African Institute for Mathematical Sciences (AIMS)
Mbour - Senegal

2015/2016 ACADEMIC SESSION

Topology-Functional Analysis

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In this course, we present fundamental theorems of topology and functional analysis which are of paramount importance from the point of view of applications.

It is well known that "compactness" is central in mathematical analysis and its numerous applications. In fact, one characterization of finite dimensional spaces is a classical result known as the *Heine–Borel Theorem* which states that "a normed linear space E is finite dimensional if and only if the unit ball in E is compact". The importance of this result in mathematics and in several applications is common knowledge. Another very important classical result connected with compactness is the well known *Bolzano-Weierstrass Theorem* which states that any *bounded sequence* in $I\!R^N$ has a convergent subsequence.

The Heine–Borel Theorem and the Bolzano-Weierstrass Theorem have a sort of analogues in *certain infinite dimensional spaces.* The difficulty in extending them to *all* infinite dimensional spaces is that there are too many "open sets" to contend with when a space is infinite dimensional; more precisely, the "topology" of infinite dimensional spaces is "too big" to be able to give us "compactness". Thus, in order to obtain some form of compactness it is necessary to "cut down" the number of open sets under consideration, i.e., it is necessary to reduce the size of the topology of the infinite dimensional space E. This brings us to the idea of *weak* topology on E. With respect to this topology we have what can be regarded as analogues of the Heine–Borel and Bolzano-Weierstrass Theorems in some large class of infinite dimensional spaces: A Banach space is *reflexive* if and only if the unit ball in E is *weakly* compact (this is a theorem of Kakutani which is the analogue of Heine-Borel Theorem); Any *bounded* sequence in a *reflexive* space has a *weakly* convergence subsequence (a theorem of Eberlin- Smul'yan which is the infinite dimentional analogue of the Bolzano-Wierstrass theorem).

Reflexive spaces are encountered in numerous applications. For example, the Sobolev spaces $W^{m,p}(\Omega), 1 < p < \infty$, in which most PDE's are done is reflexive. In particular, all Hilbert spaces are reflexive. Consequently, these analogues have wide variety of applications.

The aim of this course is, in summary, to present the following indispensable tools for any research in reflexive real Banach spaces.

First: A set in a real normed space that is (norm) closed and convex is weakly closed (i.e., closed in the weak topology).

Second: (Eberlein-Smul'yan Theorem). Any bounded sequence in a reflexive real Banach space has a subsequence which converges *weakly.*

In order to understand these theorems of Kakutani and Eberlein-Smul'yan, one requires a reasonably good knowledge of the weak topology and to understand the weak topology and prove some of its basic theorems, one requires virtually all the *fundamental theorems* of functional analysis.

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L'objectif de ce cours est de présenter les théorèmes fondamentaux de la Topologie et de l ' Analyse Fonctionelle.

En analyse, La compacité joue un rôle centrale. En fait, d' après un célèbre théorème de Heine-Borel, *un espace vectoriel normé E est de dimension finie si et seulement si sa boule unité fermé est compact.* L'importance de résultat n'est plus à démontrer. Un autre résultat classique en analyse, en connection avec la compacité est du à Bolzano-Weierstrass: *toute suite bornée de $\mathbb{R}^N$ admet une suite extraite convergente.*

Des théorèmes analogues à ceux de Heine-Borel et de Bolzano-Weierstrass existent dans certains espaces de Banach de dimension infinie. En fait, la difficulté liée à leur extension en dimension infinie est due au fait que la topologie usuelle (topologie forte) associée à un espace de Banach de dimension infinie est trop grande et du coup ceci réduit ses ensembles compacts. Ceci nous amène donc à introduire la notion de topologie faible. Cette topologie nous permet d'avoir des résultats analogues a ceux de Heine-Borel et de Bolzano-Weierstrass dans une large classe d'espaces de Banach. En effet, *Un espace de Banach est reflexif si et seulement si sa boule unité fermée est compacte pour la topolie faible* (un théorème de Kakutani qui est l'homologue en dimension infinie de celui de Heine-Borel); *dans un espace de Banach reflexif, toute suite bornée admet une suite extraite faiblement convergente* (ce théorème de Eberlein-Smul'yan est l'homologue en dimension infinie de celui de Bolzano-Weierstrass).

La classe des espaces de Banach reflexifs contient les espaces de Hilbert et tous les espaces L^p et $W^{m,p}$, $1 < p < \infty$, espaces pleins d'applications.

On notera qu'une bonne compréhension de la topologie faible et des théorèmes de base de l'analyse fonctionnelle (théorème de Hann Banach, théorème de Banach steinhauss, théorème de l'application ouverte et du théorème de graphe fermé) est nécessaire.