$ up to the action of $\mathrm{PGL}(4)$, which has dimension 24, i.e. the dimension of the moduli of this kinds of curves of genus 5 is $14 + 14 + 8 - 24 = 12$.

(b) If X has g_3^1, we may take a $D \in g_3^1$ and have $l(D) = 2$. Then $\deg(K - D) = 8 - 3 = 5$. By Riemann-Roch, we have $l(K-D) = l(D) + g - \deg(D) - 1 = 3$. So $K - D \in g_5^2$, which maps X to $\mathbb{P}^2$. Since $5 = \frac{1}{2}(5-1)(5-2) - 1$, clearly X is represented as a plane quintic curve with one node. Conversely, if $f : X \to \mathbb{P}^2$ with the image as a plane quintic curve with one node, we have $\mathscr{O}_X(E) = f^*\mathscr{O}_{\mathbb{P}^2}(1)$ for some divisor E. So $\deg(E) = 5$, and $\deg(K - E) = 8 - 5 = 3$. Then by Riemann-Roch, we have $l(E) - l(K - E) = 1$. And $h^0(f^*\mathscr{O}_{\mathbb{P}^2}(1)) \geq h^0(\mathscr{O}_{\mathbb{P}^2}(1)) = 3$, i.e. $l(K - E) \geq 2$. So E is special, and by Clifford's theorem, $\dim |E| \leq \frac{1}{2}\deg(E) = 2.5$. So $3 \leq l(E) = \dim |E| + 1 \leq 3.5$, i.e. $l(E) = 3$, and $l(K - E) = 2$. Hence $K - E$ is a g_3^1.

Since the moduli of quintic in $\mathbb{P}^2$ has dimension 20, any quintic with a node satisfies a equation more, i.e. has dimension $20 - 1 = 19$. So up to the action of $\mathrm{PGL}(2)$, which has dimension 8, the moduli of quintic with one node has dimension $19 - 8 = 11$.

(c) Define V as the blow-up of $\mathbb{P}^2$ at the node. Then by 2.7.7. V is a cubic surface in $\mathbb{P}^4$. So trivially $X \subset V$ and V is the union of all trisecant of X. Hence V and g^1_3 are both unique.

Solution 4.5.6. Suppose X has g^1_3, and $D \in g^1_3$. Then by Riemann-Roch we have $\dim|K-D| = \dim|D| - \deg D - 1 + g = 3$. But $|K-D|$ is the linear system of conics in $\mathbb{P}^2$, which gives a divisor $> D$ on C. Since the linear system of conics in $\mathbb{P}^2$ has dimension 5, and as the condition we impose yields 3 linearly independent conditions, $|K-D|$ has dimension at most 2, which makes a contradiction.

Consider a plane sextic curve Y with four nodes. Then Y has genus 6. And X is the normalization of Y. By definition, we have a closed embedding $Y \to \mathbb{P}^2 \to \mathbb{P}^3$. Hence X has a $D \in g^3_7$. So $\deg(K-D) = 10 - 7 = 3$, and $\dim|K-D| = \dim|D| - \deg D - 1 + g = 1$, i.e. $K - D \in g^1_3$. So by above, X is a nonhyperelliptic curve of genus 6 which cannot represent as a smooth plane quintic curve.

Solution 4.5.7. (a) Consider the canonical divisor K of the curve X, it induces the canonical embedding $\phi : X \to \mathbb{P}^2$. Then for any $\sigma \in \mathrm{Aut}(X)$, we have $\sigma^*(K) \sim K$. Hence K and $\sigma^*(K)$ induce the same embedding, which differs by an automorphism of $\mathbb{P}^2$, i.e. σ is induced by an automorphism of $\mathbb{P}^2$.

(b) By Klein's paper as in the hint, $\mathrm{Aut}X \cong \mathrm{PGL}(2, \mathbb{Z}/7\mathbb{Z})$, which is a simple group of order 168.

(c) For any $n > 1$, if X is a curve with an automorphism σ or order n, we may assume σ is induced by $\tau \in \mathrm{Aut}\mathbb{P}^2$. By linear algebra, τ is similar to a diagonal matrix in the form $\begin{bmatrix} \zeta_1 & 0 & 0 \\ 0 & \zeta_2 & 0\ 0 \\ 0 & 0 & 1 \end{bmatrix}$ for some roots ζ_1, ζ_2 of unity of order n. So changing the coordinates, we may assume σ can be written as $x \mapsto \zeta_1 x$, $y \mapsto \zeta_2 y$, $z \mapsto z$. So if X has the form $f(x, y, z)$ in $\mathbb{P}^2$ for some homogeneous polynomial of degree 4, the term of $x^i y^j z^k$ must have coordinate $a \sum_{i=1}^n (\zeta_1^i \zeta_2^j)^n$ for some $a \in k$. Since σ is not identity, ζ_1 and ζ_2 cannot be 1 simultaneously. So at least one of the coordinates of xz^3 and yz^3 must be zero. So curves with such automorphism induced by τ form a family of dimension≤ 8. And changing coordinates there is a 4-dimensional family of such X. Hence curves having such automorphism induced by τ form a family of dimension≤ 12 inside the 14-dimensional family of all plane curves of degree 4.

4.6 Classification of Curves in $\mathbb{P}^3$

Solution 4.6.1. Denote X as the curve we need. Consider an exact sequence $0 \to \mathscr{I}_X(2) \to \mathscr{O}_{\mathbb{P}^3}(2) \to \mathscr{O}_X(2) \to 0$. Denote D as the hyperplane section of X. Then $\dim|2D| = \dim|K-2D| + 9$ by Riemann-Roch. Since $\deg(K - 2D) < 0$, we have $\dim|K-2D| = 0$. So $h^0(\mathscr{O}_X(2)) = 9$. Since $h^0(\mathbb{O}_{\mathbb{P}^3}(2)) = \binom{5}{3} = 10$, we have $h^0(\mathscr{I}_X(2)) \geq 1$, i.e. X lies on a quadric surface Q. If X is contained in two quadric surfaces, by 2.8.4. we have $g(X) = 1 \neq 0$, which contradicts, i.e. the quadric surface is unique. Finally, since X is a rational curve, it has $n+1$ linearly independent points in $\mathbb{P}^n$. So Q is nondegenerate, hence nonsingular.

Solution 4.6.2. Denote X as the curve we need. Consider an exact sequence $0 \to \mathscr{I}_X(3) \to \mathscr{O}_{\mathbb{P}^3}(3) \to \mathscr{O}_X(3) \to 0$. Denote D as the hyperplane section of X. Then $\dim|3D| = \dim|K-3D| + 16$ by Riemann-Roch. Since $\deg(K-3D) < 0$, we have $\dim|K-3D| = 0$. So $h^0(\mathscr{O}_X(3)) = 16$. Since $h^0(\mathbb{O}_{\mathbb{P}^3}(3)) = \binom{6}{3} = 20$, we have $h^0(\mathscr{I}_X(3)) \geq 1$, i.e. X lies on a cubic surface S.

Consider the curve $X = \{(s^5 : s^4 t : st^4 + s^3t^2 : t^5) \mid (s : t) \in \mathbb{P}^1\}$. If quadric surface $Q = (u_{11}x_1^2 + u_{22}x_2^2 + u_{33}x_3^2 + u_{44}x_4^2 + u_{12}x_1x_2 + u_{13}x_1x_3 + u_{14}x_1x_4 + u_{23}x_2x_3 + u_{24}x_2x_4 + u_{34}x_3x_4$ contains X, we have

$$u_{11}s^{10} + u_{12}s^9t + (u_{13} + u_{22})s^8t^2 + u_{23}s^7t^3 + (u_{13} + u_{33})s^6t^4 + (u_{14} + u_{23})s^5t^5 + (u_{24} + 2u_{33})s^4t^6 + u_{34}s^3t^7 + u_{33}s^2t^8 + u_{34}st^9 + u_{44}t^{10} = 0$$

So if $X \subset Q$, we have $u_{11} = u_{12} = u_{13} + u_{22} = u_{23} = u_{13} + u_{33} = u_{14} + u_{23} = u_{24} + 2u_{33} = u_{34} = u_{33} = u_{34} = u_{44} = 0$, i.e. all $u_{ij} = 0$, which contradicts. So X is not contained in any quadric surface.

Solution 4.6.3. Denote X as the curve we need. Consider an exact sequence $0 \to \mathscr{I}_X(2) \to \mathscr{O}_{\mathbb{P}^3}(2) \to \mathscr{O}_X(2) \to 0$. Denote D as the hyperplane section of X. Then $\dim|2D| = \dim|K-2D| + 9$ by Riemann-Roch. Since

$\deg(K-2D) < 0$, we have $\dim|K-2D| = 0$. So $h^0(\mathscr{O}_X(2)) = 9$. Since $h^0(\mathbb{O}_{\mathbb{P}^3}(2)) = \binom{5}{3} = 10$, we have $h^0(\mathscr{I}_X(2)) \geq 1$, i.e. X lies on a quadric surface Q. If this Q is nonsingular, then by remark 6.4.1.(d), we know X is a curve on $xy = zw$ of the type $(2,3)$. So X is projectively normal. Then by 2.5.14.(d), Q is unique. If Q is singular, then clearly X is passing through the vertex of the cone. So if X is lying on another quadric Q', it must be a cone. Since X is smooth, Q and Q' must coincide, i.e. $Q = Q'$.

If X is an abstract curve of genus 2, we have $\deg K = 2$. So consider the divisor D corresponding to $\mathscr{O}_X(1)$. Then by 4.3.1., $|D|$ induces an embedding $X \to \mathbb{P}^r$ iff $\deg D \leq 5$. So for any D with degree 5, we have $\deg(D-2K) = 1$. If $D-2K$ is not effective, we have $\dim|D-2K| = 0$, hence $E = D-K$ has degree 2, and $l(E) = 2$. So divisors in $|E|$ are contained in two planes, hence spans a line which is the intersection of two planes. So this line is contained in Q since it meets X in 3 points by Bézout's theorem. Since $|D-2E| = |2K-D|$ is empty, this two different lines will not meet. So Q is a smooth quadric. If $D-2K$ is effective, we must have $|D-2K|$ is just a point, this is the vertex of the Q.

Solution 4.6.4. If X is a curve of degree 9 and has genus 11 in $\mathbb{P}^3$. Consider the exact sequence $0 \to \mathscr{I}_X(2) \to \mathscr{O}_{\mathbb{P}^3}(2) \to \mathscr{O}_X(2) \to 0$. By Riemann-Roch, we have $h^0(2D)-h^0(K-2D) = 8$, where D is the hyperplane section. If $2D$ is nonspecial, then X is clearly contained in a quadric surface. If $2D$ is special, we have $h^0(2D) > 8$. And by Clifford's theorem, $\dim|2D| \leq \frac{1}{2}\deg 2D = 9$. So $h^0(2D) < 10$, and X is lying on a quadric surface Q. So if Q is nonsingular, by remark 6.4.1., the equations $9 = a+b$, $11 = ab-a-b+1$ have no integer solution. If Q is the product of two hyperplanes, then X must be lying on one hyperplane, i.e. a plane curve. So $g(X) = \frac{1}{2}(9-1)(9-2) = 28 \neq 11$, which makes a contradiction. If Q is a quadric cone, by remark 6.4.1.(d), X must have the form $(4,5)$, which has genus $12 \neq 11$ and makes a contradiction.

Solution 4.6.5. Since X is smooth, it is normal, hence projectively normal. So by 2.5.14., $H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(l)) \to H^0(X, \mathscr{O}_X(l))$ is surjective for all $l \geq 0$, and $H^1(X, \mathscr{O}_X(l)) = 0$. So by 2.8.4., $h^0(\mathscr{O}_X(l)) = \binom{l+3}{3} - \binom{l+3-a}{3} - \binom{l+3-b}{3} + \binom{l+3-a-b}{3}$. Since $m < \min\{a,b\}$, we have $\binom{m+3-a}{3} = \binom{m+3-b}{3} = \binom{m+3-a-b}{3} = 0$, i.e. $h^0(\mathscr{O}_X(l)) = \binom{m+3}{3} = h^0(\mathscr{O}_{\mathbb{P}^3}(l))$. Hence we have $h^0(\mathscr{I}_X(m)) = 0$.

Solution 4.6.6. Case $d = 6$: By Castelnuovo's bound, we have $g \leq 4$. If $g = 0$, X is a rational curve, so in a plane. If $g = 1$, X is clearly a plane cubic curve. If $g = 2$, by proposition 6.3., the hyperplane section D is nonspecial. So $h^0(\mathscr{O}_X(1)) = \deg D + 1 - g = 5$. But $h^0(\mathscr{O}_{\mathbb{P}^3}(1)) = \binom{4}{1} = 4$, which contradicts to the fact that $\mathscr{O}_{\mathbb{P}^3}(1) \to \mathscr{O}_X(1)$ is surjective. Hence by all above, $g = 3, 4$.

Case $d = 7$: By Castelnuovo's bound, we have $g \leq 6$. If $g = 0$, X is a rational curve, so in a plane. If $g = 1, 2, 3$, by proposition 6.3., the hyperplane section D is nonspecial. So $h^0(\mathscr{O}_X(1)) = \deg D + 1 - g = 8 - g$. But $h^0(\mathscr{O}_{\mathbb{P}^3}(1)) = \binom{4}{1} = 4$, which contradicts to the fact that $\mathscr{O}_{\mathbb{P}^3}(1) \to \mathscr{O}_X(1)$ is surjective. If $g = 4$, we may consider the divisor $2D$, which has degree 12, hence nonspecial. So $h^0(\mathscr{O}_X(2)) = \deg D + 1 - g = 9$. But $h^0(\mathscr{O}_{\mathbb{P}^3}(2)) = \binom{5}{2} = 10$. Hence X is lying on a quadric surface Q. So it has type (a,b) on Q. Since $a+b = d = 7$ and $ab-a-b+1 = g = 4$, we have $(a,b) = (2,5)$. But by 3.5.6.(b), X is not projectively normal, which contradicts. Hence by all above, $g = 5, 6$.

Solution 4.6.7. If X is a line or a conic, or a twisted cubic curve in $\mathbb{P}^3$, it is clearly have no multisecants. If X is a elliptic quartic curve in $\mathbb{P}^3$, we pick a point $P \in X$ and denote the projection from P to $\mathbb{P}^2$ as π. Consider the $\pi(X)$. It is a plane curve of degree 3 and genus 1, so clearly smooth. Hence X have no multisecants.

For any other smooth curve X in $\mathbb{P}^3$, if X is a plane curve, it must have degree≥ 3 since it is not a line or a quadric curve. So by Bézout's theorem, any line in $\mathbb{P}^2$ no a tangent line is a multisecant. If X is not a plane curve has have no multisecants, for any point $P \in X$ and the corresponding projection π, we know $\pi(X)$ is a smooth plane curve of degree $d-1$, hence has genus $g = \frac{1}{2}(d-2)(d-3)$. So X has the same genus. Then by Castelnuovo's bound, we have $\frac{1}{2}(d-2)(d-3) \leq \frac{1}{4}d^2 - d + 1$, i.e. $2 \leq d \leq 4$. If $d = 2$ or 3, we have $g = 0$, i.e. a conic or a twisted cubic curve, then we have a contradiction. If $d = 4$, we have $g = 0$ or 1. If $g = 1$, we know

X is a elliptic quartic curve, then contradicts. If $g = 0$, it is a rational quartic as the type of $(1,3)$ on a smooth quadric surface, hence clearly has infinitely trisecants as lines.

Solution 4.6.8. ($\Rightarrow$) If X has a nonspecial divisor D of degree d with no base points, we have $\dim|D| = d - g$ by Riemann-Roch. And since D has no base points, we have $\dim|D| > 0$, hence $d \geq g + 1$.

($\Leftarrow$) If E is a effective special divisor of degree $d - 1$, since $\dim|K| = g - 1$, and E is a subset of K, we know the choices of E form a subset of X^{d-1} of dimension$\leq g - 1$. Hence $\{E + P\}$ for any point of $P \in X$ form a subset of X^d of dimension $\leq g$. Since all effective divisors of degree $d \geq g + 1$ form the whole set of X^d, which has dimension $d \geq g + 1$, we can pick an effective divisor D of degree d such that for all $P \in X$, we have $D - P$ is an effective nonspecial divisor of degree $d - 1$. So clearly D is nonspecial, and by Riemann-Roch, $\dim|D - P| = \dim|D| - 1$ for all P, i.e. D have no base points.

Solution 4.6.9. Consider the blow-up $\pi : \tilde{\mathbb{P}} \to \mathbb{P}^3$ of $\mathbb{P}^3$ at X. Denote $Y = \pi^{-1}(X)$. Consider the sheaf $\mathscr{I}_Y \otimes \pi^*\mathscr{O}_{\mathbb{P}^3}(m)$ on $\tilde{\mathbb{P}}$. We have $\pi_*(\mathscr{I}_Y \otimes \pi^*\mathscr{O}_{\mathbb{P}^3}(m)) \cong \pi_*(\mathscr{I}_Y)(m)$. So by Serre's theorem, it has global section for $m \gg 0$, hence $\mathscr{I}_Y \otimes \pi^*\mathscr{O}_{\mathbb{P}^3}(m)$ has global section for $m \gg 0$. For any $s \in \Gamma(\tilde{\mathbb{P}}, \mathscr{I}_Y \otimes \pi^*\mathscr{O}_{\mathbb{P}^3}(m))$, we have $Y \subset \operatorname{Supp} s$. Hence $S = \pi(\operatorname{Supp} s)$ is a surface of degree m in $\mathbb{P}^3$, and $X = \pi(Y) \subset S$.

5 Surfaces

5.1 Geometry on a Surface

Solution 5.1.1. By Riemann-Roch, we have $\chi(\mathscr{L}^{-1}) = \frac{1}{2}C.(C-K) + \chi(\mathscr{O}_X)$, $\chi(\mathscr{M}^{-1}) = \frac{1}{2}D.(D-K) + \chi(\mathscr{O}_X)$ and $\chi(\mathscr{L}^{-1}\otimes\mathscr{M}^{-1}) = \frac{1}{2}(C+D).(C+D-K) + \chi(\mathscr{O}_X)$. So $\chi(\mathscr{O}_X) - \chi(\mathscr{L}^{-1}) - \chi(\mathscr{M}^{-1}) + \chi(\mathscr{L}^{-1}\otimes\mathscr{M}^{-1}) = \frac{1}{2}((C+D).(C+D-K) - C.(C-K) - D.(D-K)) = C.D$.

Solution 5.1.2. By Riemann-Roch, we have $\chi(nH) = \frac{1}{2}(nH).(nH-K) + 1 + p_a$. So by definition of Hilbert polynomial, we have $P(n) = \frac{1}{2}n^2H^2 - \frac{1}{2}nH.K + 1 + p_a$ for $n >> 0$, hence $a = H^2$, $c = 1 + p_a$, i.e. the degree of X in $\mathbb{P}^N$ is H^2. For b, by adjunction formula, we have $H.(H+K) = 2\pi - 2$, i.e. $H.K = 2\pi - 2 - H^2$. So $b = \frac{1}{2}H^2 + 1 - \pi$. For any curve C in X, by Bertini's theorem, there exists a hyperplane S satisfying S intersect C and X transversally, hence $C.H = \sharp(C\cap H) = \sharp(C\cap(X\cap S)) = \sharp(C\cap S) = \deg C$ by Bézout's theorem.

Solution 5.1.3. (a) By Riemann-Roch, $\chi(\mathscr{O}_D^{-1}) = \frac{1}{2}(-D).(-D-K) + 1 + p_a$. Hence $\chi(\mathscr{O}_D) = \chi(\mathscr{O}_X) - \chi(\mathscr{O}_D^{-1}) = -\frac{1}{2}D.(D+K)$, i.e. $2p_a(D) - 2 = D.(D+K)$.

(b) Since $\chi(\mathscr{O}_D)$ depends only on thel inear equivalent class of D on X, so does $p_a(D)$.

