AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
MBOUR, SENEGAL

Academic Session 2015/2016

Topology-Functional Analysis : Tutorial-Set I

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Ex. 1 Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all } X\}$$

a topology on X ?

Ex. 2

1. If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X ?
2. Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collection $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$
3. If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \text{ and } \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

Ex. 3 Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ is the smallest topology containing $\mathcal{A}$.

Ex. 4 Show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, \ a, b \in \mathbb{Q}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

Ex. 5 Let X be a topological space. Let A be a subset of X. Suppose that for each $x \in A$, there exists an open set U such that $x \in U \subset A$. Show that A is open in X.

Ex. 6 In the spaces on real line obtained by giving it the indiscrete topology, the discrete topology, the usual metric topology, and the finite-complement topology, what is $\bar{A}$ if

1. $A = (0, 1]$
2. $A = [0, 1]$
3. $A = (0, 1)$

Ex. 7 Let A, B and A_α denote subsets of a space X. Prove the following :

1. If $A \subset B$, then $\bar{A} \subset \bar{B}$.
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
3. $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$; give an example where equality fails.

Ex. 8 Let A, B and A_α denote subsets of a space X. Determine whether the following equations hold ; if an equality fails, determine whether one of the inclusion $\supset$ or $\subset$ holds.

1. $\overline{A \cap B} = \bar{A} \cap \bar{B}$.
2. $\overline{\bigcap A_\alpha} = \bigcap \bar{A}_\alpha$
3. $\overline{A - B} = \bar{A} - \bar{B}$.

Ex. 9 Show that if A is closed in X and if B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Ex. 10 Let $A \subset X$ and $B \subset Y$. Show that in the product topology on $X \times Y$, $\overline{A \times B} = \bar{A} \times \bar{B}$.

Ex. 11 Show that X is Hausdorff if and only if the **diagonal** $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

Ex. 12 In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = 1/n$ converge ?

Ex. 13 If $A \subset X$, we define the **boundary** of A, ∂A or $\mathrm{Bd}\, A$ by :

$$\partial A = \bar{A} \cap \overline{X - A}.$$

Show that

1. $\mathring{A}$ and ∂A are disjoint, and $\bar{A} = \mathring{A} \cup \partial A$.
2. $\partial A = \emptyset$ if and only if A is both open and closed.
3. U is open if and only if $\partial U = \bar{U} - U$.

Ex. 14 Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$:

1. $A = \{(x, y) \mid y = 0\}$.
2. $B = \{(x, y) \mid x > 0 \text{ and } y \neq 0\}$.
3. $C = A \cup B$.
4. $D = \{(x, x) \mid x \text{ is rational}\}$.

Ex. 15
1. A topological space X satifies the *first separation axiom* T_1 if each one of any two points of X has a neighborhood that does not contain the other point. More formally : $\forall\, x, y \in X,\ x \neq y\ \exists U_y \in N(y) :\ x \notin U_y$.

(a) Show that X satisfies T_1 if and only if all one-point sets in X are closed.

(b) Show that X satisfies T_1 if and only if every point is the intersection of all of its neighborhoods.

(c) Show that any Hausdorff space is T_1.

(d) Find an example showing that the T_1-axiom does not imply the hausdorff axiom.

2. topological space X satifies the *Kolmogorov axiom* T_0 if at least one of any two distinct points of X has a neighborhood that does not contain the other point.

(a) Show that X satisfies T_0 if and only if any two different points of X has different closures.

(b) Show that if X is T_1 then X is T_0. Find an example showing that the T_0-axiom does not imply the T_1-axiom.

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