Thursday, October 29, 2015: Games and Decisions

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DALHOUSIE UNIVERSITY Inspiring Minds
Faculty of Management

This recreational lecture is based on my presentation "Mathematical Games and Random Walks" for the Undergraduate Research Program at the International College Beijing on April 25, 2013.

A Number Game (Unique Bid Auction)

1. Everybody get a pen and piece of paper.
2. Every player write down a positive integer $\{1, 2, 3, \ldots\}$.
3. Lowest Unique Bid Auction: The player with the smallest number that has not been picked by somebody else wins the game (and a prize).
4. Highest Unique Bid Auction: The highest unique bid buys CFA 1000.

The Factor Game (Taxman Problem)

1. Playing the Taxman: A player successively chooses numbers from

$$\{1, 2, 3, \ldots, n\}.$$

   After each move, the Taxman collects all the divisors of this number.

2. Paying the Taxman: The Taxman must always be paid: If there are no more numbers with divisors, the Taxman gets all remaining numbers.

Can you beat the Taxman? $n = 10$ $n = 15$

Num	Div
1	
2	1
3	1
4	1 2
5	1

Num	Div
6	1 2 3
7	1
8	1 2 4
9	1 3
10	1 2 5

Num	Div
11	1
12	1 2 3 4 6
13	1
14	1 2 7
15	1 3 5

Your Score
0

The Taxman
0

Many People Can Beat The Taxman: Best Scores for $n = 10$ to $n = 50$

Maximum number	Best score since 2009 October 1st		Best score on 2009 October 1st	
10	40 *	Brandon (USA)	40 *	Bill (USA)
11	44 *	Brandon (USA)	44 *	Mastadon (USA)
12	50 *	Brandon (USA)	50 *	Esvin (US)
13	52 *	Brandon (USA)	52 *	Nicole (United States)
14	66 *	Peter (USA)	66 *	Joshua (United States)
15	81 *	Mike (Canada)	81 *	Mark (U.S.A.)
16	89 *	Jayel4 (U. S.)	89 *	Joshua (United States)
17	93 *	Nick (America)	93 *	Joshua (United States)
18	111 *	Nicolo (UK)	111 *	Joshua (United States)
19	113 *	Popo (USA)	113 *	Joshua (United States)
20	124 *	Ryan (Australia)	124 *	Andrew (United States)
21	144 **	Nick (America)	144 **	Seth (USA)
22	166 **	Nicolo (UK)	166 **	Joshua (U.S.)
23	170 **	Nick (America)	170 **	Joshua (U.S.)
24	182 **	Keli (Benin)	182 **	Daniel (United Kingdom)
25	198 **	Alvin (USA)	198 **	Joshua (U.S.)
26	224 **	Nicolo (UK)	224 **	Joshua (U.S.)
27	251 *	Nick (America)	251 *	Joyce (USA)
28	279 *	Nick (America)	279 *	Camron (USA)
29	285 *	Nick (America)	285 *	Camron (USA)
30	301 *	Jayel4 (U. S.)	301 *	Camron (USA)
31	303 *	Nick (America)	303 *	Camron (USA)
32	319 **	Nicolo (UK)	319 **	Camron (USA)
33	352 **	Nick (America)	352 **	Camron (USA)
34	386 **	Nick (America)	386 **	Tom (PA, USA)
35	418 **	Alan (Ireland)	418 **	David (USA)
36	442 *	Nick (America)	442 *	Hilliary (USA)
37	448 *	Nick (America)	448 *	Gabbi/Brittany (USA)
38	486 *	Nick (America)	486 *	Brittany (US)
39	503	Nick (America)	503	Alex (Germany)
40	525	Sarah (USA)	525	Alex (Germany)
41	529	Joel (USA)	529	Alex (Germany)
42	565	Hasan (Turquey)	571	Alex (Germany)
43	512	Samson (United States)	573	Alex (Germany)
44	595	Samson (United States)	617	Alex (Germany)
45	609	Samson (United States)	660	David (USA)
46	655	Samson (United States)	706	Alex (Germany)
47	710	Wibstr (USA)	710	David (USA)
48	681	Nicolo (UK)	734	Alex (Germany)
49	758	Nicolo (UK)	758	Alex (Germany)
50	808	Matt (USA)	808	Joshua (U.S)

* The highest score you can get
** I think it's the highest score you can get

http://pagesperso-orange.fr/jeux.lulu/html/anglais/nbmyster/taxateur.htm (2013)

Can You Meet Alex (Germany) or David (USA)? There are 6 slots left!