(c) For the first one, $p_a(-D) = 1 + \frac{1}{2}(-D).(-D+K) = 1 + D^2 - \frac{1}{2}D.(D+K) = D^2 - p_a(D) + 2$. For the second one, $p_a(C+D) = 1 + \frac{1}{2}(C+D).(C+D+K) = 1 + \frac{1}{2}(C.(C+K) + D.(D+K)) + C.D = p_a(C) + p_a(D) + C.D - 1$.

Solution 5.1.4. (a) Since we have an exact sequence $0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_{\mathbb{P}^3} \to \Omega_X \to 0$, where $\mathscr{I}$ is the ideal sheaf of X in $\mathbb{P}^3$, we have $\omega_X = \omega_{\mathbb{P}^3}\otimes\det(\mathscr{I}/\mathscr{I}^2)'$, i.e. $K_X = (K_{\mathbb{P}^3} + X)|_X$. Since X has degree d, we have $X \sim dH$, and obviously $K_{\mathbb{P}^3} = -4H$, we have $K_X = (d-4)H$, where H is a hyperplane in $\mathbb{P}^3$. So by adjunction formula, we have $-2 = 2g(C) - 2 = C.(C+K_X)$. And by 5.1.1., $C.H = \deg C = 1$. So $C^2 = 2 - d$.

(b) Consider the surface X defined by $f(x,y,z,w) = xz^{d-1} + x^{d-1}z - yw^{d-1} - y^{d-1}w$ in $\mathbb{P}^3$. Then X is nonsingular, and contains the line $x = y = 0$.

Solution 5.1.5. (a) As we've done in 5.1.3., $K \sim (d-4)H$. So $K^2 = (d-4)^2H^2$. And by 5.1.2., $H^2 = d$. Hence $K^2 = d(d-4)^2$.

(b) Denote p_1 and p_2 are the projections from $X = C\times C'$ to C or C'. Then by 2.8.3., $K_X = p_1^*K_C + p_2^*K_{C'}$. Since C has genus g, we have $\deg K_C = 2g-2$, i.e. K_C is the sum of $2g-2$ points on C. So $p_1^*K_C \sim (2g-2)(C'\times\{\mathrm{pt}\}) = (2g-2)C'$, and similarly $p_2^*K_{C'} = (2g'-2)C$. Clearly by definition, we have $C^2 = C'^2 = 0$, and $C.C' = 1$. Hence $K_X^2 = (p_1^*K_C)^2 + 2p_1^*K_C.p_2^*K_{C'} + (p_2^*K_{C'})^2 = (2g-2)^2(C^2) + 2(2g-2)(2g'-2)(C.C') + (2g'-2)^2(C'^2) = 8(g-1)(g'-1)$.

Solution 5.1.6. (a) Clearly $\Delta.(C\times\{\mathrm{pt}\}) = \Delta.(\{\mathrm{pt}\}\times C) = 1$. So as we've done in 5.1.5.(b), $\Delta.K_X = \Delta.(p_1^*K_C + p_2^*K_C) = 4g-4$. Hence by adjunction formula, $2g-2 = \Delta.(\Delta+K_X)$, i.e. $\Delta^2 = 2g-2-(4g-2) = 2-2g$.

(b) By 5.1.5. and (a), $l^2 = m^2 = 0$, $l.m = l.\Delta = m.\Delta = 1$, $\Delta^2 = 2-2g$. So if $al + bm + c\Delta \sim_{\mathrm{Num}} 0$ for some constants a,b,c, we have $l.(al+bm+c\Delta) = m.(al+bm+c\Delta) = \Delta.(al+bm+c\Delta) = 0$, i.e. $b+c = a+c = a+b+c(2-2g) = 0$. Hence $a = b = c = 0$, which means l, m, Δ are linearly independent.

Solution 5.1.7 (Algebraic Equivalence of Divisors)**.** (a) If $D \sim_{\mathrm{Alg}} 0$ and $E \sim_{\mathrm{Alg}} 0$, we have two sequence $0 = D_0, D_1, \ldots, D_n = D$ and $0 = E_0, E_1, \ldots, E_m = E$. Then we have a sequence $0 = D_0, \ldots, D_n = D_n + 0 = D_n + E_0, D_n + E_1, \ldots, D_n + E_m = D + E$, i.e. $D + E \sim_{\mathrm{Alg}} 0$. Moreover, since $D_i \sim_{\mathrm{PreAlg}} D_{i+1}$, by definition we have $D_{i+1} \sim_{\mathrm{PreAlg}} D_i$, we have a sequence $0 = -D_0, -D_1, \ldots, -D_n = -D$. So all divisors$\sim_{\mathrm{Alg}} 0$ form a subgroup of $\mathrm{Div}\, X$.

(b) We just need to prove that for all $f \in K(X)^*$, $(f) \sim_{\mathrm{Alg}} 0$. Consider the divisor $(tf-u)$ on $X\times\mathbb{P}^1$, where t, u are the homogeneous coordinates on $\mathbb{P}^1$. Then if we write $(f) = D_f^+ - D_f^-$ and $(tf-u) = D_{(tf-u)}^+ - D_{(tf-u)}^-$, by example 9.8.5. on chapter II, $D_{(tf-u)}^+$ and $D_{(tf-u)}^-$ are flat over T. Since $D_{(tf-u)}^+$ and $D_{(tf-u)}^-$ restricts to D_f^+ and D_f^- over $(1,0)$ and 0 over $(0,1)$, we have $(f) \sim_{\mathrm{Alg}} 0$.

(c) As the proof of theorem 1.1., every divisor is the difference of two very ample divisors. So we just need to prove that if $D \sim_{\text{Alg}} D'$, we have $D.H = D'.H$. And by definition, we only need to consider the case $D \sim_{\text{PreAlg}} D'$. For any very ample divisor H of X, it induces an embedding $X \to \mathbb{P}^n$. Then we have an embedding $X \times T \to \mathbb{P}^n_T$. For any divisor $E \subset X \times T$ with fibres $D = E_0$ and $D' = E_1$, since E is flat over T, by theorem 9.9. in chapter III, $\deg D = \deg D'$, i.e. $D.H = D'.H$.

Solution 5.1.8 (Cohomology Class of a Divisor)**.** (a) Clearly, any element in Pic X is an equivalent class, so we may assume D and E are meeting transversally. And similarly we may assume D is very ample since every effective divisor is the difference of two very ample divisor. Fix a very ample divisor D. Then we have a morphism $f : H^1(X, \Omega_X) \to H^1(D, \Omega_D) \cong k$ as $f(s) = i^*(s)$, where $i : D \to X$ is the embedding. So by Serre duality and the definition of c, we have $f(\mathscr{F}) = \langle c(D), \mathscr{F}\rangle$. Hence we have a commutative diagram

$$\begin{tikzcd} \operatorname{Pic} X \arrow[r, "c_X"] \arrow[d, "\text{res}"'] & H^1(X, \Omega_X) \arrow[d, "f"] \\ \operatorname{Pic} D \arrow[r, "c_D"] & H^1(D, \Omega_D) \end{tikzcd}$$

Since $\text{res}(\mathscr{L}(E)) = \mathscr{L}(E) \otimes \mathscr{O}_D$, and by 3.7.4., we have $c_D(\mathscr{F}) = (\deg \mathscr{F}) \cdot 1$, we have $\langle c(D), c(E)\rangle = f(c_X(E)) = c_D(g(E)) = c_D(\mathscr{L}_E \otimes \mathscr{O}_D) = \deg_D(\mathscr{L}_E \otimes \mathscr{O}_D) = (D.E) \cdot 1$ by lemma 1.3.

(b) Suppose $D_1, \dots, D_n \in$ Pic X are numerically independent. If there exists $i_1, \dots, i_n$ such that $c(D_1)^{i_1} \otimes \dots \otimes c(D_n)^{i_n} = 1$, i.e. $\langle c(D_1)^{i_1} \otimes \dots \otimes c(D_n)^{i_n}, \mathscr{F}\rangle = 0$ for all $\mathscr{F}$. Take $\mathscr{F} = c(E)$ for any divisor E of X. Then we have $0 = \langle c(D_1)^{i_1} \otimes \dots \otimes c(D_n)^{i_n}, c(E)\rangle = (\sum i_k D_k).E$. Since $D_1, \dots, D_n \in$ Pic X are numerically independent, we have $i_k = 0$ for all k. So $c(D_1), \dots, c(D_n) \in H^1(X, \Omega_X)$ are linearly independent. Since $\dim_k H^1(X, \Omega_X) < \infty$, we have Num$X < \infty$.

Solution 5.1.9. (a) Define $E = (H^2)D - (H.D)H$. Then we have $E.H = (H^2)(D.H) - (H.D)(H^2) = 0$. So by Hodge index theorem, we have $0 \geq E^2 = (H^2)^2(D^2) - (H^2)(H.D)^2$. Since $H^2 > 0$, we have $D^2H^2 \leq (H.D)^2$. And the equality holds iff $D \sim_{\text{Num}} 0$.

(b) Define $H = l + m$ and $E = l - m$. Then we have $D.H = a + b$, $D.E = a - b$, $H^2 = 2$, $E^2 = -2$ and $H.E = 0$. For any curve $C'' \subset X$, if C'' do not intersect any $C \times \{\text{pt}\}$ for any pt $\in C'$, clearly $C'' \subset C \times \{\text{pt}\}$ for some pt $\in C'$. Then C'' must intersect $\{\text{pt}\} \times C'$ for some pt $\in C$. Since $H^2 = 2$, by Nakar Moishezon's theorem we have H is ample. So if we define $D' = (H^2)(E^2)D - (E^2)(D.H)H - (H^2)(D.E)E = -4D + 2(a+b)H - 2(a-b)E$, then $D'.H = -4(a+b) + 4(a+b) - 0 = 0$. Hence by (a) we have $(D'^2)(H^2) \leq (D'.H)^2 = 0$. So $0 \geq D'^2 = 16D^2 - 32ab$, i.e. $D^2 \leq 2ab$. And moreover $D^2 = 2ab$ iff we have $D' \sim_{\text{Num}} 0$, i.e. $D \sim_{\text{Num}} bl + am$.

Solution 5.1.10 (Weil's Proof (2) of the Analogue of the Riemann Hypothesis for Curves)**.** Since Γ is the preimage of Δ by morphism Frob $\times$ id, we have $\Gamma^2 = \Delta^2 \cdot \deg \text{Frob} \cdot \deg \text{id} = q(2 - 2g)$. And by definition we clearly have $\Gamma.\Delta = N$. Moreover, define $l = C \times \{\text{pt}\}$ and $m = \{\text{pt}\} \times C$. Then Γ intersect l at $f^{-1}(\text{pt}) \times \text{pt}$, and intersect m at pt $\times f(\text{pt})$. So $\Gamma.l = q$ and $\Gamma.m = 1$. And clearly $\Delta.l = \Delta.m = 1$. Define $D = r\Gamma + s\Delta$, then $D.l = rq + s$ and $D.m = r + s$. So by 5.1.9., $2(rq + s)(r + s) \geq D^2 = rq(2 - 2g) + s^2(2 - 2g) + 2rsN$, i.e. $|N - q - 1| \leq \frac{r}{s}gq + \frac{s}{r}g$ for all r, s. So $|N - q - 1| \leq \frac{r}{s}gq + \frac{s}{r}g \leq 2\sqrt{\frac{r}{s}gq \cdot \frac{s}{r}g} = 2g\sqrt{q}$.

Solution 5.1.11. (a) If $d \leq 0$, $D = C_1 + \ldots + C_n$ such that $D.H = d \leq 0$, by Nakai-Morshezon theorem, we have $C_i.H > 0$, hence $D.H > 0$, which makes a contradiction. So we may assume $d > 0$. Since Num$X$ has finite rank, and so does $H^\perp$. Then take an orthogonal basis of $H^\perp$ as $R_1, \dots, R_k$. Fix an effective divisor D with $D.H = d$. Then for any $a_i \in \mathbb{Z}$, we have $(D + \sum a_i R_i).H = d$. Denote n as the number of curves involved in the sum of D and R_i. In order to make $D + \sum a_i R_i$ to be an effective divisor, we may assume $D + \sum a_i R_i = \sum C_i$ for finite sum of$\leq n \sum a_i = N$ terms. Then $(D + \sum a_i R_i).(D + \sum a_i R_i + K) \geq C_i.(C_i + K) \geq -2N$. Hence $\sum_{i,j} a_i a_j R_i.R_j$+lower terms> 0. Since $R_i.R_j = 0$ if $i \neq j$, and $R_i^2 < 0$ because H is ample, the inequality above has only finite solution of a_i. So the number of D such that $D.H$ is finite.

(b) For any $\sigma, \sigma' \in \mathrm{Aut}C$, we denote Γ, Γ' are their graphs. Since $\sigma \neq \sigma'$, we have $\Gamma \neq \Gamma'$, i.e. $\Gamma.\Gamma' \geq 0$. Since σ and σ' are both automorphism, we have $\Delta \to \Gamma$ as $\mathrm{id} \times \sigma$ or $\Delta \to \Gamma'$ as $\mathrm{id} \times \sigma'$ has degree 1, hence $\Gamma^2 = \Gamma'^2 = \Delta^2 = 2 - 2g < 0$. So $\Gamma^2 \neq \Gamma.\Gamma'$, i.e. Γ is not numerical equivalent to Γ. Define $H = l + m$ from 5.1.9., we have $\Gamma.H = 2$ for all graph of automorphism Γ. Then the number of choice of Γ is finite by (a), hence $\mathrm{Aut}C$ is finite.

Solution 5.1.12. The first proposition is trivial by Nakai-Moishezon criterion. For second, consider a curve C with genus> 2. A divisor D with $\deg D = 2g$ is very ample iff for any $P, Q \in C$ we have $l(D - P - Q) = l(D) - 2$. Then by Riemann-Roch, $l(D) - l(K - D) = g + 1$, $l(D - P - Q) - l(K - D + P + Q) = g - 1$. So D is very ample iff D is not of the form $K + P + Q$. So there exists D is very ample and D' is just ample with $\deg D = \deg D' = 2g$. Then in $X = C \times \mathbb{P}^1$, the divisor $D \times \mathbb{P}^1$ and $D' \times \mathbb{P}^1$ are numerically equivalent, and the first is very ample but second is just ample.

5.2 Ruled Surfaces

Solution 5.2.1. Since X is a birationally ruled surface, there exists a function field L such that $K(X) = L \otimes_k k(x) = L(x)$. Then by corollary 6.12. in chapter I, there exists a unique smooth projective curve C such that $K(C) = L$, i.e. C is unique one such that X is birationally equivalent to $C \times \mathbb{P}^1$.

Solution 5.2.2. ($\Rightarrow$) If $\mathscr{E}$ is decomposable, i.e. $\mathscr{E} \cong \mathscr{L} \oplus \mathscr{L}'$, clearly the sections corresponding to $\mathscr{L}$ and $\mathscr{L}'$ has no intersection.

($\Leftarrow$) If X has two sections C and C' such that $C \cap C' = \emptyset$, we have two exact sequences $\mathscr{E} \to \mathscr{L} \to 0$ and $\mathscr{E} \to \mathscr{L}' \to 0$ corresponding to C and C' with kernels $\mathscr{K}$ and $\mathscr{K}'$. Then since $C \cap C' = \emptyset$, we have $\mathscr{K} \oplus \mathscr{K}' \to \mathscr{E}$ is injective. Since the rank is the same, we have $\mathscr{E} \cong \mathscr{K} \oplus \mathscr{K}'$.

Solution 5.2.3. (a) Consider $X = \mathbb{P}(\mathscr{E})$ and $\pi : X \to C$. The generic fibre $X_\eta \cong \mathbb{P}_K^{r-1}$, hence has a rational point. Let ξ be a rational point in the generic fibre, and define $S = \overline{\{\xi\}} \subset X$. Then $\pi|_S : S \to C$ is clearly a proper birational morphism. Since C is nonsingular, we have $S \cong C$. Consider the ideal sheaf $\mathscr{I}$ of S, we have $0 \to \mathscr{I}(1) \to \mathscr{O}_X(1) \to \mathscr{O}_S(1) \to 0$. So $0 \to \pi_*(\mathscr{I}(1)) \to \mathscr{E} \to \mathscr{O}_S(1)$ on C. Denote $\psi : \mathscr{E} \to \mathscr{O}_S(1)$. For any open subset $U \subset C$ such that $\mathscr{E}|_U$ is free, clearly $\psi|_U$ is surjective. Hence ψ is surjective. Then we define $\mathscr{E}_{r-1} = \pi_*(\mathscr{I}(1))$ and $\mathscr{L}_r = \mathscr{O}_S(1)$, we have $0 \to \mathscr{E}_{r-1} \to \mathscr{E}_r \to \mathscr{L}_r \to 0$. Clearly $\mathscr{E}_{r-1}$ and $\mathscr{L}_r$ are locally free of rank $r - 1$ and 1. So by induction we have a sequence $0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \ldots \subset \mathscr{E}_r = \mathscr{E}$.

(b) If Ω is an extension of invertible sheaves, we may assume $0 \to \mathscr{O}_X(a) \to \Omega \to \mathscr{O}_X(b) \to 0$. Since $H^1(X, \mathscr{O}_X(a)) = H^1(X, \mathscr{O}_X(b)) = 0$, we have $h^0(\Omega) = h^0(\mathscr{O}_X(a)) + h^0(\mathscr{O}_X(b))$, $h^1(\Omega) = 0$, and $h^2(\Omega) = h^2(\mathscr{O}_X(a)) + h^2(\mathscr{O}_X(b))$. By theorem 8.13. in chapter II, we have $0 \to \Omega \to \mathscr{O}_X(-1)^3 \to \mathscr{O}_X \to 0$, hence similarly, $h^0(\Omega) + h^1(\Omega) = h^0(\mathscr{O}_X(-1)^3) - h^0(\mathscr{O}_X) = 0 - 1 = -1$. So $h^0(\Omega) = -1$, which is impossible.

Solution 5.2.4. (a) Since $X = C \times \mathbb{P}^1$, by example 2.11.1., we have $e = 0$ and $C_0 = C$, hence by corollary 2.11., we have $K \sim_{\mathrm{Num}} -2C + (2g - 2)f$. For any section D, we may assume $D = C + rf$. Then by adjunction formula, $D^2 = 2g - 2 - D.K = 2g - 2 - (2g - 2 - 2r) = 2r$. So D^2 is always an even integer. If $r = 0$, we just take $D = C$. If $d \geq g + 1$, by 4.6.8., there exists a nonspecial divisor E on C such that $|E|$ has no base points. Then $D = C + Ef$ is a section with $D^2 = 2r$.

(b) (**Here I think if C is hyperelliptic, $r = 2$ or 3 are both possible.**) If X has a section $D \sim_{\mathrm{Num}} C + bf$, then the composition $C \to D \to \mathbb{P}^1$ gives a morphism $g_D : C \to \mathbb{P}^1$. And by definition $\deg g_D = C.D = b$. Conversely, if we have a morphism $g : C \to \mathbb{P}^1$ with $\deg g = d$, the graph $D_g = \Gamma_g \subset X$ is a section, and $D_g.C = d$, hence $D_g = C + df$. Since $g(C) = 1$, there are no morphism $g : C \to \mathbb{P}^1$ with $\deg g = 1$, hence $r \neq 1$. If C is hyperelliptic, i.e. C has a g_2^1, hence $r = 2$ is possible. And C has a g_3^1 by adding a point in g_2^1, and $r = 3$

is also possible. And if C is nonhyperelliptic, by example 5.5.2., C has a g_3^1 but no g_2^1, hence $r = 3$ is possible, but 2 is not.

Solution 5.2.5 (Values of e)**.** (a) As we've done in theorem 2.12., we only need to prove $\operatorname{Ext}^1(\mathscr{L}, \mathscr{O}_C)$ is nontrivial for $\deg \mathscr{L} = -e$ when $0 \leq e \leq 2g-2$. Since $\operatorname{Ext}^1(\mathscr{L}, \mathscr{O}_C) \cong H^1(C, \check{\mathscr{L}}) \cong H^0(X, \omega_C \otimes \mathscr{L})$, and $\deg \omega_C \otimes \mathscr{L} = 2g-2-e \geq 0$, hence $\operatorname{Ext}^1(\mathscr{L}, \mathscr{O}_C)$ is nontrivial. So we may pick a nontrivial extension $\mathscr{E}$, which is indecomposable.

