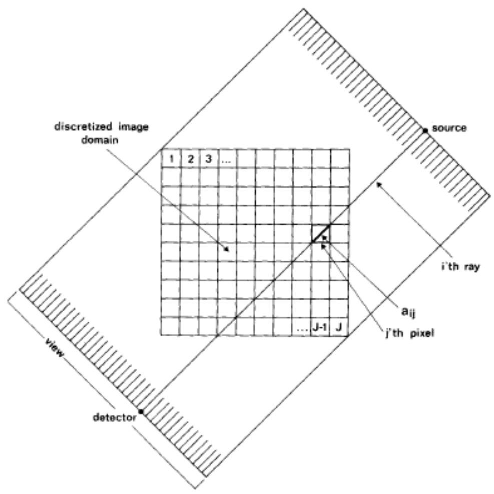
Lecture notes on topological spaces



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It would be fair to say that until you understand compactness you do not understand topology!

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Contents

CHAPTER 1

Topological Spaces

The concept of topological spaces grew out of the study of the real line and euclidean spaces. In this chapter, we define what a *topological space* is, and we study a number of ways of constructing a topology on a set so as to make it a topological space. We also consider some of the elementary concepts associated with topological spaces. *Open* and *closed* sets, *interior* and *closure* of a set, *limit points* are introduced as natural generalizations of the corresponding ideas for the real line and euclidian spaces.

1.1 Definition and Examples

Definition A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset$ and X are in $\mathcal{T}$.
2. An arbitrary union of elements of $\mathcal{T}$ is in $\mathcal{T}$.
3. A Finite intersection of elements of $\mathcal{T}$ is in $\mathcal{T}$.

A set X with a topology $\mathcal{T}$ on X is called a *a topological space.* Properly speaking, a topological space is an ordered pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X, but we often omit specific mention of $\mathcal{T}$ if no confusion will arise.

If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an open set of X if U belongs to the collection $\mathcal{T}$. Using this terminology, one can say that a topological space is a set

X together with a collection of subsets of X, called *open sets*, such that $\emptyset$ and X are both open, and such that arbitrary unions and finite intersections of open sets are open.

Example 1.1 If X is a set, take $\mathcal{T}$ to be $P(X)$, the power set of X. $\mathcal{T}$ is clearly a topology on X, it is called the discrete topology. In the discrete topology all subsets of X are open.

The other extreme is the indiscrete topology. In this case, $\mathcal{T} = \{\emptyset, X\}$. Again, it is clear that $\mathcal{T}$ is a topology on X.

Example 1.2 Let X be a three-element set, $X = \{a, b, c\}$. There are many possible topologies on X. For example, let $\mathcal{T}$ defined by $\mathcal{T} = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$. Then $\mathcal{T}$ is a topology on X.

Example 1.3 Let X be a set, let $\mathcal{T}_f$ be the collection of all subsets U of X such that $X - U$ either is finite or is X. Then $\mathcal{T}_f$ is a topology on X, called the finite complement topology.

Let $\mathcal{T}_c$ be the collection of all subsets U of X such that $X - U$ either is countable or is X. Then $\mathcal{T}_c$ is a topology on X.

Definition Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X. If $\mathcal{T} \subset \mathcal{T}'$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$. If $\mathcal{T}'$ properly contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly finer* than $\mathcal{T}$.

1.2 Basis for Topology

For each of the examples in the preceding section, we were able to specify the topology by describing the entire collection $\mathcal{T}$ of open sets. Usually this is too dificult. In most cases, one specifies instead a smaller collection of subsets and defines the topology in terms of that.

Definition If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

1. For each $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$, or equivalently $X = \cup_{B \in \mathcal{B}} B$.
2. If $x \in X$ and $B_1, B_2 \in \mathcal{B}$ are such that $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satifies these two conditions, then we define the **topology $\mathcal{T}$ generated by $\mathcal{B}$** as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. Note that each basis element is open.

Ex 1.4 *Show that $\mathcal{T}$ is a topology on X.*

Example 1.5 If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology.

Lemma 1.6 *Let X be a set, let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. Let $(B_i)_{i\in I}$ be a collection of elements of $\mathcal{B}$. Then for each $i \in I$, $B_i \in \mathcal{T}$. Because $\mathcal{T}$ is a topology, their union is in $\mathcal{T}$. Conversely, let $U \in \mathcal{T}$, choose for each $x \in U$ an element $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x\in U} B_x$, so U is a union of elements of $\mathcal{B}$.

This lemma states that every open set U can be expressed as a union of basis elements.

Remark 1.7 We have described how to go from a basis to the topology it generates. Sometimes we need to go in the reverse direction, from a topology to a basis generating it. Here is one way of obtaining a basis for a given topology; we shall use it frequently.

Proposition 1.8 *Let X be a topological space. Suppose that C is a collection of open subsets of X such that for each open set U of X and each $x \in U$, there exists $C \in C$ such that $x \in C \subset U$. Then C is a basis for the topology of X.*

Proof. Exercice.

If Topologies are given by basis, it is useful to have a criterion in terms of the bases for determining whether one topology is finer than an other. One such criterion is the following:

Proposition 1.9 *Let $\mathcal{B}$ and $\mathcal{B}'$ be basis for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:*

1. *$\mathcal{T}'$ is finer than $\mathcal{T}$.*
2. *For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there exists a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.*

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and $B \in \mathcal{B}$ such that $x \in \mathcal{B}$. We know that $B \in \mathcal{T}$ by definition and $\mathcal{T} \subset \mathcal{T}'$ by condition (1); therefore, $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, then there exists an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

(1) $\Rightarrow$ (2). Given an element $U \in \mathcal{T}$. We wish to show that $U \in \mathcal{T}'$. Let $x \in U$. Since $\mathcal{B}$ generate $\mathcal{T}$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$. By contidion (2) there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. Then $x \in B' \subset U$, so $U \in \mathcal{T}'$, by definition.

Definition If $\mathcal{B}$ is the collection of all open intervals in the real line $\mathbb{R}$,

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\},$$

the topology generated by $\mathcal{B}$ is called the **standard topology** on $\mathbb{R}$. Whenever, we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise.

If $\mathcal{B}'$ is the collection of all half-open intervals of the

$$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\},$$

where $a < b$, the topology generated by $\mathcal{B}'$ is called the *lower limit topology* on $\mathbb{R}$. When $\mathbb{R}$ is given with the lower limit topology, we denote it by $\mathbb{R}_l$.

Ex 1.10 Show that these two collections are basis of topology on $\mathbb{R}$.

Proposition 1.11 *The Topology of $\mathbb{R}_l$ is strictly finer than the standard topology on $\mathbb{R}$.*

Proof. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies of $\mathbb{R}$ and $\mathbb{R}_l$, respectively. Given a basis element (a, b) for $\mathcal{T}$ and a point $x \in (a, b)$, the basis element $[x, b)$ for $\mathcal{T}'$ contains x and lies in (a, b). On the other hand, given the basis element $[x, d)$ for $\mathcal{T}'$, there is no open interval (a, b) that contains x and lies in $[x, d)$. Thus $\mathcal{T}'$ is strictly finer than $\mathcal{T}$

A question may occur to you at this point. Since the topology generated by basis $\mathcal{B}$ may be described as the collection of arbitrary unions of elements of $\mathcal{B}$, what hapens if you start with a given collection of sets and take intersections of them as well as arbitrary unions? This question leads to the notion of subbasis for topology.

Definition A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union is X. The **topology generated by subbasis** $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

Ex 1.12 Prove that the collection of all finite intersections of elements of $\mathcal{S}$ is a basis.

Exercices

Ex 1.13 Let X be a topological space. Let A be a subset of X. Suppose that for each $x \in A$, there exists an open set U such that $x \in U \subset A$. Show that A is open in X.

Ex 1.14 Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all } X\}$$

a topology on X?

Ex 1.15

1. If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?
2. Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collection $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$
3. If $X = \{a, b, c\}$, let
$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \text{ and } \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$
Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

Ex 1.16 Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ is the smallest topology containing $\mathcal{A}$.

Ex 1.17 Show that the countable collection

$$\mathcal{B} = \{(a,b) \mid a < b,\ a,b \in \mathbb{Q}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

1.2.1 The Order Topology

Definition A relation $\mathcal{R}$ on a set A is called an order(or a simple order) if it has the following properties:

1. Comparability: For every x and y in A for which $x \neq y$, either $x\mathcal{R}y$ or $y\mathcal{R}x$.
2. Nonreflexivity: For no $x \in A$ does the relation $x\mathcal{R}x$ hold.
3. Transitivity: If $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$.

Note that properties (2) and (3) exclude the possibility for some pair of elements x and y of A, both $x\mathcal{R}y$ and $y\mathcal{R}x$ hold.

Example 1.18 Consider the relation on the real line consisting of all pairs (x,y) of real line such that $x < y$. It is an order relation.

Example 1.19 Consider on the real line the relation $\mathcal{R}$ defined by: $x\mathcal{R}y$ if $x^2 < y^2$ or if $x^2 = y^2$ and $x < y$. Then $\mathcal{R}$ is an order relation on $\mathbb{R}$.

If X is a simpler ordered set, there is a standard topology for X, defined using the order relation. It is called the *order topology*. In this section, we consider it and study some of its properties.

Suppose that X is a set having a simpler order $<$. Given elements a and b of X such that $a < b$, there are four subsets of X that are called intervals determined by a and b. The are the following:

$$(a,b) = \{x \mid a < x < b\},$$

$$(a,b] = \{x \mid a < x \leq b\},$$

$$[a,b) = \{x \mid a \leq x < b\},$$

$$[a,b] = \{x \mid a \leq x \leq b\}.$$

Remark 1.20 The notation used here is familiar to you already in the case where X is the real line, but these intervals are in arbitrary ordered set. A set of the first type is called an **open interval** in X, a set of the last type is called a **closed interval** in X, and sets of the second and third types ares called **half-open intervals**. The use of the term **open** in this connction suggests that open intervals in X should turn out to be open sets when we put a topology on X. And so they will.

Definition Let X be a set with a simple order relation; assume X has more than one element. Let $\mathcal{B}$ the collection of all sets of the following types:

1. All open intervals (a, b) in X.
2. All intervals of the form $[a_0, b)$, where a_0 is the smallest element(if any) of X.
3. All intervals of the form (a, b_0), where b_0 is the largest element(if any) of X.

The collection $\mathcal{B}$ is a basis for a topology on X, which is called the *order topology*.

Remark 1.21 If X has no smallest element, there are no sets of type (2), and X has no largest element, there are no sets of type (3).

Ex 1.22 Show that $\mathcal{B}$ is a basis for a topology on X.

Example 1.23 The standard topology on $\mathbb{R}$, as defined in the preceding section is just the order topology derived from the usual order on $\mathbb{R}$.

Example 1.24 The positive integer $\mathbb{N}$ form an ordered set with a smallest element. The order topology on $\mathbb{N}$ is the discrete topology. Every one-point set is open: If $n > 0$, then the one-point set $\{n\} = (n-1, n+1)$ is a basis element; and if $n = 0$, the one-point set $\{0\} = [0, 1)$ is a basis element.

Definition If X is an ordered set, and a is an element of X, there are four subsets of X that are called the rays dtermined by a. They are the following:

$$(a, +\infty) = \{x \mid x > a\},$$

$$(-\infty, a) = \{x \mid x < a\},$$

$$[a, +\infty) = \{x \mid x \geq a\},$$

$$(-\infty, a] = \{x \mid x \leq a\}.$$

Sets of the first two types are called **open rays**, and sets of the last two types are called **closed rays**.

Ex 1.25 Let X be an ordered set. Show that $(a, +\infty)$ and $(-\infty, a)$ are open for the order topology.

1.2.2 The Metric Topology

One of the important and frequently used ways of imposing a topology on a set is to define the topology in terms of a metric on a set. Topologies given in this way lie at the heart of modern analysis, for example. In this section, we shall define the metric topology and shall give a number of examples.

Definition A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ having the following properties:

1. $d(x,y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.

2. $d(x,y) = d(y,x)$ for all $x, y \in X$.

3. $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$ (Triangular inequality).

Given a metric d on X, the number $d(x,y)$ is called the *distance* between x and y in the metric d.

Let $\varepsilon > 0$, consider the set

$$B_d(x,\varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\}$$

of all points y whose distance from x is less than ε. It is called the *ε-ball* centered at x. Sometimes we omit d from the notation when no confusion will arise.

Lemma 1.26 *Let d be a metric on the set X. Then the collection of all ε-balls $B_d(x,\varepsilon)$, for $x \in X$ and $\varepsilon > 0$ is a basis for a topology on X, called the* **metric topology** *induced by d.*

Proof. The first condition for a basis is trivial since $x \in B(x,\varepsilon)$ for any $\varepsilon > 0$. Before checking the second condition for a basis, we prove the following fact: if $y \in B(x,\varepsilon)$ for some $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $B(y,\delta) \subset B(x,\varepsilon)$. Define $\delta = \varepsilon - d(x,y)$ then by triangular inequality, if $z \in B(y,\delta)$, then $d(x,z) \leq d(x,y) + d(y,z) < \varepsilon$. Now to check the second condition for basis, let B_1 and B_2 two basis elements and let $y \in B_1 \cap B_2$. Choose δ_1 and δ_2 such that $B(y,\delta_1) \subset B_1$ and $B(y,\delta_2) \subset B_2$. Let $\delta = \min(\delta_1,\delta_2)$, we have $B(y,\delta) \subset B_1 \cap B_2$. Using what we have just proved, we can rephrase the definition of the metric topology as follows:

Definition A set U is open in the metric topology induced by d if and only if for each $x \in U$, there exist $\varepsilon > 0$ such that

$$B_d(x,\varepsilon) \subset U.$$

Example 1.27 Given a set X, define

$$d(x,y) = 1 \ \ if \ \ x \neq y,$$

$$d(x,y) = 0 \ \ if \ \ x = y$$

It is easy to check that d is a metric. The topology it induces is the discrete topology; the basis element $B(x,1)$, for example, consists of the point x alone.

Example 1.28 The standard metric on the real numbers $\mathbb{R}$ is defined by $d(x,y) = |x-y|$. It is easy to check that d is a metric. The topology it induces is the same as the order topology; Each basis element (a,b) for the order topology is a basis element for the metric topology; indeed,

$$(a,b) = B(x,\varepsilon),$$

where $x = (a+b)/2$ and $\varepsilon = (b-a)/2$. And conversely, each *ε-ball* $B(x,\varepsilon)$ equals an open interval: the interval $(x-\varepsilon, x+\varepsilon)$.

Ex 1.29 In $\mathbb{R}^N$, consider

$$\begin{aligned} d_1(x,y) &= \sum_{i=1}^{N} |x_i - y_i|, \\ d_2(x,y) &= \Big(\sum_{i=1}^{N} |x_i - y_i|^2\Big)^{1/2}, \\ d_\infty(x,y) &= \max_{1\le i\le N} |x_i - y_i|. \end{aligned}$$

where $x = (x_1, \cdots, x_N)$ and $y = (y_1, \cdots, y_N)$. Show that d_1, d_2 and d_∞ are metrics on $\mathbb{R}^n$ and they generate the same topology on $\mathbb{R}^n$.

Ex 1.30 Let X be a metric space with metric d. Let $\bar{d} : X \times X \to \mathbb{R}$ defined by:

$$\bar{d}(x,y) = \min\{d(x,y), 1\}.$$

Show that $\bar{d}$ is metric that induces the same topology as d on X. The metric $\bar{d}$ is called the **standard bounded metric** corresponding to d.

1.2.3 Product Topology

If X and Y be topological spaces, there is a standard way of defining a topology on the cartesian product $X \times Y$. We consider this topology now and study some of its properties.

Lemma 1.31 *Let X and Y are topological spaces. Let $\mathcal{B}$ be the collection of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y. Then $\mathcal{B}$ is basis for a topology on $X \times Y$.*

Definition Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having the collection $\mathcal{B}$ as basis.

Proof of the lemma. The first condition is trivial, since $X \times Y$ is itself a basis element. The second condition almost as easy, since the intersection of any two basis elements $U_1 \times V_1$ and $U_2 \times V_2$ is an other basis element. For

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2),$$

and the later set is a basis element because $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in X and Y, respectively.

Ex 1.32 Prove that the collection $\mathcal{B}$ is not a topology on $X \times Y$.

What can one say if the topologies on X and Y are given by basis? The answer is as follows:

Theorem 1.33 *If $\mathcal{B}$ is basis for the topology on X and $\mathcal{C}$ is basis for the topology on Y, then the collection*

$$\mathcal{D} = \{B \times C \;\|\; B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topolgy on $X \times Y$.

Proof. We apply proposition 1.8.Given an open set W of $X \times Y$ and a point $(x, y) \in X \times Y$ of W, by definition of the product topology, there exists a basis element $U \times V$ such that $(x, y) \in U \times V \subset W$. Since $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y, respectively, we can choose an element $B \in \mathcal{B}$ such that $x \in B \subset U$, and an element $C \in \mathcal{C}$ such that $x \in C \subset V$. So $(x, y) \in B \times C \subset U \times V \subset W$. Thus the collection $\mathcal{D}$ meets the criterion of proposition 1.8, so $\mathcal{D}$ is a basis fro $X \times Y$.

