Problem Solving No 2

Name of the students in this assignemnt[1]

1. Student 1 El Hadji NGOM No 1637
2. Student 2 Mamadou Lamine THIAM No 1640
3. Student 3 Djibril GUEYE No 1629

1 Exercice 1:

Soit la fonction
$$f(x) = \frac{X-1}{2x+1}$$
Soient $X_1 = -1$; $X_2 = 0; X_3 = 1$

1. Construction du polynôme de Lagrange:
   $P(X) = \sum_{i=1}^{n+1} l_i(X)Y_i$ avec $l_i(x) = \prod_{j=1,j\neq i}^{n+1} \frac{X-X_j}{X_i-X_j}$.

   Calcul des différentes valeurs de Y:
   $Y_1 = f(-1) = 2$; $Y_2 = f(X_2) = -1$; $Y_3 = f(X_3) = 0$

   Cherchons les différentes l_i:
   $l_1(X) = (\frac{X-X_2}{X_1-X_2})(\frac{X-X_3}{X_1-X_3})$
   $l_1(X) = \frac{X^2-X}{2}$

   $l_2(x) = (\frac{X-X_1}{X_2-X_1})(\frac{X-X_3}{X_2-X_3})$
   $l_2(x) = -X^2 + 1$

   $l_3(x) = (\frac{X-X_1}{X_3-X_1})(\frac{X-X_2}{X_3-X_2})$
   $l_3(x) = (\frac{X+1}{2})(X) = \frac{X^2+X}{2}$

   Le polynôme de Lagrange est donnée par:
   $$P(X) = l_1(X)Y_1 + l_2(X)Y_2 + l_3(x)Y_3$$
   $P(X) = 2(\frac{X^2-X}{2}) - (-X^2+1)$
   $P(X) = X^2 - X + X^2 - 1$
   $$\boxed{P(X) = 2X^2 - X - 1}$$

[1]Don't write the name of any student that didn't work with the group, cheating is stricly forbidden

2. le polynôme de Newton s'écrit
$P(X) = \lambda_1 + \lambda_2(X - X_1) + \lambda_3(X - X_1)(X - X_2)$

$\lambda_1 = Y_1 = 2$

$\lambda_2 = Y[X_1, X_2] = \frac{Y[X_2]-Y[X_1]}{X_2-X_1}$
$\lambda_2 = \frac{Y_2-Y_1}{X_2-X_1} = \frac{-1-2}{0+1} = -3$

$\lambda_3 = Y[X_1, X_2, X_3]$

$\lambda_3 = \frac{Y[X_2,X_3]-Y[X_1,X_2]}{X_3,X_1}$

$\lambda_3 = \frac{\frac{Y_3-Y_2}{X_3-X_2} - \frac{Y_2-Y_1}{X_2-X_1}}{X_3-X_1}$

$\lambda_3 = \frac{\frac{0+1}{1-0} - \frac{-3}{0+1}}{1+1} = 2$

Le polynôme de Newton est donné par:
$P(X) = \lambda_1 + \lambda_2(X - X_1) + \lambda_3(X - X_1)(X - X_2)$ $P(X) = 2 - 3(X + 1) + 2(X + 1)(X)$

$$\boxed{P(X) = 2X^2 - X - 1}$$

3. La droite de régression au sens des moindres carrés des points $(-1, 2), (0, -1), (1, 0)$

Une équation de la droite de régression par la méthodes moindres carrés est:

$$y = ax + b$$

Avec $a = \dfrac{\displaystyle\sum_{i=0}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\displaystyle\sum_{i=0}^{n}(X_i - \overline{X})^2}$ et $b = \overline{Y} - a\overline{X}$

Calcul des moyennes $\overline{X}$ et $\overline{Y}$:

$\overline{X} = \frac{1}{3}(-1 + 0 + 1) = 0$
$\overline{Y} = \frac{1}{3}(+2 - 1 + 0) = \frac{1}{3}$

$a = \frac{(X_1-\overline{X})(Y_1-\overline{Y})+(X_2-\overline{X})(Y_2-\overline{Y})+(X_3-\overline{X})(Y_3-\overline{Y})}{(X_1-\overline{X})^2+(X_2-\overline{X})^2+(X_3-\overline{X})^2}$

