7: Binary

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We’ll start with a review of our earlier work.

```
(define-struct Z ())
(define-struct S (pred))
```

Data definition: a Nat is either `(make-Z)` or it is `(make-S p)`, where `p` is a Nat.

`(make-Z)` represents 0; `(make-S p)` represents $n + 1$, where `p` is a Nat representing n.

We will call this definition the **unary definition of Nat**, because we will shortly introduce another one.

The unary representation of the natural number 3 is
`(make-S (make-S (make-S (make-Z))))`.

Imagine the unary representation of 1000. It wouldn’t fit on this slide. Intuitively speaking, this means it takes up a lot of space in the memory of a computer.

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Recall our implementation of addition.

```
(define (plus nat1 nat2)
  (cond
    [(Z? nat1) nat2]
    [(S? nat1) (make-S (plus (S-pred nat1) nat2))]))
```

Imagine how long the trace would be if we added the representations of 1000000 and 2000000. This corresponds to a long running time on a computer.

We could try to interpret `make-S` as something other than “successor”.
So `(make-S (make-Z))` could be a number different from one.

$$0 \xrightarrow{S} 1 \xrightarrow{S} 2 \xrightarrow{S}$$

Towards a better representation

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The simplest change we can make is to use two data constructors.

```
(define-struct Z ())
(define-struct A (way))
(define-struct B (gone))
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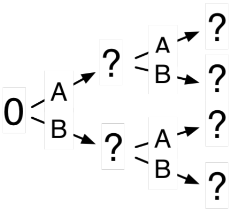
```

Unlike with the unary definition, there is no significance to the names `A` and `B`, and the names of the fields are chosen to indicate “removal”.



Desired properties of our definition

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- Every natural number should be representable.
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- The representation should be unique.
  Each natural number should have exactly one representation.
- Computation using the representation should be relatively straightforward and efficient.
- It would be nice if there were a “natural” interpretation of the representation, and if small numbers had a small representation.

The binary definition of Nat, completed

```
(define-struct Z ())
(define-struct A (way))
(define-struct B (gone))
```

A Nat is either `(make-Z)`, or it is `(make-A p)`, or it is `(make-B p)`, where `p` is a Nat.

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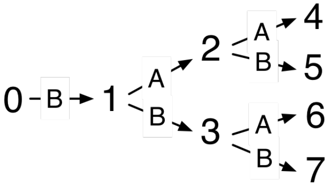
We add the rule: `make-A` cannot be applied to `(make-Z)`.

The representation of 5 is
`(make-B (make-A (make-B (make-Z))))`.



A Nat is either `(make-Z)`, or it is `(make-A p)`, or it is `(make-B p)`, where `p` is a Nat.

```
(define (my-nat-fn nat)
  (cond
    [(Z? nat) ...]
    [(A? nat) ...]
    [(B? nat) ...]))
```

```
(define (my-nat-fn nat)
  (cond
    [(Z? nat) ...]
    [(A? nat) ... (my-nat-fn (A-way nat)) ...]
    [(B? nat) ... (my-nat-fn (B-gone nat)) ...]))
```

```
(define (my-nat-fn nat)
  (cond
    [(Z? nat) ...]
    [(A? nat) ... (my-nat-fn (A-way nat)) ...]
    [(B? nat) ... (my-nat-fn (B-gone nat)) ...]))

(define (from-Nat nat)
  (cond
    [(Z? nat) 0]
    [(A? nat) (* 2 (from-Nat (A-way nat)))]
    [(B? nat) (add1 (* 2 (from-Nat (B-gone nat))))]))
```

`(make-Z)` represents 0.

`(make-A p)` represents $2n$, where `p` represents n.

