8: Trees

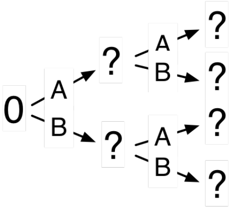
Lessons learned from unary and binary

Our unary representation of Nats was inefficient because we only had a choice of a single data constructor.

This conceptual diagram showed the relationships between different natural numbers in terms of the relationships between their representations.

$$0 \xrightarrow{S} 1 \xrightarrow{S} 2 \xrightarrow{S}$$

By using two data constructors instead of one, we could do better.



Using two data constructors allowed us to create small representations of many different natural numbers.

We'll adapt this idea to create a data structure holding many different values that allows quicker access to each of them.

Inefficiencies of lists

A list exhibits a linear structure similar to our conceptual diagram of our unary representation of Nats.

```
(cons 'A (cons 'B (cons 'C (cons 'D empty))))
```

Getting the value of the first element of a list is easy and quick.

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more code and more time.

To be clear about what we are looking for, let's return to the mathematical concept of sequences, as discussed in lecture module 05.

Our implementations of sequences in lecture module 05 (S-lists and Racket's built-in lists) resulted in implementations of the following operations:

- **extend** consumed an element e and a sequence S and produced the new sequence e, S.
- **head** consumed a non-empty sequence S and produced the element e, where $S = e, S'$.
- **tail** consumed a non-empty sequence S and produced the sequence S', where $S = e, S'$.

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For Racket lists, the implementations of these operations are `cons`, `first`, and `rest`.

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Here is an implementation for Racket lists.

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(define (list-ref i lst)
  (cond
    [(empty? lst) (error "no element of that index")]
    [(zero? i) (first lst)]
    [else (list-ref (sub1 i) (rest lst))]))
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Our goal is to create a representation of sequences that supports the four operations (extend, head, tail, index) but improves the running time for index.

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Our goal is to create a representation of sequences that supports the four operations (extend, head, tail, index) but improves the running time for index.

We'll take as a starting point our development of S-lists from lecture module 05, which was made obsolete by Racket's built-in list operations.

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(define-struct Empty ())
(define-struct Cons (fst rst))
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Data definition: an S-list is either a `(make-Empty)` or it is `(make-Cons v slist)`, where `v` is a Racket value and `slist` is an S-list.

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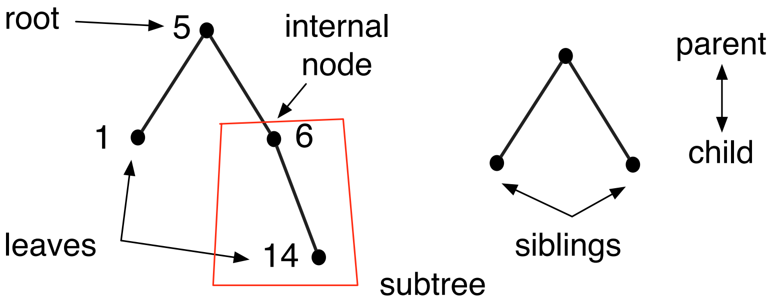
Examples:

```
(make-Node 14 (make-Empty) (make-Empty))
(make-Node 6
           (make-Empty)
           (make-Node 14 (make-Empty) (make-Empty)))
```


root
5
internal
node
1
6
leaves
14
subtree
parent
child
siblings

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Representing sequences using Braun trees