Example 1.34 We have a standard topology on $\mathbb{R}$: the order topology. The product topology of this topology with itself is called the **the standard topology** on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. It has as basis the collection of all products open sets of $\mathbb{R}$, but the theorem just proved tells us that the much smaller collection of all products $(a, b) \times (c, d)$ of open intervals in $\mathbb{R}$ will also serve as a basis for the topology of $\mathbb{R}^2$. Each such set can be pictured as the interior of a rectangle in $\mathbb{R}^2$.

It is sometimes useful to express the product topology in terms of a subbasis. To do this, we just define certain functions called projections.

Definition Let $\pi_1 : X \times Y \to X$ and Let $\pi_2 : X \times Y \to Y$ be defined by

$$\pi_1(x, y) = x \;\text{ and }\; \pi_2(x, y) = y.$$

The maps π_1 and π_2 are called projection of $X \times Y$ onto it first and second factors, respectively.

We use the word onto because they are surjective(unless one of the spaces X or Y hapens to be empty, in which case $X \times Y$ is empty and our whole discussion is empty as well).

If U is an open subset of X, then $\pi_1^{-1}(U)$ is precisely the set $U \times Y$, which is open in $X \times Y$. Similarly, if V is open in Y, then $\pi_2^{-1}(V) = X \times V$, which is also open in $X \times Y$. The intersection of these two sets is the set $U \times V$. This fact leads to the following theorem:

Theorem 1.35 *The collection*

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{T}$ denote the product topology on $X \times Y$, let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Since $\mathcal{S} \subset \mathcal{T}$ then arbitrary unions of finite intersections of elements of $\mathcal{S}$ stay in $\mathcal{T}$. Thus $\mathcal{T}' \subset \mathcal{T}$. On the other hand, every basis element $U \times V$ for the topology $\mathcal{T}$ is a finite intersection of elements of $\mathcal{S}$, since

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V).$$

There for $U \times V$ belongs to $\mathcal{T}'$, so $\mathcal{T} \subset \mathcal{T}'$ as well.

1.2.4 The Subspace Topology

Definition Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the **subspace topology**. With this topology, Y is called a subspace of X; its open sets consist of all intersections of open sets of X with Y.

Ex 1.36 Prove that $\mathcal{T}_Y$ is a topology on Y.

Lemma 1.37 *If $\mathcal{B}$ is a basis for the topology of X the collection*

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. Let U be an open set of X and $y \in U \cap Y$. By definition of basis, there exists $B \in \mathcal{B}$ such that $y \in B \subset U$. Then $y \in B \cap Y \subset U \cap Y$. It follows from propositin 1.8 that $\mathcal{B}_Y$ is a basis for the subspace topology on Y.

When dealing with a space X and a subspace Y of X, one needs to be careful when one uses the term **open set**. Does one mean an element of the topology of Y or an element of the topology of X? We make the following definition: If Y is a subspace of X, we say that a set U is open in Y (or open relative to Y) if it belongs to the topology of Y; this implies in particular it is a subspace of Y.

There is a special situation in which every set open in Y is also open in X.

Lemma 1.38 *Let Y be a subspace of X. If U is open in Y and Y open in X then U is open in X.*

Proof. Since U is open in Y, $U = V \cap Y$ for some V open in X. Since Y and V are both open in X, so is $V \cap Y$.

Ex 1.39 Let A be a subspace of X and B be a subspace of Y. Show that the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Ex 1.40 1. Let $Y = [0, 1]$. Show that the subspace topology on Y ($\mathbb{R}$ is with the standard topology) and the order topology on Y are the same.
2. Let $Y = [0, 1) \cup \{2\}$. Show that the subspace topology on Y and the order topology on Y are different.

Theorem 1.41 *Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.*

Exercices

Ex 1.42 Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Ex 1.43 Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? which are open in $\mathbb{R}$?

$$A = \{x \mid \frac{1}{2} < |x| < 1\},\ \ B = \{x \mid \frac{1}{2} < |x| \leq 1\},\ \ C = \{x \mid \frac{1}{2} \leq |x| < 1\}.$$
$$D = \{x \mid \frac{1}{2} \leq |x| \leq 1\},\ \ E = \{x \mid 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{N}\}.$$

1.3 Closed Sets and Limit Points

Now that we have a few examples at hand, we can introduce some of the basic concepts associated with topological spaces. In this section, we treat the notions of *closed set, interior of set, closure of a set, and limit point*. These lead naturally to consideration of a certain axiom for topological spaces called the *Hausdorff axiom.*

1.3.1 Closed Sets

A subset A of a topological space X is said to be **closed** if $X - A$, the complement of A is open.

Example 1.44 The subset $[a, b]$ of $\mathbb{R}$ is closed because its complement

$$\mathbb{R} - [a, b] = (-\infty, a) \cup (b, \infty),$$

is open. Similarly, $[a, +\infty)$ is closed.

Example 1.45 Consider the following subset of the real line: $Y = [0, 1] \cup (2, 3)$, in the subspace topology. In this space, the set $[0, 1]$ is open, since it is the intersection of the open set $(-\frac{1}{2}, \frac{3}{2})$ of $\mathbb{R}$ with Y. Similarly, $(2, 3)$ is open as subset of Y. Since $[0, 1]$ and $(2, 3)$ are complement in Y of each other, we conclude that both are closed as subset of Y.

> **These examples suggest that an answer to the mathematician's riddle: How is a set different from a door? should be: A door must be either open or closed, and cannot be both, while a set can be open, or closed, or both, or neither**

The collection of closed subsets of a space X has properties similar to those satisfied by the collection of open subsets of X.

Theorem 1.46 *Let X be a topological space. Then the followin conditions hold:*

1. *$\emptyset$ and X are closed.*

2. *Arbitrary intersections of closed sets are closed.*

3. *Finite unions of closed sets are closed.*

Proof. Apply DeMorgan's laws:

$$X - \bigcap_{\alpha\in I} A_\alpha = \bigcup_{\alpha\in I}(X - A_\alpha).$$

$$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n}(X - A_i).$$

Remark 1.47 Instead of using open sets, one could just as well specify a topology on a space by giving a collection of sets(to be called closed sets) satisfying the three properties of this theorem. One could then define open sets as the complements of closed sets and proceed just as before. This procedure has no particular advantage over the one we have adopted, and most mathematicains prefer to use open sets to define topologies.

Now when dealing with subspaces, one needs to be careful in using the term closed set. We have the following theorem:

Theorem 1.48 *Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.*

Proof. Assume that $A = C \cap Y$, where C is closed in X. Then $X - C$ is open in X, so that $(X - C) \cap Y$ is open in Y, by definition of the subspace topology. But $(X - C) \cap Y = Y - A$. Hence $Y - A$ is open in Y, so that A is closed in Y. Conversely, assume that A is closed in Y. Then $Y - A$ is open in Y, so that by definition it equals the intersection of an open set U of X with Y. The set $X - U$ is closed in X, and $A = Y \cap (X - U)$, so that A equals the intersection of a closed set of X with Y, as desired.

Remark 1.49 A set A that is closed in Y may or may not be closed in X. As was the case with open sets, there is a criterion for A to be closed in X.

Theorem 1.50 *Let Y be a subspace of X. If A is closed in Y and Y closed in X, then A is closed in X.*

1.3.2 Closure and Interior of a Set

Definition Let A be a subset of a topological space X.

The **interior** of A denoted by $\mathrm{Int}(A)$ or $\mathring{A}$ is defined as the union of all open sets contained in A.
The **closure** of A denoted by $\mathrm{Cl}(A)$ or $\bar{A}$ is defined as the intersection of closed sets containing A.

Clearly the interior of A is an open set and the closure of A is a closed set; futhermore,

$$\mathring{A} \subset A \subset \bar{A}.$$

If A is open, $A = \mathring{A}$; while if A is closed, $A = \bar{A}$.

Ex 1.51 Let Y be a subspace of X; let A be a subset of Y. Let $\bar{A}$ denote the closure of A in X. Show that the closure of A in Y is $\bar{A} \cap Y$.

The definition of the closure of a set does not give us a convenient way for actually finding the closure of specific sets, since the collection of all closed sets in X, like the collection of all open sets, is usually too big to work with. Another way of describing the closure of a set, useful because it involves only a basis for the topology of X, is given in the following theorem:

Theorem 1.52 *Let A be a subset of the topological space X.*

(a) Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.

(b) Supposing the topology of X is given by a basis, then $x \in \bar{A}$ if and only if every basis element B containing x intersects A.

Proof. Consider the statement in (a). It is a statement of the form $P \Leftrightarrow Q$. Let us transform each implication to its contrapositive, thereby obtaining the logically equivalent statement (not P) $\Leftrightarrow$ (not Q). Written out, it is the following:

$x \notin \bar{A}$ if and only if there exists an open set U containing x that does not intersect A.

In this term our theorem is easy to prove. If x is not in $\bar{A}$, the set $X - \bar{A}$ is an open set containing x that does not intersect A, as desired. Conversely, if there is an open set U containing x which does not intersect A, then $X - A$ is a closed set containing A. By definition of the closure $\bar{A}$, the set $X - U$ must contain $\bar{A}$; therefore, $x \notin \bar{A}$.
Part (b) follows from the definition of basis.

Definition Let X be a topological space. Let $x \in X$ and V be a subset of X containing x. We say that V is a **neighborhood** of x if there exists an open set U of X containing x and contained in V. We denote by $\mathcal{N}(x)$, the collection of all neighborhood of x.

Ex 1.53 Let X be a topological space and $x \in X$. Prove:

1. $\mathcal{N}(x)$ is nonempty;
2. If $V \in \mathcal{N}(x)$ and $V \subset A$ then $A \in \mathcal{N}(x)$.
3. A finite intersection of neighborhoods of x is a neighborhood of x.

Ex 1.54 Let X be a topological space. Let U be a subset of X. Prove that U is open if and only if, $U \in \mathcal{N}(x)$ for every $x \in U$.

Lemma 1.55 *If A is a subset of a topological space X, then $x \in \bar{A}$ if and only if every neighborhood of x intersects A.*

Example 1.56 Let X be the real line $\mathbb{R}$. if $A = (0,1]$, then $\bar{A} = [0,1]$, $B = \{1/n \mid n \geq 1\}$ then $\bar{B} = B \cup \{0\}$. If $C = \{0\} \cup (1,2)$ then $\bar{C} = \{0\} \cup [1,2]$, $\bar{\mathbb{Q}} = \mathbb{R}$.

Example 1.57 Consider the subspace $Y = (0,1]$ of the real line $\mathbb{R}$. The set $A = (0, \frac{1}{2})$ is a subset of Y. its closure in $\mathbb{R}$ is the set $A = [0, \frac{1}{2}]$, and its closure in Y is the set $A = [0, \frac{1}{2}] \cap Y = A = (0, \frac{1}{2}]$.

1.3.3 Limit Points

Definition If A is a subset of the topological space X and if x is a point of X, we say that x is **limit point** (or cluster point or point of accumulation) of A if every neighborhood of x intersects A in some point other that x itself. Said differently, x is a limit point of A if x belongs to the closure of $A - \{x\}$. The point x may lie in A or not.

Theorem 1.58 *Let A be a subset of the topological space X. Let A' be the set of all limit points of A. Then*

$$\bar{A} = A \cup A'.$$

Proof. Clearly $A \cup A' \subset \bar{A}$. To prove the reverse inclusion, let $x \in \bar{A}$. If x happens to be in A, it is trivial that $x \in A \cup A'$; suppose that $x \notin A$. Since $x \in \bar{A}$, we know that every neighborhood U of x intersects A. Because $x \notin A$, the set U intersects A in a point different from x. Then $x \in A'$, so that $x \in A \cup A'$ as desired.

Corollary 1.59 *A subset of a topological space is closed if and only if it contains all its limit points.*

Proof. The set A is closed if and only if $A = \bar{A}$, and the later holds if and only if $A' \subset A$.

1.4 Hausdorf Spaces

One's experience with open and closed sets and limit points in the real line and the plane can be misleading when one consider more general topological spaces. For example, in the spaces $\mathbb{R}$ and $\mathbb{R}^2$, each one-point set is closed. But this fact is not true for arbitrary topological spaces. For example on the three-point set $X = \{a, b, c\}$, consider the topology $\mathcal{T}$ defined by

$$\mathcal{T} = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}.$$

In this space the one-point set $\{b\}$ is not closed, because it complement $\{a, c\}$ is not open.

Similarly, one's experience with the properties of convergence of sequences in $\mathbb{R}$ and $\mathbb{R}^2$ can be misleading when one deals with more general topological spaces. In $\mathbb{R}$ and $\mathbb{R}^2$ a sequence cannot converge to more than one point, but in general topology it can.

Topologies in which one-point sets are not closed, or in which sequences can converge to more than one point are considered by many mathematicians to be somewhat stange. They are not really very interesting. And the theorems that one can prove about topological spaces are rather limited if such examples are allowed. Therefore, one often imposes an additional condition what will rule out examples like this one, bringing the class of spaces under consideration closer to those to which one's geometric intuition applies. The condition was suggested by the mathematician Felix Hausdorff, so mathematicians have come to call it by his name.

Definition A topological space is called a **Hausdorff space**, if for each x_1, x_2 of distinct points of X, there exists neighborhood U_1 and U_2 of x_1 and x_2, respectively, that are disjoint.

Example 1.60 Every metric topology is Hausdorff.

Example 1.61 Every order topology is Hausdorff.

Example 1.62 The real line $\mathbb{R}$ in the finite complement topology is not Hausdorff.

Theorem 1.63 *Every finite point set in a Hausdorff space X is closed.*

Proof. It suffices to prove that every one-point set $\{x_0\}$ is is closed. If x is a point of X different from x_0, then x and x_0 have dijoints neighborhood U and V, respectively. Since U does not intersect $\{x_0\}$, the point point x cannot belong to the closure of the set $\{x_0\}$. As a result, the closure of the set $\{x_0\}$ is $\{x_0\}$ itself, so that it is closed.
The condition that finite point sets be closed is in fact weaker than the Hausdorff condition.b For example, the real line $\mathbb{R}$ in the finite complement topology is not a Hausdorff space, but it is a space in which finite point sets are closed. The condition that finite point sets be closed has been given a name by its own: T_1-**axiom**

Theorem 1.64 *Let X be a space satisfying the* T_1-**axiom**; *let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. If every neighborhood of x intersects A in infinitely many points, it certainly intersects A in some point different from x, so that x is a limit point of A.

Conversely, suppose that x is a limit point of A, and suppose that some neighborhood U of X intersects A in only finitely many points . Then U also intersects $A - \{x\}$ in finitely many points; let $\{x_1, \cdots, x_n\}$ be the points of $A \cap (A - \{x\})$. The set $X - \{x_1, \cdots, x_n\}$ is an open set of X, since the finite set $\{x_1, \cdots, x_n\}$ is closed; then

$$U \cap (X - \{x_1, \cdots, x_n\})$$

is a neighborhood of x that intersect the set $A - \{x\}$ not at all. This is contradicts the assumption that x is a limit point of A.

Definition Let $\{x_n\}$ be a sequence in a topological space X. We say that the sequence $\{x_n\}$ converges to $x \in X$ if for every neighborhood V of x, there exists $N \geq 1$, such that $x_n \in V$ whenever $n \geq N$.

Theorem 1.65 *If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.*

Proof. Suppose that $\{x_n\}$ is a sequence of X that converges to x. If $y \neq x$, let U and V be disjoint neighborhoods of x and y, respectively. Since U contains x_n for all but finitely many values of n, the set V cannot. Therefore $\{x_n\}$ cannot converege to y.

Ex 1.66 On the three-point set $X = \{a, b, c\}$, consider the topology $\mathcal{T}$ defined by

$$\mathcal{T} = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}.$$

Show that the sequence defined by $x_n = b$ for all n converges not only to b, but also to the point a and to the point c.

Exercices

Ex 1.67 Let C be a collection of subsets of the set X. Suppose that $\emptyset$ and X are in C, and that finite unions and arbitrary intersections of elements of C are in C. Show that the collection

$$\mathcal{T} = \{X - C \mid C \in C\}$$

is a topology on X.

Ex 1.68 In the spaces on real line obtained by giving it the indiscrete topology, the discrete topology, the usual metric topology, and the finite-complement topology, what is $\bar{A}$ if

1. $A = (0, 1]$
2. $A = [0, 1]$
3. $A = (0, 1)$

Ex 1.69 Let A, B and A_α denote subsets of a space X. Prove the following:

1. If $A \subset B$, then $\bar{A} \subset \bar{B}$.
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
3. $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$; give an example where equality fails.

Ex 1.70 Let A, B and A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusion $\supset$ or $\subset$ holds.

1. $\overline{A \cap B} = \bar{A} \cap \bar{B}$.
2. $\overline{\bigcap A_\alpha} = \bigcap \bar{A}_\alpha$
3. $\overline{A - B} = \bar{A} - \bar{B}$.

Ex 1.71 Show that if A is closed in X and if B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Ex 1.72 Let $A \subset X$ and $B \subset Y$. Show that in the product topology on $X \times Y$, $\overline{A \times B} = \bar{A} \times \bar{B}$.

Ex 1.73 Show that X is Hausdorff if and only if the **diagonal** $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

Ex 1.74 In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = 1/n$ converge?

