AIMS-Sénégal
MBour, February 2014

Assignment: Counting Rooted Plane Trees and Melonic Graphs

The goal of this assignment is to enumerate some simple graphs.

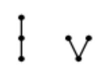
1 Rooted plane trees

Definition Rooted plane trees are defined pages 45-46 of the Lecture Notes. Please read these pages before starting the assignment...

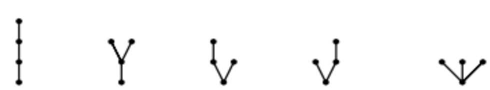
1) Draw all rooted plane trees with 4 edges.

Answer:

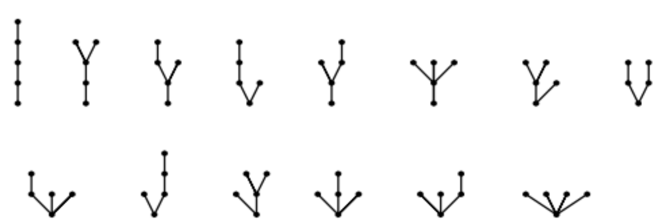
For $n = 1$, there is 1 such rooted plane tree:

for $n = 2$ there are 2 such rooted plane trees: 

For $n = 3$, there are 5 such rooted plane trees:



For $n = 4$, there are 14 such rooted plane trees:



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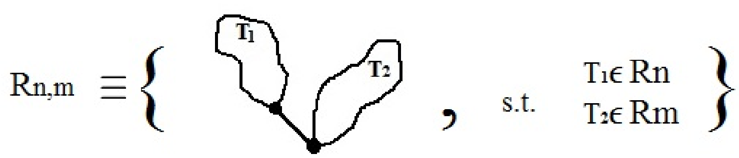
2) Prove that the number C_n of rooted plane trees with n edges obeys the following recursion:

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \quad ; \quad C_0 = 1. \tag{1}$$

Answer:
2) Define the set R_n to be the set of rooted plane trees with n vertices (i.e. $|R_n| = C_n$). We define the sets $R_{n,m}$ as the rooted plain trees created by taking a tree T_1 from R_n and a tree T_2 from R_m and joining their roots by an edge. We require that T_1 is to the left of T_2 and that the root of the new root is the root of T_2 (Using the convention that we list the edges by rotating around the graph clockwise). Heuristically:



Note that $R_{n,m}$ is not usually equal to $R_{m,n}$. We will prove that

$$R_{n+1} = \bigcup_{i=0}^{n} R_{i,n-i}, \quad \text{and that} \quad R_{i,n-i} \cap R_{j,n-j} = \emptyset \quad \forall i \neq j \tag{2}$$

We can see that the sets $\{R_{i,n-i}\}$ are disjoint by removing the additional edge. This results in two trees that belong to R_i and R_{n-i}. These trees cannot belong to R_j and R_{n-j} respectively for $j \neq i$, since they don't have j and $n-j$ edges.

To show the first part of equation 2, we take a tree $T \in R_{n+1}$. Removing the first edge we encounter on the left, will result in two connected component trees T_1 and T_2, where T_1 is to the left of T_2. If we assume the number of edges in T_1 is i then we must have that the number of edges in T_2 is $n-i$ (since the total number of edges must sum to $n+1$). From here it's clear by adding the removed edge back, that $T \in R_{i,n-i}$.
Now assume we take a tree from $\bigcup_{i=0}^{n} R_{i,n-i}$, i.e $T \in R_{i,n-i}$ for some i. From the definition of $R_{i,n-i}$ we see that T is a rooted plane tree with $i + (n-i) + 1 = n+1$ edges. Thus, $T \in R_{n+1}$. This finishes the proof of equation 2.
We can now get the desired result by comparing the cardinality of these sets.
Recall that $|R_i| = C_i$. We note that $|R_{i,n-i}| = C_i C_{n-i}$. This is true since every two trees $T_1 \in R_i$ and $T_2 \in R_{n-i}$ create a unique rooted plain tree. We then get:

$$|R_{n+1}| = C_{n+1} = |\bigcup_{i=0}^{n} R_{i,n-i}| = \sum_{i=0}^{n} |R_{i,n-i}| = \sum_{i=0}^{n} C_i C_{n-i} \tag{3}$$

Where the third equality used the fact that the union is over disjoint sets.[1] ■

[1] If you don't know why this is true look up the INCLUSION/EXCLUSION principle.