Maximum number	Best score since 2009 October 1st		Best score on 2009 October 1st	
36	442 *	Nick (America)	442 *	Hilliary (USA)
37	448 *	Nick (America)	448 *	Gabbi/Brittany (USA)
38	486 *	Nick (America)	486 *	Brittany (US)
39	503	Nick (America)	503	Alex (Germany)
40	525	Sarah (USA)	525	Alex (Germany)
41	529	Joel (USA)	529	Alex (Germany)
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http://pagesperso-orange.fr/jeux.lulu/html/anglais/nbmyster/taxateur.htm (2013)

A New Casino Game (Factor-Game Roulette)

1. Imagine a roulette wheel with numbers

$$\{1, 2, 3, \ldots, 36\}.$$

2. You can bet any amount on any number.
3. If your bet divides the winning number, you get back your amount times the bet.

What is your optimal playing strategy?

- Are there any 'fair' numbers?
- Are there any losing numbers?
- Are there any winning numbers?

A Winning Strategy in Factor-Game Roulette

1. Bet CFA 1 on 2; if the winning number is even, you get $2 \times$ CFA 1:

$$\text{Win} = \text{Get} - \text{Bet} = 2 - 1 = 1$$

2. Every time the winning number is odd, double your bet on 2:

$$1 \xrightarrow{\text{2nd bet}} 2 \xrightarrow{\text{3rd bet}} 4 \xrightarrow{\text{4th bet}} 8 \longrightarrow \cdots \xrightarrow{k\text{th bet}} 2^{k-1} \longrightarrow \cdots$$

3. Eventually, a winning number will be even; let this happen in round k:

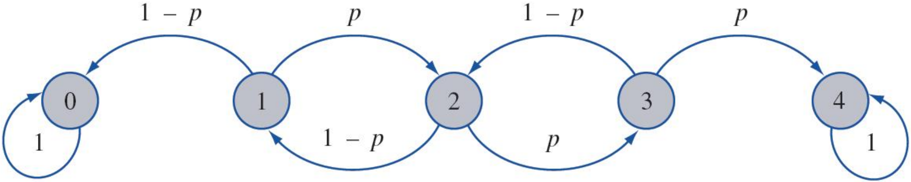
$$\begin{aligned}\text{Win} = \text{Get} - \text{Bet} &= 2^k - \left(1 + 2 + 4 + 8 + \ldots + 2^{k-1}\right) \\ &= 2^k - \left(2^k - 1\right) = 1\end{aligned}$$

Warning: This strategy may not work for everybody!

- Do you have enough money to double your bet k times?
- What is the probability that you have to double your bet k times?

The Gambler's Ruin

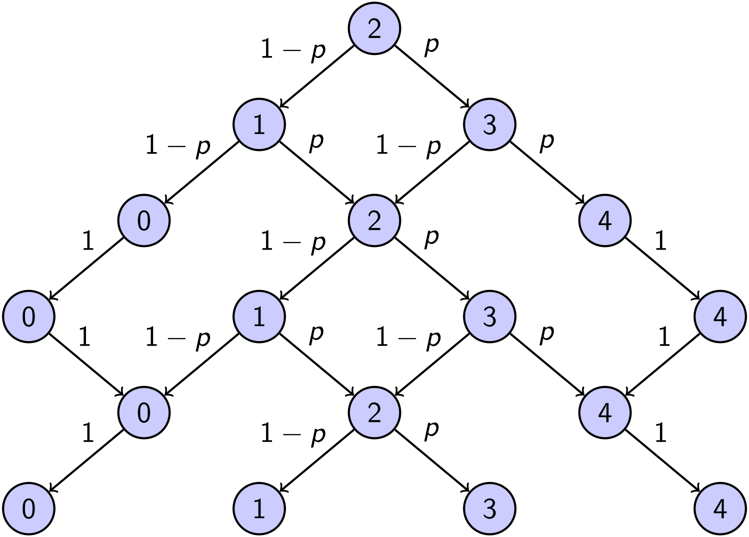
Starting with an initial capital of CFA 2, you play a game and in every round either win or lose CFA 1 with a known probability of p or $1 - p$. The game ends either if you reach CFA 4, or if you are out of money.