Ex 1.75 If $A \subset X$, we define the **boundary** of A, ∂A or $\mathrm{Bd}\, A$ by:

$$\partial A = \bar{A} \cap \overline{X - A}.$$

Show that

1. $\mathring{A}$ and ∂A are disjoint, and $\bar{A} = \mathring{A} \cup \partial A$.
2. $\partial A = \emptyset$ if and only if A is both open and closed.
3. U is open if and only if $\partial U = \bar{U} - U$.

Ex 1.76 Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$:

1. $A = \{(x, y) \mid y = 0\}$.
2. $B = \{(x, y) \mid x > 0 \text{ and } y \neq 0\}$.
3. $C = A \cup B$.
4. $D = \{(x, x) \mid x \text{ is rational}\}$.

Ex 1.77
1. A topological space X satifies the *first separation axiom* T_1 if each one of any two points of X has a neighborhood that does not contain the other point. More formally: $\forall\ x, y \in X,\ x \neq y\ \exists U_y \in N(y):\ x \notin U_y$.

(a) Show that X satisfies T_1 if and only if all one-point sets in X are closed.

(b) Show that X satisfies T_1 if and only if every point is the intersection of all of its neighborhoods.

(c) Show that any Hausdorff space is T_1.

(d) Find an example showing that the T_1-axiom does not imply the hausdorff axiom.

2. topological space X satifies the *Kolmogorov axiom* T_0 if at least one of any two distinct points of X has a neighborhood that does not contain the other point.

(a) Show that X satisfies T_0 if and only if any two different points of X has different closures.

(b) Show that if X is T_1 then X is T_0. Find an example showing that the T_0-axiom does not imply the T_1-axiom.

CHAPTER 2

Continuous Functions

The concept of continuous function is basic in mathematics. Continuous functions on the real line appear in the first page of any caculus book, and continuous functions in the plane and in space follow not far behind. More general kinds of continuous functions arise as one goes futher in mathematics. In this chapter, we shall formulate a definition of continuity that will include all these as special cases, and we shall study various properties of continuities functions. Many of these properties are direct generalizations of things you learned about continuous functions in calculus and analysis.

2.1 Continuity of Function

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y , the set $f^{-1}(V)$ is an open subset of X, where

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}.$$

Continuity of a function depends not only upon the function f itself, but also on the topologies specified for its domain and range.

Theorem 2.1 *If the topology on the range Y is given by a basis $\mathcal{B}$, then f is continuous if and only if for any basis elemnt $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X.*

Proof. $\Leftrightarrow$ is obvious, because any basis element is an open subset of Y. Conversely, let V be an arbitrary open subset of Y. Then V can be written as a union of basis elements

$$V = \bigcup_{i \in I} B_i.$$

Therefore

$$f^{-1}(V) = \bigcup_{i \in I} f^{-1}(B_i).$$

So that $f^{-1}(V)$ is open since each $f^{-1}(B_i)$ is open.

Theorem 2.2 *If the topology on the range Y is given by a subbasis $\mathcal{S}$, then f is continuous if and only if for any subbasis element $S \in \mathcal{S}$, the set $f^{-1}(S)$ is open in X.*

Proof. Exercice. [Hint: use theorem 2.1].

Example 2.3 Let X and Y be topological spaces and $f : X \to Y$ be a function defined by $f(x) = y_0, \ \forall x \in X$, for some $y_0 \in Y$. Then f is continuous. [i.e. Any constant function is continuous].

Ex 2.4 Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x, & \text{if } x \leq 1, \\ \\ x+2, & \text{if } x > 1. \end{cases}$$

Is f continuous?(Justify your answer).

Ex 2.5 Let X be the subspace of $\mathbb{R}$ given by $X = [0,1] \cup [2,4]$. LDefine $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1], \\ \\ 2, & \text{if } x \in [2,4]. \end{cases}$$

Prove that f is continuous. (Does this surprise you?)

Example 2.6 Consider a real valued function of real variable $f : \mathbb{R} \to \mathbb{R}$. In Analysis one defines continuity via the ϵ-δ definition. As one would expect, the ϵ-δ definition and our are equivalent.

Given $x_0 \in \mathbb{R}$, and given $\epsilon > 0$, the interval $V = (f(x_0) - \epsilon, f(x_0) + \epsilon)$ is an open set of the range space $\mathbb{R}$. Therefore $f^{-1}(V)$ is an open set in the domain space $\mathbb{R}$. Since $x_0 \in f^{-1}(V)$, then there exists a basis element (a,b) containing x_0 such that $(a,b) \subset f^{-1}(V)$. Let $\delta = \min(x_0 - a, b - x_0)$. Thus if $|x - x_0| < \delta$, the point x must be in (a,b), so that $f(x) \in V$, and $|f(x) - f(x_0)| < \epsilon$, as desired.

Ex 2.7 Prove that the ϵ-δ definition implies our definition.

Example 2.8 Let $\mathbb{R}$ denote the set of real numbers in the usual topology, and let $\mathbb{R}_l$ denote the same set in the lower limit topology. Let $f : \mathbb{R} \to \mathbb{R}_l$ be the identity function: $f(x) = x$ for every $x \in \mathbb{R}$. Then f is not a continuous function, the inverse image of the open set $[a,b)$ of $\mathbb{R}_l$ equals itself is not open in $\mathbb{R}$. On the other hand the identity function $f : \mathbb{R}_l \to \mathbb{R}$ is continuous, because the inverse image of any basis element of $\mathbb{R}$ is itself, which is open in $\mathbb{R}_l$.

Theorem 2.9 *Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:*

(1) f is continuous.

(2) For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$.

(3) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

(4) For each $x \in X$ and each neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$.

If the condition in (4) holds for the point x, we say that f is **continuous at** *x.*

Proof. We show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and that (1) $\Rightarrow$ (4) $\Rightarrow$ (1)

(1) $\Rightarrow$ (2). Assume that f is continuous. Let A be a subset of X. We show that if $x \in \bar{A}$, then $f(x) \in \overline{f(A)}$. Let $x \in \bar{A}$ and let V be an open neighborhood of $f(x)$. Then $f^{-1}(V)$ is an open subset of X containing x, so $f^{-1}(V) \cap A \neq \emptyset$ because $x \in \bar{A}$. Let $y \in f^{-1}(V) \cap A$, then $f(y) \in V \cap f(A)$, thus $f(x) \in \overline{f(A)}$,as desired.

(2) $\Rightarrow$ (3). Let B be a closed subset of Y and let $A = f^{-1}(B)$. We wish to show that A is closed in X. We show that $\bar{A} = A$. By elementary set theory, we have $f(A) = f(f^{-1}(B)) \subset B$. Therefore, if $x \in \bar{A}$,

$$f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B,$$

so that $f(x) \in B$, thus $x \in f^{-1}(B) = A$, as desired.

(3) $\Rightarrow$ (1). Let V be an open subset of Y. Set $B = Y - V$. Then $f^{-1}(B) = X - f^{-1}(V)$. Now B is a closed set of Y, then $f^{-1}(B)$ is closed in X by hypothesis, so that $f^{-1}(V)$ is open in X, as desired.

(1) $\Rightarrow$ (4). Let $x \in X$ and let V be an open neighborhood of $f(x)$. Then the set $U = f^{-1}(V)$ is an open neighborhood of x such that $f(U) \subset V$.

(4) $\Rightarrow$ (1). Let V be an open set of Y. Let $x \in f^{-1}(V)$. Then $f(x) \in V$, so that by hypothesis there is an open neighborhood U_x of x such that $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$. It follows that $f^{-1}(V)$ can be written as the union of the open sets U_x, so that it is open.

2.2 Homeomorphisms

Definition Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and its inverse $f^{-1} : Y \to X$ are continuous, then f is called and homeomorphism.

Remark 2.10 The condition that f^{-1} be continuous says that for each open set U of X, the inverse image under f^{-1} is open in Y. But the inverse image of U under f^{-1} is the same as the image of U under f. So f is a homeomorphism if it is bijective and $f(U)$ is open in Y if and only if U is open in X.

This remark shows that a homeomorphism $f : X \to Y$ gives us a bijective correspondence not only between X and Y but between the collection of open sets of X and of Y.

Suppose that $f : X \to Y$ is an injective continuous map, where X and Y are topological spaces. Let Z be the image set $f(X)$, considered as a subspace of Y; then the function $f' : X \to Z$ obtained by restricting the range of f is bijective.

Definition If $f' : X \to Z$ is an homeomorphism, we say that the map $f : X \to Y$ is a **topological imbedding** or simply an **imbedding** of X in Y.

Example 2.11 The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 1$ is a homeomorphism.

Example 2.12 The function $F : (-1, 1) \to \mathbb{R}$ given by

$$F(x) = \frac{x}{1 - x^2}$$

is a homeomorphism.

Example 2.13 The identity map $g : \mathbb{R}_l \to \mathbb{R}$ is bijective and continuous, but it is not a homeomorphism.

Example 2.14 Let S^1 denote the unit circle,

$$S^1 = \{(x, y) \mid x^2 + y^2 = 1\},$$

considered as a subspace of the plane $\mathbb{R}^2$, and let

$$F : [0, 1] \to S^1$$

be the map defined by $f(t) = (cos 2\pi t, \sin 2\pi t)$. The map F is bijective and continuous, but F^{-1} is not continuous.

Theorem 2.15 *Let X, Y and Z be topological spaces.*

1. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.*
2. *(Restricting the domain). If $f : X \to Y$ is continuous, and if A is a subspace of X, then the restricted function $f|A : A \to Y$ is continuous.*
3. *(Restricting or expanding the range). Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the map $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.*

Proof. Exercice.

Theorem 2.16 (The pasting lemma) *Let $X = AUB$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then the fonction $h : X \to Y$ defined by*

$$h(x) = \begin{cases} f(x) \;\; if \;\; x \in A, \\ \\ g(x) \;\; if \;\; x \in B, \end{cases}$$

is continuous.

Proof. Let F be a closed subset of Y. Now

$$h^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$$

by elementary set theory. Since f and g are continuous, $f^{-1}(F)$ and $g^{-1}(F)$ are closed in A and B respectively, therefore they are also closed in X, because A and B are closed in X. So their union $h^{-1}(F)$ is closed in X, as desired.

Remark 2.17 This theorem also holds if A and B are open in X.

Example 2.18 Let us define $h : \mathbb{R} \to \mathbb{R}$ by:

$$h(x) = \begin{cases} x \;\text{ for }\; x \leq 0, \\ x/2 \;\text{ for }\; x \geq 0, \end{cases}$$

Each of the pieces of this function is continuous and they agree on the overlapping part of their domain, which is the singleton $\{0\}$. Since their domains are continous in $\mathbb{R}$, the functionnis continuous.

One needs the pieces of the function to agree on the overlapping part of their domains in order to have a function at all. For instance

$$k(x) = \begin{cases} x - 2 \;\text{ for }\; x \leq 0, \\ x + 2, \;\text{ for }\; x \geq 0, \end{cases}$$

do not define a function.

On the other hand, one needs some limitations on the sets A and B to guarantee continuity. For instance

$$l(x) = \begin{cases} x - 2 \;\text{ for }\; x < 0, \\ x + 2, \;\text{ for }\; x \geq 0, \end{cases}$$

define a function mapping $\mathbb{R}$ to $\mathbb{R}$, and both of the pieces are continuous. But l is not continuous. The inverse image of the open interval $(1, 3)$, for instance, is the set $[0, 1)$.

Theorem 2.19 (Maps in products) *Let $f : Z \to X \times Y$ be given by*

$$f(z) = (f_1(z), f_2(z)).$$

Then f is continuous if and only if the functions

$$f_1 : Z \to X \quad \text{and} \quad f_2 : Z \to Y$$

are continuous.

The maps f_1 and f_2 are called coordinate functions of f.

Proof. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be projections maps. These maps are continuous. Note that for each $z \in Z$,

$$f_1(z) = \pi_1(f(z)) \ \text{ and } \ f_2(z) = \pi_1(f(z)).$$

If the function f is continuous then f_1 and f_2 are continuous as composites of continuous functions.

Conversely, suppose that f_1 and f_2 are continuous. Let $U \times V$ a basis element for the product topology in $X \times Y$. A point z is in $f^{-1}(U \times V)$ if and only if $f(z) \in U \times V$, that is, if and only if $f_1(z) \in U$ and $f_2(z) \in V$. Therefore

$$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(U).$$

Since both of the sets $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open, so is their intersection.

2.2.1 Exercices

Ex 2.20 Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

Ex 2.21 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous from the rigfht, that is,

$$\lim_{x \to a^+} f(x) = f(a),$$

for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R}_l$ to $\mathbb{R}$.

Ex 2.22 Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by

$$f(x) = (x, y_0) \ \text{ and } \ g(y) = (x_0, y)$$

are imbeddings.

Ex 2.23 Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous functions.

(a) Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X.

(b) Let $h : X \to Y$ be the function

$$h(x) = \min\{f(x), g(x)\}.$$

Show that h is continuous. [Hint: Use the pasting lemma.]

2.3 The Product Topology

Definition Let $(X_i)_{i\in I}$ be an indexed family of topological spaces. Let

$$X = \prod_{i\in I} X_i$$

be the cartesian product of the spaces X_i. The collection of all sets of the form

$$\prod_{i\in I} U_i$$

where U_i is an open subset of X_i is a basis for a topology on X. The topology generated by this basis is called the **box topology**.

Let $\pi_i : X \to X_i$ be the function defined by $\pi_i(x = (x_i)) = x_i$, for every $x \in X$. The function π_i is called the ith-projection mapping. Let $\mathcal{S}_i$ denote the colection

$$\mathcal{S}_i = \{\pi_i^{-1}(U_i) \mid U_i \text{ open in } X_i\}.$$

and let

$$\mathcal{S} = \{\pi_i^{-1}(U_i) \mid U_i \text{ open in } X_i, \text{ and } i \in I\} = \bigcup_{i\in I} \mathcal{S}_i.$$

Lemma 2.24 *The collection $\mathcal{S}$ is a subbasis for a topology on the cartesian product X and the topology generated by $\mathcal{S}$ is called the* ***product topology****.*

To compare these two topologies, we consider the basis $\mathcal{B}$ genetared by $\mathcal{S}$. The collection $\mathcal{B}$ consists of all finite intersections of elements of $\mathcal{S}$. Let B and B' elements of $\mathcal{S}$, if they both come from $\mathcal{S}_i$ for some i, then

$$B \cap B' = \pi_i^{-1}(U_i) \cap \pi_i^{-1}(V_i) = \pi_i^{-1}(U_i \cap V_i),$$

which is again an element of $\mathcal{S}_i$, so we do not get anything new.

Now let $i_1, i_2, \cdots, i_n$ be different indices from the index set I, and let $U_{i_1}, U_{i_2}, \cdots U_{i_n}$ be open sets in X_{i_k} for $k = 1, \cdots, n$. Then

$$B = \pi_{i_1}^{-1}(U_{i_1}) \cap \pi_{i_2}^{-1}(U_{i_2}) \cap \cdots \pi_{i_n}^{-1}(U_{i_n})$$

is the typical element of $\mathcal{B}$. A point $x = (x_i)$ is in B if and only if $x_{i_k} \in U_{i_k}$ for $k = 1, 2, \cdots, n$. There is no restriction whatever on the *ith* coordinate of x if i is not one of the indices $i_1, i_2, \cdots, i_n$. As a result, we can write B as the product

$$B = \prod_{i\in I} U_i,$$

where $U_i = X_i$ if $i \neq i_1, i_2, \cdots, i_n$. All this is summarized in the following theorem.

Theorem 2.25 (Basis for the product topology) *The product topology has basis all sets of the form $\prod U_i$, where U_i is open in X_i for each i and U_i is X_i except for finitely many values of i.*

Two things are immediately clear. First, for finite product $\prod X_i$ the two topologies are precisely the same. Second, the box topology is in general finer than the product topology.

What is not so clear is why we prefer the product topology to the box topology. The answer will appear as we continue our study. We shall find that a number of important theorems about finite products will also hold for arbitrary products if we use the product topology, but not if we use the box topology. As a result, the product topology is extremely important in mathematics. The box topology is not so important; we shall use it primarily for constructing contreexamples. Therefore, we make the following convention:

> *Whenever we consider the product $\prod X_i$, we shall assume it is given the product topology unless we specifically state otherwise.*

Some of the theorems, we proved for the product $X \times Y$ hold for the product $\prod X_i$ no matter which topology we use. We list them here.

Theorem 2.26 *Suppose the topology on each X_i is given by a basis $\mathcal{B}_i$. The collection of all sets of the form*

$$\prod_{i\in I} B_i,$$

where $B_i \in \mathcal{B}_i$, will serve as basis for the box topology on $\prod_{i\in I} X_i$.
The collection of all sets of the same form, where $B_i \in \mathcal{B}_i$ for finitely many indices i and $B_i = X_i$ for all remaining indices, will serve as a basis for the produc topology on $\prod_{i\in I} X_i$.

Example 2.27 Consider the euclidean n-space $\mathbb{R}^n$. A basis for $\mathbb{R}$ consists of all open intervals in $\mathbb{R}$; hence a basis for the topology of $\mathbb{R}^n$ consists of all products of the form

$$(a_1, b_1) \times (a_2, b_2) \cdots (a_n, b_n).$$

Since $\mathbb{R}^n$ is a finite product, the box and product topologies agree. Whenever we consider $\mathbb{R}^n$, we will assume that it is given this topology, unless we specifically state othrwise.

