AIMS-Sénégal
Year 2015-16
MBour, February 2016

Graphs and Applications
Max-Flow Min-Cut & Potts Model

Correction of the Assignment 2

Part 1: Study of a Network.

Consider the network $\mathcal{N}(G, c, s, t)$ with edge capacities, source and target as shown in Figure 1 below.

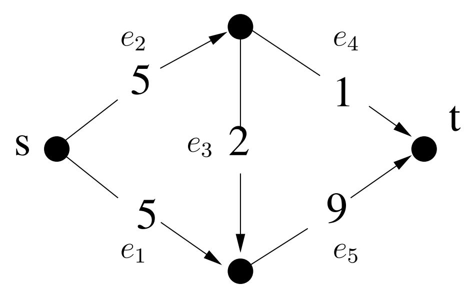

Figure 1: A network and its capacities, the source s and target t.

1) • List of the cuts of $\mathcal{N}$ and their capacities.

$$C_1 = \{e_1, e_2\}, \qquad c(C_1) = 5 + 5 = 10 \tag{1}$$
$$C_2 = \{e_2, e_3, e_5\}, \qquad c(C_2) = 5 + 9 = 14 \tag{2}$$
$$C_3 = \{e_1, e_3, e_4\}, \qquad c(C_3) = 5 + 1 + 2 = 8 \tag{3}$$
$$C_4 = \{e_4, e_5\}, \qquad c(C_4) = 1 + 9 = 10 \tag{4}$$

• The min-cut capacity for this network is C_3 and it is unique.

2) A max-flow for this network is given in Figure 2. The total flow is 8. There is no other max-flow.

3) Consider the same network $\mathcal{N}$ but with capacities a, b, c, d, and e as shown in Figure 3. Assume that $a + b = c + d$.

Denote the flow in e_3 by e', the flow in e_5 by c' and the flow in e_4 by d'.

If this network transport a total flow $a + b = c + d$, this means that $c' + d' = c + d$, this implies $c = c'$ and $d = d'$ and that the flow is maximal.

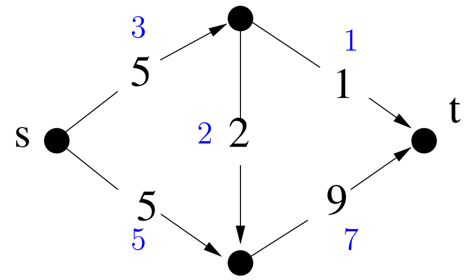

Figure 2: A max-flow of $\mathcal{N}$ (in blue) of total flow 8.

Then, for this to hold, we find a condition on b, c and e as

$$c = c' = b + e' \leq b + e \qquad \Rightarrow c - b \leq e \tag{5}$$

and consequently

$$0 \leq c - b \leq e \qquad \Leftrightarrow \qquad 0 \leq a - d \leq e \tag{6}$$

with $a - d = c - b$.

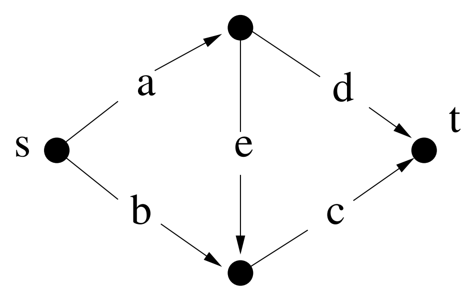

Figure 3: A network and its capacities a, b, c, d and e.

4) Consider finally the same network with capacities a, b, c, d, e but no conditions on the capacities. Using the max-flow-min-cut theorem, to find a max-flow with total flow $\min(a+b, c+d)$, we must impose

$$\min(a+b, a+c, b+e+d, d+c) = \min(a+b, c+d)\,. \tag{7}$$

We can discuss two main cases.

A) $\min(a+b, c+d) = a+b$**.** Then

$$\min(a+b, a+c, b+e+d, d+c) = \min(a+b, a+c, b+e+d) = a+b \tag{8}$$

which implies

$$\begin{cases} a+b \leq a+c \\ a+b \leq b+e+d \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} 0 \leq c-b \\ a-d \leq e \end{cases} \tag{9}$$

B) $\min(a+b, c+d) = c+d$**.** Then

$$\min(a+b, a+c, b+e+d, d+c) = \min(a+c, b+e+d, c+d) = c+d \tag{10}$$

which implies

$$\left\{\begin{array}{l} c+d \leq a+c \\ c+d \leq b+e+d \end{array}\right. \qquad \Leftrightarrow \qquad \left\{\begin{array}{l} 0 \leq a-d \\ c-b \leq e \end{array}\right. \tag{11}$$

Then if $a+b = c+d$, recover from (9) and (11),

$$0 \leq c-b \leq e\,, \qquad 0 \leq a-d \leq e\,. \tag{12}$$

which is (5) or equivalently (6).

