Lecture 23.
Elementary Graph Algorithms

Introduction to Algorithms
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Graphs

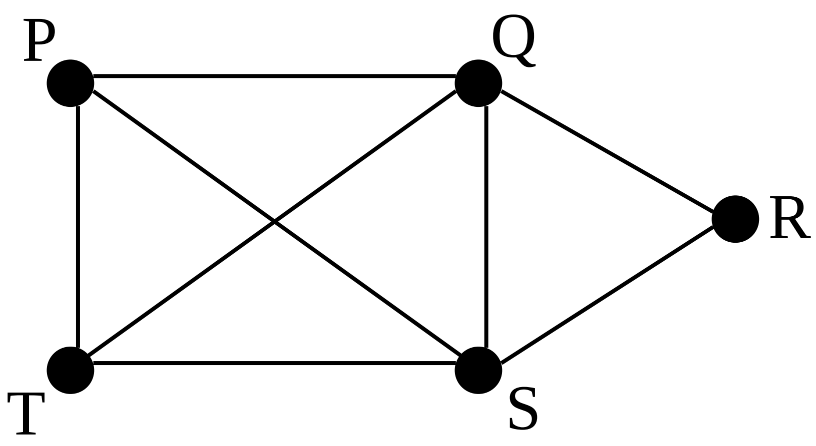
- *Graph* $G = (V, E)$
  - V is set of vertices
  - E is set (family) of edges $\subseteq (V \times V)$
    - edge or link or arc
- Types of graphs
  - Undirected: edge $(u, v) = (v, u)$
    - for all v, $(v, v) \notin E$ (No self loops)
  - Directed: (u, v) is edge from u to v, denoted as $u \rightarrow v$
    - self loops are allowed
  - Weighted: each edge has an associated weight, given by a weight function $w : E \rightarrow \mathrm{R}$
  - Dense: $|E| \approx |V|^2$
  - Sparse: $|E| << |V|^2$
- $|E| = O(|V|^2)$

Graphs

- If $(u, v) \in E$, then vertex v is adjacent to vertex u
- Adjacency relationship is:
  - Symmetric if G is undirected
  - Not necessarily so if G is directed
- If G is connected:
  - There is a path between every pair of vertices
  - $|E| \geq |V| - 1$
  - Furthermore, if $|E| = |V| - 1$, then G is a tree
- Other definitions in Appendix B (B.4 and B.5) as needed
  - On pages 1080-1093

Eulerian and Hamiltonian Graphs

- A graph containing paths that include every 'edge' exactly once and end at the initial vertex is called an Eulerian graph
- A graph containing paths that include every 'vertex' exactly once and end at the initial vertex is called a Hamiltonian graph



Hamiltonian but not Eulerian

Trees and Planar Graphs

- A connected graph with only one path between each pair of vertices is called a <u>tree</u>
- A tree can also be defined as a connected graph containing no cycles (figure 1)
- A graph that can be redrawn without crossings is called a <u>planar graph</u> (figures 2 and 3)

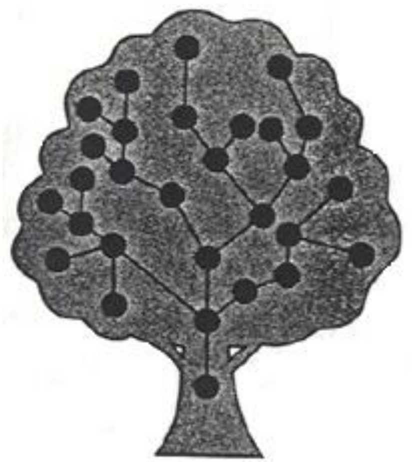


Fig. 1

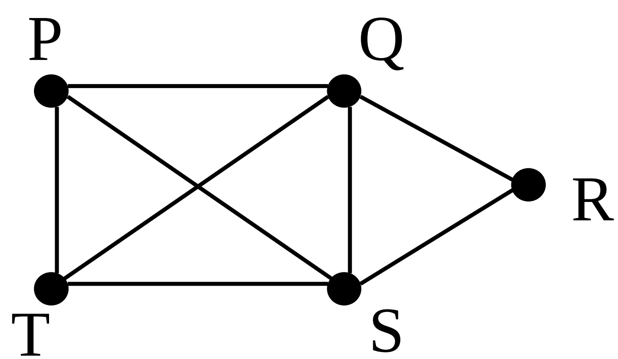


Fig. 2

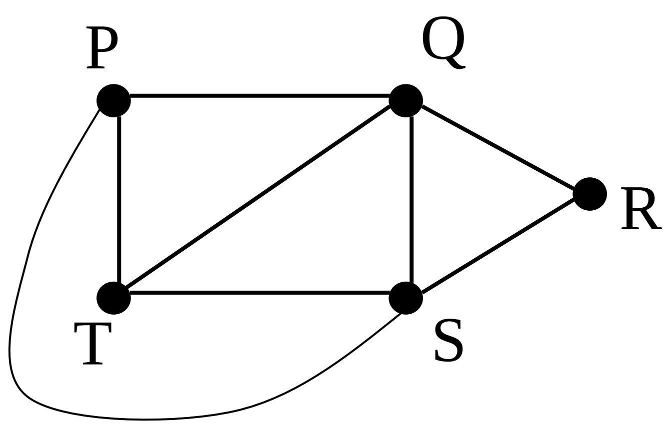


Fig. 3

Other Definitions

- Two digraphs are <u>isomorphic</u> if there is an isomorphism between their underlying graphs that preserves the ordering of the vertices in each arc
  - See figures 4 and 5 : not isomorphic
  - See figures 6 and 7 : isomorphic

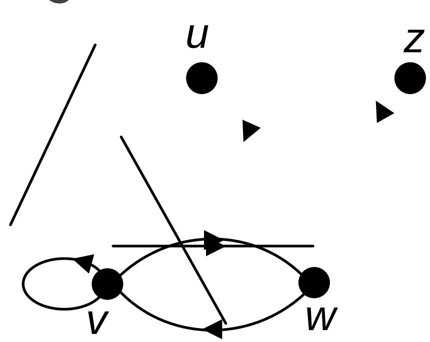


Fig. 4

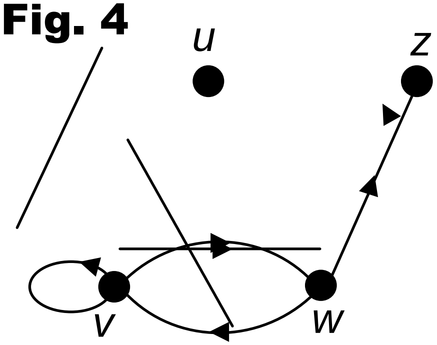


Fig. 5

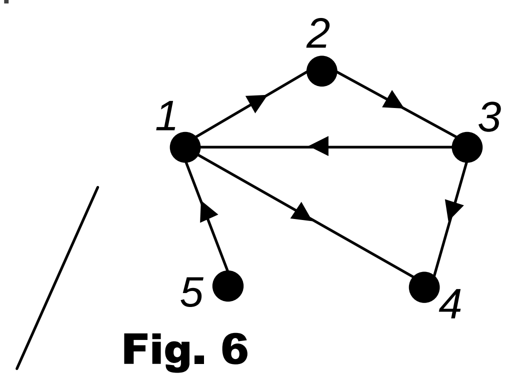


Fig. 6

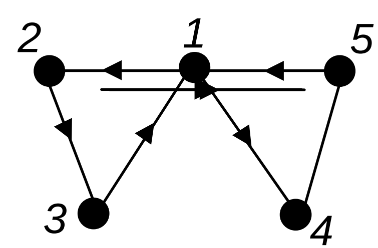
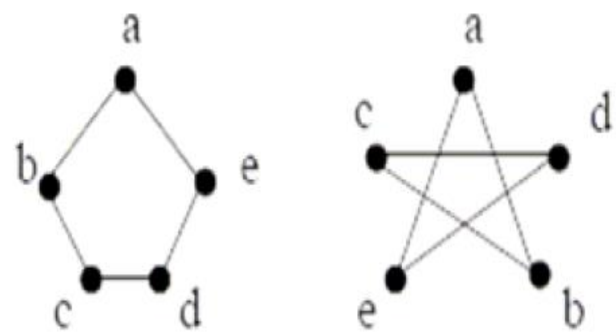