(b) ($\Rightarrow$) If $\mathscr{E}$ is normalized, the morphism $\gamma : H^0(\mathscr{L}(D+K-E)) \to H^1(\mathscr{L}(-E))$ is injective. So for the $\xi \in H^1(\mathscr{L}(-D))$, we have

$$\begin{tikzcd} H^0(C,\mathscr{O}_C) \arrow[r, "\delta"] \arrow[d, "\alpha"'] & H^1(C,\mathscr{L}(-D)) \arrow[d, "\beta"] \\ H^0(C,\mathscr{L}(D+K-E)) \arrow[r, "\gamma"] & H^1(C,\mathscr{L}(-E)) \end{tikzcd}$$

where $\delta(1) = \xi$, $\alpha(1) = t$ and t is the section defining the divisor $D+K-E$, β is induced from α. So β is dual to $\beta' : H^0(C, \mathscr{L}(E)) \to H^0(C, \mathscr{L}(D+K))$ induced by t. So for any nonzero element $e \in H^0(C, \mathscr{L}(E))$, $\beta'(e)$ is a section of $H^0(C, \mathscr{L}(D+K))$ corresponding to $D+K = E+(D+K-E)$. So varying E and $(D+K-E)$, we can take all divisors in $|D+K|$. So β' is surjective, hence $L_E \subset |D+K|$. If $L_E \subset H$, we have $\beta(\xi) = 0$, which contradicts to the injectivity of γ. Hence $L_E \not\subseteq H$.

($\Leftarrow$) If $\mathscr{E}$ is not normalized, there exists some E such that $H^0(\mathscr{E} \otimes \mathscr{L}(-E)) \neq 0$. Hence the map γ is not injective, i.e. $L_E \subset \ker(\xi) = H$.

(c) By (b), we only need to find a ξ such that $H = \ker(\xi)$ does not contain any $|D+K-E|+E$. By Riemann-Roch, we have $h^0(D+K) = d+g-1$, and $h^0(D+K-E) = g$. Then consider $B \subset C^{d-1} \times H^1(C, \mathscr{L}(-D))$ consisting all pair (E, ξ) such that $L_E \subset \ker(\xi)$. Then $B \to C^{d-1}$ is surjective, and for any effective divisor E of degree $d-1$, $\dim B_E = (d+g-1) - g = d-1$. So B has dimension $(d-1)+(d-1) = 2d-2 < d+g-1$, hence there must exist some $\xi \in H^0(C, \mathscr{L}(D+K))'$ such that $\xi \notin B_E$ for any effective divisor E.

(d) If $e < -g = -2$, by definition, we know if $E \neq E'$, we have $L_E \neq L_{E'}$ since $\dim L_E = \dim L_{E'} = 1 \leq d-2$. So all L_E's form the Grassmannian G_{d+1}^2, which has dimension $2(d-1)$ since $d > 2$. Since all $\ker(\xi)$ form the Grassmiannian G_{d+1}^d, which has dimension d. By theorem 2.12., we have $e \geq -2g = -4$, i.e. $3 \leq d \leq 4$. So we have $d \leq 2(d-1)$. So for any ξ, there exists at least one E such that $L_E \subset \ker(\xi)$. Hence no one $\mathscr{E}$ is normalized, which is impossible.

Solution 5.2.6. For some locally free sheaf $\mathscr{E}$ of rank r on $\mathbb{P}^1$, by theorem 8.8.(c) in chapter III we have $H^1(\mathbb{P}^1, \mathscr{E}(n)) = 0$ for $n \gg 0$. Then by Serre duality, we know $H^0(\mathbb{P}^1, \mathscr{E}(-n)) = 0$ for $n \gg 0$. So we may assume i is the largest integer such that $\mathscr{E}(-i)$ has a global section, which induces a morphism $0 \to \mathscr{O} \to \mathscr{E}(-i)$. Hence $0 \to \mathscr{O}(i) \to \mathscr{E} \to \mathscr{F} \to 0$ for some cokernel $\mathscr{F}$ locally free of rank $r-1$. By induction, we may assume $\mathscr{F} = \bigoplus_{j=1}^{r-1} \mathscr{O}(n_j)$. If $n_j > i$, there exists a surjection $\mathscr{E}(-n_j) \to \mathscr{O} \to 0$, which means $\mathscr{E}(-n_j)$ has a global section and contradicts to the definition of i. Hence all $n_j \leq i$. Since $\operatorname{Ext}^1(\mathscr{F}, \mathscr{O}(i)) = \bigoplus \operatorname{Ext}^1(\mathscr{O}(n_j), \mathscr{O}(i)) = 0$, the extension is trivial, i.e. $\mathscr{E} = \mathscr{O}(i) \oplus (\bigoplus \mathscr{O}(n_j)))$.

Solution 5.2.7. By the proof in theorem 2.12., $\mathscr{E}$ is indecomposable with $e = -1$. And by theorem 2.15., this kind of ruled surface X is unique. Fix the normalized $\mathscr{E}$ and a point $P_0 \in C$ with section $\mathscr{E} \to \mathscr{L}(P_0) \to 0$. Then any section corresponds to a surjection $\mathscr{E} \to \mathscr{L}(\mathfrak{d}) \to 0$. Then by theorem 2.15., $\mathfrak{d}$ has degree 1, i.e. $\mathfrak{d} = P$ for some $P \in C$. Conversely, by corollary 2.16., there is a 1-1 correspondence between all $\mathscr{E}$'s and points in C. Then any $\mathscr{E}' \to \mathscr{L}(P) \to 0$ corresponding to a section $\mathscr{E} \to \mathscr{L}(\frac{P+P_0}{2}) \to 0$, since by the proof of theorem 2.15. we have $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L}(\frac{P-P_0}{2})$. And $\mathscr{L}(P+Q) = \mathscr{L}(R)$ for some $R \in C$ since C is an elliptic curve. Moreover, if two section C_0 and C_0' are linearly equivalent, we have $C_0 \sim C_0' \sim C_0 + (P-Q)f$, i.e. $Pf \sim Qf$, so $\mathscr{L}(P) = \mathscr{L}(Q)$ on C. Since C is an elliptic curve, we have $P = Q$.

Solution 5.2.8. Here we denote $\mu(\mathscr{E}) = \deg \mathscr{E}/\text{rank}\, \mathscr{E}$ for any locally free sheaf $\mathscr{E}$ of finite rank. Then clearly, if rank $\mathscr{E} > 1$, we have $\mu(\mathscr{E}) = \mu(\mathscr{E} \otimes \mathscr{L})$ or any invertible sheaf $\mathscr{L}$. And if we have $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$, we have $\deg \mathscr{E} = \deg \mathscr{E}' + \deg \mathscr{E}''$. So $\mu(\mathscr{E}) = \frac{\text{rank } \mathscr{E}'}{\text{rank } \mathscr{E}' + \text{rank } \mathscr{E}''}\mu(\mathscr{E}') + \frac{\text{rank } \mathscr{E}''}{\text{rank } \mathscr{E}' + \text{rank } \mathscr{E}''}\mu(\mathscr{E}'')$. So the definition of (semi)stability is equal to for any $0 \to \mathscr{F} \to \mathscr{E}$, we have $\mu(\mathscr{F}) < (\leq)\mu(\mathscr{E})$.

(a) If $\mathscr{E} = \mathscr{F} \oplus \mathscr{G}$ is stable, we may assume $\mu(\mathscr{F}) \leq \mu(\mathscr{G})$. Then consider the morphism $\mathscr{E} \to \mathscr{F} \to 0$, we have $\mu(\mathscr{E}) < \mu(\mathscr{F})$. Since $\mu(\mathscr{F}) \leq \mu(\mathscr{E}) \leq \mu(\mathscr{G})$, we have $\mu(\mathscr{E}) < \mu(\mathscr{F}) \leq \mu(\mathscr{E})$, which is impossible.

(b) ($\Rightarrow$) We may assume $\mathscr{E}$ is normalized. Then since we have $0 \to \mathscr{O}_C \to \mathscr{E} \to \mathscr{L} \to 0$. Then $\mu(\mathscr{E}) > (\geq)\mu(\mathscr{O}_C) = 0$, i.e. $\deg \mathscr{E} > (\geq)0$.

($\Leftarrow$) Again, we may assume $\mathscr{E}$ is normalized. If $\deg \mathscr{E} > (\geq)0$ and decomposable, for any $\mathscr{F} \subset \mathscr{E}$, by theorem 2.12.(a), we have $\deg \mathscr{F} \leq 0$. Hence $\mu(\mathscr{F}) < (\leq)\mu(\mathscr{E})$, i.e. (semi)stable. If $\deg \mathscr{E}) > (\geq)0$ and indecomposable, then the case $g(C) = 0$ is clear by 5.2.6. So if $\mathscr{E}$ is not (semi)stable, there must exist $\mathscr{F} \subset \mathscr{E}$ such that $\deg \mathscr{F} \geq (>)\frac{1}{2}\deg \mathscr{E} > (\geq)0$, i.e. $\deg \mathscr{F} > 0$. So $\mathscr{E} \otimes \check{\mathscr{F}}$ has a section. Since $\deg \check{\mathscr{F}} < 0$, but since $\mathscr{E}$ is normalized, we have $h^0(\mathscr{E} \otimes \check{\mathscr{F}}) = 0$, which makes a contradiction.

(c) If $\mathscr{E}$ is not semistable, by (b) we have $\deg \mathscr{E} < 0$. By the proof of theorem 2.12., we have $0 \to \mathscr{O}_C \to \mathscr{E} \to \mathscr{L} \to 0$. Then $\deg \mathscr{L} = \deg \mathscr{E} < 0$. So by theorem 2.12., $e > 0$, i.e. $0 < e \leq 2g - 2$, and $\deg \mathscr{L} = -e$. Moreover, $\mathscr{E}$ corresponds to a non-trivial element in $\text{Ext}^1(\mathscr{L}, \mathscr{O}_C) = H^1(C, \check{\mathscr{L}})$ up to scalar. Hence all indecomposable locally free sheaves $\mathscr{E}$ of rank 2 that are not semistable are classified, up to isomorphism, by these three conditions.

Solution 5.2.9. If Y is the complete intersection of X_0 with a surface of degree $a \geq 1$, by 2.8.4., we have $\deg Y = 2a$ and $g(Y) = \frac{1}{2} \times 2 \times a(a + 2 - 4) + 1 = (a - 1)^2$. For general case, we may treat Y_0 as a divisor on X_0. Blow up X_0 at the vertex, we get a ruled surface X over $\mathbb{P}^1$ with projection $\pi : X \to X_0$. Moreover, by example 2.11.4., the e of X is -2. And Y_0 has a preimage Y. Clearly, the preimage of the ruling of X_0 in X is the section C_0, so we may assume $Y \sim aC_0 + \mathfrak{b}f$. Clearly, $b = 2a + 1$ since Y is not the complete intersection. Then for any hyperplane H in $\mathbb{P}^3$, we may denote $\tilde{H} = \pi^{-1}(H \cap X_0)$ then $\tilde{H}.C_0 = 1$ and $\tilde{H}.f = 2$, i.e. $\tilde{H} \sim_{\text{Num}} C_0 + 2f$. So $\deg Y_0 = (H \cap X_0).Y_0 = -2a + 2a + b = 2a + 1$. And $K_X \sim_{\text{Num}} -2C_0 - 4f$, $C_0^2 = -2$. So by adjunction formula, $2g - 2 = (aC_0 + (2a + 1)f)((a - 2)C_0 + (2a - 3)f) = -2a(a - 2) + a(2a - 3) + (2a + 1)(a - 2) = 2(a^2 - a - 1)$, i.e. $g = a^2 - a$.

Solution 5.2.10. By the proof of corollary 2.19., the embedding is induced by the very ample divisor $D = C_0 + nf$. Consider the divisor $3C_0 + af$ for some a. Then it can be written as a curve iff $C.C_0 \geq 0$, i.e. $a \geq 3e$. So if $a = n + e + 2$, we have $a \geq 3e$, then there is a curve $C \sim_{\text{Num}} 3C_0 + (n + e + 2)f$. Since $K \sim_{\text{Num}} -2C_0 - (e + 2)f$, by adjunction formula, $2g - 2 = C.(C + K) = 4n - 2e + 2$, i.e. $g = 2n - e + 2$. And clearly, C has a g^1_3 since $C.f = 3$.

Solution 5.2.11. (a) Consider $0 \to \mathscr{L}_X(-C_0) \to \mathscr{O}_X \to \mathscr{O}_C \to 0$. Since $\mathfrak{b}$ is nonspecial, we have $h^1(\mathscr{L}_X(\mathfrak{b}f)) = h^2(\mathscr{L}_X(\mathfrak{b}f)) = 0$. Since $h^i(\mathscr{L}_X(\mathfrak{b}f)) = h^i(\mathscr{L}_C(\mathfrak{b}))$ and $h^i(\mathscr{L}_X(C_0 + \mathfrak{b}f)) = h^i(\mathscr{L}_C(\mathfrak{b} + \mathfrak{e}))$. So we have $h^i(\mathscr{L}_X(C_0 + \mathfrak{b}f)) = h^i(\mathscr{L}_C(\mathfrak{b})) + h^i(\mathscr{L}_C(\mathfrak{b} + \mathfrak{e}))$. Denote $D = C_0 + \mathfrak{b}f$. For any $P \in C$, we have $h^0(\mathscr{L}_X(D - Pf)) = h^0(\mathscr{L}_C(\mathfrak{b} - P)) + h^0(\mathscr{L}_C(\mathfrak{b} + \mathfrak{e} - P)) = h^0(\mathscr{L}_C(\mathfrak{b})) - 1 + h^0(\mathscr{L}_C(\mathfrak{b} + \mathfrak{e})) - 1 = h^0(D) - 2$. But as the same proof of theorem 3.1. in chapter IV, we know D is base points free on Pf for some $P \in C$ iff $h^0(\mathscr{L}_X(D - Pf)) = h^0(\mathscr{L}_X(D)) - 2$, i.e. D is base points free.

(b) For any $P, Q \in C$, $h^0(\mathscr{L}_X(D - Pf - Qf)) = h^0(\mathscr{L}_C(\mathfrak{b} - P - Q)) + h^0(\mathscr{L}_C(\mathfrak{b} + \mathfrak{e} - P - Q)) = h^0(\mathscr{L}_X(D)) - 4$. Then similarly as the same proof of theorem 3.1. in chapter IV, D is very ample.

Solution 5.2.12. If $e \geq 0$, then by corollary 3.2. in chapter IV and 5.2.11., this problem is trivial. Hence we only need to consider the case $e = -1$.

(a) Here $b = \deg \mathfrak{b} \geq 1$. If $b \geq 2$, then by corollary 3.2. in chapter IV and 5.2.11., trivial. So we only need to consider the case $b = 1$. In this case, we may assume $\mathfrak{e} = E$, and $\mathfrak{b} = B$. Then $h^0(\mathscr{L}_X(C_0 + Bf)) =$

$h^0(\mathscr{L}_C(E+B)) = 2$. And for any $P \in X$, $h^0(\mathscr{L}_X(C_0+Bf-Pf)) = h^0(\mathscr{L}_C(E+B-P)) - h^1(\mathscr{L}_X(Bf-Pf)) = 1-1 = 0$. Hence by the same reason with 5.2.11.(a), $C_0 + \mathfrak{b}f$ has no base points.

(b) Here $b = \deg \mathfrak{b} \geq 2$. If $b \geq 3$, then by corollary 3.2. in chapter IV and 5.2.11., trivial. So we only need to consider the case $b = 2$. In this case, we may assume $\mathfrak{b} = A + B$. Then $h^0(\mathscr{L}_X(C_0 + (A+B)f)) = h^0(\mathscr{L}_X(C_0 + Bf)) + 2 = 4$ by (a). And for any $P, Q \in X$, $h^0(\mathscr{L}_X(C_0 + (A+B-P-Q)f)) = 0$. Hence by the same reason with 5.2.11.(a), $C_0 + \mathfrak{b}f$ is very ample.

Solution 5.2.13. By 5.2.12.(b), if $n \geq e+3$, any divisor $D \sim_{\text{Num}} C_0 + nf$ is very ample. Since $D.f = 1$ and $D^2 = 2n - e$, we know the embedding corresponding to D is an elliptic scroll of degree $d = 2n - e$. Moreover, $h^0(\mathscr{L}_X(D)) = h^0(\pi_*\mathscr{L}_X(D)) = h^0(\mathscr{O}_C(n) + \mathscr{O}_C(n-e)) = 2n - e$. So the embedding is in $\mathbb{P}^{d-1}$. Moreover, if we take $e = -1$ and $n = 2$, we have $d = 2n - e = 5$, i.e. there is an elliptic scroll of degree 5 in $\mathbb{P}^4$.

Solution 5.2.14. (a) Denote $\tilde{Y}$ as the normalization of Y with natural map $\varpi : \tilde{Y} \to Y$, and the projection $\pi : Y \to C$. If $\pi \circ \varpi$ has degree d, we have $2g(\tilde{Y}) - 2 \geq d(2g-2) + \deg R$. By 4.1.8., we have $g(Y) \geq g(\tilde{Y})$, i.e. $2g(Y) - 2 \geq d(2g-2)$. Moreover, by adjunction formula, $2g(Y) - 2 = Y.(Y+K) = (aC_0 + bf).((a-2)C_0 + (b + 2g - 2 - e)f) = -ae(a-2) + a(b + 2g - 2 - e) + b(a-2)$, i.e. $ae(1-a) + 2b(a-2) \geq (d-a)(2g-2)$. If $2 \leq a \leq p-1$, we have $d = a$, i.e. $b(a-1) \geq \frac{1}{2}ae(a-1)$, i.e. $b \geq \frac{1}{2}ae$. If $a \geq p$, we have $d = a - np$ for some n, then $(d-a)(2g-2) = 2np(1-g)$. So $b(a-1) \geq \frac{1}{2}ae(a-1) + np(1-g)$. Since $a - d \leq a - 1$, we have $b \geq \frac{1}{2}ae + 1 - g$. Finally, if $a = 1$, Y is a section corresponding to $\mathscr{E} \to \mathscr{L} \to 0$, we have $\deg \mathscr{L} \geq \deg \mathscr{E}$ since $\mathscr{E}$ is normalized. Since $\deg \mathscr{L} = C_0.Y$, we have $b - e \geq -e$, i.e. $b \geq 0$.

(b) If D is ample, by Nakai-Noishezon criterion we have $D.f > 0$m i.e. $a > 0$, and $D^2 > 0$, so $-a^2e + 2ab > 0$, i.e. $b > \frac{1}{2}ae$. Conversely, suppose $a > 0$ and $b > \frac{1}{2}ae + \frac{a}{p}(g-1)$. Since $g > 2$, $a > 0$ and $b > \frac{1}{2}ae$, we have $D.f > 0$ and $D^2 > 0$. And $D.C_0 = -ae + b > -\frac{1}{2}ae > 0$. So for any irreducible curve $Y \neq C_0, f$, we may denote $Y \sim_{\text{Num}} \alpha C_0 + \beta f$, then $D.Y = -a\alpha e + a\beta + \alpha b$. If $\alpha = 1$, by (a) we have $\beta \geq 0$. So $D.Y = -ae + a\beta + b > -ae + b > 0$. If $2 \leq \alpha \leq p-1$, by (a) we have $\beta \geq \frac{1}{2}\alpha e$, so $D.Y \leq -a\alpha e + \frac{1}{2}a\alpha e + \frac{1}{2}ae\alpha = 0$. If $\alpha \geq p$, by (a) we have $\beta \geq \frac{1}{2}\alpha e + 1 - g$, so similarly $D.Y = -a\alpha e + a\beta + \alpha b \geq -a\alpha e + \frac{1}{2}a\alpha e + a - ga + \frac{1}{2}a\alpha e + \frac{a}{p}\alpha(g-1) = a(1-g)(1 - \frac{\alpha}{p}) \geq 0$. Hence by Nakai-Moishezon criterion, D is ample.