$a = \frac{(-1)(2-\frac{1}{3})+(0)(-1-\frac{1}{3})+(1)(-\frac{1}{3})}{(-1)^2+(0)^2+(1)^2} = -1$

On trouve: $\boxed{a = -1}$ et $\boxed{b = \dfrac{1}{3}}$

Ainsi l'équation est donnée par: $\boxed{y = -x + \dfrac{1}{3}}$

2 Exercise 2

a.) The Exact Solution

$$f(x) = x^3 - 3x^2 + 2$$

$$\begin{aligned}
I &= \int_1^4 = f(x)\,dx \\
&= \int_1^4 \left(x^3 - 3x^2 + 2\right) dx \\
&= \left[\frac{1}{4}x^4 - x^3 + 2x\right]_1^4 \\
&= \left(\frac{1}{4}(4)^4 - (4)^3 + 2(4)\right) - \left(\frac{1}{4}(1)^4 - (1)^3 + 2(1)\right) \\
&= \left(\frac{256}{4} - 64 + 8\right) - \left(\frac{1}{4} - 1 + 2\right) \\
&= (64 - 64 + 8) - \left(\frac{1}{4} + 1\right) \\
&= \frac{28 - 1}{4} \\
&= \frac{27}{4}
\end{aligned}$$

b. By subdividing the interval $[1, 4]$ into 3 intervals.

$$f(x) = x^3 - 3x^2 + 2$$

$$x_1 = 1; \qquad x_2 = 2; \qquad x_3 = 3; \qquad x_4 = 4$$

i.) Using the Trapezium method

$$
\begin{aligned}
I &= \int_a^b f\left(x\right)dx = \left(f\left(a\right) + f\left(b\right)\right) * \frac{b-a}{2} \\
&= \sum_{k=1}^{N} \int_a^b f\left(x\right)dx \approx \frac{h}{2}\left(f(x(k)) + f(x(k+1))\right) \\
h &= \frac{\left(x_n - x_0\right)}{N} = \frac{4-1}{3} = 1 \\
f\left(x_1\right) &= 1^3 - 3\left(1\right)^2 + 2 = 1 - 3 + 2 = 0 \\
f\left(x_2\right) &= 2^3 - 3\left(2\right)^2 + 2 = 8 - 12 + 2 = -2 \\
f\left(x_3\right) &= 3^3 - 3\left(3\right)^2 + 2 = 27 - 27 + 2 = 2 \\
f\left(x_4\right) &= 4^3 - 3\left(4\right)^2 + 2 = 64 - 48 + 2 = 18 \\
k &= 1 \\
\frac{h}{2}\left[f\left(x_k\right) + f\left(x_{k+1}\right)\right] &= \frac{1}{2}\left[f\left(x_1\right) + f\left(x_2\right)\right] = \frac{1}{2}\left(0-2\right) = -1 \\
k &= 2 \\
\frac{h}{2}\left[f\left(x_k\right) + f\left(x_{k+1}\right)\right] &= \frac{1}{2}\left[f\left(x_2\right) + f\left(x_3\right)\right] = \frac{1}{2}\left(-2+2\right) = 0 \\
k &= 3 \\
\frac{h}{2}\big[f(x_k) + f(x(k+1))\big] &= \frac{1}{2}\big[f(x_3) + f(x_4)\big] = \frac{1}{2}\big[2+18\big] = 10 \\
\int_1^4 f(x)dx &= -1 + 0 + 10 = 9
\end{aligned}
$$

ii. Using the Simpson's method

$$\int_a^b f(x)dx = \sum_{k=1}^{4} \frac{h}{6}\big[f(x_k) + 4f\big(\frac{x_k + x_{k+1}}{2}\big) + f(x_{k+1})\big]$$