Theorem 2.28 *Let A_i be a subspace of X_i, for each $i \in I$. Then $\prod A_i$ is a subspace of $\prod X_i$ if both products are given the box topology, or both products are given the product topology.*

Proof. Exercice.

Theorem 2.29 *If each space X_i is a Hausdorff space, then $\prod X_i$ is a Hausdorff space in both the box and product topologies.*

Proof. We prove the theorem in the case of product topology. For this end, assume that each X_i is Hausdorff. Let $x = (x_i)$ and $y = (y_i)$ be two distinct points in $\prod X_i$. Then for some indice

i_0, $x_{i_0} \neq y_{i_0}$. Since X_{i_0} is Hausdorff there exists disjoint open sets U and V such that $x_{i_0} \in U$ and $y_{i_0} \in V$. Define

$$U_i = \begin{cases} X_i & \text{if } i \neq i_0 \\ U & \text{if } i = i_0 \end{cases} \quad \text{and} \quad V_i = \begin{cases} X_i & \text{if } i \neq i_0 \\ V & \text{if } i = i_0 \end{cases}$$

Then $\prod U_i$ is a neighborhood of x, $\prod V_i$ is a neighborhood of y and they are disjoint. Therefore $\prod X_i$ is Hausdorff.

Theorem 2.30 *Let X_i be an indexed family of spaces. Let $A_i \subset X_i$ for each i. If $\prod X_i$ is given either the product or the box topology, then*

$$\overline{\prod A_i} = \prod \bar{A}_i.$$

So far, no reason has appeared for preferring the product to the box topology. It is when we try to generalize our previous theorem about continuity of maps into product spaces that a difference first arises. Here is a theorem that does not hold if $\prod X_i$ is given the box topology.

Theorem 2.31 *Let $f : Z \to \prod X_i$ be given by $f(z) = (f_i(z))$, where $f_i : Z \to X_i$ for each i. Then the function f is continuous if and only if each function f_i is continuous.*

Proof. Let π_i be the *ith* projection map. The function π_i is continuous. Indeed if U_i is open in X_i, the set $\pi_i^{-1}(U_i)$ is a subbasis element for the product topology.

Now suppose that f is continuous. We have $f_i(z) = \pi_i(f(z))$. So f_i is continuous as the composite of two continuous functions.

Conversely, suppose that each f_i is continuous. To prove that f is continuous, it suffices to prove that the inverse image under f of each subbasis element is open in Z. For this, let U_i be open in X_i, we have

$$f^{-1}(\pi_i^{-1}(U_i)) = f_i^{-1}(U_i),$$

because $f_i = \pi_i \circ f$. Since f_i is continuous, this set is open in Z, as desired.

`Why does this theorem fail if we use the box topology? Probably the most convicing thing to do is to look at an example.`

Example 2.32 Consider $\mathbb{R}^\omega$, the countable infinite product of $\mathbb{R}$ with itself. Recall that

$$\mathbb{R}^\omega = \prod_{n \in \mathbb{N}} X_n,$$

where $X_n = \mathbb{R}$ for each n. Let us define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = (t, t, t, \cdots, t, \cdots).$$

The nth coordinate function of f is the function $f_n(t) = t$. Each of the coordinate functions $f_n : \mathbb{R} \to \mathbb{R}$ is continuous; therefore the function f is continuous if $\mathbb{R}^\omega$ is given the product topology. But f is not continuous if $\mathbb{R}^\omega$ is given the box topology. Consider for example, the basis element

$$B = \prod_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \cdots$$

for the box topology. We assert that $f^{-1}(B)$ is not open in $\mathbb{R}$. If $f^{-1}(B)$ were open in $\mathbb{R}$, it would contain some interval $(-\delta, \delta)$ about the point 0. This would mean that $f((-\delta.\delta)) \subset B$, so that, applying π_n to both side of the inclusion,

$$f_n((-\delta, \delta)) = (-\delta, \delta) \subset (-1/n, 1/n)$$

for all n, a contradiction.

2.4 Continuity on Metric Sapces

In this section, we discuss the relation of the metric topology to the concepts we have previously introduced.

Subspaces of metric spaces behave the way one would wish them to; If A is a subspace of the topological space X and d is a metric for X, then the restriction of d to $A \times A$ is a metric for the topology of A.

Ex 2.33 Check this fact.

About order topology, some are metrizable (for instance $\mathbb{R}$) and others are not.

The Hausdorff axiom is satisfied by every metric topology. If x and y are distinct points of the metric space (X, d) we let $\epsilon = \frac{1}{2}d(x, y)$; then the triangular inequality implies that $B_d(x, \epsilon)$ and $B_d(y, \epsilon)$ are disjoint.

The product topology we have already considered in special cases; We will prove that $\mathbb{R}^\omega$ is metrizable. It is true in general that countable product of metrizable spaces is metrizable.

About continuity functions there is a good deal to be said. Consideration of this topic will occupy the remainder of this section.

When we study continuous functions on metric spaces, we are about as close to the study of calculus and analysis.

We want to show that the **familiar ϵ-δ definition of continuity** carries over to general metric spaces, and so does the **convergence sequence definition** of continuity.

Theorem 2.34 *Let $f : X \to Y$, where X and Y are metrizable spaces with metrics d_X and d_Y. Then continuity of f is equivalent to the requirement that given $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d_X(x,y) < \delta \;\Rightarrow\; d_Y(f(x), f(y)) < \epsilon.$$

Proof. Suppose that f is continuous. Given x and ϵ, consider the set

$$f^{-1}(B(f(x), \epsilon)),$$

which is open and contains the point x. It contains some δ-ball at x. If $y \in B(x, \delta)$, then $f(y) \in B(f(x), \epsilon)$, as desired.

Ex 2.35 *Prove the reverse direction.*

Now we turn to the convergence sequence definition of continuity. We begin by considering the relation between convergence sequences and closures of sets. It is certainly believable, from one's experience in analysis, that if x lies in the closure of a subset A, of the space X, then there should exist a sequence of points of A converging to x. This is not true in general, but it is true for metrizable spaces.

Lemma 2.36 (Sequence Lemma) . *Let X be a topological space. Let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is metrizable.*

Proof. Suppose that $x_n \to x$, where $x_n \in A$. Then every neighborhood U of x contains a point of A, so $x \in \bar{A}$. Conversely, suppose that X is metrizable and $x \in \bar{A}$. Let d be a metric for the topology of X. For each $n \geq 1$, the neighborhood $B(x, 1/n)$ intersects A. Choose x_n to be a point in the intersection. Then the sequence converges to x. Check it.

Theorem 2.37 *Let $f : X \to Y$. If the function f is continuous, then for every convergence sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.*

Proof. Assume that f is continuous. Given $x_n \to x$, we wish to show that $f(x_n) \to f(x)$. Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is a neighborhood of x, and so there is $N \geq 1$ such that $x_n \in f^{-1}(V)$ for $n \geq N$. Then $f(x_n) \in V$ for $n \geq N$, as desired.

Conversely, assume that the convergence sequence condition is satisfied. Let A be a subset of X. We show that $f(\bar{A}) \subset \overline{f(A)}$. If $x \in \bar{A}$, there is a sequence x_n of point of A converging to x (by sequence lemma). By assumption the sequence $f(x_n)$ converges to $f(x)$. Since $f(x_n) \in f(A)$, the sequence lemma implies that $f(x) \in \overline{f(A)}$, as desired.

Definition Given an index set J, and given points $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ of $\mathbb{R}^I$, let us define a metric $\bar{\rho}$ by:

$$\bar{\rho}(x,y) = \sup_{i \in I} \bar{d}(x_i, y_i),$$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$. It is easy to check that $\bar{\rho}$ is indeed a metric; it is called the uniform metric on $\mathbb{R}^I$, and the topology it induces is called the **uniform topology**.

The relation between this topology, the product and box topologies is the following.

Theorem 2.38 *The uniform topology is* ***finer*** *that the product topology and* ***coarser*** *than the box topology; these three topologies are all different if I is infinite.*

Proof. Suppose that we are given a point $x = (x_i)$ and product topology basis element $\prod U_i$ containing x. Let $i_1, i_2, \cdots, i_n$ be the indices for which $U_i \neq \mathbb{R}$. Then for each k, choose $\epsilon_k > 0$ such that $B(x_{i_k}, \epsilon_k) \subset U_{i_k}$, this we can do because U_{i_k} is open in $\mathbb{R}$. Let $\epsilon = \min \epsilon_k$; then $B_{\bar{\rho}}(x, \epsilon) \subset \prod U_i$. It follows that the uniform topology is finer that the product topology.

On the other hand let $x \in \mathbb{R}^I$ and $\epsilon > 0$. Then

$$U = \prod_{i \in I} (x_i - \frac{1}{2}\epsilon, x_i + \frac{1}{2}\epsilon,) \subset B_{\bar{\rho}}(x, \epsilon).$$

As desired.

In the case where I is infinite, we still have not determined whether $\mathbb{R}^I$ is metrzable in either the box or the product topology. It turns out that only one of these cases where R^I is metrizable is the case where I is countable and $\mathbb{R}^I$ has the product topology.

Theorem 2.39 *Let $\bar{d}(a, b) = \min(1, |a - b|)$ be the standard bounded metric on $\mathbb{R}$. If x and y are two points of $\mathbb{R}^\omega$, define*

$$D(x, y) = \sup_{i \geq 1} \Big\{ \frac{\bar{d}(x_i, y_i)}{i} \Big\}.$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$.

Proof. The properties of a metric are satisfied trivially. The fact that D gives the product topology requires a little more work.

First let U be open in the metric topology and let $x \in U$. Then there exists $\epsilon > 0$ such that $B_D(x, \epsilon) \subset U$. Choose N large enough such that $1/N < \epsilon$. Finally, let V be the basis element for the product topology

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \cdots.$$

We assert that $V \subset B_D(x, \epsilon)$. Given $y \in \mathbb{R}^\omega$,

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{N}, \quad \text{for } i \geq N.$$

Therefore,

$$D(x, y) \leq \max \Big\{ \frac{\bar{d}(x_1, y_1)}{1}, \cdots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N} \Big\}.$$

If $y \in V$, this expression is less that ϵ, so $V \subset B_D(x, \epsilon)$, as desired.

Conversely, consider a basis element

$$U = \prod_{n \geq 1} U_n$$

for the product topology, where U_n is open in $\mathbb{R}$ for $i = i_1, \cdots, i_m$ and $U_n = \mathbb{R}$ for all other indices. Given $x \in U$, we wish to find an open set V of the metric topology such that $x \in V \subset U$. Choose an interval $(x_i - \epsilon, x_i + \epsilon)$ in $\mathbb{R}$ lying in U_i, for $i = i_1, \cdots, i_m$; and choose each $\epsilon_i \leq 1$. Then define

$$\epsilon = \min \epsilon_i/i \;\mid\; i = i_1, \cdots, i_m.$$

It is now easy to check that $B_D(x, \epsilon) \subset U$, as desired.

Theorem 2.40 $\mathbb{R}^\omega$ *in the box topology is not metrizable.*

Proof. We shall show that the sequence lemma does not hold for $\mathbb{R}^\omega$. Let A be the subset of $\mathbb{R}^\omega$ defined by

$$A = \{x = (x_1, x_2, \cdots, x_n, \cdots) \mid x_n > 0 \;\; \forall\, n\}.$$

Let $0 = (0, 0, \cdots, 0, \cdots)$ be the origine in $\mathbb{R}^\omega$. We assert that $0 \in \bar{A}$. For this, let

$$B = (a_1, b_1) \times (a_2, b_2) \times \cdots$$

be any basis element containing 0, then B intersects A. For instance, the point

$$(\frac{1}{2}b_1, \frac{1}{2}b_2, \cdots) \in B \cap A.$$

But there is no sequence of points of A converging to 0. For this, let $\{a_n\}$ be a sequence of points of A, where

$$a_n = (x_{1n}, x_{2n}, \cdots, x_{nn}, \cdots).$$

Let

$$B' = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \cdots \times (-x_{nn}, x_{nn}) \cdots .$$

Then B' is a neighborhood of 0 but it contains no term of the sequence $\{a_n\}$. Hence the sequence $\{a_n\}$ cannot converge to 0 in the box topology.

Theorem 2.41 *If I is infinite and uncountable, then $\mathbb{R}^I$ in the product topology is not metrizable.*

Proof. We shall show that $\mathbb{R}^I$ with the product topology does not satisfy the sequence lemma. Let A be the subset of $\mathbb{R}^I$ consisting of all points $x = (x_i)$ such that $x_i = 1$ for all except finitely many values of i. Let 0 be the origine of $\mathbb{R}^I$.

We assert that $0 \in \bar{A}$. Let $U = \prod U_i$ be a basis element containg 0. Let $i_1, \cdots, i_n$ such that $U_{i_k} \neq \mathbb{R}$. Let $x = (x_i)$ defined by $x_i = 0$ for $i = i_1, \cdots, i_n$ and $x_i = 1$ for all other indices. Then $x \in U \cap A$, therefore $0 \in \bar{A}$.

We assert that there is no sequence of points of A converging to 0. For this, let $\{a_n\}$ be sequence of points of A. For each n, let I_n denote the subset of I consisting of those indices i for which the *ith* coordinate of a_n is different from 1. The union of all sets I_n is a countable union of finite sets and therefore countable. Since I itself is uncountable, there is an indice in I, say j, such that $j \notin I_n$ for each n. This means $a_{jn} = 1$ for each n. Now let U_j be the open interval $(-1, 1)$ in $\mathbb{R}$ and let $U = \pi_j^{-1}(U_j)$. The set U is a neighborhood of 0 in the product topology that contains none of the pints a_n; therefore, the sequence $\{a_n\}$ cannot converge to 0.

CHAPTER 3

Connected and Compact Spaces

In the study of calculus, there are three basic theorems about continuous functions, and on these theorems the rest of claculus depends. They are the following:

1. Intermediate value theorem. *If $f : [a, b] \to \mathbb{R}$ is continuous and r is a real number between $f(a)$ and $f(b)$, then there exists an element $c \in [a, b]$ such that $f(c) = r$.*

2. Maximum value theorem. *If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists an element $c \in [a, b]$ such that $f(x) \leq f(c)$ for every $x \in [a, b]$.*

3. Uniform continuity theorem. *If $f : [a, b] \to \mathbb{R}$ is continuous, given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ for every x_1, x_2 in $[a, b]$ such that $|x_1 - x_2| < \delta$.*

These theorems are used in a number of places. The intermediate value theorem is used for instance in constructing inverse functions; and the maximum value theorem is used for proving the mean value theorem for derivatives, upon which the two fundamental thorems of calculus depend. The uniform continuity theorem is used, among other things, for proving that every continuous function is integrable.

We can spoken of these theorems as theorems about continuous functions. But they can also be considered as theorems about the closed interval $[a, b]$ of real numbers. The theorems depends not only on the continuity of f but also on the properties of the topological spaces $[a, b]$.

The property of the space $[a, b]$ on which the intermediate value theorem depends is the property called *connectedness*, and the property on which the other two theorems depend is the property called *compactness*. In this chapter, we shall define these properties for arbitrary topological spaces, and shall prove the appropriate generalized versions of these theorems.

As the three quote theorems are fundamental for the theory of calculus, so are trhe notions of connectedness and compactness fundamental in higher analysis, geometry, and topology, indeed, in almost any subject for which the notion of topological space itself is relevant.

3.1 Connected Spaces

Definition Let X be a topological space. A separation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be connected if there does not exist a separation of X.

Connectedness is obviously a topological property, since it is formulated entirely in terms of the collection of open sets of X. If X is connected, so is any space homeomorphic to X.

Another way of formulating the definition of connectedness is the following:

Theorem 3.1 *A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.*

Proof. If A is a nonempty proper subset of X that is both open and closed in X, then the sets $U = A$ and $V = X - A$ constitue a separation of X, for they are open, disjoint, and nonempty, and their union is X.

Concersely, if U and V form a separation of X, then U is nonempty and different from X and it is both open and closed in X.

For a subspace Y of a topological space X, there is another useful way of formulating the definition of connectedness:

Lemma 3.2 *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.*

Proof. Suppose first that A and B form a separation of Y. Then A is both open and closed in Y. The closure of A in Y is the set $\bar{A} \cap Y$ (where $\bar{A}$ is the closure of A in X); since A is closed in Y, $A = \bar{A} \cap Y$, which implies that $\bar{A} \cap B = \emptyset$. Since $\bar{A}$ is the union of A and its limit points, B contains no limit points of A. A similar argument shows that A contains no limit points of B.

Conversely, suppose that A and B are disjoint nonempty sets whose union is Y, neither of whichj contains a limit point of the other. Then $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$; therefore we conclude that $A = \bar{A} \cap Y$ and $B = \bar{B} \cap Y$. Thus A and B are closed in Y, and since $A = Y - B$ and $B = Y - A$, they are open in Y, as desired.

Example 3.3 let X denote a two-point set in the indiscrete topology. Obviously there is no separation of X, so X is connected.

Example 3.4 Let Y denote the subspace $[-1,0)\cup(0,1)$ of the real line. Each of the set $[-1,0)$ and $(0,1)$ is nonempty and open in Y; therefore, they form a separation of Y, so Y is not connected.

Example 3.5 Let X denote the subspace $[-1,1]$ of the real line. The sets $[-1,0]$ and $(0,1]$ are disjoint and nonempty, but they do not form a separation of X, because the first set is not open in X.