Part 2: Potts Model.

Preliminary remarks. To evaluate the multivariate polynomial on the complete graph K_4, we can proceed by the sum over subgraphs or the contraction deletion.

The multivariate polynomial satisfies the factorization property along connected components and one-point join operations:

$$Z_{G_1 \cup G_2} = Z_{G_1} Z_{G_2}\,, \qquad Z_{G_1 \cdot G_2} = \frac{1}{q} Z_{G_1} Z_{G_2}\,, \tag{13}$$

where $(\cdot)$ is the one-point join or vertex-union of two graphs G_1 and G_2.

The multivariate polynomial of K_4 will be evaluated using contractions and deletions up to a point where the resulting graph reduces to one of the below cases:

(i) A tree T_n made with $n \geq 1$ edges labelled by $e_1, \dots, e_n$ with weights $y_1, \dots, y_n$, respectively. Consider a tree T_n, and the edge e_n of T_n. The graph $T_n - e_n$ disconnects in two connected components T_1 and T_2 which are trees themselves and $T_n - e_n = T_1 \cup T_2$. Then

$$\begin{aligned} Z_{T_n} &= Z_{T_{n_1} \cup T_{n_2}} + y_n Z_{T_{n-1}} = Z_{T_{n_1}} Z_{T_{n_2}} + y_n Z_{T_{n-1}} \\ &= q Z_{T_{n_1} \cdot T_{n_2}} + y_n Z_{T_{n-1}} = q Z_{T_{n-1}} + y_n Z_{T_{n-1}} = (q + y_n) Z_{T_{n-1}} \\ &= q \prod_{k=1}^{n} (q + y_k)\,; \end{aligned} \tag{14}$$

where $n = n_1 + n_2 + 1$.

(ii) The n-bouquet graph F_n made with one vertex and n loops incident to it has a multivariate polynomial such that

$$Z_{F_n} = (1 + y_n) Z_{F_{n-1}} = q \prod_{k=1}^{n} (1 + y_k)\,; \tag{15}$$

Both formulas (14) and (15) can be proved by recurrence on the number n of edges.

(iii) Consider now a connected graph $G_{n,m}$ made only with n loops with m bridges. It is clear that $G_{n,m}$ is made of a tree (by definition connected) with n bridges and on which are incident m loops. This means that $G_{n,m}$ can be obtained by taking a tree graph T_n and successive one-point join operations of the tree and m times the graph F_1 made with one vertex and one loop. Let us label for simplicity the n edges of T_n by $\{e_1, \dots, e_n\}$ and the m loops by $\{e_{n+1}, \dots, e_{n+m}\}$. Therefore,

$$G_{n,m} = ((((T_n \cdot F_1) \cdot F_1) \cdots) \cdot F_1) \tag{16}$$

where the number of vertex union factors is m. We can evaluate using (13)

$$\begin{aligned}
&Z_{G_{n,1}}(q, \{y_k\}_{k=1}^{n+1}) = Z_{T_n \cdot F_1} = \frac{1}{q}\, Z_{T_n}(q; \{y_k\}_{k=1}^{n})\, Z_{F_1}(q; y_{n+1}) \\
&\vdots \\
&Z_{G_{n,m}}(q, \{y_k\}_{k=1}^{n+m}) = Z_{((((T_n \cdot F_1)\cdot F_1)\cdots)\cdot F_1)} \\
&= \frac{1}{q^m} Z_{T_n}(q, \{y_k\}_{k=1}^{n}) \prod_{k=n+1}^{n+m} Z_{F_1}(q, y_k) \\
&= \frac{1}{q^m}\Big(q \prod_{k=1}^{n}(q + y_k)\Big)\Big(\prod_{k=n+1}^{n+m} q(1+y_k)\Big) \\
&= q \prod_{k=1}^{n}(q+y_k) \prod_{k=n+1}^{n+m}(1+y_k)\,,
\end{aligned} \tag{17}$$

where we have used (14) and (15) for evaluating Z_{T_n} and Z_{F_1}. Hence, up to the pre-factor q, we have shown that for a "terminal form" formed with n bridges and m loops we can take the products of indexed terminal form contributions very similar to the Tutte polynomial.