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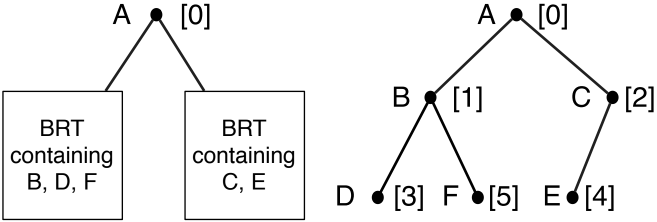
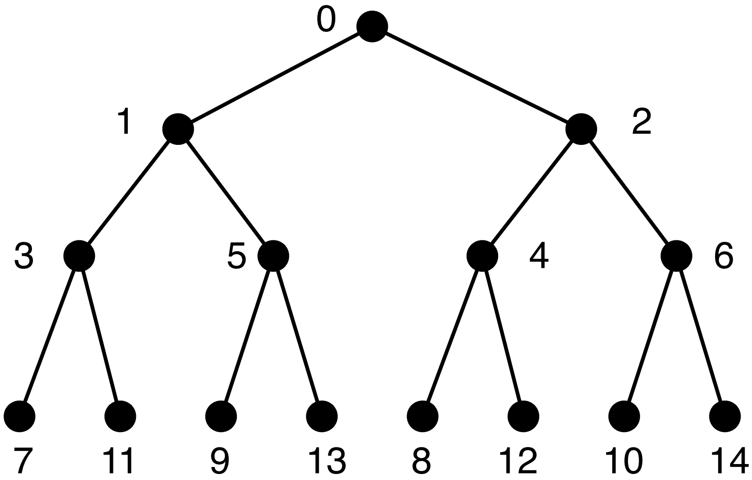
That is: the left and right subtrees will also be Braun trees.




```
(make-Node 'A
  (make-Node 'B
    (make-Node 'D (make-Empty) (make-Empty))
    (make-Node 'F (make-Empty) (make-Empty)))
  (make-Node 'C
    (make-Node 'E (make-Empty) (make-Empty))
    (make-Empty)))
```


0
1
2
3
5
4
6
7
11
9
13
8
12
10
14

The index operation in a Braun tree

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(define (brt-ref i brt)
  (cond
    [(Empty? brt) (error "index too large")]
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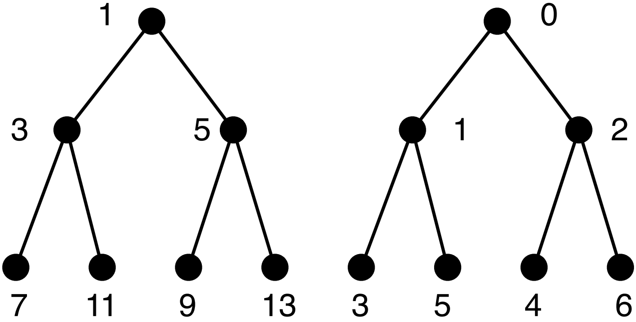
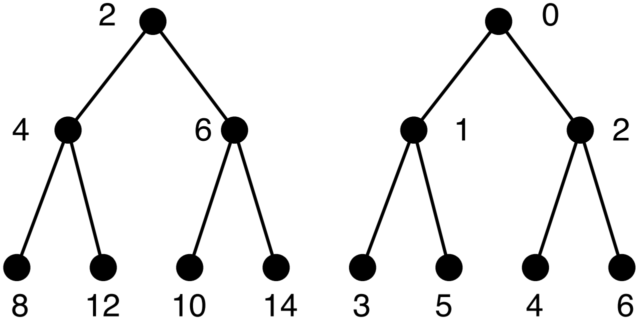
We must compute, from the index of the desired element in the whole sequence, its index in the odd-index or even-index subsequences that are stored in the left or right subtree respectively.

Here is the left subtree.



The element at index i in the whole sequence (for odd i) is found at index $(i-1)/2$ in the odd-indexed sequence (stored in the left subtree).

Here is the right subtree.



The element at index i in the whole sequence (for nonzero even i) is found at index $(i/2) - 1$ in the even-indexed sequence (stored in the right subtree).