Example 3.6 The subspace $\mathbb{Q}$ of $\mathbb{R}$ is not connected.Indeed, the only connected subspace of $\mathbb{Q}$ are the one-point sets. If Y is a subspace of $\mathbb{Q}$ containing two points p and q, one can choose an irrational number a lying between p and q, and write Y as the union of the open sets

$$Y\cap(-\infty,a) \text{ and } Y\cap(a,+\infty).$$

Ex 3.7 Consider the following subset of $\mathbb{R}^2$

$$X=\{(x,y) \mid y=0\}\cup\{(x,y) \mid x>0 \text{ and } y=1/x\}.$$

Show that X is not connected.

> **We have given several examples of spaces that are not connected. How we can one construct spaces that are connected? We shall now prove several theorems that tell how to form new connected spaces from given ones. In the next section, we shall apply these theorems to show some specific spaces, such as intervals in $\mathbb{R}$, and balls and cubes in $\mathbb{R}^n$, are connected. First, a lemma**.

Lemma 3.8 *If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either on C or D.*

Proof. Since C and D are both open in X, the set $C\cap Y$ and $D\cap Y$ are open in Y. These two sets are disjoint and their union is Y; if they are both nonempty, they constitute a separation of Y. Therefore one of them is empty. Hence Y must lie entirely in C or in D.

Theorem 3.9 *The Union of a collection of connected subspaces of X that have a point in common is connected.*

Proof. Let (A_i) be a collection of connected spaces of X; let p be a point of $\cap A_i$. We prove that the space $Y=\cup A_i$ is connected. Suppose that $Y=C\cup D$ is a separation of Y. The point p is in one of the sets C or D; suppose $p\in C$. Since A_i is connected, it must lie entirely in either C or D, and it cannot lie in D because it contains the point p of C. Hence $A_i\subset C$ for every i, so $\cup A_i\subset C$, contradicting the fact that D is nonempty.

Theorem 3.10 *Let A be a connected subspace of X. If $A\subset B\subset\bar{B}$, then B also is connected.*

Proof. Let A be connected and let $A \subset B \subset \bar{B}$. Suppose that $B = C \cup D$ is a separation of B, by lemma 3.8, the set A lies entirely in C or in D. Suppose that $A \subset C$, then $\bar{A} \subset \bar{C}$; since $\bar{C} \cap D = \emptyset$, B cannot intersect D, this contradicts the fact that D is a nonempty subset of B.

Theorem 3.11 *The image of a connected space under a continuous map is connected.*

Proof. Let $f : X \to Y$ be a continuous map; let X be coonected. We wish to prove that the space $Z = f(X)$ is connected. Since the map obtained from f by restricting its range to the space Z is also continuous, it suffices to consider the case of a continuous surjective map

$$g : X \to Z.$$

Suppose $Z = A \cup B$ is a separation of Z into two disjoint nonempty open sets. Then $g^{-1}(A)$ and $g^{-1}(B)$ form a separation of X, contradicting the assumption that X is connected.

Theorem 3.12 *A finite cartesian product of connected spaces is connected.*

Proof. We prove the theorem first for the product of two connected spaces X and Y. Choose a point (a, b) in $X \times Y$. Note that the **horizontal slice** $X \times \{b\}$ is connected, being homeomorphic with X, and each **vertical slice** $\{x\} \times Y$ is coonected, being homeomorphic with Y. As a result each **T-shaped space**

$$T_x = (X \times \{b\}) \cup (\{x\} \times Y)$$

is connected, being the union of two connected spaces that the point (x, b) in common.
Now form the union $\cup_{x \in X} T_x$ of all these T-shaped spaces. The union is connected because it is the union of a collection of connected spaces that have the point (a, b) in common. Since this union equals $X \times Y$, the space $X \times Y$ is connected.
The proof for any finite product of connected spaces follows by induction.

It is natural to ask whether this theorem extends to arbitrary products of connected spaces. The answer depends on which topology is used for the product, as the following theorems show

Theorem 3.13 *The product space $\mathbb{R}^\omega$ is not connected in the box topology and is connected in the product topology.*

Proof. Consider the cartesian product $\mathbb{R}^\omega$ in the box topology. We write $\mathbb{R}^\omega$ as the union of the set A of all bounded sequences of real numbers, and the set B of all unbounded sequencess. These sets are disjoint, and each is open in the box topology. Indeed if $a = (a_n) \in \mathbb{R}^\omega$, the open set

$$U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \cdots$$

is A if $a \in A$ and is in B if $a \in B$.

Now consider the cartesian product $\mathbb{R}^\omega$ in the product topology. Assuming that $\mathbb{R}$ is connected, we show that $\mathbb{R}^\omega$ is connected. Let A_n denote the subspace of $\mathbb{R}^\omega$ consisting of all sequences

$x = (x_1, x_2, \cdots)$ such that $x_i = 0$ for $i > n$. The space A_n is homeomorphic to $\mathbb{R}^n$, so that is connected, by the preceding theorem. It follows that the space A_∞ that is the union of the spaces A_n is connected, for these spaces have the point $0 = (0, 0, \cdots)$ in common. We show that the closure of A_∞ is $\mathbb{R}^\omega$, from which it follows that $\mathbb{R}^\omega$ is connected as well. For this end, let $a = (a_1, a_2, \cdots)$ be a point $\mathbb{R}^\omega$. Let $U = \prod U_i$ be a basis element containing a. We show that $U \cap A_\infty \neq \emptyset$. From the definition of basis element of the product topology, there exists n such that $U_i = \mathbb{R}$ for $i > n$. Then the point

$$x = (a_1, a_2, \cdots a_n, 0, 0, \cdots) \in U \cap A_\infty,$$

as, desired.

The argument just given generalizes to show that an arbitrary product of connected spaces is connected in the product topology.

3.1.1 Connected Subspaces of the real line

The preceding theorems show us how to construct new connected spaces out of given ones. But where can we find some connected spaces to start with? The best place to begin is the real line. We shall prove that $\mathbb{R}$ is connected, and so are the intervals and rays in $\mathbb{R}$.

One application is the intermediate value theorem of calculus, suitably generalized. An other is the result that such spaces as balls and spheres in euclidean space are connected. The fact that intervals and rays in $\mathbb{R}$ are connected may be familiar to you from analysis. We prove it again here, in generalized form. It turns out that this fact does not depend on the algebraic properties of $\mathbb{R}$, but only on its order properties. To male this clear, we shall prove the theorem for an arbitrary ordered set that has the order properties of $\mathbb{R}$. Such a set is called a *linear continuum*.

Definition A simply ordered set L having more than one element is called **linear continuum** if the following hold:

1. L has the least upper bound property.
2. If $x < y$, there exists z such that $x < z < y$.

Theorem 3.14 *If L is a linear continuum in the order topology, then L is connected, and so are intervals and rays in L.*

Proof. Recall that a subspace Y of L is said to be convex if for each pair of points a, b of Y with $a < b$, one has the interval $[a, b]$ lies in Y. We prove that if Y is a convex subspace of L, then Y is connected.

Suppose that $Y = A \cup B$ is a separation of Y. Choose $a \in A$ and $b \in B$, suppose that $a < b$. The interval $[a, b]$ of points of L is the union of the disjoint sets

$$A_0 = A \cap [a, b] \quad \text{and} \quad B_0 = B \cap [a, b],$$

each is open in $[a, b]$ in the subspace topology, which is the same as the order topology. The sets A_0 and B_0 are nonempty because $a \in A_0$ and $b \in B_0$. Thus A_0 and B_0 constitute a separation of $[a, b]$.
Let $c = \sup A_0$. We show that c belongs neither to A_0 nor to B_0, which contradicts the fact that $[a, b]$ is the union of A_0 and B_0.

Case1: Suppose that $c \in B_0$. Then $c \neq a$, so either $c = b$ or $a < c < b$. In either case; it follows from the fact that B_0 is open in $[a, b]$ that there exist some interval of the form $(d, c]$ contained in B_0. If $c = b$, we have a contradiction at once, for d is a smaller upper bound on A_0 than c. If $c < b$, we note that $(c, b]$ does not intersect A_0 (because c is an upper bound on A_0). Then

$$(d, b] = (d, c] \cup (c, b]$$

does not intersect A_0. Again, d is a smaller upper bound on A_0 than c, contrary to construction.

Case2: Suppose that $c \in A_0$ then $c \neq b$, so either $c = a$ or $a < c < b$. Because A_0 is open in $[a, b]$, there must be some interval of the form $[c, e)$ contained in A_0. Because of order property (2) of the linear continuum L, we can choose a point $z \in L$ such that $c < z < e$. Then $z \in A_0$, contrary to the fact that c is an upper bound for A_0.

Corollary 3.15 *The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.*

As an application, we prove the intermediate value theorem of calculus, suitably generalized.

Theorem 3.16 (Intermediate value theorem) . *Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c in X such that $f(c) = r$.*

The intermediate value theorem of calculus is the special case of this theorem that occurs when we take X to be a closed interval in $\mathbb{R}$ and Y to be $\mathbb{R}$.

Proof. Assume the hypotheses of the theorem. The sets

$$A = f(X) \cap (-\infty, r) \quad \text{and} \quad B = f(X) \cap (r, +\infty)$$

are disjoint, and they are nonempty because one contains $f(a)$ and the other contains $f(b)$. Each is open in $f(X)$. If there is no point $c \in X$ such that $f(c) = r$, then A and B form a separation of $f(X)$ which connected. Contradiction.

Definition Given points x and y of the topological space X, **a path** in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space is said to be **path connected** if every pair of points of X can be joigned by a path in X.

Theorem 3.17 *If X is a path connected space then X is connected.*

Proof. Suppose $X = A \cup B$ is a separation of X. Let $x \in A$ and $y \in B$. Choose a path $f : [a, b] \to X$ joigning x and y. The subspace $f([a, b])$ of X is connected as a continous image of a connected space. Therefore it lies entirely in either A or B which contradictc the fact that A and B are disjoint.

Example 3.18 Define the unit ball B^n in $\mathbb{R}^n$ by

$$B = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\},$$

where

$$\|x\|(x_1^2 + \cdots + x_n^2)^{1/2}.$$

The unit ball B^n is path connected; given any two points x, y in B^n, the straight-line path $f : [0, 1] \to \mathbb{R}^n$ defined by

$$f(t) = (1 - t)x + ty$$

lies in B^n.

Ex 3.19

1. Show that if $n > 1$ then $\mathbb{R}^n - \{0\}$ is path connected.
2. Show that a continuous image of a path connected space is path connected.
3. Show that the unit sphere S^{n-1} in $\mathbb{R}^n$ defined by

$$\{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

is path connected.

Ex 3.20 Let U be an nonempty open connected subset of $\mathbb{R}^2$. Let $x_0 \in U$, define P_{x_0} to be the set of points in U that can be joined to x_0 by a path in U.

1. Show that P_{x_0} is both open and closed.
2. Show that U is path connected.

Ex 3.21 If A is a connected subspace of X, does it follows that $\mathring{A}$ and ∂A are connected? Does the converse holds?

Ex 3.22 Let X be a topological space. Prove that

1. For every $x \in X$, there exists a unique maximal connected subset C_x containing x.
2. $C_x = C_y$ or $C_x \cap C_y = \emptyset$ for every pair x, y in X.
3. Each nonempty connected subspace of X intersects only one of them.
4. For each x, C_x is closed in X.

Where C_x maximal means C_x is not properly contained in any other connected subset of X. [Hint: Zorn lemma.]

3.1.2 Components and local Connectedness

Definition Given X, define an equivalence relation on X by setting $x \sim y$ if there exists a connected subspace of X containing x and y. The equivalence classes are called **connected components** of X.

Theorem 3.23 *The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only of them.*

Proof. Being equivalence classes, the components of X are disjoint and their union is X. Each connected subspace A of X intersects only one of them. For if A intersects the components C_1 and C_2 of X, say in points x_1 and x_2 respectively, then $x_1 \sim x_2$ by definition; this cannot happen unless $C_1 = C_2$. To show the component C is connected, choose apoint x_0 of C. For each point x of C, we know that $x_0 \sim x$, so there is a connected subspace A_x containing x_0 and x. By the result just proved $A_x \subset C$. Therefore

$$C = \bigcup_{x \in C} A_x.$$

since the subspaces A_x are connected and have the point x_0 in common, their union is connected.

Ex 3.24 Let X be a topological space. Prove that

1. Each components of X are closed.

2. If X has only finite many components, then each component is also open.

Ex 3.25 Prove that each component of $\mathbb{Q}$ is a singleton.

Ex 3.26 What are the components of $\mathbb{R}_l$? what are the continuous maps $f : \mathbb{R} \to \mathbb{R}_l$?

CHAPTER 4

Compactness

The most important topological property is compactness. It plays a key role in many branches of mathematics. **It would be fair to say that until you understand compactness you do not understand topology**! (James R. Munkres)

So what is compactness? It could be described as the topologist generalization of finitness. The formal definition says that a topological space is compact if **whenever it is a subset of a union of an infinite number of open sets, then it is also a subset of a union of a finite number of these open sets**. Obviously every finite subset of a topological space is compact. And quickly see that in a discrete space a set is compact if and only if it is finite. When we move to topological spaces with richer topological structures , such as $\mathbb{R}$, we discover that infinite sets can be compact. Indeed all closed bounded intervals $[a, b]$ in $\mathbb{R}$ are compact. But intervals of this type are the only ones which are compact.

So we are led to ask: precisely which subsets of $\mathbb{R}$ are compact? The **Heine Borel Theorem** will tell us that the compact subsets of $\mathbb{R}$ are precisely the sets which are both **closed** and **bounded**.

As we go farther into our study of topology, we shall see that compactness plays a crucial role. This is so of applications of topology to analysis.

Definition 4.1 Compact Spaces

A collection $\mathcal{A}$ of subsets of a space X is said to be a covering of X, if the union of the elements of $\mathcal{A}$ is X. It is called open covering if its elements are open subsets of X.

Definition A space X is said to be compact if every open covering of X contains a finite subcovering.

Example 4.1 The real line $\mathbb{R}$ is not compact, for the open covering of $\mathbb{R}$ by open intervals

$$\mathcal{A} = \{(-n, n) \mid n \geq 1\}$$

contains no finite subcovering.

Example 4.2 The following subspace of $\mathbb{R}$ is compact

$$X = \{0\} \cup \{1/n \mid n \geq 1\}.$$

Example 4.3 Any space containing only finitely many points is compact because in this case every open covering is finite.

Example 4.4 The interval $(0, 1]$ is not compact, the open covering

$$\mathcal{A} = \{(1/n, 1) \mid n \geq 1\}$$

contains no finite subcovering.

Definition If Y is a subspace of X, a collection $\mathcal{A}$ of subsets of X is said to cover Y if the union of its elements contains Y.

Proposition 4.5 *Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by open sets in X contains a finite subcovering of Y.*

Proof. Exercice

Theorem 4.6 *Every closed subspace of a compact space is compact.*

Proof. Let Y be a closed subspace of the compact space X. Given a open covering $\mathcal{A} = (O_i)_{i\in I}$ of Y by open sets in X. Then

$$X = \bigcup_{i\in I} O_i \cup (X - Y).$$

Only finitely many of them cover X (because X is compact). If this subcollection contains the set $X - Y$, discard $X - Y$; otherwise, leave the subcollection alone. The resulting subcollection is a subcovering of Y, as desired.

Theorem 4.7 *Every compact subspace of a Hausdorff space is closed.*

Proof. Let Y be a compact subspace of a Hausdorff space. We shall prove that $X - Y$ is closed. For this end, let $x \in (X - Y)$. For each $y \in Y$, let us choose disjoint open neighborhoods U_y and V_y of the points x and y respectively (using Hausdorff condition). The collection $\{V_y \mid y \in Y\}$ is an open covering of Y by sets open in X. Therefore, finitely many of them V_{y_i} for $i = 1, \cdots, m$ cover Y. The open set

$$V = V_{y_1} \cup \cdots \cup V_{y_m}$$

contains Y and it is disjoint to the open set

$$U = U_{y_1} \cap \cdots \cap U_{y_m}$$

formed by taking the intersection of the corresponding neighborhoods of x. Then U is a neighborhood of x disjoint from Y, as desired.

In the course of our proof, we showed the following

Lemma 4.8 *If Y is a compact space of a Hausdorff space and x is not in Y, then there exists disjoint open sets U and V containing x and Y respectively.*

Example 4.9 It follows from the theorem 4.7 that the intervals (a, b) and $(a, b]$ are not compact in $\mathbb{R}$ because they are not closed in $\mathbb{R}$.

Remark 4.10 One needs the Hausdorff condition in the hypothesis of theorem 4.7. Consider for example, the finite complement topology on the real line. The only proper subsets of $\mathbb{R}$ that are closed in this topology are the finite sets. But every subset of $\mathbb{R}$ is compact in this topology, as you can check.

Theorem 4.11 *The image of a compact space under a continuous map is comapct.*

Proof. Let $f : X \to Y$ be continuous; let X compact. Let $\mathcal{A}$ be an open covering of the set $f(X)$ by sets open in Y. The collection

$$\{f^{-1}(V) \mid V \in \mathcal{A}\}$$

is an covering of X, these sets are open in X because f is continuous. Hence finitely many of them say

$$f^{-1}(V_1), \cdots, f^{-1}(V_m),$$

cover X. Then the sets $V_1, \cdots, V_m$ cover $f(X)$, as desired.