Solution 5.2.15 (Funny Behavior in Characteristic p)**.** (a) Denote $f = x^3y + y^3z + z^3x$. Since C is smooth, we have $g(C) = \frac{1}{2}(d-1)(d-2) = 3$. Consider the exact sequence $0 \to \mathscr{O}_{\mathbb{P}^2}(-4) \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_C \to 0$, we have $0 \to H^1(C, \mathscr{O}_C) \to H^2(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}) \to 0$, i.e. $H^1(C, \mathscr{O}_C) \cong H^2(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2})$. Hence $h^1(\mathscr{O}_C) = h^2(\mathscr{O}_{\mathbb{P}^2}(-4)) = h^0(\mathscr{O}_{\mathbb{P}^2}(1)) = \binom{3}{1} = 3$. Denote Frob as the Frobenius morphism on $\mathbb{P}^2$. Then we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_{\mathbb{P}^2}(-4) \arrow[r] \arrow[d, "\text{Frob}^*"] & \mathscr{O}_{\mathbb{P}^2} \arrow[r] \arrow[d, "\text{Frob}^*"] & \mathscr{O}_C \arrow[r] \arrow[d, "\text{Frob}^*"] & 0 \\
0 \arrow[r] & \mathscr{O}_{\mathbb{P}^2}(-4p) \arrow[r] & \mathscr{O}_{\mathbb{P}^2} \arrow[r] & \mathscr{O}_{C^p} \arrow[r] & 0
\end{tikzcd}$$

where C_p is defined as $x^9y^3 + y^9z^3 + z^9x^3 = f^3 = 0$. And we have

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_{\mathbb{P}^2}(-4p) \arrow[r] \arrow[d, "\times f^2"] & \mathscr{O}_{\mathbb{P}^2} \arrow[r] \arrow[d] & \mathscr{O}_{C^p} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}_{\mathbb{P}^2}(-4) \arrow[r] & \mathscr{O}_{\mathbb{P}^2} \arrow[r] & \mathscr{O}_C \arrow[r] & 0
\end{tikzcd}$$

So combining these two diagram, we have

$$\begin{tikzcd} H^1(C,\mathscr{O}_C) \arrow[r, "\cong"] \arrow[d, "\mathrm{Frob}^*"] & H^2(\mathbb{P}^2,\mathscr{O}_{\mathbb{P}^2}(-4)) \arrow[d, "\mathrm{Frob}^*"] \\ H^1(C^p,\mathscr{O}_C^p) \arrow[r, "\cong"] \arrow[d] & H^2(\mathbb{P}^2,\mathscr{O}_{\mathbb{P}^2}(-4)) \arrow[d] \\ H^1(C,\mathscr{O}_C) \arrow[r, "\cong"] & H^2(\mathbb{P}^2,\mathscr{O}_{\mathbb{P}^2}(-4)) \end{tikzcd}$$

Since $H^2(\mathbb{P}^2,\mathscr{O}_{\mathbb{P}^2}(-4))$ has a basis $\{x^{-2}y^{-1}z^{-1}, x^{-1}y^{-2}z^{-1}, x^{-1}y^{-1}z^{-2}\}$ as a free $\mathbb{F}_3[x^{-1},y^{-1},z^{-1}]$-module. So $\mathrm{Frob}^*(x^{-2}y^{-1}z^{-1}) = x^{-6}y^{-3}z^{-3}$ and same for two others, and the image in $H^2(\mathscr{O}_{\mathbb{P}^2}(-4))$ at the bottom row is $f^2\cdot\mathrm{Frob}^*(x^{-2}y^{-1}z^{-1}) = \frac{y^3}{x^6z}+\frac{2z}{x^5}+\frac{z^3}{x^4y^3}+\frac{2y}{x^3z^2}+\frac{2}{x^2y^2}+\frac{1}{yz^3}$ and same for two others. Then any monomial with non-negative exponent on x,y,z is 0, so each of the above three expression is 0, i.e. Frob^* is identically 0.

(b) Denote H as the hyperplane section. Then we have

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{O}_C(-1) \arrow[r] \arrow[d, "\mathrm{Frob}^*"] & \mathscr{O}_C \arrow[r] \arrow[d] & \mathscr{O}_H \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{O}_C(-3) \arrow[r] \arrow[d, "\times f^2"] & \mathscr{O}_C \arrow[r] \arrow[d] & \mathscr{O}_{H^3} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{O}_C(-1) \arrow[r] & \mathscr{O}_C \arrow[r] & \mathscr{O}_H \arrow[r] & 0 \end{tikzcd}$$

Hence for any $\xi\in H^1(\mathscr{L}(-P))$, $\mathrm{Frob}^*(\xi)=\xi^3$, and its image in $H^1(C,\mathscr{O}_C(-1))$ is $f^2\cdot\xi^3$. Since every element in $H^1(C,\mathscr{O}_C)$ has the form $\sum a_{ij}x^iy^j$ for $1\le i<4$ and $-i<j<0$ on the affine open subset $U_z=(z\neq 0)$. So ξ has the form $(ax+by+c)^{-1}(\sum a_{ij}x^iy^j)$ on U_z. Hence $(ax+by+c)^{-1}(\sum a_{ij}x^iy^j)$ has non-negative exponent for some suitable (a_{ij}), i.e. $\mathrm{Frob}^*\xi=0$.

(c) Clearly $\deg\pi = Y.f = 3$. So π is not separable with inseparable degree 3, i.e. separable degree 1, hence purely inseparable. Then π is an isomorphism on the underlying topological space, i.e. Y is a set-theoretic section $\tilde{Y}$ of C lying on the surface. Hence $Y=\tilde{Y}_p$ is a curve contained in X. Moreover, $e=-1$ since $\deg\mathscr{L}(P)=1$. Then $D=2C_0$ satisfies $a>0$ and $b>\frac{1}{2}ae$, and $C_0.D=(3C_0-3f).(2C_0)=6-6=0$, hence D is not ample by Nakai-Moishezon criterion.

Solution 5.2.16. Denote $R=\mathscr{O}_C(C)$. Since C is a nonsingular affine curve, R is a PID. So for any locally free sheaf $\mathscr{E}$, it must have the form $\tilde{M}$ for some finitely generated projective R-module M. Since R is a PID, M is free. So every two locally free sheaves $\mathscr{E}$ and $\mathscr{E}'$ are the same, hence clearly in the same classes in the Grothendieck group.

Solution 5.2.17. (a) Clearly the 3-uple embedding is $\phi:\mathbb{P}^1\to\mathbb{P}^3$, $(t,u)\mapsto(x,y,z,w)=(t^3,t^2u,tu^2,u^3)$, and we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathscr{O}_C \arrow[r, "(2)"] \arrow[d, "(t,u)"'] & \mathscr{O}_C \arrow[d, "(x,y,z,w)"] & & \\
 & \mathscr{O}_C(1)^2 \arrow[r, "\mathscr{J}"] \arrow[d, "\binom{u}{-t}"'] & \mathscr{O}_C(4)^3 \arrow[d, "\kappa"] & & \\
0 \arrow[r] & \mathscr{T}_C \arrow[r, "\phi"] \arrow[d] & \mathscr{T}_{\mathbb{P}^3}|_C \arrow[r] \arrow[d] & \mathscr{N}_C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where $\mathscr{J} = \frac{\partial(x,y,z,w)}{\partial(t,u)}$. Consider the functor $F = \sum_n H^0(C, \mathscr{O}_C(n))$. Denote $R(n)$ as the free $k[t,u]$-module whose generator has degree n. Then $F(\kappa)$ is just $R(-4) \oplus R(-4) \oplus R(-4) \to R(-3)^4$ given by matrix

$$\begin{bmatrix} u & 0 & 0 \\ -t & 0 & 2u \\ 0 & u & -2t \\ 0 & -t & 0 \end{bmatrix}$$

And $F(\phi)$ is $R(-4) \oplus R(-4) \oplus R(-4) \to R(-2)$ given by $(t^2, u^2, 2tu)$. So $\ker F(\phi) = R(-5) \oplus R(-5)$, i.e. $\varphi^*(\mathscr{N}_C) = \mathscr{O}_{\mathbb{P}^1}(5) \oplus \mathscr{O}_{\mathbb{P}^1}(5)$. So $\varphi^*(\mathscr{I}/\mathscr{I}^2) = \mathscr{O}_{\mathbb{P}^1}(-5) \oplus \mathscr{O}_{\mathbb{P}^1}(-5)$.

(b) If $\mathrm{char}(k) \neq 2, 3$, all things are like (a). So $0 \to \mathscr{T}_C \xrightarrow{\phi} \mathscr{T}_{\mathbb{P}^3}|_C \to \mathscr{N}_C \to 0$, and $F(\phi) : R(-5) \oplus R(-5) \oplus R(-6) \to R(-2)$ given by matrix $(t^3, u^3, 6t^2u^2)$. So here $\ker F(\phi) = R(-7) \oplus R(-7)$, i.e. $\varphi^*(\mathscr{I}/\mathscr{I}^2) = \mathscr{O}_{\mathbb{P}^1}(-7) \oplus \mathscr{O}_{\mathbb{P}^1}(-7)$. If $\mathrm{char}(k) = 2, 3$, $F(\phi)$ is given by $(t^3, u^3, 0)$. So $\ker F(\phi) = R(-8) \oplus R(-6)$, i.e. $\varphi^*(\mathscr{I}/\mathscr{I}^2) = \mathscr{O}_{\mathbb{P}^1}(-8) \oplus \mathscr{O}_{\mathbb{P}^1}(-6)$.

5.3 Monoidal Transformations

Solution 5.3.1. Denote $\pi : \tilde{X} \to X$ as the blow-up of X at Y. Then if $\mathscr{I}$ is the ideal sheaf of Y in X, we have $E := \pi^{-1}(Y) = \mathbf{Proj}\ \mathscr{I}/\mathscr{I}^2$ by theorem 8.24. in chapter II. As the proof of theorem 3.4., we only need to prove that $H^i(E, \mathscr{O}_E(n)) = 0$ for all $i > 0$, $n > 0$.

Since $\mathrm{codim}(Y, X) = 1$, $\mathscr{I}/\mathscr{I}^2$ is an invertible sheaf on Y. So we can take an affine open covering $\mathfrak{U} = \{U_i\}$ of Y such that $\mathscr{I}/\mathscr{I}^2$ is free on each U_i. If $U_i = \mathrm{Spec}\ A_i$, we have $\pi^{-1}(U_i) = \mathbf{Proj}\ A_i[x] = \mathbb{P}^1_{A_i}$. Since Y is separated, we know any $U = U_{i_1} \cap \ldots \cap U_{i_k}$ is affine as some $\mathrm{Spec}\ A$, hence $\pi^{-1}(U) = \mathbb{P}^1_A$ also. Since $\mathfrak{V} = \{V_i\}$, where $V_i = \pi^{-1}(U_i)$, is an open covering of E. And for any $V = V_{i_0} \cap \ldots \cap V_{i_k}$, if $U = U_{i_0} \cap \ldots \cap U_{i_k} = \mathrm{Spec}\ A$, we have $V = \mathbb{P}^1_A$. So for any $V = V_{i_0} \cap \ldots \cap V_{i_k}$, we have $H^i(V, \mathscr{O}_E(n)|_V) = H^i(\mathbb{P}^1_A, \mathscr{O}_{\mathbb{P}^1_A}(n)) = 0$. So by 3.4.11., $H^i(E, \mathscr{O}_E(n)) = \check{H}^i(\mathfrak{V}, \mathscr{O}_E(n))$. Clearly, by calculating the Čech complex, we have $\check{H}^i(\mathfrak{V}, \mathscr{O}_E(n)) = 0$, we have $H^i(E, \mathscr{O}_E(n)) = 0$ for all $i > 0$, $n > 0$. Then by the process of the proof of theorem 3.4., $h^i(\tilde{X}, \mathscr{O}_{\tilde{X}}) = h^i(X, \mathscr{O}_X)$, i.e. $p_a(\tilde{X}) = p_a(X)$.

Solution 5.3.2. Since $\tilde{C} = \pi^*C + \mu_P(C)E$ and $\tilde{D} = \pi^*D + \mu_P(D)E$, we have $\tilde{C}.\tilde{D} = C.D + \mu_p(C) \cdot \mu_p(D)E^2 = C.D - \mu_p(C) \cdot \mu_p(D)$. By definition of blow-up and intersection number, we clearly have $\tilde{C}.\tilde{D} = 0$. So $C.D = \mu_p(C) \cdot \mu_p(D)$.

Solution 5.3.3. Since D is ample, we have $D^2 > 0$, i.e. $D^2 \geq 1$. So $(2\pi^*D - E)^2 = 4D^2 - 1 \geq 3 > 0$. Moreover, since D is very ample, for any curve C on X, we have $D.C = \deg C$. For any curve $\tilde{C}$ on $\tilde{X}$ with $\pi(\tilde{C}) = C$, we have $\tilde{C} = \pi^*C - rE$ with $r = \mu_P(C) \leq \deg C$. So $(2\pi^*D - E).\tilde{C} = (2\pi^*D - E).(\pi^*C - rE) = 2\deg C - r \geq \deg C > 0$. So by Nakai-Noishezon criterion, D is ample.

Solution 5.3.4 (Multiplicity of a Local Ring)**.** (a) Consider the graded ring $M = \mathrm{Gr}_{\mathfrak{m}} A = \bigoplus_{d \geq 0} \mathfrak{m}^d/\mathfrak{m}^{d+1}$. By theorem 7.5 in chapter I, there exists a Hilbert polynomial ϕ_M of M such that $\phi_M(l) = \dim_k \mathfrak{m}^{l-1}/\mathfrak{m}^l$ for $l \gg 0$. Since A is a noetherian local ring, we have length $A/\mathfrak{m}^l = \dim \mathfrak{m}^{l-1}/\mathfrak{m}^l$. So we may just take $P_A(z) = \phi_M(l)$, then we have $P_A(l) = \phi_M(l) = \dim_k \mathfrak{m}^{l-1}/\mathfrak{m}^l = \text{length } A/\mathfrak{m}^l = \psi(l)$ for $l \gg 0$.

(b) Take the system of parameters of $\mathfrak{m}$ as $\{x_1, \ldots, x_n\}$. Then $\mathrm{Gr}_{\mathfrak{m}} A \cong (A/\mathfrak{m})[x_1, \ldots, x_n]$. Then $\deg P_A = \dim Z(\mathrm{Ann}\,(A/\mathfrak{m}[x_1, \ldots, x_n])$. Since $A/\mathfrak{m}$ is a field, we have $\mathrm{Ann}\,(A/\mathfrak{m}[x_1, \ldots, x_n]) = 0$, i.e .$\deg P_A = \dim Z(0) = \dim \mathbb{P}^n = n$. Moreover, since $\dim A$ equals to the length of system of parameters, i.e. $\dim A = n$. So $\deg P_A = \dim A$.

(c) **Every Chapter**.

(d) Take a small neighbourhood U or P on X. Denote $A = \mathscr{O}_{X,P}$ and $\mathfrak{m} = \mathfrak{m}_P$. Then we may assume C has local equation f on U. If C has multiplicity r on P, we have $f \in \mathfrak{m}^r$ but $f \notin \mathfrak{m}^{r+1}$. Since $\mathscr{O}_{C,P} = A/(f)$, if P has local parameter x, y on U such that f has a term x^r, for any $l > r$, we have a basis of $(A/(f))(l)$ as $x^i y^j$ for $0 \leq i < r$ and $0 \leq j \leq l - i$, i.e. $\dim(A/(f))(l) = lr - \frac{r^2}{2} + \frac{3r}{2}$. Then $P_A(l) = lr - \frac{r^2}{2} + \frac{3r}{2}$. Hence $\mu_P(C) = \mu(\mathscr{O}_{C,P}) = r$.

(e) Denote X as the cone of Y with vertex P. Then we may take a L^{n-r} passing through P meets X only at P. Then $\deg X = \sum_{x \in X \cap L} i(y, X \cap L) = i(P, X \cap L) = \mu_P(X)$. But $\deg X = \deg Y = d$, i.e. $\mu_P(X) = d$.

Solution 5.3.5. Since C has equation $z^{r-2}y^2 = x^r + b_1 x^{r-1} z + \ldots + b_r z^r$ for some coefficients b_i. Then at the infinity point $(0:1:0)$, the Jacobian is $(rx^{r-1} + \ldots + b_{r-1} z^{r-1}, -2yz^{r-2}, -(r-2)z^{r-3}y^2 + b_1 x^{r-1} + \ldots + b_r z^{r-1})|_\infty = (0,0,0)$. Hence ∞ is a singular point. Moreover, on the affine plane $(y \neq 0)$, we the equation is $z^{r-2} = x^r + b_1 x^{r-1} z + \ldots + b_r z^r$. Then clearly r_∞ =the lowest term= $r - 2$, so $\delta_\infty = \frac{1}{2} r_\infty (r_\infty - 1) = \frac{1}{2}(r-2)(r-3)$. And consider the morphism $\pi : \tilde{C} \to C \to \mathbb{P}^1$, $(x : y : z) \mapsto (x : z)$. Then clearly $\deg \pi = 2$, and all a_i's are branch points. If r is odd, the point $(0:1:0)$ has just one preimage in $\tilde{C}$, hence a branch point. But if r is ever, the point $(0:1:0)$ has two, so not a branch point. Then by Hurwitz's formula, $2g(\tilde{C}) - 2 = 2(2g(\mathbb{P}^1) - 2) + \deg R$, where $\deg R = r$ if r is even, and $\deg R = r + 1$ if r is odd. Then $g(\tilde{C}) = \frac{1}{2} r - 1$ if r is even, and $g(\tilde{C}) = \frac{1}{2}(r-1)$ if r is odd.

Solution 5.3.6. For any singular point P on curve C, we may denote $\tilde{C} = \mathrm{Bl}_P C$, and P has preimage $Q_1, \ldots, Q_r$. Then by definition of blow-up, we may take sufficiently small affine neighbourhood $U = \mathrm{Spec}\, A$ of P on C, such that the preimage of U, i.e. $\mathrm{Bl}_P U = \coprod_{i=1}^r V_i$, such that $V_i = \mathrm{Spec}\, B_i$ is a neighbourhood of Q_i. Hence we have $\mathrm{Bl}_{\mathfrak{p}} A = \bigoplus B_i$. So we may take a inverse limit for such A, and have $\mathrm{Bl}_{\mathfrak{m}_P} \mathscr{O}_{P,C} = \bigoplus \mathscr{O}_{Q_i, \tilde{C}}$. Since completion is flat, we have $\mathrm{Bl}_{\hat{\mathfrak{m}}_P} \hat{\mathscr{O}}_{P,C} = \bigoplus \hat{\mathscr{O}}_{Q_i, \tilde{C}}$. So the resolution is determined by the completion of local ring, i.e. analytically isomorphic implies equivalent.

Here is a counterexample of inverse direction. Consider two curves $C = (x^3 + y^7 = 0)$ and $C' = (x^3 + xy^5 + y^7 = 0)$ over $\mathbb{C}$. They are equivalent but not analytically isomorphic.

Solution 5.3.7. (a) $x^3 + y^5 = 0$. Take $x = x_1 y_1$ and $y = y_1$, we have $x_1^3 + y_1^2 = 0$ with exceptional curve $y_1 = 0$, and the singular point has multiplicity 2. Then take $x_1 = x_2$ and $y_1 = x_2 y_2$, we have $x_2 + y_2^2 = 0$ with exceptional curve $x_2 = 0$, and this is a smooth curve, intersecting $(x_2 = 0)$ at a double point. Hence we may take $x_2 = x_3 y_3$ and $y_2 = y_3$, we have $x_3 + y_3 = 0$ with exceptional curve $x_3 = 0$, which intersect $x_2 = 0$ and the curve at the same point. So we may take one more blow-up to get the normal crossing.