Fig. 7

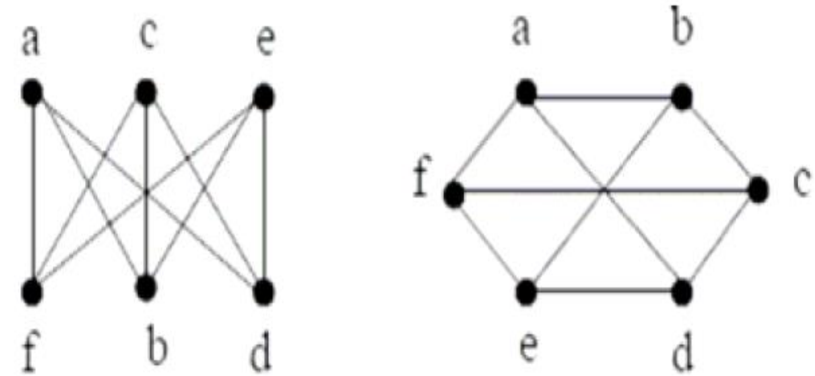
Practice Problems

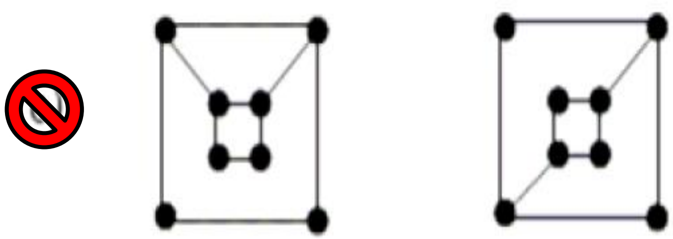
- Which of the following pairs are isomorphic graphs:

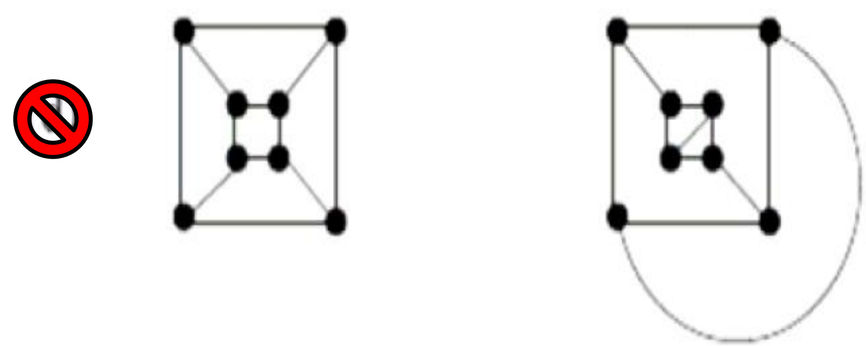
(a)



(b)







Connected and Strongly Connected

- A digraph D is <u>connected</u> if it cannot be expressed as the union of two digraphs
  - This is equivalent to saying that the underlying graph of D is a connected graph
- D is <u>strongly connected</u> if, for any two vertices v and w of D, there is a path from v to w
- Every strongly connected digraph is connected, but not all connected digraphs are strongly connected
  - See figure 4

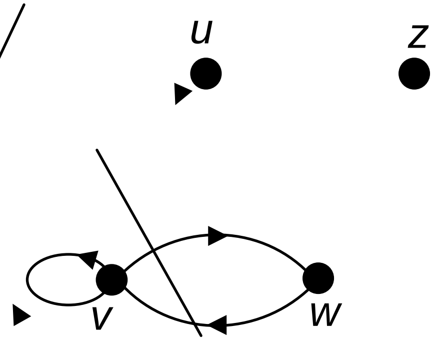


Fig. 4

Connected and Strongly Connected

- We define a graph *G* to be orientable if each edge of *G* can be directed so that the resulting digraph is strongly connected
  - See figures 8 and 9
- Any Eulerian graph is orientable

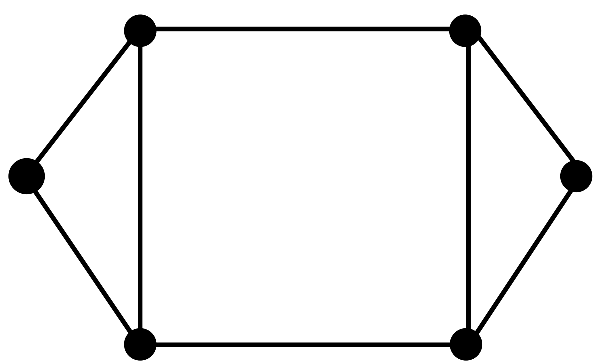


Fig. 8

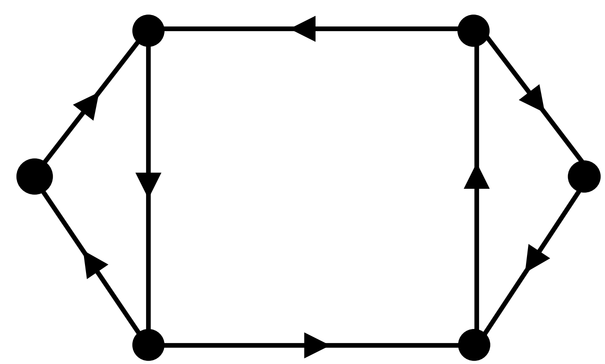
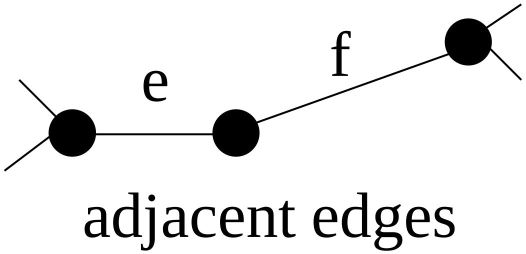


Fig. 9

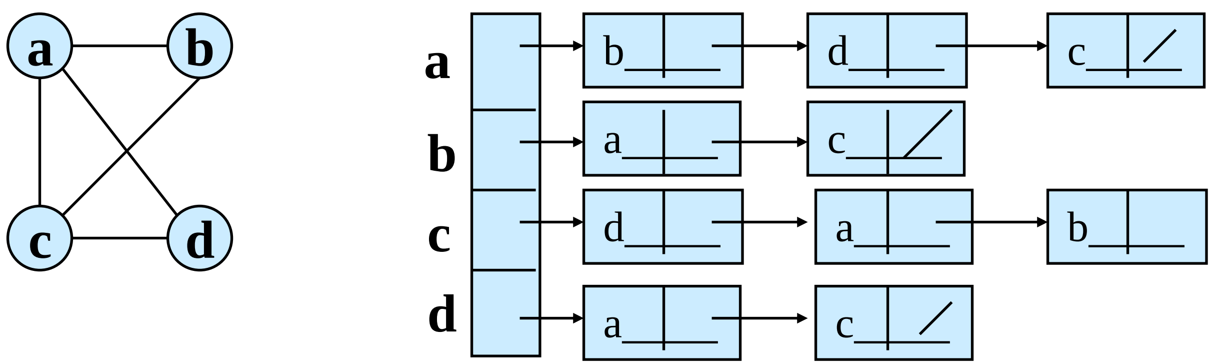
Adjacency

- Two vertices v and w are adjacent if there is an edge (v,w) joining them, and the vertices v and w are then incident with such an edge
- Two distinct edges e and f are adjacent if they have a vertex in common

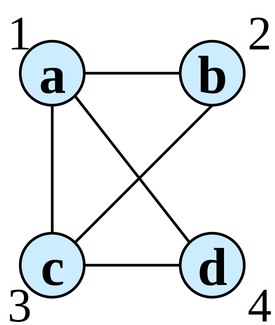


Representation of Graphs

- Two standard ways
  - Adjacency lists



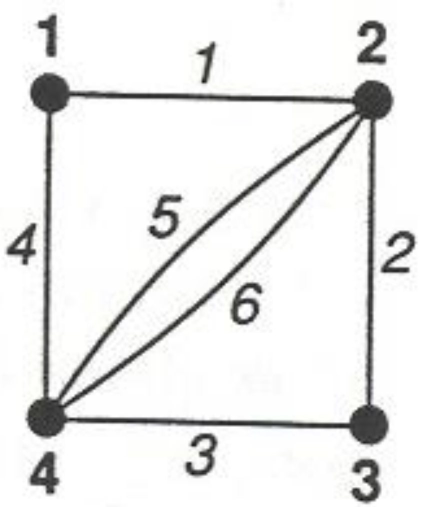
  - Adjacency matrix



	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

Representation of Graphs

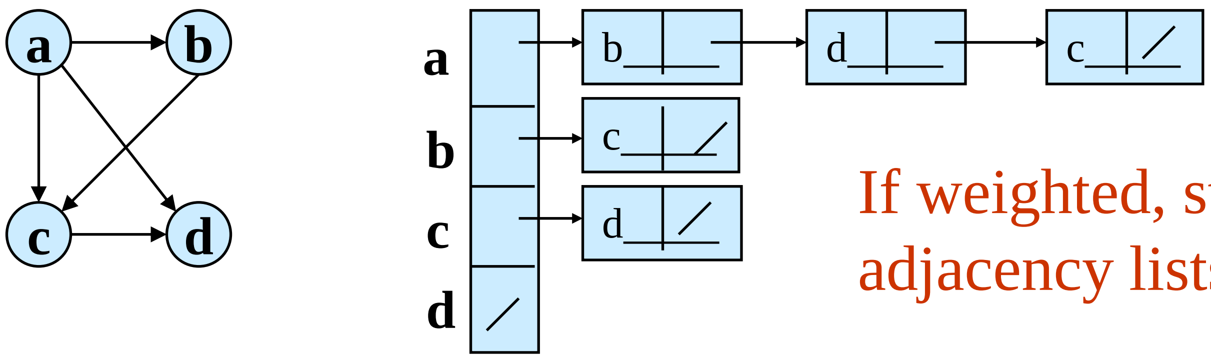
- Adjacency matrix A
- Incidence matrix M



