Proof of Cycle Adjacency Matrix [1][2]

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1D Cycle with 3 Vertices

Analytical Method

By $v_i(j) = cos(\frac{2\pi ij}{N})$ and $i = 0, 1, 2, j = 0, 1, 2, N = 3$, the eigenvectors are

$$\vec{v}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (\lambda_0 = 2) \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -0.5 \\ -0.5 \end{pmatrix} (\lambda_1 = -1) \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -0.5 \\ -0.5 \end{pmatrix} (\lambda_2 = -1)$$

Since two of the vectors are identical, Gram-Schmidt procedure is applied to find the third vector, and the orthonormal set is

$$\{\begin{pmatrix} \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}\}$$

By $A = \sum_{i=0}^{n-1} \lambda_i |\tilde{v}_i\rangle\langle\tilde{v}_i|$,

$$\sum_{i=0}^{n-1} \lambda_i |\tilde{v}_i\rangle\langle\tilde{v}_i| = (-1) \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} + (2) \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} + (-1) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{pmatrix}$$

$$\sum_{i=0}^{n-1} \lambda_i |\tilde{v}_i\rangle\langle\tilde{v}_i| = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Numerical Method

The matrix generated by the code is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The eigenvectors generated by the code are

$$\vec{v}_0 = \begin{pmatrix} -0.81649658 \\ 0.40824829 \\ 0.40824829 \end{pmatrix} \approx \begin{pmatrix} \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \end{pmatrix} (\lambda_0 = -1)$$

$$\vec{v}_1 = \begin{pmatrix} 0.57735027 \\ 0.57735027 \\ 0.57735027 \end{pmatrix} \approx \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} (\lambda_1 = 2)$$

$$\vec{v}_2 = \begin{pmatrix} -0.09265789 \\ -0.65620994 \\ 0.74886783 \end{pmatrix} \approx -0.113 \begin{pmatrix} \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \end{pmatrix} - 0.994 \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} (\lambda_2 = -1)$$

Therefore, the results by the code and the analytical method agree with each other.

2D Cycle with 9 Vertices

Analytical Method

By $B = A \square A$,

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

We know that the eigenstates of B must be $|v_i\rangle \otimes |v_j\rangle$, so

$$|v_0\rangle \otimes |v_0\rangle = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} (\lambda = -1 - 1 = -2) \quad |v_0\rangle \otimes |v_1\rangle = \begin{pmatrix} \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \end{pmatrix} (\lambda = -1 + 2 = 1)$$

$$|v_0\rangle\otimes|v_2\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ 0 \\ -\frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{pmatrix} (\lambda = -1-1 = -2) \quad |v_1\rangle\otimes|v_0\rangle = \begin{pmatrix} \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \end{pmatrix} (\lambda = 2-1 = 1)$$

$$|v_1\rangle \otimes |v_1\rangle = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} (\lambda = 2+2 = 4) \quad |v_1\rangle \otimes |v_2\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix} (\lambda = 2-1 = 1)$$

$$|v_2\rangle\otimes|v_0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{pmatrix} (\lambda = -1-1 = -2) \quad |v_2\rangle\otimes|v_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix} (\lambda = -1+2 = 1)$$

$$|v_2\rangle \otimes |v_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ -0.5 \\ 0 \\ -0.5 \\ 0.5 \end{pmatrix} (\lambda = -1-1 = -2)$$

Numerical Method

The matrix generated by the code is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The eigenvectors generated by the code are

$$\vec{v}_0 = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} (\lambda_0 = -2)$$

$$\vec{v}_1 = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{6} \\ -\frac{1}{6} \\ -\frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = -\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \end{pmatrix} (\lambda_1 = 1)$$

$$\vec{v}_2 = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} = -1 \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} (\lambda_2 = 4)$$

$$\vec{v}_3 = \begin{pmatrix} 0.11324429 \\ -0.53493029 \\ -0.37400418 \\ 0.51108938 \\ -0.1370852 \\ 0.0238409 \\ 0.51108938 \\ -0.1370852 \\ 0.0238409 \end{pmatrix} \approx -0.573 \begin{pmatrix} \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \end{pmatrix} + 0.803 \begin{pmatrix} \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \end{pmatrix} - 0.197 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix} (\lambda_3 = 1)$$

