

Graphs and Applications

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Abstract

In this course we present the basics of graph theory and some of its applications. We introduce the basic notions for a fixed graph (trees, forests, Laplacian matrix). Then we give some applications, in particular we treat in detail the Kirchhoff-Maxwell solution of electric circuits. We consider then the notion of random motion on a graph, and give the random path representation of the inverse of its Laplacian matrix (with mass regulator). In the case of a regular lattice of finer and finer mesh, we show the continuous limit which expresses the inverse of the continuous Laplacian in term of Brownian paths. Random trees are defined to model growth phenomena. Ribbon graphs and triangulations are introduced to study two-dimensional geometry. The course ends on the notions of Gaussian measures, Feynman graphs, and the Wick theorem explained in the case of a simple finite quantum field toy model.

I Introduction

Graph theory is not part of the regular mathematics undergraduate curriculum and this is particularly true in Africa. This is unfortunate since graphs are perhaps the most ubiquitous concept of discrete mathematics. They require little mathematical background and have plenty of applications.

The interplay between discrete and continuous mathematics propels science forward, in cycles which progressively uncover the dual, digital/analogic and global/local aspects of reality. After the triumph of analysis and partial differential equations in the XIXth and early XXth century, the return of discrete concepts has reshaped forcefully mathematics, following the emergence of quantum mechanics and of the digital computers. We expect this trend to continue well within the XXIst century.

The ambition of these lecture notes is to provide the students at Aims-Sénégal with a concise and useful introduction to graphs, geometry and physics. In graph theory there are few basic concepts (vertices, edges...) and at least the initial levels of the theory (cycles, trees, complexity etc) are in principle accessible to high school students

without any mathematical background. The subject can be both destabilizing at first sight but ultimately very rewarding. It leads students to new ways of reasoning and widens the horizons of their traditional mathematical undergraduate education. Indeed graph theory does not fit well into the narrow view of mathematics as a collection of tricks (derivation, integration by parts etc...) to compute quantities through sequences of equalities.

Graph theory can also reconcile students with mathematical reasoning based on spatial intuition. Many students were told that mathematics is the art of correct reasoning even on wrong figures. This is a dangerous statement, especially in graph theory where correct drawings of good examples is the best and most natural way to progress. Graph theory could therefore play today the role that traditional planar geometry played for generations of students in the past. It provides them with a fertile ground to use their spatial intuition to boost their mathematical understanding and creativity.

The plan of the lectures is to emphasize the applications of graph theory to geometry and physics, from ordinary graphs to ribbon graphs and Riemann surfaces and from the electric circuits of the XIXth century to the Feynman graphs which support the modern intuition in particle physics. We do not treat the computer science applications of graph theory for lack of time and competence; this could be the subject of another book.

We also include plenty of exercises, to help assimilate the “rules of the game” of the subject. It is our hope that in this way many Aims-Sénégal students will dig further into graph theory and its applications and discover new fruitful and fulfilling ways of doing mathematics and science in general.

II Generalities

II.1 Definitions

Graphs are fundamental discrete mathematical structures designed to model pairwise relations between very different types of objects. Hypergraphs, which model higher-than-binary relations, are also interesting, but not studied in this book.

To model the binary relation we define a graph as a set of vertices, a set of edges, and an incidence relation between them. This relation expresses the fact that edges connect either two different vertices or a single vertex with itself.

Definition II.1. A (unoriented, labeled)¹ graph $G = (V, E, R)$ is a set of vertices V and of edges E together with a relation R (called incidence), which maps E to the target set $F = P(V) \cup \bar{V}$. The set $P(V)$ is the set of unordered pairs of distinct vertices $u \in V$, $v \in V$, $u \neq v$, and the set \bar{V} is the set of diagonal pairs (v, v) in V^2 .

Hence $|P(V)| = |V|(|V| - 1)/2$, $|\bar{V}| = |V|$ and $|F| = |V|(|V| + 1)/2$ ². The map R should be thought as associating to any edge “ends” which are vertices. When $R(e) = \{u, v\} \in P(E)$, these ends are a pair $\{u, v\}$ of two distinct vertices. In that case we call

¹We often forget the words *unoriented* or *labeled* when there is no ambiguity.

²For any finite set S we note $|S|$ the number of elements in S .

the edge e a *link* joining u to v . When $R(e) = (v, v) \in \bar{V}$ we consider that the two ends of e coincide, and that e joins v to itself. In that case we call the edge e a *tadpole*³.

In physics, vertices are sometimes called interactions, and edges sometimes lines, wires, propagators, ... (depending on context). Usually in this book we label vertices with roman letters $\{a, b, c, \dots\}$ and edges with positive integers $\{1, 2, \dots\}$.

As is usual in mathematics, a graph should in fact be thought as an equivalence class of graphs up to isomorphism. These isomorphisms $G = (V, E, R) \rightarrow G' = (V', E', R')$, are bijections $V \rightarrow V'$ and $E \rightarrow E'$ which transform the relation R into the relation R' ; in that case we consider G and G' to be *the same graph*.

Each edge e of a graph can be considered also as the fusion of two half-edges, one attached to u , the other to v . This is an important remark which becomes even fundamental in the theory of Feynman graphs (see section VIII), in which half-edges correspond to *fields*. In particular it leads to the idea that half-edges, also called *flags* in a mathematical context, could be added as decorations in arbitrary number at any vertex in a graph. This produces the interesting extended category of flagged graphs:

Definition II.2. A flagged graph (G, F, R_F) is a graph $G = (V, E, R)$ plus a set F of flags and a map R_F which to each flag $f \in F$ associates a vertex of G , $v = R_F(f) \in V$.

Hence graphs without flags correspond to flagged graphs with $F = \emptyset$. They are called vacuum graphs in the field theory context. They are the ones we consider by default in this book, that is unless specified otherwise.

A graph should be understood first as a simplified drawing. To picture the graph, we usually represent vertices as black dots. Edges are represented as lines (straight or curved, long or short, it does not matter) joining vertices. What matters is that a line e with $R(e) = \{u, v\}$ has u and v as its ends, and that a tadpole goes from a vertex v to itself. Any flag can be pictured as a half-line attached to a vertex v with a single end: there is no vertex at its other end.

If we try to draw a graph on a sheet of paper or on the blackboard, this representation is implicitly *planar*, because the piece of paper or the blackboard are parts of the *two-dimensional plane*. In the case of complicated graphs with many edges, the drawing becomes complicated, and often it cannot be done without *crossings*. We shall return later to this very interesting issue of how to distinguish the graphs (called *planar*) which can be drawn on a sheet of paper without crossings from those which cannot. For the moment let us accept drawing with crossings. To remember however that these crossings “do not mean anything” we represent true vertices as *fat* black dots as in Figure II.1. In this representation, remember also that the cyclic orderings of the edges arriving at any vertex does not mean anything. Permuting the half-lines incident at any vertex in an arbitrary way leads to an equivalent representation of the *same* graph, although possibly with different number of crossings.

An arbitrary number of tadpoles can be attached at any vertex v (see Figure 1.2). Also we allow for the possibility of many edges with the same ends: $R(e_1) = R(e_2) = \dots = R(e_k)$. Such a family of k edges with the same ends is sometimes called a multi-edge

³ Mathematicians usually call tadpoles “loops” or “self-loops”. The alternate name *tadpole* comes from Feynman graph terminology, hence from particle physics, see chapter VIII). We shall use it in this book and urge more mathematicians to adopt it, as it is more original than “loop” or “self-loop” and cannot be confused with anything else.

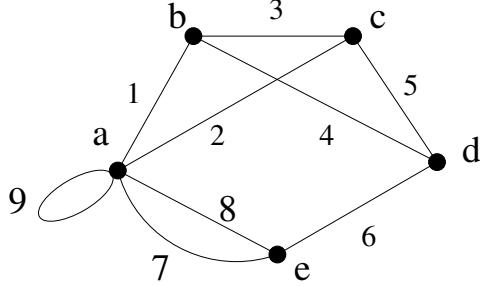


Figure 1: A graph with $V = \{a, b, c, d, e\}$ and $E = \{1, 2, \dots, 9\}$. The adjacency relation is $R(1) = \{a, b\}$, $R(2) = \{a, c\}$, $R(3) = \{b, c\}$, $R(4) = \{b, d\}$, $R(5) = \{c, d\}$, $R(6) = \{d, e\}$, $R(7) = \{e, a\}$, $R(8) = \{e, a\}$, $R(9) = \{a, a\}$. Although this graph is actually planar (one can see this by moving edge 4 to pass above vertex c), the drawing chosen has a crossing in the middle which is *not* a vertex. This graph has a multi-edge of order 2 (7 and 8) and one tadpole (9).

of order k . In this case we should *not* write $e_1 = e_2 = \dots = e_k$, since the set of all edges of the graph is *not* defined as a subset of the set of pairs of vertices, but is an independent set with its own labeling.

The reader should be fully aware of the many different terminologies in the literature on graph theory. There is some consensus between many mathematicians and graph theorists to get rid of tadpoles and of multi-edges, considered unessential complications. Graphs without tadpoles and multi-edges are therefore commonly just called graphs by this community and they call *multigraphs* the more general definition which allows for tadpoles and multi-edges. However, tadpoles are essential in Feynman graphs and multi-edges occur naturally in most applications of graph theory, from electric circuits and traveling salesmen to edge-colorings. They were even present in the Königsberg bridge problem, the birth problem of graph theory (see Figure 4). Hence we prefer to include them in our definition of graphs. We shall instead call *simple graphs* the restricted graphs without tadpoles and multi-edges.

Definition II.3. A *simple graph* is a graph G without any tadpole or multi-edge. A tadpole-free graph can have multi-edges but no tadpole.

The set of edges E of a simple graph can be conveniently defined simply as a subset of the set $P(V)$ of pairs of vertices. Remark that the relation R is then redundant and can be omitted.

Erasing all tadpoles of a general graph G produces a unique tadpole-free graph G° . Collapsing the k edges to a single one in any multi-edge of order k of a tadpole-free graph produces an underlying simple graph \hat{G} .

Until now we did not attempt to *order* the pair of ends of any edge. Doing this leads to *oriented graphs*, also called directed graphs or simply *digraphs*. There is some problem to orient tadpoles, since their two ends are the same. Hence the theory of digraphs works best for tadpole-free graphs.

Definition II.4. An oriented graph or digraph $G = (E, V, R)$ is made of a set V of vertices, a set E of edges plus a binary relation R which maps E to $V^2 - \bar{V}$, the set of *ordered pairs* of distinct vertices. The initial vertex is called the start and the final one the end of the edge.

Equivalently we can define an oriented graph as a tadpole-free (unoriented) graph, plus the choice of an orientation on each edge. One can picture this orientation by adding an arrow on each edge. A tadpole-free graph has therefore exactly $2^{|E|}$ distinct orientations or corresponding digraphs (see Figure 1.2). Digraphs are also sometimes called *quivers* in algebra.

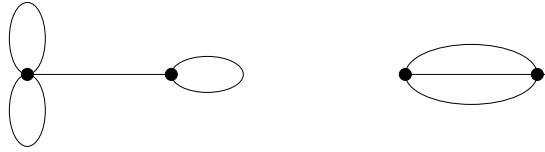


Figure 2: G_1 has three tadpoles and no multi-edge; G_2 is a tadpole-free graph with three edges which form a multi-edge of order 3. It has 8 different orientations.

Definition II.5 (Degree). The *degree* of a vertex v is the number of half-edges which attach to v . It is an integer noted $d_G(v)$. It is also sometimes called the coordination number at v . A tadpole counts for 2 in the degree of its vertex.

Vertices of degree 0 are called *isolated* or *bare*. They are not very interesting! A vertex of degree 1 is called *extremal* or a *leaf*. It has a single edge $e \in E$ attached to it, which moreover is not a tadpole.

Definition II.6. The *complete graph* K_n on n vertices is the unique simple graph with $|V| = n$ in which every distinct pair of vertices is joined by an edge.

The complete graph K_n has therefore $n(n - 1)/2$ edges and each of its n vertices has degree $n - 1$.

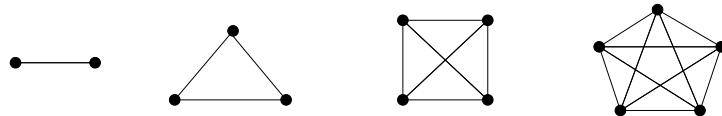


Figure 3: The complete graphs K_2 , K_3 , K_4 and K_5 .

Exercise II.7. An *irreducible graph* is a graph in which there is no vertex of degree exactly 2. Define precisely the natural reduction process which to any graph associates a single irreducible graph.

The reverse of reduction is called *subdivision*. A *subdivision* of a graph G is a graph G' formed by subdividing its edges, that is by adding on each of them an arbitrary number of vertices of degree 2 (possibly none).

Definition II.8 (Bipartite Graphs). A graph is called bipartite if its vertices can be partitioned into two sets such that each edge has one endpoint in each set. A bipartite graph is therefore tadpole-free.

Definition II.9. A complete bipartite graph, $G = (V_1 \cup V_2, E = V_1 \times V_2)$, is a bipartite graph which has exactly one edge for every pair (v_1, v_2) , $v_1 \in V_1$ and $v_2 \in V_2$. The complete bipartite graph with $|V_1| = m$ and $|V_2| = n$, is denoted $K_{m,n}$ and has mn edges.

Exercise II.10. Prove that a graph is bipartite if and only if it does not contain any odd cycle.

II.2 Paths and Subgraphs

Definition II.11. A path ω of G (also called a walk) is an alternating sequence of vertices and edges, beginning and ending with a vertex. Hence $\omega = \{v_0, e_1, v_1, e_2, \dots, e_n, v_n\}$, where $n \in \mathbb{N}$ and the two ends of e_i are v_{i-1} and v_i . n is called the length of the path, e_i is the i -th *step* of the path. A vertex v is said to be visited at time k if $v_k = v$.

A path is called closed if its first and last vertices v_0 and v_n are the same, and open if they are different. Recall that there is no constraint that the edges of the path should be distinct. Any edge can be traveled in both directions any number of times. The total number of visits of the path ω at vertex v is noted $n(\omega, v)$ (beware however that in the case of a closed path, the convention is always to identify the initial and end visit: hence they count only as one). Hence the total number of visits of an open path of length n is $n + 1$, whether the total number of visits of a closed path of length n is n^4 . A path which visits each vertex at most once, hence such that $\forall v, n(\omega, v) \leq 1$ is called a *self-avoiding path* or a *polymer*⁵.

A *trail* is a path in which all the edges are distinct (but not necessarily all vertices, hence a trail is not necessarily self-avoiding). A closed trail is called a *circuit*, an open trail is called a *chain*.

A *cycle* (also sometimes called elementary cycle) is a self-avoiding circuit, hence it is a closed path for which both all edges used and all visited vertices are disjoint.

Hence any cycle is made of a set of distinct edges

$$C = \{e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\} \dots e_n = \{v_n, v_1\}\} \quad (\text{II-1})$$

such that v_1, \dots, v_n are all distinct. The number n of edges is called the length of the cycle. Tadpoles are exactly the cycles of length 1, and all cycles of longer length cannot contain tadpoles.

A self-avoiding chain is made of a set of distinct edges

$$C = \{e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\} \dots e_{n-1} = \{v_{n-1}, v_n\}\} \quad (\text{II-2})$$

such that v_1, \dots, v_n are all distinct.

⁴The zero-length path $\{v_0\}$ is defined for all $v_0 \in V$ and can be considered as an open path from v_0 to v_0 with a single visit at v_0 .

⁵These names come from physics and chemistry.

Example

Consider the graph G of Figure 1. The path $\omega_1 = (a, 1, b, 4, d, 4, b, 3, c)$ is a path of length 4 which starts at a and ends at c . It visits twice b and once a , d and c . It is not self-avoiding since it visits b twice and it is not a trail since it uses edge 4 twice. The path $\omega_2 = (a, 7, e, 8, a, 9, a)$ is a closed path of length 3 which visits twice a and once e . It is a trail and a circuit but it is not self-avoiding as a is visited twice. Hence it is not a cycle.

Exercise II.12. How many paths from a to d in the graph of Figure II.1 are made of exactly 4 steps? How many of them are trails?

The Königsberg bridges problem played a historical role in mathematics. Leonhard Euler's solution in 1735 was the starting point of graph theory and perhaps also of topology.

The city of Königsberg (now Kaliningrad) was set on two large islands and the two river banks of the Pregel River, hence on four different land masses. They were connected to each other through seven bridges. The problem was to find a walk through the city that would cross each bridge once and only once. The walk needed not to start and end at the same point. Euler proved that the problem has no solution by introducing the notion of Eulerian path.

Definition II.13 (Eulerian Path). An Eulerian path is a path in a connected graph which uses every edge exactly once (hence it must also visit all vertices). The path can be open or closed (in which case it is a circuit). A graph that contains a Eulerian circuit is called an Eulerian graph. A graph that contains an open Eulerian path is called an open Eulerian graph.

Clearly Eulerian graphs (open or closed) are also those that can be fully drawn without “lifting the pencil from the paper”.

Theorem II.14. *A connected graph G is Eulerian if and only if all its vertices have even degree.*

Proof It is easy to check that if G is Eulerian, the degree of every vertex is even because an Eulerian circuit entering k times at v must also exit k times, hence must use $2k$ half-edges. For the reciprocal, choose any starting vertex v_0 , and follow a path of edges from that vertex until returning to v_0 , never taking twice the same edge. It is not possible to get stuck at any vertex other than v_0 , because the even degree of all vertices ensures that when the path enters any other vertex v there must be still an unused half-edge leaving v . The path formed in this way when returning at v_0 is a circuit C_0 , but may not cover all the vertices and edges of the initial graph. If there exists a vertex v_1 that belongs to C_0 and has still adjacent edges not part of C_0 , start another path from v_1 , following unused edges until returning to v_1 . This creates a circuit C_1 . Then insert or glue C_1 into C_0 by considering the part of the circuit C_0 from V_0 to the first visit of C_0 at v_1 , then the circuit C_1 and then the rest of C_0 . This creates a circuit C_{01} which is strictly larger than C_0 . Iterate from v_2 if there remains still edges out of C_{01} and so on. Since the graph is connected, this process cannot stop until all edges have been used, hence an Eulerian circuit has been built. \square

Corollary II.15. *A connected graph G is an open Eulerian graph if and only if all its vertices have even degree except two which have odd degree.*

Proof An open Eulerian path uses an even number of half-edges (to enter and exit) at every vertex, except for the initial and final one. Hence for an Eulerian open path to exist, the degree of every vertex must be even, with exactly two exceptions, the entrance and exit vertices a and b . If this is the case the open Eulerian path can be built starting at a by the same induction than in the previous theorem, and it can end only at b . \square

The Königsberg bridges problem reduces to find whether there is an Eulerian (open or closed) path in the graph of Fig 4, which is a simplified picture of the town, with its four vertices representing the two islands and the two river banks and its seven edges representing the seven bridges.

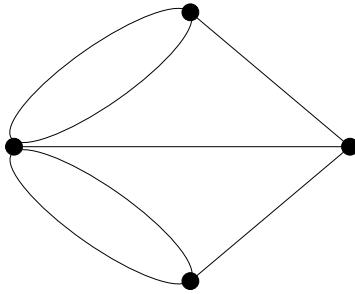


Figure 4: The graph for the Königsberg bridges problem

Since the graph of the Königsberg bridges problem has four vertices of odd degree, Euler concluded that the problem has no solution.

Definition II.16 (Hamiltonian path). A Hamiltonian path is a path (open or closed) that visits each vertex of the graph exactly once. A graph that contains a Hamiltonian path is called a Hamiltonian graph (respectively open or closed).

For instance complete graphs are obviously Hamiltonian (closed) for the circuit $\{(v_1, v_2), \dots, (v_n, v_1)\}$.

Definition II.17 (Connected Components). A graph is said to be *connected* if for any pair of its vertices $\{u, v\}$ there is an open path starting at u and ending at v . In particular a bare vertex is connected.

Definition II.18. A *subgraph* S of a graph $G = (V, E, R)$ is a graph (V_S, E_S, R_S) whose vertex set is a subset of that of G , whose edge set is a subset of that of G , and such that the ends of any edge in E_S all lie in V_S . The incidence relation R_S is then required to be the restriction of R to V_S .

A subgraph S of a graph G for which $V_S = V$ is called an edge-subgraph or in short *e-subgraph* of G . It is entirely defined by its edge subset $E_S \subset E$. Hence a graph has

always exactly $2^{|E|}$ e-subgraphs. Remark that since we always keep all the vertices of G in an e-subgraph, but not all edges, many vertices which were not isolated in G can become isolated in S . The simplified subgraph associated to E , S_E° is then defined as S_E minus its isolated vertices.

Conversely to any subset $V' \subset V$ is associated a subgraph $S_{V'}$ of G called a vertex-induced (or full) subgraph. It has V' as vertex set and an edge set made of exactly those edges of G with both ends in V' .

A *connected component* of a graph is a maximal subset of edges and vertices which is connected. Isolated vertices (those which are not at the end of any edge) count as connected components. Any graph is naturally partitioned as the disjoint union of its connected components. The *connexity* $k(G)$ of a graph G is the number of connected components of G .

The connexity of any edge-subgraph S is greater or equal to that of G . An edge whose removal strictly increases the number of connected components of the graph must increase that number by exactly 1 and is called a *bridge* (in physics it may be called a one-particle-reducible line).

An edge which is neither a bridge nor a tadpole is called regular. We shall call *semi-regular* an edge which is not a tadpole, hence which joins two distinct vertices. Edges in tadpole-free graphs are all semi-regular.

II.3 Trees and Forests

Trees and forests are essential combinatorial objects which appear in a myriad of different problems. They index algorithms in computer science and they index the iterative solutions of algebraic or differential equations. They also play a key role for the physics applications of graph theory, starting with the Kirchhoff-Maxwell solution of electric circuits up to the computation of Feynman's amplitudes in quantum field theory.

- Definition II.19.**
- A forest is an acyclic⁶ graph.
 - A tree is a connected acyclic graph. A rooted tree is a tree in which a single particular vertex has been marked as the root.
 - A forest of a graph G is an acyclic e-subgraph of G . A *spanning* forest of G is a forest of G which has as many connected components as G itself.
 - A (spanning) tree of a graph G is a forest of G which is connected. Hence only a connected graph can have spanning trees, which then coincide with its spanning forests and must connect all vertices of G . Most of the time we forget the word “spanning” where there is no ambiguity.

All these definitions apply by default to labeled unoriented objects, but extend to other categories such as digraphs, which lead to oriented trees and forests, flagged graphs which lead to flagged trees and forests and so on.

The connected components of a forest are trees, hence a forest is “made of trees” as common sense requires.

⁶acyclic means “without cycles”.

A tree with 1 vertex and no edge is called the empty tree. The forest of G with no edges, hence only the bare vertices of V is called the empty forest of G . Here the word “empty” refers to *edges*.

Trees and forests are simple graphs, hence their edges can be identified with pairs of their vertices, and the relation R is redundant for them and can be omitted.

Exercise II.20. Prove that a tree $T = (V, E)$ with $|V| \geq 2$ always has leaves and that $|E| = |V| - 1$. Prove that conversely if G is connected and $|E| = |V| - 1$, then G is a tree.