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Converting built-in natural numbers to binary Nats

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`(make-B p)` represents $2n + 1$, where `p` represents n.

```
(define (to-Nat k)
  (cond
    [(zero? k) (make-Z)]
    [(even? k) (make-A (to-Nat (/ k 2)))]
    [(odd? k)  (make-B (to-Nat (/ (sub1 k) 2)))]))
```

```
(define (to-Nat k)
  (cond
    [(zero? k) (make-Z)]
    [(even? k) (make-A (to-Nat (quotient k 2)))]
    [(odd? k)  (make-B (to-Nat (quotient k 2)))]))
```

Implementing addition for binary Nats

```
(define (my-nat-fn nat)
  (cond
    [(Z? nat) ...]
    [(A? nat) ... (my-nat-fn (A-way nat)) ...]
    [(B? nat) ... (my-nat-fn (B-gone nat)) ...]))

(define (plus nat1 nat2)
  (cond
    [(Z? nat1) nat2]
    [(A? nat1) ... (plus (A-way nat1) nat2) ...]
    [(B? nat1) ... (plus (B-gone nat1) nat2) ...]))
```

```
(define (my-nat-fn nat)
  (cond
    [(Z? nat) ...]
    [(A? nat) ... (my-nat-fn (A-way nat)) ...]
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```
(define (plus nat1 nat2)
  (cond
    [(Z? nat1) nat2]
    [(A? nat1) ... (plus (A-way nat1) nat2) ...]
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```

If `nat1` represents $2x$ and `nat2` represents y,
then `(A-way nat1)` represents x.

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(define (my-nat-fn nat)
  (cond
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(define (plus nat1 nat2)
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```

If `nat1` represents $2x$ and `nat2` represents y,
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What does $x + y$ tell us about $2x + y$?

```
(define (my-nat-fn nat)
  (cond
    [(Z? nat) ...]
    [(A? nat) ... (my-nat-fn (A-way nat)) ...]
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(define (plus nat1 nat2)
  (cond
    [(Z? nat1) nat2]
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```

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then `(A-way nat1)` represents x.

What does $x + y$ tell us about $2x + y$?

This is too simplistic an approach.

Using structural recursion on two binary Nats

```
(define (my-nat-fn nat1 nat2)
  (cond
    [(Z? nat1) ...]
    [(Z? nat2) ...]
    [(and (A? nat1) (A? nat2))
      ... (my-nat-fn (A-way nat1) (A-way nat2))]
    [(and (A? nat1) (B? nat2))
      ... (my-nat-fn (A-way nat1) (B-gone nat2))]
    [(and (B? nat1) (A? nat2))
      ... (my-nat-fn (B-gone nat1) (A-way nat2))]
    [(and (B? nat1) (B? nat2))
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      ... (my-nat-fn (A-way nat1) (B-gone nat2))]
    [(and (B? nat1) (A? nat2))
      ... (my-nat-fn (B-gone nat1) (A-way nat2))]
    [(and (B? nat1) (B? nat2))
      ... (my-nat-fn (B-gone nat1) (B-gone nat2))]))
```

```
(define (plus nat1 nat2)
  (cond
    [(Z? nat1) nat2]
    [(Z? nat2) nat1]
    [(and (A? nat1) (A? nat2)) ... (plus (A-way nat1) (A-way nat2))]
    [(and (A? nat1) (B? nat2)) ... (plus (A-way nat1) (B-gone nat2))]
    [(and (B? nat1) (A? nat2)) ... (plus (B-gone nat1) (A-way nat2))]
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```

More details of addition

```
(define (plus nat1 nat2)
  (cond
    [(Z? nat1) nat2]
    [(Z? nat2) nat1]
    [(and (A? nat1) (A? nat2)) ... (plus (A-way nat1) (A-way nat2))]
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```

```
(define (plus nat1 nat2)
  (cond
    [(Z? nat1) nat2]
    [(Z? nat2) nat1]
    [(and (A? nat1) (A? nat2))
       (make-A (plus (A-way nat1) (A-way nat2)))]
    [(and (A? nat1) (B? nat2))
       (make-B (plus (A-way nat1) (B-gone nat2)))]
    [(and (B? nat1) (A? nat2))
       (make-B (plus (B-gone nat1) (A-way nat2)))]
    [(and (B? nat1) (B? nat2))
       ... (plus (B-gone nat1) (B-gone nat2)) ...]))
```

