Proof of H [7]

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2D Cycle Matrix with 9 Vertices

Analytical Method

For $matrix\_up\_up$,

$$\begin{aligned}
H_{up,up}\exp\left[\frac{2\pi}{N}(mj_x + +nj_y)\right] &= (-t+t_z)\exp\left\{\frac{2\pi i}{N}\left[m(j_x+1)+n(j_y)\right]\right\} \\
&+ (-t+t_z)\exp\left\{\frac{2\pi i}{N}\left[m(j_x-1)+n(j_y)\right]\right\} \\
&+ (-t+t_z)\exp\left\{\frac{2\pi i}{N}\left[m(j_x)+n(j_y+1)\right]\right\} \\
&+ (-t+t_z)\exp\left\{\frac{2\pi i}{N}\left[m(j_x)+n(j_y-1)\right]\right\} \\
&+ m_z\exp\left\{\frac{2\pi i}{N}\left[m(j_x)+n(j_y)\right]\right\} \\
H_{up,up} &= (-t+t_z)\exp\left[\frac{2\pi i}{N}(m)\right] + (-t+t_z)\exp\left[\frac{2\pi i}{N}(-m)\right] \\
&+ (-t+t_z)\exp\left[\frac{2\pi i}{N}(n)\right] + (-t+t_z)\exp\left[\frac{2\pi i}{N}(-n)\right] + m_z \\
H_{up,up} &= (-t+t_z)\left\{2\cos\left[\frac{2\pi}{N}(m)\right] + 2\cos\left[\frac{2\pi}{N}(n)\right]\right\} + m_z
\end{aligned}$$

For *matrix_down_down*,

$$\begin{aligned}
H_{down,down}\exp\left[\frac{2\pi}{N}(mj_x+nj_y)\right] &= (-t-t_z)\exp\left\{\frac{2\pi i}{N}\left[m(j_x+1)+n(j_y)\right]\right\} \\
&+ (-t-t_z)\exp\left\{\frac{2\pi i}{N}\left[m(j_x-1)+n(j_y)\right]\right\} \\
&+ (-t-t_z)\exp\left\{\frac{2\pi i}{N}\left[m(j_x)+n(j_y+1)\right]\right\} \\
&+ (-t-t_z)\exp\left\{\frac{2\pi i}{N}\left[m(j_x)+n(j_y-1)\right]\right\} \\
&- m_z\exp\left\{\frac{2\pi i}{N}\left[m(j_x)+n(j_y)\right]\right\} \\
H_{down,down} &= (-t-t_z)\exp\left[\frac{2\pi i}{N}(m)\right]+(-t-t_z)\exp\left[\frac{2\pi i}{N}(-m)\right] \\
&+ (-t-t_z)\exp\left[\frac{2\pi i}{N}(n)\right]+(-t-t_z)\exp\left[\frac{2\pi i}{N}(-n)\right]-m_z \\
H_{down,down} &= (-t-t_z)\left\{2\cos\left[\frac{2\pi}{N}(m)\right]+2\cos\left[\frac{2\pi}{N}(n)\right]\right\}-m_z
\end{aligned}$$

For $matrix\_up\_down$,

$$
\begin{aligned}
H_{up,down} \exp\left[\frac{2\pi}{N}(mj_x + +nj_y)\right] &= t_{so} \exp\left\{\frac{2\pi i}{N}[m(j_x+1)+n(j_y)]\right\} + t_{so} \exp\left\{\frac{2\pi i}{N}[m(j_x-1)+n(j_y)]\right\} \\
&- t_{so} \exp\left\{\frac{2\pi i}{N}[m(j_x)+n(j_y+1)]\right\} - t_{so} \exp\left\{\frac{2\pi i}{N}[m(j_x)+n(j_y-1)]\right\} \\
&+ it_{so} \exp\left\{\frac{2\pi i}{N}[m(j_x+1)+n(j_y+1)]\right\} + it_{so} \exp\left\{\frac{2\pi i}{N}[m(j_x-1)+n(j_y-1)]\right\} \\
&- it_{so} \exp\left\{\frac{2\pi i}{N}[m(j_x+1)+n(j_y-1)]\right\} - it_{so} \exp\left\{\frac{2\pi i}{N}[m(j_x-1)+n(j_y+1)]\right\} \\
H_{up,down} &= t_{so} \exp\left[\frac{2\pi i}{N}(m)\right] + t_{so} \exp\left[\frac{2\pi i}{N}(-m)\right] - t_{so} \exp\left[\frac{2\pi i}{N}(n)\right] - t_{so} \exp\left[\frac{2\pi i}{N}(-n)\right] \\
&+ it_{so} \exp\left[\frac{2\pi i}{N}(m+n)\right] + it_{so} \exp\left[\frac{2\pi i}{N}(-m-n)\right] \\
&- it_{so} \exp\left[\frac{2\pi i}{N}(m-n)\right] - it_{so} \exp\left[\frac{2\pi i}{N}(-m+n)\right] \\
H_{up,down} &= t_{so}\left\{2\cos\left[\frac{2\pi}{N}(m)\right] - 2\cos\left[\frac{2\pi}{N}(n)\right]\right\} \\
&+ it_{so}\left\{2\cos\left[\frac{2\pi}{N}(m+n)\right] - 2\cos\left[\frac{2\pi}{N}(m-n)\right]\right\}
\end{aligned}
$$

