AIMS-Sénégal
Année 2015-16
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Graph Theory and Applications
Problem Solving: Solution

Solution of Exercise 1. • "The complexity of a graph G is the number of its spanning trees."

• The complexity of the graph G_1 is

$$\chi(G) = 3 \tag{1}$$

which is the number of its spanning trees listed as follows: $\{e_1, e_2\}$, $\{e_1, e_3\}$, $\{e_2, e_3\}$.

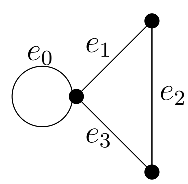
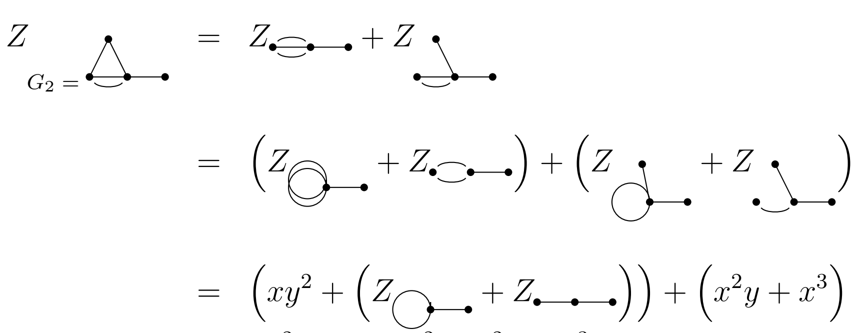


Figure 1: The graph G_1.

Solution of Exercise 2. 1) The Tutte polynomial T_{G_2} of the graph G_2 can be calculated using a contraction/deletion procedure. We have:



$$= xy^2 + xy + x^2 + x^2y + x^3\,. \tag{2}$$

2) The complexity of G_2 is given by

$$\chi(G_2) = T_{G_2}(1,1) = 1 + 1 + 1 + 2 = 5\,. \tag{3}$$

3) Two spanning trees of G_2 in red.

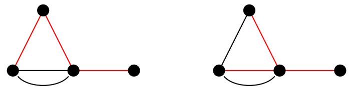


Figure 2: Two spanning trees (in red) of the graph G_2.

Solution of Exercise 3. 1) We evaluate the partition function Z_{G_3} of the Potts model on the graph G_3 by contraction/deletion:

$$
\begin{aligned}
Z_{G_3=} &= (q+y_4)\, Z = (q+y_4)\Big(Z + y_1 Z \Big)\\
&= (q+y_4)\Big((q+y_2) Z + y_1 \Big(Z + y_2 Z \Big)\Big)\\
&= (q+y_4)\Big(q(q+y_2)(q+y_3) + y_1 (q(q+y_3) + q y_2 (1+y_3))\Big)\,. \qquad (4)
\end{aligned}
$$

2) Consider $\beta = \frac{1}{kT}$, $J<0$, $y_e = e^{-\beta J} - 1$.

[2a] The probability $P_=$ that all vertices have the same color is

$$
P_= = \frac{q e^{-\beta J \times 4}}{Z_{G_3}} \qquad (5)
$$

At the 0-temperature limit, the dominant term in

$$
Z_{G_3} = (q+y)\Big(q(q+y)(q+y) + y(q(q+y) + qy(1+y))\Big) \qquad (6)
$$

is $qy^4 = qe^{-4\beta J}$ and so we have

$$
\lim_{T\to 0} P_= = \frac{qe^{-4\beta J}}{qe^{-4\beta J}} = 1\,. \qquad (7)
$$

[2b] At the infinite-temperature limit, $\beta \to 0$, then $e^{-\beta J} \to 1$, and all $y_e \to 0$. Thus, we obtain $\lim_{T\to\infty} Z_{G_3} = q^4$, such that

$$
\lim_{T\to\infty} P_= \frac{q}{q^4} = q^{-3}\,. \qquad (8)
$$

3) Consider $\beta = \frac{1}{kT}$, $J_e = J < 0$, $y_e = e^{-\beta J} - 1$. The 0-temperature limit of the probability that 3 vertices have the same color and the remaining vertex has a different color is 0. This is because, at the limit $T \to 0$, the event “all vertices have the same color” is of probability $\lim_{T\to 0} P_= = 1$ as shown in (7). This implies that this is only event which exists for this system at 0-temperature. Therefore any over event (or possibility) must be of vanishing probability.

Solution of Exercise 4 (Bonus). Consider the same graph G_3 with all $J_e = J > 0$.

At 0-temperature, $y_e = -1$, and thus

$$\lim_{T \to 0} Z_{G_3} = q(q-1)^2(q-2) \tag{9}$$

An event on the graph with non zero probability $P_{\neq}$ at 0-temperature limit is

"all vertices have a different color"

$$\begin{aligned} P_{\neq} &= \frac{q(q-1)^2(q-2)}{Z_{G_3}} \\ \lim_{T \to 0} P_{\neq} &= \frac{q(q-1)^2(q-2)}{q(q-1)^2(q-2)} = 1\,. \end{aligned} \tag{10}$$

This event is unique because it is the entire universe of events. However, there exist other events which have non zero probability at 0-temperature.