We now address the questions.

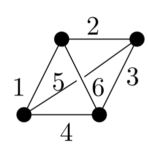

Figure 4: The complete graph $G_2 = K_4$ and its edge labels

(0) The partition function $Z_{G_2}^{\text{Potts}}(q; \{y_e\}_{e\in E})$ of the Potts model with q-states (or colors) on G_2 is given by a sum of contributions as shown in Figure 5. We then write:

$$\begin{aligned}
Z_{G_2}(q; \{y_e\}_{e\in E}) \;&=\; Z_A + y_5(Z_B + y_3 Z_C) + \\
&+\; y_2\Big(Z_D + y_3 Z_E + y_5 y_3 Z_F + y_6 Z_G + y_5 y_6 Z_H\Big) \\
&+\; y_1\Bigg(Z_I + y_6 Z_J + y_5\Big(Z_K + y_6 Z_L\Big) + y_2\Big(Z_M + y_6 Z_N + y_3 Z_O\Big)\Bigg) \\
&=\; Z_A + (e^{-\beta J_5} - 1)(Z_B + (e^{-\beta J_3} - 1) Z_C) + \\
&+\; (e^{-\beta J_2} - 1)\Big(Z_D + (e^{-\beta J_3} - 1) Z_E + (e^{-\beta J_4} - 1)(e^{-\beta J_3} - 1) Z_F \\
&+\; (e^{-\beta J_6} - 1) Z_G + (e^{-\beta J_5} - 1)(e^{-\beta J_6} - 1) Z_H\Big)
\end{aligned}$$

$$+ \quad (e^{-\beta J_1}-1)\bigg(Z_I + (e^{-\beta J_6}-1)Z_J + (e^{-\beta J_5}-1)\Big(Z_K + (e^{-\beta J_6}-1)Z_L\Big)$$

$$+ \quad (e^{-\beta J_2}-1)\Big(Z_M + (e^{-\beta J_6}-1)Z_N + (e^{-\beta J_3}-1)Z_O\Big)\bigg),$$