$$f\left(x_1\right) = 1^3 - 3\left(1\right)^2 + 2 = 1 - 3 + 2 = 0$$

$$f\left(x_2\right) = 2^3 - 3\left(2\right)^2 + 2 = 8 - 12 + 2 = -2$$

$$f\left(x_3\right) = 3^3 - 3\left(3\right)^2 + 2 = 27 - 27 + 2 = 2$$

$$f\left(x_4\right) = 4^3 - 3\left(4\right)^2 + 2 = 64 - 48 + 2 = 18$$

$$f(\frac{x_1 + x_2}{2}) = f(\frac{1+2}{2}) = f(\frac{3}{2}) = \frac{27}{8} - 3(\frac{9}{4}) + 2 = \frac{27}{8} - \frac{27}{4} + 2 = \frac{-11}{8}$$

$$f(\frac{x_2 + x_3}{2}) = f(\frac{2+3}{2}) = f(\frac{5}{2}) = \frac{125}{8} - 3(\frac{25}{4}) + 2 = \frac{125}{8} - \frac{75}{4} + 2 = \frac{-9}{8}$$

$$f(\frac{x_3 + x_4}{2}) = f(\frac{3+4}{2}) = f(\frac{7}{2}) = \frac{343}{8} - 3(\frac{49}{4}) + 2 = \frac{343}{8} - \frac{147}{4} + 2 = \frac{65}{8}$$

$$k = 1$$

$$\frac{1}{6}\big[f(x_1) + 4f\big(\frac{x_1 + x_2}{2}\big) + f(x_2)\big] = \frac{1}{6}\big[0 + 4(\frac{-11}{2}) - 2\big] = \frac{1}{6}\big[\frac{-11}{2} - 2\big] = \frac{-15}{12}$$

$$k = 2$$

$$\frac{1}{6}\big[f(x_2) + 4f\big(\frac{x_2 + x_3}{2}\big) + f(x_3)\big] = \frac{1}{6}\big[-2 + 4(\frac{-9}{8}) + 2\big] = \frac{1}{6}\big[\frac{-9}{2}\big] = \frac{-9}{12}$$

$$k = 3$$

$$\frac{1}{6}\big[f(x_3) + 4f\big(\frac{x_3 + x_4}{2}\big) + f(x_4)\big] = \frac{1}{6}\big[2 + 4(\frac{65}{8}) + 18\big] = \frac{1}{6}\big[20 + \frac{65}{2}\big] = \frac{105}{12}$$

$$\int_1^4 f(x)dx = \frac{-15}{12} - \frac{9}{12} + \frac{105}{12} = \frac{81}{12} = \frac{27}{4}$$

3.) Comparing the results with the exact value.

$$for\ Trapezium\ method$$

$$Error = \left|\frac{I_{trap} - I_{exa}}{Itrap}\right| \times 100 = \left|\frac{9 - \frac{27}{4}}{9}\right| \times 100$$

$$= \frac{1}{4} \times 100 = 25\%$$

$$for\ Simpson\ method$$

$$Error = \left|\frac{I_{trap} - I_{exa}}{Itrap}\right| \times 100 = \left|\frac{\frac{27}{4} - \frac{27}{4}}{\frac{27}{4}}\right| \times 100$$

$$= 0 \times 100 = 0\%$$

Hence, the Simpson method is more accurate than the Trapezoid method.

Programme scinotes (Exercise 2)

3 Exercise 3

Voir Programme fichier scinotes

4 Exercice 4

Soit

$$A = \begin{pmatrix} 4 & 2 & 1 \\ -1 & 2 & 0 \\ 2 & 1 & 4 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 2 \\ 9 \end{pmatrix} \qquad X_0 = \begin{pmatrix} 4 \\ 2 \\ 9 \end{pmatrix}$$