Exercise II.21. Prove that a graph G is a tree if and only if any pair of distinct vertices $u \neq v$ can be joined by a *single* trail of edges in G , which must then also be self-avoiding.

Exercise II.22. Prove that trees and forests are bipartite graphs.

Examples

A *linear graph* is a particularly simple example of a tree. It is a tree that is not branched at all. Hence it is either an isolated vertex or contains two leaves plus an arbitrary number of vertices of degree 2 in between. Linear graphs are Eulerian and Hamiltonian open graphs, and are the only type of trees with this property.

A *star graph* S_n is somewhat the opposite of a linear graph: it is a tree with a single internal vertex of degree $n - 1$ and n leaves, and is also the complete bipartite graph $K_{1,n-1}$.

Using Exercise II.21 we see that any rooted tree is naturally equipped with a partial order on its vertices. We say that a vertex v is a descendant of u , and write $u \leq v$ if the unique path from v to the root goes through u . If v is a descendant of u with an edge between u and v , v is called a *child* of u .

An n -ary tree is a rooted tree for which each vertex has at most n children. A n -ary tree is called full if every vertex has exactly 0 or n children. 2-ary trees and 3-ary trees are often called respectively binary and ternary trees.

Exercise II.23. What is your genealogy tree? is it really a tree?

An ordered tree or plane tree is a rooted tree where the vertex set is the set of integers $\{1, \dots, n\}$, the root is at 1, and the labels of the vertices respect the partial order of the tree, that is if $u \leq v$ for two vertices u and v , then the label of u is smaller than the label of v .

Exercise II.24. Define a unique natural way to draw a plane tree on the oriented plane. (Hint: suppose the orientation is clockwise, draw the root, then its children with increasing clockwise labels, then grandchildren ...)

Exercise II.25 (Mazes). A *maze* is a connected graph with two marked entries, entrance and exit. A perfect maze is a maze without cycles, hence is a tree. To go out of a usual (bidimensional) maze made of walled passageways there is the famous solution to “always follow the left wall with your hand”. Can you explain this solution in more mathematical terms, and why it works? Why is this solution often far from optimal?

Plane trees have many interesting statistical applications, in particular to search algorithms and random generation processes which are sketched in chapter VI.5.

Definition II.26 (Complexity). The *complexity* of a connected graph $G = (V, E)$, noted $\chi(G)$, is defined as the number of (spanning) trees of G . The complexity of a non-connected graph is defined as the product of the complexities of its connected components, hence as the number of its spanning forests.

Exercise II.27 (Complexity of K_4 and K_5). Describe and draw the 16 trees of K_4 by organizing them into 2 families. Describe and draw the three families that make up the 125 trees of K_5 . (Hint: classify trees according to the number of their leaves).

Suppose from now on that the vertex set of a tree is $V = \{1, \dots, n\}$. Since a tree cannot have tadpoles nor multi-edges, the set of edges E of the tree can be identified with a subset of the set P_n of the $n(n - 1)/2$ unordered pairs of integers between 1 and n .

Counting trees may be tricky. We can, among others, count labeled trees (also called Cayley trees), unlabeled trees, in which labelings are erased, also called tree shapes, and plane trees which are trees with a particular embedding in the plane. The next exercises are devoted to explore this issue.

Exercise II.28. How many *labeled* linear graphs have n vertices? Same question with star graphs.

Exercise II.29 (Caterpillars). A *caterpillar tree* is a tree for which removing all the leaves together with their single incident edges produces a linear graph.

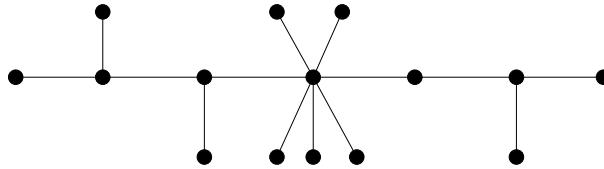


Figure 5: A caterpillar

- 1) Draw all caterpillar (unlabeled) *shapes* up to 7 vertices.
- 2) Count how many *labeled caterpillars* correspond to each shape.
- 3) Prove the following equivalent characterizations
 - Caterpillars are the trees in which there exists a path that contains every vertex of degree two or more.
 - Caterpillars are the trees in which every vertex of degree at least three has at most two neighbors who are not leaves.
 - Caterpillars are the trees that do not contain as a subgraph the graph of Fig 6, which we call the double-3-star.

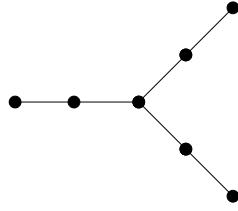


Figure 6: The double-3-star

- Caterpillars are the connected graphs that can be drawn with their vertices on two parallel straight lines, with edges represented as non-crossing line segments that have one endpoint on each line.

4) Check that the number of caterpillar *shapes* with n *unlabeled* vertices is $2^{n-4} + 2^{\lfloor n/2 \rfloor - 2}$, where $\lfloor \cdot \rfloor$ means the integer part, for $n = 4, 5, 6$ and 7 . Prove this formula holds for any $n \geq 4$.

II.4 Incidence, Laplacian, Rank and Nullity

It is not practical to rely solely on drawings to describe graphs. For instance we may want to store graphs in a computer as a list of numbers. Hence mathematicians have described graphs through various algebraic representations, such as matrices. It works optimally in the case of oriented tadpole-free graphs.

Definition II.30 (Incidence Matrix). The *incidence matrix* for an oriented tadpole-free graph G is the rectangular $|E|$ by $|V|$ matrix with indices running over vertices and edges respectively, such that

- $\epsilon_G(v, e)$ is $+1$ if e ends at v ,
- $\epsilon_G(v, e)$ is -1 if e starts at v ,
- $\epsilon_G(v, e)$ is 0 otherwise.

Any oriented tadpole-free graph is fully characterized through its incidence matrix $\epsilon_G(v, e)$. Remark that the sum of any column in the incidence matrix is 0.

It is sometimes also useful to introduce the positive incidence matrix $\eta_G(v, e) = |\epsilon_G(v, e)|$. This matrix can be then generalized to unoriented graphs including tadpoles by defining $\eta_G(e, v) = 2$ for a tadpole attached at vertex v . With that convention we have

$$d_G(v) = \sum_{e \in E} \eta_G(e, v). \quad (\text{II-3})$$

Definition II.31 (Degree and Adjacency Matrices). The diagonal $|V|$ by $|V|$ square matrix with entries $d_G(v)$ on the diagonal is called the *degree matrix* D_G . Hence $D_G(v, v) = d_G(v) \forall v \in V$, and $D_G(v, w) = 0$ if $v \neq w$. The *adjacency matrix* of a

tadpole-free graph G is a symmetric square matrix $A = A_G$ indexed by $V \times V$. It is made of zeroes on the diagonal: $A_G(v, v) = 0 \quad \forall v \in V$, and if $v \neq w$ then $A_G(v, w)$ is the number of edges of G which have vertices v and w as their ends.

Definition II.32 (Laplacian). The *Laplacian matrix* of a tadpole-free G is defined to be $L_G = D_G - A_G$.

For example, the Laplacian matrix of the graph pictured in Figure 4 is

$$\begin{pmatrix} 5 & -2 & -2 & -1 \\ -2 & 3 & 0 & -1 \\ -2 & 0 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}. \quad (\text{II-4})$$

We can extend the definition of a Laplacian matrix to general graphs G with tadpoles by erasing all the tadpoles, hence by defining $L_G = L_{\hat{G}}$.

We begin by expressing the Laplacian matrix of a tadpole-free graph in a different form. Consider a connected tadpole-free graph $G = (V, E)$, and orient each edge of G arbitrarily. For any matrix M we denote the conjugate transpose of M by M^* . It is easy to check that:

Lemma II.33. *Let $G = (V, E)$ be a tadpole-free graph, orient G arbitrarily, and let ϵ_G be the incidence matrix of G . Then $\epsilon_G \cdot \epsilon_G^* = L_G$, where the product is the product of rectangular matrices.*

Proof We have

$$L_G(v, v) = d_G(v) = \sum_{e \in E} |\epsilon_G(v, e)| = \sum_{e \in E} \epsilon_G(v, e) \epsilon_G^*(e, v). \quad (\text{II-5})$$

Similarly for $v \neq w$

$$L_G(v, w) = -A_G(v, w) = -\sum_{e \in E} |\epsilon_G(v, e) \epsilon_G(w, e)| = \sum_{e \in E} \epsilon_G(v, e) \epsilon_G^*(e, w). \quad (\text{II-6})$$

□

Although the incidence matrix depends on the orientations of the edges, the Laplacian matrix does not, as the changes of signs in some terms of ϵ_G and ϵ_G^* cancel out in $\epsilon_G \cdot \epsilon_G^* = L_G$.

Exercise II.34 (Connected components as maximal blocks of the incidence matrix). Prove that the decomposition of a tadpole-free graph into connected components corresponds algebraically to the decomposition of its incidence matrix as a maximal set of factorized blocks.

Definition II.35 (Rank, Nullity). We define $r(S) = |V| - k(S)$ as the *rank* of the subgraph S and $n(S) = |S| + k(S) - |V|$ as its *nullity* or *cyclomatic number*. In loose language $n(S)$ is the number of *independent cycles* in S .

Exercise II.36. Prove that the rank and nullity of a tadpole-free graph are respectively the rank and the dimension of the kernel of its incidence matrix (hint: if you do not find, read Proposition IV.2 below).

For the moment, remark that $r(S)$ is the number of edges in any spanning forest of S , and that $n(S)$ is the number of the remaining edges in S when a spanning forest is suppressed. This allows a quick visual identification of the rank and nullity of simple graphs.

Independent cycles, often simply called *loops* by physicists, is an important concept for the many applications of graph theory in which one deals with conserved flows or currents. In this case an algebraic number or a vector is attached to each (oriented) edge of the graph to represent the *flow* of some quantity through that edge, and there is a conservation law, namely the algebraic sum of the flows at every vertex is 0. Electric currents in electric circuits is an important example treated in detail in section IV.2. But there are many other conservative flows, which can measure quantities as diverse as the amount of people and goods in and out of a city, the megabits flowing along internet edges into or out of an internet node, or the momenta of particles at a collision in a particle accelerator.

To parametrize the values of conservative flows or currents at every edge of the graph, we need only to know the flows through all edges associated to independent cycles, as the others can be deduced from these ones by using the conservation rules at every vertex. Be careful however that this parametrization is not canonical. It requires the choice of a particular spanning forest to define the complement as a particular representation of the independent cycles. Ultimately the parametrization of the flow therefore depends on the particular choice of the spanning forest, although the physical flow which is modeled in this way obviously did not depend on this choice. Here is an occasion to recall that one should always distinguish a real phenomenon from the particular convention used to parametrize it, and to remember that ultimately the phenomenon does not *depend* on this convention.

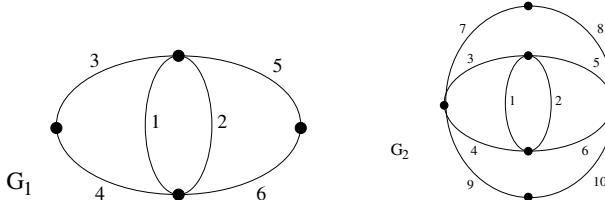


Figure 7: The graphs G_1 and G_2

Exercise II.37. What are the connexity, rank and nullity of graphs G_1 and G_2 ?

Find $\chi(G_1)$ and $\chi(G_2)$ by listing all the trees of graph G_1 , and then of graph G_2 .

How many forests are there in graphs G_1 and G_2 ?

Exercise II.38. Orient the graph G_1 in an arbitrary way. Write some conserved scalar flow for this graph, that is attribute to each oriented edge an algebraic number such that flows are conserved at each vertex. For a particular choice of a spanning tree in G_1 , compute the flows of the edges of the tree in terms of the flows of the edges not in the tree. Repeat all these operations with G_2 .

Why are the values of the flow independent of the choice of the tree, although the formulas themselves change?

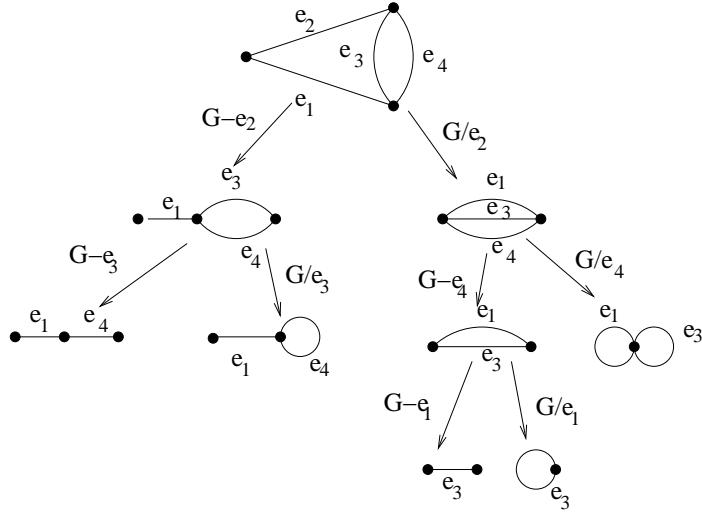


Figure 8: The contraction-deletion of a graph

Can you abstract at this stage the general rules to compute a conserved flow for a general spanning forest of a general oriented graph in terms of the flow of the edges which do not belong to the forest?

Try to relate what you have learnt from this example to even more general scientific principles in science. Why do we need particular coordinate systems in science in general, although ultimately physical realities cannot depend on our choice of coordinates?

III Tutte Polynomial and Applications

III.1 Deletion, Contraction

There are two natural operations associated to an edge e of a graph G , pictured in Figure III.1:

- the deletion, which leads to a graph noted $G - e$,
- the contraction, which leads to a graph noted G/e . If e is not a tadpole, it identifies the two vertices v_1 and v_2 at the ends of e into a new vertex v_{12} , attributing all the flags (half-edges) attached to v_1 and v_2 to v_{12} , and then it removes e . If e is a tadpole, G/e is by definition the same as $G - e$.

A terminal form for the deletion-contraction process is a connected graph made solely of bridges and tadpoles, hence an end point in the process pictured in Figure III.1.

There are also two natural unions operations on a pair of *disjoint* graphs G_1 and G_2 (where disjoint means that V_1, V_2, E_1 and E_2 are all disjoint sets)

- the disjoint union $G = G_1 \cup G_2$ is defined with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. In particular there is no edge in G between vertices of G_1 and vertices of G_2 and the number of connected components of G is:

$$k(G_1 \cup G_2) = k(G_1) + k(G_2). \quad (\text{III-7})$$

- the vertex-union $G = G_1 \star_{v_1, v_2} G_2$. It is defined for two *rooted* graphs G_1 and G_2 , that is for graphs in which a special vertex v_1 has been picked in G_1 and a special vertex v_2 has been picked in G_2 . The vertex union collapses the two vertices v_1 and v_2 into a single one, without modifying edges. It can be also defined by first adding an edge between v_1 and v_2 and then contracting it. Hence

$$\begin{aligned} k(G) &= k(G_1 \star_{v_1, v_2} G_2) = k(G_1) + k(G_2) - 1, \\ |V(G_1 \star_{v_1, v_2} G_2)| &= |V(G_1)| + |V(G_2)| - 1. \end{aligned} \quad (\text{III-8})$$

A function F_G defined for any graph is called factorized over disjoint and vertex unions if it obeys the properties

$$F_{G_1 \cup G_2} = F_{G_1} F_{G_2} \quad (\text{III-9})$$

and

$$F_{G_1 \star_{v_1, v_2} G_2} = F_{G_1} F_{G_2}, \quad (\text{III-10})$$

for all possible choices of $G_1, G_2, v_1 \in V(G_1), v_2 \in V(G_2)$.

Definition III.1. A function $F_G(a, b, \dots)$ is said to obey a linear deletion-contraction relation with coefficients a and b if for any G connected and e regular

$$F_G(a, b, \dots) = aF_{G-e}(a, b, \dots) + bF_{G/e}(a, b, \dots). \quad (\text{III-11})$$

Lemma III.2. *The complexity is factorized over disjoint and vertex unions and obeys a linear deletion contraction relation with coefficients 1 and 1:*

$$\chi_G = \chi_{G-e} + \chi_{G/e}. \quad (\text{III-12})$$

Proof Factorization III-9-III-10 is easy to check. The deletion-contraction linear relation is due to the fact that the trees in G can be classified into trees not containing e (hence trees of $G - e$) and trees containing e , which are in bijection with those of G/e by contracting e . \square

Since the complexity of any terminal form is 1, the complexity of a (connected) graph counts the number of terminal forms under any full deletion contraction such as the process of Figure III.1. This proves that the *number* of such terminal forms is independent of the order in which regular edges are deleted or contracted: it is a function of G alone.

III.2 Tutte Polynomial

Developing this idea in more generality leads to a two-variable generalization of complexity, called the Tutte polynomial. It is one of the most interesting polynomial associated to a graph.

Definition III.3 (Sum overs subsets). If $G = (V, E)$ is a graph, the Tutte polynomial of G , $T_G(x, y)$ is a polynomial in two variables x and y defined by

$$T_G(x, y) = \sum_{A \subset E} (x - 1)^{r(G) - r(A)} (y - 1)^{n(A)}, \quad (\text{III-13})$$

where the sum over A runs over the $2^{|E|}$ subgraphs of G (in the usual sense of edge subgraphs), $r(A) = |V| - k(A)$ is the rank of the subgraph A and $n(A) = |A| + k(A) - |V|$ is its nullity.

Theorem III.1. The Tutte polynomial obeys a linear deletion-contraction relation III-14 with coefficients 1 and 1, just as the complexity does. If $G = (V, E)$ is a connected graph, and e is any regular edge, then

$$T_G(x, y) = T_{G/e}(x, y) + T_{G-e}(x, y). \quad (\text{III-14})$$

Furthermore its value on terminal forms with m bridges and n tadpoles is

$$T_G(x, y) = x^m y^n. \quad (\text{III-15})$$

The Tutte polynomial factorizes over disjoint and vertex unions. It is the unique polynomial with all these properties.

Proof First it is easy to check that definition (III-13) factorizes over disjoint and vertex unions. Then for a terminal form, it is also possible to check that III-13 equals III-15, by organizing the sum over A in III-13 according to the number of bridges and tadpoles in A :

$$\begin{aligned} \sum_{A \subset E} (x - 1)^{r(G) - r(A)} (y - 1)^{n(A)} &= \sum_{b=0}^m \sum_{s=0}^n \binom{m}{b} \binom{n}{s} (x - 1)^{m-b} (y - 1)^s \\ &= (1 + (x - 1))^m (1 + (y - 1))^n = x^m y^n, \end{aligned} \quad (\text{III-16})$$

where $\binom{p}{q} = p! / q!(p - q)!$ is the binomial coefficient counting the number of subsets with q elements in a set of p elements. We now prove the theorem by induction on N , the number of regular edges in a connected graph G . We know already that it is true for $N = 0$, that is for terminal forms. We consider a connected graph G with $N + 1$ regular edges and choose any of them e . Organizing the sum over A in III-13 according to whether e belongs to A or not, we can check that III-13 leads to III-14. Then we apply the induction hypothesis.

There are still other definitions of the Tutte polynomial which we do not study here.

The Tutte polynomial at particular values of x and y count interesting quantities related to a graph G :

Lemma III.4. $T_G(1, 1)$ counts the complexity or the number of spanning forests in G , and in particular the number of (spanning) trees in G if G is connected: $\chi(G) = T_G(1, 1)$. $T_G(2, 1)$ counts the number of forests, i.e., the number of acyclic edge subsets. $T_G(1, 2)$ counts the number of connected spanning subgraphs. $T_G(2, 0)$ counts the number of acyclic orientations of G , ie orientations which do not allow any consistent oriented cycle.

Proof We have already seen the first statement. To check the other formulas one checks them on terminal forms, then establish the deletion/contraction rule with coefficients 1 and 1.

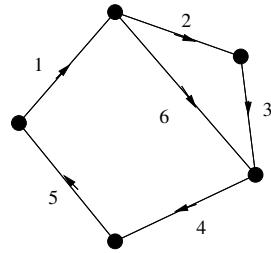


Figure 9: Another graph

Exercise III.5. Compute the incidence ϵ_G and Laplacian matrix L_G of the graph of Figure 9. Check the relation $L_G = \epsilon_G \epsilon_G^*$.

Compute the Tutte polynomial of that same graph

- by a deletion-contraction process
- by the global formula $T_G(x, y) = \sum_{A \subseteq E} (x - 1)^{r(G) - r(A)} (y - 1)^{n(A)}$.

Check that both expressions agree.

The Tutte polynomial is the *universal* solution of linear deletion-contraction rules with general coefficients. More precisely

Theorem III.2 (Universality). Suppose a function $F_G(a, b, x, y)$

- factorizes over disjoint and vertex unions:

$$F_{G_1 \cup G_2}(a, b, x, y) = F_{G_1}(a, b, x, y) \cdot F_{G_2}(a, b, x, y) \quad (\text{III-17})$$

and

$$F_{G_1 \star_{v_1, v_2} G_2}(a, b, x, y) = F_{G_1}(a, b, x, y) \cdot F_{G_2}(a, b, x, y), \quad (\text{III-18})$$

- obeys a linear deletion contraction relation with coefficients a and b

$$F_G(a, b, x, y) = aF_{G-e}(a, b, x, y) + bF_{G/e}(a, b, x, y), \quad (\text{III-19})$$

- has terminal forms $F_{bridge}(a, b, x, y) = x$, $F_{tadpole}(a, b, x, y) = y$.

Then the function F is the Tutte polynomial, up to rescalings. More precisely

$$F_G(a, b, x, y) = a^{n(G)} b^{r(G)} T_G(x/b, y/a) \quad (\text{III-20})$$

where T_G is the Tutte polynomial, r is the rank and n the nullity of G .

Proof The proof proceeds by induction as for the uniqueness theorem above. By the disjoint union property it is sufficient to check the theorem for connected graphs. By the vertex-factorization property, III-20 holds for terminal forms, hence connected graphs without regular edges. Finally III-20 is proved in the general case by induction on the number of regular edges using the linear deletion-contraction rule III-19.

III.3 Reliability

Networks are increasingly important concepts in our daily life. Water, food, electricity, information are all brought to us through networks, and social networks flourish on the internet, which now connects all countries of the world. More abstractly and mathematically we can represent a *network* as a connected graph G . The vertices (also called terminals or nodes) are the units to relate, whether they are cities, houses, computers or else, and edges are their connections.

Exercise III.6. Describe some networks you know in real life. Why is their reliability important?

Consider a network (i.e. a graph) and suppose at initial time $t = 0$ all connections work. Suppose also that during a fixed given amount of time (eg a month) every connection has a probability p to fail. The reliability of the network is the probability that at the end of that period of time all terminals are still connected together through the set of connections that have not failed. It is a function $R_G(p)$, called the reliability of the network.

Exercise III.7. Explain without actually computing it, why $R_G(p)$ must be a polynomial in p for every graph G .

At $x = 1, y = 1/p$, the Tutte polynomial computes, up to an overall factor, the all-terminal reliability polynomial. Indeed the reliability polynomial is a function $R_G(p)$ that gives the probability that every pair of vertices in G remains connected after the edges have had a chance of failing with probability p . The connection to the Tutte polynomial is given by

Theorem III.3 (Network Reliability).

