Joint GEE is a GEE - based modelling approach for joint mean and over dispersion analysis.

# 1 Joint Structure - Formulas

Input Data: for each subject i $X_i, W_i, Y_i$

The following is joint estimating equations

$$S(\hat{\beta}, \hat{\gamma} = \sum_{i=1}^{N} D_i^T V_i^{-1} r_i = 0_{p+q}$$

where

$$D_i = \begin{pmatrix} D_{i1} & 0 \\ D_{i21} & D_{i2} \end{pmatrix}_{2T\times(p+q)} = \begin{pmatrix} \frac{\partial \mu_i}{\partial \beta} & 0 \\ \frac{\partial m_i}{\partial \beta} & \frac{\partial m_i}{\partial \gamma} \end{pmatrix}_{2T\times(p+q)}$$

$$V_i = \begin{pmatrix} V_{i1} & 0 \\ V_{i21} & V_{i2} \end{pmatrix}_{2T\times 2T}$$

$$r_i = \begin{pmatrix} y_i - \mu_i \\ y_i^2 - m_i \end{pmatrix}_{2T\times(p+q)}$$

- Formula for mean and over dispersion

$$\mu_{it} = E(Y_{it}) = exp(X_{it}^T \beta)$$

$$\phi_{it} = exp(W_{it}^T \gamma)$$

$$m_{it} = E(Y_{it}^2) = \mu_{it} + (\phi_{it} + 1)\mu_{it}^2$$

- Formula for Di

$$D_{i1}[t,k] = X_i[t,k] * \mu[t]$$

$$D_{i2}[t,k] = Wi[t,k] * \mu[t]^2 * \phi[t]$$

According to Prentice - Zhao $D_{i21} = 0$

- Formula for Vi

The exchangeable structure

$$R_i \rho_i = \begin{pmatrix} 1 & \rho_i & ...\rho_i & \\ \rho_i & 1 & ...\rho_i & \\ & & & \\ & & & \\ \rho_i & \rho_i & ... & 1 \end{pmatrix}$$

$$V_{i1} = A_i^{\frac{1}{2}} R_1(\rho_1) A_i^{\frac{1}{2}}$$

$$\hat{\rho}_1 = \frac{\sum_{i=1}^{N} \sum_{j>j*} \hat{r}_{ij} \hat{r}_{ij*}}{\frac{N}{2}(T-1)T - p}$$

$$r_{ij} = \frac{y_{ij} - \hat{\mu}_{ij}}{\sqrt{\mu_{ij} + \phi_{ij}\mu_{ij}^2}}$$

$$V_{i2} = H_i^{\frac{1}{2}} R_2(\rho_2) H_i^{\frac{1}{2}}$$

$$\hat{\rho}_2 = \frac{\sum_{i=1}^{N} \sum_{j>j*} \hat{s}_{ij}\hat{s}_{ij*}}{\frac{N}{2}(T-1)T - p - q}$$

$$s_{ij} = \frac{Y_{ij}^2 - m_{ij}}{\sqrt{var(Y_{ij}^2)}}$$

$$var(Y_{ij}^2) = \hat{\mu}_{ij} + (6+7\hat{\phi}_{ij})\hat{\mu}_{ij}^2 + (4+16\hat{\phi}_{ij}+12\hat{\phi}_{ij}^2)\hat{\mu}_{ij}^3 + (4\hat{\phi}_{ij}+10\hat{\phi}_{ij}^2+6\hat{\phi}_{ij}^3)\hat{\mu}_{ij}^4$$

$$V_{i21} = cov(Y, Y^2) = 0$$

## 2 Simulation Study

We need to generate a longitudinal count data Input

$$N = 260$$

$$\beta = 1, 1$$

$$\gamma = 0.1, 0.1$$

$$\rho = 0.7$$

$$X_{it}0 = \begin{cases} -1: & i = 1, 2.., N/4 \\ 0: & i = N/4 + 1, ..., 3N/4 \\ 1: & i = 3N/4 + 1, ..., N \end{cases} \tag{1}$$

$$X_{it}1 \sim \text{Uniform}(min, max)$$

in our case: I set min = 0.4 , max = 0.9

$$W_{ij}0 = 1$$

$$W_{ij}1 \sim \text{Uniform}(min, max)$$

For each subject i:

$$\mu_{it} = exp(X_{it}^T \beta)$$

$$\phi_{it} = exp(W_{it}^T \gamma)$$

Generate $Y_{it}$ using method developed by McKenzie(1986)

$$Y_{it} = \varphi_{it} * Y_{i0} + e_{it}$$

initial

$$\mu_{i0} = mean(\mu_{it})$$

$$\phi_{i0} = mean(\phi_{it})$$

We generate

$$Y_0 = NB(\mu_{i0}, \phi_{i0})$$

For each $Y_{it}$

$$Y_{it} = d_{it} + e_{it}$$

where

$$d_{it} \sim NB(\sqrt{\rho}\mu_{i0}, \phi_{i0}/\sqrt{\rho})$$

$$e_{it} \sim NB((1 - \sqrt{\rho}\mu_{i0}), \phi_{i0}/(1 - \sqrt{\rho}))$$

# 3 Result

- file 1: NB.exchange.R - Generate a sample data.
- file 2 : JGEE1 algorithm to solve
- file 3: TestModel to run simulation

result for 150 simulations for real data:

$$N = 260, T = 20, \beta = (1, 1), \gamma = (0.5, 0.1)$$

Table 1:

| $\rho$ | $\beta_0$ | $\beta_1$ | $\gamma_0$ | $\gamma_1$ |
|---|---|---|---|---|
| $\rho = 0.1$ mean | 1.03562864 | 0.92102625 | 0.7615614 | 0.07134913 |
| std | 0.04110263 | 0.03761612 | 0.3814256 | 0.58182775 |
| $\rho = 0.3$ mean | 1.04507964 | 0.9166358 | 0.7838658 | 0.0371248 |
| std | 0.04016281 | 0.0358529 | 0.3695337 | 0.5724225 |
| $\rho = 0.7$ mean | 1.0403979 | 0.91589818 | 0.7277560 | 0.1264163 |
| std | 0.0353597 | 0.03716551 | 0.3369449 | 0.5212272 |