For $matrix\_down\_up$,

$$\begin{aligned}
H_{down,up}\exp\left[\frac{2\pi}{N}(mj_x++nj_y)\right] &= t_{so}\exp\left\{\frac{2\pi i}{N}[m(j_x+1)+n(j_y)]\right\}+t_{so}\exp\left\{\frac{2\pi i}{N}[m(j_x-1)+n(j_y)]\right\}\\
&-t_{so}\exp\left\{\frac{2\pi i}{N}[m(j_x)+n(j_y+1)]\right\}-t_{so}\exp\left\{\frac{2\pi i}{N}[m(j_x)+n(j_y-1)]\right\}\\
&-it_{so}\exp\left\{\frac{2\pi i}{N}[m(j_x+1)+n(j_y+1)]\right\}-it_{so}\exp\left\{\frac{2\pi i}{N}[m(j_x-1)+n(j_y-1)]\right\}\\
&+it_{so}\exp\left\{\frac{2\pi i}{N}[m(j_x+1)+n(j_y-1)]\right\}+it_{so}\exp\left\{\frac{2\pi i}{N}[m(j_x-1)+n(j_y+1)]\right\}\\
H_{down,up} &= t_{so}\exp\left[\frac{2\pi i}{N}(m)\right]+t_{so}\exp\left[\frac{2\pi i}{N}(-m)\right]-t_{so}\exp\left[\frac{2\pi i}{N}(n)\right]-t_{so}\exp\left[\frac{2\pi i}{N}(-n)\right]\\
&-it_{so}\exp\left[\frac{2\pi i}{N}(m+n)\right]-it_{so}\exp\left[\frac{2\pi i}{N}(-m-n)\right]\\
&+it_{so}\exp\left[\frac{2\pi i}{N}(m-n)\right]+it_{so}\exp\left[\frac{2\pi i}{N}(-m+n)\right]\\
H_{down,up} &= t_{so}\left\{2\cos\left[\frac{2\pi}{N}(m)\right]-2\cos\left[\frac{2\pi}{N}(n)\right]\right\}\\
&+it_{so}\left\{-2\cos\left[\frac{2\pi}{N}(m+n)\right]+2\cos\left[\frac{2\pi}{N}(m-n)\right]\right\}
\end{aligned}$$

For a 2×2 matrix, the corresponding eigenvalues are

$$\lambda_{\pm}=\frac{1}{2}(a+d)\pm\sqrt{4bc+(a–d)^2}$$

By substituting

$$a=(-t+tz)\left\{2\cos\left[\frac{2\pi}{N}(m)\right]+2\cos\left[\frac{2\pi}{N}(n)\right]\right\}+m_z$$

$$b=t_{so}\left\{2\cos\left[\frac{2\pi}{N}(m)\right]-2\cos\left[\frac{2\pi}{N}(n)\right]\right\}+it_{so}\left\{2\cos\left[\frac{2\pi}{N}(m+n)\right]-2\cos\left[\frac{2\pi}{N}(m-n)\right]\right\}$$

$$c=t_{so}\left\{2\cos\left[\frac{2\pi}{N}(m)\right]-2\cos\left[\frac{2\pi}{N}(n)\right]\right\}+it_{so}\left\{-2\cos\left[\frac{2\pi}{N}(m+n)\right]+2\cos\left[\frac{2\pi}{N}(m-n)\right]\right\}$$

$$d=(-t-tz)\left\{2\cos\left[\frac{2\pi}{N}(m)\right]+2\cos\left[\frac{2\pi}{N}(n)\right]\right\}-m_z$$

and putting $t=1, t_z=0.5, m_z=0.1, t_{so}=1, N=3$,

$$\begin{aligned}
\lambda=-1.9\quad -6.1\quad 2.06\ (Multiplicity=4)\quad -4.06\ (Multiplicity=4)\\
-1.13(Multiplicity=4)\ 5.13(Multiplicity=4)
\end{aligned}$$