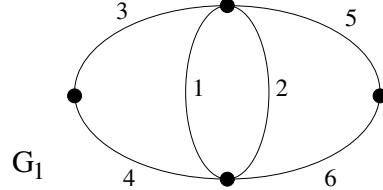
$$R_G(p) = (1 - p)^{|V|-1} p^{|E|-|V|+1} T_G(1, 1/p). \quad (\text{III-21})$$

Proof One can check the deletion contraction property

$$R_G(p) = p R_{G-e}(p) + (1 - p) R_{G/e}(p), \quad (\text{III-22})$$

the factorization properties III-9-III-10 and the value $1 - p$ on a bridge and 1 on a tadpole. Then one concludes by the universality theorem.

Exercise III.8. Compute the Tutte polynomial of the following graph:



Deduce the reliability $R_G(p)$ of that network. What is the reliability of the network for $p = 1/10$? Can you find a more reliable network with the same number of terminals and connections?

Exercise III.9. Can you characterize the graphs G for which $R_G(p) = 1 + O(p^q)$ where q is any given positive integer (ie the graphs for which the polynomial $R_G(p) - 1$ has valuation exactly q)?

III.4 Coloring

Suppose we have been given an integer c . We consider a set of c colors. There are two distinct main problems of coloring for a graph, namely vertex coloring and edge coloring.

In the vertex coloring problem we are interesting in coloring the vertices of the graph, that is assigning a color to each of them, so that no vertices joined by an edge share the same color. Such a coloring is also called a *proper* vertex coloring. The (vertex) chromatic polynomial $P_G(c)$ counts the number of such proper vertex colorings of G with c colors. Obviously there are no such colorings if G has a tadpole, hence in that case $P_G(c) = 0$.

The *edge coloring* problem asks for the number of ways, if any, of giving a color to each edge of the graph in such a way that no half-edges hooked to a common vertex share the same color. Obviously again there are no such colorings if G has a tadpole.

The vertex coloring problem is particularly famous in mathematics because there was for a long time an important conjecture which stated that any planar graph, that is any graph which can be drawn on a plane without crossings could be colored with at most 4 colors. This conjecture has been finally proved, but with the help of a large computer. The problem was first reduced to testing whether a large but finite number of complicated planar graphs could be colored with four colors. The computer then searched and found that each of these particular graphs could be colored with at most four colors. Hence:

Theorem III.10 (4 Colors Theorem). *The chromatic polynomial $P_G(4)$ is strictly positive for any planar graph G .*

It is an interesting problem to search for a simpler proof of this theorem that may not require a computer. We wont dig into this subject further but instead prove that the chromatic polynomial $P_G(c)$ is the Tutte polynomial at $y = 0$, up to an overall factor. Indeed

Theorem III.4 (Vertex Coloring).

$$P_G(c) = (-1)^{|V|-k(G)} c^{k(G)} T_G(1 - c, 0). \quad (\text{III-23})$$

Proof The first thing to remark is that the chromatic polynomial satisfies the linear deletion-contraction rule

$$P_G(c) = P_{G-e}(c) - P_{G/e}(c) \quad (\text{III-24})$$

for any regular edge e . Indeed if e joins v_1 to v_2 , the colorings of $G - e$ can be partitioned into those in which v_1 and v_2 have distinct colors, which are those of G , and those in which v_1 and v_2 have the same color, which are those of G/e . Hence $P_{G-e}(c) = P_G(c) + P_{G/e}(c)$.

However we cannot apply the universality theorem because the factorization property does not hold. For instance taking $G_1 = G_2$ to be the simple bridge, $P_{G_1}(c) = P_{G_2}(c) = c(c-1)$, but the star union of two bridges is a linear graph $G_3 = G_1 \star G_2$ of length 2 hence $P_{G_3}(c) = c(c-1)^2 \neq P_{G_1}(c) \cdot P_{G_2}(c)$. But this can be easily corrected. Invariance of colorings under the permutation group can be fixed by considering colorings with one given vertex or root per connected component of a fixed color, say color 1. The number of such colorings is $Q_G(c) = P_G(c)/c^{k(G)}$. Then it is easily checked that Q obeys the same deletion contraction relation than P , but has the additional advantage of factorization under disjoint and vertex union. The terminal values of $Q_G(c)$ are $c-1$ for a bridge and 0 for a tadpole. Hence by the universality theorem

$$Q_G(c) = (-1)^{|V|-k(G)} T_G(1 - c, 0). \quad (\text{III-25})$$

from which III-23 follows.

Again one can conclude through the universality theorem.

Recalling the definition II.8 we have obviously that the 2-colorable graphs are exactly the bipartite graphs.

Exercise III.11. Draw a planar graph which cannot be properly (vertex)-colored with three colors.

III.4.1 Cliques, Perfect Graphs, Edge Colorings

Definition III.12 (Cliques, Covers, Matchings). Consider a graph G .

- A clique of G is a subgraph of G isomorphic to the complete graph K_n . n is called the order of the clique.
- A *vertex cover* of G is a subset of vertices that includes at least one endpoint of each edge. A vertex cover is *minimal* if no other vertex cover has fewer vertices.
- A *matching* of G is a set of edges, no two of which share an endpoint. A matching is *maximal* if no other matching has more edges. It is perfect if it joins all vertices of G .

A typical theorem on matchings and vertex covers is

Theorem III.13 (König Theorem). *In any bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.*

Proof Suppose that $G = (V, E, R)$ is a bipartite graph, where $V = A \cup B$. Let M be a matching for G of size m . We must show that either G has a vertex cover C of size m , or M is not a maximum matching.

First, if M is a perfect matching, then M is maximum. In this case, every edge is incident to exactly one vertex on either side, so each side is a vertex cover of size $|M|$ and we are done. Otherwise, use an alternating path argument. An alternating path is a path where the edges alternate between M and $E - M$. Partition the vertices of G into subsets S_i as follows. Let S_0 consist of all vertices unmatched by M . For integer $j \geq 0$, let S_{2j+1} be the set of vertices that are adjacent to vertices in S_{2j} via some edge $e \in E - M$ and have not been included in any previously-defined set S_k , where $k < j$.

Each vertex $v \in S_{2j+1}$ must be adjacent to another vertex u via an edge $e \in M$ (otherwise, v is unmatched by M and would have been placed in S_0). If the vertex u has not yet been included in a set S_i , put u in S_{2j+2} . If there are no vertices adjacent to S_{2j} , arbitrarily pick an unused vertex and continue in S_{2j+1} .

Each vertex in S_i has an edge to a vertex in S_{i-1} . An alternating path can be formed from a given vertex, going up one level at a time, ending either at an unmatched vertex at level S_0 , or at some level S_{2j+1} containing a single (matched) vertex.

Note that there cannot be any edges in M between vertices in an even level subset S_{2j} because each of the vertices is connected by its single matched edge to a vertex in the previous level. If there exists any unmatched edge between two vertices u and v in the same even-level subset S_{2j} , or if there exists any matched edge between two vertices u and v in the same odd-level subset S_{2j+1} , then the two alternating paths for u and v can be connected via that edge uv into a longer alternating path. This path cannot have any repeated vertices since G is bipartite, so its two end vertices must be distinct vertices from S_0 . Since the path contains one edge less in M than in $E - M$, swapping the matched and unmatched edges along the path gives a larger matching than M . Thus, if M is maximum, each edge in M has a single endpoint in an odd-level subset S_{2j+1} , and each edge in $E - M$ has at least one endpoint in an odd-level subset S_{2j+1} . The union of the odd-level subsets forms a vertex cover of size $|M|$. Since no smaller set of vertices could cover every edge in M , it must be a minimum vertex cover. Therefore, a bipartite graph G with maximum matching M and minimum vertex cover C satisfies $|M| = |C|$. \square

Definition III.14 (Line Graph). The *line graph* $L(G)$ is the graph obtained in the following way: to each edge of G we associate a vertex of $L(G)$, then we put an edge between two vertices of $L(G)$ if and only if their corresponding edges in G share a common endpoint, hence are adjacent in G .

Exercise III.15. What is the line graph of a tadpole? of a path-graph? of the complete graph K_n ? Prove that the line graph of a connected graph is connected. Is the converse true?

A (proper) *edge coloring* of a graph attributes a color (in a finite set $\{1, \dots, c\}$) to each edge of the graph in such a way that two edges sharing an end have different

colors. The edge chromatic number of a graph G , $P_G^{\text{edge}}(c)$ is again a polynomial in c . Indeed, just observe that it is equal to the vertex chromatic number $P_{L(G)}(c)$ of its line graph $L(G)$. Hence it is also a polynomial in c :

$$P_G^{\text{edge}}(c) = P_{L(G)}(c). \quad (\text{III-26})$$

This observation does not mean that edge-coloring completely reduces to vertex-coloring, in particular because not every graph is the line-graph of an other graph. A graph G is the line graph of some other graph G' if and only if it is possible to find a set of cliques of G partitioning the edges of G such that each vertex of G belongs to exactly two of the cliques. In order to do this, it may be necessary for some of the cliques to be single vertices. This is a necessary and sufficient condition because cliques of this partition clearly are then associated to the vertices of G' , and vertices of G are associated to edges of G' .

Definition III.16. A perfect graph is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph.

In particular the line graph of a bipartite graph is perfect (see König's theorem).

Exercise III.17. A *minimal cycle* is a cycle $v_1, e_1, \dots, v_n, e_n, v_1$ for which there is no shortcut, that is no cycle on a strict subset of the vertices v_1, \dots, v_n . A *triangulated graph* is a graph in which any minimal cycle is the triangle K_3 .

Prove that complete graphs are triangulated, and that triangulated graphs are perfect.

IV Circuits and Flows

IV.1 Matrix-Tree Theorem

The deletion-contraction relation and Tutte polynomial is powerful but typically its computation (eg through deletion contractions or through subset sums) requires an exponential number of operations as the size of the graph increase, of the order of $2^{|E|}$. In fact the computation of the Tutte polynomial is known to be NP -complete. Hence it cannot be solved in polynomial time, unless $P = NP$, that is unless every problem whose solution can be checked in polynomial time can also be solved in polynomial time. The answer to the $P = NP$ question is not known yet and is one of the most famous problem in mathematics. But whatever the answer may be we are interested in faster and simpler methods to compute simpler quantities such as the complexity of a graph.

In that case there is indeed an older and faster method, namely the Tree-Matrix Theorem. This method, which was introduced by Kirchoff and Maxwell, allows to compute the equivalent conductance of electric circuits of a general type.

If M is a matrix and i is a row index for M and j is a column index for M , let $M_{i,j}$ denote the submatrix of M obtained by deleting row i and column j from M . More generally for subsets S and T of the lines indices and of the columns indices $M_{\bar{S},\bar{T}}$ means that in the matrix M we have deleted the lines in S and the columns in T , $M_{\bar{S},\bar{T}}$

means that in the matrix M we have deleted the lines in S and kept the columns in T etc...

The Matrix-Tree Theorem computes the complexity of a connected tadpole-free graph in terms of its Laplacian matrix. The complexity of more general graphs follow easily by erasing the tadpoles and working connected component by connected component.

Theorem IV.1 (Matrix-Tree Theorem). Let $G = (V, E)$ be a tadpole-free connected graph, and let L_G be the Laplacian matrix of G . Then for any $v \in V$,

$$\chi(G) = \det L_{G, \bar{v}, \bar{v}}. \quad (\text{IV-27})$$

Note that this determinant can be evaluated quickly (as we remarked earlier, this is not the case for the Tutte polynomial in general). Indeed the Laplacian matrix L_G is easy to construct from G , and determinants are fast to compute (by creating lots of zeros before expanding). Thus, the Matrix-Tree Theorem is much more effective than the deletion/contraction recursion for computing $\chi(G)$ in the case of large graphs.

The evaluation of $\det L_{G, \bar{v}, \bar{v}}$ proceeds in two steps: first we find a combinatorial meaning of the $|V| - 1$ by $|V| - 1$ subdeterminants of the incidence matrix ϵ_G , and then we feed this information into a general fact of linear algebra, the Binet-Cauchy Formula.

Proposition IV.2. Let $G = (V, E)$ be a connected tadpole-free oriented graph and let ϵ_G be its incidence matrix. Let $v \in V$ and $S \subset E$ be such that $|S| = |V| - 1$. Then $\epsilon_{G, \bar{v}, S}$ is a $|V| - 1$ by $|V| - 1$ square matrix, and

- $\det \epsilon_{G, \bar{v}, S} = \pm 1$ if S is a tree,
- $\det \epsilon_{G, \bar{v}, S} = 0$ otherwise.

Proof Clearly if S is not a tree it has to contain a cycle C . Orient the edges of C consistently in one of the two consistent orientations of C , and for each $e \in E(C)$, let $\eta_C(e) = +1$ if the orientations of e in C and G agree and $\eta_C(e) = -1$ if they differ, Then

$$\sum_{e \in E(C)} \eta_C(e) \epsilon_G(v, e) = 0, \quad (\text{IV-28})$$

the zero vector. Thus, the columns of $\epsilon_{\bar{v}, S}$ indexed by $E(C)$ are linearly dependent. Since $E(C) \subset S$, the columns of ϵ_G indexed by S are linearly dependent. Therefore the columns of $\epsilon_{G, \bar{v}, S}$ are linearly dependent. Therefore $\det \epsilon_{G, \bar{v}, S} = 0$.

Now suppose S is a tree. We can prove that $\det \epsilon_{G, \bar{v}, S} = \pm 1$ by induction. It is obvious for $|V| = 2$. Suppose it is true up to $|V| = n$ vertices. If S is a tree with n edges on $n + 1$ vertices, it has at least two leaves, hence one leaf v' not equal to v . The line with index v' in $\epsilon_{G, \bar{v}, S}$ has a single non zero element which is ± 1 (corresponding to the unique edge e touching the leaf v'). Expanding $\det \epsilon_{G, \bar{v}, S}$ along that line and using the induction hypothesis on $S - e$ we obtain $\det \epsilon_{G, \bar{v}, S} = \pm 1$. \square

Let M be an r by m matrix, and let P be an m by r matrix. The product MP is then a square r by r matrix, so its determinant is defined. The Binet-Cauchy Formula generalizes the ordinary product formula for square determinants.

Proposition IV.3 (The Binet-Cauchy Formula). Let M be an m -by- n matrix, and let P be an n -by- m matrix. Then

$$\det MP = \sum_{S, |S|=m} \det M_{:,S} \det P_{S,:}, \quad (\text{IV-29})$$

where the point \cdot means we delete nothing. The sum is over all m -element subsets of the n column indices of M (which are the same as the row indices of P).

Proof Obviously the formula is true for $n < m$, since both sides are zero, for $n = m$ (where it reduces to the ordinary product formula for determinants) and for $m = 1$ where it is easy. In more general cases the proof is tedious. One writes the columns of MP as linear combinations of columns of M with coefficients in P , use multilinearity to expand the determinant and gather all terms corresponding to $\det M_{:,S}$ for a given set S of columns of M . Then one checks that the coefficient, a combination of products of terms in P , is nothing but $\det P_{S,:}$. \square

Proof of the Matrix-Tree Theorem. Return to $G = (V, E)$ a tadpole-free oriented connected graph, and to L_G , its Laplacian matrix. Let $v \in V$ be any vertex. Using Lemma II.33, we evaluate $\det L_{G,\bar{v},\bar{v}}$ through the Binet-Cauchy formula:

$$\begin{aligned} \det L_{G,\bar{v},\bar{v}} &= \det \epsilon_{G,\bar{v}}, [\epsilon^\dagger]_{G,:,:} \\ &= \sum_{S \subset E, |S|=|V|-1} \det \epsilon_{G,\bar{v},S} \cdot \det [\epsilon]_{G,S,\bar{v}}^\dagger \\ &= \sum_{S \subset E, |S|=|V|-1} [\det \epsilon_{G,\bar{v},S}]^2 \\ &= \sum_{S \text{ tree of } G} 1 = \chi(G). \end{aligned} \quad (\text{IV-30})$$

The second equality holds by the Binet-Cauchy Formula, the third equality follows since $\det M^\dagger = \det M$ for any real square matrix M , and the fourth equality follows from proposition IV.2.

Exercise IV.1. What is $\chi(K_n)$? (how many trees are there on n distinct labeled vertices)?

IV.2 Electric Circuits

In this section we follow the presentation of [4], see also [3]-[5].

Electric circuits were first studied in full generality by Kirchhoff and Maxwell. In 1847, Kirchhoff gave a formula for the effective conductance of linear resistive electrical networks. The solution relies on the inversion of the Laplacian matrix of the circuit (with one line and column deleted) hence the essential step is the computation of determinants through the Matrix-Tree Theorem or of minors of the Laplacian matrix through some slight generalization thereof.

Let $G = (V, E)$ be a finite, connected, undirected graph (which may be tadpole-free or not). It can be considered representing an electrical network: the edges are

wires and the vertices are junctions at which the wires are connected with one another. Each edge $e \in E$ is assigned an electrical resistance $R_e > 0$, a positive real number. Given two distinct vertices $a, b \in V$, we pass an electric current through the graph G by attaching the vertices a and b to the poles of an external current source. By measuring the difference in electric potential between the vertices a and b we can then determine the effective resistance of the network G between the terminals a and b , by Ohm's Law. This calculation always leads to an effective resistance which is a rational fraction of the wire resistances and encodes a great deal of combinatorial information about the graph G .

Kirchhoff's formula is more naturally expressed in terms of conductance rather than resistance: conductance is merely the reciprocal of resistance. Resistance and conductance are real-valued quantities, but Kirchhoff's formula remains valid for complex quantities, which appear physically in the case of LRC networks with periodic currents. The analogue of resistance is the impedance and the analogue of conductance is referred to as admittance. An electrical network is therefore a pair $(G, \{y\})$ in which $G = (V, E)$ is a graph as above, and $\{y\} = \{y_e : e \in E\}$ is the set of conductances (or admittances) of all edges e .

In order to derive Kirchhoff's Formula we need to specify the behavior of an electrical network precisely. This is accomplished by Ohm's Law, Kirchhoff's Current Law, and Kirchhoff's Voltage Law.

Proposition IV.4 (Ohm's Law). In a wire e with ends v and w , the current j_e flowing through e from v to w is directly proportional to the difference in electric potential $U(v) - U(w)$ between the ends. The constant of proportionality is the admittance y_e of the wire e . That is, $j_e = y_e(U(v) - U(w))$.

Proposition IV.5 (Kirchhoff's Current Law). In an electrical network (G, y) , at every vertex v the amount of current flowing in equals the amount of current flowing out.

Proposition IV.6 (Kirchhoff's Voltage Law). In an electrical network (G, y) , there is a potential function $U : V \rightarrow K$ such that Ohm's Law is satisfied for every wire $e \in E$, and such that the currents determined by Ohm's Law also satisfy Kirchhoff's Current Law.

Ohm's Law and Kirchhoff's Current Law specify the system of equations governing the currents and electrical potentials in the network. Mathematically Kirchhoff's Voltage Law is in fact a lemma stating that this system of equations is consistent. Physically it is a consequence of the fact that currents are made of charges which move under the action of electric fields and that electric fields derive from a potential; this can be checked through Coulomb's law and the superposition principle. Clearly only differences of potentials enter the physical laws. Hence the potential is defined up to an overall constant, and can be grounded (put to zero) at a fixed arbitrary point.

To measure the effective admittance of an electrical network $(G, \{y\})$ between the vertices $a, b \in V$, we can connect a and b to an external source of current and force one ampere of current through the network from b to a . Ground the vertex a so that its electric potential is $U(a) = 0$. The electric potential $U(b)$ is then inversely proportional to the effective admittance of G , by Ohm's Law. So a calculation of the quantity $y_{ab}(G; \{y\}) = \frac{1}{U(b)}$ is what we seek.

Let us define the *weighted* tree polynomial

$$F_G(y) = \sum_{T \text{ tree of } G} \prod_{e \in T} y_e. \quad (\text{IV-31})$$

It generalizes the complexity since $\chi(G) = F_G(y)|_{y_e=1 \forall e}$.

For a graph $G = (V, E)$ and vertices $a, b \in V$, we let G/ab denote the graph obtained by merging the two vertices a and b together into a single vertex. Another way to define this graph is to add between a and b and extra edge e_{ab} , then contract this edge: $G/ab = G/e_{ab}$.

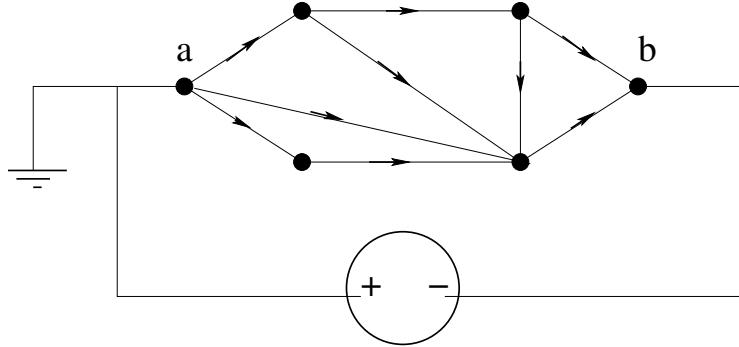


Figure 10: An electric circuit

Kirchoff's solution of electric circuits can be expressed as the following theorem:

Theorem IV.7. Let $(G, \{y\})$ be an electrical network, and let $a, b \in V$. The effective admittance between a and b in G is

$$y_{ab}(G; \{y\}) = F(G; \{y\}) / F(G/ab; \{y\}). \quad (\text{IV-32})$$

Proof We begin by translating the physical laws above into linear algebra. To do this we fix an arbitrary orientation of $G = (V, E)$. The choice of orientation does not affect the final answer, but the orientation is needed in order to write down the equations corresponding to Ohm's Law.

Next we consider the V by E incidence matrix ϵ_G of G with respect to this orientation. Also, let $\mathbf{j} = \{j_e : e \in E\}$ be the E -indexed column vector of currents, let $Y = \text{diag}(y_e : e \in E)$ be the diagonal matrix of admittances, and let $U = U(v) : v \in V$ be the V -indexed column vector of voltages, normalized so that $U(a) = 0$. We can translate the physical laws as follows:

Ohm's Law

$$\mathbf{j} = -Y\epsilon_G^\dagger U. \quad (\text{IV-33})$$

This is the statement of Ohm's Law for every wire in the network.

Kirchhoff's Current Law:

$$\epsilon_G \mathbf{j} = \delta_a - \delta_b. \quad (\text{IV-34})$$

Here δ_v is the V -indexed column vector given by the Kronecker rule $\delta_v(v) = 1$ and $\delta_v(w) = 0$ if $v \neq w$.

The reason that the currents are not all zero is that there is a source in IV-34: one ampere of current is being supplied to b externally and removed from a externally. The currents internal to the network G must compensate for this external driving current.