(b) $x^3 + x^4 + y^5 = 0$. Take $x = x_1 y_1$ and $y = y_1$, we have $x_1^3 + x_1^4 y_1 + y_1^2 = 0$ with exceptional curve $y_1 = 0$, and the singular point has multiplicity 2. Then take $x_1 = x_2$ and $y_1 = x_2 y_2$, we have $x_2 + x_2^3 y_2 + y_2^2 = 0$ with exceptional curve $x_2 = 0$, and this is a smooth curve, intersecting $(x_2 = 0)$ at a double point. Hence we may take $x_2 = x_3y_3$ and $y_2 = y_3$, we have $x_3 + x_3^3y_3^3 + y_3 = 0$ with exceptional curve $x_3 = 0$, which intersect $x_2 = 0$ and the curve at the same point. So we may take one more blow-up to get the normal crossing.

(c) $x^3 + y^4 + y^5 = 0$. Take $x = x_1y_1$ and $y = y_1$, we have $x_1^3 + y_1 + y_1^2 = 0$ with exceptional curve $y_1 = 0$, and this is a smooth curve, which intersect ($y_1 = 0$ at two distinct points. Hence we get the normal crossing.

(d) $x^3 + y^5 + y^6 = 0$. Take $x = x_1y_1$ and $y = y_1$, we have $x_1^3 + y_1^2 + y_1^3 = 0$ with exceptional curve $y_1 = 0$, and the singular point has multiplicity 2. Then take $x_1 = x_2$ and $y_1 = x_2y_2$, we have $x_2 + y_2^2 + x_2y_2^3 = 0$ with exceptional curve $x_2 = 0$, and this is a smooth curve, intersecting $(x_2 = 0)$ at a double point. Hence we may take $x_2 = x_3y_3$ and $y_2 = y_3$, we have $x_3 + y_3 + x_3y_3^3 = 0$ with exceptional curve $x_3 = 0$, which intersect $x_2 = 0$ and the curve at the same point. So we may take one more blow-up to get the normal crossing.

(e) $x^3 + xy^3 + y^5 = 0$. Take $x = x_1y_1$ and $y = y_1$, we have $x_1^3 + x_1y_1 + y_1^2 = 0$ with exceptional curve $y_1 = 0$, and the singular point has multiplicity 2. Then take $x_1 = x_2$ and $y_1 = x_2y_2$, we have $x_2 + y_2 + y_2^2 = 0$ with exceptional curve $x_2 = 0$, and this is a smooth curve, intersecting $(x_2 = 0)$ at two distinct points. Hence we get the normal crossing.

So (a), (b), (d) are equivalent to each other. And (c), (e) are not equivalent to any others.

Solution 5.3.8. (a) $x^4 - xy^4 = 0$. Take $x = x_1y_1$ and $y = y_1$, we have $x_1^4 - x_1y_1 = 0$, with exceptional curve $y_1 = 0$, and the singular point has multiplicity 2. Then take $x_1 = x_2$ and $y_1 = x_2y_2$, we have $x_2^2 - y_2 = 0$, with exceptional curve $x_2 = 0$, and this is a smooth curve, intersecting $(y_1 = 0)$ and $(x_2 = 0)$ at one point. Hence one more blow-up, we can get the normal crossing.

(b) $x^4 - x^2y^3 - x^2y^5 + y^8 = 0$. Take $x = x_1y_1$ and $y = y_1$, we have $x_1^4 - x_1^2y_1 - x_1^2y_1^3 + y_1^4 = 0$, which exceptional curve $y_1 = 0$, and the singular point has multiplicity 3. Then take $x_1 = x_2$ and $y_1 = x_2y_2$, we have $x_2 - y_2 - x_2^2y_2^3 + x_2y_2^4 = 0$, with exceptional curve $x_2 = 0$, and this curve is nonsingular at $(0, 0)$.

So on these two curves, $\delta_{(0,0)}$ are the same. But two curves are not equivalent.

5.4 The Cubic Surface in $\mathbb{P}^3$

Solution 5.4.1. By theorem 4.2., $\dim \mathfrak{d} = 3$. Then by theorem 4.1., $\mathfrak{d}'$ has no base points, so it determines a morphism $\psi : X' \to \mathbb{P}^3$. Changing the coordinates, we may assume $P_1 = (1, 0, 0)$ and $P_2 = (0, 1, 0)$. Then $V \subset H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2))$ corresponding to $\mathfrak{d}$ is spanned by $x_0x_1, x_1x_2, x_2x_0, x_2^2$, so this is the space of all conics passing through P_1 and P_2. Then we may assume $\psi(x_2^2) = y_0$, $\psi(x_0x_1) = y_1$, $\psi(x_1x_2) = y_2$ and $\psi(x_2x_0) = y_3$. So the $\mathrm{Im}(\psi)$ satisfies the equation $y_0y_1 = y_2y_3$, which is a quadric surface $Y = (xy = zw)$. Conversely, we clearly have $Y \subset \mathrm{Im}(\psi)$, hence equal.

Moreover, define $\pi : Y \to \mathbb{P}^2$ to be the projection $(x, y, z, w) \mapsto (x, z, w)$. So $\pi \circ \psi = \mathrm{id}_{\mathbb{P}^2}$, and $\psi \circ \pi = \mathrm{id}_Y$. Then we denote Γ as the graph of π. By definition, $\Gamma = \tilde{Y}$ is the blow-up of Y at Q. Since $(x_0x_1, x_1x_2, x_2x_0, x_2^2)$ generate an ideal with saturation the homogeneous ideal of $I(P_1, P_2)$, so Γ is also the blow-up of $\mathbb{P}^2$ at P_1 and P_2, i.e. $X' \cong \tilde{Y}$.

Solution 5.4.2. Changing coordinates, we may assume $P_1 = (1, 0, 0)$, $P_2 = (0, 1, 0)$ and $P_3 = (0, 0, 1)$. And we may assume C is defined by $f(x, y, z) = 0$ for some homogeneous polynomial f with $\deg f = d$. Since $\mu_{P_1}(C) = r_1$, we have $f(x, y, z) = f_{r_1}(y, z)x^{d-r_1} + \ldots + f_d(y, z)$ for some homogeneous polynomial $f_i(y, z)$ of degree i. So $f(yz, zx, xy) = f_{r_1}(zx, xy)(yz)^{d-r_1} + \ldots + f_d(zx, xy) = x^{r_1}\left((yz)^{d-r_1} f_{r_1}(z, y) + \ldots + x^{d-r_1} f_d(z, y)\right)$. Hence clearly C' has equation $f'(x, y, z) = \frac{f(yz,zx,xy)}{x^{r_1}y^{r_2}z^{r_3}}$. Clearly $\deg C' = \deg f' = 2d - r_1 - r_2 - r_3$. And for $Q_1 = (1, 0, 0)$, we have $y^{r_2}z^{r_3}f'(x, y, z) = (yz)^{d-r_1} f_{r_1}(z, y) + \ldots + x^{d-r_1} f_d(z, y)$. Hence $\mu_{C'}(Q_1) = d - r_2 - r_3$. And similarly for Q_2 and Q_3.

Solution 5.4.3. For any singular point P_1 on C, we may pick any two points P_2, P_3 on $\mathbb{P}^2$ such that P_1P_i intersects C at P_1 transitively. Then we blow up $\mathbb{P}^2$ at P_1, P_2, P_3 to get a $\tilde{X}$. And as the notation of example 4.2.3., we may assume $\tilde{C} \cap E_1 = \{R_1, \ldots, R_n\}$. By choosing of P_2 and P_3, R_i is not on $\tilde{L}_{12}$ nor $\tilde{L}_{13}$. Since

$\mu(R_i) < \mu(P_1)$, the image of quadratic transformation of the curve $C' \cong \psi(\tilde{C})$ splits P_1 into some point R_i, some of which are smooth, and some of which has multiplicity less than P_1. Since $\sum_{P\in C}\mu(P) - 1 < \deg C$ is finite, this process will stop when C' has only ordinary singularities.

Solution 5.4.4. (a) Since $(\overline{PP'} + \overline{QQ'} + \overline{RR'}).C = P + Q + R + P' + Q' + R' + P'' + Q'' + R''$, and $(L + L' + \overline{P''Q''})$ intersects C with $P, Q, R, P', Q', R', P'', Q''$, by corollary 4.5., $(L + L' + \overline{P''Q''}).C = (\overline{PP'} + \overline{QQ'} + \overline{RR'}).C$, i.e. $R'' \in \overline{P''Q''}$).

(b) If $\overline{PQ}$ meet C at R, $\overline{P_0R}$ meet C at T, $\overline{ST}$ meet C at U, $\overline{P_0U}$ meet C at V, by definition we have $P + Q = T$ and $T + S = V$. If $\overline{SQ}$ meet C at B, $\overline{PU}$ meet C at A, consider the line $L = \overline{QPR}$ and $L' = \overline{SUT}$, then by (a), B, A, P_0 are collinear. Hence by definition $S + Q = A$, and $P + A = V$. So we have the associativity $(P + Q) + S = P + (Q + S)$.

Solution 5.4.5. Denote the conic as S. Since $(\overline{AB'} + \overline{BC'} + \overline{CA'}).(\overline{A'B} + \overline{B'C} + \overline{C'A}) = A + B + C + A' + B' + C' + P + Q + R$, and $(S + \overline{PQ})$ meets $(\overline{AB'} + \overline{BC'} + \overline{CA'})$ at $A, B, C, A', B', C', P, Q$, by corollary 4.5., $(S + \overline{PQ}).(\overline{AB'} + \overline{BC'} + \overline{CA'}) = A + B + C + A' + B' + C' + P + Q + R$, i.e. P, Q, R are collinear.

Solution 5.4.6. Since $H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2)) = 15$, the complete linear system $|D|$ of all quartic curves has dimension 14. For any $P_i = (x_i, y_i, z_i)$, we define a vector $v_i = (x_i^4, y_i^4, z_i^4, x_i^3y_i, x_iy_i^3, y_i^3z_i, y_iz_i^3, z_i^3x_i, z_ix_i^3, x_i^2y_i^2, y_i^2z_i^2, z_i^2x_i^2, x_i^2y_iz_i, y_i^2z_ix_i, z_i^2x_iy_i)$. Then if the matrix $(v_1, \ldots, v_{13})$ has rank 13, clearly $\dim|D - P_1 - \ldots - P_{13}| = 1$. So if $C, C' \in |D - P_1 - \ldots - P_{13}|$, C meets C' at $P_1, \ldots, P_{13}$ and three more points P_{14}, P_{15}, P_{16}. For any $C'' \in |D - P_1 - \ldots - P_{13}|$, we must have $C'' = \lambda C + \mu C'$, hence C'' pass through P_{14}, P_{15}, P_{16}.

Solution 5.4.7. If $D = al - \sum b_ia_i$, then $d = 3a - \sum b_i$. Then $p_a(D) = \frac{1}{2}(a-1)(a-2) - \frac{1}{2}\sum b_i^2 + \frac{1}{2}\sum b_i = \frac{1}{18}(d - 3 + \sum b_i)(d - 6 + \sum b_i) - \frac{1}{2}(\sum b_i^2) + \frac{1}{2}\sum b_i = \frac{1}{18}(d-3)(d-6) + \frac{1}{9}d\cdot\sum b_i + \frac{1}{18}(\sum b_i)^2 - \frac{1}{2}(\sum b_i^2) \le \frac{1}{18}d^2 - \frac{1}{2}d + 1 - \left((\frac{1}{6}\sum b_i)^2 - \frac{1}{9}d\cdot\sum b_i + (\frac{1}{3}d)^2\right) + \frac{1}{9}d^2 \le \frac{1}{6}d^2 - \frac{1}{2}d + 1 = \frac{1}{6}(d-1)(d-2) + \frac{2}{3}$, where the first inequality is the Schwartz's inequality, and the second is $(\frac{1}{6}\sum b_i)^2 - \frac{1}{9}d\cdot\sum b_i + (\frac{1}{3}d)^2 = (\frac{1}{6}\sum b_i - \frac{1}{3}d)^2 \ge 0$. So if $d \equiv 1, 2 \mod 3$, since $p_a(D)$ is an integer, we have $p_a(D) \le \frac{1}{6}(d-1)(d-2)$.

For achievement, when $d \equiv 0 \mod 3$, we just take $D = dl + \frac{d}{3}(\sum b_i)$, then $d = 3d - 6 \times \frac{d}{3} = d$, and $p_a(D) = \frac{1}{2}(d-1)(d-2) - 3\times\frac{d^2}{9} + 3\times\frac{d}{3} = \frac{1}{6}(d-1)(d-2) + \frac{2}{3}$. When $d \equiv 1 \mod 3$, we may assume $d = 3k+1$, then clearly the divisor $D = dl - k(e_1 + e_2 + e_3 + e_4) - (k+1)(e_5 + e_6)$ achieve the bound. When $d \equiv 2 \mod 3$, we may assume $d = 3k + 2$, then clearly the divisor $D = dl - k(e_1 + e_2) - (k+1)(e_5 + e_6 + e_3 + e_4)$ achieve the bound.

Solution 5.4.8. We firstly consider the surface X_r forming by blow-up r points $P_1, \ldots, P_r$ on $\mathbb{P}^2$ with $2 \le r \le 5$. Then similarly to proposition 4.8., we have Pic $X_r = \mathbb{Z}^{r+1}$ generated by $l, e_1, \ldots, e_r$ with $l^2 = 1$, $e_i^2 = -1$, $l.e_i = 0$ for any i, and $e_i.e_j = 0$ for any $i \ne j$, where l is the pullback of a line on $\mathbb{P}^2$ to X_r. Then also, similarly to theorem 4.9., if $r < 5$, then X_r contains exactly lines E_i and F_{ij}, where E_i is the exception line of P_i, and F_{ij} is the strict line of P_iP_j on X_r. Moreover, $E_i = e_i$, $F_{ij} = l - e_i - e_j$. And if $r = 5$, X_r contains exactly lines E_i, F_{ij} and G, where G is the strict curve of the conic passing through $P_1, \ldots, P_5$ on $\mathbb{P}^2$. And $G = 2l - e_1 - \ldots - e_5$. Similar to lemma 4.12., we may define $D_0 = l$, $D_1 = l - e_1$, $D_2 = 2l - e_1 - e_2$, $D_3 = 2l - e_1 - e_2 - e_3$ if $r \ge 3$, $D_4 = 2l - e_1 - e_2 - e_3 - e_4$ if $r \ge 4$, and $D_5 = 3l - e_1 - e_2 - e_3 - e_4 - e_5$ if $r \ge 5$. Then $|D_0|$, $|D_1|$ correspond to the linear systems of lines in $\mathbb{P}^2$ passing through or not passing through P_1, hence very ample. And by proposition 4.1., D_2, D_3, D_4 are very ample on X_r if r is big enough, by proposition 4.3., D_5 is very ample on X_5. So on X_r, $\sum_{i=0}^r n_iD_i$ is very ample, i.e. all divisor $D = al - \sum b_ie_i$ satisfying $b_1 \ge \ldots \ge b_5 > 0$, and $a > b_1 + b_2 + b_5$ if $r = 5$, or $a > b_1 + b_2$ if $r < 5$, is very ample on X_r. Conversely, as the proof of theorem 4.13., if $D = al + \sum b_ie_i$ is very ample, we have $a > b_1 + b_2$ when $r < 5$ or $a > b_1 + b_2 + b_5$ when $r = 5$, where $b_1 \ge \ldots \ge b_5$ is a sorting. In the language of intersecting, D is very ample iff $D.L > 0$ for all line L on X_r. Moreover, as Bertini's theorem, if $D.L > 0$ for all line L, there exists an irreducible nonsingular curve $C \subset X_r$ such that $C \sim D$. (*)

Back to the question.

(i)⇒(iii). If D contains the line E_i for some i, this is the condition (a) and we've done. If D does not contain and E_i, and contains a curve C, then for any X_r for $0 \leq r \leq 5$ (here we assume $X_0 = \mathbb{P}^2$) and $\pi_r : X \to X_r$, $\pi_r(C)$ is a curve on X_r. If $D = al + \sum b_i e_i$, we may sort i as $b_1 \geq \ldots \geq b_r > 0 = b_{r+1} = \ldots = b_6$. In the case $r = 0$, $D = al$, then clearly $D.L > 0$. In the case $r = 1$, we have $D = al - b_1 e_1$. Then we may consider the curve $D' = \pi_1(D) \in X_1$. By example 2.11.5., X_1 is a ruled surface over $\mathbb{P}^1$ with $e = 1$, and clearly $D' \sim_{\mathrm{Num}} (a - b_1)C_0 + af$. Then by corollary 2.18., we have $a - b_1 = 1$ and $a = 0$, or $a = 1$ and $a - b_1 = 0$, or $a - b_1 > 0$ and $a \geq a - b_1$. Hence $a = 0, b_1 = -1$, which is just E_1, or $a = 1, b_1 = 1$, which is a conic, or $a > b_1 \geq 0$, which is the condition (c). In the case $r \geq 2$, we may consider $\pi_r : X \to X_r$, $\pi_r^*(D) = al + \sum_{i=1}^{r} b_i e_i$. Then $D.L > 0$ for all line $L \subset X_r$. Hence clearly $D.L \geq 0$ for all line $L \subset X$, which is the condition (c).

(ii)⇒(i). Trivial.

(iii)⇒(ii). The condition (a) and (b) are trivial. In the condition (c), we may assume $D = al - \sum b_i e_i$. If $D.L \geq 0$ for all line L, we have $D.E_i = b_i \geq 0$. Then rearranging P_i, we may assume $b_1 \geq \ldots \geq b_r > 0 = b_{r+1} \geq \ldots$. In the case $r = 0$, $D = al$, then by Bertini's theorem, there exists nonsingular irreducible curve C of degree a in $\mathbb{P}^2$, hence $\pi^*(C)$ is an irreducible nonsingular curve on X. In the case $r = 1$, consider $D' = \pi_1^*(D)$ as a divisor on X_1. Then by corollary 2.18., there exists an irreducible nonsingular curve C on X_1. Then $\pi_1^*(C)$ is an irreducible nonsingular curve on X. In the case $r \geq 2$, consider $D' = \pi_r^*(D)$ similarly, then by (*) or corollary 4.11., there exists an irreducible nonsingular curve C on X_r, then $\pi_r^*(C)$ is an irreducible nonsingular curve on X.

Solution 5.4.9. We may assume $b_1 \geq \ldots \geq b_6$. Then by 5.4.8., we have $b_6 \geq 0$, $a - b_1 - b_2 \geq 0$ and $2a - b_1 - \ldots - b_5 \geq 0$. So adding these conditions and $a^2 - \sum b_i^2 \geq 3a - \sum b_i$ and $d = 3a - \sum b_i \geq 8$ if d is even, or $d = \geq 13$ if d is odd, the question is equivalent to the inequality $a^2 - 6a + 14 \geq \sum (b_i - 1)^2$ for such $a, b_1, \ldots, b_6$, which likes a crazy IMO question.

So we may fix $a \geq 0$ and $b = \sum b_i \geq 0$, i.e. fix the value of d, and find the minimal value of g. Then $\sum b_i^2 = b^2 - 2b(b_2 + \ldots + b_6) + (b_2^2 + \ldots + b_6^2) + (b2 + \ldots + b_6)^2$. So for the minimal of $\sum b_i^2$, we must take $(b_2^2 + \ldots + b_6^2) + (b2 + \ldots + b_6)^2$ to be small enough. If we make it be zero, we have $g = \frac{1}{2}(a(a-3) - b(b-1))$. So for the minimal value of g, we must take b to be big enough. So we may assume $b = a$ by the condition, and we have $g = 1 - a < 1$, which makes a contradiction. So we cannot make the value $(b_2^2 + \ldots + b_6^2) + (b2 + \ldots + b_6)^2$ to be zero. Then if we make $b_2 = 1$ and $b_1 = b - 1$, we have $g = \frac{1}{2}(a(a-3) - b(b-1))$. For b big enough, we may assume $b = a$, and $g = 0$, i.e. we cannot take $b = a$. If $b = a - 1$, we have $d = 2a + 1$, and $g = a - 2 = \frac{1}{2}(d - 5)$. This is the minimal value of g in the case d is an odd number. We try to amplify the value $(b_2^2 + \ldots + b_6^2) + (b2 + \ldots + b_6)^2$ a little, i.e. we may assume $b_2 = 2$, and $b_1 = b - 2$, then $g = \frac{1}{2}(a(a-3) - b(b-5) - 6)$. So we make $b = a$ to be big enough, and we have $d = 2a$, $g = a - 3 = \frac{1}{2}(d - 6)$, which is the limit of the bound in the case d is an even number. Clearly calculating, this two value is the minimal bound of g when $g > 1$.