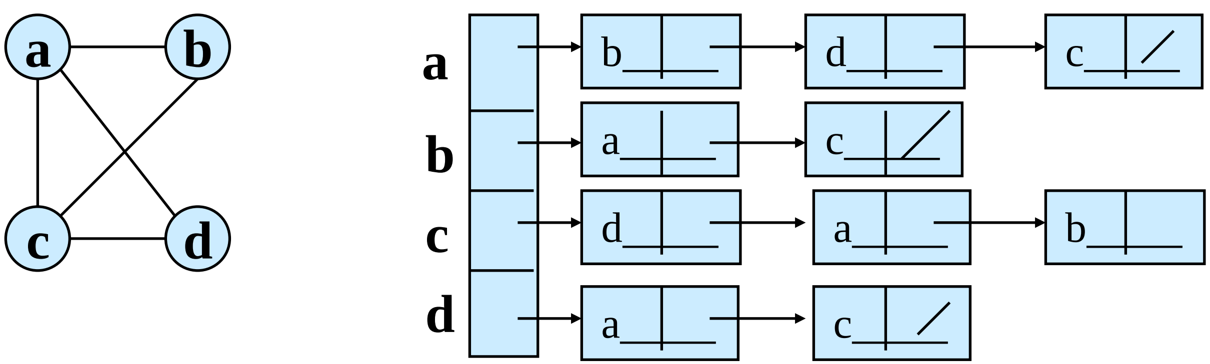
$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Adjacency Lists

- Consists of an array *Adj* of $|V|$ lists
- One list per vertex
- For $u \in V$, $Adj[u]$ consists of all vertices adjacent to u

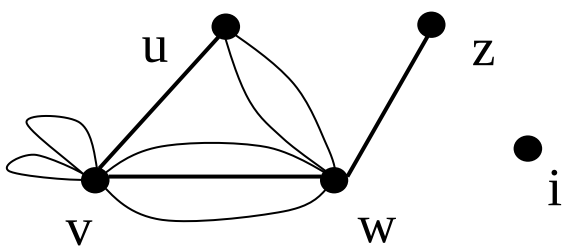


If weighted, store weights also in adjacency lists



Degree

- The degree of a vertex v is the number of edges incident with v, and is written deg(v)
- A loop at v contributes 2 to the degree of v
- A vertex of degree 0 is an isolated vertex
- A vertex of degree 1 is an end-vertex
- Remember the handshaking lemma and its corollary



Handshaking Lemma

- Handshaking Lemma
  - If several people shake hands, then the total number of hands shaken must be even
  - In any graph the sum of all the vertex degrees is an even number
    - in fact, twice the number of edges
- Corollary
  - In any graph the number of vertices of odd degree is even

Handshaking Dilemma

- The out-degree of a vertex v of G is the number of arcs of the form (v,w), and is denoted by outdeg(v)
- The in-degree of a vertex v of G is the number of arcs of the form (w,v), and is denoted by indeg(v)
- The sum of the out-degrees of all the vertices of G is equal to the sum of their in-degrees
- We call this result the <u>handshaking dilemma</u>
- A <u>source</u> of G is a vertex with in-degree 0
- A <u>sink</u> of G is a vertex with out-degree 0

Storage Requirement

- For directed graphs
  - Sum of lengths of all adj. lists is

    $$\sum_{v \in V} \mathrm{outdeg}(v) = |E|$$

  - Total storage: $\Theta(V+E)$
- For undirected graphs
  - Sum of lengths of all adj. lists is

    $$\sum_{v \in V} \deg(v) = 2|E|$$

  - Total storage: $\Theta(V+E)$

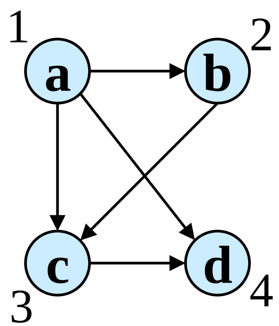
Pros and Cons: Adj List

- Pros
  - Space-efficient, when a graph is sparse
  - Can be modified to support many graph variants
- Cons
  - Determining if an edge $(u,v) \in E$ is not efficient
    - Have to search in u's adjacency list, $\Theta(\deg(u))$ time
    - $\Theta(V)$ in the worst case

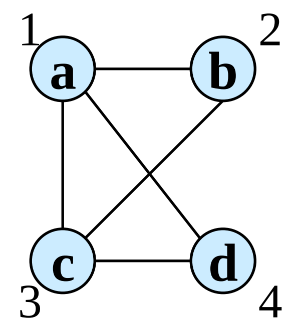
Adjacency Matrix

- $|V| \times |V|$ matrix A
- Number vertices from 1 to $|V|$ in some arbitrary manner
- A is then given by:

$$A[i, j] = a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$



	1	2	3	4
1	0	1	1	1
2	0	0	1	0
3	0	0	0	1
4	0	0	0	0



	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

$A = A^{\mathrm{T}}$ for undirected graphs

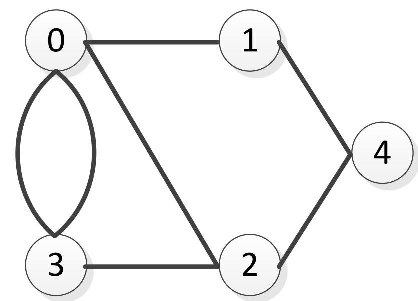
Space and Time

- Space: $\Theta(V^2)$
  - Not memory efficient for large graphs
- Time: to list all vertices adjacent to u
  - $\Theta(V)$
- Time: to determine if $(u, v) \in E$
  - $\Theta(1)$
- Can store weights instead of bits for weighted graph

Practice Problems

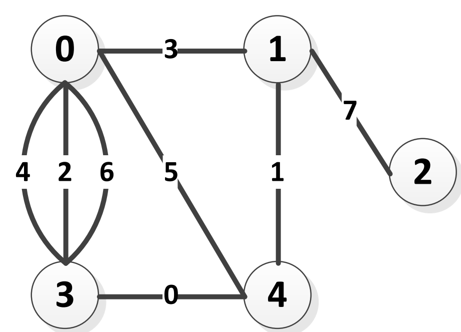
- Draw the graph whose adjacency matrix is given as follows:

$$\begin{pmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$



- Draw the graph whose incidence matrix is given as follows:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



Graph-Searching Algorithms

- Searching a graph
  - Systematically follow the edges of a graph to visit the vertices of the graph
- Used to discover the structure of a graph
- Standard graph-searching algorithms
  - Breadth-First Search (BFS)
  - Depth-First Search (DFS)