Let us define the $|V|$ by $|V|$ weighted Laplacian matrix $L_{G,y} = \epsilon_G Y \epsilon_G^\dagger$. Combining IV-33 and IV-34 we get the equation for U

$$L_{G,y} U = \delta_b - \delta_a. \quad (\text{IV-35})$$

Since

$$L_{G,y}(v, v) = \sum_{e \text{ incident to } v} y_e, \quad L_{G,y}(v, w) = - \sum_{e \text{ incident to } v \text{ and } w} y_e, \quad (\text{IV-36})$$

the matrix $L_{G,y}$ does not depend on the choice of orientation used to define ϵ_G , and its rows and columns sum to zero. Hence the image of the associated linear map $L_{G,y}$ is included in the $|V| - 1$ dimension hyperplane of vectors whose components also sum to zero, and an equation such as IV-35 can have solutions only for externally supplied currents which sum to zero; that's what we assumed by considering the current entering a to be equal to the current exiting b .

Kirchhoff's voltage law states that under this condition, a solution U exists. This is equivalent to state that the rank of $L_{G,y}$ is indeed $|V| - 1$, like the rank of L_G . Then in fact an infinite number of solutions exist for U , which differ by a constant, as expected. Let us prove now this statement.

The Matrix-Tree Theorem generalizes easily to

Theorem IV.8 (Weighted Matrix-Tree Theorem).

$$\det L_{G,y,\bar{v},\bar{v}} = \sum_{T \text{ tree of } G} \prod_{e \in T} y_e = F_G(y). \quad (\text{IV-37})$$

Proof Almost strictly identical to that of the Matrix-Tree Theorem. \square

For strictly positive admittances the rank of $L_{G,y}$ is therefore $|V| - 1$, as announced.

We also have the following generalization to the Laplacian matrix with more lines and columns deleted:

Theorem IV.9 (Principal Minors Weighted Matrix-Tree Theorem). For any subset $R \subset V$ of vertices, we have

$$\det L_{G,y,\bar{R},\bar{R}} = \sum_{\mathcal{F} \text{ R-forest of } G} \prod_{e \in \mathcal{F}} y_e \quad (\text{IV-38})$$

in which the sum over R -forests means a sum over all maximal forests \mathcal{F} of G for which each component of \mathcal{F} contains exactly one vertex of R .

Proof Exercise! \square

To find a unique solution for U we have to fix the potential at a point. This was done by choosing $U(a) = 0$. Under this condition IV-35 becomes

$$[L_{G,y}]_{\bar{a},\bar{a}} U_{\bar{a}} = (\delta_b)_{\bar{a}}. \quad (\text{IV-39})$$

hence we work in the space of column vectors indexed by $V - \{a\}$.

Since we only want the value of $U(b)$, Cramer's rule is enough. By the weighted Matrix-Tree Theorem IV.8,

$$\det[L_{G,y}]_{\bar{a},\bar{a}} = \det(\epsilon_{G,\bar{a}} Y \epsilon_{G,\bar{a}}^*) = F_G(y), \quad (\text{IV-40})$$

is nonzero for positive admittances $\{y\}$. Replacing column b of $[L_{G,y}]_{\bar{a},\bar{a}}$ by δb , we obtain a matrix M with $M_{b,b} = 1$ being the only nonzero entry in column b . The expansion of $\det M$ along column b then shows that

$$\det M = \det(\epsilon_{G,\bar{a}\bar{b}} Y \epsilon_{G,\bar{a}\bar{b}}^*) = \det[L_{G,y}]_{\bar{a}\bar{b},\bar{a}\bar{b}}. \quad (\text{IV-41})$$

By Theorem IV.9 this determinant is the generating series for spanning forests of G which have exactly two components, one containing a and one containing b . These forests of G correspond bijectively to spanning trees of G/ab , so that $\det M = F_{G/ab}(y)$. Cramer's rule thus implies that $U(b) = F_{G/ab}(y)/F_G(y)$. Since the effective admittance was defined to be $y_{ab}(G; \{y\}) = 1/U(b)$, this completes the proof. \square

Exercise IV.2. a) Compute the electric conductance of the “Wheatstone bridge” graph of Figure 11 between sources a and b in terms of the conductances y_e of each wire, using the tree polynomial formulas.

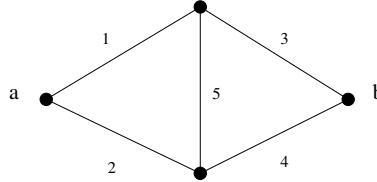


Figure 11: The Wheatstone bridge

b) Can you find the same result by the usual method of parallel/series equivalence?

Exercise IV.3. Find the conductance between any pair of vertices for the complete graph K_n on n vertices with equal conductance on every edge.

Exercise IV.4. Find the conductance between a black and a white vertex for the complete bipartite graph $K_{m,n}$ on m black vertices and n white vertices with equal conductance on every edge.

IV.3 The Max-Flow-Min-Cut Theorem

In the previous section the total current between sources is proportional to the voltage difference. The solution is obtained essentially by inverting the Laplacian matrix,

with the subtlety that one row and one column has to be deleted because of the gauge invariance (the potential is defined only up to an arbitrary additive constant).

In practice the linear response regime computed in this way is only valid for currents up to a certain intensity. If one tries to force too much current through a circuit, at some point some wire will melt and the circuit will break.

Similar effects occur in any network in real life: whether links are electric wires, optic fibers, roads, subway lines etc... there is always a maximum quantity per edge, beyond which the edge might break, called its *capacity*.

To model this very important practical limitation one considers networks.

Definition IV.5 (Network). A network \mathcal{N} is a quadruplet $\mathcal{N} = (G, c, s, t)$ made of a connected *directed tadpole-free graph* G and a function c which to each edge e of G associates a positive number $c(e) \geq 0$, called its *capacity*, plus two particular vertices s and t , often called source and sink, but that we prefer to call source and target in these notes. We also assume (without significant loss of generality), that all edges touching the source are oriented so as to leave the source and that all edges touching the target are oriented so as to enter the target.

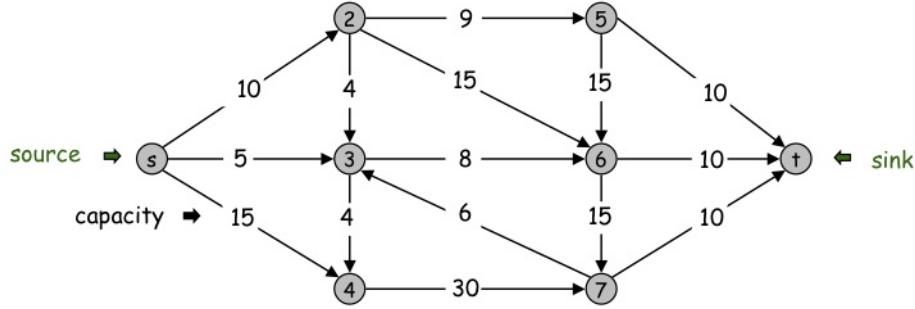


Figure 12: A network

Definition IV.6 (Flow). A flow f is a function from the set of edges of a network \mathcal{N} to the positive real numbers which to each edge associates the flow $f(e)$ through this edge and satisfies the two properties

- Respect capacities

$$0 \leq f(e) \leq c(e) \quad \forall e. \quad (\text{IV-42})$$

- Conserve quantities at every vertex other than the source and target:

$$\sum_{e \rightarrow v} f(e) - \sum_{v \rightarrow e} f(e) = 0 \quad \forall v \neq s, v \neq t. \quad (\text{IV-43})$$

where the notation $\sum_{e \rightarrow v}$ stands for the sum over all edges e entering v and $\sum_{v \rightarrow e}$ stands for the sum over all edges e exiting v .

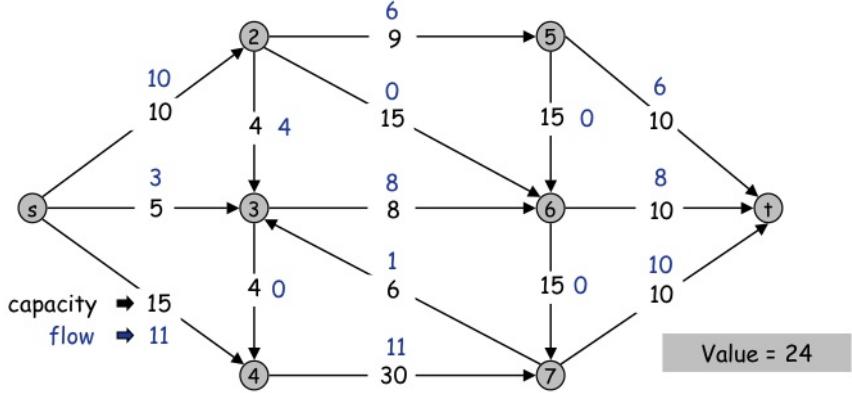


Figure 13: The same network with a flow such that $F(f) = 24$

For any given flow f , the total flow from source to target, also called the value of the flow, and noted $F(f)$, is defined as $\sum_{s \rightarrow e} f(e)$. By the conservation rule, it must be equal to $\sum_{e \rightarrow t} f(e)$.

$$F(f) = \sum_{s \rightarrow e} f(e) = \sum_{e \rightarrow t} f(e). \quad (\text{IV-44})$$

The maximum flow problem is to maximize $F(f)$ over all possible flows f . Hence the mathematics required are more the ones of a variational problem, and inequalities rather than equalities play a fundamental role (see (IV-42)). We say that f is a max-flow if it realizes the maximal flow

The supremum over all flows of a given network \mathcal{N} is in fact a maximum

$$Mf(\mathcal{N}) = \max_{f \text{ flow of } \mathcal{N}} (F(f)). \quad (\text{IV-45})$$

It means this supremum is reached for at least one flow. This is because a network is finite, hence the finite set of inequalities constraining the flows defines a bounded closed set, over which any continuous function must reach its supremum. Every flow which attains this maximum, hence for which $F(f) = Mf(\mathcal{N})$ is called a max-flow.

Definition IV.7 (Cut). A cut C (or $s - t$ cut) is a partition of the vertices of G into two subsets V_1 and V_2 with $s \in V_1$ and $t \in V_2$ such that the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are both connected, where E_1 is the set of edges of G joining two vertices in V_1 and E_2 is the set of edges of G joining two vertices in V_2 . The subset of edges $E(C)$ complement of $E_1 \cup E_2$, hence joining a vertex of V_1 to a vertex of V_2 , is called the cut-set of C .

The capacity of a cut C is defined by the sum of the capacities over all edges leaving G_1 , that is $c(C) = \sum_{e: G_1 \rightarrow G_2} c(e)$, where the sum $\sum_{e \in C, G_1 \rightarrow e}$ is performed over all edges

$e \in C$ which *exit* a vertex of G_1 . The min-cut capacity of a network $mc(\mathcal{N})$ is defined as the infimum over all cuts of the network of the capacity of the cut

$$mc(\mathcal{N}) = \min_{C \text{ cut of } \mathcal{N}} c(C). \quad (\text{IV-46})$$

Again this infimum is indeed a minimum and every cut for which this minimum is reached is called a min-cut.

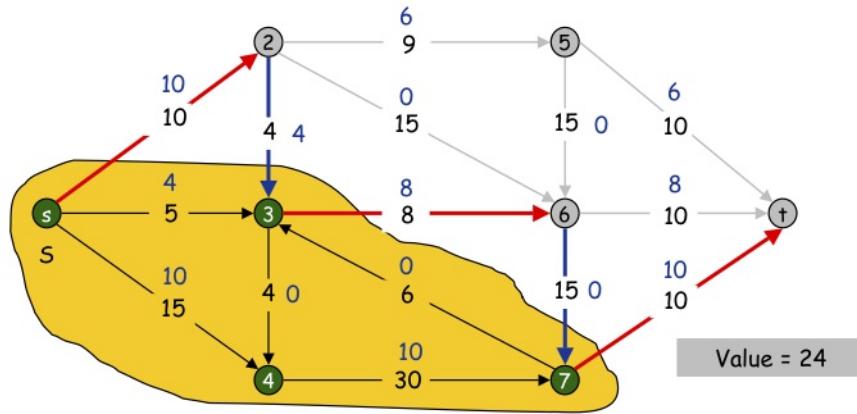


Figure 14: A cut with capacity 28.

The flow across a cut C , $F(C)$ is the sum over all edge flows exiting G_1 towards G_2 minus the sum over all edge flows entering G_1 coming from G_2 :

$$F(C) = \sum_{e:G_1 \rightarrow G_2} f(e) - \sum_{e:G_2 \rightarrow G_1} f(e). \quad (\text{IV-47})$$

Lemma IV.8. *For any cut C , $F(C) = F(f)$.*

Proof Using the conservation rule at every internal vertex of G_1 , the total flow exiting s , namely $F(f)$ must be equal to the total flow exiting G_1 minus the total flow entering G_1 , namely $F(C)$. \square

The Max-Flow-Min-Cut Theorem (Ford-Fulkerson, 1956), a cornerstone of optimization theory, relates the minimum cut problem to the maximum flow problem.

Theorem IV.9 (Max-Flow-Min-Cut Theorem). *The maximum value of an $F(f)$ over all flows f is equal to the minimum capacity $c(C)$ over all cuts C :*

$$Mf(\mathcal{N}) = mc(\mathcal{N}) \quad \forall \mathcal{N}. \quad (\text{IV-48})$$

This theorem is an example of duality in linear programming. The proof proceeds in several steps.

Lemma IV.10. Let f be a flow and C be a cut. The total flow $F(f)$ is at most the capacity of the cut:

$$F(f) \leq c(C), \quad \forall f, C. \quad (\text{IV-49})$$

Proof By Lemma IV.8, the flow $F(f)$ is the flow across C . By (IV-47), this flow is $F(f) \leq \sum_{e:G_1 \rightarrow G_2} f(e)$. By (IV-42) this is bounded by $\sum_{e:G_1 \rightarrow G_2} c(e)$, which is nothing but $c(C)$. \square

Corollary IV.11. Taking the infimum over cuts and the supremum over flows of the previous Lemma, we get

$$Mf(\mathcal{N}) \leq mc(\mathcal{N}). \quad (\text{IV-50})$$

Lemma IV.12. Let f be a flow of a network \mathcal{N} and C be a cut. If the capacity of C equals the total flow $F(f)$, then f is a max-flow and C is a min-cut, hence that network \mathcal{N} obeys the Max-Flow-Min-Cut Theorem, namely $Mf(\mathcal{N}) = mc(\mathcal{N})$.

Proof Using the conservation rule at every internal vertex of G_1 , the total flow exiting s , namely $F(f)$ must be equal to the total flow exiting G_1 minus the total flow entering G_1 , which is the flow $F(C)$ across C . Hence using the previous Lemma

$$mc(\mathcal{N}) \leq F(f) = F(C) \leq Mf(\mathcal{N}). \quad (\text{IV-51})$$

Combining with (IV-50) achieves the proof. \square

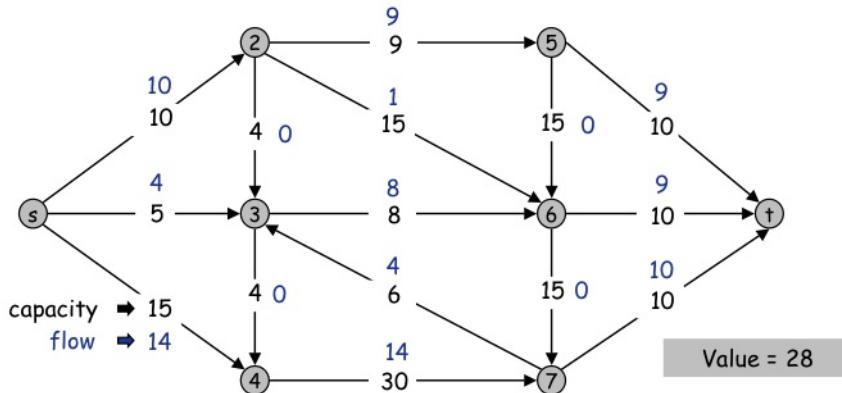


Figure 15: A flow f with total flow $F(f) = 28$.

Let us now describe a naive attempt to find a max-flow in a given network. It will fail but in a way which will be instructive.

An $s - t$ path γ in the network is an oriented path (respecting the orientations of the network) which goes from s to t . Its capacity $c(\gamma)$ is defined as the minimum of the capacity of its edges. Since no edge has zero capacity, we have $c(\gamma) > 0$ for any $s - t$ path. We could start from the trivial zero flow $f(e) = 0$, pick a path and push a maximal

flow $c(\gamma)$ through all the edges of γ . Then we could search for a second path γ' and try to push an additional flow through it, and repeat until we are stuck. The problem with this strategy is that we could easily get stuck (i.e no $s - t$ path can be augmented) before having reached the maximal flow.

In the example below, after pushing a flow of 10 through the first $s - t$ path, none of the three $s - t$ paths of the graph can be augmented, (although the true maximal flow for this network is in fact 14, see Figure 18).

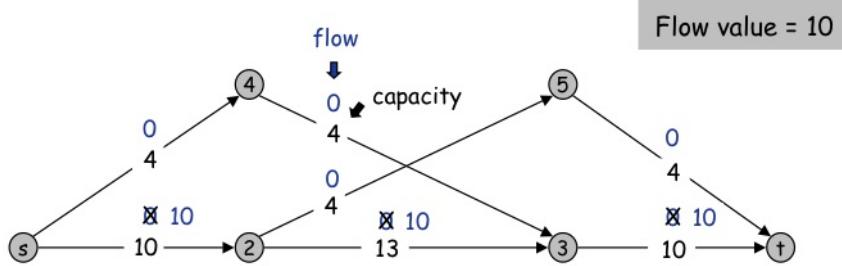


Figure 16: A flow stuck at a non-maximal value.

Therefore we should design a better strategy to reach a max-flow, which allows to backtrack, i.e. sometimes *decrease* the former flow of some edge (although this may seem counterintuitive) to unlock a better flow.

To complete the proof of the theorem, we need the notions of residual graph and of augmenting paths of a flow. The residual network $\mathcal{R}(f)$ of a flow is the network obtained by keeping all edges with 0 flow, reverting all edges with maximal flow $f(e) = c(e)$ flows into "backward edges", and doubling the edges e with $0 < f(e) < c(e)$ into two edges: a residual forward edge with capacity $c(e) - f(e)$, and a residual "backward" edge with capacity $f(e)$. The backward edges allows to decrease the flow f . Beware that therefore, in spite of its name, the *residual* network has typically *more* edges than the initial network.

An augmenting path for a flow f is then defined as an $s - t$ path in the residual graph (see Figure 17). It is then easy to see that if an augmenting path γ (which has therefore a strictly positive capacity $c(\gamma) > 0$) exists in the residual graph, the initial flow is not a max-flow since the total flow can be augmented by pushing an additional flow $c(\gamma)$ across all edges of the augmenting path. This does not exceed the capacity of the initial network.

Lemma IV.13. *If the residual graph of a flow f has no augmenting path, then there exists a cut of the initial network whose capacity equals the total flow $F(f)$.*

Proof Consider a flow of $\mathcal{N} = (G, c, s, t)$ with no augmenting path in its residual graph. Define the subgraph G_1 made of the set V_1 of vertices of G which can be reached from s in the residual graph of f , plus all edges of the network between vertices of V_1 . It contains s but not t since there is no augmenting path in the residual graph. We then

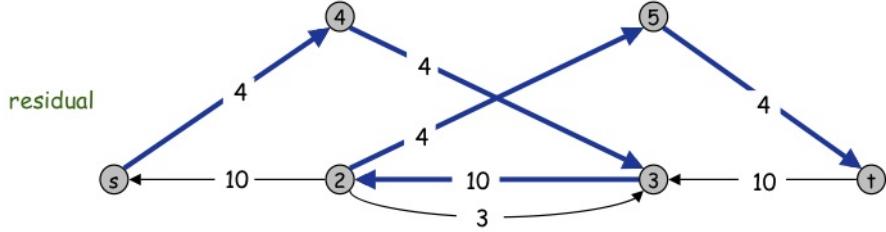


Figure 17: The residual graph for the previous flow has an augmenting path.

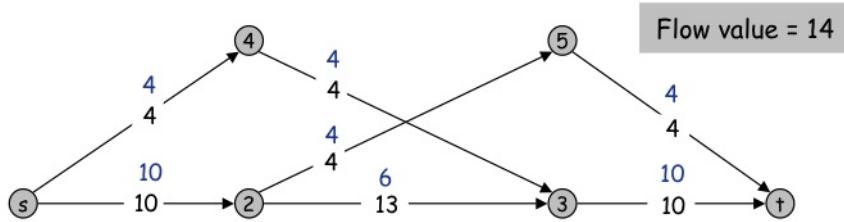


Figure 18: A better flow using the previous augmenting path in the residual graph.

define the subgraph G_2 made of the complement set of vertices $V_2 = V(G) - V_1$ (which contains t), and of all edges of the network between vertices of V_2 .

Let us check that the set E of all edges of G joining a vertex of V_1 to a vertex of V_2 is a cut. We only have to check that G_1 is a connected component of $G - \{e \in E\}$, which is obvious since by definition each vertex of V_1 is joined to a source by a path visiting only vertices of V_1 .

Now remark that all edges e leaving G_1 towards G_2 must have $f(e) = c(e)$, and all edges e entering G_1 from G_2 must have $f(e) = 0$, otherwise there would be some exiting edge from V_1 to V_2 in the residual graph, hence from the source we could reach strictly more vertices than those of V_1 in the residual graph, which would contradict the definition of V_1 . Hence

$$F(f) = F(C) = \sum_{e:G_1 \rightarrow G_2} f(e) - \sum_{e:G_2 \rightarrow G_1} f(e) = \sum_{e:G_1 \rightarrow G_2} c(e) = c(C). \quad (\text{IV-52})$$

We can now complete the proof of the theorem. Consider a network \mathcal{N} and one of its max-flow f . There cannot be any augmenting path in its residual graph, as remarked above. Therefore by Lemma IV.13 there is a cut whose capacity equals its total flow, and by Lemma IV.12, the network obeys the theorem. \square

Example of Application: Project selection

Suppose a company has several projects, forming a set P . Each project requires some equipment in a set Q . Each project $p \in P$ yields some revenue $r(p)$ and each equipment $q \in Q$ has some cost $c(q)$ to purchase. Each equipment can be shared by several projects. The problem is to determine which projects and equipments should be selected so that the earning, that is the revenues minus the costs, is maximal. If the subset of projects implemented is $S \subset P$ and the set of equipments purchased is $T \subset Q$, this earning is

$$g = \sum_{p \in S} r(p) - \sum_{q \in Q} c(q) = \sum_{p \in P} r(p) - \sum_{p \notin S} r(p) - \sum_{q \in Q} c(q). \quad (\text{IV-53})$$

Hence this maximization problem can be also formulated as a minimization problem for the quantity

$$h = \sum_{p \notin S} r(p) + \sum_{q \in Q} c(q). \quad (\text{IV-54})$$

This is a min-cut problem for the graph made of a source connected to each project p by a link of capacity $r(p)$, a sink connected to each equipment q by a link of capacity $c(q)$, and edges of infinite capacity between each project and the equipments it requires. The $s - t$ cut-set represents respectively the projects not selected and the equipments selected. By the max-flow min-cut theorem, one can solve the minimization problem for h as the search for the maximum flow of this graph.

Menger's theorem relates the size of minimal edge or vertex cuts in a graph that disconnect two vertices u and v to the number of disjoint paths from u to v .