Addition for binary Nats, completed

```
(define (plus nat1 nat2)
  (cond
    [(Z? nat1) nat2]
    [(Z? nat2) nat1]
    [(and (A? nat1) (A? nat2))
       (make-A (plus (A-way nat1) (A-way nat2)))]
    [(and (A? nat1) (B? nat2))
       (make-B (plus (A-way nat1) (B-gone nat2)))]
    [(and (B? nat1) (A? nat2))
       (make-B (plus (B-gone nat1) (A-way nat2)))]
    [(and (B? nat1) (B? nat2))
       (make-A (plus-one (plus (B-gone nat1) (B-gone nat2))))]))

(define (plus-one nat)
  (cond
    [(Z? nat) (make-B (make-Z))]
    [(A? nat) (make-B (A-way nat))]
    [(B? nat) (make-A (plus-one (B-gone nat)))]))
```

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A natural interpretation of Nat

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A 32-bit word representing the natural number 4 would be filled with zeroes on the left:
`00000000000000000000000000000100`

Reasons for the use of binary in computers

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Our code essentially uses the addition table for binary digits.

+	0	1
0	0	1
1	1	10

The correspondence between the binary addition table and our algorithm becomes clearer if we rewrite the table slightly.

+	A	B
A	A	B
B	B	A +1

In contrast, the addition table for decimal digits is much larger, and would lead to more complex hardware.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

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Space analysis of binary Nats

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For the time being, we'll take the number of constructors used in a Nat as a measure of memory use.

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```
(define (size nat)
  (cond
    [(Z? nat) 1]
    [(A? nat) (add1 (size (A-way nat)))]
    [(B? nat) (add1 (size (B-gone nat)))]))
```

Let's define $S(n)$ as `(size nat)`, where `nat` is the binary representation of n.

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n	`nat`	$S(n)$
0	`(make-Z)`	1
1	`(make-B (make-Z))`	2
2	`(make-A (make-B (make-Z)))`	3
3	`(make-B (make-B (make-Z)))`	3
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Let's define $S(n)$ as `(size nat)`, where `nat` is the binary representation of n.

We're interested in how quickly $S(n)$ grows.

n	`nat`	$S(n)$
0	`(make-Z)`	1
1	`(make-B (make-Z))`	2
2	`(make-A (make-B (make-Z)))`	3
3	`(make-B (make-B (make-Z)))`	3
4	`(make-A (make-A (make-B (make-Z))))`	4
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Conjecture: For $2^k \leqslant n < 2^{k+1}$, $S(n) = k + 2$.

Or: For $k \leqslant \log_2 n < k + 1$, $S(n) = k + 2$.

Or: For $n > 0$, $S(n) = \lfloor \log_2 n \rfloor + 2$.

```
(define (plus-one nat)
  (cond
    [(Z? nat) (make-B (make-Z))]
    [(A? nat) (make-B (A-way nat))]
    [(B? nat) (make-A (plus-one (B-gone nat)))]))
```

Time analysis of plus-one

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(define (plus-one nat)
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```

The recursive application of `plus-one` is on an argument whose size is one smaller than `nat`.

Thus, the number of applications of `plus-one` starting with an argument `nat` is bounded above by `(size nat)`.

Problem with analysis of plus

Our implementation of `plus` also reduces the size of its arguments in a similar fashion, but the last case is problematic:

```
[(and (B? nat1) (B? nat2))
   (make-A (plus-one (plus (B-gone nat1) (B-gone nat2))))]
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The code we developed is actually efficient, but it is difficult to see why.