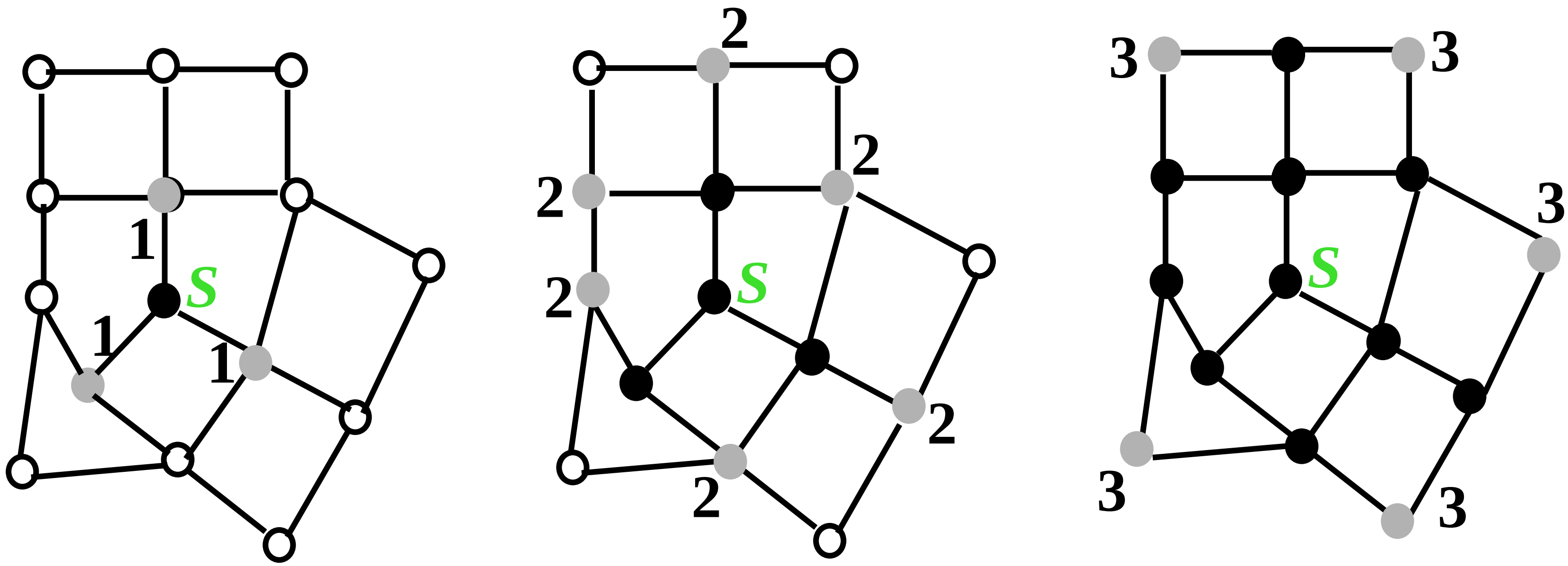
Breadth-First Search

- Input
  - Graph $G = (V, E)$, either directed or undirected, and *source vertex* $s \in V$
- Output
  - $d[v]$ = distance (smallest number of edges, or shortest path) from s to v,
    for all $v \in V$
  - $d[v] = \infty$ if v is not reachable from s
  - $\pi[v] = u$ such that (u, v) is last edge on shortest path $s \leadsto v$
    - u is v's predecessor
  - Builds Breadth-First Tree with root s that contains all reachable vertices

Breadth-First Search

- Expands the frontier between discovered and undiscovered vertices uniformly across the breadth of the frontier
  - A vertex is "discovered" the first time it is encountered during the search
  - A vertex is "finished" if all vertices adjacent to it have been discovered
- Colors the vertices to keep track of progress
  - White – Undiscovered
  - Gray – Discovered but not finished
  - Black – Finished
    - Colors are required only to reason about the algorithm. Can be implemented without colors.

Breadth-First Search



● **Finished** ● **Discovered** ○ **Undiscovered**

Pseudo Code

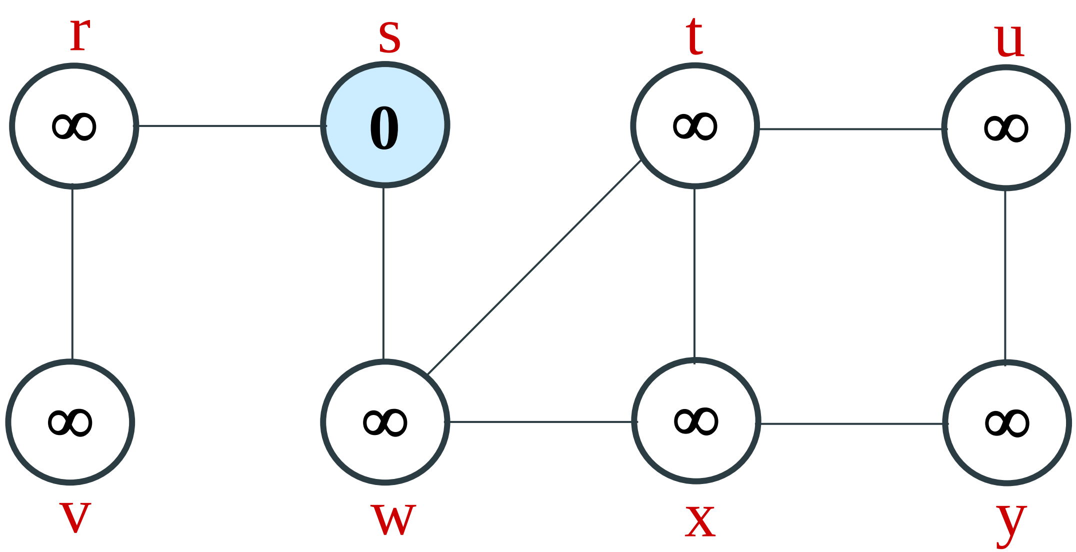
<u>BFS(G,s)</u>

```
for each vertex u in V[G] – {s}
  do color[u] ← white
       d[u] ← ∝
       π[u] ← nil
color[s] ← gray
d[s] ← 0
 π[s] ← nil
Q ← Φ
enqueue(Q,s)
while Q ≠ Φ
  do u ← dequeue(Q)
       for each v in Adj[u]
            do if color[v] = white
                 then color[v] ← gray
                      d[v] ← d[u] + 1
                      π[v] ← u
                      enqueue(Q,v)
       color[u] ← black
```

white: undiscovered
gray: discovered
black: finished

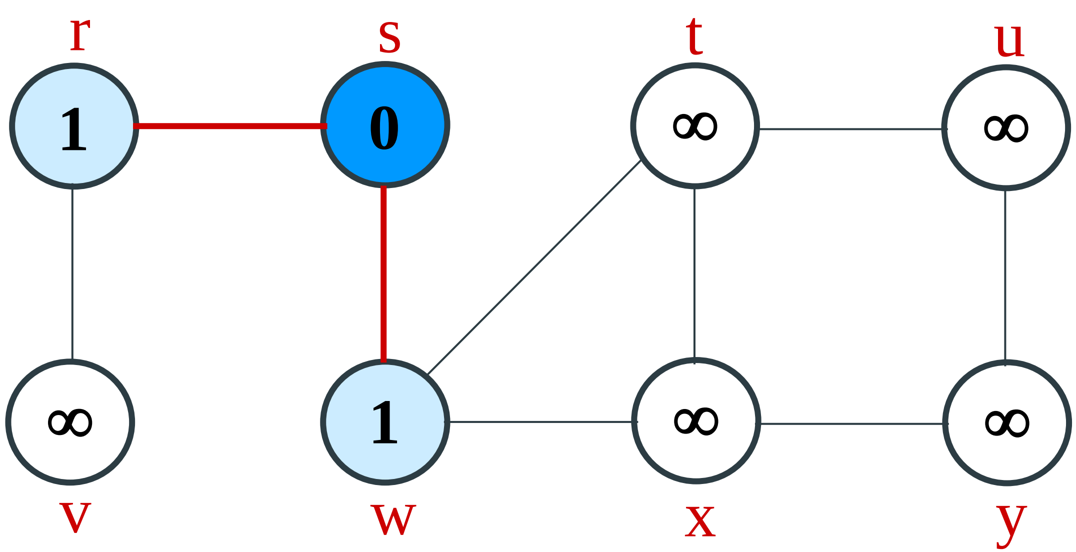
Q: a queue of discovered vertices
color[v]: color of v
d[v]: distance from s to v
π[u]: predecessor of v