Theorem IV.14 (Menger's Theorem for Edges). *Let G be a finite connected graph and u, v two distinct vertices. The minimum number of edges whose removal disconnects u from v is equal to the maximum number of edge-disjoint paths from u to v .*

There is also a vertex version of Menger's theorem:

Theorem IV.15 (Menger's Theorem for Vertices). *Let G be a finite connected graph and u, v two distinct nonadjacent vertices. The size of the minimum vertex-cut for u and v (the minimum number of vertices whose removal disconnects u and v) is equal to the maximum number of pairwise vertex-disjoint paths from u to v .*

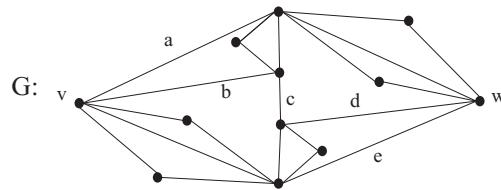


Figure 19: A graph with $v - w$ minimum vertex-cut 3. Find the three disjoint paths!

Extended to subgraphs, Menger's theorem means that a maximal subgraph disconnected by no less than a k -edge cut is identical to a maximal subgraph with a minimum number k of edge-independent paths between any pairs of vertices u and v in the subgraph.

IV.4 Shortest Distance, Traveling Salesman

Suppose we are given a weighted graph, where each edge e carry a positive number ℓ_e . It could represent for instance a distance associated to that edge, or any kind of cost to use that edge (gas, toll on a highway, and so on).

A very general problem is to find the shortest or most economical path, that is to find the minimum distance or traveling cost between two vertices a and b on such a weighted graph. Hence we should minimize the functional $d_{G,\{\ell\}}(a,b) \sum_{e \in C:a \rightarrow b} \ell_e$ over the paths starting at a and ending at b . This functional is indeed a distance: it is positive, symmetric and satisfies the triangular inequality, hence it induces a metric structure on the set of vertices of G . When all edge lengths are equal to 1, this distance is simply called the graph distance between vertices.

We do not need to consider graphs with tadpoles, nor with multi-edges (since we can replace them by the shortest one) so G is always a simple graph in this kind of problems.

In many problems of real life the graph itself representing the problem and its weights can change with time but slowly; the problem is then called quasi *static*. This is for instance the case for eg the road system of a country in which new roads are added or old roads deleted only rarely. It is easy to preprocess in advance a lot of information on a fixed graph. One can then retrieve very fast the smallest distance between two locations, as everybody can experience using mapping programs available on the web (mappy, gmaps etc).

Exercise IV.16. Let us study in some detail Dijkstra's algorithm [10], which finds the shortest path from a root vertex v_0 to any other vertex v_f in a weighted graph G with positive edge lengths ℓ_e . The algorithm is the following:

- Assign to every vertex v an initial tentative distance $d_0(v) = 0$ if $v = v_0$ and $d_0(v) = \infty$ if $v \neq v_0$.
- Consider v_0 as the initial *current vertex* and define the initial set of *unvisited* vertices $V_0 = \{v, v \neq v_0\}$.
- At step i consider all unvisited neighbors w (in V_i) of the current vertex v_i and calculate their new tentative distances, by adding their real distance $\ell(w, v_i)$ to the tentative distance $d_i(v)$ of v_i to the root. If this new tentative distance is less than the previously recorded tentative distance of w , then overwrite that distance. Even though a neighbor has been examined, it is not marked as “visited” at this time, and it remains in the unvisited set.
- When we are done considering all of the neighbors of the current vertex, mark the current vertex as visited and remove it from the unvisited set. A visited vertex will never be checked again.

- If the destination vertex v_f has been marked visited, then stop. The algorithm has finished.
- Otherwise select as unvisited vertex one that is marked with the smallest tentative distance, and set it as the new “current vertex” then go back to step 3.

Understand and explain why this algorithm indeed computes the shortest distance between v_0 and any v_f .

An other famous and more complicated problem of this kind is the so-called travelling salesman problem (TSP). This is a problem in combinatorial optimization, operations research and theoretical computer science.

Given a list of cities and their pairwise distances, the traveling salesman problem is to find the shortest possible circuit that visits each city exactly once and returns to the origin city. The problem consists therefore to minimize the functional $\sum_{e \in H} \ell_e$ over all *Hamiltonian* paths H on the graph.

This problem is one of the most intensively studied problems in optimization. It is used as a benchmark for many optimization methods. Even though the problem is computationally difficult, a large number of heuristics and exact methods are known, so that some instances with tens of thousands of cities can be solved. This problem has many applications even in its purest formulation, such as planning, logistics, and the manufacture of microchips. Slightly modified, it appears as a sub-problem in many areas, such as DNA sequencing. In these applications, the concept city represents, for example, customers, soldering points, or DNA fragments, and the concept distance represents traveling times or cost, or a similarity measure between DNA fragments. In many applications, additional constraints make the problem considerably harder.

We do not discuss further this important class of problems. But let us mention an important theorem giving a characterization of Hamiltonian graphs.

Given a graph G with n vertices, the closure $C_l(G)$ is uniquely built from G by repeatedly adding a new edge $e = (uv)$ connecting a non-adjacent pair of vertices u and v with $d_G(v) + d_G(u) \geq n$ until no more pairs with this property can be found.

Theorem IV.17 (Bondy-Chvátal). *A graph is Hamiltonian if and only if its closure is Hamiltonian.*

In particular all graphs whose closure is complete are Hamiltonian, which is the content of earlier theorems by Dirac and Ore.

Ant Colonies and DNA at the rescue

In order to solve complex problems in graph theory, sometimes analytic solutions are not available or practical. Weird ideas may then come from mother nature.

For instance an ant colony is made of many ants which randomly forage for foods. They leave on their trail a pheromone. Their fellow ants have a tendency to follow paths marked with a lot of pheromone. The pheromone marker however fades with time, and sometimes an ant walks away even of a well-marked trail. When an ant has found food it returns with it to the ant colony.

In this way ant colonies can be quite effective in finding and retrieving food. Inspired by their behavior, computer scientists may study complicated graphs by launching artificial “ants” which walk randomly on the graph keeping some markers and follow preferably (but not always) the best marked trails.

Another non standard idea to solve TSP like problems is to ask some analogical physical system to automatically perform a search for an approximate minimum of the desired functional. For instance if we want to connect cities in an optimal way, we may build a scaled physical map of these cities with particular proteins positioned at the cities sites and throw on this map some molecules such as DNA that tend to attach to the markers and otherwise to minimize their lengths, acting like springs. After waiting for a while, one could take a picture of the system, looking at what the molecules did and perhaps find a better solution than by running for a long time a computer program.

V Multivariate Polynomials

V.1 Sokal Polynomial

In the previous section we saw that electric circuits or networks with flows do not simply reduce to their underlying graph. They were represented essentially as *weighted graphs*. A weighted graph corresponds to a graph plus a variable for each edge of the graph, which can represent conductance, capacity, length, travel cost and so on. We now generalize the Tutte polynomial to weighted graphs by defining *multivariate graph polynomials*.

Multivariate graph polynomials can be defined like the Tutte polynomial through a global “sum over subsets” formula. They are also the unique ones to satisfy certain deletion-contraction rules with some specified terminal values. They bridge the gap between the ordinary Tutte polynomial and the weighted Matrix-Tree theorem of the previous section. They occur in many physics problems, such as statistical models on graphs or the parametric representation of Feynman amplitudes in quantum field theory.

The simplest of such multivariate graph polynomials $Z_G(q, \{y\})$, also often noted $Z_G(q, y)$ for simplicity, has a different variable y_e for each edge e , plus another variable q to count vertices. It is defined as a sum over subsets of edges:

Definition V.1 (Sum over subsets).

$$Z_G(q, y) = \sum_{A \subset E} q^{k(A)} \prod_{e \in A} y_e, \quad (\text{V-55})$$

where we recall that $k(A)$ is the number of connected components in the subgraph (V, A) .

It obeys also a completely general linear deletion-contraction relation:

Definition V.2 (Deletion-Contraction). For any edge e (not necessarily regular)

$$Z_G(q, \{y\}) = y_e Z_{G/e}(q, \{y - \{y_e\}\}) + Z_{G-e}(q, \{y - \{y_e\}\}). \quad (\text{V-56})$$

This relation together with the evaluation on terminal forms also defines $Z_G(q, y)$, since the result is again independent of the order of suppression of edges. The terminal forms are graphs without edges, and with v vertices; for such graphs $Z_G(q, y) = q^v$.

In [5] this multivariate polynomial is discussed in detail. Here are some examples.

Example V.3 (Trees). For any tree $T = (V, E)$ we have $Z_T(q, y) = q \prod_{e \in E} (q + y_e)$. This follows immediately from V-55, and expansion of the product $\prod_{e \in E} (q + y_e)$.

Example V.4 (Cycles). Let $G = (V, E)$ be a cycle with n lines C_n . Then $Z_{C_n}(q, y) = \prod_{e \in E} (q + y_e) + (q - 1) \prod_{e \in E} y_e$. This follows again quite easily from V-55.

To understand the relation between this multivariate Sokal polynomial and the ordinary Tutte polynomial with two variables we multiply Z_G by q^{-V} , we set $y_e = y - 1$ and $q = (x - 1)(y - 1)$ and get

$$[q^{-V} Z_G(q, y)]|_{y_e=y-1, q=(x-1)(y-1)} = (x - 1)^{k(E)-|V|} T_G(x, y). \quad (\text{V-57})$$

V.2 Spanning Polynomials

We consider now

$$q^{-k(G)} Z_G(q, y). \quad (\text{V-58})$$

Taking the limit $q \rightarrow 0$ that is retaining only the constant term in q we obtain a sum over maximally spanning subgraphs A , that is subgraphs with $k(A) = k(G)$:

$$S_G(y) = \sum_{A \text{ maximally spanning } E} \prod_{e \in A} y_e. \quad (\text{V-59})$$

If we now retain only the lowest degree of homogeneity in y we obtain a sum over maximally spanning graphs with lowest number of edges, ie maximally spanning acyclic graphs or *spanning forests* of G .

$$F_G(y) = \sum_{\mathcal{F} \text{ spanning forest of } G} \prod_{e \in \mathcal{F}} y_e. \quad (\text{V-60})$$

This polynomial in the case of a connected graph is nothing but the weighted tree polynomial of IV-31. It satisfies the factorization properties III-9-III-10 on disjoint unions and on vertex-unions.

Let us call an edge *semi-regular* if it is either regular, or a bridge. Tadpole-free graphs have only semi-regular edges. The polynomial F satisfies the deletion contraction-recursion

$$F_G(y) = F_{G-e}(y) + y_e F_{G/e}(y) \quad (\text{V-61})$$

for any semi-regular edge e , together with the terminal form evaluation

$$F_G(y) = 1 \quad (\text{V-62})$$

on graphs made solely of tadpoles.

Similarly we can generalize the terms appearing in the right hand side of IV-38. We define, for a graph G and a spanning subset R of vertices of G , that is a subset of vertices containing *at least one vertex per connected component of G* , a generalization of F_G called the R -forest polynomial:

$$F_G^R(y) = \sum_{\mathcal{F} \text{ R-forest of } G} \prod_{e \in \mathcal{F}} y_e. \quad (\text{V-63})$$

where we recall that an R -forest is a maximal forest of G containing exactly one vertex of R per connected component of the forest.

In particular we have for a connected graph G and two distinguished vertices a and b

$$F_G^{ab}(y) = \sum_{\mathcal{F} \text{ ab-forest of } G} \prod_{e \in \mathcal{F}} y_e = F_{G/ab}(y). \quad (\text{V-64})$$

V.3 The Potts Model

In this section we outline briefly a statistical mechanics application: the Sokal polynomial computes the partition function of the q -states Potts model on a general graph.

Let q be a positive integer; then the q -state Potts-model partition function for the graph G is defined by

$$Z_G^{Potts}(q, \{y\}) = \sum_{\sigma: V \rightarrow \{1, 2, \dots, q\}} \prod_{e \in E} [1 + y_e \delta(\sigma_{a(e)}, \sigma_{b(e)})]. \quad (\text{V-65})$$

Here the sum runs over all maps $\sigma: V \rightarrow \{1, 2, \dots, q\}$, and we sometimes write σ_a as a synonym for $\sigma(a)$; the δ is the Kronecker delta $\delta(x, y) = 1$ if $x = y$, $\delta(x, y) = 0$ if $x \neq y$. In (V-65) $a(e), b(e) \in V$ are the two endpoints of the edge e (in arbitrary order).

We usually consider y_e to be a real or complex variable. In statistical physics, this formula arises as follows: in the Potts model, an atom (or spin) at the site $a \in V$ can exist in any one of q different states. The energy of a configuration is the sum, over all edges $e \in E$, of 0 if the spins at the two endpoints of that edge are unequal and $-J_e$ if they are equal. The Boltzmann weight of a configuration is then $e^{-\beta H}$, where H is the energy of the configuration and $\beta \geq 0$ is the inverse temperature. The partition function is the sum, over all configurations, of their Boltzmann weights. Clearly this is just a rephrasing of V-65, with $y_e = e^{\beta J_e} - 1$. A coupling J_e (or y_e) is called ferromagnetic if $J_e \geq 0$ ($y_e \geq 0$), as it is then favored for adjacent spins to take the same value; antiferromagnetic if $-\infty \leq J_e \leq 0$ ($-1 \leq y_e \leq 0$), as it is then favored for adjacent spins to take different values; and unphysical if $y_e \notin [-1, \infty)$, as the statistical weights are then no longer positive.

It is far from obvious that $Z_G^{Potts}(q, \{y\})$, which is defined separately for each positive integer q , is in fact the restriction to $q \in \mathbb{N}$ of a *polynomial* in q . But this is in fact the case, and indeed we have:

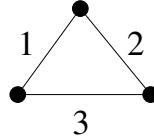
Theorem V.1 (Fortuin-Kasteleyn representation of the Potts model). The Potts-model partition function is simply the specialization of the multivariate Tutte polynomial to $q \in \mathbb{N}$.

Proof In V-65, expand out the product over $e \in E$, and let $A \subset E$ be the set of edges for which the term $y_e \delta(\sigma_{a(e)}, \sigma_{b(e)})$ is taken. Now perform the sum over configurations $\{\sigma_a\}_{a \in V}$: in each component of the subgraph (V, A) the color σ must be constant, and there are no other constraints. Therefore

$$Z_G(q, \{y\}) = \sum_{A \subset E} q^{k(A)} \prod_{e \in A} y_e, \quad (\text{V-66})$$

as was to be proved. \square

An important special case arises when $y_e = -1$ for all edges e : then Z_G^{Potts} gives weight 1 to each proper vertex coloring and weight 0 to each improper vertex coloring, and so counts the proper vertex colorings. It follows from Theorem V.1 that the number of vertex q -colorings of G is in fact the restriction to $q \in \mathbb{N}$ of the chromatic polynomial of III-23: $P_G(q) = Z_G(q, -1)$. The chromatic polynomial thus corresponds to the zero-temperature ($\beta \rightarrow +\infty$) limit of the antiferromagnetic ($J_e < 0$) Potts model. Many properties of the chromatic polynomial in fact extend to the entire antiferromagnetic region (i.e., $-1 \leq y_e \leq 0$ for all e).



Exercise V.5. Compute the partition function of the Potts model with q -states (or colors) on the triangular graph K_3 above, as a function of $\beta = 1/kT$ and of the activities $y_e = e^{-\beta J_e} - 1$.

What is the probability that the three summits all have different colors?

Check that if $J_e > 0$ and $T \rightarrow 0$ (or $\beta \rightarrow +\infty$), this probability tends to 1.

What is the probability that the three summits all have the same color?

Check that if $J_e < 0$ and $T \rightarrow 0$ (or $\beta \rightarrow +\infty$), this probability tends to 1.

VI Moving on a Graph

VI.1 Laplacian and Random Walks

For any graph G , the Laplacian is a positive symmetric bilinear form. Indeed denoting $\langle \cdot, \cdot \rangle$ the usual scalar product on $\mathbb{R}^{|V|}$:

$$\langle f, L_G f \rangle = \sum_{e \in E} [f(a(e)) - f(b(e))]^2. \quad (\text{VI-67})$$

However the Laplacian is not positive definite, because it vanishes on functions constant on the connected components of G . As we already saw, its rank $r(G)$ is always smaller than its dimension $|V|$.

The Laplacian itself is therefore *never invertible*. But we have seen in the previous section that problems such as finding the equivalent conductance of a connected electric

circuit can be solved by inverting not exactly the Laplacian matrix on a connected graph but a closely related matrix, obtained by erasing the row and column corresponding to a fixed vertex.

Another good idea to “almost invert” the Laplacian matrix is to first make it positive definite by adding a term proportional to the identity, $\mu\mathbf{1}$ ⁷.

Hence we can define the regularized Laplacian with mass

$$D = L_G + \mu\mathbf{1} \quad (\text{VI-68})$$

where L_G is the “Laplacian plus mass” matrix of the graph G . The covariance, also called propagator is then $C = D^{-1}$. There is a representation of the matrix elements of $C(a, b)$ as sums over all paths from a to b pondered by a certain weight. This is called the random path representation of C , or resolvent expansion of D :

Theorem VI.1 (Random Path Representation). $C = D^{-1}$ exists and is given by the convergent random walk representation on the graph G :

$$C(a, b) = \sum_{\omega: a \rightarrow b} \prod_{i \in V_G} \left[\frac{1}{d_G(i) + \mu} \right]^{n(\omega, i)} \quad (\text{VI-69})$$

where the sum over ω is the sum over the paths going from a to b of all lengths $n \in \mathbb{N}$, which start at a hence $\omega(0) = a$, then at each successive step $k \in [0, 1, \dots, N]$ jump from one vertex $i = \omega(k)$ to an adjacent vertex $j = \omega(k+1)$, the jump being performed along any of the edges $e = (i, j)$ hooked to i , and which ends at b , that is such that $\omega(n) = b$. $n(\omega, i)$ is the total number of times the path ω visits i , hence the number of times $k \in [0, 1, \dots, n]$ such that $\omega(k) = i$.

Proof We introduce $B = (D_G + \mu\mathbf{1})^{-1}$ which is a diagonal matrix $B_{ij} = B(i)\delta_{ij}$ and write

$$\begin{aligned} C &= (D_G + \mu\mathbf{1} - A)^{-1} = [B^{-1} - A]^{-1} \\ &= [B^{-1}(1 - BA)]^{-1} = (1 - BA)^{-1}B \\ &= \sum_{n=0}^{\infty} (BA)^n B. \end{aligned} \quad (\text{VI-70})$$

Hence using matrix multiplication

$$\begin{aligned} C(a, b) &= \sum_{n=0}^{\infty} \sum_{i_1, j_1, \dots, i_n, j_n} B_{ai_1} A_{i_1 j_1} B_{j_1 i_2} A_{i_2 j_2} \cdots B_{j_{n-1} i_n} A_{i_n j_n} B_{j_n b} \\ &= \sum_{n=0}^{\infty} \sum_{j_1, \dots, j_{n-1}} B(a) A_{aj_1} B(j_1) A_{j_1 j_2} \cdots B(j_{n-1}) A_{j_{n-1} b} B(b) \\ &= \sum_{\omega: a \rightarrow b} \prod_{i \in V_G} [B(i)]^{n(\omega, i)} \end{aligned} \quad (\text{VI-71})$$

where ω is a sequence of arbitrarily many choices of edges $\omega(1), \dots, \omega(n)$ in the graph such that each edge hooks j_k to j_{k+1} , the total path starting at $a = j_0$ and ending at $b = j_n$.

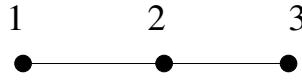
⁷Such a term is called a mass term in quantum field theory, since $\mu = m^2$, m being the particle’s rest mass.

It remains to check convergence of this expansion. This is essentially because for each visit at i we get a factor $\frac{1}{d_G(i)+\mu}$ and have only $d_G(i)$ possibilities for the next step. More precisely the full sum over all random walks ω can be bounded as

$$\begin{aligned} \sum_{\omega: a \rightarrow b} \prod_{i \in V_G} [B(i)]^{n(\omega,i)} &= \sum_{n=0}^{\infty} \frac{1}{d_G(b) + \mu} \sum_{\omega(1)} \frac{1}{d_G(a) + \mu} \cdots \sum_{\omega(n)} \frac{1}{d_G(j_{n-1}) + \mu} \\ &\leq \frac{1}{d_G(b) + \mu} \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \frac{1}{d_G(b) + \mu}, \end{aligned} \quad (\text{VI-72})$$

where $r = \max_{i \in V} \frac{d_G(i)}{d_G(i) + \mu} = \frac{\max_{i \in V} d_G(i)}{\max_{i \in V} d_G(i) + \mu} < 1$.

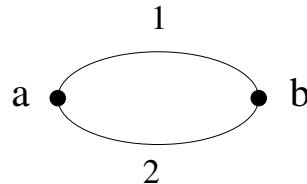
Exercise VI.1. Consider the following graph Γ , with three vertices numbered 1, 2, 3.



- 1) Write the 3 by 3 Laplacian matrix with mass, namely $L_\Gamma + \mu \mathbf{1}$, where $\mathbf{1}$ is the identity, and $\mu > 0$.
- 2) Compute the propagator $C(i, j)$ $i = 1, 2, 3$, $j = 1, 2, 3$ which is the inverse of this matrix.
- 3) Check that the element $C(2, 2)$ is the sum over all random paths from 2 to 2:

$$C(2, 2) = \sum_{\omega: 2 \rightarrow 2} \prod_i \left[\frac{1}{d(i) + \mu} \right]^{n(\omega,i)}. \quad (\text{VI-73})$$

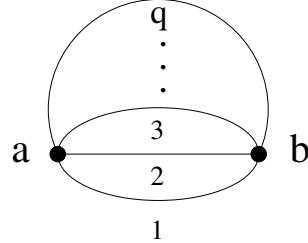
Exercise VI.2. Consider the following graph Γ , with two vertices a and b .



- 1) Write the 2 by 2 Laplacian matrix with mass, namely $L_\Gamma + \mu \mathbf{1}$, where $\mathbf{1}$ is the identity and $\mu > 0$. Invert it.
- 2) Compute $C(a, a)$ and $C(a, b)$ as sum over random paths:

$$\begin{aligned} C(a, a) &= \sum_{\omega: a \rightarrow a} \left[\frac{1}{d(a) + \mu} \right]^{n(\omega,a)} \left[\frac{1}{d(b) + \mu} \right]^{n(\omega,b)}, \\ C(a, b) &= \sum_{\omega: a \rightarrow b} \left[\frac{1}{d(a) + \mu} \right]^{n(\omega,a)} \left[\frac{1}{d(b) + \mu} \right]^{n(\omega,b)}. \end{aligned}$$

3) Do all these computations again on the graph G_q below, with q edges between a and b .



VI.2 The Regular Lattice Case

We want now to consider very large and very regular graphs, called lattices. A standard regular cubic d -dimensional lattice (of finite size) consists in a graph G which is the intersection of a lattice of mesh $\kappa > 0$ with a large box of side size L . It has vertex set $V = (\kappa\mathbb{Z})^d \cup [-L/2, L/2]^d$, and edges between “nearest neighbor” vertices, which correspond to pairs of vertices (v, v') with $|v - v'| = \kappa$. Each vertex inside the lattice has then exactly $2d$ nearest neighbors; on the frontier of the box it has less than $2d$ such neighbors. Then one can study two basic interesting limits.