3) Compute the numbers C_n up to C_{10}. In view of question 2, explain a systematic method to draw the 42 rooted plane trees with 5 edges.

Answer :

n	1	2	3	4	5	6	7	8	9	10
C_n	1	2	5	14	42	132	429	1430	4862	16796

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4-5) Introducing

$$f(x) = \sum_{n=0}^{\infty} C_n x^n, \tag{4}$$

prove that

$$f(x) = 1 + x f^2(x), \tag{5}$$

and conclude that

$$f(x) = \frac{1 - \sqrt{1-4x}}{2x}. \tag{6}$$

Answer :

We have

$$\sum_{n=0}^{\infty} C_{n+1} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^n$$

Then

$$\frac{1}{x} \sum_{n=0}^{\infty} C_{n+1} x^{n+1} = f^2(x),$$

hence

$$\sum_{n=1}^{\infty} C_n x^n = x f^2(x)$$

$$\left(\sum_{n=0}^{\infty} C_n x^n - C_0 \right) = x f^2(x),$$

finally, we get

$$f(x) - 1 = x f^2(x) \implies x f^2(x) - f(x) + 1 = 0$$

hence, solving the quadratic equation above gives

$$f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

Note that if we choose the solution $f(x) = \frac{1+\sqrt{1-4x}}{2x}$, we will get $f(x) \to \infty$, when $x \to 0$, but we have $f(0) = C_0 = 1$, therefore we must choose the other solution $f(x) = \frac{1-\sqrt{1-4x}}{2x}$, which by a Taylor expansion of the square root, tends to 1 as $x \to 0$. Hence we conclude that

$$f(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

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6) Prove the asymptotic behavior

$$C_n \simeq_{n\to\infty} \frac{4^n}{\sqrt{\pi} n^{3/2}}.$$

Answer :
Using Stirling's formula $n! \simeq n^n e^{-n}\sqrt{2\pi n}$. Now we expand $f(x)$ using the binomial formula on

$$\sqrt{(1-4x)} = (1-4x)^{1/2}$$

Recalling that $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^n b^{n-k}$. Hence

$$(1-4x)^{1/2} = 1 - \frac{1}{1!}2x - \frac{1}{2!}4x^2 - \frac{3.1}{3!}8x^3 - \frac{5.3.1}{4!}16x^4 - \ldots$$

Using the fact that $f(x) = \frac{1-\sqrt{1-4x}}{2x}$, we get

$$f(x) = 1 + \frac{1}{2}\frac{2!}{1!}x + \frac{1}{3}\frac{4!}{2!2!}x^2 + \frac{1}{4}\frac{6!}{3!3!}x^3 = \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{2i}{i} z^i.$$

Comparing with the generating function of Catalan numbers, we get

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{(n+1)}\frac{2n!}{n!n!},$$

hence using Stirling's formula $n! \simeq n^n e^{-n}\sqrt{2\pi n}$, we get

$$C_n \simeq_{n\to\infty} \frac{4^n}{\sqrt{\pi} n^{3/2}}.$$

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2 Melonic Graphs

We now want to count slightly more complicated graphs called melonic graphs [2]. All graphs considered from now on are assumed to be *connected*.

We consider a fixed set of $d \geq 2$ different colors, labeled as $\{1, ...d\}$. A bipartite d-regular edge-colored graph (in short a d-BREC) is a graph in which

- vertices are either black and white and have equal degree d, and

- every edge joins a black and a white vertex (bipartite graph) and has a color label such that all edges meeting at a vertex have *different* colors.

Hence in a d-BREC all colors are represented exactly once at each vertex.

An *open* d-BREC (in short d-OBREC) is a graph obtained by deleting a single edge of a d-BREC graph. If the deleted edge has color i, the d-OBREC is said to be of color i (see Figure 1).