One important use of the preceding theorem is as a tool for verifying that a map is a homeomorphism:

Theorem 4.12 *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y Hausdorff, then f is a homeomorphism.*

Proof. We shall prove that the image of closed sets of X under f are closed in Y; this will prove continuity of the map f^{-1}. If A is closed in X, then A is compact by theorem 4.6. Therefore, by theorem 4.11 just proved, $f(A)$ is compact. Since Y is Hausdorff, $f(A)$ is closed in Y, by theorem 4.7, as desired.

Theorem 4.13 (Tube Lemma) *Consider the product space $X \times Y$, where Y is compact. Let W be an open set of $X \times Y$ containing the vertical slice $\{x\} \times Y$ for some $x \in X$. Then W contains some tube $U \times Y$ about $\{x\} \times Y$, where U is an open neighborhood of x.*

Proof. Let $y \in Y$, then $(x, y) \in W$. Therefore there exists open neighborhoods U_y and V_y of x and y respectiveley such that $(x, y) \subset U_y \times V_y \subset W$. It follwos that the collection $\{U_y \times V_y, \ \ y \in Y\}$ is an open cover of $\{x\} \times Y$ which is compact, being homeomorphic to Y. Therefore there is a finite subcover $\{U_{y_1} \times V_{y_1}, \cdots, U_{y_n} \times V_{y_n}\}$ which cover $\{x\} \times Y$. Now define

$$U = U_{y_1} \cap \cdots U_{y_n}.$$

U is an open set containing x and $U \times Y \subset W$.

Example 4.14 The tube lemma is certainly not true if Y is not compact. For example, let Y be the y-axis in $\mathbb{R}^2$, and let

$$W = \{(x, y) \ | \ |y| < 1/|x|, \ x \neq 0\} \cup \{(0, y) \ | \ y \in \mathbb{R}\}.$$

Then W is an open set containing the set $\{0\} \times \mathbb{R}$, but it contains no tube about $\{0\} \times \mathbb{R}$.

Example 4.15 Let Y be the y-axis in $\mathbb{R}^2$, and let

$$W = \{(x, y) \ | \ |x| < 1/(y^2 + 1)\}.$$

Then W is an open set containing the set $\{0\} \times \mathbb{R}$, but it contains no tube about $\{0\} \times \mathbb{R}$.

The tube lemma can be used to prove tha a **finite product** of compact spaces is compact.

There is one final criterion for a space to be compact, a criterion that is formulated in terms of closed sets rather than open sets. It does not look very natural nor very useful at first glance, but it in fact proves to be useful on a number of ocasions. First we make the following definition.

Definition Let X be a topological space. A collection C of subsets of X satisfies the **finite intersection property** if the intersection of any finite subcollection of C is nonempty.

Theorem 4.16 *A topological space X is compact if and only if every collection of closed sets of X satisfying the finite intersection property has nonempty intersection.*

Proof. Given a collection $\mathcal{A}$ of subsets of X, let

$$C = \{X - A \mid A \in \mathcal{A}\}$$

be the collection of their complements. Then the following statements hold:

(1) $\mathcal{A}$ is a collection of open sets if and only if C is a collection of closed sets.

(2) The collection $\mathcal{A}$ covers X if and only if the intersection of all the elements of C is empty.

(3) The finite subcolection $\{A_1, \cdots, A_n\}$ covers X if and only if the intersection of the corresponding elements $C_i = X - A_i$ is empty.

The first statement is trivial, while the second and third follows from DeMorgan's law:

$$X - \bigcup_{i\in I} A_i = \bigcap_{i\in I}(X - A_i).$$

The proof of the theorem now proceeds in two steps: taking the contrapositive of the theorem, and then the complements of the sets.

A special case of this theorem occurs when we have a **nested sequence** $C_1 \supset C_2 \supset \cdots C_n \supset C_{n+1} \supset \cdots$ of closed sets in a compact space X. If each of the sets C_n is nonempty, the collection $C = \{C_n\}$ automatically has the finite intersection property. The intersection all elements of the collection is nonempty.

We shall use the closed set criterion for compactness in the next section to prove the uncountability of the set of real numbers.

4.2 Compact Spaces of the Real Line

Theorem 4.17 *If X is an ordered set satisfying the* ***least upper bound property****, then any closed interval $[a, b]$ in X is compact.*

Proof. Let $(U_i)_{i\in I}$ be an open cover of $[a, b]$.

Step1: Suppose $a \le x < b$. Then there exists $y > x$ such that $[x, y]$ can be covered by at most two $U_i's$. For this end, if x has an immediate successor y, then the interval $[x, y]$ has only two elements, so it can be covered by at most two $U_i's$. If x does not have an immediate successor, find U_i containing x. Pick $z > x$ such that $[x, z[\subset U_i$; this is possible because U_i is open. Since x does not have an immediate successor, there is y such that $x < y < z$. Then $[x, y] \subset U_i$.

Step2: Not let

$$A = \{y \in (a, b] \mid [a, y] \text{ can be covered by finitely many } U_i\}$$

By step1, there exists an element $y > a$ such that $[a, y]$ can be covered at most by two $U_i's$. Therefore A is nonnempty and boubded above. Let $c = \sup A$.

Step3: Claim: $c \in A$.
Let i such that $c \in U_i$. Since U_i is open and $c > a$, there exists an interval $(d, c] \subset U_i$. Since d cannot be an upper bound for A, there is an element of A larger than d. Let z such that $d < z < c$. Then $[a, c']$ can be covered by finitely many $U_i's$ and $[c', c] \subset U_i$. Therefore $[a, c] = [a, c'] \cup [c'c]$ can be covered by finitely many $U_i's$. Hence $c \in A$.

Step4: Calim: $c = b$.
Suppose $c < b$. By step1, there is $y > c$ such that $[c, y]$ can be covered at most by tow $U_i's$. Since

$c \in A$, $[a, c]$ can be covered by finitely many $U_i's$. So $[a, y] = [a, c] \cup [c, y]$ can be covered by finitely many U_i' and therefore $y \in A$. This contradicts the fact that $c = \sup A$. Hence $c = b$.

Corollary 4.18 *A closed interval $[a, b]$ in $\mathbb{R}$ is compact.*

Theorem 4.19 (Hein-Borel) *A subset A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded relative to the standard metric d_2 or the square metric d_∞.*

Proof. It will suffice to consider only the metric d_∞; the inequalities

$$d_\infty(x, y) \leq d(x, y) \leq \sqrt{n} d_\infty(x, y)$$

imply that A is bounded under d if and only if it is bounded under d_∞.

Suppose that A is compact. Then by theorem 4.6, it is closed. Consider the collection of open ball

$$\{B_\infty(0, m) \mid m \geq 1\},$$

whose union is all $\mathbb{R}^n$. Some finite subcollection covers A. It follows that there exists $M > 0$ such that $A \subset B_\infty(0, M)$. This implies that A is bounded.

Conversely, suppose that A is closed and bounbded. There exists $R > 0$ such that $A \subset [-R, R]^n$, which is compact as a finite product of compact sets. Therefore A is compact as closed set in a compact space.

Warning: **Students often remember this theorem as stating the collection of compact sets in metric space equal to the collection of closed and bounbded sets. This statement is clearly ridiculous as it stands, because the question as to which sets are bounded depends for its answer on the metric, whereas which sets are compact depends only on the topology of the space.**

Example 4.20 The unit sphere S^{n-1} and the unit ball B^n in $\mathbb{R}^n$ are compact because they are both closed and bounded.

Example 4.21 The set

$$A = \{(x, 1/x) \mid 0 < x \leq 1\}$$

is closed in $\mathbb{R}^2$, but it is not compact because it is not bounded.

Example 4.22 The set

$$A = \{(x, \sin(1/x)) \mid 0 < x \leq 1\}$$

is bounded in $\mathbb{R}^2$, but it is not compact because it is not closed.

Let us now prove the extrem value theorem of calculus, in suitably generalized form.

Theorem 4.23 *Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If Let K be a compact subset of X, then there exist points $\bar{c}$ and $\underline{c}$ in K such that*

$$f(\underline{c}) = \min_{x\in K} f(x) \quad and \quad f(\bar{c}) = \max_{x\in K} f(x).$$

Remark 4.24 The extrem value theorem of calculus is the special case of this theorem that occurs when we take X to be $\mathbb{R}$, K to be a closed bounded interval and Y to be $\mathbb{R}$.

Proof. Since f is continuous and K compact, the set $A = f(K)$ is compact. We show that A has a largest element M and a smallest element m. Then since m and M belongs to A, we must have $m = f(\underline{c})$ and $M = f(\bar{c})$ for some points $\underline{c}$ and $\bar{c}$ in K.

By contradiction, assume that A has no largest element, then the collection

$$\{(-\infty, a) \mid a \in A\}$$

forms an open cover of A. Since A is compact, some finite subcover $(-\infty, a_1), \cdots, (-\infty, a_n)$ covers A. If a_{i_0} is the larest of the elements $a_1, \cdots, a_n$ then a_{i_0} belongs to none of these sets, contrary to the fact that they cover A (because $a_{i_0} \in A$). A similar argument shows that A has a smallest element.

Theorem 4.25 *Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof.
Step1: We shall first show that given any nonempty open set of X and any point x of X, there exists a nonempty open set V contained in U such that $x \notin \bar{V}$.

Choose a point $y \in U$ different from x, this is possible if x is in U because x is not an isolated point of X and it is also possible if x is not in U simply because U is nonempty. Now choose disjoint neighborhood W_1 and W_2 of x and y respectively. Then take $V = U \cap W_2$.

Step2: Let $f : \mathbb{N} \to X$. We show that f is not surjective.

Let $x_n = f(n)$. Apply step1 to the nompty open set $U = X$ to choose a nonempty open set V_1 such that $x_1 \notin \bar{V}_1$. In general, given V_{n-1}, an nonempty open set, choose V_n to be a nonempty open set such that $V_n \subset V_{n-1}$ and $x_n \notin \bar{V}_n$. Consider the nested sequence $\{\bar{V}_n\}$ of nonempty closed sets of X. Since X is compact, there exists a point $x \in \cap\bar{V}_n$. Now If f is surjective, then there n $f(n) = x_n = x$, which implies that $x_n \in \bar{V}_n$. Contradiction.

Corollary 4.26 *Every closed and bounded interval in $\mathbb{R}$ is uncountable.*

Now we are going to prove the **uniform continuity theorem of calculus**. In the process, we are led to introduce a new notion that will prove to be surprisingly useful, that of a **Lebesgue number** for an open covering of a metric space.

Lemma 4.27 (The lebesgue number lemma) . *Let $\mathcal{A}$ be an open cover of the metric space (X, d). If X is* ***compact****, there is a $\delta > 0$ such that for each subset of X* ***having diameter less than*** *δ, there exists an element of containing it.*

The number δ is called a **Lebesgue number** for the open cover $\mathcal{A}$. **Proof.** Let $\mathcal{A}$ be an open cover of X. If X itself is an element of $\mathcal{A}$, then any positif number is a Lebesgue number for $\mathcal{A}$. So assume X is not in $\mathcal{A}$. Choose a finite collection $\{U_1, \cdots, U_n\}$ of $\mathcal{A}$ that covers X. For each i, set $C_i = X - U_i$, and define $f : X \to \mathbb{R}$ as follows

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, C_i).$$

We show that $f(x) > 0$ for all x. Given x, choose i such that $x \in U_i$. Then there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset U_i$. Therefore $d(x, C_i) \geq \epsilon$, so that $f(x) \geq \epsilon/n$.

Since f is continuous, it has a minimum value $\delta > 0$. We show that δ is our required Lebesgue number. For this let A be a subset of X of diameter less than δ. Choose a point x_0 in A, then $A \subset B(x_0, \delta)$. Now

$$\delta \leq f(x_0) \leq d(x_0, C_m)$$

where $d(x_0, C_m)$ is the largest of the number $d(x_0, C_i)$. Then $B(x_0, \delta) \subset U_m$, as desired.

Definition A function from the metric space (X, d_X) to the metric space (Y, d_Y) is said to be **uniformly continuous** if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_1, x_2 of X,

$$d_X(x_1, x_2) < \delta \ \Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon.$$

Proof. Given $\epsilon > 0$, take the open covering of Y by balls $B(y, \epsilon/2)$ of radius $\epsilon/2$. Let $\mathcal{A}$ be the open covering of X by the inverse images of these balls under f. Choose $\delta > 0$ to be a Lebesgue number for the covering $\mathcal{A}$. Then if x_1 and x_2 are two points of X such that $d_X(x_1, x_2) < \delta$, the two-point set $\{x_1, x_2\}$ has diameter less than δ, so that its image $\{f(x_1), f(x_2)\}$ lies in some ball $B(y, \epsilon/2)$. Then $d_Y(f(x_1), f(x_2)) < \epsilon$, as desired.

Ex 4.28 Prove that if X is an ordered set in which every closed interval is compact, then X has the least upper bounded property.

Ex 4.29 Let (X, d) be a metric space.

1. Show that every compact subset of X is closed and bounded.
2. Let A be a subset of X. Let $x \in X$. If A is compact show that $d(x, A) = d(x, a_0)$ for some $a_0 \in A$.
3. We define the ε-*neighborhood* of A in X to be the set

   $$U(A, \varepsilon) = \{x \in X \mid d(x, A) < \varepsilon\}.$$

   Show that $U(A, \varepsilon)$ is the union of the open balls $B_d(x, a)$ for $a \in A$.

4. Assume that A is compact; let U be an open set containing A. Show that some ε-*neighborhood* of A is contained in A.

Ex 4.30 Let X be a compact Hausdorff space; let $\{A_n\}$ be a countable collection of closed sets of X. Show that if each set A_n has empty interior in X, then the union $\cup A_n$ has empty interior in X. This is a special case of the **Baire Category theorem**.

Ex 4.31 Let A_0 be the closed interval $[0, 1]$ in $\mathbb{R}$. Let A_1 be the set obtained from A_0 by deleting its **middle third** $(\frac{1}{3}, \frac{2}{3})$. Let A_2 be the set obtained from A_1 by deleting its **middle thirds** $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. In general, define A_n by the equation

$$A_n = A_{n-1} - \bigcup_{k=0}^{\infty}(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}).$$

The intersection

$$\mathbb{K} = \bigcap_{n\in\mathbb{N}} A_n$$

is called the **Cantor**, it is a subset of $[0, 1]$.

(a) Show that $\mathbb{K}$ is compact.

(b) Show that each set A_n is a union of finitely many disjoint closed intervals of lenght $1/3^n$; and show that the end points of these intervals lie in $\mathbb{K}$.

(c) Show that $\mathbb{K}$ has no isolated points.

(d) Conclude that $\mathbb{K}$ is uncountable.

4.3 Limit Point Compactness

Definition A space X is said to be **limit point compact** if every infinite subset of X has a limit point.

Theorem 4.32 *Any compact space is limit point compact, but not conversely.*

Proof. Let X be a compact space. Given a subset A of X, we wish to prove that if A is infinite, then A has a limit point. We prove the contrapositive. If A has no limit point, then A must be finite.

Suppose tha A has no limit point. Then A is closed. Since X is compact, it follows that A is compact. Futhermore, for each $a \in A$ we can choose an open neighborhood U_a of a such that U_a intersects A in the point A alone. The subspace A is covered by the open cover $\{U_a, \mid a \in A\}$; being compact, it can be covered by finitely many of these sets. Each U_a contains only one point of A, the set A must be finite.

We now show that these two versions of compactness coincide for metrizable spaces. For this purpose, we introduce yet another version of compactness called **sequentially compactness**.

Definition A topological space X is said to be **sequentially compact** if every sequence of points of X has a convergence subsequence.

Theorem 4.33 *Let X be a metrizable space. Then the following are equivalent:*

1. *X is compact.*
2. *X is limit point compact.*
3. *X is sequentially compact.*

Proof. We have already proved that (1) $\Rightarrow$ (2). To prove that (2) $\Rightarrow$ (3), assume that X is limit point compact. Given a sequence (x_n) of points of X, consider the set $A = \{x_n \mid n \geq 1\}$. If the set A is finite, then there is a point x such that $x_n = x$ for infinitely many values of n. In this case, the sequence (x_n) has a subsequence that is constant, and therefore converges. On the other hand, if A is infinite, then A has a limit point x. We define a subsequence of (x_n) converging to x as follows. First choose n_1 so that

$$x_{n_1} \in B(x, 1).$$

Then suppose the positive integer n_{i-1} is given. Because the ball $B(x, 1/i)$ intersects A in infinitely many points, we can choose an index $n_i > n_{i-1}$ such that

$$x_{n_i} \in B(x, 1/i).$$

Then the subsequence (x_{n_k}) converges to x.

Finally we show that (3) $\Rightarrow$ (1). This is the hardest part of the proof. First, we show that if X is sequentially compact, then the Lebesgue number holds for X.(This would from compactness, but compactness is what we are trying to prove!) Let $\mathcal{A}$ be an open cover of X. We assume that there is no $\delta > 0$ such that each set of diameter less than δ has an element of $\mathcal{A}$ containing it.