In statistical mechanics one wants to let the size of the box, hence the number L to go to infinity. This is called the thermodynamic limit. In this thermodynamic limit a very interesting problem is to consider the inverse of the regularized Laplacian with mass μ defined in (VI-68) and to study its behavior in the “critical limit” in which the parameter μ goes to zero. This type of limit occurs in so-called mean-field critical statistical mechanics at a second order phase transition.

In (Euclidean) quantum field theory one is interested to add to the thermodynamic limit another limit, namely $\kappa \rightarrow 0$ which is called the ultraviolet limit, in order to describe random fields on a continuous space such as \mathbb{R}^d . Quantum fields are defined on space-time. Space has three dimensions and time is one dimensional, hence the most interesting case is $d = 4$.

Let us now consider the case of a fixed infinite cubic lattice Γ of mesh 1. Strictly speaking, this is an infinite graph, but we can still treat it with the same methods as previously. The vertices are the infinite set $V = \mathbb{Z}^d = \{(m_1, \dots, m_d)\}$, where each $m_j \in \mathbb{Z}$. There are $2d$ edges touching m , namely the links $e(m, m \pm e_j)$ where $m \in \mathbb{Z}^d$, $j \in \{1, \dots, d\}$ and e_j , $j = 1, \dots, d$ are the canonical basis vectors of \mathbb{R}^d ; hence the graph has constant degree $2d$ at each vertex.

The adjacency matrix is $A_{m,m \pm e_j} = 1$, and $A = 0$ otherwise.

Hence the Laplacian matrix Δ on \mathbb{Z}^d acts on a function f on \mathbb{Z}^d through the following formula:

$$(\Delta f)(m) = 2df(m) - \sum_{j=1}^d [f(m + e_j) + f(m - e_j)]. \quad (\text{VI-74})$$

Using Fourier series we can write the Laplacian in a different representation. We know that any square summable series of coefficients $f(m)$ correspond to a unique square

integrable complex function f on the d dimensional torus $U(1)^d$, where $U(1)$ is the unitary group of complex numbers of modulus 1. It can be seen also as a function on \mathbb{R}^d periodic in each argument:

$$\hat{f}(\theta_1, \dots, \theta_d) = \sum_{m \in \mathbb{Z}^d} f(m) e^{im \cdot \theta}. \quad (\text{VI-75})$$

We have Fourier reciprocity law:

$$f(m_1, \dots, m_d) = \frac{1}{(2\pi)^d} \int_0^{2\pi} \dots \int_0^{2\pi} d\theta_1 \dots d\theta_d \hat{f}(\theta) e^{-im \cdot \theta}. \quad (\text{VI-76})$$

Let us compute the operator Δ in the continuous representation θ . This means that we should compute the function \hat{g} whose Fourier coefficients are the $(\Delta f)(m)$ in terms of the initial function \hat{f} . We have

$$\begin{aligned} \hat{g}(\theta_1, \dots, \theta_d) &= \sum_{m \in \mathbb{Z}^d} \left[2df(m) - \sum_{j=1}^d [f(m + e_j) + f(m - e_j)] \right] e^{im \cdot \theta} \\ &= 2d\hat{f}(\theta) - \sum_{j=1}^d [e^{-i\theta_j} \hat{f}(\theta) + e^{+i\theta_j} \hat{f}(\theta)] \\ &= (2d - 2 \sum_{j=1}^d \cos \theta_j) \hat{f}(\theta) \end{aligned} \quad (\text{VI-77})$$

where in the second line we just use a relabeling $m \rightarrow m \pm e_j$. From VI-77 we conclude that Δ is a multiplication operator in the θ continuous representation, that is it is *diagonal*. This also means we can easily invert $\Delta + \mu \mathbf{1}$ in that representation:

$$[(\Delta + \mu \mathbf{1})^{-1} \hat{f}](\theta) = (2d + \mu - 2 \sum_{j=1}^d \cos \theta_j)^{-1} \hat{f}(\theta). \quad (\text{VI-78})$$

It remains to Fourier transform back to the initial \mathbb{Z}^d representation to find the kernel or matrix elements of the operator $C = [(\Delta + \mu \mathbf{1})^{-1}$ in that representation. That kernel can be computed as an scalar product. Indeed for a and b in \mathbb{Z}^d , and any operator H , the matrix elements $H(a, b)$ is $\langle \delta_a, H \delta_b \rangle$, where δ_a is the characteristic function of site a and δ_b the characteristic function of site b .

The Fourier transform of δ_a is

$$\hat{\delta}_a(\theta) = e^{ia \cdot \theta} \quad \hat{\delta}_b(\theta) = e^{ib \cdot \theta} \quad (\text{VI-79})$$

and since the L^2 scalar product is invariant under Fourier transform (Parseval identity):

$$\begin{aligned} C(a, b) = \langle \delta_a, C \delta_b \rangle &= \frac{1}{(2\pi)^d} \int_0^{2\pi} \dots \int_0^{2\pi} d\theta \overline{\hat{\delta}_a(\theta)} \hat{C}(\theta) \delta_b(\theta) \\ &= \frac{1}{(2\pi)^d} \int_0^{2\pi} \dots \int_0^{2\pi} d\theta \frac{e^{i\theta(b-a)}}{2d + \mu - 2 \sum_{j=1}^d \cos \theta_j}. \end{aligned} \quad (\text{VI-80})$$

Remark that this kernel is translation invariant: the result depends only on the difference between a and b .

On the other hand we know from the random path representation on the \mathbb{Z}^d graph that

$$C(a, b) = \sum_{\omega: a \rightarrow b} \prod_v \frac{1}{(d_v + \mu)^{n(\omega, v)}} \quad (\text{VI-81})$$

but the degree of all the sites in \mathbb{Z}^d is $2d$. Hence

$$C(a, b) = \sum_{n=0}^{\infty} (2d + \mu)^{-n-1} \sum_{\substack{\omega: a \rightarrow b \\ \ell(\omega)=n}} 1 = \sum_{n=0}^{\infty} \frac{N(a, b; n)}{(2d + \mu)^{n+1}} \quad (\text{VI-82})$$

where $N_n(a, b) = \#\{\omega : a \rightarrow b, \ell(\omega) = n\}$ is the number of random paths going from a to b in n steps on the \mathbb{Z}^d lattice, hence by jumping n times from one site to a nearest neighbor.

Equating the two expression for $C(a, b)$ we find an integral representation of the sum over random paths:

Lemma VI.3.

$$\sum_{n=0}^{\infty} \frac{N_n(a, b)}{(2d + \mu)^{n+1}} = \frac{1}{(2\pi)^d} \int d\theta \frac{e^{i\theta(b-a)}}{2d + \mu - 2 \sum_{j=1}^d \cos \theta_j}. \quad (\text{VI-83})$$

VI.3 Continuum limit: Brownian Motion

We can now reproduce all these computations on a lattice $(\kappa\mathbb{Z})^d$ with mesh κ .

Exercise VI.4. In the case of the lattice $(\kappa\mathbb{Z})^d$ with mesh κ , consider the suitably rescaled (minus) lattice Laplacian and mass term

$$C_{\kappa, \mu}(x, y) = \int_{-\pi/\kappa}^{\pi/\kappa} \cdots \int_{-\pi/\kappa}^{\pi/\kappa} dp_1 \cdots dp_d \frac{\kappa^2 e^{ip \cdot (x-y)}}{2d + \mu\kappa^2 - 2 \sum_{a=1}^d \cos \kappa p_a}. \quad (\text{VI-84})$$

Prove that the ultraviolet limit of this propagator as $\kappa \rightarrow 0$ is the distribution with integral representation

$$C_\mu(x, y) = \int_0^\infty e^{-\alpha\mu-(x-y)^2/4\alpha} \frac{d\alpha}{(4\pi\alpha)^{d/2}} \quad (\text{VI-85})$$

for $x \neq y$. This α -representation has also an interpretation in terms of Brownian motion which generalizes VI-69:

$$C_\mu(x, y) = \int_0^\infty d\alpha \exp(-\mu\alpha) P(x, y; \alpha) \quad (\text{VI-86})$$

where $P(x, y; \alpha) = (4\pi\alpha)^{-d/2} \exp(-|x - y|^2/4\alpha)$ is the heat kernel, hence the probability distribution of a continuous Brownian path going from x to y in time α .

Exercise VI.5. Find for $d > 2$ a convergent integral representation of $C_0(x, y) = \lim_{\mu \rightarrow 0} C_\mu(x, y)$, for $|x - y| \neq 0$. What is the asymptotic behavior of $C_0(x, y)$ as $|x - y| \rightarrow \infty$? What about the $d = 2$ case?



Figure 20: A Random Path on the regular square 2D Lattice

VI.4 Random Trees

In the previous sections the graph G was *fixed*. We would like now to understand what is a typical *random graph*. We start with the simpler case of *trees*, since as we saw already in the previous chapters the concept of tree is truly central and ubiquitous in graph theory. This chapter is therefore dedicated to introduce the notion of random trees.

In Chapter 1, section II.3, we defined *labeled trees*, also called Cayley trees, in which the set of vertices is made of distinct elements and in which no root is distinguished. We should not confuse these labeled trees with the unlabeled trees obtained by erasing the labels at the vertices. We call the latter *tree shapes* (see in particular Exercise II.29 and its solution in chapter ?? to understand this distinction in a simplified case).

We also introduced the natural partial ordering of a rooted tree, and defined a *plane rooted tree* as a rooted tree with a labeling of its n vertices which respect that partial ordering. Turning clockwise around that plane rooted tree provides not only the list of children, grand-children etc of any vertex but also the *ordering* in which the children

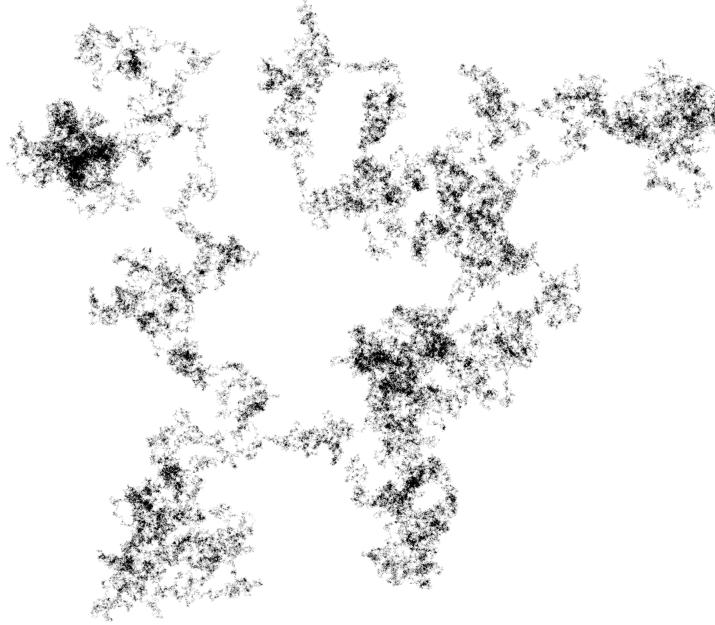


Figure 21: Brownian Motion as Very Large Random Paths

were born.

These *plane rooted trees* are the natural combinatoric structures to represent the growth in time of the tree away from its root. Apart from the natural biological applications (genealogy, population growth, epidemiology etc) plane trees are also important in computer science for search algorithms and many other applications.

Since such processes as population growth may be governed by many unknown complicated factors we are lead to the notion of random processes and of *random trees*.

To understand this concept of random trees, careful counting is crucial. Remember how combinatorics is central to probability theory. If we throw two dices the probability to get a 7 is much much higher than to get 2 or 12 (in fact six times higher!) This is revealed by combinatorics and careful counting.

VI.5 Plane Trees and Catalan Numbers

Labeled trees on n vertices can be labeled in a universal way by considering them as subsets of the complete graph K_n . We can define the vertex set of K_n as $V_n = [1, \dots, n]$, and the edge set of K_n as $E_n = \{(i, j), 1 \leq i < j \leq n\}$. The n^{n-2} labeled trees on n vertices are then the acyclic subsets of E_n which connects all vertices of V_n ; each of which has $n - 1$ edges.

The tree is called a rooted tree if one vertex has been designated the root. In any such rooted tree the edges can be given one of two canonical orientations, towards or away from the root.

To each rooted tree is also canonically associated the partial ordering on the vertices of a tree, which represents the filiation or descent if the tree is interpreted as a genealogy. In that interpretation each vertex is a descendant of the root. To each vertex v is associated a unique path to the root; and the number of edges in this unique path can be interpreted as the order of descent. Vertices at distance 1 from the root, hence related to the root by an edge are the first order descendants of the root, or its children, and so on. We say that $u \leq v$, or that v is a descendant of u if and only if the unique path from the root to v passes through u . Orders of descent compose in a natural way: any p -th order descendant of a q -th order descendant of a vertex u is a $p + q$ th-order descendant of u . There is also a natural notion of distance between vertices u and v , namely the number of edges in the unique path of the tree which connects u to v . This distance corresponds to the degree of parenthood. Brothers are at distance 2, an uncle and his nephew are at distance three and so on.

Definition VI.6. An ordered tree or plane tree is a rooted tree for which an ordering is specified for the children of each vertex. This is called a “plane tree” because an ordering of the children is equivalent to an embedding of the tree in the plane (up to homotopy through embeddings). Given an embedding of a rooted tree in the plane, if one fixes a direction of children (starting from root, then first child, second child, etc.), say counterclockwise, then an embedding gives an ordering of the children. Conversely, given an ordered tree, one can conventionally draw the root at the bottom, then the child nodes in the order of the tree can be drawn left-to-right, yielding an essentially unique planar embedding (up to embedded homotopy, i.e., moving the edges and nodes without crossing).

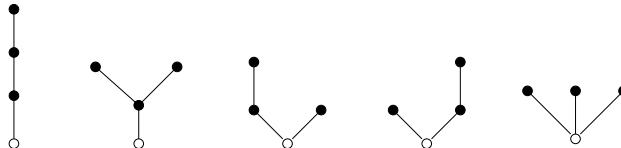


Figure 22: The five rooted plane trees at order 3

It is not obvious at first sight to count such plane trees. However it is relatively easy to find a recursion between them. Indeed consider a plane tree of order $n + 1$. It has n edges. We can order canonically the lines by turning around the tree and labeling the edges in the order we meet them (note that in this process we meet each edge twice). Cutting the edge with first label, we obtain two plane trees of smaller size, and the following recursion relation holds

$$C_{n+1} = \sum_{i=0}^n C_i C_{n-i} \quad ; \quad C_0 = 1. \quad (\text{VI-87})$$

*Catalan numbers*⁸ form the sequence of numbers given by the recursion VI-87. The n -th number in the sequence can be computed directly in terms of binomial coefficients

⁸These numbers were used before Catalan by the Mongolian mathematician Minggantu by 1730.

by

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \prod_{k=2}^n \frac{n+k}{k} \quad (\text{VI-88})$$

$$= \binom{2n}{n} - \binom{2n}{n+1}. \quad (\text{VI-89})$$

To prove this statement let us introduce the method of the generating function. It associates the formal power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad (\text{VI-90})$$

to a sequence of numbers a_n corresponding to an enumeration problem.

This is a very powerful method especially when the a_n obey a recursion rule which correspond to an algebraic or differential equation for f . It allows typically to find closed formulas for a_n or at least their asymptotic behavior as n gets large. In the particular case of the Catalan numbers, we get from the recursion relation (VI-87)

$$f(x) = 1 + xf^2(x) \quad (\text{VI-91})$$

hence

$$f(x) = \frac{1 - \sqrt{1 - 4x}}{2x}. \quad (\text{VI-92})$$

Since

$$\sqrt{1+y} = 1 - 2 \sum_{n=1}^{\infty} \binom{2n-2}{n-1} \frac{(-y)^n}{n4^n}, \quad (\text{VI-93})$$

setting $y = 4x$, substituting this power series into the expression for $c(x)$ and shifting the summation index n by 1 leads to the result.

Because the recursion relation is quite simple, the Catalan numbers arise in many enumeration problems related to trees. For instance

Exercise VI.7. Compute the first Catalan numbers 1, 2, 5, 14, 42, 132, ... Write a program to get further terms.

Exercise VI.8. Prove the asymptotic behavior

$$C_n \underset{n \rightarrow \infty}{\approx} \frac{4^n}{\sqrt{\pi n^{3/2}}}. \quad (\text{VI-94})$$

Exercise VI.9 (Full Binary Trees). A full binary tree is a rooted binary tree such that each vertex has either 2 or 0 children. Prove that C_n is the number of full binary trees with $n+1$ leaves. Draw the five full binary trees with 4 leaves.

Exercise VI.10 (Matched Parentheses). In computer languages such as Latex it is well known that every open parenthesis must close and that one cannot close more parentheses than already open. A sequence of parentheses with these properties is called

matched. Prove that the number of n pairs of such matched parentheses is again C_n . For instance if $n = 3$, there are five such expressions:

$$\{\{\}, \{\{\}\}, \{\}\{\}, \{\}\{\}, \{\{\}\}, \quad (\text{VI-95})$$

Exercise VI.11 (Planar Matchings). Prove that C_n is the number of ways that the vertices of a convex $2n$ -gon can be paired so that the line segments joining paired vertices do not intersect. Draw the five planar matchings of the hexagon.

VI.6 Galton-Watson Trees

Galton and Watson wrote an 1874 paper entitled *On the probability of extinction of families* in which they studied the long time survival chance of family names assuming, as was taken for granted in their time that family names are patrilineal, which means passed from father to children.

Suppose the number of a man's sons to be a random variable distributed on the set $\{0, 1, 2, 3, \dots\}$. Further suppose the numbers of different men's sons to be independent random variables, all having the same distribution. Then the simplest substantial mathematical conclusion is that if the average number of a man's sons is 1 or less, then their surname will almost surely die out, and if it is more than 1, then there is more than zero probability that it will survive for any given number of generations. But if the average number of a man's sons (surviving childhood) is more than one, and if there is equal average number of girls than boys surviving childhood, then the average number of descendants of any individual is more than 2. In that case world population grows exponentially, something which cannot be sustained for ever (unless perhaps we conquer the galaxy, and then an infinite universe....)

Hence the main conclusion of the study is that indeed, in first approximation and in a stationary world, surnames tend to rarefy. We can also arrive to this conclusion through a much less rigorous and more naive reasoning: if surnames are strictly passed on to all male children by their father, none of them are created, and the total number of surnames can only decrease with time; since it seems some names indeed by (mis)chance die from time to time we should not be too surprised that the total number of names should tend to either zero (population extinction by average number of descendants being less than 2) or to one, which then never dies (if average number of descendants is *exactly* 2). One observes indeed that in modern China, which has a long history of patrilineal names, there are only about 3,100 surnames now in current use out of about 12000 two thousand year ago under the Han dynasty; and the three most common surnames in Mainland China, Li, Wang and Zhang, total more than 20% of the population hence close to 300 million people.

The more rigorous definition of a Galton-Watson random tree is based on the probabilistic notion of branching processes. Modern applications of such branching processes range from the initiation of a nuclear chain reaction, to epidemiology (the dynamics of disease outbreaks) and ecology (risks of extinction of small population of organisms). Since human Y-chromosome is transmitted only from male to male, its DNA structure also gives a kind of biological marker analog to the male-transmitted family name. It may explain why there are indeed few Y-chromosome DNA types in

mankind, just as there are few Chinese family names; it seems that only a handful of males in the deep past of humanity now have any surviving male-line descendants.

Exercise VI.12. Criticize the Galton-Watson implicit hypotheses. Are all family names really patrilinear (in XIXth century England? or now?) Are no new family names ever created? What is the analog of family name creation for Y-chromosome DNA?

Exercise VI.13 (Poisson's distribution). Suppose the probability $P(k)$ of having k sons is the Poisson distribution:

$$P(k) = \frac{\lambda^k}{k!} e^{-\lambda}. \quad (\text{VI-96})$$

What is the mean of P ? Compute its probability-generating function

$$G(z) = \sum_{k=0}^{\infty} P(k) z^k. \quad (\text{VI-97})$$

Prove the recursion relation $x_{n+1} = G(x_n)$ for the total extinction probability x_n of a process starting with a single individual at time $n = 0$. Plot it on a computer for various values of λ and conclude.

As often, the most interesting problem is the critical case: what happens if we are very very close to a stationary population, hence if the average number of a man's sons (and daughter's) is 1? How should one describe the structure and shape of genealogy trees in that case? We now turn to this type of questions.

To vary our examples with fathers and sons suppose we now study the equivalent problem with mothers and daughters.

The simplest case leading to interesting critical behavior can be obtained by introducing a further simplification. Rather than modeling the number of daughters through a Poisson distribution as in exercise , suppose that at each generation each woman can have either two daughters with probability p or zero daughters with probability $1 - p$. If it has two daughters, they are ordered: the elder one is pictured as a left branch on the tree and the younger one as a right branch.

In this way the offspring distribution after a finite number of generations must be a full binary plane tree. Each vertex of the tree represents a woman; it is an internal vertex if it has daughters and it is a leaf if it has not.

The average number of daughters being $2p$, the critical case is $p = 1/2$. In that case the population in average is neither increasing exponentially fast, nor dying exponentially fast. The work of Aldous [8] and many others then leads to a picture of how after a long time the genealogy tree looks like. It is neither thin nor bush like, but intermediate and it can be essentially mapped to what is called a brownian excursion.

This allows for instance to answer such typical questions: in that critical case and for a population of n descendants when n gets large what is the average degree of parenthood between two randomly picked people?

Before we enter this question, let us recall some elementary notions of probability theory. A key idea is conditional probability. Suppose we have an initial probability measure μ and a set S of non-zero measure $\mu(S) > 0$ according to μ . The characteristic function of S is denoted as χ_S : $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ if $x \notin S$. Then

Definition VI.14. The probability measure μ_S conditioned by S is

$$\mu_S = \frac{\chi_S}{\mu(S)} \cdot \mu \quad (\text{VI-98})$$

For example for the throw of a dice, the total set is made of six possible elementary events (the six possible outcomes of throwing the dice). If the dice is fair, each elementary event has probability $1/6$. Suppose the set S is made of the three even results. The conditioned probability of throwing the dice under the condition that the result is even, is a new set of 3 elementary events each with probability $1/3$.

We have now to formalize the idea of repeating random experiments in time (such as throwing a dice many times). This leads to the notion of stochastic process, or random process.

Definition VI.15. A random process is a collection of random variables indexed by a continuous or discrete time.

Putting both ideas together we get the notion of a conditioned process. For instance we can throw a fair dice twice, which is a very simple process with two times, and condition the process under the condition that the sum obtained is a multiple of 3, in which case we have $2+5+4=11$ elementary events kept out of the initial 36. Under this conditioned process, the probability of having eg an even sum is therefore $5/11$.

Having recalled these elementary notions we can understand what is $GW_{\mu,n}$, the branching process for a Galton-Watson tree conditioned so that the total population size is n . It only depends on the choice of a distribution μ for the elementary branching process, and the set of complete events is then the set of all plane trees with exactly n vertices, distributed with a factor at each node which depends on μ .