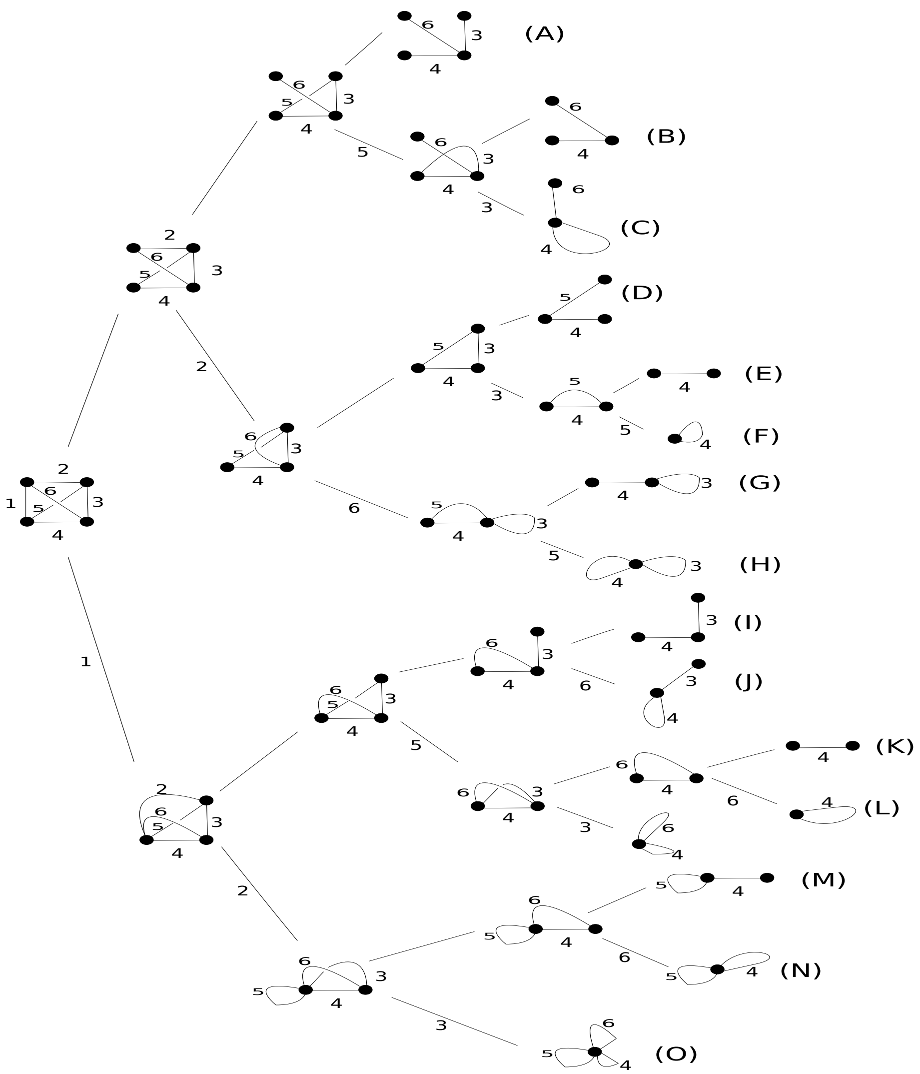

Figure 5: Contractions and deletions of K_4: we stop at the particular graphs (A) to (O) that we can compute separately.

where, at the last stage, we replace y_e by its value $e^{-\beta J_e}-1$. We can evaluate Z_A, …, Z_O using the formula (17), we have

$$Z_A = q(q+y_3)(q+y_4)(q+y_6) = q(q+e^{-\beta J_3}-1)(q+e^{-\beta J_4}-1)(q+e^{-\beta J_6}-1)$$
$$Z_B = q(q+y_4)(q+y_6) = q(q+e^{-\beta J_4}-1)(q+e^{-\beta J_6}-1)$$
$$Z_C = q(1+y_4)(q+y_6) = q(e^{-\beta J_4})(q+e^{-\beta J_6}-1)$$
$$Z_D = q(q+y_4)(q+y_5) = q(q+e^{-\beta J_4}-1)(q+e^{-\beta J_5}-1)$$
$$Z_E = q(q+y_4) = q(q+e^{-\beta J_4}-1)$$

$$
\begin{aligned}
&Z_F = q(1+y_4) = qe^{-\beta J_4} \\
&Z_G = q(1+y_3)(q+y_4) = q(e^{-\beta J_3})(q+e^{-\beta J_4}-1) \\
&Z_H = q(1+y_3)(1+y_4) = qe^{-\beta J_3}e^{-\beta J_4} \\
&Z_I = q(q+y_3)(q+y_4) = q(q+e^{-\beta J_3}-1)(q+e^{-\beta J_4}-1) \\
&Z_J = q(q+y_3)(1+y_4) = q(q+e^{-\beta J_3}-1)e^{-\beta J_4} \\
&Z_K = q(q+y_4) = q(q+e^{-\beta J_4}-1) \\
&Z_L = q(1+y_4) = qe^{-\beta J_4} \\
&Z_M = q(q+y_4)(1+y_5) = q(q+e^{-\beta J_4}-1)e^{-\beta J_5} \\
&Z_N = q(1+y_4)(1+y_5) = qe^{-\beta J_4}e^{-\beta J_5} \\
&Z_O = q(1+y_4)(1+y_5)(1+y_6) = qe^{-\beta J_4}e^{-\beta J_5}e^{-\beta J_6}\,.
\end{aligned} \tag{18}
$$

(1) At $J_e = J$ and $q < 4$:

(1.1) The probability that the 4 vertices all have different colors is 0 because this even cannot be realized:

$$P_{q<4}(\text{“the 4 vertices all have different colors”}) = 0\,. \tag{19}$$

(1.2) The probability that the 4 vertices all have the same color is

$$P_{q<4}(\text{“the 4 vertices all have the same color”}) = \frac{qe^{-6\beta J}}{Z_{G_2}}\,. \tag{20}$$

(2) At $J_e = J$ and $q \geq 4$:

(2.1) The probability $P_{\neq}$ that the 4 vertices all have different colors is

$$P_{\neq} = \frac{q(q-1)(q-2)(q-3)}{Z_{G_2}}\,. \tag{21}$$

(2.2) If $J_e > 0$, $\forall e$, and $T \to 0$, i.e. $\beta \to \infty$, we have

$$
\begin{aligned}
&\lim_{T\to\infty} Z_A = q(q-1)^3 \\
&\lim_{T\to\infty} Z_B = q(q-1)^2 \\
&\lim_{T\to\infty} Z_C = 0 \\
&\lim_{T\to\infty} Z_D = q(q-1)^2 \\
&\lim_{T\to\infty} Z_E = q(q-1) \\
&\lim_{T\to\infty} Z_F = 0 \\
&\lim_{T\to\infty} Z_G = 0 \\
&\lim_{T\to\infty} Z_H = 0 \\
&\lim_{T\to\infty} Z_I = q(q-1)^2 \\
&\lim_{T\to\infty} Z_J = 0 \\
&\lim_{T\to\infty} Z_K = q(q-1) \\
&\lim_{T\to\infty} Z_L = 0
\end{aligned}
$$

$$
\begin{aligned}
\lim_{T\to\infty} Z_M &= 0 \\
\lim_{T\to\infty} Z_N &= 0 \\
\lim_{T\to\infty} Z_O &= 0\,.
\end{aligned}
\tag{22}
$$