BFS Example



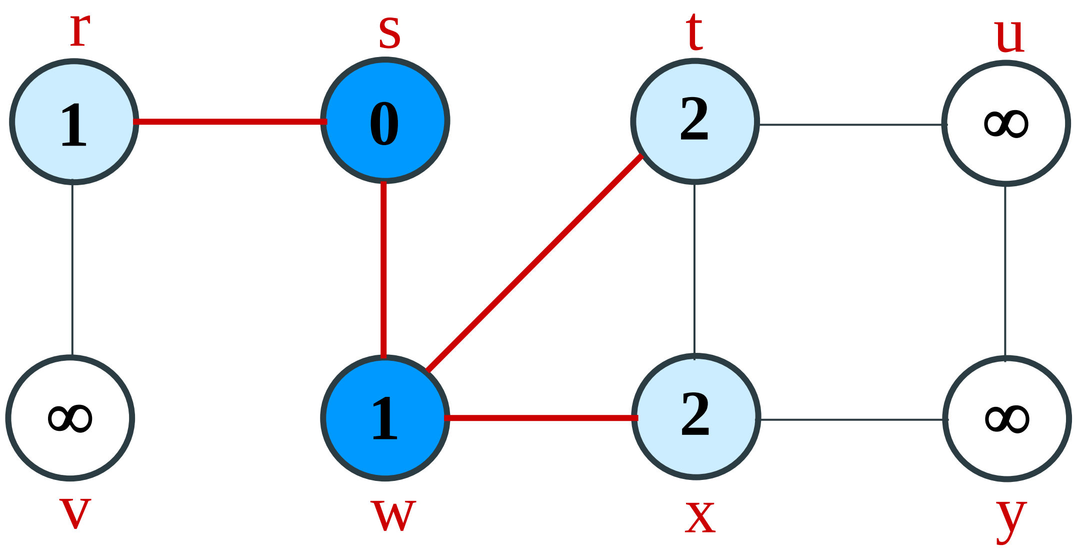
Q: s
0

BFS Example



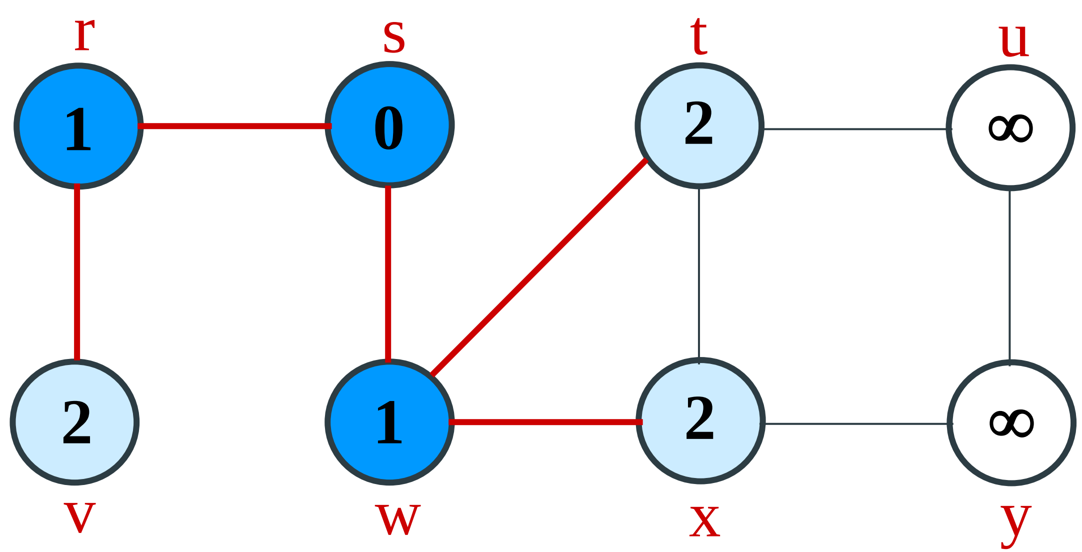
Q: w r
1 1

BFS Example



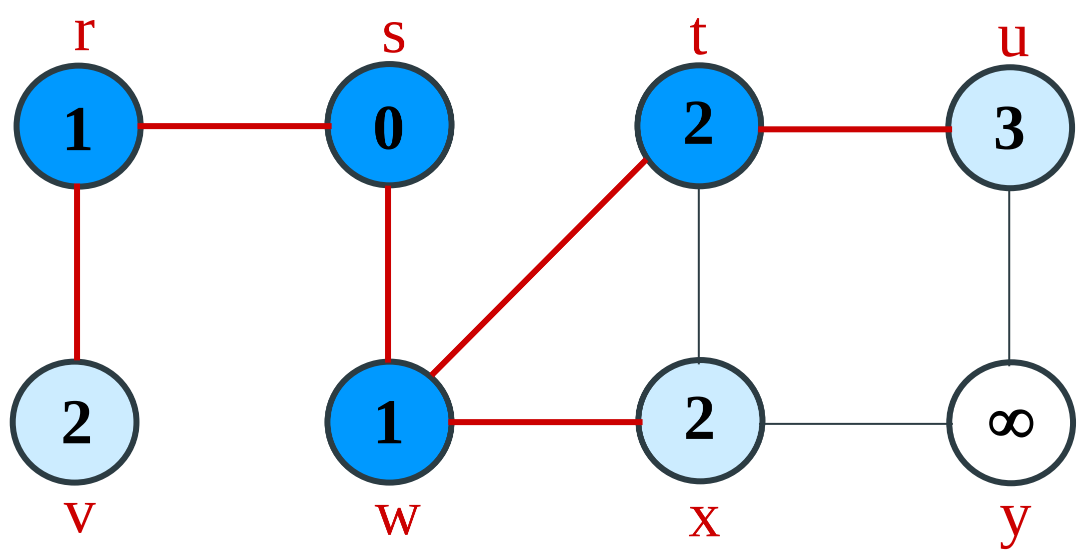
Q:	r	t	x
	1	2	2

BFS Example



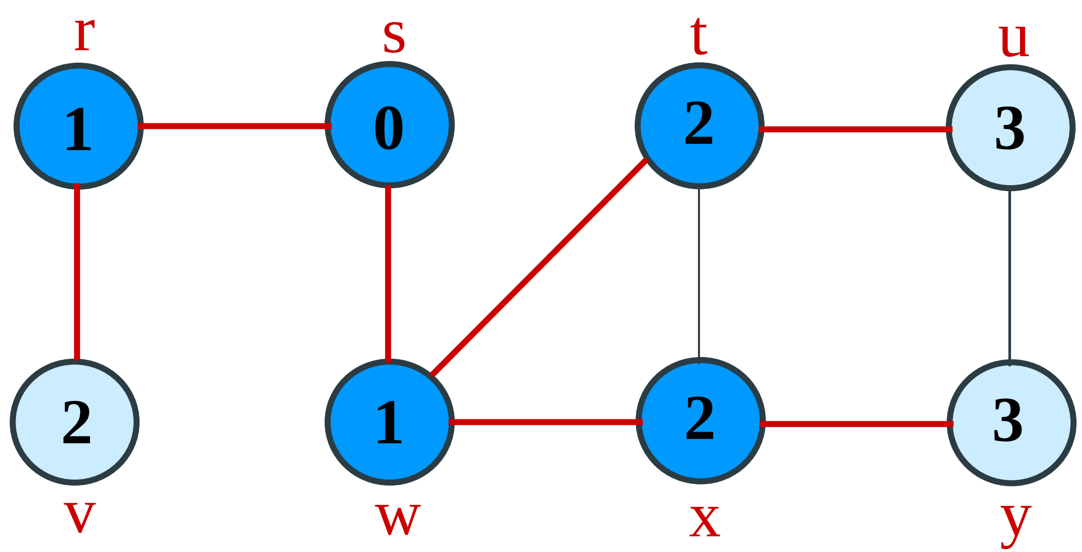
Q: t x v
2 2 2

BFS Example



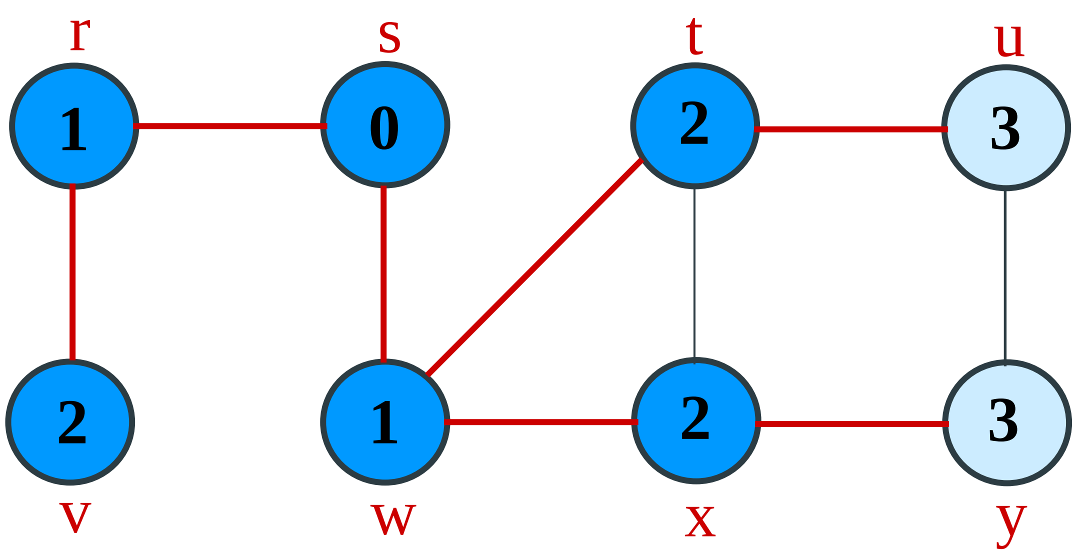
Q:	x	v	u
	2	2	3

BFS Example



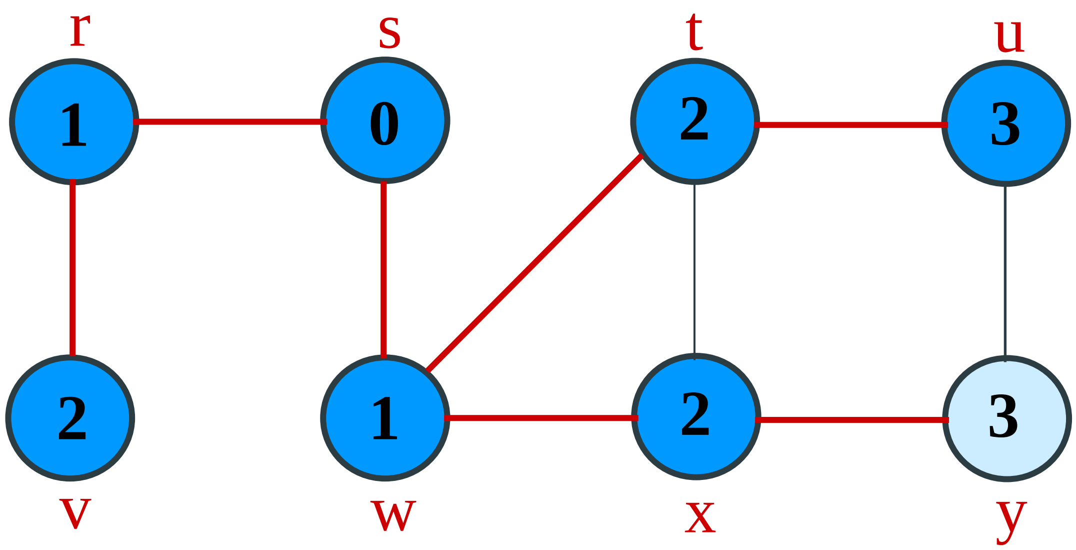
Q:	v	u	y
	2	3	3

BFS Example



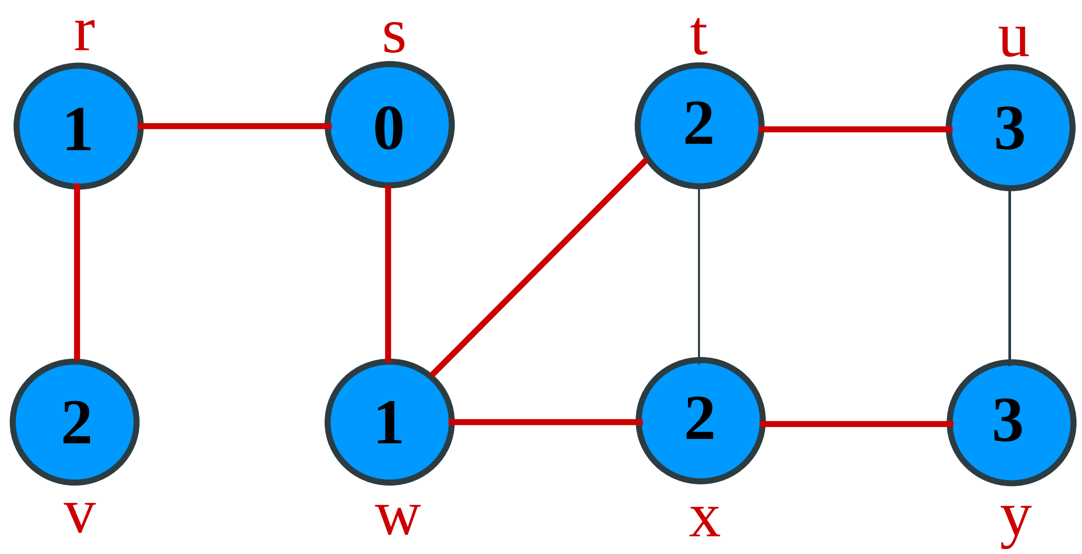
Q:	u	y
	3	3

BFS Example



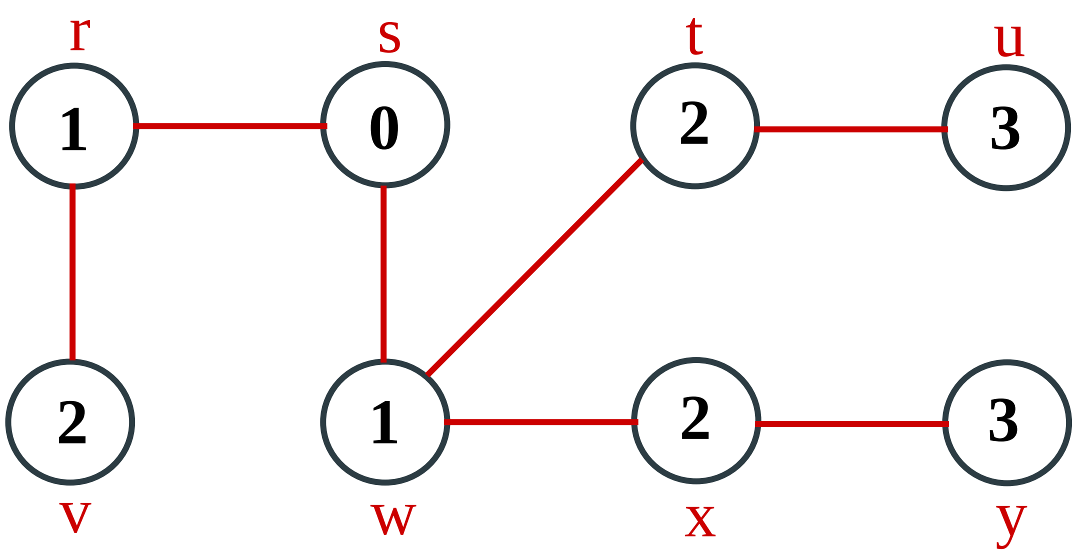
Q: y
3

BFS Example



Q: $\varnothing$

BFS Example



BF Tree

Analysis of BFS

- Initialization takes $O(V)$
- Traversal Loop
  - After initialization, each vertex is enqueued and dequeued at most once, and each operation takes $O(1)$. So, total time for queuing is $O(V)$
  - The adjacency list of each vertex is scanned at most once. The sum of lengths of all adjacency lists is $\Theta(E)$
- Summing up over all vertices
  - total running time of BFS is $O(V+E)$, linear in the size of the adjacency list representation of graph

Breadth-First Trees

- For a graph $G = (V, E)$ with source s, the predecessor subgraph of G is $G_\pi = (V_\pi, E_\pi)$ where
  - $V_\pi = \{v \in V : \pi[v] \neq \text{NIL}\} \cup \{s\}$
  - $E_\pi = \{(\pi[v], v) \in E : v \in V_\pi - \{s\}\}$
- The predecessor subgraph G_π is a breadth-first tree if:
  - V_π consists of the vertices reachable from s and