$$\vec{v}_4 = \begin{pmatrix} -0.24595211 \\ -0.41364672 \\ 0.65959883 \\ 0.12297605 \\ 0.20682336 \\ -0.32979941 \\ 0.12297605 \\ 0.20682336 \\ -0.32979941 \end{pmatrix} \approx -0.369 \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} - 0.929 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ 0 \\ -\frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{pmatrix} - 3.70 \times 10^{-17} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ -0.5 \\ 0 \\ -0.5 \\ 0.5 \end{pmatrix} (\lambda_4 = -2)$$

$$\vec{v}_5 = \begin{pmatrix} 0.0090808 \\ 0.02226828 \\ -0.12961424 \\ 0.45668828 \\ 0.46987576 \\ 0.31799324 \\ -0.34026152 \\ -0.32707404 \\ -0.47895656 \end{pmatrix} \approx -0.0695 \begin{pmatrix} \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \end{pmatrix} + 0.0887 \begin{pmatrix} \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \end{pmatrix} + 0.186 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix} + 0.976 \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix} (\lambda_5 = 1)$$

$$\vec{v}_6 = \begin{pmatrix} -0.03474412 \\ 0.08808924 \\ -0.05334512 \\ 0.40567488 \\ -0.60218226 \\ 0.19650738 \\ -0.37093076 \\ 0.51409302 \\ -0.14316226 \end{pmatrix} \approx -0.0521 \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} + 0.122 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ 0 \\ -\frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{pmatrix} + 0.673 \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{pmatrix} - 0.728 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ -0.5 \\ 0 \\ -0.5 \\ 0.5 \end{pmatrix} (\lambda_6 = -2)$$

$$\vec{v}_7 = \begin{pmatrix} 0.04458823 \\ -0.0808107 \\ 0.03622247 \\ -0.08066776 \\ 0.56331065 \\ -0.48264289 \\ 0.03607953 \\ -0.48249995 \\ 0.44642042 \end{pmatrix} \approx 0.0669 \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} - 0.101 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ 0 \\ -\frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{pmatrix} - 0.101 \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{pmatrix} + 0.987 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ -0.5 \\ 0 \\ -0.5 \\ 0.5 \end{pmatrix} \; (\lambda_7 = -2)$$

$$\vec{v}_8 = \begin{pmatrix} -0.00405038 \\ 0.22633531 \\ -0.46641618 \\ -0.07495955 \\ 0.15542613 \\ -0.53732536 \\ 0.31099005 \\ 0.54137573 \\ -0.15137575 \end{pmatrix} \approx -0.173 \begin{pmatrix} \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \end{pmatrix} + 0.164 \begin{pmatrix} \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ \frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \end{pmatrix} + 0.848 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix} - 0.473 \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix} \; (\lambda_8 = 1)$$

Therefore, the results by the code and the analytical method agree with each other.

1D Cycle with 4 Vertices

Analytical Method

By $v_i(j) = cos(\frac{2\pi ij}{N})$ and $i = 0, 1, 2, 3, j = 0, 1, 2, 3, N = 4$, the eigenvectors are

$$\vec{v}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (\lambda_0 = 2) \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} (\lambda_1 = 0) \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} (\lambda_2 = -2) \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} (\lambda_3 = 0)$$

Since two of the vectors are identical, Gram-Schmidt procedure is applied to find the third vector, and the orthonormal set is

$$\{ \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \}$$

By $A = \sum_{i=0}^{n-1} \lambda_i |\tilde{v}_i\rangle\langle\tilde{v}_i|$,

$$\sum_{i=0}^{n-1} \lambda_i |\tilde{v}_i\rangle\langle\tilde{v}_i| = (2) \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix} + (-2) \begin{pmatrix} 0.25 & -0.25 & 0.25 & -0.25 \\ -0.25 & 0.25 & -0.25 & 0.25 \\ 0.25 & -0.25 & 0.25 & -0.25 \\ -0.25 & 0.25 & -0.25 & 0.25 \end{pmatrix}$$

$$\sum_{i=0}^{n-1} \lambda_i |\tilde{v}_i\rangle\langle\tilde{v}_i| = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Numerical Method

The matrix generated by the code is

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

The eigenvectors generated by the code are

$$\vec{v}_0 = \begin{pmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{pmatrix} (\lambda_0 = -2)$$

$$\vec{v}_1 = \begin{pmatrix} -0.5 \\ -0.5 \\ -0.5 \\ -0.5 \end{pmatrix} = -1 \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} (\lambda_1 = 2)$$

$$\vec{v}_2 = \begin{pmatrix} 0 \\ -0.707106781 \\ 0 \\ 0.707106781 \end{pmatrix} \approx \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = -1 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} (\lambda_2 = 0)$$

$$\vec{v}_3 = \begin{pmatrix} 0 \\ -0.707106781 \\ 0 \\ 0.707106781 \end{pmatrix} \approx \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = -1 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} (\lambda_3 = 0)$$

Therefore, the results by the code and the analytical method agree with each other.