[2] These graphs occur in the study of triangulations of spaces of dimension d, hence are of interest to understand quantum gravity.

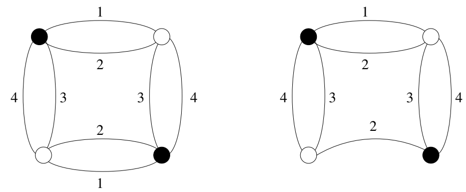

Figure 1: A 4-BREC (left) and a 4-OBREC of color 1 (right)

1) Prove that any d-BREC or d-OBREC has an even number of vertices. What is the number of edges in a d-BREC with $n = 2p$ vertices? and in a d-OBREC with $n = 2p$ vertices?

Answer: Each edge of a d-BREC or d-OBREC has a color i. The number of half-edges of color i is 2 times the number of edges and each vertex has only one of these half-edges. The number of vertices is then even since the number of half-edges is even and there is no tadpole in the graph. The number of edges E in a d-BREC with $n = 2p$ vertices is given by

$$2E = dv = 2dp. \tag{7}$$

Hence $E = dp$. The number of edges in a d-OBREC is $E = dp - 1$. ■

2) For $d = 2$, find the 2-BRECs and 2-OBRECs with $n = 2p$ vertices. What do you remark?

Answer: If $d = 2$, the number of vertices is equal to the number of edges in the 2-BRECs. Hence there is only one such graph for any given p, namely the closed chain-graph with $2p$ vertices C_{2p}. The 2-OBRECs are the linear trees with $2p$ vertices T_{2p} obtained by removing an edge in the closed chain C_{2p}. Hence there is again only one such graph for any given p. ■

From now on we therefore consider $d \geq 3$. We study now a particular class of d-OBRECs, called melonic graphs.

We call *d-melon* the unique d-BREC with two vertices. The *open d-melon* of color i is defined as the d-OBREC obtained by deleting the single edge of color i of the d-melon.

When an open d-melon of color i occurs as a strict edge-subgraph $S \subset G$ of a d-OBREC G, there is an associated contraction called *melonic contraction*. It contracts S and one of its attached edges of color i to a single vertex, resulting in a contracted graph G/S (see Figure 3).

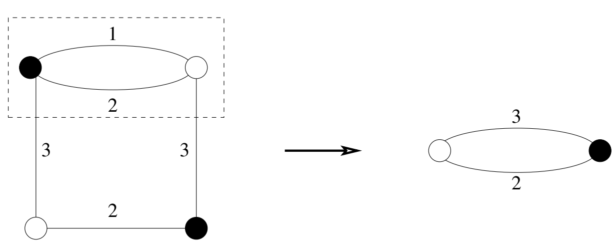

Figure 2: Example of a melonic contraction: the contraction (left) of an open 3-melon of color 3 (dashed box) in a 3-OBREC of color 1 gives a reduced 3-OBREC of color 1 (right)

3) Check that G/S does not depend on which of the two edges of color j attached to S we choose for the contraction (if there are two such edges), and is still a d-OBREC (of same color than G).

Answer: Let us remark that the contraction will not affect the structure of the graph. We will only concentrate on the color of the edges. The graph S (with a color j) can appear in two different ways in G. Else there are two edges of color j (as illustrated in Figure 2) attached to S or there is only one kind of such edge.

In the second case, the contraction will elliminate S and this edge. The graph G/S still misses only on edge with color j to be a d-BREC and is a d-OBREC with color j.

In the first case the contraction does not depend on which of the two edges of color j attached to S we choose since the two edges have the same color and after the contraction the remaining uncontracted edge will join the same two vertices. The graph G is obtained by removing an edge of color j from a d-BREC and this vertex where not attached to the vertices of S else S will miss two edges to be a d-BREC. After contraction the graph G/S will miss the same edge of color j to be a d-BREC. Finally the graph G/S has the same color as G. ■

4) A d-OBREC is called *melonic* if it reduces to an open d-melon (of the same color) through a sequence of melonic contractions. Count and draw the melonic d-OBRECs of a given fixed color (say 1) with 4 vertices. Count and draw the melonic 3-OBRECs of a given fixed color (say 1) with 6 vertices.