  - for all $v \in V_\pi$, there is a unique simple path from s to v in G_π that is also a shortest path from s to v in G.
- The edges in E_π are called tree edges
  - $|E_\pi| = |V_\pi| - 1$

Depth-First Search

- Explore edges out of the most recently discovered vertex *v*
- When all edges of *v* have been explored, backtrack to explore other edges leaving the vertex from which *v* was discovered (its *predecessor*)
- "Search as deep as possible first"
- Continue until all vertices reachable from the original source are discovered
- If any undiscovered vertices remain, then one of them is chosen as a new source and search is repeated from that source

Depth-First Search

- Input
  - $G = (V, E)$, directed or undirected. No source vertex given!
- Output
  - 2 timestamps on each vertex. Integers between 1 and $2|V|$.
    - $d[v]$ = *discovery time* (v turns from white to gray)
    - $f[v]$ = *finishing time* (v turns from gray to black)
  - $\pi[v]$: predecessor of $v = u$, such that v was discovered during the scan of u's adjacency list.
- Uses the same coloring scheme for vertices as BFS
  - White – Undiscovered
  - Gray – Discovered but not finished
  - Black – Finished

Pseudo Code

DFS(*G*)

```
for each vertex u ∈ V[G]
      do color[u] ← white
            π[u] ← NIL
time ← 0
for each vertex u ∈ V[G]
      do if color[u] = white
               then DFS-Visit(u)
```

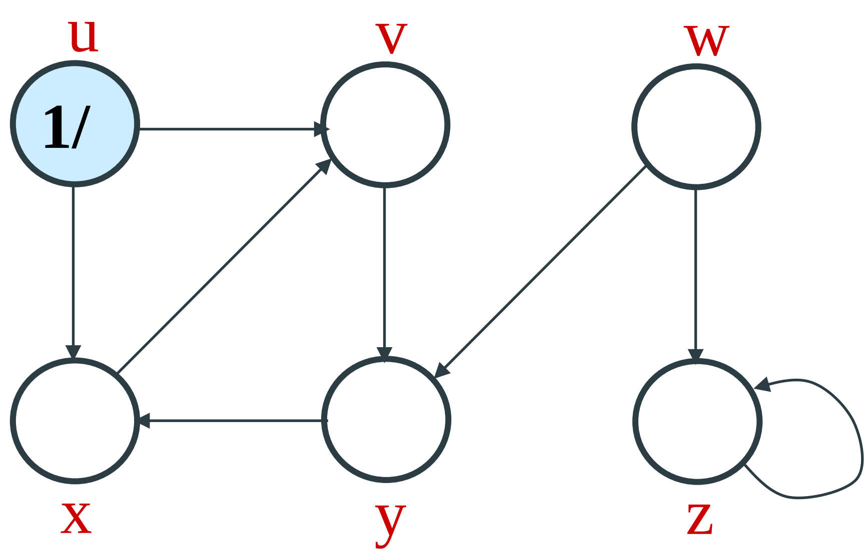
Uses a global timestamp ***time***

DFS-Visit(*u*)