Therefore:

$$
\begin{aligned}
\lim_{T\to 0} P_{\neq} &= \frac{q(q-1)(q-2)(q-3)}{q\Big[(q-1)^3-(q-1)^2-(q-1)^2+(q-1)-(q-1)^2+(q-1)\Big]} \\
&= \frac{q(q-1)(q-2)(q-3)}{q(q-1)\Big[(q-1)^2-3(q-1)+2\Big]} = 1\,.
\end{aligned}
\tag{23}
$$

(2.3) The probability $P_{=}$ that the 4 vertices all have the same color is

$$
P_{=} = \frac{qe^{-6\beta J}}{Z_{G_2}}\,. \tag{24}
$$

(2.4) Assume that $J_e < 0$, $T \to 0$, so that $\beta \to \infty$, then

$$
\begin{aligned}
\lim_{T\to 0} Z_A &= qe^{-3\beta J} \\
\lim_{T\to 0} Z_B &= qe^{-2\beta J} \\
\lim_{T\to 0} Z_C &= qe^{-2\beta J} \\
\lim_{T\to 0} Z_D &= qe^{-2\beta J} \\
\lim_{T\to 0} Z_E &= qe^{-\beta J} \\
\lim_{T\to 0} Z_F &= qe^{-\beta J} \\
\lim_{T\to 0} Z_G &= qe^{-2\beta J} \\
\lim_{T\to 0} Z_H &= qe^{-2\beta J} \\
\lim_{T\to 0} Z_I &= qe^{-2\beta J} \\
\lim_{T\to 0} Z_J &= qe^{-2\beta J} \\
\lim_{T\to 0} Z_K &= qe^{-\beta J} \\
\lim_{T\to 0} Z_L &= qe^{-\beta J} \\
\lim_{T\to 0} Z_M &= qe^{-2\beta J} \\
\lim_{T\to 0} Z_N &= qe^{-2\beta J} \\
\lim_{T\to 0} Z_O &= qe^{-3\beta J}\,.
\end{aligned}
\tag{25}
$$

At the limit $T \to 0$, the term $y_1 y_2 y_3 Z_O \to e^{-3\beta J} e^{-3\beta J}$ becomes dominant in the denominator. We get:

$$
\lim_{T\to 0} P_{=} = \frac{qe^{-6\beta J}}{qe^{-3\beta J}e^{-3\beta J}} = 1\,. \tag{26}
$$

(2.5) We have $P_= + P_{\neq} \neq 1$ because there are other possible events which corresponds neither to "vertices all have the same color" or "vertices all have different colors".

The probability $1 - P_= - P_{\neq}$ corresponds to the event

"almost 3 vertices have the same color and at least two vertices have the same color".

- **Numerical application:** Evaluation of $Z_{G_2}^{\text{Potts}}(q;\{y_e\})$, $P_=$ and $P_{\neq}$ with two digits precision:

  - $\{q=3,\ \beta=1,\ J_e=J=1\}$:

$$\begin{aligned}&Z_{G_2}[q=3;\beta=1;J_e=J=1]=163.72\\&P_= = 0 \qquad (\sim 4.10^{-5})\\&P_{\neq}=0\,.\end{aligned} \tag{27}$$

  - $\{q=4,\ \beta=1,\ J_e=J=1\}$:

$$\begin{aligned}&Z_{G_2}[q=4;\beta=1;J_e=J=1]=437.01\\&P_= = 0 \qquad (\sim 2.10^{-5})\\&P_{\neq}=0.05\,.\end{aligned} \tag{28}$$

  - $\{q=4,\ \beta=10,\ J_e=J=1\}$:

$$\begin{aligned}&Z_{G_2}[q=4;\beta=10;J_e=J=1]=256.01\\&P_= = 0 \qquad (10^{-28})\\&P_{\neq}=0.09\,.\end{aligned} \tag{29}$$