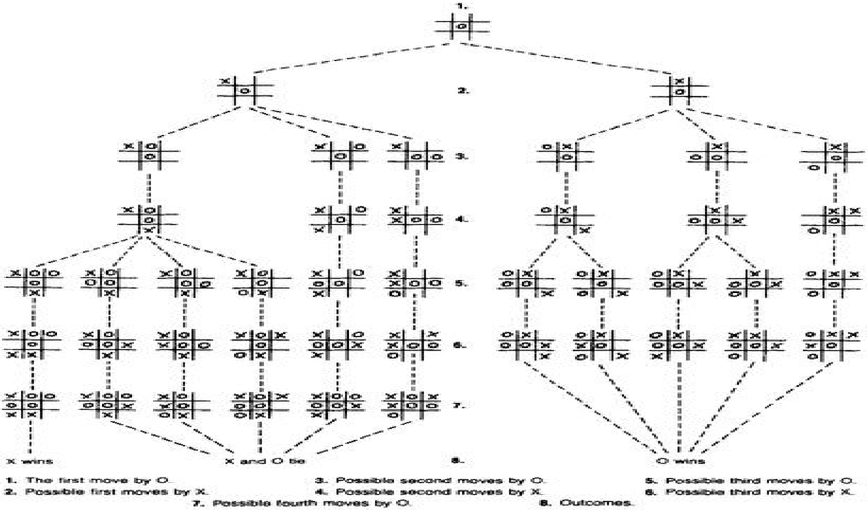
This can be called Stochastic Process, Random Walk, or Markov Chain.

- What is the probability to win CFA 4, or to run out of money?
- What is the probability that this game will last forever?
- What is the expected duration of this game?

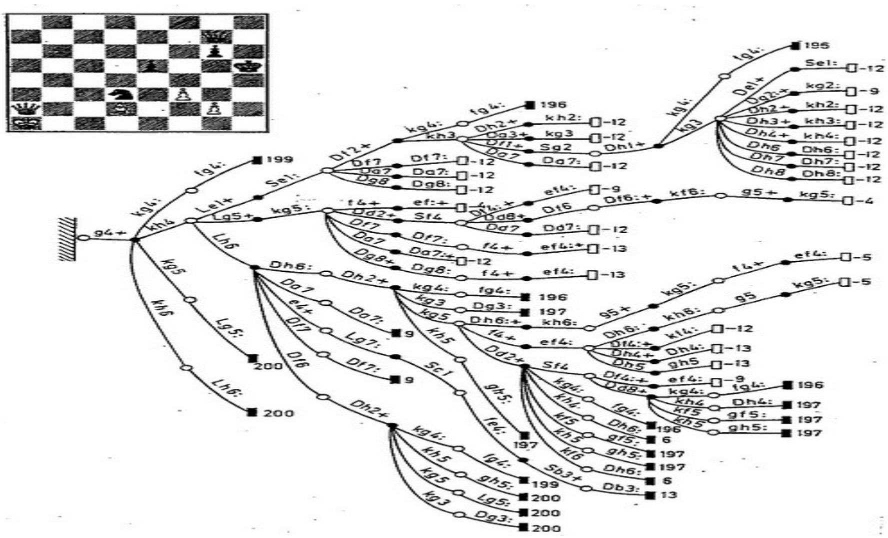
The Gambler's Ruin Scenario Tree



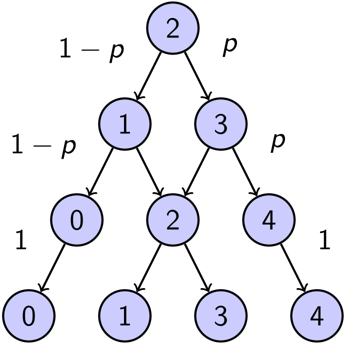
TicTacToe Decision Tree



Chess Endgame Decision Tree



Gambler's Ruin Scenario Tree Analysis



1. Probability to reach 0 or 4?

$$\begin{aligned} p_0 &= (1-p)^2 + 2p(1-p)p_0 \\ &= \frac{(1-p)^2}{(1-p)^2+p^2} \\ p_4 &= p^2 + 2p(1-p)p_4 \\ &= \frac{p^2}{(1-p)^2+p^2} \end{aligned}$$