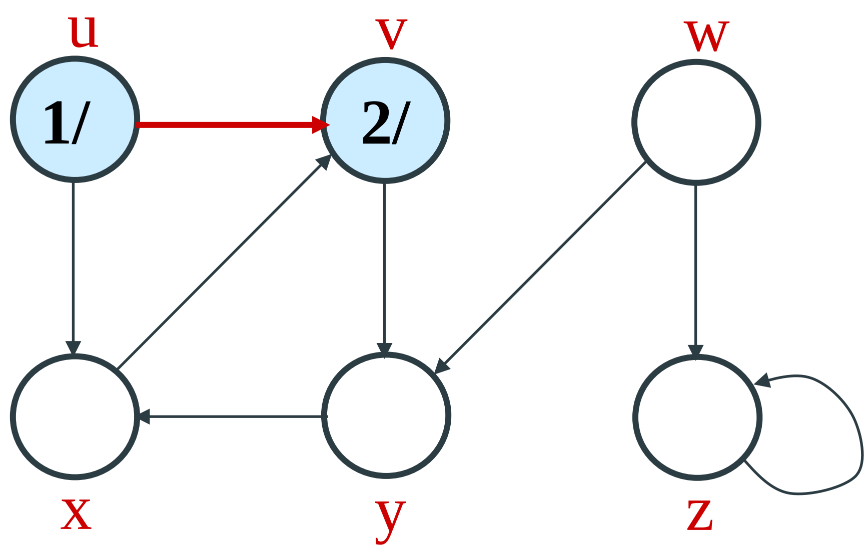
```
color[u] ← GRAY
  ∇ White vertex u has been discovered
time ← time + 1
d[u] ← time
for each v ∈ Adj[u]
      do if color[v] = WHITE
               then π[v] ← u
                    DFS-Visit(v)
color[u] ← BLACK
∇ Blacken u;  it is finished.
f[u] ← time ← time + 1
```