2D Cycle with 16 Vertices

Analytical Method

By $B = A \square A$,

$$B = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

We know that the eigenstates of B must be $|v_i\rangle \otimes |v_j\rangle$, so

$$|v_0\rangle \otimes |v_0\rangle = \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix} (\lambda = 2 + 2 = 4) \quad |v_0\rangle \otimes |v_1\rangle = \begin{pmatrix} \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \end{pmatrix} (\lambda = 2 + 0 = 2)$$

$$|v_0\rangle \otimes |v_2\rangle = \begin{pmatrix} 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \end{pmatrix} (\lambda = 2 - 2 = 0) \quad |v_0\rangle \otimes |v_3\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \end{pmatrix} (\lambda = 2 + 0 = 2)$$

$$|v_1\rangle \otimes |v_0\rangle = \begin{pmatrix} \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} (\lambda = 0 + 2 = 2) \quad |v_1\rangle \otimes |v_1\rangle = \begin{pmatrix} 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} (\lambda = 0 + 0 = 0)$$

$$|v_1\rangle \otimes |v_2\rangle = \begin{pmatrix} \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} (\lambda = 0 - 2 = -2) \quad |v_1\rangle \otimes |v_3\rangle = \begin{pmatrix} 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} (\lambda = 0 + 0 = 0)$$

$$|v_2\rangle \otimes |v_0\rangle = \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \end{pmatrix} (\lambda = 2-2 = 0) \quad |v_2\rangle \otimes |v_1\rangle = \begin{pmatrix} \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \end{pmatrix} (\lambda = -2+0 = -2)$$

$$|v_2\rangle\otimes|v_2\rangle = \begin{pmatrix} 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \end{pmatrix} (\lambda = -2-2 = -4) \quad |v_2\rangle\otimes|v_3\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \end{pmatrix} (\lambda = -2+0 = -2)$$

$$|v_3\rangle\otimes|v_0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \end{pmatrix} (\lambda = 0+2 = 2) \quad |v_3\rangle\otimes|v_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \end{pmatrix} (\lambda = 0+0 = 0)$$

$$|v_3\rangle \otimes |v_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \end{pmatrix} (\lambda = 0 - 2 = -2) \quad |v_3\rangle \otimes |v_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \end{pmatrix} (\lambda = 0 + 0 = 0)$$

Numerical Method

The matrix generated by the code is

$$\begin{pmatrix}
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{pmatrix}$$

The eigenvectors generated by the code are

$$\vec{v}_0 = \begin{pmatrix} -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \end{pmatrix} = -1 \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix} (\lambda_0 = 4)$$

$$\vec{v}_1 = \begin{pmatrix} -0.5 \\ -0.25 \\ 0 \\ -0.25 \\ -0.25 \\ 0 \\ 0.25 \\ 0 \\ 0 \\ 0.25 \\ 0.5 \\ 0.25 \\ -0.25 \\ 0 \\ 0.25 \\ 0 \end{pmatrix} \approx -\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} (\lambda_1 = 2)$$

$$\vec{v}_2 = \begin{pmatrix} 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \end{pmatrix} (\lambda_2 = -4)$$

$$\vec{v}_3 = \begin{pmatrix} -0.0117062164 \\ 0.0991742826 \\ 0.465960423 \\ 0.355079924 \\ -0.238833320 \\ -0.127952821 \\ 0.238833320 \\ 0.127952821 \\ -0.465960423 \\ -0.355079924 \\ 0.0117062164 \\ -0.0991742826 \\ -0.238833320 \\ -0.127952821 \\ 0.238833320 \\ 0.127952821 \end{pmatrix} \approx -0.676 \begin{pmatrix} \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \end{pmatrix} - 0.362 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \end{pmatrix} + 0.642 \begin{pmatrix} \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} (\lambda_3 = 2)$$