2. Probability to play forever?

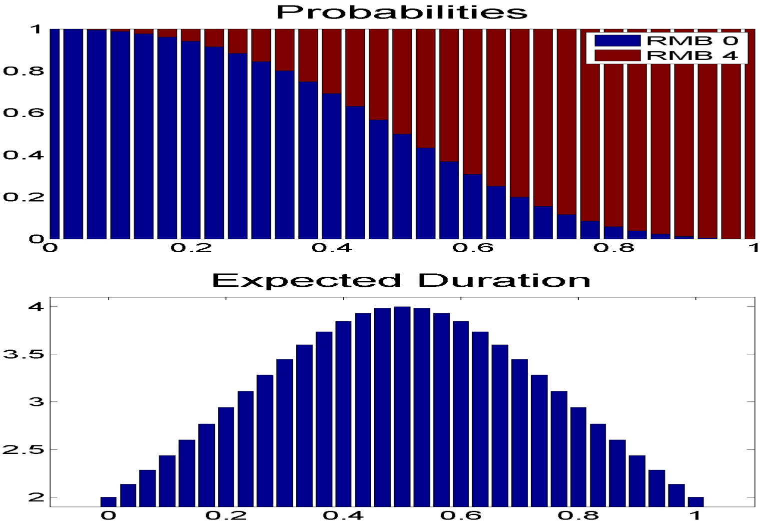
$$\lim_{n\to\infty} [2p(1-p)]^n = 0$$

(also note that $p_0 + p_4 = 1$)

3. Expected duration of game?

$$\sum_{k=1}^{\infty} \left[2k\,(2p(1-p))^{k-1}\left((1-p)^2+p^2\right) \right] = \frac{2}{(1-p)^2+p^2}$$

Probabilities and Expected Duration for Different p Values



Gambler's Ruin Markov Chain Analysis

Let $S = \{0, 1, 2, 3, 4\}$ and P_{ij} be the probability to go from state i to j:

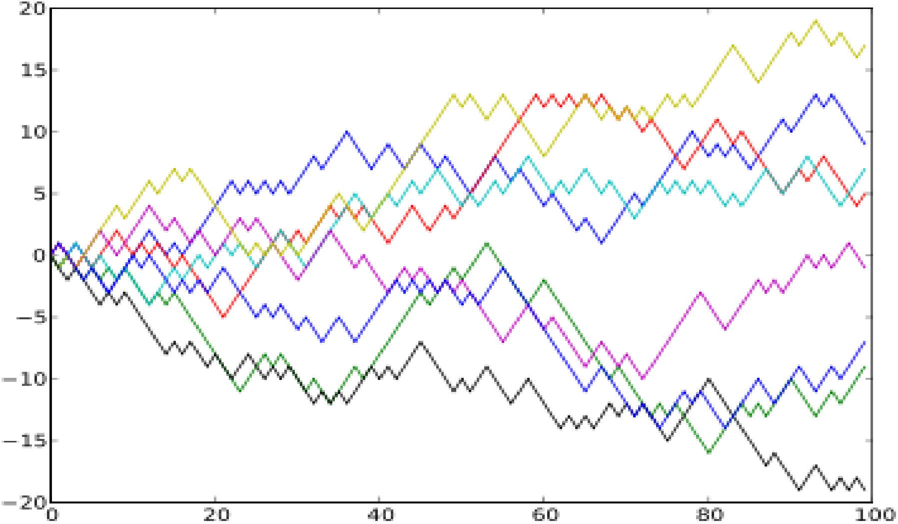
$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We can compute matrix powers P^n to find n-step transition probabilities:

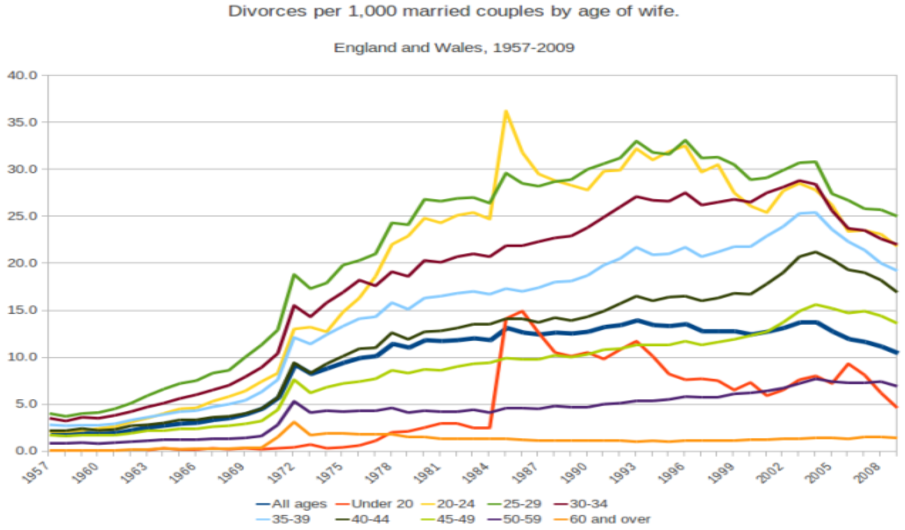
$$P^2 = P \times P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-p & p(1-p) & 0 & p^2 & 0 \\ (1-p)^2 & 0 & 2p(1-p) & 0 & p^2 \\ 0 & (1-p)^2 & 0 & (1-p)p & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This may seem complicated but only uses a little bit of linear algebra $\ddot\smile$