Resolvons $Ax = b$ par la méthode de:
<u>Jacobi:</u>

$$X_1 \begin{cases} x_1^1 &= & \frac{1}{4}(4 - 2 \times 0 - 0) &= & 1 \\ x_1^2 &= & \frac{1}{2}(2 + 1 \times 0 - 0) &= & 1 \\ x_1^3 &= & \frac{1}{4}(9 - 2 \times 0 - 1 \times 0) &= & \frac{9}{4} \end{cases}$$

$$X_2 \begin{cases} x_2^1 &= & \frac{1}{4}(4 - 2 \times 1 - \frac{9}{4}) &= & -\frac{1}{16} \\ x_2^2 &= & \frac{1}{2}(2 + 1 \times 1 - 0) &= & \frac{3}{2} \\ x_3^3 &= & \frac{1}{4}(9 - 2 \times 1 - 1 \times 1) &= & \frac{3}{2} \end{cases}$$

$$X_3 \begin{cases} x_3^1 &= & \frac{1}{4}(4 - 2 \times \frac{3}{2} - \frac{3}{2}) &= & -\frac{1}{8} \\ x_3^2 &= & \frac{1}{2}(2 + 1 \times \frac{-1}{16} - 0) &= & \frac{31}{32} \\ x_3^3 &= & \frac{1}{4}(9 - 2 \times \frac{-1}{16} - 1 \times \frac{3}{2}) &= & \frac{61}{32} \end{cases}$$

$$X_4 \begin{cases} x_4^1 &= & \frac{1}{4}(4 - 2 \times \frac{31}{32} - \frac{61}{32}) &= & -\frac{5}{128} \\ x_4^2 &= & \frac{1}{2}(2 + 1 \times \frac{-1}{8} - 0) &= & \frac{15}{16} \\ x_4^3 &= & \frac{1}{4}(9 - 2 \times \frac{-1}{8} - 1 \times \frac{31}{32}) &= & \frac{265}{128} \end{cases}$$

On obtient les vecteurs suivants aprés 4 itrérations:

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ \frac{9}{4} \end{pmatrix} \qquad X_2 = \begin{pmatrix} -\frac{1}{16} \\ \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} \qquad X_3 = \begin{pmatrix} -\frac{1}{8} \\ \frac{31}{32} \\ \frac{61}{32} \end{pmatrix} \qquad X_4 = \begin{pmatrix} \frac{5}{128} \\ \frac{15}{16} \\ \frac{265}{128} \end{pmatrix}$$

Ainsi on peut conclure la méthode converge vers une solution

$$X = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

<u>Gauss-Seidel:</u>

$$X_1 \begin{cases} x_1^1 &= & \frac{1}{4}(4 - 2 \times 0 - 0) &= & 1 \\ x_1^2 &= & \frac{1}{2}(2 + 1 \times 0 - 0) &= & 1 \\ x_1^3 &= & \frac{1}{4}(9 - 2 \times 0 - 1 \times 0) &= & \frac{9}{4} \end{cases}$$

$$X_2 \begin{cases} x_2^1 &= & -\frac{1}{16} & & \\ x_2^2 &= & \frac{1}{2}(2 + 1 \times \frac{1}{16} - 0) &= & 0.968 \\ x_2^3 &= & \frac{1}{4}(9 - 2 \times \frac{-1}{16} - 0.968) &= & 2.039 \end{cases}$$

$$X_3 \begin{cases} x_3^1 &= & -\frac{1}{8} & & \\ x_3^2 &= & \frac{1}{2}(2 + 1 \times \frac{-1}{8} - 0) &= & 0.937 \\ x_3^3 &= & \frac{1}{4}(9 - 2 \times \frac{-1}{8} - 0.937) &= & 1.953 \end{cases}$$

$$X_4 \begin{cases} x_4^1 &= & \frac{5}{128} & & \\ x_4^2 &= & \frac{1}{2}(2 + \times \frac{5}{128} - 0) &= & 2.039 \\ x_4^3 &= & \frac{1}{4}(9 - 2 \times \frac{5}{128} - 2.039) &= & 1.720 \end{cases}$$

On obtient les vecteurs suivants aprés 4 itrérations:

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ \frac{9}{4} \end{pmatrix} \qquad X_2 = \begin{pmatrix} -\frac{1}{16} \\ 0.968 \\ 2.039 \end{pmatrix} \qquad X_3 = \begin{pmatrix} -\frac{1}{8} \\ 0.937 \\ 1.953 \end{pmatrix} \qquad X_4 = \begin{pmatrix} \frac{5}{128} \\ 2.039 \\ 1.72 \end{pmatrix}$$

Ainsi on peut conclure la méthode converge vers une solution:

$$X = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

5 Exercise 5