$$\vec{v}_4 = \begin{pmatrix} 0.0641749974 \\ 0 \\ -0.0213916658 \\ 0.574171147 \\ -0.287085574 \\ -0.0213916658 \\ -0.287085574 \\ -0.213916658 \\ -0.0213916658 \\ 0.574171147 \\ 0.0641749974 \\ 0 \\ -0.287085574 \\ -0.0213916658 \\ -0.287085574 \\ -0.0213916658 \end{pmatrix} \approx -0.531 \begin{pmatrix} 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \end{pmatrix} + 0.0856 \begin{pmatrix} 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 0.574 \begin{pmatrix} 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0.617 \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \end{pmatrix} (\lambda_4 \approx 0)$$

$$\vec{v}_5 = \begin{pmatrix} -0.609377507 \\ 0 \\ 0.203125836 \\ 0.0570306275 \\ -0.0285153137 \\ 0.203125836 \\ -0.0285153137 \\ 0.203125836 \\ 0.203125836 \\ 0.0570306275 \\ -0.609377507 \\ 0 \\ -0.0285153137 \\ 0.203125836 \\ -0.0285153137 \\ 0.203125836 \end{pmatrix} \approx -0.463 \begin{pmatrix} 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \end{pmatrix} - 0.813 \begin{pmatrix} 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 0.0570 \begin{pmatrix} 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 0.349 \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \end{pmatrix} (\lambda_5 \approx 0)$$

$$\vec{v}_6 = \begin{pmatrix} -0.5 \\ 0.25 \\ 2.29288391 \times 10^{-16} \\ 0.25 \\ 0.25 \\ -5.36293023 \times 10^{-18} \\ -0.25 \\ -1.99973883 \times 10^{-17} \\ -1.05874935 \times 10^{-16} \\ -0.25 \\ 0.5 \\ -0.25 \\ 0.25 \\ -2.99001684 \times 10^{-17} \\ -0.25 \\ -6.61718391 \times 10^{-19} \end{pmatrix} \approx -\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \end{pmatrix} (\lambda_6 = -2)$$

$$\vec{v}_7 = \begin{pmatrix} 0.0515911993 \\ -0.130291045 \\ 0.463600414 \\ -0.384900568 \\ 0.206004607 \\ -0.127304762 \\ -0.206004607 \\ 0.127304762 \\ -0.463600414 \\ 0.384900568 \\ -0.0515911993 \\ 0.130291045 \\ 0.206004607 \\ -0.127304762 \\ -0.206004607 \\ 0.127304762 \end{pmatrix} \approx 0.729 \begin{pmatrix} \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 0.583 \begin{pmatrix} \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \end{pmatrix} + 0.360 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \end{pmatrix} (\lambda_7 = -2)$$

$$\vec{v}_8 = \begin{pmatrix} -0.0120276016 \\ 0.424236960 \\ -0.211193092 \\ -0.201016266 \\ -0.0296470031 \\ -0.382562355 \\ 0.169518488 \\ 0.242690871 \\ 0.211193092 \\ 0.201016266 \\ 0.0120276016 \\ -0.424236960 \\ -0.169518488 \\ -0.242690871 \\ 0.0296470031 \\ 0.382562355 \end{pmatrix} \approx -0.316 \begin{pmatrix} \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0.282 \begin{pmatrix} \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \end{pmatrix} + 0.884 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \end{pmatrix} + 0.198 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \end{pmatrix} (\lambda_8 = -2)$$

$$\vec{v}_9 = \begin{pmatrix} 0.00773428834 \\ -0.243352817 \\ 0.0977089513 \\ 0.137909577 \\ 0.334566885 \\ -0.0989483562 \\ 0.244592222 \\ -0.480210750 \\ -0.0977089513 \\ -0.137909577 \\ -0.00773428834 \\ 0.243352817 \\ -0.244592222 \\ 0.480210750 \\ -0.334566885 \\ 0.0989483562 \end{pmatrix} \approx 0.149 \begin{pmatrix} \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 0.127 \begin{pmatrix} \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \end{pmatrix} - 0.539 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \end{pmatrix} + 0.819 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \end{pmatrix} (\lambda_9 = -2)$$

$$\vec{v}_{10} = \begin{pmatrix} -0.0431564630 \\ 0.0552000308 \\ -0.120307047 \\ -0.0757813827 \\ -0.463857133 \\ 0.00534579813 \\ 0.484438484 \\ 0.158117712 \\ -0.120307047 \\ -0.0757813827 \\ -0.0431564630 \\ 0.0552000308 \\ 0.484438484 \\ 0.158117712 \\ -0.463857133 \\ 0.00534579813 \end{pmatrix} \approx -0.136 \begin{pmatrix} 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \end{pmatrix} + 0.08 \begin{pmatrix} 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0.134 \begin{pmatrix} 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 0.184 \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \end{pmatrix}$$