Our assumption implies in particular that for each positive intege n, there exists a set of diameter less $1/n$ that is not contained in any element of $\mathcal{A}$. Let C_n be such set. Choose a point $x_n \in C_n$, for each n. By hypothesis, some subsequence (x_{n_k}) of the sequence (x_n) converges, say to the point a. Now a is in some element U of the open cover $\mathcal{A}$. Because U is open, we may choose an $\epsilon > 0$ such that $B(a, \epsilon) \subset U$. Let k large enough such that $1/n_k < \epsilon/2$ and $d(x_{n_k}, a) < \epsilon/2$, then the set $C_{n_k} \subset B(a, \epsilon)$. Contradiction.

Second, we show that if X is sequentially compact, then given ϵ, there exists a finite cover of X ϵ-ball. Once again, we proceed by contradiction. Assume that there exists an $\epsilon > 0$ such that X cannot be covered by finitely many ϵ-ball. Construct a sequence of points x_n as follows: First, choose x_1 to be any point of X. Noting that the ball $B(x_1, \epsilon) \neq X$ (otherwise X could be covered by a single ϵ-ball), choose x_2 to be a point of X not in $B(x_1, \epsilon)$. In general, given $x_1, \cdots, x_n$, choose x_{n+1} to be a point of X not in the union

$$B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon),$$

using the fact that these ball do not cover X. By construction $d(x_{n+1}, x_i) \geq \epsilon$ for $i = 1, \cdots, n$. Therefore, the sequence (x_n) can have no convergence subsequence. In fact any ball of radius $\epsilon/2$ can contain x_n for at most one value of n.

Finally, we show that if X is sequentially compact, then X is compact. Let $\mathcal{A}$ be an open cover of X. Because X is sequentially compact, then the open cover $\mathcal{A}$ has a Lebesgue number δ. Let $\epsilon = \delta/3$; using sequentially compact of X to find a finite cover of X by ϵ-balls. Each of these balls has diameter at most $2\delta/3$, so it lies in an element of $\mathcal{A}$. Choosing one such element of $\mathcal{A}$ for each of these ϵ-balls, we obtain a finite subcollection of $\mathcal{A}$ that covers X.

Ex 4.34 Let A be a subset of $\mathbb{R}^n$. Prove that A is compact if and only if each continuous numerical function on A is bounded.

Ex 4.35 Prove that if A and B are disjoint subsets of a metric space, A closed and B compact, then $d(A, B) > 0$.

Ex 4.36 Prove that if a First countable space is sequentially compact then it is compact.

Ex 4.37 Show that if Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map.

Ex 4.38 Prove that any compact metric space is separable.

CHAPTER 5

Selected Exercices

Ex 5.1 In the spaces on real line obtained by giving it the indiscrete topology, the discrete topology, the usual metric topology, and the finite-complement topology, what is $\bar{A}$ if

1. $A = (0, 1]$
2. $A = [0, 1]$
3. $A = (0, 1)$

Ex 5.2 Instead of giving you a specific problem, this exercice asks you to formulate the problems and solve them. Answer the following questions as completely as you can

1. How does the closure and the interior behave with respect to unions
2. How does the closure and the interior behave with respect to intersections
3. How does the closure and the interior behave with respect to complemts

Ex 5.3 Let X be a topological space. Let D be a non-empty seubset of X. Show that the follwing are equivalent

1. D is dense in X;
2. Every non-empty open set intersects D;
3. For every point $x \in X$, for every neighborhood V of x, $V \cap D \neq \emptyset$.

Ex 5.4 Let X be a first countable space. Let A be a subset of X. Let $x \in X$. Show that $x \in \bar{A}$ if and only if there exists a sequence $\{x_n\}$ of points in A that converges to x.

Ex 5.5 Let (X, T) be a topological space and let $\mathbb{B}$ be abasis for T. for $x \in X$, let

$$\mathbb{B}_x = \{B \in \mathbb{B} \ : x \in B\}.$$

1. Show that $\mathbb{B}_x$ is a local neighborhood basis at x.
2. Show that if X is *second countable* then it is *first countable*.

Ex 5.6 Let (X, T) be a *second countable* topological space. Let P be the set of points x in X such that x has no countable neighborhoud. Show that P is *perfect* (P is closed and P has no islated point).

Ex 5.7 Let (X, T) be a *second countable* topological space. Show that X is separable.

Ex 5.8

1. A topological space X satifies the *first separation axiom* T_1 if each one of any two points of X has a neighborhood that does not contain the other point. More formally: $\forall\ x, y \in X,\ x \neq y\ \exists U_y \in N(y):\ x \notin U_y$.
   (a) Show that X satisfies T_1 if and only if all one-point sets in X are closed.
   (b) Show that X satisfies T_1 if and only if every point is the intersection of all of its neighborhoods.
   (c) Show that any Hausdorff space is T_1.
   (d) Find an example showing that the T_1-axiom does not imply the hausdorff axiom.
2. A topological space X satifies the *Kolmogorov axiom* T_0 if at least one of any two distinct points of X has a neighborhood that does not contain the other point.
   (a) Show that X satisfies T_0 if and only if any two different points of X has different closures.
   (b) Show that if X is T_1 then X is T_0. Find an example showing that the T_0-axiom does not imply the T_1-axiom.
3. A topological space X satifies the *third separation axiom* T_3 if every closed set in X and every point of its complement have disjoint neighborhoods.**i.e.** for every closed set $F \subset X$ and every point $b \in F^c$, there exists open sets U and V such that $U \cap V = \emptyset$, $F \subset U$, and $b \in V$. A space is *regular* if it satisfies T_1 and T_3.
   (a) Show that a regular space is Hausdorff.
   (b) Find a Hausdorff space which is not regular.
   (c) Prove that a space X satisfies T_3 if and only if every neighborhood of every point x contains the closure of a neighborhood of x.

Ex 5.9 Let X be a topological space. Consider the following assertions:

(P_1): X is *second countable*.

(P_2): X is *separable*.
(P_3): if $\{O_i, i \in I\}$ is a family of non empty open sets pairewise disjoints then I is *countable*.
(P_4): if A is a subset of X such that all its points are isolated then A is *countable*.

Show that

1. $(P_1) \Rightarrow (P_2) \Rightarrow (P_3)$.
2. $(P_1) \rightarrow (P_4)$.

Ex 5.10 Let $(X, \mathcal{T})$ a toplogical space and (Y, d) a metric space. Let $f : X \to Y$ be a function. For $x_0 \in X$, we define $\omega(f; x_0)$, the osciallation of f at x_0 as follows:

$$\omega(f; x_0) = \inf_{V \in \mathcal{N}(x)} \text{diameter}\big(f(V)\big).$$

1. Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0; \\ 0 & \text{if } x < 0. \end{cases}$$

Compute $\omega(f; 0)$
2. Show that $f : (X, \mathcal{T} \to (Y, d)$ is continuous at $x_0 \in X$ if and only if $\omega(f; x_0) = 0$.

Ex 5.11
1. Prove that map $f : X \to Y$ is continuous iff $f(\bar{A}) \subset \overline{f(A)}$ for any $A \subset X$
2. Prove that a map $f : X \to Y$ is continuous iff $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ for any $B \subset Y$.

Ex 5.12 Prove that the image of a dense subset under a surjective continuous map is dense.

Ex 5.13 Let (X_1) and X_2 be topological spaces and let $X_1 \times X_2$ have the product topology.

1. Show that if F_1 is closed in X_1 and F_2 is closed in X_2, then $F_1 \times F_2$ is closed in $X_1 \times X_2$.
2. Show by finding a contre-example that a closed subset of $X_1 \times X_2$ need not be the product of two closed sets.
3. Prove that for any $A \subset X_1$ and $A_2 \subset X_2$, we have $\overline{A_1 \times A_2} = \bar{A}_1 \times \bar{A}_2$.
4. Prove that for any $A \subset X_1$ and $A_2 \subset X_2$, we have $(\overset{\circ}{\overline{A_1 \times A_2}}) = \overset{\circ}{A}_1 \times \overset{\circ}{A}_2$.

Ex 5.14 Prove that the projection maps π_1 and π_2 are open but not closed.

Ex 5.15 Let X be a topological space and Y be a Hausdorff topological space. Let $f, g : X \to Y$ be continuous functions. Show that if f and g are equal in a dense subset of X, then they are equal in X.

Ex 5.16 Let X be a topological space and Y be a Hausdorff topological space. Let $f : X \to Y$ be function.

1. Show that if f is continuous on X, then its graph is closed in $X \times Y$.
2. Show by finding a contre-example that the converse is not true.

Ex 5.17 Let $\{X_i\}_{i\in I}$ be a family of topological spaces and let

$$X = \Pi_{i\in I} X_i$$

. Let $\mathbb{B}$ be the collection of subsets of X defined by

$$\mathbb{B} = \{B = \Pi_{i\in I} O_i, \ \text{ where } \ O_i \ \text{ is open in } \ X_i\}.$$

1. Show that $\mathbb{B}$ is a basis for a topology on X. The topology generated by $\mathbb{B}$ is called the *box topology*.
2. Show that the *box topology* is finer (bigger) than the product topology on X.
3. Let $I = \mathbb{N}$, $X_n = \mathbb{R}$. We denote by $\mathbb{R}^\omega$, the countable infinite product of $\mathbb{R}$ with itself. i.e.

$$\mathbb{R}^\omega = \Pi_{n\in\mathbb{N}} X_n.$$

Let $f : \mathbb{R} \to \mathbb{R}^\omega$ be the function defined by $f(t) = (t, t, t, \cdots)$. Show that f is continuous if in $\mathbb{R}^\omega$ we have the product topology and not continuous if in $\mathbb{R}^\omega$ we have the box topology.

Ex 5.18 Let $d(a, b) = \min\{1, |a-b|\}$ be the standard bounded metric on $\mathbb{R}$. If $x = (x_n)$ and $y = (y_n)$ are two points of $\mathbb{R}^\omega$, define

$$\rho(x, y) = \sup_{n\geq 1}\Big\{\frac{d(x_n, y_n)}{n}\Big\}.$$

1. Show that ρ is a metric that induces the product topology on $\mathbb{R}^\omega$. ρ is the uniform metric on $\mathbb{R}^\omega$
2. For $\epsilon > 0$ and $x \in R^\omega$, let

$$U(x, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \times \cdots$$

1. Show that $U(x, \epsilon)$ is not equal to the ϵ-ball $B_\rho(x, \epsilon)$.
2. Show that $U(x, \epsilon)$ is not even open in the uniform topology.
3. Show that

$$B_\rho(x, \epsilon) = \bigcup_{\delta<\epsilon} U(x, \delta).$$

3. Let $\mathbb{R}^\omega$ with the Box topology. We whish to show that it is not metrizable. For this, let A be the subset of $\mathbb{R}^\omega$ consisting of those points all whose coordinates are positive:

$$A = \{(x_1, x_2, \cdots) \ | \ x_i > 0\}.$$

1. Show tha the point $(0, 0, \cdots)$ belongs to $\bar{A}$, the closure of A.
2. Show there is no sequence of points of A converging to 0. Use Sequence lemma to conclude.

4. Let J be an uncountable index set. We want to show that $R^J := \Pi_{i\in J}\mathbb{R}$ with the product topology is not metrizable. Let A be the subset of $\mathbb{R}^J$ consisting of all points $x = (x_i)$ such that $x_i = 1$ for all but finitely many values of i.

1. Show tha the point 0 belongs to $\bar{A}$, the closure of A.
2. Show there is no sequence of points of A converging to 0. Use Sequence lemma to conclude.

For reminding

Lemma 5.19 (Sequence lemma) Let X be a topological space; let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$. The converse holds if X is *First countable* (in particular if X is metrizable).

Ex 5.20 Let $\mathbb{R}$ with the finite complement topology.

1. Show that $\mathbb{R}$ is not Hausdorff.
2. Show that any subset of $\mathbb{R}$ is compact. Prove that in this topology, a compact set need not be closed.

Ex 5.21 Let (X, d) be a metric space.

1. Show that every compact subset of X is closed and bounded.
2. Let A be a subset of X. Let $x \in X$. If A is compact show that $d(x, A) = d(x, a_0)$ for some $a_0 \in A$.
3. We define the *ε-neighborhood* of A in X to be the set
$$U(A, \varepsilon) = \{x \in X \mid d(x, A) < \varepsilon\}.$$
Show that $U(A, \varepsilon)$ is the union of the open balls $B_d(x, a)$ for $a \in A$.
4. Assume that A is compact; let U be an open set containing A. Show that some *ε-neighborhood* of A is contained in A.

Ex 5.22 Let X be a Hausdorff topological space.

1. Let $x \in X$ and K a compact subset of X such that $x \in A$. Show that there exists an open set U containing x, un open set V containing K such that $U \cap V = \emptyset$.
2. Let K_1 and K_2 be two disjoint compact subsets of X. Show that there exists an open set U containing K_1, un open set V containing K_2 such that $U \cap V = \emptyset$.

Ex 5.23 Let X be a Hausdorff space. Show that X is *locally compact* if and only if for each $x \in X$, and each neighborhood U of x, there exists a neighborhood V of x such that $\bar{V} \subset U$.

Ex 5.24 *Let X be a compact topological space. We assume that X is Hausdorff. Show that every point $x \in X$ has a local compact neigborhood basis.*

Ex 5.25 Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for each $x \in X$. Assume that f is continuous, the sequence $\{f_n\}$ is monotone increasing ($fn(x) \leq f_{n+1}(x)$ for all n and x) and X is compact. Show that the convergence is uniform.

Ex 5.26 Let $f : X \to Y$ be a function. Assume that Y is compact and Hausdorff. Show that f is continuous if and only if the graph of f, G_f,

$$G_f = \{(x, f(x)) \mid x \in X\},$$

is closed in $X \times Y$.

Ex 5.27 Let Y be the y-axis in $\mathbb{R}^2$, and let

$$W = \{(x, y) \in \mathbb{R}^2 \mid |x| < \frac{1}{y^2 + 1}\}.$$

Show that W is an open set containing $\{0\} \times \mathbb{R}$ and it contains no tube about $\{0\} \times \mathbb{R}$.

Ex 5.28 *A space X is said to be* countably compact *if every countable open cover of X contains a finite subcollection that covers X. Show that for a* Hausdorff *space,* countable *compacteness is equivalent to* limit point *compactness.*

Ex 5.29 A function $\rho : X \times X \to \mathbb{R}+$ is an *assymmetric* in a set X if

1. $\rho(x, y) = 0$ and $\rho(y, x) = 0$, iff $x = y$;
2. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for any $x, y, z \in X$

Prove that if $\rho : X \times X \to \mathbb{R}+$ is an *assymmetric*, then the function d defined on $X \times X$ by:

$$d(x, y) = \rho(x, y) + \rho(y, x)$$

is a metric on X.

Ex 5.30 Let (X, T) be a topological space and $\mathbb{B}$ a basis for T. Show that T is the smallest topology on X containing $\mathbb{B}$.

Ex 5.31 Prove that a subset A of a metric space (X, d) is bounded if and only if A is contained in a ball.

Ex 5.32 Let (X, T) be a *first countable* topological space. Show that a subset F of X is closed if and only if whenever $\{x_n\}$ is a sequence of points of F that converges to x, we have $x \in F$.

Ex 5.33 Let (X, T) be a topological space. Show that

1. If (X, T) is *Hausdorff* then all one-point sets are closed.
2. If (X, T) is *Hausdorff* then any sequence has at most one limit.
3. (X, T) is *Hausdorff* if and only if for each $x \in X$ we have

$$\{x\} = \bigcap_{V \in N(x)} \bar{V}$$

where $\bar{V}$ is the closure of V and $N(x)$ is the collection of neighborhoods of x.

Ex 5.34 Show that $\mathbb{R}$ with the finite complement topology is not *Hausdorff*.

Ex 5.35 Let X be a topological space. Let $f, g : X \to \mathbb{R}$ be continuous functions.

1. Show that the set $\{x \in X \mid f(x) \leq g(x)\}$ is closed in X.
2. Let $h : X \to \mathbb{R}$ be the function defined by:

$$h(x) = \min\{f(x), g(x)\}.$$

Show that h is continuous. [**Hint:** Use the *pasting lemma*]

Ex 5.36 Let X and Y be topological spaces and let $f, g : X \to Y$ be functions. The diagonal of Y denoted by Δ is the subset of $Y \times Y$ defined by:

$$\Delta = \{(y, y) \mid y \in Y\}.$$

1. Show that Y is *Hausdorff* if and only if Δ is closed in $Y \times Y$ with the product topology.
2. Show that if f and g are continuous and Y *Hausdorff*, then the set $\{x \in X \;\; f(x) = g(x)\}$ is closed in X.
3. Let $h : X \times Y \to Y \times Y$ defined by $h(x, y) = (f(x), y)$. Show that if f is continous then h is continuous.
4. Show that if f is continous and Y *Hausdorff*, then the graph of f, G_f,

$$G_f = \{(x, f(x)) \mid x \in X\},$$

is closed in $X \times Y$.

Ex 5.37 Let X be a topological space. If $A \subset X$, we define the *boundary* ∂A of A by:

$$\partial A = \bar{A} \cap (\overline{X - A})$$

1. Show that $\mathring{A}$ and ∂A are disjoint, and $\bar{A} = \mathring{A} \cup \partial A$.

2. Show that $\partial A = \emptyset$ if and only if A is both open and closed.

3. Show that U is open if and only if $\partial U = \bar{U} - U$.

Ex 5.38 Let $f : X \to Y$ be a surjective continuous open map. Show that:

1. If X is *first countable*, then Y is *first countable*.

2. If X is *second countable*, then Y is *second countable*.

Ex 5.39 Let X, Y be topological spaces and let $f : X \to Y$ be a bijective continuous function. Assume that X is compact and Y *Hausdorff*. Show that f is a homeomorphism.