We will rewrite the code to make it clearer.

Rewriting plus

```
[(and (B? nat1) (B? nat2))
      (make-A (plus-one (plus (B-gone nat1) (B-gone nat2))))]
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```
[(and (B? nat1) (B? nat2))
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Instead of using `plus-one`, we will develop a helper function `plus-s`, which computes the successor of the sum of its two arguments.

```
[(and (B? nat1) (B? nat2))
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Instead of using `plus-one`, we will develop a helper function `plus-s`, which computes the successor of the sum of its two arguments.

```
[(and (B? nat1) (B? nat2))
      (make-A (plus-s (B-gone nat1) (B-gone nat2)))]
```

```
[(and (B? nat1) (B? nat2))
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`plus` has recursive applications of itself, and an application of `plus-s`.

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It will turn out that `plus-s` has recursive applications of itself, and applications of `plus`.

`plus` and `plus-s` are **mutually recursive**.

```
(define (plus-s nat1 nat2)
  (cond
    [(Z? nat1) ...]
    [(Z? nat2) ...]
    [(and (A? nat1) (A? nat2))
       ... (plus??? (A-way nat1) (A-way nat2)) ...]
    [(and (A? nat1) (B? nat2))
       ... (plus??? (A-way nat1) (B-gone nat2)) ...]
    [(and (B? nat1) (A? nat2))
       ... (plus??? (B-gone nat1) (A-way nat2)) ...]
    [(and (B? nat1) (B? nat2))
       ... (plus??? (B-gone nat1) (B-gone nat2)) ...]))
```

plus-s completed

```
(define (plus-s nat1 nat2)
  (cond
    [(Z? nat1) (plus-one nat2)]
    [(Z? nat2) (plus-one nat1)]
    [(and (A? nat1) (A? nat2))
       (make-B (plus (A-way nat1) (A-way nat2)))]
    [(and (A? nat1) (B? nat2))
       (make-A (plus-s (A-way nat1) (B-gone nat2)))]
    [(and (B? nat1) (A? nat2))
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    [(and (B? nat1) (B? nat2))
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```

Analysis of the new version of plus

The mutually-recursive `plus` and `plus-s` always reduce the size of their arguments in recursive applications.

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This matches our intuition that the work done in adding two numbers should be proportional to the number of digits, not the value of the numbers (as it was for the unary definition of Nat).

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First, we will try to augment with a sign, as we do with decimal numbers. We can view -8 as a negative sign attached to the natural number 8.

```
(define-struct Pos (nat))
(define-struct Neg (nat))
```

Data definition: an Int is either a `(make-Pos n)` or a `(make-Neg n)`, where `n` is a Nat.

Our first problem is that `(make-Pos (make-Z))` and `(make-Neg (make-Z))` both represent the integer 0.

Problems with augmenting with a sign

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A bigger problem arises when we try to implement addition.
Some cases are easy.

```
(define (iplus int1 int2)
  (cond
    [(and (Pos? int1) (Pos? int2))
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        ???]
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This is too complicated.

Instead of augmenting with a sign, we will augment the structures we had for Nat with a new structure to represent the quantity -1.

A better idea: augmenting with negative one

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`(make-B p)` represents $2i + 1$, where `p` represents i.

An immediate advantage is that our representation of non-negative integers (natural numbers) is the same in Int as it is in Nat.

All our code for `plus` is still useful. We only have to add cases that involve `(make-N)`.

We still have the rule that we can't apply `make-A` to `(make-Z)`.

Ensuring unique representation

We still have the rule that we can't apply `make-A` to `(make-Z)`.

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One way to enforce these rules is to create **smart constructors**, which are replacements for `make-A` and `make-B` with the rules built in.

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One way to enforce these rules is to create **smart constructors**, which are replacements for `make-A` and `make-B` with the rules built in.