$$- 0.944 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \end{pmatrix} - 0.15 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \end{pmatrix} (\lambda_{10} \approx 0)$$

$$\vec{v}_{11} = \begin{pmatrix} -0.0441148669 \\ 0.0547842201 \\ 0.182068152 \\ -0.0805484500 \\ -0.405474694 \\ -0.287521159 \\ 0.431238924 \\ 0.149567874 \\ 0.182068152 \\ -0.0805484500 \\ -0.0441148669 \\ 0.0547842201 \\ 0.431238924 \\ 0.149567874 \\ -0.405474694 \\ -0.287521159 \end{pmatrix} \approx 0.163 \begin{pmatrix} 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \end{pmatrix} - 0.222 \begin{pmatrix} 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0.129 \begin{pmatrix} 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0.121 \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \end{pmatrix}$$

$$- 0.841 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \end{pmatrix} + 0.121 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \end{pmatrix} (\lambda_{11} \approx 0)$$

$$\vec{v}_{12} = \begin{pmatrix} 0.00208064714 \\ 0.00791473549 \\ -0.0262089826 \\ -0.0320430710 \\ 0.366643359 \\ 0.372477447 \\ 0.338353729 \\ 0.332519641 \\ 0.0262089826 \\ 0.0320430710 \\ -0.00208064714 \\ -0.00791473549 \\ -0.338353729 \\ -0.332519641 \\ -0.366643359 \\ -0.372477447 \end{pmatrix} \approx 0.0400 \begin{pmatrix} \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \end{pmatrix} + 0.0565 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \end{pmatrix} - 0.0341 \begin{pmatrix} \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0.997 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \end{pmatrix} (\lambda_{12} = 2)$$

$$\vec{v}_{13} = \begin{pmatrix} 0.00695375899 \\ 0.308521083 \\ 0.135390932 \\ -0.166176392 \\ -0.308095701 \\ -0.00652837667 \\ -0.179658528 \\ -0.481225852 \\ -0.135390932 \\ 0.166176392 \\ -0.00695375899 \\ -0.308521083 \\ 0.179658528 \\ 0.481225852 \\ 0.308095701 \\ 0.00652837667 \end{pmatrix} \approx -0.182 \begin{pmatrix} \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \end{pmatrix} + 0.671 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \\ 0 \\ \frac{1}{\sqrt{8}} \\ 0 \\ -\frac{1}{\sqrt{8}} \end{pmatrix} + 0.201 \begin{pmatrix} \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 0.690 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} \end{pmatrix} (\lambda_{13} = 2)$$

$$\vec{v}_{14} = \begin{pmatrix} 0.00667900842 \\ 0.378838404 \\ 0.374582861 \\ 0.0718584089 \\ -0.216104762 \\ -0.325808407 \\ -0.234592051 \\ -0.0554534627 \\ 0.374582861 \\ 0.0718584089 \\ 0.00667900842 \\ 0.378838404 \\ -0.234592051 \\ -0.0554534627 \\ -0.216104762 \\ -0.325808407 \end{pmatrix} \approx -0.068 \begin{pmatrix} 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \end{pmatrix} - 0.368 \begin{pmatrix} 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0.308 \begin{pmatrix} 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0.832 \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \end{pmatrix}$$

$$+ 0.01 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \end{pmatrix} - 0.278 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \end{pmatrix} (\lambda_{14} \approx 0)$$

$$\vec{v}_{15} = \begin{pmatrix} -0.00521694869 \\ -0.209121227 \\ -0.258192128 \\ -0.329073893 \\ 0.457251164 \\ 0.00727469378 \\ 0.0809439563 \\ 0.256134382 \\ -0.258192128 \\ -0.329073893 \\ -0.00521694869 \\ -0.209121227 \\ 0.0809439563 \\ 0.256134382 \\ 0.457251164 \\ 0.00727469378 \end{pmatrix} \approx 0.278 \begin{pmatrix} 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \end{pmatrix} + 0.26 \begin{pmatrix} 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0.118 \begin{pmatrix} 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 0.798 \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \end{pmatrix}$$

$$+ 0.376 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \\ 0 \end{pmatrix} - 0.246 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0.5 \end{pmatrix} \quad (\lambda_{15} \approx 0)$$

Therefore, the results by the code and the analytical method agree with each other.