Solution 5.4.10. We may assume $b_1 \geq b_2 \geq \ldots \geq b_6 > 0$. Then the condition is reduced to $a - b_1 - b_2 > 0$ and $2a - b_1 - \ldots - b_5 > 0$. Then we split this question in two cases very freshly:

Case I. $b_1 \geq b_2 + b_5$. In this case, we have $2b_1 b_2 \geq 2b_2^2 + 2b_2 b_5 \geq b_3^2 + b_4^2 + b_5^2 + b_6^2$. Hence by $a \geq b_1 + b_2$, we have $a^2 > b_1^2 + b_2^2 + 2b_1 b_2 \geq \sum b_i^2$.

Case II. $b_1 < b_2 + b_5$. In this case, $a > b_1 + b_2 + b_3 + b_4 + b_5 > 2b_1 + b_3 + b_4$. Hence $a^2 > 4b_1^2 + b_3^2 + b_4^2 + 4b_1 b_3 + 4b_1 b_4 + 2b_3 b_4$. Since $b_1^2 \geq b_i^2$ for $i \geq 2$, we have $a^2 > \sum b_i^2$.

Solution 5.4.11 (The Weyl Groups)**.** (a) Define the morphism $\phi : \mathbb{A}_n \to \Sigma_n$ as $x_i \mapsto (i, i+1)$. Then clearly $\phi(x_i)^2 = (i, i+1)^2 = 1$. If $i < j - 1$, we have $\phi(x_i x_j)^2 = ((i, i+1)(j, j+1))^2 = 1$. And $\phi(x_i x_{i+1})^2 = ((i, i+1)(i+1, i+2))^3 = (i, i+2, i+1)^3 = 1$. Hence ϕ is a group morphism. And since Σ_n is generated by $(12), (23), \ldots, (n-1, n)$, this morphism is surjective. Moreover, we may define $\ell : \mathbb{A}_n \to \mathbb{Z}$ as following:

for any element $a \in \mathbb{A}_n$, a clearly can be written as the form $x_{i_1}x_{i_2}\ldots x_{i_k}$ for some finite k, then we define $\ell(a)$ as the minimal value of such k. Consider $\mathbb{A}_{n-1}$ as a subgroup of $\mathbb{A}_n$ generated by $x_1,\ldots,x_{n-2}$. Then for any $x \in \mathbb{A}_n\mathbb{A}_{n-1}$, we may define $p = \min\{\ell(a) : a \in \mathbb{A}_{n-2}x\}$. By definition of $\mathbb{A}_n$, we know $x_{n-1}\ldots x_{n-p} \in \mathbb{A}_n x$. So $\mathbb{A}_n = \mathbb{A}_{n-1} \cup \mathbb{A}_{n-1}x_{n-1} \cup \mathbb{A}_{n-1}x_{n-1}x_{n-2} \cup \ldots \cup \mathbb{A}_{n-1}x_{n-1}\ldots x_1$. Hence $|\mathbb{A}_n| = n\cdot|\mathbb{A}_{n-1}|$. Since $\mathbb{A}_1 \cong \mathbb{Z}/2\mathbb{Z}$, i.e. $|\mathbb{A}_2| = 2$. So by induction, $|\mathbb{A}_n| = n!$. Since $|\Sigma_n| = n!$, we know ϕ is an isomorphism, i.e. $\mathbb{A}_n \cong \Sigma_n$.

(b) Define the morphism $\phi : \mathbb{E}_6 \to G$ as in the theorem. Then since the the square of quadratic transformation is the identity, we have $\phi(y)^2 = 1$. And since quadratic transformation is symmetric for E_1, E_2, E_3 and E_4, E_5, E_6, we have $\phi(x_i)\phi(y) = \phi(y)\phi(x_i)$, i.e. $(\phi(x_i)\phi(y))^2 = 1$. And by calculation, $y(E_1) = F_{23}$, $y(E_2) = F_{13}$, $y(E_3) = F_{12}$, $y(E_i) = E_i$ for $4 \le i \le 6$, $y(F_{ij}) = F_{ij}$ if $1 \le i \le 3$ and $4 \le j \le 6$, $y(F_{ij}) = G_k$, if $\{i,j,k\} = \{4,5,6\}$, and $y(G_i) = G_i$ for $1 \le i \le 3$. So we clearly have $(\phi(x_3)\phi(y))^3 = 1$. Hence ϕ is a morphism. Moreover, as in the proof of proposition 4.10., ϕ is a surjection.

(c) As in remark 4.10.1., $|G| = 51,840$. Consider the subgroup $\mathbb{E}_5$ generated by x_1, x_2, x_3, x_4, y, and the subgroup $\mathbb{A}_5$ generated by x_1, x_2, x_3, y. By definition and calculate by Mathematica, we have $|\mathbb{E}_6/\mathbb{E}_5| = 27$, and $|\mathbb{E}_5/\mathbb{A}_5| = 16$, and by (a), we have $|\mathbb{A}_5| = 5! = 120$. Hence we have $|\mathbb{E}_6| = 120\times16\times27 = 51,840$. Hence ϕ is an isomorphism.

Solution 5.4.12. Clearly we have $H^0(X,\mathscr{O}_X) = k$, $H^1(X,\mathscr{O}_X) = 0$. Then for any ample divisor D, it is very ample by theorem 4.11., we have $H^0(X,\mathscr{L}(D)) = k$. So we may consider the exact sequence $0 \to \mathscr{L}(-D) \to \mathscr{O}_X \to \mathscr{L}(D) \to 0$, i.e. $0 \to H^0(X,\mathscr{L}(-D)) \to k \to k \to H^1(X,\mathscr{L}(-D)) \to 0$. So we only need to prove $H^0(X,\mathscr{L}(-D)) = 0$. If not, there must exist $s \in H^0(X,\mathscr{L}(-D))$ such that $C = \operatorname{Supp} s$, and there exists at least a line E such that $C.E \ge 0$, hence $D.E \le 0$, which contradicts with theorem 4.11.

Solution 5.4.13. (a) As we've done in 5.4.8., the 16 lines contained in X are E_i, $1 \le i \le 5$, which are the exceptional curve, and F_{ij}, $1 \le i < j \le 5$, which are the preimage of line $\overline{P_iP_j}$, and G, which are the preimage of the conic passing through all P_i's.

(b) Similarly to theorem 4.7., we have $\mathscr{O}_X(1) = \omega_X^{-1}$, and X is embedded in $\mathbb{P}^4$ by a linear system of cubics passing through all P_i. So $\mathscr{L}(-2K) \cong \mathscr{O}_X(2)$. Then we may consider the exact sequence $0 \to \mathscr{I}_X(2) \to \mathscr{O}_{\mathbb{P}^4}(2) \to \mathscr{O}_X(2) \to 0$, so we have $0 \to H^0(\mathbb{P}^4,\mathscr{I}_X(2)) \to H^0(\mathbb{P}^4,\mathscr{O}_{\mathbb{P}^4}(2)) \to H^0(X,\mathscr{O}_X(2)) \to 0$. Since $h^0(\mathscr{O}_{\mathbb{P}^4}(2)) = 15$. And by Riemann-Roch on X, $h^0(-2K) - h^1(-2K) = \frac{1}{2}(-2K-K).(-2K) + 1 = 3K^2 + 1 = 3\times(3^2-5)+1 = 13$. Then since $\mathscr{L}(-2K) \cong \mathscr{O}_X(2)$, we have $H^1(X,\mathscr{L}(-2K)) = 0$. So $h^0(\mathscr{I}_X(2)) = 2$, i.e. X is a complete intersection of two quadric hypersurfaces in $\mathbb{P}^4$.

Solution 5.4.14. As following.

a	b_1	b_2	b_3	b_4	b_5	b_6	d	g
8	3	3	3	3	2	2	8	7
8	4	3	3	2	2	2	8	6
9	4	3	3	3	3	2	9	9
9	4	4	3	3	2	2	9	8
9	4	4	4	2	2	2	9	7
10	5	3	3	3	3	3	10	11
11	5	5	4	3	3	3	10	10
12	5	5	5	5	3	3	10	9
10	6	3	3	3	3	2	10	8

Solution 5.4.15. (a) ($\Rightarrow$) If $P_1,\ldots,P_6$ are in general position, clearly any three of them are not collinear. And we may assume $P_1 = (1,0,0)$, $P_2 = (0,1,0)$ and $P_3 = (0,0,1)$. If $P_1,\ldots,P_6$ all lie on a conic, we may assume

this conic is $axy + byz + czx = 0$. Then if $P_i = (x_i, y_i, z_i)$ for $i = 4, 5, 6$, the image of P_i is $(y_i z_i, z_i x_i, x_i y_i)$ for $i = 4, 5, 6$. They are lying on the line $bx + cy + ax = 0$, which contradicts to the definition of general position.

($\Leftarrow$) If $P_1, \ldots, P_6$ are not collinear and not lying on a conic, we may assume $P_1 = (1, 0, 0)$, $P_2 = (0, 1, 0)$ and $P_3 = (0, 0, 1)$ also. Then just de a quadratic transformation for P_1, P_2, P_3, if the image of P_4, P_5, P_6 are lying on a line $ax + by + cz = 0$, similarly $P_1, \ldots, P_6$ are lying on a conic $cxy + ayz + bzx = 0$, which contradicts.

(b) Trivial, because if new $P'_1, \ldots, P'_r$ are no in general position, three of points P', Q', R' will be collinear in furthermore a finite sequence of admissible transformation, i.e. P, Q, R will be collinear in a finite sequence of admissible transformation.

(c) Consider a variety T as the union of all line $\overline{P_i P_j}$ and the conic $P_{i_1} P_{i_2} P_{i_3} P_{i_4} P_{i_5}$. We have $\dim T = 1$. Since k is uncountable, $V = \mathbb{P}^2 - T$ is a open dense subset of $\mathbb{P}^2$. Then by (a), any point $P_{r+1} \subset V$ and $P_1, \ldots, P_{r+1}$ are in general position.

(d) Similar to 5.4.8., we may assume $C = al - \sum_{i=1}^r b_i e_i$. In the case of $r = 7$, by Schwartz's inequality, $g = \frac{1}{2}(a-1)(a-2) - \sum b_i(b_i - 1) = 0$ and $C^2 = a^2 - \sum b_i^2 = -1$ has solution $(a, b_1, \ldots, b_7) = (0, 1, 0, 0, 0, 0, 0, 0)$ and rest 6, or $(2, 1, 1, 0, 0, 0, 0, 0)$ and rest 20, or $(2, 1, 1, 1, 1, 1, 0, 0)$ and rest 20, or $(3, 2, 1, 1, 1, 1, 1, 1)$ and rest 6. The total number$= 7 + 21 + 21 + 7 = 56$. In the case of $r = 8$, the solution are like $(0, 1, 0, 0, 0, 0, 0, 0, 0)$ for 8, and $(2, 1, 1, 0, 0, 0, 0, 0, 0)$ for 28, and $(2, 1, 1, 1, 1, 1, 0, 0, 0)$ for 56, and $(3, 2, 1, 1, 1, 1, 1, 1, 0)$ for 56, and $(4, 2, 2, 2, 1, 1, 1, 1, 1)$ for 56, and $(5, 2, 2, 2, 2, 2, 2, 1, 1)$ for 28, and $(6, 3, 2, 2, 2, 2, 2, 2, 2)$ for 8. And the total number$= 8 + 28 + 56 + 56 + 56 + 28 + 8 = 240$.

(e) In the case $r > 9$, clearly $C = e_1$ is a line on X. Moreover, for any line $C = al - \sum b_i e_i$, we have $d = 1$, $g = 0$, $C^2 = -1$. Then for admissible transformation σ about P_1, P_2, P_3, $\sigma(C) = (a+c)l - c(e_1 + e_2 + e_3) - \sum b_i e_i$, where $c = a - b_1 - b_2 - b_3$. Suppose $b_1 \le \ldots \le b_r$, or we just need to rearrange the index. Then since $d = 3a - \sum b_i \ge 1 > 0$, we have $3a > \sum b_i \ge 3(b_1 + b_2 + b_3)$, i.e. $a > b_1 + b_2 + b_3$, then $\sigma(C).l = a + (a - b_1 - b_2 - b_3) > a = C.l$. So there exists infinitely many line C.

Solution 5.4.16. And line L in $\mathbb{P}^3$ has the form $a_0 x_2 + b_0 x_3 - x_0 = a_1 x_2 + b_1 x_3 - x_1 = 0$. So if $L \subset X$ the Fermat cubic curve, we have $(a_0 x_2 + b_0 x_3)^3 + (a_1 x_2 + b_1 x_3)^3 + x_2^3 + x_3^3 = 0$, i.e. $a_0^3 + a_1^3 = -1$, $b_0^3 + b_1^3 = -1$, $a_0^2 b_0 + a_1^2 b_1 = 0$, $a_0 b_0^2 + a_1 b_1^2 = 0$. Suppose $a_0 = 0$, we have $b_1 = 0$, $b_0^3 = a_1^3 = -1$. Denote $\omega = \exp(\frac{2\pi i}{3})$, then the 27 lines on X are $x_0 + x_3\omega^k = x_1 + x_2\omega^j = 0$ for $0 \le j, k \le 2$, and $x_0 + x_2\omega^k = x_3 + x_1\omega^j = 0$ for $0 \le j, k \le 2$, and $x_0 + x_1\omega^k = x_3 + x_2\omega^j = 0$ for $0 \le j, k \le 2$.

5.5 Birational Transformations

Solution 5.5.1. Denote $(f) = \sum n_i C_i$. And consider the curve $Y = \bigcup C_i$, and the embedded resolution as theorem 3.9. $\pi : X' \to X$ such that $f^{-1}(Y)$ is normal crossing. So for any C_i, C_j such that $n_i n_j < 0$, if $C_i \cap C_j = \sum P_k$ on X' for some finitely many points P_k, we may blow-up all P_k on X' to separate C_i and C_j. Since (f) is a finite sum, we just need a finite sequence of blow-up on X' to separate zeros and poles at X''. Hence any point on X'', it cannot be a zero and a pole of f at same time, i.e. we resolve the singularities of f.

Solution 5.5.2. Denote $m = -Y^2$. By theorem 5.2. in chapter III, there exists a very ample divisor H on X such that $H^1(X, \mathscr{L}(H)) = 0$. Denote $k = H.Y$, and we may assume $k \ge 2$ or just change H to $2H$. Then we may prove $H^1(X, \mathscr{L}(mH + iY)) = 0$ for $i = 0, \ldots, k$. For $i = 0$, trivial. Then for $i \ge 1$, we may consider the exact sequence $0 \to \mathscr{L}(mH + (i-1)Y) \to \mathscr{L}(mH + iY) \to \mathscr{O}_Y \otimes \mathscr{L}(mH + iY) \to 0$. Since $Y \cong \mathbb{P}^1$, we have $(mH + iY).Y = m(k-i)$. So $\mathscr{O}_Y \otimes \mathscr{L}(mH + iY) \cong \mathscr{O}_{\mathbb{P}^1}(m(k-i))$. So $H^1(X, \mathscr{L}(mH + (i-1)Y)) \to H^1(X, \mathscr{L}(mH + iY)) \to H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(m(k-i)) \to 0$. So by induction, we have $H^1(X, \mathscr{L}(H + iY)) = 0$.

Denote $\mathscr{M} = \mathscr{L}(mH + kY)$. Since H is very ample, $|mH + kY|$ has no base points out of Y. And on Y, we have $\mathscr{M} \otimes \mathscr{I}_Y \cong \mathscr{L}(mH + (k-1)Y)$. Since $H^1(X, \mathscr{L}(mH + (k-1)Y) = 0$, and $(mH + kY).Y = mk - km = 0$, we have $\mathscr{M} \otimes \mathscr{O}_Y \cong \mathscr{O}_{\mathbb{P}^1}$, which is generated by global section i. So lift the 1 to $H^0(X, \mathscr{M})$ and use the Nakayama lemma, we know $\mathscr{M}$ is generated by global section on all X.

So $\mathscr{M}$ defines a morphism $f_1 : X \to \mathbb{P}^N$ with image X_1. Since $f_1^*\mathscr{O}(1) \cong \mathscr{M}$ by theorem 11.7. in chapter III, and $\deg(\mathscr{M} \otimes \mathscr{O}_Y) = \deg \mathscr{O}_{\mathbb{P}^1} = 0$, we know $f_1(Y) = P_1$ for some point P_1. And the rest are all same with the proof of Castelnuovo's theorem.

Solution 5.5.3. By 3.8.1. and 3.8.3., we have $H^i(\tilde{X}, \pi^*\Omega_X) \cong H^i(X, \pi_*\pi^*\Omega_X) \cong H^i(X, \Omega_X)$. Consider the exact sequence $0 \to \pi^*\Omega_X \to \Omega_{\tilde{X}} \to \Omega_{\tilde{X}/X} \to 0$. Sonce $\Omega_{\tilde{X}/X}$ is supported on ramified points, we have $\Omega_{\tilde{X}/X} \cong \Omega_E$. Then $0 \to H^0(X, \Omega_X) \to H^0(\tilde{X}, \Omega_{\tilde{X}}) \to H^0(E, \Omega_E) \to H^1(X, \Omega_X) \to H^1(\tilde{X}, \Omega_{\tilde{X}}) \to H^1(E, \Omega_E) \to H^2(X, \Omega_X) \to H^2(\tilde{X}, \Omega_{\tilde{X}}) \to 0$ since E has dimension 1. Since $H^0(E, \Omega_E) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2)) = 0$, we have $H^1(E, \Omega_E) \cong H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(2) = k$. Moreover, by Serre's duality, we have $h^2(\Omega_X) = h^1(\Omega_X^2) = h^1(\mathscr{O}_X)$. Since $h^1(\mathscr{O}_X)$ is a birational invariant, we know $H^2(X, \Omega_X) \to H^2(\tilde{X}, \Omega_{\tilde{X}})$ is an isomorphism. Hence the exact sequence is $0 \to H^1(X, \Omega_X) \to H^1(\tilde{X}, \Omega_{\tilde{X}}) \to k \to 0$. Since k is a field, we have $H^1(\tilde{X}, \Omega_{\tilde{X}}) \cong H^1(X, \Omega_X) \oplus k$.

Solution 5.5.4. (a) Take a very ample divisor H on X'. Then f^*H is ample on X. Since $f^*H.Y = H.f_*Y = H.0 = 0$. Then by Hodge index theorem, we have $Y^2 < 0$.

(b) By (a), $Y_i^2 < 0$. Moreover, we may take further quadratic transformation if necessary and assume all Y_i are nonsingular, and if $i \neq j$, $Y_i \cap Y_j \neq \varnothing$, then $Y_i \cap Y_j$ is just one point. Take H_1' as a hyperplane section in X' passing through P and H_2' a hyperplane section in X' not passing through P. And suppose $(f) = H_1' - H_2'$. Take H_i as the strict transform of H_i' on X. Then we have $H_2 \sim H_1 + \sum m_i Y_i$ for some $m_i = \mathrm{ord}_{Y_i}(f) > 0$. So if $S = (Y_i.Y_j)_{i,j}$, and $S' = (m_iY_i.m_jY_j)_{i,j}$, we have $S' = MSM$ for diagonal matrix $M = \mathrm{diag}\{m_1, \ldots, m_r\}$. Hence $S_{ij} \geq 0$ if $i \neq j$. And for any j we have $\sum_i S'_{ij} = \sum_i (m_iY_i.m_jY_j) = -H_1.m_jY_j \leq 0$. So S is negative indefiniteness. Moreover, since H_1 passes through some Y_j, we have $\sum_i S'_{ij} < 0$ for this j. So if for some real vector $v = (v_1, \ldots, v_r)^T$, we have $0 = \sum_{i,j} v_iv_jS'_{ij} = \sum v_i^2 S'_{ii} + 2\sum_{i<j} v_iv_jS'_{ij} = \sum_j (\sum_i S'_{ij})v_j^2 - \sum_{i<j} S'_{ij}(v_i - v_j)^2$. Since v is real, we have $v_j = 0$ for some j. Since $\bigcup Y_i$ is connected, clearly we cannot split $(1, \ldots, r)$ into $(i_1, \ldots, i_k) \cap (j_1, \ldots, j_{r-k})$ such that $S'_{i_aj_b} = 0$ for any a, b. So the vector v must have $v_i = v_j$ for all i, i.e. $v = 0$, hence S' is negative definite, and so is S.