Answer: We have d d-OBRECs of a given fixed color (say 1) with 4 vertices and 12 3-OBRECs of a given fixed color (say 1) with 6 vertices. See Figures 3 and 4

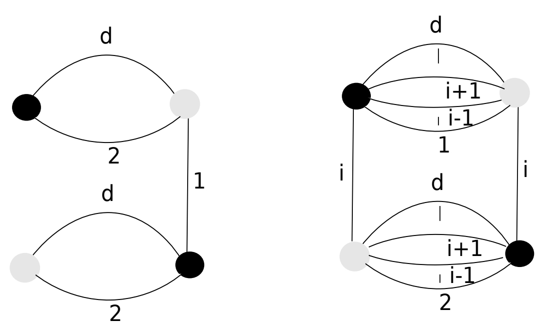

Figure 3: There are two d-OBRECs of color 1 ($i = 2, \cdots, d$)

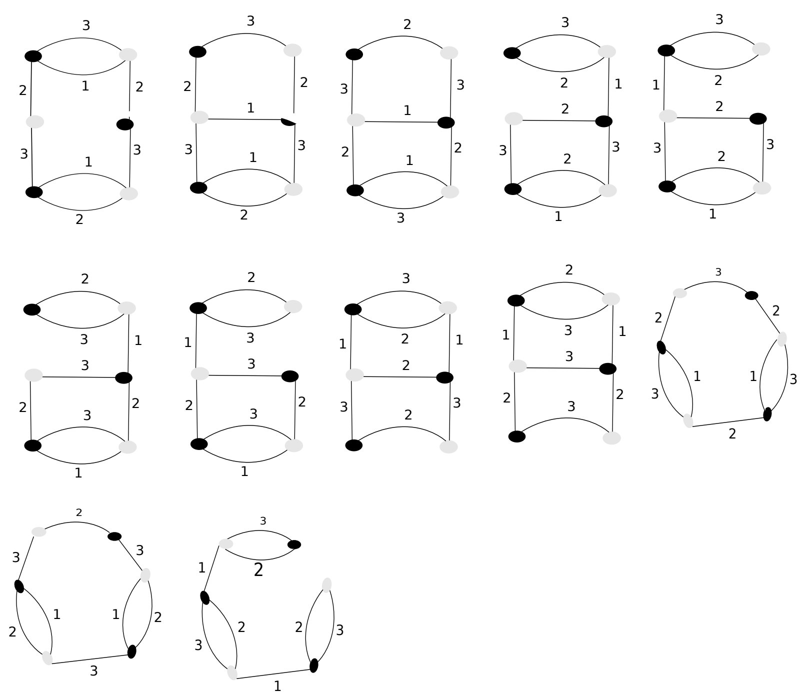

Figure 4: There are 12 3-OBRECs of color 1 and 6 vertices

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3 Facultative: Counting Melonic Graphs

The goal in this last section is to compute the number $N_d(p)$ of melonic d-OBRECs of a fixed color (say 1) with $2p$ vertices.

1) Defining the power series

$$f_d(x) = 1 + \sum_{p=1}^{\infty} N_d(p) x^p, \tag{8}$$

prove[3] that it satisfies the equation

$$f_d(x) = 1 + x[f_d(x)]^d \tag{9}$$

2) Check that $N_d(p) = \frac{(dp)!}{p![(d-1)p+1]!}$ is the solution of equation (9)

3) What is the radius of convergence of the power series (8)? (Hint: you can use Stirling's formula to approximate $k!$ at large k).

[3]Hint: you may introduce the number $M_d(p)$ of melonic d-OBRECs of a fixed color (say 1) with $2p$ vertices and *no bridge*. Defining the series $g_d(x) = \sum_{p=1}^{\infty} M_d(p) x^p$ you may prove first that $f_d(x) = \frac{1}{1-g_d(x)}$, and then that $g_d(x) = x[f_d(x)]^{d-1}$.