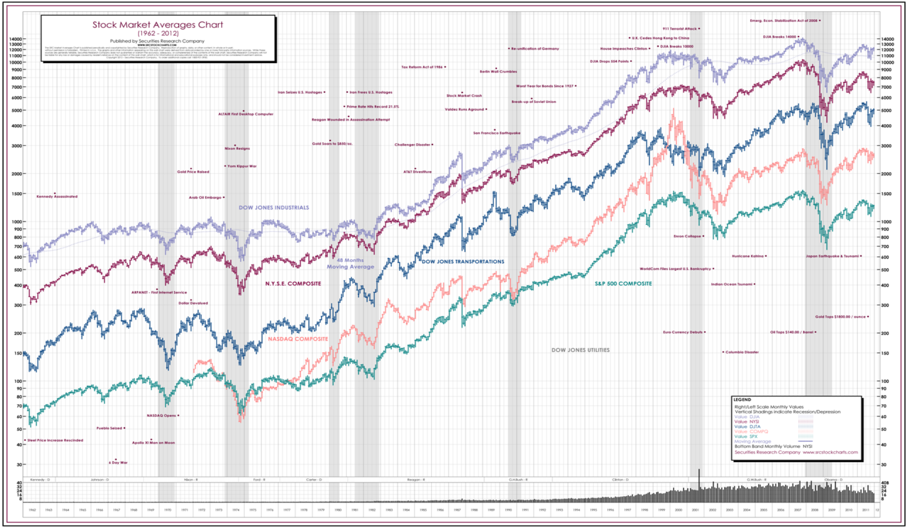
Gambler's Random Walk (Monte Carlo Simulation)



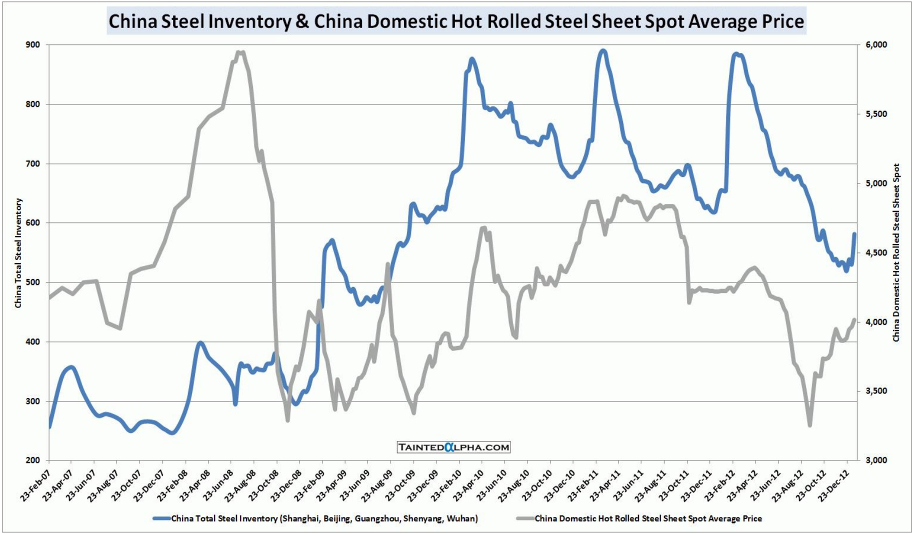
Social Random Walk (Divorces in England and Wales, 1957-2009)



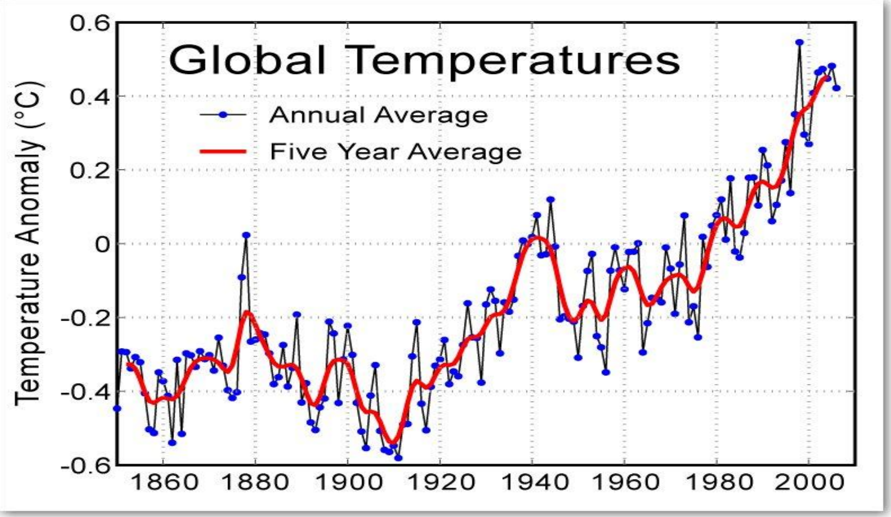
Financial Random Walk (Stock Market Averages Charts, 1962-2012)