The index operation in a Braun tree, completed

```
(define (brt-ref i brt)
  (cond
    [(Empty? brt) (error "index too large")]
    [(zero? i) (Node-val brt)]
    [(odd? i)  (brt-ref (/ (sub1 i) 2) (Node-left brt))]
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Let $T_1(i)$ be the number of applications of `brt-ref` on a Braun tree and a valid index `i`.

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$$
\begin{aligned}
T_1(0) &= 1 \\
T_1(2k+1) &= 1 + T_1(k) \\
T_1(2k+2) &= 1 + T_1(k)
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$$

n	$T_1(n)$
0	1
1	2
2	2
3	3
4	3
5	3
6	3
7	4

Running time of brt-ref on a valid index

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Conjecture: For $2^k \leqslant i + 1 < 2^{k+1}$, $T_1(n) = k + 1$.

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Or: For $i \geqslant 0$, $T_1(i) = \lfloor \log_2(i + 1) \rfloor + 1$.

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We conclude that the number of applications of `brt-ref` on a Braun tree representing a sequence of length n and an index i is $\lfloor \log_2(\min\{i+1, n+1\}) \rfloor + 1$.

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Reasoning about height works because the recursion is done on only one subtree. For a task where recursion may be performed on both subtrees (for example, an "element of" computation), the number of recursive applications can be as big as the size.

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The key is to ensure that recursion is done on at most one subtree.

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Similar reasoning works for the tail operation. The details are left as an exercise.

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Braun trees have logarithmic height, but it is more difficult or even impossible to ensure this for trees used for other purposes.

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Web pages are specified in a language (HTML) which is tree-structured.

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We will take a look at one line of development for which Racket is well-suited.

Consider a Racket expression using only numbers, and addition and multiplication with two arguments.

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(* (+ 3 4) (+ 1 6))
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While this works, we can also use lists, which are made more convenient for this purpose because of quote notation.

```
'(* (+ 3 4) (+ 1 6))
```

```
(list '* (list '+ 3 4) (list '+ 1 6))
```

Data definition: a **binary arithmetic S-expression** (BAS-exp) is either a number or a `(list op lft rgt)`, where `op` is either `'+` or `'*`, and `lft` and `rgt` are BAS-exps.

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```
(define op first)
(define left second)
(define right third)

(define (my-base-fn base)
  (cond
    [(number? base) ...]
    [(list? base) ... (op base) ...
                  ... (my-base-fn (left base)) ...
                  ... (my-base-fn (right base)) ...]))
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(define (eval base)
  (cond
    [(number? base) base]
    [(list? base) (apply (op base)
                         (eval (left base))
                         (eval (right base)))]))

(define (apply op val1 val2)
  (cond
    [(symbol=? op '+) (+ val1 val2)]
    [(symbol=? op '*) (* val1 val2)]))
```

Stepping with expression trees

In DrRacket's Stepper, `'(* (+ 3 4) (+ 1 6))` steps to `'(* 7 (+ 1 6))`.

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```
(define (reducible? base) (not (number? base)))

(define (one-step base)
  (cond
    [(and (number? (left base)) (number? (right base)))
       (apply (op base) (left base) (right base))]
    [(number? (left base))
       (list (op base) (left base) (one-step (right base)))]
    [else
       (list (op base) (one-step (left base)) (right base))]))

(define (eval2 base)
  (cond
    [(reducible? base) (eval2 (one-step base))]
    [else base]))
```

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Generalizing binary arithmetic S-expressions

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We can also allow symbols as well as numbers, letting us represent algebraic expressions with variables.

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This opens up many possibilities for programs that write or compute with other programs.

It also lets us prove a surprising result about the limits of computation.

Programs that don't terminate

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When this program is run, it does not produce a value, but runs forever.

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Let’s start by specifying clearly what we are going to prove cannot be computed.

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; halting?: (listof sexp) any -> boolean

(define (halting? sxl arg) ...)
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The halting function

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Suppose someone gave us code for `halting?`. We can then write code that uses it.

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We are still waiting for a similar proof to show that some "natural" problem cannot be computed *efficiently*.