Ex 5.40 Let X, Y be topological spaces. Let A, B be subsets of X and Y, respectively; let W be an open subset of $X \times Y$ containing $A \times B$. Show that if A and B are compact, then there exist open sets U and V in X and Y respectively, such that

$$A \times B \subset U \times V \subset W.$$

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Ex 5.41 Let $f : X \to Y$; let Y be compact Hausdorff. Show that if the graph G_f of f is closed in $X \times Y$, then f is continuous. Show by a contre-example that the assumption [X compact is crucial].

$$G_f = \{(x, y) \in X \times Y \mid y = f(x)\}.$$

Ex 5.42 Let $(X, \mathcal{T})$ a toplogical space and (Y, d) a metric space. Let $f : X \to Y$ be a function. For $x_0 \in X$, we define $\omega(f; x_0)$, the osciallation of f at x_0 as follows:

$$\omega(f; x_0) = \inf_{V \in \mathcal{N}(x)} \text{diameter}\big(f(V)\big).$$

1. Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 \text{ if } x \geq 0; \\ 0 \text{ if } x < 0. \end{cases}$$

Compute $\omega(f; 0)$
2. Show that $f : (X, \mathcal{T} \to (Y, d)$ is continuous at $x_0 \in X$ if and only if $\omega(f; x_0) = 0$.

Ex 5.43 Prove that the projection maps π_1 and π_2 are open but not closed.

Ex 5.44 Let X be a topological space and A be a subset of X. We denote by $\mathcal{X}_A$ the function defined by

$$\mathcal{X}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

1. Show that the function $\mathcal{X}_A$ is continuous at x if and only if $x \notin \partial A$.
2. In which condition $\mathcal{X}_A$ is continuous on X?
3. In which condition $\mathcal{X}_A$ is continuous on X for every subset A of X?

Ex 5.45 Let X and Y be topological spaces. Let $f : X \to Y$ be a function. Show that f is continuous if and only if for every subset B of Y, one has

$$f^{-1}(\mathrm{int}(B)) \subset \mathrm{int}(f^{-1}(B)).$$

Ex 5.46 Let X be a topological space and Y be a Hausdorff topological space. Let $f, g : X \to Y$ be continuous functions.
1. Show that the set

$$A = \{x \in X \mid f(x) = g(x)\}$$

is closed in X.
2. Show that if $f = g$ in a dense subset of X, then tey are equal in X.

Ex 5.47 Let X be a second countable space. Let P be the set of points of X having no countable neighborhood.
1. Show that P is perfect (closed and no isolated point).
2. Show tha $X\mathbb{P}$ is countable.

Ex 5.48 Let X be a topological space. Consider the following assertions:

(a) X is second countable.

(b) E is separable

(c) Every subset A of X such that all its points are isolated is countable.

(d) Every family of nonempty pairwise disjoint open sets is countable.

Show that

$$(a) \Rightarrow (b), \quad (a) \Rightarrow (c), \quad (c) \Rightarrow (d), \quad (b) \Rightarrow (d).$$

Ex 5.49 Let X be a topological space. Let A be a subset of X. Prove that if C is a connected subspace of X that intersects A and $X - A$, then it intersects ∂A the boundary of A.

Ex 5.50 Let X be a topological space. Prove that

1. For every $x \in X$, there exists a unique maximal connected subset C_x containing x.
2. $C_x = C_y$ or $C_x \cap C_y = \emptyset$ for every pair x, y in X.
3. Each nonempty connected subspace of X intersects only one of them.
4. For each x, C_x is closed in X.

Where C_x maximal means C_x is not properly contained in any other connected subset of X. [Hint: Zorn lemma.]

Ex 5.51 1. Show that if $n > 1$ then $\mathbb{R}^n - \{0\}$ is path connected.

2. Show that a continuous image of a path connected space is path connected.

3. Show that the unit sphere S^{n-1} in $\mathbb{R}^n$ defined by

$$\{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

is path connected.

Ex 5.52 Let U be an nonempty open connected subset of $\mathbb{R}^2$. Let $x_0 \in U$, define P_{x_0} to be the set of points in U that can be joined to x_0 by a path in U.

1. Show that P_{x_0} is both open and closed.
2. Show that U is path connected.

Ex 5.53

Ex 5.54 Let X be a topological space. Prove that

1. Each component of X is closed.
2. If X has only finite many components, then each component is also open.

Ex 5.55 Prove that each component of $\mathbb{Q}$ is a singleton.

Ex 5.56 What are the components of $\mathbb{R}_l$? what are the continuous maps $f : \mathbb{R} \to \mathbb{R}_l$?

Ex 5.57 Let X be a Hausdorff topological space.

1. Let $x \in X$ and K a compact subset of X such that $x \notin K$. Show that there exists an open set U containing x, an open set V containing K such that $U \cap V = \emptyset$.
2. Let K_1 and K_2 be two disjoint compact subsets of X. Show that there exists an open set U containing K_1, un open set V containing K_2 such that $U \cap V = \emptyset$.

Ex 5.58 Let X be a compact topological space. We assume that X is Hausdorff. Show that every point $x \in X$ has a neigborhood basis of compact sets.

Ex 5.59 Show that if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is a closed map [that is f carries closed sets to closed sets].

Ex 5.60 Show that if Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map.

Ex 5.61 Let $f : X \to Y$ be a function. Assume that Y is compact and Hausdorff. Show that f is continuous if and only if the graph of f, G_f,

$$G_f = \{(x, f(x)) \mid x \in X\},$$

is closed in $X \times Y$.

Ex 5.62 Let Y be the y-axis in $\mathbb{R}^2$, and let

$$W = \{(x, y) \in \mathbb{R}^2 \mid |x| < \frac{1}{y^2 + 1}\}.$$

Show that W is an open set containing $\{0\} \times \mathbb{R}$ and it contains no tube about $\{0\} \times \mathbb{R}$.

Ex 5.63 (Generalized Tube lemma) Let A and B be subspaces of X and Y, respectively. Let W be an open set inn $X \times Y$ containing $A \times B$. Prove that if A and B are compact, there exists open sets U and V in X and Y such that

$$A \times B \subset U \times V \subset W.$$

Ex 5.64 Let X be a Hausdorff space. Show that X is *locally compact* if and only if for each $x \in X$, and each neighborhood U of x, there exists a neighborhood V of x such that $\bar{V} \subset U$.

Ex 5.65 Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for each $x \in X$. Assume that f is continuous, the sequence $\{f_n\}$ is monotone increasing ($fn(x) \leq f_{n+1}(x)$ for all n and x) and X is compact. Show that the convergence is uniform.

Ex 5.66 Let A be a subset of $\mathbb{R}^n$. Prove that A is compact if and only if each continuous numerical function on A is bounded.

Ex 5.67 Prove that if A and B are disjoint subsets of a metric space, A closed and B compact, then $d(A, B) > 0$.

Ex 5.68 Prove that if a First countable space is sequentially compact then it is compact.

Ex 5.69 A space X is said to be *countably* compact if every countable open cover of X contains a finite subcollection that covers X. Show that for a *Hausdorff* space, *countable* compacteness is equivalent to *limit point* compactness.

Ex 5.70 Let (X, d) be a metric space.

1. Let A be a subset of X. Let $x \in X$. If A is compact show that $d(x, A) = d(x, a_0)$ for some $a_0 \in A$.
2. We define the *ε-neighborhood* of A in X to be the set

$$U(A, \varepsilon) = \{x \in X \mid d(x, A) < \varepsilon\}.$$

   Show that $U(A, \varepsilon)$ is the union of the open balls $B_d(x, a)$ for $a \in A$.
3. Assume that A is compact; let U be an open set containing A. Show that some *ε-neighborhood* of A is contained in U.

Ex 5.71 Let (X, d) be a metric space. If $f : X \to X$ satisfies the condition

$$d(f(x), f(y)) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$, then f is called a **shrinking map**. If there is $0 < k < 1$ such that

$$d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in X$, then f is called a **contraction**. A **fixed point** of f is a point $x \in X$ such that $f(x) = x$.

1. If f is a contraction and X is compact, show that f has a unique fixed point. [Hint: Define $f^1 = f$ and $f^{n+1} = f \circ f^n$. Consider the intersection A of the sets $A_n = f^n(X)$.]

2. Show more generally that if f is shrinking map and X compact, then f has a unique fixed point.[Hint: Let A as before. Given $x \in A$, choose x_n so that $x = f^{n+1}(x_n)$. Consider the sequence $y_n = f^n(x_n)$]

3. Let $X = [0, 1]$. Show that $f(x) = x - x^2/2$ maps X into X and is a shrinking map that is not a contraction.

Ex 5.72 Let Ω be an open subet of $\mathbb{R}^n$. For $n \geq 1$, define

$$K_n = \{x \in \mathbb{R}^n \mid \|x\| \leq n \text{ and } d(x, \Omega^c) \geq \frac{1}{n}\}$$

where d is the usual distance in $\mathbb{R}^n$. Prove that

1. Each K_n is compact, $K_n \subset \Omega$ and $K_n \subset \mathring{K}_{n+1}$.

2. $\Omega = \bigcup_{n=1}^{\infty} K_n = \bigcup_{n=2}^{\infty} \mathring{K}_n$.

3. For any compact K in Ω, there exists $N \geq 1$ such that $K \subset \mathring{K}_N$.

Ex 5.73 Prove that if a First countable space is sequentially compact then it is compact.

Ex 5.74 rove that any compact metric space is separable.

Ex 5.75
1. Let X equal the countable union $\bigcup B_n$. Show that if X is a nonempty Baire space, at least one of the sets $\bar{B}_n$ has a nonempty interior.
2. The Baire Category theorem implies that $\mathbb{R}$ cannot be written as a countable union of closed sets having empty interior(why?). Show that this fails if the sets are not required to be closed.
3. Show that every locally Hausdorff space is a Baire space.

Ex 5.76
1. Show that if $f : \mathbb{R} \to \mathbb{R}$, then the set C of points at which f is continuous is a G_δ set in $\mathbb{R}$ [Hint: Let U_n be the union of all open sets U of $\mathbb{R}$ such that diameter$(f(U)) < 1/n$. Show that $C = \cap U_n$].
2. Let D be a countable dense subset of $\mathbb{R}$. Show that there is no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous precisely at the points of D.

Ex 5.77 If f_n is a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n(x) \to f(x)$ for each $x \in \mathbb{R}$. Show that f is continuous at uncountable many points of $\mathbb{R}$.

Ex 5.78 Let $g : \mathbb{N} \to \mathbb{Q}$ be a bijection function. Let $x_n = g(n)$. Define $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x_n) = \frac{1}{n} \text{ for } x_n \in \mathbb{Q},$$

$$f(x_n) = 0 \text{ for } x_n \notin \mathbb{Q}.$$

Show that f is continuous at each irrational number and discontinuous at each rational.

MA : Teste2: Topolgy
October, 6, 2009

✍ **Prof.** N. Djitte [Gaston Berger university, Saint Louis, Senegal].
☛ **Attempt ALL Questions.**. Time allowed: 3 hours.

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Ex 5.79 Let X be a First countable topological space and A be a subset of X. Prove the following facts:

(a) Every point x of X has a countable neighborhood basis $W = \{V_n,\ n \geq 1\}$ satisfying

$$V_{n+1} \subset V_n,\ \ \forall\, n \geq 1.$$

==

Ex 5.80 Let $A \subset \mathbb{R}$ be a subset of $\mathbb{R}$. We say that A is at most countable it it is either finite or countable.

Let T_c be the collection of subsets of $\mathbb{R}$ defined by

$$T_c = \{A \subset \mathbb{R} \mid A = \emptyset \text{ or } C_{\mathbb{R}}A \text{ is at most countable}\}$$

(a) Prove that T_c is a topology on $\mathbb{R}$.

(b) Show that a countable intersection of elements of T_c is an element of T_c.

(c) Show that T_c is not a Hausdorff space.

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Ex 5.81 Suppose that in $\mathbb{R}$ is given the discrete topology. Let $\{x_n\}$ be sequence of real numbers. Show that the sequence $\{x_n\}$ converge if and only if it is stationary.

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Ex 5.82 In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = 1/n$ converge?

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Ex 5.83 In the spaces on real line obtained by giving it the indiscrete topology, the discrete topology, the standard topology, and the finite-complement topology, what is $\bar{A}$, $\mathring{A}$ and the boundary ∂A of A if:

1. $A = (0, 1]$
2. $A = [0, 1]$

GOOD LUCK

✍ **Prof.** N. Djitte

MA : Teste2: Topolgy

October, 6, 2009

✍ **Prof.** N. Djitte [Gaston Berger university, Saint Louis, Senegal].

☛ **Attempt ALL Questions.**. Time allowed: 2 hours.

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Ex 5.84 Show that if a product space is separable, then each factor is separable.

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Ex 5.85 Let X be the set of polynomial functions of degree less than one defined in $[0, 1]$. for $f(x) = a + bx$ and $g(x) = a' + b'x$ in X, we define

$$d(f, g) = |a - a'| + |b - b'|.$$

(a) Show that d is a metric in X.

(b) Show that (X, d) is separable.

==

Ex 5.86 Let X be a **First countable** topological space. Prove the following facts:

(a) Every point x of X has a countable neighborhood basis $W = \{V_n,\ n \geq 1\}$ satisfying

$$V_{n+1} \subset V_n, \quad \forall\, n \geq 1.$$

(b) $x \in \bar{A}$ if and only if there exists a sequence $\{x_n\}$ a sequence of points of A converging to x.

(c) A subset F of X is closed if and only if every convergence sequence of points of F has its limit in F.

(d) A function $f : X \to Y$ is continuous at $\bar{x}$ if and only if for every convergence sequence to $\bar{x}$, one has the sequence $\{f(x_n)\}$ converges to $f(\bar{x})$.[Hint: you may try a proof by contradiction].

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Ex 5.87 Let X be the set $(\mathbb{R} setminus \mathbb{N}) \cup \{1\}$. Define the function $f : \mathbb{R} \to X$ by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{R} setminus \mathbb{N} \\ 1, & \text{if } x \in \mathbb{N}. \end{cases}$$

and let $\mathcal{T}$ be defined as follows

$$\mathcal{T} = \{U \subset X \mid f^{-1}(U) \text{ is open in } \mathbb{R}\}.$$

(a) Show that $\mathcal{T}$ is a topology on X.

(b) Show that f is continuous.

(c) Show that $\mathcal{T}$ is Hausdorff.

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GOOD LUCK

✍ **Prof.** N. Djitte

THE AFRICAN UNIVERSITY OF SCIENCE AND TECHNOLY
ABUJA - NIGERIA

2009/2010 ACADEMIC SESSION

MA : Teste3: Topolgy
October, 17, 2009

✍ **Prof.** N. Djitte [Gaston Berger university, Saint Louis, Senegal].
☛ **Attempt ALL Questions.**. Time allowed: 3 hours.

Ex 5.88 Consider the cartesian product $\mathbb{R}^w$, the set of sequences of real numbers.
1. Let A denote the set of all bounded sequences in $\mathbb{R}$ and B denote the set of all unbounded sequences in $\mathbb{R}$. Prove that A and B are open in the box topology. Hence $\mathbb{R}^w$ is not connected in the **box topology**.

2. Now, we consider the cartesian product $\mathbb{R}^w$ in the **product topology**. For $n \geq 1$, let A_n denote the set of all sequences $x = (x_1, x_2, \cdots)$ such that $x_i = 0$ for $i > n$.

(a) Prove that for each n, A_n is connected. [Hint: `you may show that` A_n `is homeomorphic to` $\mathbb{R}^n$].

(b) Show that the set A_∞ defined by

$$A_\infty = \bigcup_{n=1}^{\infty} A_n$$

is connected.

(c) Show that the closure of A_∞ is $\mathbb{R}^w$. Finally, show that $\mathbb{R}^\omega$ is connected in the product topology.

Ex 5.89 Let X be a topological space.

(a) Prove that each component of X is closed.

(b) Prove that if X has only finite many components, then each component is also open.

Ex 5.90 Let X be a compact Hausdorff topological space.

(a) Let $x \in X$ and K be a closed subset of X such that $x \notin K$. Show that there exists an open set U containing x, an open set V containing K such that $U \cap V = \emptyset$.

(b) Let K_1 and K_2 be two disjoint closed subsets of X. Show that there exists an open set U containing K_1, un open set V containing K_2 such that $U \cap V = \emptyset$.

(c) Let K be a compact subet of X and Ω be an open subset of X such that $K \subset \Omega$. Show that there exists a compact set K_1 such that

$$K \subset \mathring{K}_1 \subset K_1 \subset \Omega.$$

[Hint: `You may use` (b) `with appropriate sets`.]

(d) Explain briefely, how we can get a sequence $\{K_n\}$ of compact subsets of X such that

$$K \subset \mathring{K}_{n+1} \subset K_{n+1} \subset \mathring{K}_n \subset K_n \subset \Omega.$$

GOOD LUCK

✍ **Prof.** N. Djitte