For instance in the critical GW process with equal probability $1/2$ to have 0 or 2 daughters, the set of events is restricted to binary plane rooted trees with n vertices, which have all equal weights, as the probability to be a leaf (0 daughters) or to have two daughters are equal.

The main results of mathematicians such as Aldous [8] is that there is a universal limit, called the continuous random tree, for $n \rightarrow \infty$, which does not depend on the particular μ chosen, provided it is critical. This is a kind of branched analog of existence of a universal limit by the central theorem for the sum of n independent identically distributed variables; here again universal means that the limit does not depend on the atomic distribution of each event.

The proof of universality of the CRT is too technical to be fully detailed here but let's give an idea of it. A key simplifying tool is to encode a plane tree in terms of its Harris walk. This walk starts at the root and turns clockwise around the tree, always choosing at a node the first unexplored link on the left, see Figure VI.6.

The fact that the Harris walk encodes the tree is pictured in Fig VI.6.

We need to recall the central limit theorem: a random walk reaches typical distance \sqrt{n} after n steps. A random walk in \mathbb{R} conditioned to start and return at zero and to remain positive at all intermediate times is called a Brownian excursion. A random walk which returns to the origin can therefore be decomposed into alternating sequences of Brownian and reflected-Brownian (negative) excursions.

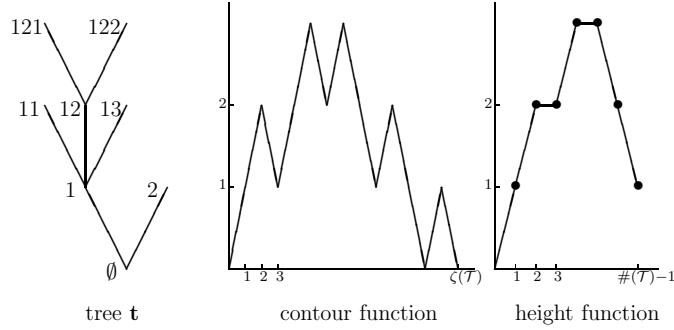


Figure 23: The contour and height function of a tree

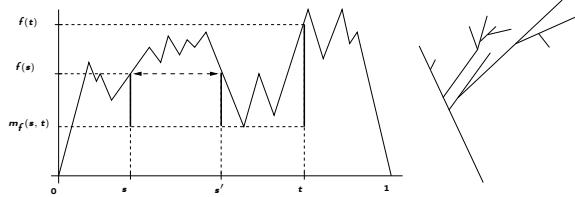


Figure 24: From Harris Walk back to the Tree

It is not too difficult to check that Brownian excursion of length n therefore reach typical height \sqrt{n} , just like ordinary unconditioned Brownian walks. Form there we obtain the fundamental result:

Theorem VI.16. *On a random critical plane tree with n vertices, the average degree of parenthood (or tree distance) between two vertices scales as $C\sqrt{n}$ when n gets large.*

For more details on this subject we refer to [8].

One could believe that it means that in a family tree such as the one of mankind with roughly 10^{10} nodes representing 10 billion people (present and past), any two randomly chosen people are cousins typically at distance $10^5 = 100000$. Again this is not true; the family relations of mankind do not form a tree. They contain many many cycles, which act as shortcuts, shortening the tree distance. Hence the typical parenthood distance is much smaller: two randomly chosen people are probably cousins to order between 100 and 1000. Mankind is truly a single family!

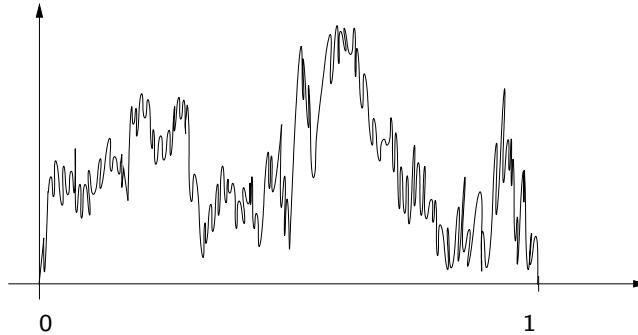


Figure 25: Brownian Excursion

VII Ribbon Graphs and Triangulations

VII.1 Planar Graphs

Just as graph theory started with the Königsberg bridge problem, an entrance door to the subject of ribbon graphs and triangulation is the simple puzzle known as the three cottage problem. Suppose there are three cottages on a plane and each needs to be connected to the gas, water and electric companies. Is there a way to make all nine connections without any of the lines crossing each other?

The answer is no. It is impossible to connect the three cottages with the three different utilities without at least one of the connections crossing another. To prove that statement, we have first to remark that the graph formed by the cottages, the companies and the connections is the complete bipartite graph $K_{3,3}$. Hence in mathematical terms, the problem asks whether $K_{3,3}$ is *planar*.

A *planar graph* is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn so that no edges cross each other. Such a drawing is called a *planar embedding* of the graph. Every graph that can be drawn on a plane can be drawn on the sphere as well, and vice versa.

Any such planar embedding of a connected graph splits the complement of the graph in the infinite plane into connected components, called the *faces* of the embedding. They form a set F , and exactly one of them is non compact and called the exterior face.

Proposition VII.1 (Euler's Formula). *There is a relation between the number of vertices $|V|$, of edges $|E|$ and of faces $|F|$ of a connected planar graph, namely*

$$|V| - |E| + |F| = 2. \quad (\text{VII-99})$$

Proof One can prove Euler's formula for instance by an induction on the number of faces. If there is only one face, that face must turn around the graph, which must be a tree, and for a tree $|V| = |E| + 1$, and $|F| = 1$ so that Euler's relation is true. If the graph

has more than one face, opening one edge bordering an inner face we obtain a graph with as many vertices but one edge less and one face less and we can use the induction hypothesis to conclude. \square

Using this proposition we can prove that

Proposition VII.2. *The complete graphs K_5 and $K_{3,3}$ are not planar.*

Proof K_5 has 5 vertices and 10 edges. Suppose it would be a planar embedding. Each edge belongs to exactly two faces, hence the number of pairs (e, f) such that e belongs to f is $p = 2|E| = 20$. Since each face is at least of length 3 (because K_5 has no multiedge), this number is also at least $3|F|$. Hence $20 \geq 3|F|$, which implies $|F| \leq 6$, since $|F|$ is an integer. On the other hand by Euler's relation we would have $|F| = 2 + 10 - 5 = 7$, hence we reach a contradiction.

Similarly for the utility graph $K_{3,3}$, which has 6 vertices and 9 edges, each face has at least four edges (in bipartite graphs, faces have even length) hence here $p = 2|E| = 18 \geq 4|F|$, hence $|F| \leq 4$. On the other hand by Euler's relation we would have $|F| = 2 + 9 - 6 = 5$, again a contradiction. \square

Recall that a *subdivision* of a graph G is a graph G' formed by subdividing its edges, that is by adding on each of them a certain number vertices of degree 2 (possibly none). An important theorem by Kuratowski states that

Theorem VII.3 (Kuratowski). *A finite graph is planar if and only if it does not contain a subgraph which is a subdivision of K_5 or of $K_{3,3}$.*

Euler's relation extends to graphs drawn on a compact Riemann surface with g handles, in which case it becomes

$$|V| - |E| + |F| = 2 - 2g. \quad (\text{VII-100})$$

However one can wonder if there is a canonical way to define the faces of a graph *without referring to any embedding*. This is indeed possible but requires to enlarge the category of graphs to new objects, called ribbon graphs, which have a specific cyclic ordering of half-edges at each vertex.

VII.2 Ribbon graphs

A ribbon graph G is a (not necessarily orientable) surface with boundary represented as the union of two sets of closed topological discs called vertices $V(G)$ and edges $E(G)$. These sets satisfy the following:

- vertices and edges intersect by disjoint line segment,
- each such line segment lies on the boundary of precisely one vertex and one edge,
- every edge contains exactly two such line segments.

Figure 26 shows an example of a ribbon graph. Note that we allow the edges to twist (giving the possibility to the surfaces associated to the ribbon graphs to be non-orientable). A priori an edge may twist more than once but the polynomials we are

going to consider only depend on the parity of the number of twists (this is indeed the relevant information to count the boundary components of a ribbon graph) so that we will only consider edges with at most one twist.

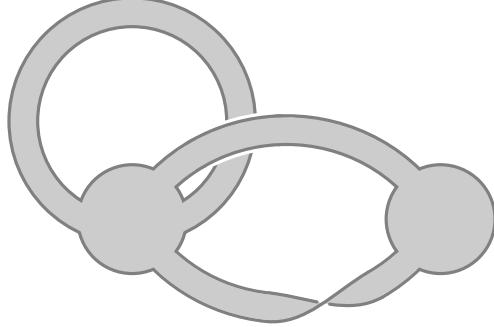


Figure 26: A ribbon graph

Two ribbon graphs are isomorphic if there is a homeomorphism from one to the other mapping vertices to vertices and edges to edges. A ribbon graph is a graph with a fixed cyclic ordering of the incident half-edges at each of its vertices.

A face of a ribbon graph is a connected component of its boundary as a surface. If we glue a disk along the boundary of each face we obtain a closed Riemann surface whose genus is also called the genus of the graph. The ribbon graph is called planar if that Riemann surface has genus zero.

Generalized ribbon graphs that can also incorporate Moebius strips and correspond to non-orientable surfaces can be defined but will not be considered here.

There is a duality on ribbon graphs which preserves the genus but exchanges faces and vertices, keeping the number of edges fixed. It simply considers the disks glued along faces as the vertices of a dual graph and changes the ends of each ribbon into borders of the dual ribbon.

Extended categories of ribbon graphs with flags can be defined. Flags can be represented as ribbons bordered by dotted lines to distinguish them from ordinary edges (see Figures 27-28). Beware that the cyclic ordering of flags and half-edges at each vertex is very important and must be respected under isomorphisms. The genus of an extended graph is defined as the genus of the graph obtained by removing the flags and closing the corresponding segments on their vertices. The number of broken faces is the number of faces which do contain at least one flag.

We define for any ribbon graph

- $V(G)$ as the number of vertices;
- $E(G)$, the number of edges,
- $k(G)$, the number of connected components,
- $r(G) = V(G) - k(G)$, the rank of G ,
- $n(G) = E(G) - r(G)$, the nullity of G ,

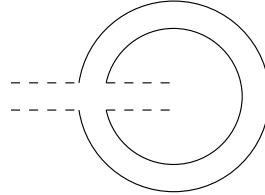


Figure 27: A planar ribbon graph with $V = E = 1$, $bc = 2$ and two flags.

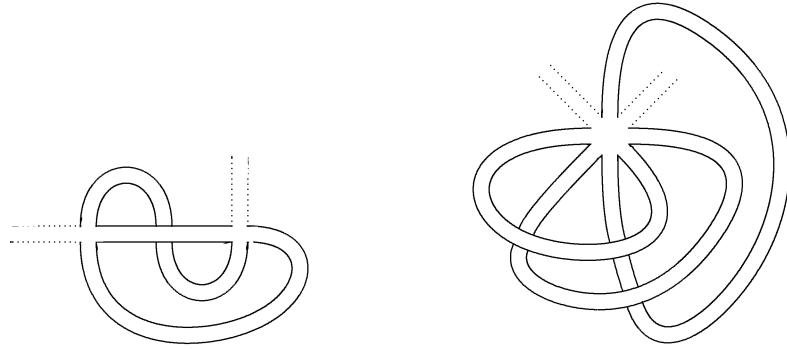


Figure 28: A non-planar ribbon graph without flags, with $V = 2$, $E = 3$, $bc = 1$, $g = 1$, $f = 2$, and its dual graph with $V = 1$, $E = 3$, $bc = 2$, $g = 1$, $f = 2$.

- $bc(G) = F(G)$, the number of components of the boundary of G^9 ,
- $g(G) = k - (V - E + bc)/2$, the genus of the graph,
- $f(G)$, the number of flags of the graph.

A graph with a single vertex hence with $V = 1$ is called a *rosette*.

A subgraph H of a ribbon graph G is a subset of the edges of G .

The Bollobás-Riordan polynomial, which is a generalization of the Tutte polynomial, is a polynomial that is used to incorporate new topological information specific to ribbon graphs, such as the genus and the number of “broken” or “external” faces. It is a polynomial invariant of the ribbon graph.

Definition VII.4 (Bollobás-Riordan). The Bollobás-Riordan polynomial is defined by:

$$R_G = R_G(x, y, z) = \sum_{H \subset G} (x - 1)^{r(G) - r(H)} y^{n(H)} z^{k(H) - bc(H) + n(H)}. \quad (\text{VII-101})$$

The relation to the Tutte polynomial for the underlying graph \tilde{G} is $R_G(x - 1, y - 1, 1) = T_{\tilde{G}}(x, y)$. Remark also that if G is planar we have $R_G(x - 1, y - 1, z) = T_{\tilde{G}}(x, y)$.

⁹This is the number of *faces* of G when G is connected.

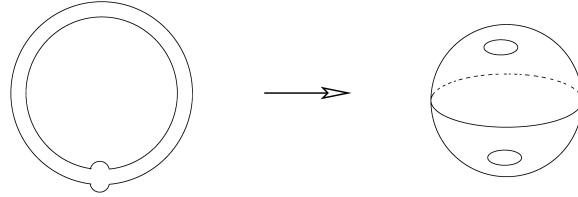


Figure 29: Contraction of the single tadpole G_1 .

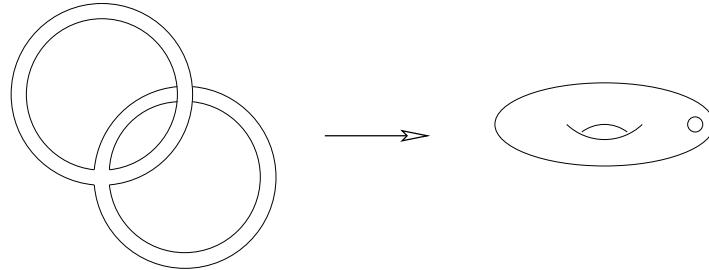


Figure 30: Contraction of the two self loops non-planar G_2 .

When H is a spanning graph of G , we have $k(H) - k(G) = r(G) - r(H)$. So we can rewrite the R polynomial as:

$$R_G = (x - 1)^{-k(G)} \sum_{H \subset G} M(H), \quad (\text{VII-102})$$

where

$$M(H) = (x - 1)^{k(H)} y^{n(H)} z^{k(H) - bc(H) + n(H)} \quad (\text{VII-103})$$

so that $M(H)$ depends only on H but not on G .

The deletion and contraction of edges in a ribbon graph are defined quite naturally: the deletion removes the edge and closes the two scars at its end; the contraction of a semi-regular edge creates a new disk out of the two disks at both ends of the ribbon with a new boundary which is the union of the boundaries of the two disks and of the ribbon (see Figure 32). An interesting property is that deletion and contraction of edges are exchanged in the dual graph.

The deletion of a tadpole is standard. However the natural contraction of a tadpole creates a surface with a new border. Iterating, we may get surfaces of arbitrary genus with an arbitrary number of disks removed, a category also called disk-punctured surfaces. The ribbons can now join any puncture to any other. For instance the contraction of the tadpole on the graph G_1 of Figure 29 leads to a cylinder ie to a single vertex which is a sphere with two disks removed. The contraction of the two tadpoles in graph G_2 of Figure 30 corresponds to the cylinder with a ribbon gluing the two ends, hence to a torus.

Deletion and contraction defined in this extended category of graphs can be iterated until the graph has no longer any edge, ie is a collection of disk-punctured Riemann

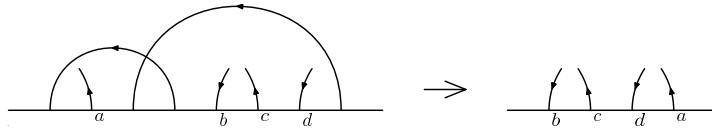


Figure 31: When deleting the two edges of a nice pair crossing on some contracted vertex, one also needs to interchange the half-edges encompassed by the first edges with those encompassed by the second one. Beware that the horizontal line in this picture is a part of the rosette cycle.

surfaces. These punctured Riemann surfaces are used in theoretical physics. But let us remain in the category of ordinary ribbon graphs with disk-like vertices. The contraction/deletion of semi-regular edges leads to rosettes as terminal forms. To treat them we introduce the notion of *double contraction on nice crossings*:

Definition VII.5. A nice crossing pair of edges in a rosette is a pair of crossing edges e_1 and e_2 which are adjacent on the cycle of the rosette. Adjacency means that one end of e_1 is consecutive with an end of e_2 (see Figure 31).

Exercise VII.6. Prove that any rosette \mathcal{R} of genus $g > 0$ contains at least one nice crossing.

The double contraction of such a nice crossing pair consists in deleting e_1 and e_2 and interchanging the half-edges encompassed by e_1 with the ones encompassed by e_2 , see Figure 31. This *double contraction* decreases the genus by one and the number of edges by 2.

In the next section iterating this double contraction until we reach planarity allows us to compute a new kind of U^* Symanzik polynomial by remaining in the category of ordinary ribbon graphs.

Theorem VII.1 (Bollobás-Riordan polynomial, contraction/deletion). Let G be any ribbon graph, then

$$R_G = R_{G/e} + R_{G-e} \quad (\text{VII-104})$$

for every ribbon graph G and any regular edge e of G and

$$R_G = xR_{G/e} \quad (\text{VII-105})$$

for every bridge of G .

Therefore the R polynomial satisfy contraction-deletion relations as the Tutte polynomial. However to complete its definition we also need to define the R polynomial for single vertex graphs, namely the rosettes, which can be read off from (VII-103). For such a rosette \mathcal{R} , $k(\mathcal{R}) = V(\mathcal{R}) = k(H) = V(H) = 1$, so that the R polynomial does not depend on x and

$$R_{\mathcal{R}}(y, z) = \sum_{H \subset \mathcal{R}} y^{E(H)} z^{2g(H)}. \quad (\text{VII-106})$$

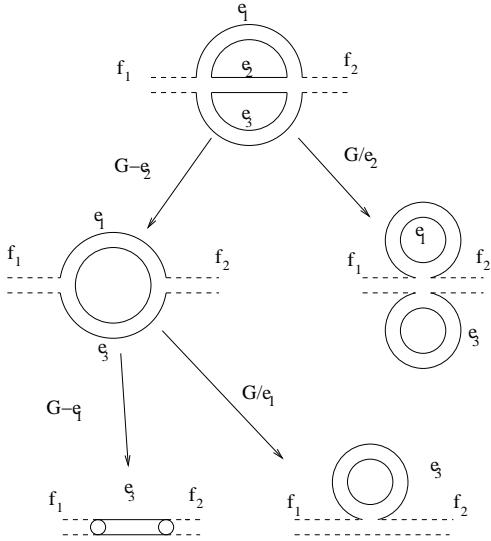


Figure 32: The contraction-deletion for a ribbon graph.

For $z = 1$ we recover $R_R(y - 1, 1) = y^{E(\mathcal{R})}$.

Like in the case of Tutte polynomial, we can generalize the Bollobás-Riordan polynomial to a multivariate case. As before, we associate to each edge e a variable y_e .

Definition VII.7. The multivariate Bollobás-Riordan polynomial of a ribbon graph analog of the multivariate polynomial (V-56) is:

$$Z_G(x, \{y_e\}, z) = \sum_{H \subset G} x^{k(H)} \left(\prod_{e \in H} y_e \right) z^{bc(H)}. \quad (\text{VII-107})$$

It obeys again a deletion/contraction relation similar to Theorem (VII.1) for any semi-regular edge.

VII.3 Triangulations and Schaeffer Map

Consider a triangulation of a given Riemann surface. An arbitrary number of triangles meet at a given vertex.

Exercise VII.8. Suppose all triangles are equilateral and the Riemann surface is the *flat* plane. How many triangles must meet at each vertex? Can one pave the sphere in this way? What happens when a lower or higher number triangles meet at a vertex?

The dual ribbon graph of a ribbon graph is obtained by exchanging the role of vertices and faces and drawing a ribbon edge across each initial edge. In this way a ribbon graph is obtained which has degree 3 at each vertex.

Exercise VII.9. Suppose G is a ribbon graph and e an edge. What is the dual of $G - \{e\}$? What is the dual of $G/\{e\}$?

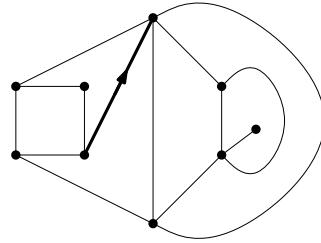
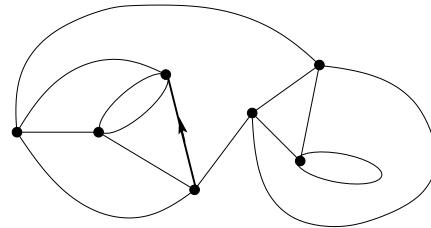


Figure 33: A rooted planar quadrangulation...



...and its dual regular graph of degree 4

A rooted planar map is a planar graph with one particular edge oriented (see Figure 33). The map is called a quadrangulation if every face, including the external one is of length 4.

To any rooted planar map is associated a bipartite quadrangulation by introducing one (white) vertex in each face, joining them to the ordinary (black) vertices bordering the face by dotted lines, and then erasing the ordinary (black) edges. Since every black edge has two ends and borders two faces, every dotted face must be of length 4.

Tutte (again!) was the first to count exactly rooted planar quadrangulations with n

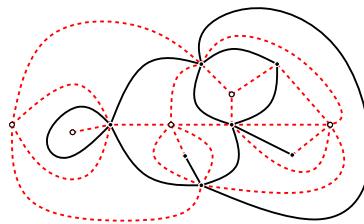


Figure 35: The quadrangulation associated to a general planar map

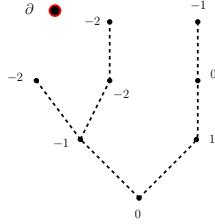


Figure 36: A well-labeled tree plus the extra “sink” vertex

faces. He found their number to be

$$Q_n = 3^n \frac{2}{n+2} C_n \quad (\text{VII-108})$$

where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is again the n -th Catalan number! This suggests a beautiful relation between rooted planar quadrangulations and rooted plane trees.

This relation was then made more precise by french mathematicians Cori, Vauquelin and Schaeffer. They found an explicit two-to-one map, now called the Schaeffer map, between pointed rooted quadrangulations and well-labeled plane trees.

Definition VII.10. A well-labeled plane tree is a rooted plane tree with an integer label at each vertex, which is zero at the root and changes by at most 1 along any edge.

Clearly the number of well-labeled plane trees with n edges is $3^n C_n$, because C_n is the number of plane trees with n edges and for any edge the well-labeling has three choices: the label at one end is equal to the label at the other end plus 0, 1 or -1. Since by Euler’s relation any planar quadrangulations with n faces has $n+2$ vertices, the number of *pointed* rooted planar quadrangulations is $(n+2)Q_n$. Hence Tutte’s equation is a consequence of the existence of the two-to-one Schaeffer map. The map itself is very beautiful and interesting because the labels have geometrical meaning: the label of a vertex is the graph distance of that vertex to the pointed vertex in the quadrangulation.