Economic Random Walk (China Steel Inventory & Prices, 2007-2012)



Environmental Random Walk (Temperature Anomalies, 1850-2010)



The Mathematics of Policy and Decision-Making

- Math 3301/3302 Operations Research: develops methods to help or support policy and decision-makers in solving real-world problems
- Math 4390 Game Theory: develops mathematical models of conflict and cooperation for intelligent rational policy and decision-makers

A Simple Decision Model ("one-person game against nature")

- $A = \{a_1, a_2, a_3, \dots\}$ be a set of available actions or behaviors
- $S = \{s_1, s_2, s_3, \dots\}$ be the set of possible "states of the world"

1. A decision-maker chooses an action $a \in A$.
2. The world (or "nature") adopts a certain state $s \in S$.
3. The decision-maker earns a reward $r(a, s)$ or incurs a cost $c(a, s)$.

The goal is to choose $a \in A$ so to maximize reward and minimize cost.

The News Vendor Problem

Every day, Bobby Obama buys newspapers from a store for CFA 2,000 and sells them on the street for CFA 2,500. If daily demand is between 6 and 10, how many newspapers should Bobby supply to maximize his profit?

Rewards		States 6	7	8	9	10	$\min_{s \in S}$	$\max_{s \in S}$
Actions	**6**	3	3	3	3	3	3	3
	7	1	3.5	3.5	3.5	3.5	1	3.5
	8	-1	1.5	4	4	4	-1	4
	9	-3	-0.5	1.5	4.5	4.5	-3	4.5
	10	-5	-2.5	0	2.5	5	-5	5

- Best Worst Case: $\max_{a \in A} \min_{s \in S} r(a, s)$ (max-min criterion)
- Best Best Case: $\max_{a \in A} \max_{s \in S} r(a, s)$ (max-max criterion)

The News Vendor Regret Criterion

- For each $s \in S$, compute max possible reward $r^*(s) = \max_{a \in A} r(a, s)$
- For each $s \in S$ and $a \in A$, compute opportunity costs (or regrets):

$$c(a, s) = r^*(s) - r(a, s)$$

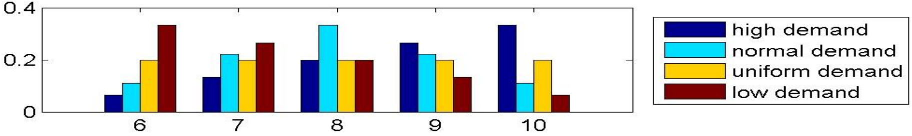
- The regret criterion minimizes disappointment of lost opportunities:

$$\min_{a \in A} \max_{s \in S} c(a, s) = \min_{a \in A} \max_{s \in S} (r^*(s) - r(a, s))$$

	Rewards $r(a,s)$					Regrets $c(a,s)$					$\max_{s \in S}$
	6	**7**	**8**	**9**	**10**	**6**	**7**	**8**	**9**	**10**	
6	3	3	3	3	3	0	0.5	1	1.5	2	2
7	1	3.5	3.5	3.5	3.5	2	0	0.5	1	1.5	2
8	-1	1.5	4	4	4	4	2	0	0.5	1	4
9	-3	-0.5	1.5	4.5	4.5	6	4	2	0	0.5	6
10	-5	-2.5	0	2.5	5	8	6	4	2	0	8

Expected Values and Demand Profiles

Suppose the news vendor can estimate probabilities $\pi(s)$ for all $s \in S$:



$$\max_{a\in A} E_r(a) = \sum_{s=6}^{10} \pi(s) r(a,s) \qquad \min_{a\in A} E_c(a) = \sum_{s=6}^{10} \pi(s) c(a,s)$$