Solution 5.5.5. For any $P \in X$, consider the ruled surface $\mathrm{elm}_P X$. For any two curves $C \sim aC_0 + bf$ and $D \sim cC_0 + df$, we have $C'.D' = C.D + ac - a \cdot \mathrm{mult}_P D - c \cdot \mathrm{mult}_P C$. So if $a = c = 1$ and $P \in C \cap D$, we have $C'.D' = C.D - 1$, and if $P \notin C \cup D$, we have $C'.D' = C.D + 1$. So for the ruled surface X with e, we pick a point $P \in C_0$, then by above we know the new ruled surface $\mathrm{elm}_P X$ has $\tilde{e} = e + 1$. Then we blow-up so many times, we may assume X and X' has $e, e' > 2g - 2$. Hence by theorem 2.12., we may assume $\mathscr{E}$ and $\mathscr{E}'$ are decomposable with $\mathscr{E} = \mathscr{O}_C \oplus \mathscr{L}$ and $\mathscr{E}' = \mathscr{O}_C \oplus \mathscr{L}'$. If $\mathscr{L} = \mathscr{L}(\mathfrak{e})$, for any $P \in C_0$, by definition, $\mathrm{elm}_P X$ has $\tilde{\mathfrak{e}} = \mathfrak{e} + P$. So we may blow-up X and X' more and assume $\mathfrak{e} = \mathfrak{e}'$. Hence $X \cong X'$. So there is a finite sequence of elementary transformations which transform X into X'.

Solution 5.5.6. If R is a valuation ring with valuation v, since X is projective, hence proper, the valuation criterion give us a morphism $\phi : \mathrm{Spec}\, R \to X$. For any closed point $\mathfrak{m}$ of $\mathrm{Spec}\, R$, $\phi(\mathfrak{m})$ is the center of v. If $P \in X$ is the center, then R dominates $\mathscr{O}_P$. So if v is nontrivial, the dimension of v, d, corresponds to $\mathrm{tr.d.}R/\mathfrak{m}_R$, which corresponds to the dimension of the center. Since X is a surface, $d = 0, 1$. If $d = 1$, R and $\mathscr{O}_P$ are discrete valuation rings. Since R dominates $\mathscr{O}_P$, $R = \mathscr{O}_P$, this is the type (1). If $d = 0$, then we may blow-up P and get an X'. So the center of v in X' is either the exceptional divisor or a point contained in the exceptional divisor. For the first case, R is the type of (2). For the second case, we just repeat this process. If in finite repeats, we get a sequence of birational morphism $X_r \to X_{r-1} \to \ldots \to X_1 \to X$, and a sequence of point $P_i \in X_i$ for $i < r$ such that P_i is in the exceptional divisor blown-up from P_{i-1}, and R corresponds to the exceptional divisor in X_r, then R is the type of (2). If we cannot get the result in finite process, then we have an infinite sequence $X \leftarrow X_1 \leftarrow \ldots$ and $(P, P_1, \ldots)$, such that P_i lies on the exceptional divisor blown-up from P_{i-1}. Then $R = \varinjlim \mathscr{O}_{P_i, X_i} = \bigcup \mathscr{O}_{P_i, X_i}$, which is the type of (3).

Solution 5.5.7. Take the very ample divisor class $|H|$ on X_0. Let $H_0 \in |H|$ be a divisor containing P, and $H_1 \in |H|$

be a divisor not containing P. Then if $\tilde{H}_0$ and $\tilde{H}_1$ are strict transform of H_0 and H_1 on X, then $\tilde{H}_1 = \tilde{H}_0 + mY$ for some m. Then $mY.mY = (mY - \tilde{H}_1).mY = -\tilde{H}_0.mY < 0$.

Solution 5.5.8 (A Surface Singularity). (a) Since $A/(z) \cong k[x,y]/(x^2+y^3)$, which is an integral domain, we know z is irreducible in A. Then consider the ring $A[t]$. We have $A[t] = k[x,y,z][t] = k[u,v][t]$, which is clearly a UFD since u, v are algebraically independent over k. Then by Nagata's criterion for factoriality, A is a UFD.

(b) First blow-up: $(x^2+y^3+z^5, xz_1 = zx_1, yz_1 = zy_1, xy_1 = yx_1)$. In the piece $x_1 = 1$, it is $(x^2+x^3y_1^3+x^5z_1^5, xz_1 = z, xy_1 = y) = (x^2, xz_1 = z, xy_1 = y) \cup (1+xy_1^3+x^3z_1^5, xz_1 = z, xy_1 = y)$. The jacobian is $(3x^2z^5 + y^3, 3xy^2, 5x^3z^4)$, hence this piece is nonsingular. In the piece $y_1 = 1$, it is $(x_1^2 + y + y^3z_1^5)$. The jacobian is $(2x, 3y^2z^5+1, 5y^3z^4)$, hence this piece is nonsingular. In the piece $z_1 = 1$, it is $(z^2x_1^2 + z^3y_1^3 + z^5, x = zx_1, y = zy_1) = (z^2, x = zx_1, y = zy_1) \cup (x_1^2 + zy_1^3 + z^3, x = zx_1, y = zy_1)$. The jacobian is $(2x, 3y^2z, y^3+3z^2)$, hence there exists a singularity at $(0,0,0)$.

Second blow-up: $(x_1^2+zy_1^3+z^3, y_2x_1 = y_1x_2, x_2z = z_2x_1, y_1z_2 = y_2z)$. In the piece $x_2 = 1$, it is $(1+z_2y_2^3x_1^2 + z_2^3x_1)$. The jacobian is $(3y_2^2z_2x_1^2, x_1(y_2^3x_1+3z_2^2), z_2(2y_2^3x_1+z_2^2))$, hence this piece is nonsingular. In the piece $y_2 = 1$, it is $(x_2^2 + y_1^2z_2 + y_1z_2^3)$. The jacobian is $(2x_2, 2y_1z_2+z_2^3, y_1^2+3z_2^2y_1)$, hence there exists a singularity at $(0,0,0)$. In the piece $z_2 = 1$, it is $(x_2^2 + z + y_2^3z^2)$. The jacobian is $(2x_2, 3y_2^2z^2, 1+2zy_2^3)$, hence this piece is nonsingular.

Third blow-up: $(x_2^2 + y_1^2z_2 + y_1z_2^3, x_2y_3 = z_2x_3, x_2z_3 = y_1x_3, z_2z_3 = y_1y_3)$. In the piece $x_3 = 1$, it is $(1+x_2^2y_3^3z_3 + x_2y_3z_3^2)$, hence clearly nonsingular. In the piece $y_3 = 1$, it is $(x_3^2 + z_2^2z_3 + z_2z_3^2)$. The jacobian is $(2x_3, 2z_2z_3+z_3^2, z_2^2+2z_2z_3)$, hence there exists a singularity at $(0,0,0)$. In the piece $z_3 = 1$, it is $(z_2^3+y_1y_3^2+y_1z_2)$. The jacobian is $(y_3^2+z_2, 2y_1y_3, y_1+3z_2^2)$, hence there exists a singularity at $(0,0,0)$.

Fourth blow-up: $(z_2^3 + y_1y_3^2 + y_1z_2, y_1y_4 = y_3x_4, y_3z_4 = z_2y_4, y_1z_4 = z_2x_4)$. In the piece $x_4 = 1$, it is $(z_4 + y_1(y_4^2+z_4^3))$. The jacobian is $(y_4^2+z_4^3, 2y1y4, 1+3y1z4^2)$, hence this piece is nonsingular. In the piece $y_4 = 1$, it is $(z_2^3 + x_4y_3(y_3^2+z_2))$. The jacobian is $(y3(y3^2+z2), 2x4y3^2 + x4(y3^2+z2), x4y3 + 3z2^2)$, which is nonsingular at $y_3 = z_2 = 0$. This is the exceptional curve of another singular point in the third blow-up. In the piece $z_4 = 1$, it is $(x_4 + z_2 + x_4y_4^2z_2)$. The jacobian is $(1+y_4^2z_2, 2x_4y_4z_2, 1+x_4y_4^2)$, hence this piece is nonsingular.

Fifth blow-up: $(x_3^2 + z_2^2z_3 + z_2z_3^2, x_3z_5 = x_5z_3, z_2z_5 = z_3y_5, x_3y_5 = x_5z_2)$. In the piece $x_5 = 1$, it is $(1 + x_3y_5^2z_5 + x_3y_5z_5^2)$, hence clearly nonsingular. In the piece $y_5 = 1$, it is $(x_5^2 + z_2z_5 + z_2z_5^2)$. The jacobian is $(2x_5, z_5+z_5^2, z_2+2z_2z_5)$, hence there exist two singularities $(0,0,0)$ and $(0,0,-1)$. The point $x_5 = z_2 = z_5 = 0$ is impossible, hence there only exists one singularities $(0,0,-1)$. In the piece $z_5 = 1$, it is $(x_5^2 + y_5(1+y_5)z_3)$, hence there exists two singularities $(0,-1,0)$ and $(0,0,0)$. Similarly, $(0,0,0)$ is impossible, and the singularity $(x_5, y_5, z_3, z_5) = (0,-1,0,1)$ is the same with $(x_5, y_5, z_3, z_5) = (0,1,0,-1)$.

Sixth blow-up: $(x_5^2 + z_2z_5 + z_2z_5^2, x_5y_6 - z_2x_6, z_2z_6 - y_6(z_5+1), x_5z_6 - x_6(z_5+1))$. In the piece $(x_6 = 1)$, it is $(1 + y_6z_6(-1+x_5z_6))$. The jacobian is $(y_6z_6^2, z_6(-1+x_5z_6), x_5y_6z_6 + y_6(-1+x_5z_6))$, hence this piece is nonsingular. In the piece $(y_6 = 1)$, it is $(x_6^2 + z_6(-1+z_2z_6))$. The jacobian is $(2x_6, z_6^2, -1+2z_2z_6)$, hence this piece is nonsingular. In the piece $(z_6 = 1)$, it is $(x_6^2 + y_6z_5)$. This jacobian is $(2x_6, z_5, y_6)$, hence there exists a singularity $(0,0,0)$.

Seventh blow-up: $(x_6^2+y_6z_5, x_6y_7 - x_7y_6, y_6z_7 - z_5y_7, z_5x_7 - x_6z_7)$. In the piece $(x_7 = 1)$, it is $(1+y_7z_7)$, hence clearly nonsingular. In the piece $(y_7 = 1)$, it is $(x_7^2 + z_7)$, hence clearly nonsingular. In the piece $(z_7 = 1)$, it is $(x_7^2 + y_7)$, hence clearly nonsingular.

And one more blow-up to separate all exceptional curves. Hence this Du Val singularity is of the type $\mathbb{E}_8$.

5.6 Classification of Surfaces

Solution 5.6.1. By 2.8.4., $\omega \cong \mathscr{O}_X(\sum d_i - n - 1)$. So if $\kappa(X) = -1$, we have $\sum d_i - n - 1 < 0$. Since $d_i \geq 2$, we have $n + 1 > \sum d_i \geq 2(n-2)$, i.e. $n < 5$. And the choices of $(n; d_1, \dots, d_{n-2})$ in the case $\kappa = -1$ are $(3; 2)$, $(3; 3)$, and $(4; 2, 2)$. If $\kappa(X) = 0$, we have $\sum d_i - n - 1 = 0$. Similarly we have $n \leq 5$. And the choices of $(n; d_1, \dots, d_{n-2})$ in the case $\kappa = 0$ are $(3; 4)$, $(4; 2, 3)$ and $(5; 2, 2, 2)$. Moreover, these three cases are all K3 surfaces. If $\kappa(X) = 1$, then X is an elliptic surface. Then $\omega^2 = 0$, i.e. $\sum d_i - n - 1 = 0$. But these solutions are all K3 surfaces, which is impossible. So almost all choices of n and d_i are of general type.

Solution 5.6.2. If H is the hyperplane section, then $\deg H = d$ and $\dim |H| = n$. So if $h^2(\mathscr{O}_X) = h^1(\mathscr{O}_H(d)) > 0$, by Clifford's theorem, we have $d \geq 2n$. So if $d < 2n$, $p_g(X) = 0$. Then if $p_g(X) \neq 0$ and $d = 2n$, by adjunction formula, $2g - 2 = d + H.K \geq d$. By Clifford's theorem again, we have $n + 1 \leq h^0(\mathscr{O}_H(d)) = 1 - g + d$. Then $2n + 2 \leq 2 - 2g + 2d \leq d$, so here we must have $d = 2g - 2$, i.e. $d = 2n$ and $g = n + 1$, and $H.K = 0$. Since X is not contained in any hyperplane, it is not a ruled surface. Then by theorem 6.2., we have $|12K| \neq 0$. So since $12K = 0$, we have $\kappa = 0$ by theorem 6.3. By Nakai Moishezon criterion, we have $H - K$ is ample, so $h^1(K - H) = 0$ by Kodaira vanishing theorem, then $h^1(H) = 0$ by Serre duality. So by 5.1.1., we have $\deg(K - H) < 0$, i.e. $h^0(K - H) = 0$. Then by 5.1.6., $p_a(X) = 1$, hence X is a K3 surface by theorem 6.3.

Appendix

A Intersection Theory

Solution Appx.A.6.1. We will call the definition of rational equivalence in section 1 as definition (a), and the new definition as (b). Moreover, we can define a rational equivalence (c) as in Fulton's *Intersection Theory* as: $Y \sim Z$ iff there exist a cycle W of codimension $r-1$ and a rational function $f \in K(W)^*$ such that Y and Z are the zeros and the poles of f, which is similar to the definition of principal divisor as we used to do.

(b$\Rightarrow$c) Consider the projection $\pi_X : W \to X$. Then $W' = \mathrm{Im}\pi_X$ is a subvariety of X with codimention $r-1$. So there exists a rational function $f(x) = (\pi_{\mathbb{A}^1}(W \cap \pi_X^{-1}(x)) - 1)^{-1} + 1 \in K(W')^{-1}$. And the zero of f is $W \cap (X \times \{0\})$, and the pole of f is $W \cap (X \times \{1\})$.

(c$\Rightarrow$b) If we have a subvariety W of codimension $r-1$ and an $f \in K(W)^*$, we may consider the subvariety W' of $X \times \mathbb{A}^1$ as $\{(x,t) \in W \times \mathbb{A}^1 \mid (f(x)-1)^{-1} + 1 = t\}$. So the zero of f is $\{(x,t) \in W \times \mathbb{A}^1 \mid (f(x)-1)^{-1} + 1 = t, f(x) = 0\} = W' \cap (X \times \{0\})$, and the pole is $W' \cap (X \times \{1\})$.

(a$\Leftrightarrow$c) If V is a subvariety of X, and $\tilde{V}$ is the normalization of V, we have $K(V)^* = K(\tilde{V})^*$. And for any $f \in K(V)^*$, we have $\mathrm{ord}_Y(f) = \sum \mathrm{ord}_{\tilde{Y}}(f) \cdot [K(\tilde{Y}) : K(Y)]$, where the sum if over all subvarieties $\tilde{Y}$ of $\tilde{V}$ which map into Y, and $[K(\tilde{Y}) : K(Y)]$ is the degree of the field extension. So clearly (a) and (c) are equivalent.

Solution Appx.A.6.2. If f is generically finite, we may shrink X and X' by deleting a closed subset of codimention≥ 2 and assume f is finite flat and surjective by 2.3.7. and 3.9.3.(a). For any effective divisor D, we have an exact sequence $0 \to \mathscr{L}(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0$. Then since f is flat, we have $0 \to f_*\mathscr{L}(-D) \to f_*\mathscr{O}_X \to f_*\mathscr{O}_D \to 0$. So the rest is same as 4.2.6.(a) and (b), if D is a principal divisor, we have $f_*(D)$ is also principal. Hence f_* is well-defined modulo rational equivalence.

Solution Appx.A.6.3. If subvariety X is a hypersurface in $\mathbb{P}^n$, then trivial. If X has codimention$-r > 1$, as in example 9.8.3. in chapter III, we can find a point $P \in \mathbb{P}^n$ not in X and consider the projection π from P to $\mathbb{P}^{n-1}$. Then $X' = \pi(X)$ is a subvariety in $\mathbb{P}^{n-1}$ of codimention $r-1$. Clearly $\pi|_X$ is finite on an open dense subset of X, and $\deg X' = \deg X / \deg(\pi|_X)$. So after finite step of projection, we have a hypersurface $X' \subset \mathbb{P}^{n-r+1}$ of degree $\deg X/\deg(\pi|_X)$, where π is the component of all projection. And $X' \sim (\deg X') \cdot H$ for hyperplane H in $\mathbb{P}^{n-r+1}$. So by the Axiom of $\pi^* : A(\mathbb{P}^{n-r+1}) \to A(\mathbb{P}^n)$, we have $X \sim \deg(\pi|_X) \cdot \pi^*((\deg X') \cdot H) = \deg X \cdot \pi^*(H)$. Clearly $\pi^*(H)$ is a linear subspace of dimension equals to $\dim X$.

Solution Appx.A.6.4. By Axiom A8, we only need to prove $A^2(X) \cong A^1(C)$. Since we have the projection $\pi : X \to C$ and the section $\sigma : C \to X$, we can just define $\pi_* : Z^2(X) \to Z^1(C)$ as $P \mapsto \pi(P)$ and $\sigma_* : Z^1(C) \to Z^2(X)$ as $P \mapsto \sigma(P)$, where Z^i is the group of i-cycles. Then for any point $P \in C$, we have $\pi_*\sigma_*(P) = P$. For any point $P \in X$, we have $\sigma_*\pi_*(P)$ and P are lying on the same ruling $\mathbb{P}^1$, hence rationally equivalent. And for any point $P, Q \in C$ with $P \sim Q$, if $P \sim Q$, there exists $f \in K(C)$ such that $(f) = P - Q$. Since f can by extended to a morphism $X \to C \to \mathbb{P}^1$, we have $\sigma_*(P) \sim \sigma_*(Q)$. Conversely, if $P, Q \in X$ with $P \sim Q$, then since $P \sim \sigma_*\pi_*(P)$, and same for Q, we have $\sigma_*\pi_*(P) \sim \sigma_*\pi_*(Q)$, hence clearly $\pi_P \sim \pi_Q$. Then so clearly $A^2(X) \cong A^1(C)$.

Solution Appx.A.6.5. By Axiom A3 and proposition 3.2. in chapter V, we have $A^1(\tilde{X}) \cong A^1(X) \oplus \mathbb{Z}$. For A^2, we can define $Z^2(\tilde{X}) \to Z^2(X)$ as $P \mapsto \pi(P)$. So this morphism is isomorphic out of P. If $P', P'' \in \tilde{X}$ such that $\pi(P') = \pi(P'') = P$, since $E \cong \mathbb{P}^1$, we clearly have $P' \sim P''$. So this morphism induces $A^2(\tilde{X}) \cong A^2(X)$. Hence we have a group isomorphism $A(\tilde{X}) \cong \pi^*A(X) \oplus \mathbb{Z}$. For the ring structure, if C is a curve in X passing through P, by proposition 3.6. in chapter V, we have $\pi^*(C).E = (\tilde{C} + rE).E = Q_1 + \ldots + Q_r - r \cdot \pi^*(P)$, where $\tilde{C}$ is the strict transformation of C in $\tilde{X}$, and Q_i are r intersection point of $\tilde{C}$ and E. Since $E \cong \mathbb{P}^1$, we have $\pi^*(C).E \sim 0$. So this induces the multiplication structure of the ring morphism $A(\tilde{X}) \to \pi^*A(X) \oplus \mathbb{Z}$, hence an isomorphism.