The Schaeffer map is built as follows. First add to a well-labeled tree (with dotted lines) an extra vertex called the “sink” and give it as label the minimum of all labels minus 1. Then turn around the tree, eg clockwise, and draw a line between every vertex v you meet, with label $h(v)$ and the next vertex around the tree that has label $h(v) - 1$. Sometimes that vertex is the sink, and then you return to the next unvisited vertex. Recall that turning around the tree some vertices are visited several times. Ultimately you exactly get a pointed quadrangulation. The first edge can serve as the root but there is still two ways to orient it. Hence the map associates two rooted pointed quadrangulations to each well-labeled plane tree. It is possible to study the inverse map and prove that the correspondence is indeed exactly two-to-one.

Exercise VII.11. Build the reverse Schaeffer map and complete the proof that the Schaeffer map is two-to-one.

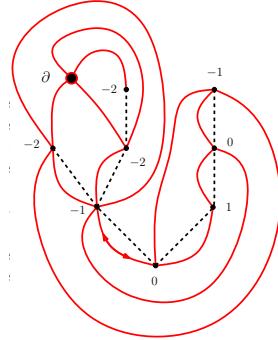


Figure 37: Reconstructing the quadrangulation from the labels

We can also work the analog of the Schaeffer map in the dual picture. Since quadrangulations are dual to graphs with degree 4 at each vertex, the dual Schaeffer map associates planar graphs with that degree and two marked half-lines to blossom trees.

We explain now why 4-valent graphs are in one to one correspondence with certain blossom trees.

Starting from a two-leg graph (a), we can apply an iterative cutting procedure, which requires turning several times around the graph. In (b), the indices indicate the order in which the edges are cut during the first turn (1, 2, 3) and second turn (4, 5, 6). Each cut edge is replaced by a black/white leaf pair (c), while the in-coming leg is replaced by a leaf and the out-coming one by a root, finally leading to a blossom tree (d). Conversely, the matching of black and white leaves of the blossom tree (d) rebuilds the edges of (a).

The Schaeffer map allows to access properties which are not easy to compute in the matrix models, such as the mean “graph distance” of a vertex to a particular root in a planar graph. Using this bijection it has been possible to define the topological limit of *large* random spherical maps and to analyze some of their statistical properties. In particular it has been proved that their Hausdorff dimension is 4, that is the number of vertices at distance d of a given vertex scales as d^4 . This gives a good example of the distance shortcuts that loops can create. If the parenthood relations of the 10^{10} people of mankind are pictured by a random planar graph rather than a random tree, which is probably much closer to the truth, we find indeed that the typical degree of parenthood between two human beings is of the order $10^{10/4} = \sqrt{10^5} \simeq 300$, a quite reasonable order of magnitude.

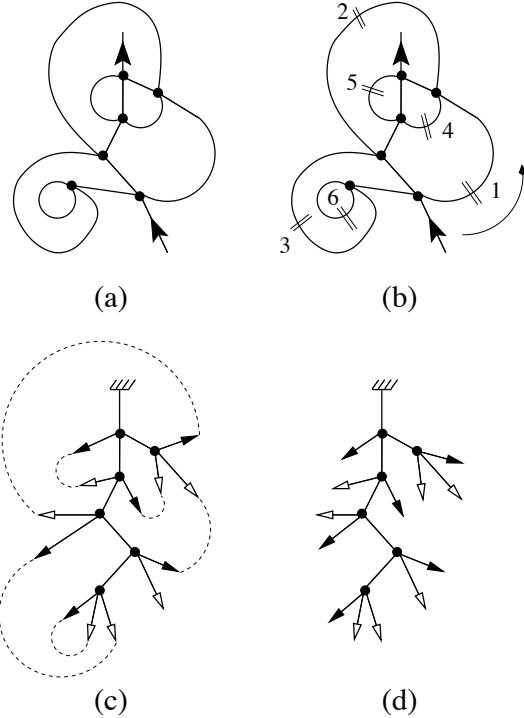


Figure 38: Schaeffer Bijection

VIII Feynman Graphs and Quantum Field Theory

VIII.1 Gaussian Measures and Wick Theorem

A finite dimensional centered normalized Gaussian measure $d\mu_C$ is defined through its covariance C . Consider a finite dimensional space \mathbb{R}^N and a symmetric positive definite N by N matrix D , called the variance. The inverse of the matrix D is also a definite positive symmetric N by N matrix $C = D^{-1}$ called the covariance associated to D . The centered normalized Gaussian measure of variance D , or of covariance C is

$$d\mu_C = (2\pi)^{-N/2} \sqrt{\det D} e^{-\frac{1}{2}\phi^T D \phi} d^N \phi, \quad (\text{VIII-109})$$

so that $\int d\mu_C = 1$.

To understand $d\mu_C$ it is better to know C than D since the moments or correlation functions of a Gaussian measure can be expressed simply as sums of monomials in C . The reader familiar with eg ordinary linear PDE's knows that the essential point is to invert the operator, hence to know the "Green's function". But quadratic forms have linear equations as their variational solutions, so both problems are linked.

Since any function can be approximated by polynomials, probability measures are

characterized by their moments, that is by the integrals they return for each polynomial of the integration variables.

The corresponding theorem which computes the moments of a Gaussian measure in terms of the covariance is fundamental in probability theory (since Gaussian measures are universal in probability through the central limit theorem) but it is also fundamental for in quantum field theory where it is known under the name of Wick's theorem. It expresses the result as the sum over all possible pairings or the variables of a product of covariances between the paired variables:

$$\int \phi_{i_1} \dots \phi_{i_n} d\mu_C = \sum_G \prod_{\ell \in G} C_{i_{b(\ell)}, i_{e(\ell)}}, \quad (\text{VIII-110})$$

where G runs over all Wick contractions or pairings of the labels $1, \dots, n$. Each pair ℓ is pictured as a line joining two labels $b(\ell)$ and $e(\ell)$ (which we call arbitrarily the “beginning” and “end” of the line).

Exercise VIII.1. Prove this formula for $n=2, 4, 6$ and then by induction for general n using integration by parts.

Exercise VIII.2. Check that the number of Wick contractions or pairings between n variables is 0 if n is odd and $n!! = (n-1)(n-3)\dots 5.3.1$ if n is even. Hence $3!!=3, 5!!=15, 7!!=105, 9!!=945, 11!!=10395$, etc....

Exercise VIII.3. Prove that Wick's theorem can also be written as

$$\int d\mu_C F(\phi) = [e^{1/2 \sum ij \frac{\partial}{\partial \phi_i} C_{ij} \frac{\partial}{\partial \phi_j}} F] |_{\phi=0}. \quad (\text{VIII-111})$$

Exercise VIII.4. What is the Gaussian measure of covariance zero? More generally show that formula VIII-111 makes perfect sense if C is non invertible, but that the corresponding measure has no density with respect to the Lebesgue measure in this case (hence formula (VIII-109) cannot be taken literally in that case).

In the following we shall restrict ourselves to models which live on a universe which is a *fixed, connected* graph $\Gamma = (V_\Gamma, E_\Gamma)$ with a finite set of vertices (also called sites) V_Γ , and a finite set of edges E_Γ between these sites. We should think to the vertices as points in a finite universe, and edges as the only possible ways to go from one point to another.

We consider Gaussian measures on vectors ϕ in $\mathbb{R}^{|V_\Gamma|}$; these vectors can be considered as functions from V_Γ into \mathbb{R} , hence they are the scalar fields of our Γ universe. The most interesting Gaussian measure would be the one in which the variance D is the Laplacian matrix of Γ , and the propagator is therefore $C = (L_\Gamma + \mu)^{-1}$.

Quantum field theory is a synthesis between quantum mechanics and *special relativity*. This synthesis is required to study eg particles in collision experiments in which some particles are created and destroyed. Around 1950 the physicist R. Feynman introduced the idea to represent the quantum probabilities for the outcome of such physical experiments as a sum over histories of particles, also called a functional integral.

Expanding this functional integral in term of the number of collisions or interactions of the particles is called *perturbative* quantum field theory. Feynman wrote the

result as a weighted sum indexed by *Feynman graphs*. In these graphs the particle collisions or interactions are represented by vertices. The particles propagation is represented by the edges joining these vertices. The outcome of any experiment is then represented by the weighted sum over infinitely many graphs of a certain type with some fixed external data. The type of graphs to be summed and the weights, also called Feynman amplitudes, are determined from the classical action of the theory by the *Feynman rules*.

Hence Feynman expressed perturbative quantum field theory as a theory of random graphs. That theory of random graphs, which in a way also generalizes the theory of random paths, has inspired many developments in other domains, hence deserves a presentation. However the full formalism of quantum field theory is clearly beyond the scope of these notes, as it involves two difficult limits, the thermodynamic or infrared limit and the continuum or ultraviolet limit.

We shall therefore present the idea of functional integral and Feynman graphs only on a simplified finite model. The universe itself will be modeled as a fixed finite graph Γ with a vertex set V . Vertices are also called sites in this context. Particles propagation on this graph will be represented by sums over random paths on Γ . Interactions will take place at the sites of Γ where several propagators meet. In what follows beware to always distinguish the fixed graph Γ which represents the simplified universe, and the Feynman graphs G which represent the particular histories that have to be summed to understand the behavior of interacting particles in that Γ -universe¹⁰.

The simplest interacting field theory is the theory of a one component scalar bosonic field ϕ with quartic interaction $\sum_{x \in V} \lambda \phi^4(x)$ at every vertex of Γ .

One is interested in the correlations or Schwinger functions of that ϕ^4 model on the universe Γ ¹¹. They are the moments of the measure:

$$d\nu_\Gamma = \frac{1}{Z} e^{-\lambda \sum_{x \in V} \phi^4(x)} d\mu_C, \quad (\text{VIII-112})$$

where

- λ is the coupling constant, assumed positive (or complex with positive real part).
- Z is a normalization factor which makes (VIII-112) a probability measure.
- C is the propagator on the graph Γ , hence the Gaussian measure $d\mu_C$ is

$$d\mu_C(\phi) = \frac{1}{Z_0} e^{-\phi(L_\Gamma + \mu \mathbf{1})\phi} D\phi, \quad (\text{VIII-113})$$

where Z_0 is again the normalization factor which makes (VIII-113) a probability measure.

This Gaussian measure is uniquely defined by its moments, or the integral of polynomials of fields. Explicitly this integral is zero for a monomial of odd degree, and for $N = 2p$ even it is equal to

¹⁰In some approaches to quantum gravity, the geometry of the universe itself is quantized, hence one should perform some summation over both Γ and G .

¹¹Interaction must be more than quadratic, and ϕ^3 is not stable, hence a quartic interaction is the simplest choice.

$$S_N(z_1, \dots, z_N) = \int \phi(z_1) \dots \phi(z_N) d\mu_C(\phi) = \sum_{\gamma} \prod_{\ell \in \gamma} C(z_{i_\ell}, z_{j_\ell}), \quad (\text{VIII-114})$$

where the sum runs over all the $2p!! = (2p-1)(2p-3)\dots$ pairings γ of the $2p$ arguments into p disjoint pairs $\ell = (i_\ell, j_\ell)$, and the external positions z_1, \dots, z_N are fixed sites (hence belong to V , without being necessarily distinct).

The Feynman graph representation expresses each pairing between variable ϕ_i and ϕ_j as an edge, usually simply called a line or propagator, going from i to j . The set of all vertices i and lines is then the associated Feynman graph. It is still quite trivial, as connected components are made of a single line and its two ends.

VIII.2 Feynman Graphs

The full interacting measure may now be defined as the multiplication of the Gaussian measure $d\mu_C(\phi)$ by the interaction factor:

$$d\nu = \frac{1}{Z} e^{-\lambda \sum_{x \in V} \phi^4(x)} d\mu_C(\phi) \quad (\text{VIII-115})$$

and the Schwinger functions are the normalized moments of this measure:

$$S_N(z_1, \dots, z_N) = \int \phi(z_1) \dots \phi(z_N) d\nu(\phi). \quad (\text{VIII-116})$$

Expanding the exponential as a power series in the coupling constant λ , one obtains a formal expansion for the Schwinger functions:

$$S_N(z_1, \dots, z_N) = \frac{1}{Z} \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int [-\lambda \sum_{x \in V} \phi^4(x)]^n \phi(z_1) \dots \phi(z_N) d\mu_C(\phi). \quad (\text{VIII-117})$$

It is now possible to perform explicitly the functional integral of the corresponding polynomial. The result gives at any order n a sum over $(4n+N-1)!!$ “Wick contractions schemes \mathcal{W} ”, i.e. ways of pairing together $4n+N$ fields into $2n+N/2$ pairs. It is important to label the n dummy variables in the n -th order product (VIII-117) as x_1, \dots, x_n in order not to confuse them. Indeed we know how dangerous it is to use the same name for the dummy variables of different sums...

At order n the result of this perturbation scheme is therefore simply the sum over all these schemes \mathcal{W} of the spatial sums over $x_1 \in V, \dots, x_n \in V$ of the integrand $\prod_{\ell \in \mathcal{W}} C(x_{i_\ell}, x_{j_\ell})$ times the factor $\frac{1}{n!} (-\lambda)^n$. These integrals then depend on the external positions z_1, \dots, z_N ¹².

To picture the underlying Feynman graph, we draw a line ℓ for each contraction of two fields. Each position x_1, \dots, x_n is then associated to a four-legged vertex and each external source z_i to a one-legged vertex, as shown in Figure 39.

¹²In the case where Γ would be replaced by \mathbb{R}^4 the spatial finite sums over x_1, \dots, x_n may diverge either as integrals over \mathbb{R}^4 which has infinite volume (infra-red problem) or because of the singularities in the propagator C at coinciding points (ultraviolet problem).

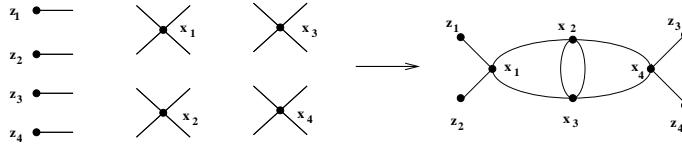


Figure 39: A possible contraction scheme with $n = N = 4$.

It is convenient to draw these Wick contractions and to regroup all contractions which give rise to the same drawing or graph. There are some subtleties about labels.

Example VIII.5. For the normalization at order 1 we have 4 fields, hence 3 Wick contractions, which all give the same graph.

Example VIII.6. For the 2 point function at order 1 we have 6 fields, and 15 Wick contractions which fall into 2 categories with weight 3 and 12.

Exercise VIII.7. For the normalization at order 2, we have 8 fields hence 105 Wick contractions. Draw all the graphs with their weights (Answer: 3 graphs, with weights 9, 24 and 72).

Exercise VIII.8. For the 4 point function at order 1 we have 8 fields, and 105 Wick contractions. Draw all the graphs with their weights (Answer: 3 graphs with weights 9, 24 and 72)

Exercise VIII.9. For the 2 point function at order 2, we have 10 fields hence 945 Wick contractions. Draw all the graphs with their weights (Answer: 7 graphs, with weights 9, 24, 72, 72, 288, 288 and 192).

Exercise VIII.10. For the 4 point function at order 2, we have 12 fields hence 10 395 Wick contractions. Draw all the graphs with their weights (Answer: 11 graphs, with weights 27, 72, 216, 1728, 432, 1728, 1152, 864, 144, 2304 and 1728).

Remark that to get smaller numbers it would be convenient to use a coupling constant $\lambda/4!$ in order to trim the permutations of the names of all fields at each vertex. This may be convenient but is not a deep observation.

Remark also that the structure of the computations is organized nicely into the connected components of the graphs. These components may or may not have external arguments. If they do not have such external arguments they are called vacuum components. It is in fact a very important observation. In the expansion for the normalized functions the vacuum graphs factor out and disappear so one gets only graphs whose connected components contain external arguments. If we further search for elementary bricks of the expansion, we can consider the truncated or connected Schwinger functions which are obtained by subtracting from the normalized functions the product over all non trivial partitions of the external arguments of normalized functions with less arguments.

In the expansion of these truncated functions only the interesting graphs with a single connected component containing all external arguments survive. For instance only 2 of the 11 graphs of exercise 4 survive in the truncated 4 point function.

The “Feynman rules” summarize how to compute the amplitude associated to a Feynman graph with its correct combinatoric factor.

Let us use the following notations for a graph G :

- $n(G)$ or simply n is the number of internal vertices of G , or the order of the graph.
- $l(G)$ or l is the number of internal lines of G , i.e. lines hooked at both ends to an internal vertex of G .
- $N(G)$ or N is the number of external vertices of G ; it corresponds to the order of the Schwinger function one is looking at. When $N = 0$ the graph is a vacuum graph, otherwise it is called an N -point graph.
- $c(G)$ or c is the number of connected components of G ,
- $L(G)$ or L is the number of independent cycles of G .

For a *regular* ϕ^4 graph, i.e. a graph which has no line hooked at both ends to external vertices, we have the relations:

$$l(G) = 2n(G) - N(G)/2, \quad (\text{VIII-118})$$

$$L(G) = l(G) - n(G) + c(G) = n(G) + 1 - N(G)/2. \quad (\text{VIII-119})$$

where in the last equality we assume connectedness of G , hence $c(G) = 1$.

A *subgraph* F of a graph G as before is a subset of internal lines of G , together with the corresponding attached vertices. Lines in the subset defining F are the internal lines of F , and their number is simply $l(F)$, as before. Similarly all the vertices of G hooked to at least one of these internal lines of F are called the internal vertices of F and considered to be in F ; their number by definition is $n(F)$. Finally one should define as external half-line of F every half-line of G which is not in F but which is hooked to a vertex of F ; it is then the number of such external half-lines which we call $N(F)$. With these conventions one has for ϕ^4 subgraphs the same relation (VIII-118) as for regular ϕ^4 graphs.

To compute the amplitude associated to a ϕ^4 graph, we have to add the contributions of the corresponding contraction schemes. This is summarized by the “Feynman rules”:

- To each line ℓ with end vertices at positions x_ℓ and y_ℓ , associate a propagator $C(x_\ell, y_\ell)$.
- To each internal vertex, associate $-\lambda$.
- Count all the contraction schemes giving this diagram. The number should be of the form $n!/S(G)$ where $S(G)$ is called the symmetry factor of the diagram.
- Multiply all these factors, divide by $n!$ and sum over the position of all internal vertices.

The formula for the bare amplitude of a graph is therefore

$$A_G(z_1, \dots, z_N) \equiv \sum_{i_1, \dots, i_n} \prod_{\ell \in G} C(x_\ell, y_\ell). \quad (\text{VIII-120})$$

The *unnormalized* Schwinger functions are therefore formally given by the sum over all graphs with the right number of external lines of the corresponding Feynman amplitudes:

$$ZS_N = \sum_{\phi^4 \text{ graphs } G \text{ with } N(G)=N} \frac{(-\lambda)^{n(G)}}{S(G)} A_G. \quad (\text{VIII-121})$$

Z itself, the normalization, is given by the sum of all vacuum amplitudes:

$$Z = \sum_{\phi^4 \text{ graphs } G \text{ with } N(G)=0} \frac{(-\lambda)^{n(G)}}{S(G)} A_G. \quad (\text{VIII-122})$$

Let us remark that since the total number of Feynman graphs is $(4n + N)!!$, taking into account Stirling's formula and the symmetry factor $1/n!$ from the exponential we expect perturbation theory at large order to behave as $K^n n!$ for some constant K . Indeed at order n the amplitude of a Feynman graph contains n independent spatial sums. It is reasonable to expect that in average it should behave as c^n for some constant c . But this means that one should expect zero radius of convergence for the series (VIII-121). This is not too surprising. Even the theory on a single-site graph Γ , given by the one-dimensional integral

$$F(g) = \int_{-\infty}^{+\infty} e^{-x^2/2 - \lambda x^4} dx \quad (\text{VIII-123})$$

is well-defined only for $\lambda \geq 0$. We cannot hope functional integrals of the same kind but corresponding to universes Γ with many more sites to behave better than this single-site theory. In mathematically precise terms, F is not analytic near $\lambda = 0$, but only Borel summable [11]. Borel summability is therefore the best we can hope for the ϕ^4 theory, and it has indeed been proved for such a quantum field theory on continuous space-times of dimensions 1, 2 and 3.

The finite Γ universe is just a toy model. Full-fledged quantum field theory consists in trying to perform the limit of such a finite-sites theory when the number of sites, like the space-time points of our universe, becomes very large. It corresponds therefore to consider the graph Γ to be for instance the regular lattice of section VI.2 with vertex set $V = (\kappa\mathbb{Z})^d \cup [-L/2, L/2]^d$. The limits to be taken are $\kappa \rightarrow 0$ and $L \rightarrow \infty$, leading to infrared ($L \rightarrow \infty$) and ultraviolet ($\kappa \rightarrow 0$) problems. With simple combinatoric verification at the level of contraction schemes we can factorize the sum over all vacuum graphs in the expansion of unnormalized Schwinger functions, hence get for the normalized functions a formula analog to (VIII-121):

$$S_N = \sum_{\substack{\phi^4 \text{ graphs } G \text{ with } N(G)=N \\ G \text{ without any vacuum subgraph}}} \frac{(-\lambda)^{n(G)}}{S(G)} A_G. \quad (\text{VIII-124})$$

Usually in (VIII-124) it is possible to pass to the thermodynamic limit, hence to perform the $L \rightarrow \infty$ limit. Indeed for a propagator C with positive mass, using the exponential decrease of the propagator C due to the mass term, each individual graph has a limit at fixed external arguments. There is no need to divide by the volume for connected functions because each connected component in (VIII-124) is tied to at least one external source. This provides the necessary breaking of translation invariance.

The ultraviolet limit $\kappa \rightarrow 0$ is harder and as soon as the dimension of space-time is 2 or more, it requires *renormalization* [11].

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References

- [1] F. Bergeron, G. Labelle and P. Leroux, Combinatoric, Species and Tree-like Structures, Cambridge University Press (1997).
- [2] W. Tutte, Graph Theory, Addison-Wesley, 1984.
- [3] Paul, Clayton R. (2001). Fundamentals of Electric Circuit Analysis. John Wiley & Sons. ISBN 0-471-37195-5.
- [4] D. Wagner, Combinatorics of Electrical Networks, Lectures given at University of Waterloo, Canada, July 2009.
- [5] A. Sokal, The multivariate Tutte polynomial (alias Potts model) for graphs and matroids, arXiv: math/0503607.
- [6] R. Diestel, Graph Theory (2005, 3rd ed.), Berlin, New York: Springer-Verlag, ISBN 978-3-540-26183-4.
- [7] Flajolet, Philippe; Sedgewick, Robert (2009), Analytic Combinatorics, Cambridge University Press, ISBN 978-0-521-89806-5
- [8] D. Aldous, The Continuum Random Tree I, II and III; The Annals of Probability, 1991, Vol 19, 1-28; in Stochastic Analysis, London Math Society Lecture Notes, Cambridge University Press 1991n eds Barlow and Bingham; The Annals of Probability, 1993, Vol 21, 248-289.
- [9] F. Harary, Graph Theory (1969), AddisonWesley, Reading, MA.
- [10] E. W. Dijkstra, A short introduction to the art of programming, 1971
<http://www.cs.utexas.edu/users/EWD/ewd03xx/EWD316.PDF>
- [11] V. Rivasseau, From Perturbative to Constructive Renormalization, Princeton University Press, 1991.