AIMS-Sénégal
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Assignment: Counting Rooted Plane Trees and Melonic Graphs

The goal of this assignment is to enumerate some simple graphs.

1 Rooted plane trees

Definition Rooted plane trees are defined pages 45-46 of the Lecture Notes. Please read these pages before starting the assignment...

1) Draw all rooted plane trees with 4 edges.

2) Prove that the number C_n of rooted plane trees with n edges obeys the following recursion:

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \quad ; \quad C_0 = 1. \tag{1}$$

3) Compute the numbers C_n up to C_{10}. In view of question 2, explain a systematic method to draw the 42 rooted plane trees with 5 edges.

4) Introducing

$$f(x) = \sum_{n=0}^{\infty} C_n x^n \tag{2}$$

prove that

$$f(x) = 1 + x f^2(x) \tag{3}$$

5) Conclude that

$$f(x) = \frac{1 - \sqrt{1-4x}}{2x}. \tag{4}$$

6) Prove the asymptotic behavior

$$C_n \simeq_{n\to\infty} 4^n \frac{\sqrt{\pi}}{n^{3/2}}. \tag{5}$$

using Stirling's formula $n! \simeq n^n e^{-n} \sqrt{2\pi n}$.

2 Melonic Graphs

We now want to count slightly more complicated graphs called melonic graphs [1]. All graphs considered from now on are assumed to be *connected*.

[1] These graphs occur in the study of triangulations of spaces of dimension d, hence are of interest to understand quantum gravity.

We consider a fixed set of $d \geq 2$ different colors, labeled as $\{1, ...d\}$. A bipartite d-regular edge-colored graph (in short a d-BREC) is a graph in which

- vertices are either black and white and have equal degree d, and

- every edge joins a black and a white vertex (bipartite graph) and has a color label such that all edges meeting at a vertex have *different* colors.

Hence in a d-BREC all colors are represented exactly once at each vertex.

An *open* d-BREC (in short d-OBREC) is a graph obtained by deleting a single edge of a d-BREC graph. If the deleted edge has color i, the d-OBREC is said to be of color i (see Figure 1).

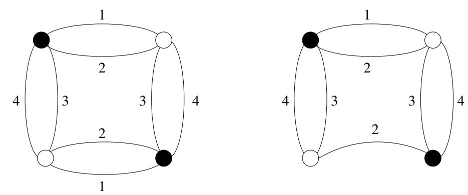

Figure 1: A 4-BREC (left) and a 4-OBREC of color 1 (right)

1) Prove that any d-BREC or d-OBREC has an even number of vertices. What is the number of edges in a d-BREC with $n = 2p$ vertices? and in a d-OBREC with $n = 2p$ vertices?

2) For $d = 2$, find the 2-BRECs and 2-OBRECs with $n = 2p$ vertices. What do you remark? From now on we therefore consider $d \geq 3$.

We study now a particular class of d-OBRECs, called melonic graphs.

We call *d-melon* the unique d-BREC with two vertices. The *open d-melon* of color i is defined as the d-OBREC obtained by deleting the single edge of color i of the d-melon.

When an open d-melon of color i occurs as a strict edge-subgraph $S \subset G$ of a d-OBREC G, there is an associated contraction called *melonic contraction*. It contracts S and one of its attached edges of color i to a single vertex, resulting in a contracted graph G/S (see Figure 3).

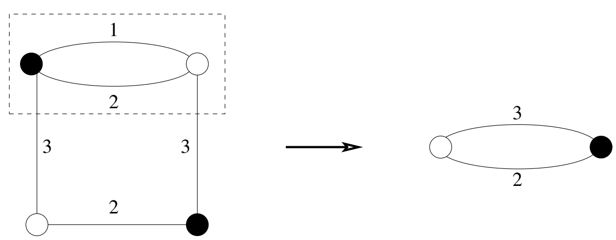

Figure 2: Example of a melonic contraction: the contraction (left) of an open 3-melon of color 3 (dashed box) in a 3-OBREC of color 1 gives a reduced 3-OBREC of color 1 (right)

3) Check that G/S does not depend on which of the two edges of color j attached to S we choose for the contraction, and is still a d-OBREC (of same color than G).

4) A d-OBREC is called *melonic* if it reduces to an open d-melon (of the same color) through a sequence of melonic contractions. Count and draw the melonic d-OBRECs of

a given fixed color (say 1) with 4 vertices. Count and draw the melonic 3-OBRECs of a given fixed color (say 1) with 6 vertices.

3 Facultative: Counting Melonic Graphs

The goal in this last section is to compute the number $N_d(p)$ of melonic d-OBRECs of a fixed color (say 1) with $2p$ vertices.

1) Defining the power series

$$f_d(x) = 1 + \sum_{p=1}^{\infty} N_d(p) x^p, \tag{6}$$

prove[2] that it satisfies the equation

$$f_d(x) = 1 + x[f_d(x)]^d \tag{7}$$

2) Check that $N_d(p) = \frac{(dp)!}{p![(d-1)p+1]!}$ is the solution of equation (7)

3) What is the radius of convergence of the power series (6)? (Hint: you can use Stirling's formula to approximate $k!$ at large k).

[2]Hint: you may introduce the number $M_d(p)$ of melonic d-OBRECs of a fixed color (say 1) with $2p$ vertices and *no bridge*. Defining the series $g_d(x) = \sum_{p=1}^{\infty} M_d(p)x^p$ you may prove first that $f_d(x) = \frac{1}{1-g_d(x)}$, and then that $g_d(x) = x[f_d(x)]^{d-1}$.