Solution Appx.A.6.6. By Axiom C7, we have $c_n(\mathscr{N}_\Delta) = \Delta^2$. Since by theorem 8.17 in chapter II, we have an exact sequence $0 \to \mathscr{T}_\Delta \to \mathscr{T}_{X\times X} \otimes \mathscr{O}_Y \to \mathscr{N}_\Delta \to 0$, and clearly $\Delta \cong X$, we have $c_t(\mathscr{N}_\Delta) = c_t(\mathscr{T}_{X\times X} \otimes \mathscr{O}_Y)/c_t(\mathscr{T}_\Delta) = c_t(\mathscr{T}_X)^2/c_t(\mathscr{T}_X) = c_t(\mathscr{T}_X)$, we have $c_n(\mathscr{T}_X) = c_n(\mathscr{N}_\Delta)$, hence $c_n(\mathscr{T}_X) = \Delta^2$.

Solution Appx.A.6.7. For any divisor D of X, we may consider the case $\mathscr{E} = \mathscr{L}(D)$. By Axiom C1 we have $c_t(\mathscr{L}(D)) = 1+Dt$, i.e. $c_1(\mathscr{L}(D)) = D$, $c_2(\mathscr{L}(D)) = 0$ and $c_3(\mathscr{L}(D)) = 0$. So $\mathrm{ch}(\mathscr{L}(D)) = 1+D+\frac{1}{2}D^2+\frac{1}{6}D^3$. Since $\mathrm{td}(\mathscr{L}(D)) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2$, by Hirzebruch-Riemann-Roch, we have $\chi(\mathscr{L}(D)) = \deg((1+D+\frac{1}{2}D^2+\frac{1}{6}D^3).(1+\frac{1}{2}c_1+\frac{1}{12}(c_1^2+c_2)+\frac{1}{24}c_1c_2))_3 = \frac{1}{6}(D^3)+\frac{1}{4}D^2.c_1+\frac{1}{12}D.(c_1^2+c_2)+\frac{1}{24}(c_1c_2)$. Since $\mathscr{T}_X$ is the dual of Ω_X, and by Axiom C5, we have $c_1(\tilde{T}_X) = -K$. So we have $\chi(\mathscr{L}(D)) = \frac{1}{12}D.(D-K).(2D-K)+\frac{1}{12}D.c_2+\frac{1}{24}c_1c_2$. In the case $D = 0$, we have $1-p_a = \chi(X) = \frac{1}{24}c_1c_2$. So we have $\chi(\mathscr{L}(D)) = \frac{1}{12}D.(D-K).(2D-K)+\frac{1}{12}D.c_2+1-p_a$.

Solution Appx.A.6.8. By theorem 8.13. in chapter II, we have $0 \to \mathscr{O} \to \mathscr{O}(1)^4 \to \mathscr{T} \to 0$, we have $c_t(\mathscr{T}) = c_t(\mathscr{O}(1))^4/c_t(\mathscr{O}) = (1+ht)^4 = 1+4ht+6h^2t^2+4h^3t^3$, where $h \in A^1(\mathbb{P}^4)$ is the class of hyperplane. So $\mathrm{td}(\mathscr{T}) = 1+2h+\frac{11}{6}h^2+h^3$. Since $\mathrm{ch}(\mathscr{E}) = 2+c_1+\frac{1}{2}(c_1^2-2c_2)+\frac{1}{6}(c_1^3-3c_1c_2)$, we have $\chi(\mathscr{E}) = 8+6c_1+2(c_1^2-2c_2)+\frac{1}{6}(c_1^3-3c_1c_2)$. Since $\chi(\mathscr{E})$ is an integer, we know $c_1^3-3c_1c_2 \equiv 0 \mod 6$, hence $c_1c_2 \equiv 0 \mod 2$.

Solution Appx.A.6.9 (Surfaces in $\mathbb{P}^4$)**.** (a) For the rational cubic scroll X in example 2.19.1. in chapter V, we have $d = 3$, $H.K = (C_0+2f).(-2C_0-3f) = -5$ and $K^2 = 8$. So $12p_a = -d^2+10d+5H.K+2K^2-12 = -9+30-25+16-12 = 0$, i.e. $p_a = 0$, which is what we have.

(b) By definition of K3 surface, we have $K = 0$, $p_a = p_g = 1$, we have $d^2-10d-0-0+12+12 = 0$, i.e. $d^2-10d+24 = 0$, so $d = 4$ or 6.

(c) By theorem 6.3. in chapter V, we have $12K = 0$, $p_a = -1$ and $p_g = 1$. So we have $d^2-10d-0-0+12-12 = 0$, i.e. $d^2-10d = 0$. Since $d > 0$, we have $d = 10$.

(d) Suppose $H \sim aC_0+bf$ is the very ample divisor of X determining the embedding $X \to \mathbb{P}^4$, so we have $H^2 = 2ab-a^2e = a(2b-ae) = 4$. Since $K = -2C_0-(e+2)f$ by corollary 2.11. in chapter V, we have $H.K = 2ae-(2b+ae+2a) = -2a-2b+ae$, and $K^2 = -4e+4(e+2) = 8$. So we have $16-40+10a-5ae+10b-16+12+0 = 0$. So $10a+10b-5ae = 28$, which is impossible, since $5 \nmid 28$.

Solution Appx.A.6.10. For an abelian 3-fold X, since $\mathscr{T}_X$ is free, we have $\mathscr{T}_X \cong \mathscr{O}_X^3$. So $c_t(\mathscr{T}_X) = c_t(\mathscr{O}_X)^4 = 1$, i.e. $c_1 = c_2 = c_3 = 0$. So $\mathrm{td}(\mathscr{T}_X) = 1$. So if we have $0 \to \mathscr{T}_X \to \mathscr{T}_{\mathbb{P}^5} \to \mathscr{N} \to 0$, we must have $1\cdot(1+c_1(\mathscr{N})t+c_2(\mathscr{N})t^2+c_3(\mathscr{N})t^3) = 1+6Ht+15H^2t^2+20H^3t^3$, i.e. $c_1(\mathscr{N}) = 6H$, $c_2(\mathscr{N}) = 15H^2$ and $c_3(\mathscr{N}) = 20H^3$. If $\mathscr{N} = \mathscr{L}(D)$ for some divisor D of X, then $c_1(\mathscr{N}) = D$, $c_2(\mathscr{N}) = D^2$ and $c_3(\mathscr{N}) = D^3$. But $D^3 = D^2.D = 15H^2.6H = 90H^3 \neq 20H^3 = D^3$, which is impossible.

B Transcendental Methods

Solution Appx.B.6.1. If the unit disc $\mathfrak{X} = \{|z| < 1\}$ is algebraic, i.e. $\mathfrak{X} = X_h$ for some scheme X of finite type over $\mathbb{C}$, we have $\dim X = 1$. Then X must has an affine open subscheme Y with dimension 1, and $Y_h \subset X_h$. We may assume $Y = \mathbb{C}[x_1,\ldots,x_n]/I$ for some ideal I, then there exists a sequence of points $\{y_n\} \subset Y_h$ such that $|y_n| \to \infty$, which contradicts to the definition of the disc.

Solution Appx.B.6.2. If there exists a sheaf of ideal $\mathscr{I} = \tilde{I}$ in $\mathrm{Spec}\,\mathbb{C}[z]$ such that $I \subset (z-z_i)$ for all i, since $\mathbb{C}[z]$ is a PID, I has a generator f. Then f has zeros at all z_i, which contradicts to the fundamental theorem of algebra. By theorem 4.1. in chapter V of Stein's *Complex Analysis*, there exists a holomorphic function f vanishing on all z_i, and other holomorphic function vanishing on all z_i has the form $f(z)\cdot\exp(g(z))$ for some entire function $g(z)$. Hence we can define a $\mathbb{C}[z]$-module generated by $f(z)$, and $\mathscr{F} = \tilde{M}$, then we have $\mathscr{F}_h \cong \mathfrak{I}$.

Solution Appx.B.6.3. We may consider a global section $\mathfrak{s}$ of $\mathfrak{L}$, such that $\mathfrak{s}|_{(z\neq 0)} = e^z$, and $\mathfrak{s}|_{(w\neq 0)} = e^{-1/w}$. Then $\mathfrak{s}$ on $\mathbb{C}^2$ has only one singular point at $(0,0)$. But for any invertible sheaf $\mathscr{L}$ of $\mathbb{A}^2 - \{(0,0)\}$, then there exists an open subset $U \subset \mathbb{A}^2$ near $(0,0)$, and $\mathscr{L}|_{U-(0,0)} \cong \mathscr{O}_{U-(0,0)}$. Then for any $s \in \mathscr{L}|_{U-(0,0)}$, s has the form $\frac{f}{g}$ such that f, g are algebraic function on $U-(0,0)$. So the pole of s must be a curve of codimension 1 in $\mathbb{A}^2 - \{(0,0)\}$. Hence there does not exists a section on $U - \{(0,0)\}$ such that $s_h \cong \mathfrak{s}|_{U-(0,0)}$. So there does not exists an invertible sheaf $\mathscr{L}$ on $\mathbb{A}^2 - \{(0,0)\}$ such that $\mathscr{L}_h \cong \mathfrak{L}$.

Solution Appx.B.6.4. Clearly $\alpha : H^0(X, \mathscr{O}_X) \to H^0(X_h, \mathscr{O}_{X_h})$ is injective since reduced. And since X is proper, X_h is compact, so by Liouville's theorem, $H^0(X_h, \mathscr{O}_{X_h}) = \mathbb{C}$. Clearly $\mathbb{C} \subset H^0(X, \mathscr{O}_X)$, hence isomorphic. Conversely, for any X not proper, then X_h is not complete. Hence there exists an entire exponent function f on X_h which has an essential pole in the compactification of X_h. So f is not algebraic, hence $H^0(X, \mathscr{O}_X) \neq H^0(X_h, \mathscr{O}_{X_h})$.

Solution Appx.B.6.5. Since $X_h \cong X'_h$, we know there exists a bijection between the set of closed points of X and X'. Since the Zariski topology on curve defines the open subset as the whole set minus finite points, so the topology on closed points of X and X' are the same. Since $X = \{\text{closed points of } X\} \cup \{\text{generic point } \eta\}$, and same for X', and moreover η and η' are not closed points, and any open subset of X or X' contains η or η', we know the topology of X and X' are the same, hence $X \cong X'$.

Solution Appx.B.6.6. Since Y are projective, we may consider the closed embedding $Y \to \mathbb{P}^n$ for some n, it induces a morphism $Y_h \to \mathbb{P}^n_h$. So $X_h \to Y_h \to \mathbb{P}^n_h$. If there exists morphisms $f : X \to \mathbb{P}^n$ and $g : Y \to \mathbb{P}^n$, such that $\mathrm{Im} f \subset \mathrm{Im} g = Y$, so actually, f is mapping to Y, it is a morphism from X to Y, and $f_h = \mathfrak{f}$.

So we only need to consider the case $Y = \mathbb{P}^n$. Since $\mathfrak{f} : X_h \to \mathbb{P}^n_f$ is a morphism, we may consider the invertible analytic sheaf $\mathfrak{L} = \mathfrak{f}^*\mathscr{O}(1)$ on X_h. Since X is projective, by GAGA, there exists an invertible sheaf $\mathscr{L}$ on X such that $\mathfrak{L} = \mathscr{L}_h$. Since $\mathfrak{L}$ is very ample, there exists global sections $\mathfrak{s}_0, \ldots, \mathfrak{s}_n$ such that they generates $\mathfrak{L}$. Then there exists global sections $s_0, \ldots, s_n$ corresponding to all $\mathfrak{s}_i$, such that $s_{i,h} = \mathfrak{s}_i$. Hence $s_0, \ldots, s_n$ generates $\mathscr{L}$, so by theorem 7.1. in chapter II, $\mathscr{L}$ is very ample, and determines a closed embedding $f : X \to \mathbb{P}^n$. Clearly $f_h = \mathfrak{f}$.

C The Weil Conjectures

Solution Appx.C.5.1. If $X = \coprod_i X_i$, we have $N_r(X) = \sum_i N_r(X_i)$, then $Z(X,t) = \exp(\sum_{r=1}^\infty N_r(X)\frac{t^r}{r}) = \exp(\sum_r \sum_i N_r(X_i)\frac{t^r}{r}) = \exp(\sum_i \sum_r N_r(X_i)\frac{t^r}{r}) = \prod_i \exp(\sum_r N_r(X_i)\frac{t^r}{r}) = \prod_i Z(X_i, t)$.

Solution Appx.C.5.2. Since $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \ldots \cup \mathbb{A}^1 \cup \{\infty\}$, we have $N_r(\mathbb{P}^n) = 1 + q^r + \ldots + q^{nr}$. So $Z(\mathbb{P}^n, t) = \exp(N_r(\mathbb{P}^n)\frac{t^r}{r}) = \prod_{i=0}^n \exp(\sum_r \frac{(q^it)^r}{r}) = \prod(1-q^it)^{-1}$. The Rationality and the RH are clearly. For the functional equation, since $c_t(\mathbb{P}^n \times \mathbb{P}^n) = c_t(\mathbb{P}^n)^2$, we have $\mathrm{ch}_n(\mathbb{P}^n \times \mathbb{P}^n) = \bigoplus_{i+j=n} \mathbb{P}^i \times \mathbb{P}^j$. Clearly, $(\mathbb{P}^i \times \mathbb{P}^j).(\mathbb{P}^{i'} \times \mathbb{P}^{j'}) = 1$ if $i + i' = j + j' = n$ and 0 otherwise. So by Appx.A.6.6., we have $E = \Delta.\Delta = n+1$. Then,

$$Z(\mathbb{P}^n, \frac{1}{q^nt}) = \prod_{i=0}^n (1-q^{i-n}t^{-1})^{-1} = \prod_{i=0}^n (1-\frac{1}{q^it})^{-1} = \frac{\prod_{i=0}^n (q^it)}{\prod_{i=0}^n (q^it-1)} = (-1)^{n+1}\frac{q^{n(n+1)/2}t^{n+1}}{\prod_{i=0}^n (1-q^it)} = (-1)^{n+1}q^{nE/s}t^EZ(\mathbb{P}^n, t)$$

For Betti number, we have $H^i(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}$ if i is even, or 0 if i is odd. So $\sum_{i=0}^{2n}(-1)^iB_i = n+1 = E$.

Solution Appx.C.5.3. Clearly $Z(X \times \mathbb{A}^1, t) = \exp(\sum_r N_r(X \times \mathbb{A}^1)\frac{t^r}{r}) = \exp(\sum_r q^rN_r(X)\frac{t^r}{r}) = Z(X, qt)$.

Solution Appx.C.5.4. Denote $N_r(x)$ as the number of points in $X(\mathbb{F}_{q^r})$ corresponding to closed point $x \in X$. Then clearly $\zeta_X(x) = \prod_x(1-N(x)^{-s})^{-1} = \prod_x(1-q^{-s\deg x})^{-1} = \exp(\sum_x \sum_{r=1}^\infty q^{-sr\deg x}r^{-1} = \exp(\sum_x \sum_{r=1}^\infty (q^{-s})^{r\deg x}r^{-1} = \exp(\sum_x \sum_{r=1}^\infty \frac{(\deg x)(q^{-s})^{r\deg x}}{r\deg x} = \exp(\sum_x \sum_{n=1}^\infty \frac{N_n(x)(q^{-s})^n}{n}) = \exp(\sum_n N_nq^{-ns}n^{-1}) = Z(X, q^{-s})$.

Solution Appx.C.5.5. By Weil conjecture, we have $\dim H^1(X, \mathbb{Q}_\ell) = B_1 = \deg P_1(t) = 2g$. So we may assume $P_1(t) = \prod_{i=1}^{2g}(1-\alpha_it)$. Then

$$Z(X,t) = \frac{\prod_{i=1}^{2g}(1-\alpha_it)}{(1-t)(1-qt)} = \exp\left(\sum_{r=1}^\infty\left(\frac{t^r}{r} + \frac{q^rt^r}{r} - \sum_{i=1}^{2g}\alpha_i^r\frac{t^r}{r}\right)\right)$$

Since $Z(X,t) = \exp(\sum_r N_r \frac{t^r}{r})$, we have $N_r = 1 + q^r - \sum_{i=1}^{2g} \alpha_i^r$ for all $r \geq 1$. Moreover, by functional equation, we have $Z(1/qt) = \pm q^{1-g} t^{2-2g} Z(t)$, i.e.

$$\frac{\prod_{i=1}^{2g}(1 - \frac{\alpha_i}{qt})}{(1 - \frac{1}{qt})(1 - \frac{1}{t})} = \frac{t^{2-2g} q^{1-g} \prod_{i=1}^{2g}(\sqrt{q}t - \frac{\alpha_i}{\sqrt{q}})}{(1-t)(1-qt)}$$

So $P_1(t) = \prod_{i=1}^{2g}(1 - \alpha_i t) = \pm \prod_{i=1}^{2g}(\sqrt{q}t - \frac{\alpha_i}{\sqrt{q}})$. Comparing the coefficient, and by the fact $|\alpha_i| = \sqrt{q}$, we know all α_i are pairwise have $\alpha_i \alpha_{2g+1-i} = \pm q$ and the $\pm$ are determined by functional equation. So if we know $N_1, \ldots, N_r$ and $\pm\alpha_1\alpha_{2g} = \ldots = \pm\alpha_g\alpha_{g+1} = q$, they determine all α_i. Hence all N_r's are determined by $N_1, \ldots, N_r$.

Solution Appx.C.5.6. By 4.4.16., we have $N_r = q^r - (f^r + \hat{f}^r) + 1$, and $f\hat{f} = q$, $a = f + \hat{f} \in \mathbb{Z}$. Then $Z(E,t) = \exp(\sum_r N_r \frac{t^r}{r}) = \exp(\sum_r (q^r - (f^r + \hat{f}^r) + 1)\frac{t^r}{r}) = \frac{(1-ft)(1-\hat{f}t)}{(1-t)(1-qt)} = \frac{1-at+qt^2}{(1-t)(1-qt)}$. So $P_1(t) = 1 - at + qt^2$. The functional equation is clearly by calculation. And from this, we have $P_1(t) = (1-ft)(1-\hat{f}t) = \pm(\sqrt{q}t - \frac{f}{\sqrt{q}})(\sqrt{q}t - \frac{f}{\sqrt{q}})$, we have $|a| = |f + \hat{f}| \leq 2q$, hence by Appx.C.5.7.(b), we have $|f| = |\hat{f}| = \sqrt{q}$.

Solution Appx.C.5.7. (a) Since $Z(X,t) = \frac{\prod_{i=1}^{2g}(1-\alpha_i t)}{(1-t)(1-qt)} = \exp(\sum_r (1 - \sum_{i=1}^{2g} \alpha_i^r + q^r) t^r r^{-1})$, then we have $N_r = 1 - \sum_{i=1}^{2g} \alpha_i^r + q^r = 1 - a_r + q^r$, hence $a_r = \sum_{i=1}^{2g} \alpha_i^r$.

(b) ($\Leftarrow$) Trivial. ($\Rightarrow$) Since $\sum_{i=1}^{2g} \frac{\alpha_i t}{1-\alpha_i t} = \sum_{r=1}^{\infty} a_r t^r$, and $|a_r| \leq 2gq^{r/2}$, the right-hand side is holomorphic in $|t| < q^{-1/2}$. So if $|\alpha_i| > q^{1/2}$ for some i, the left-hand side has a pole at $t = \alpha_i^{-1}$ in the domain $|t| < q^{-1/2}$, which makes a contradiction.

(c) Since $P_1(t) = \prod_{i=1}^{2g}(1-\alpha_i t) = \pm \prod_{i=1}^{2g}(\sqrt{q}t - \frac{\alpha_i}{\sqrt{q}})$, we have $\{\alpha_1^{-1}, \ldots, \alpha_{2g}^{-1}\} = \{\alpha_1/q, \ldots, \alpha_{2g}/q\}$. So we may assume $\alpha_i \alpha_{2g+1-i} = q$. Then the assumption $|\alpha_i| \leq \sqrt{q}$ means $|\alpha_i| \geq \sqrt{q}$, i.e. $|\alpha_i| = \sqrt{q}$.