Stochastic Simulation of Langevin Dynamics in Optical Systems

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I. INTRODUCTION

This document presents a numerical simulation of a driven, underdamped harmonic oscillator subject to two sources of stochastic perturbations:

1. Phase noise in the external driving force, modeled as a random Gaussian phase shift.
2. Thermal noise of the oscillator, introduced as a random Gaussian perturbation.

The system's dynamics are governed by the following differential equation:

$$\ddot{\zeta} + 2r\dot{\zeta} + \omega_m^2 \zeta + \frac{\xi(t)}{m} = -\frac{F(t)}{m}$$

where:

- ζ (denoted as x in the implementation) represents the oscillator's displacement along the x-axis.
- r is the damping coefficient, characterizing energy dissipation in the system.
- ω_m is the natural angular frequency of the oscillator.
- $\xi(t)$ represents the stochastic thermal force, modeled as a Gaussian white noise term.
- $F(t)$ is the external driving force, which includes a deterministic harmonic term as well as a stochastic phase noise component.

The numerical solution employs an adaptive Runge-Kutta 45 (RK45) method to integrate the system over time, capturing both deterministic oscillatory behavior and the influence of noise. Results are analyzed through Fourier transforms, phase-space visualizations, and an exponential envelope fitting to characterize the system's relaxation time.

II. FEATURES

- Phase Shift Noise: Affects the phase of the driving force $F(t)$, introducing random shifts after a few cycles.
- Gaussian Noise: Affects the damping term $\xi(t)$, adding random perturbations to the system's dynamics.
- RK45 Integration: The system is solved using the Runge-Kutta 45 method with adaptive step sizing and error control.
- Exponential Envelope Fitting: The code fits an exponential decay envelope to the oscillator's motion and calculates the relaxation time T_1.
- Visualization: The simulation generates position-time plots, Fourier Transform graphs, and phase space diagrams.

III. WORKFLOW

1. The RK45 solver is used twice to compute the system's position and velocity over time:
   - First, with no noise in the driving force ("noiseless" run).
   - Second, with noise in the driving force ("noisy" run).
2. The results from each run are interpolated to produce smooth position and velocity data for plotting.
3. A Fourier Transform is applied to the oscillator's position data in both runs to reveal frequency components.
4. The phase space diagram is constructed to visualize the system's trajectory under noisy and noiseless conditions.
5. The exponential envelope of the noiseless position data is fitted, yielding an estimate of the relaxation time T_1.
6. All plots are saved in `PDF` format without displaying them. Parallel `PNG` images are also created for use in an Excel workbook.
7. The simulation data are exported to a `CSV` file.
8. An Excel workbook is generated: one sheet holds the exported data, and separate sheets contain each of the `PNG` figures inserted as images.

IV. RELAXATION TIME CALCULATION

The exponential decay of the oscillator's motion is fitted using the envelope function:

$$Ae^{-\gamma t}$$

where γ is the damping coefficient, and the **relaxation time** T_1 is calculated as:

$$T_1 = \frac{1}{\gamma}$$

The relaxation time is printed as output:

```
Relaxation time T1: X.XXXX seconds
```

V. HOW TO RUN THE CODE

A. Prerequisites

Ensure you have the following Python packages installed:

```
pip install numpy scipy matplotlib pandas
xlsxwriter
```

B. Running the Code

To run the simulation, execute the main script. You will be prompted for a detuning factor:

```
Enter the detuning factor (e.g., 0.5 for half
a linewidth):
```

After entering a value, the script runs two simulations (with and without noise), saves all PDF figures in the `pdf` directory, and saves corresponding PNG images in the `png` directory. Data files (CSV and Excel) are stored in the `data` folder.

C. Output

- Position-Time Plots (position vs. time): Show the oscillator's position over time for noisy and noiseless simulations.
- Fourier Transform Plots (amplitude vs. frequency): Display frequency analysis of the oscillator's motion.
- Phase Space Diagrams (momentum vs. position): Visualize the system's trajectory in phase space.
- Exponential Envelope Fittings (absolute position vs. time): Fit an exponential envelope to the noiseless data and displays the relaxation time T_1.
- Data Files:
  - `simulation_data.csv`: Contains time, positions, velocities, and absolute position.
  - `simulation_output.xlsx`: Includes a data sheet and separate sheets with inserted PNG images of each plot.