```
(define (sA int)
  (cond
    [(Z? int) int]
    [else (make-A int)]))

(define (sB int)
  (cond
    [(N? int) int]
    [else (make-B int)]))
```

As before, we define $S(i)$ to be the number of constructors in the representation of i.

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i	nat	$S(i)$
0	`(make-Z)`	1
-1	`(make-N)`	1
-2	`(make-A (make-N))`	2
-3	`(make-B (make-A (make-N)))`	3
-4	`(make-A (make-A (make-N)))`	3
-5	`(make-B (make-B (make-A (make-N))))`	4
-6	`(make-A (make-B (make-A (make-N))))`	4
-7	`(make-B (make-A (make-A (make-N))))`	4
-8	`(make-A (make-A (make-A (make-N))))`	4

Addition for Ints

```
(define (plus int1 int2)
  (cond
    [(Z? int1) int2]
    [(Z? int2) int1]
    ; add N cases here
    [(and (A? int1) (A? int2))
       (sA (plus (A-way int1) (A-way int2)))]
    [(and (A? int1) (B? int2))
       (sB (plus (A-way int1) (B-gone int2)))]
    [(and (B? int1) (A? int2))
       (sB (plus (B-gone int1) (A-way int2)))]
    [(and (B? int1) (B? int2))
       (sA (plus-one (plus (B-gone int1) (B-gone int2))))]))

(define (plus-one int)
  (cond
    [(Z? int) (sB (make-Z))]
    ; add N case here
    [(A? int) (sB (A-way int))]
    [(B? int) (sA (plus-one (B-gone int)))]))
```

```
; missing cases for plus

[(and (N? int1) (N? int2)) (sA (make-N))]
[(and (B? int1) (N? int2)) (sA (B-gone int1))]
[(and (N? int1) (B? int2)) (sA (B-gone int2))]
[(and (A? int1) (N? int2)) (sB (plus (A-way int1) (make-N)))]
[(and (N? int1) (A? int2)) (sB (plus (make-N) (A-way int2)))]

; missing case for plus-one

[(N? int) (make-Z)]
```

The resulting code preserves the nice features of `plus` for Nats that made it efficient.

Subtraction for Ints

Having a representation for negative numbers allows us to think about implementing subtraction.

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We will implement subtraction in terms of negation, and negation in terms of another helper function, `flip`, which maps the representation of i to the representation of $-(i + 1)$.

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```
(define (subt int1 int2) (plus int1 (negate int2)))
(define (negate int) (plus-one (flip int)))

(define (flip int)
  (cond
    [(N? int) (make-Z)]
    [(Z? int) (make-N)]
    [(A? int) (make-B (flip (A-way int)))]
    [(B? int) (make-A (flip (B-gone int)))]))
```

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```
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(define (flip int)
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    [(Z? int) (make-N)]
    [(A? int) (make-B (flip (A-way int)))]
    [(B? int) (make-A (flip (B-gone int)))]))
```

This also makes it clear that $S(i)$ grows like the logarithm to the base 2 of the absolute value of i.

To create a more readable version of Int, as before, we erase parentheses, replace `make-A` by 0 and `make-B` by 1, and read right-to-left.

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Similarly, we replace `make-N` by the left-infinite sequence of ones, which we'll write $\ldots 1$.

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Its "readable" representation is $\ldots 1011$.

Two's complement notation

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 4 = ...0100
 3 = ...011
 2 = ...010
 1 = ...01
 0 = ...0
-1 = ...11
-2 = ...10
-3 = ...101
-4 = ...100
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For example, the representation of -5 would be `11111111111111111111111111111011`.

Interpreted as a natural number instead, this 32-bit pattern represents 4294967291.

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The short code for negation and subtraction corresponds to a small amount of additional hardware to support those features.

Goals of this module

You should be able to write space- and time-efficient code that works with the unary and binary representations of natural numbers and integers.

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You should be able to read and work with binary notation for natural numbers and two's complement notation for integers.