		Expected Rewards $\mathbf{E_r(a)}$				Expected Regrets $\mathbf{E_c(a)}$			
		high	uniform	normal	low	high	uniform	normal	low
Actions	**6**	3	3	3	3	1.3	1	1	0.7
	7	3.3	3.2	3	2.7	1	0.8	1	1
	8	3.3	2.9	2.5	1.7	1	1.1	1.5	2
	9	2.7	1.5	1.4	0.1	1.5	2.3	2.5	3.5
	10	1.7	0	0	-1.7	2.7	4	4	5.3

A Mathematical Model for Two-Person Games

- Two players Rowley and Coleen choose from a finite set of moves:

$$R = \{1, \ldots, m\} \qquad C = \{1, \ldots, n\}$$

- If Rowley and Coleen play $i \in R$ and $j \in C$ they receive r_{ij} and c_{ij}:

$$\begin{bmatrix} (r_{11}, c_{11}) & (r_{12}, c_{12}) & \ldots & (r_{1n}, c_{1n}) \\ (r_{21}, c_{21}) & (r_{22}, c_{22}) & \ldots & (r_{2n}, c_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (r_{m1}, c_{m1}) & (r_{m2}, c_{m2}) & \ldots & (r_{mn}, c_{mn}) \end{bmatrix}$$ is called the payoff matrix.

- Rowley is called row player and Coleen is called column player.

Game 1

$$\begin{bmatrix} (1,0) & (2,2) \\ (3,1) & (0,3) \end{bmatrix}$$

Equilibrium

Game 2

$$\begin{bmatrix} (1,0) & (2,2) \\ (0,3) & (3,1) \end{bmatrix}$$

No equilibrium

Game 3

$$\begin{bmatrix} (2,3) & (0,0) \\ (1,1) & (3,2) \end{bmatrix}$$

Two equilibria

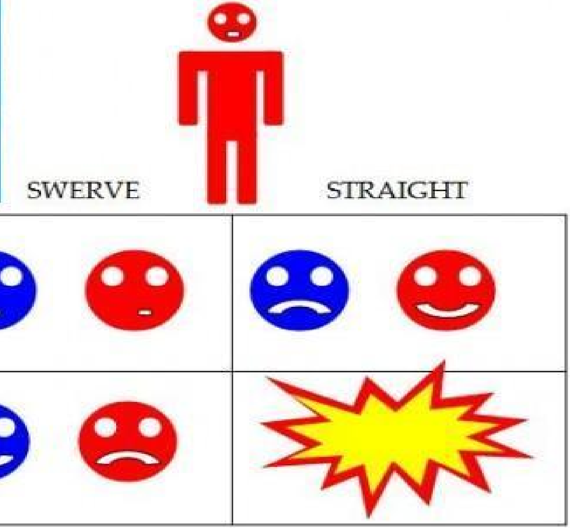
Coop Games

$$\begin{bmatrix} (0,0) & (0,5) \\ (5,0) & (1,1) \end{bmatrix}$$

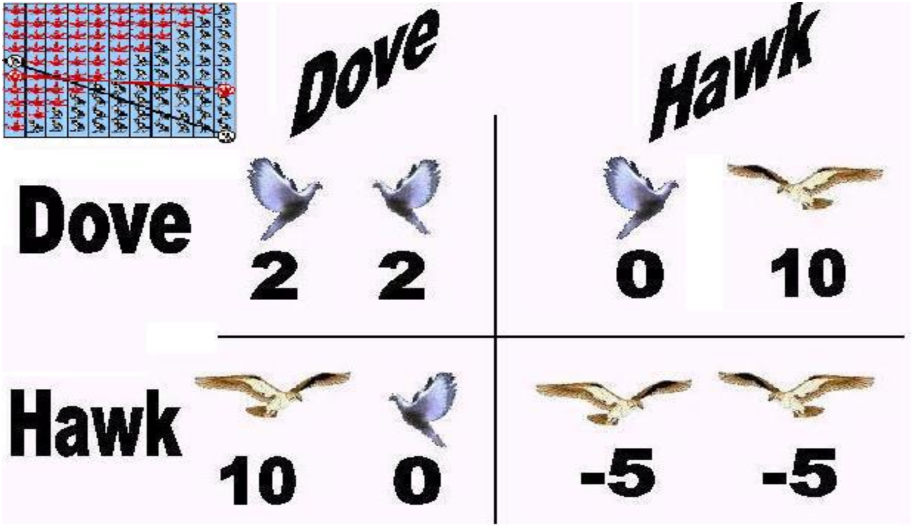
$$\begin{bmatrix} (5,0) & (0,5) \end{bmatrix}$$

