AIMS-Sénégal
Year 2015-16
MBour, February 2016

Graphs and Applications, Assignment 2
Max-Flow Min-Cut & Potts Model

Part 1: Study of a Network.

Consider the network with edge capacities, source and target as shown in Figure 1 below.

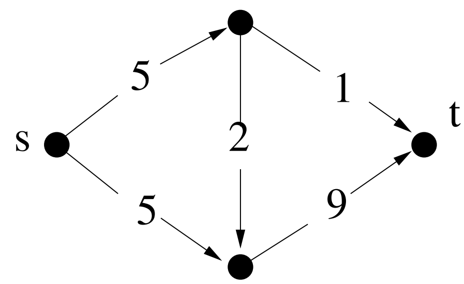

Figure 1: A network and its capacities, the source s and target t.

1) List all the cuts of this network and their capacities. Find a min-cut for this network. Are there several different min-cuts?

2) Design a max-flow for this network. What is its total flow? Are there several different flows that are max-flows?

3) We now consider the same network but with more general capacities a, b, c, d, e as shown in Figure 2 below. We suppose that $a + b = c + d$. Under which condition on b, c, e can this network transport a total flow $a + b$?

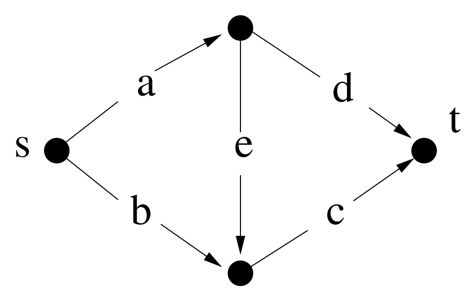

Figure 2: A network and its capacities a, b, c, d and e.

4) We consider finally the same network with capacities a, b, c, d, e as shown but no longer suppose that $a+b = c+d$. Using the max-flow-min-cut theorem, find under which conditions on a, b, c, d, e can this network transport a total flow $\min(a + b, c + d)$.

Part 2: Potts Model.

Consider the complete graph $G_2 = K_4$ (Figure 3).

(0) Compute the partition function $Z_{G_2}^{\text{Potts}}(q; \{y_e\})$ of the Potts model with q-states (or colors) on G_2, as a function of $\beta = 1/(kT)$ and of the edges activities $y_e = \exp(-\beta J_e) - 1$.

(1) Assume that all J_e are constant and fixed to $J_e = J$ and that $q < 4$:

(1.1) What is the probability that the 4 vertices all have different colors ?

(1.2) What is the probability that the 4 vertices all have the same color ?

(2) Assume that all J_e are constant and fixed to $J_e = J$ and that $q \geq 4$:

(2.1) What is the probability $P_{\neq}$ that the 4 vertices all have different colors ?

(2.2) Check that if $J_e > 0$, $\forall e$, and $T \to 0$, i.e. $\beta \to \infty$, this probability tends to 1.

(2.3) What is the probability $P_{=}$ that the 4 vertices all have the same color ?

(2.4) Check that if $J_e < 0$, $T \to 0$, i.e. $\beta \to \infty$, this probability tends also to 1.

(2.5) Why $P_{=} + P_{\neq} \neq 1$? To which event does the probability $1 - P_{=} - P_{\neq}$ correspond?

- **Numerical application:** Evaluate $Z_{G_2}^{\text{Potts}}(q; \{y_e\})$, $P_{=}$ and $P_{\neq}$ with two digits precision for the cases: $\{q = 3,\ \beta = 1,\ J_e = J = 1\}$; $\{q = 4,\ \beta = 1,\ J_e = J = 1\}$ and $\{q = 4,\ \beta = 10,\ J_e = J = 1\}$.

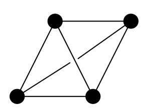

Figure 3: The complete graph $G_2 = K_4$