Battle of the Sexes

The Chicken Game

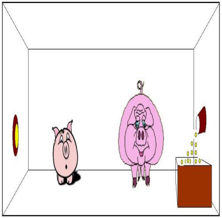


Dove and Hawk (Classic Example of Evolutionary Games)



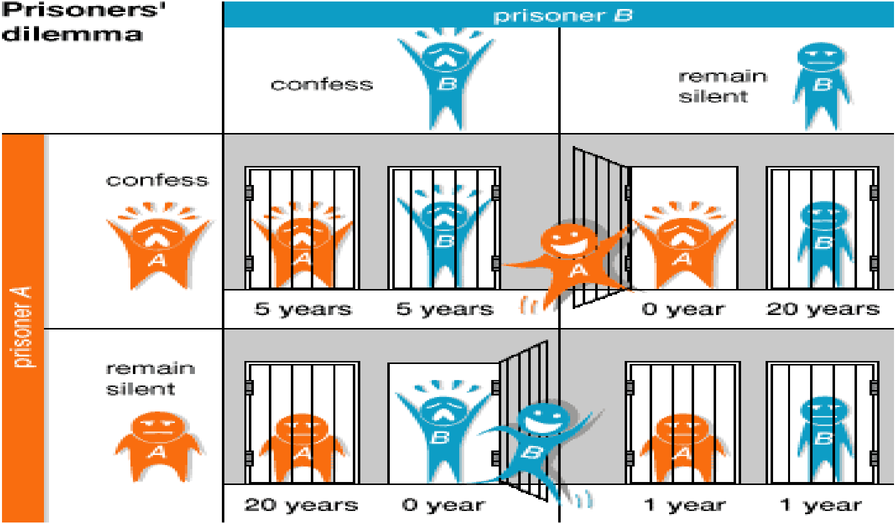
Boxed Pigs

- A litte pig and a big pig are boxed in a room.
- Pressing a button on left wall gives 10 units of food from right wall and costs 2 units of energy.
  - If little pig presses, big pig eats 9, little pig gets 1.
  - If big pig presses, little pig eats 5, big pig gets 5.
  - If both pigs press, little pig eats 3, big pig eats 7.



Big Pig \ Little Pig	Press	Wait
Press	5 , 1	4 , 4
Wait	9 , -1	0 , 0

The Prisoners Dilemma



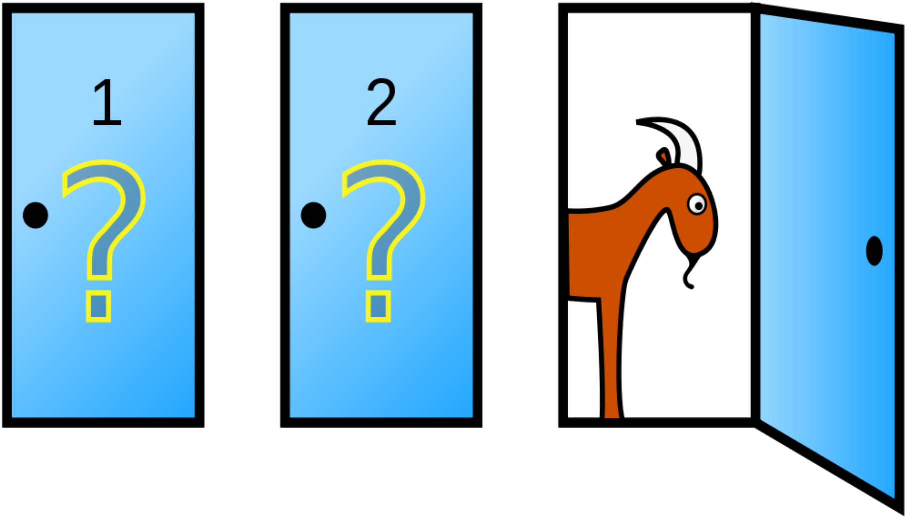
The Three Prisoners Problem

- Three prisoners A, B, and C are facing 5-year sentences in prison.
- The judge has randomly chosen one of the prisoners to be released.
- Prisoner A begs the judge to tell him which of the two other prisoners will be kept in prison (if both will be kept, he shall say at least one).
- The judge tells prisoner A that prisoner C will be kept in prison.
- What is the probability that prisoner A will be released?

The Two Goats Problem

- You play in a game show and can choose one of three closed doors.
- Behind one door is the main prize, behind the other doors are goats.
- You choose to open Door 1 and the game host opens Door 3: a goat.
- The host offers a deal: you can stay with Door 1 or switch to Door 2.

Which Door Do You Choose?



Think Critically, Study Hard, and Be Good!