white: undiscovered
gray: discovered
black: finished

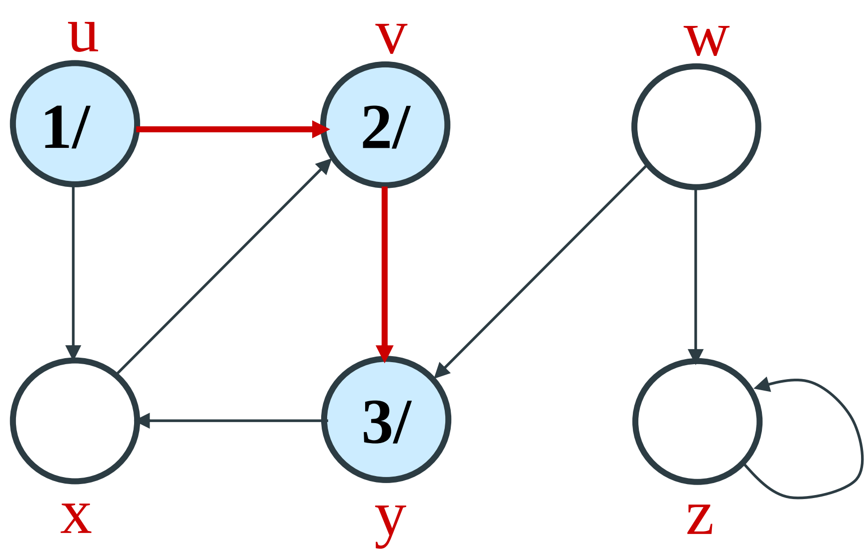
DFS Example



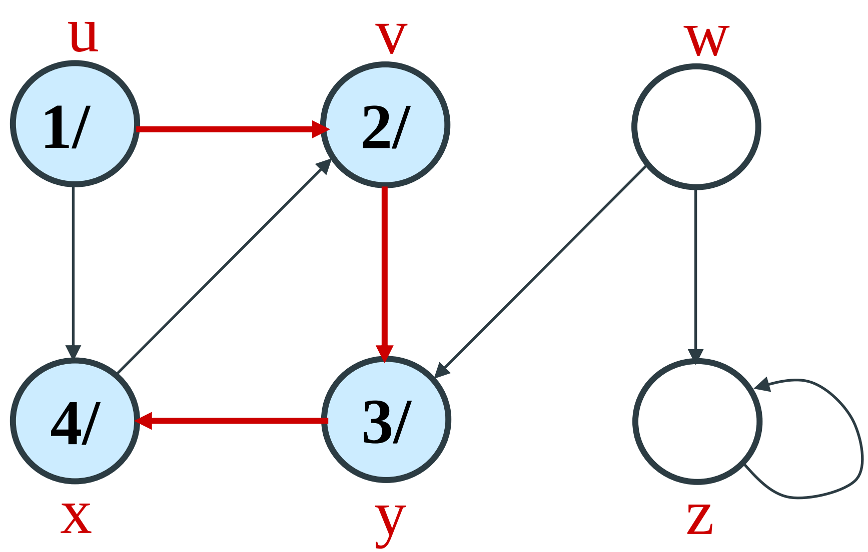
DFS Example



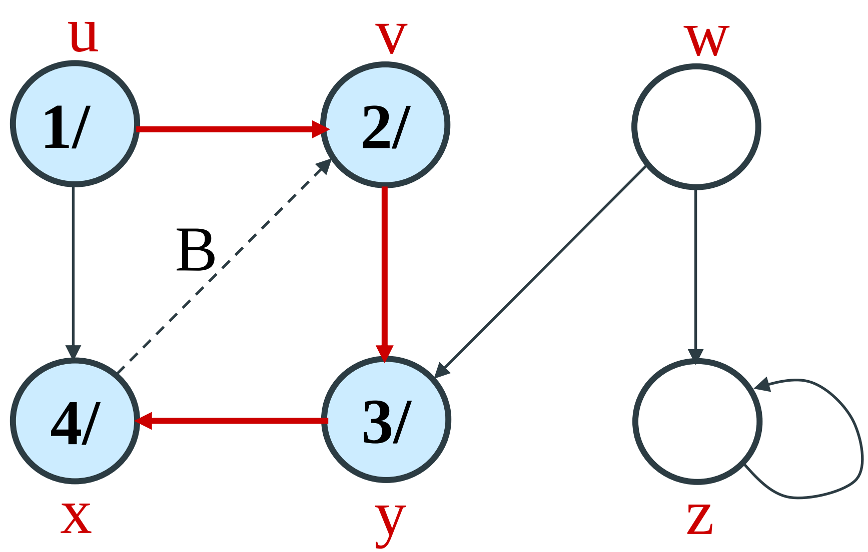
DFS Example



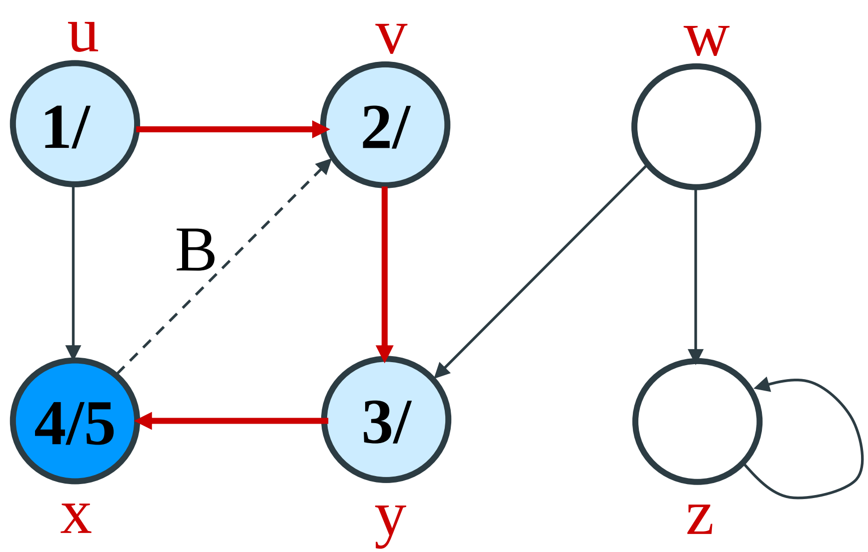
DFS Example



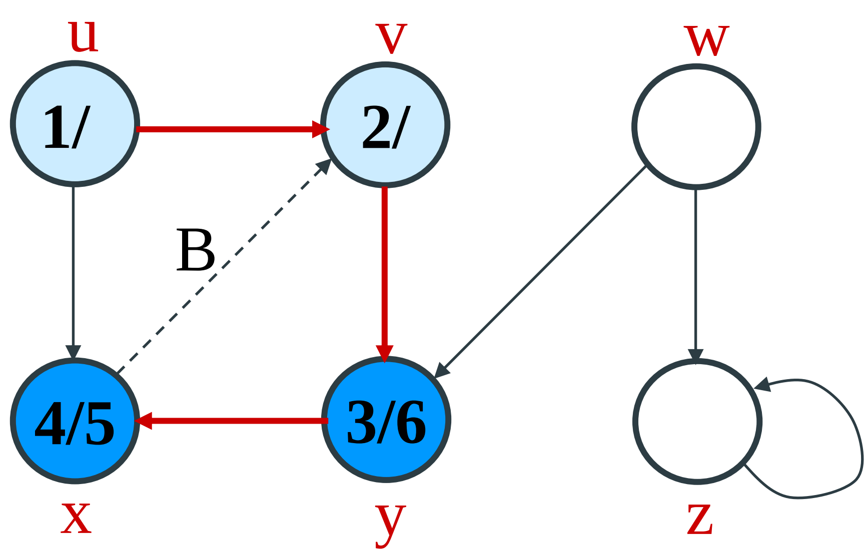
DFS Example



DFS Example



DFS Example



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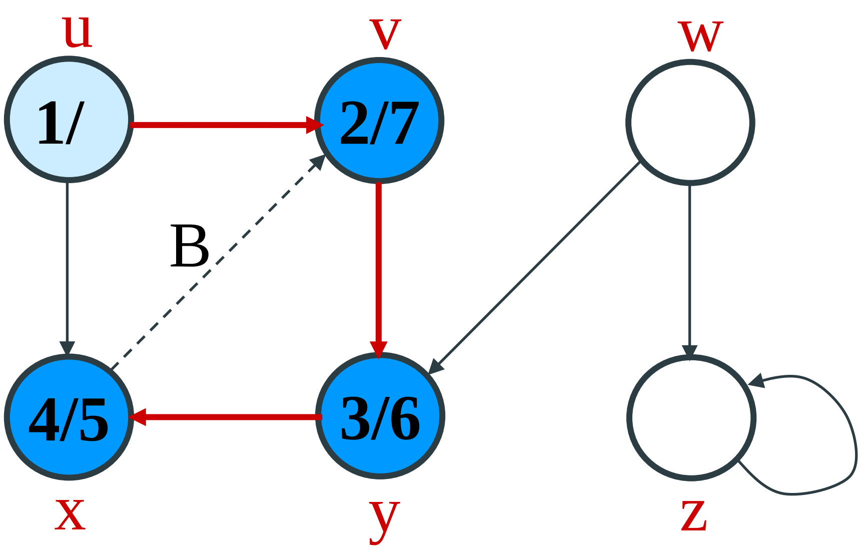
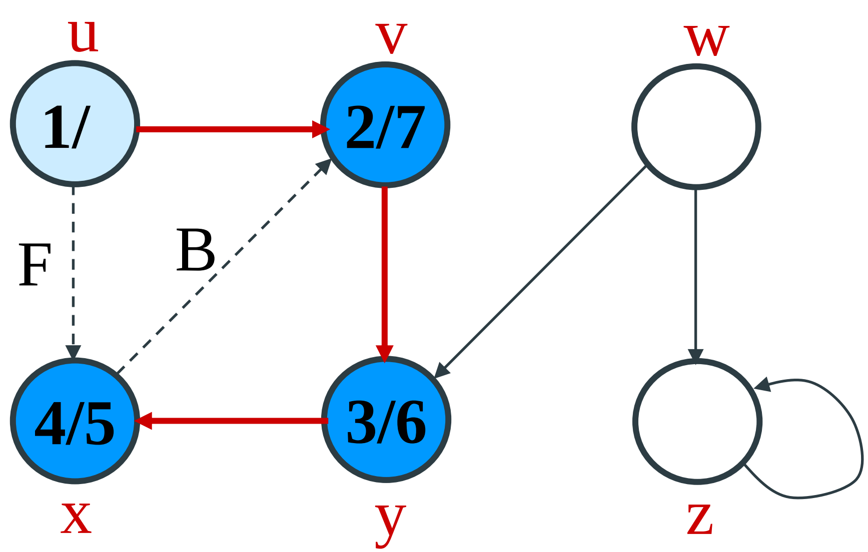
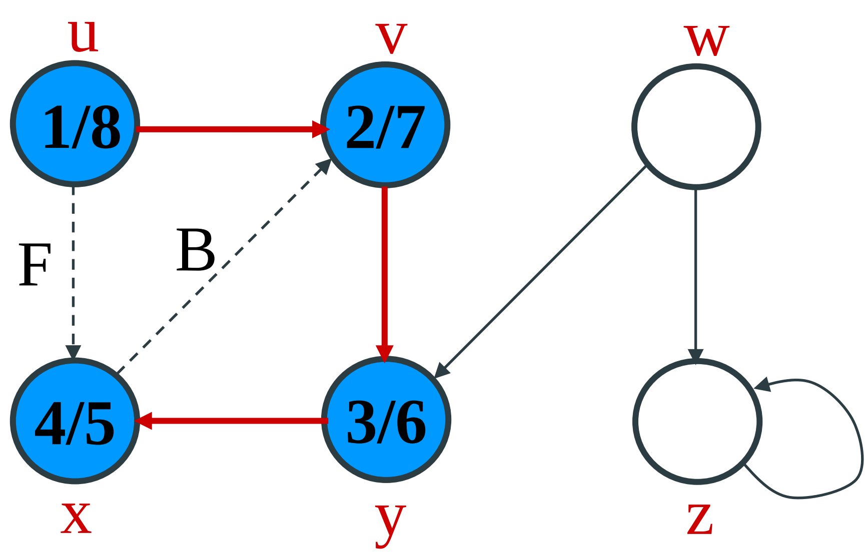


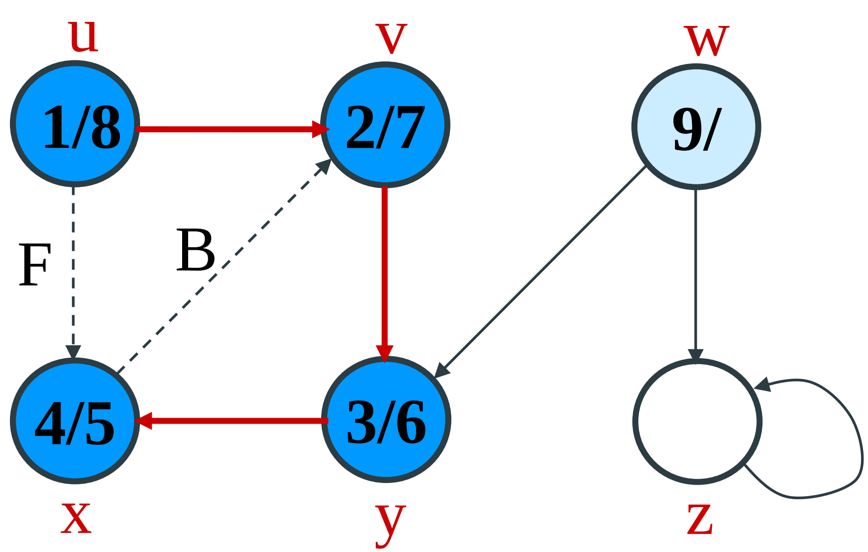
DFS Example



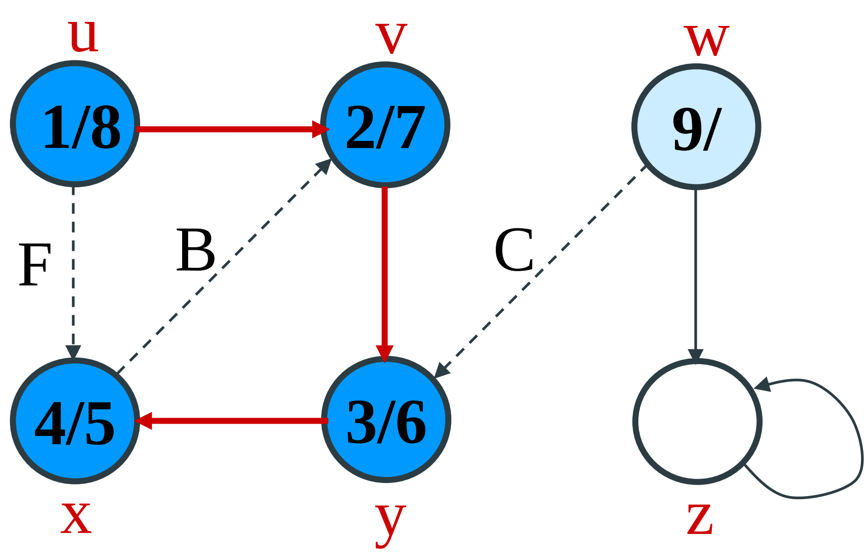
DFS Example



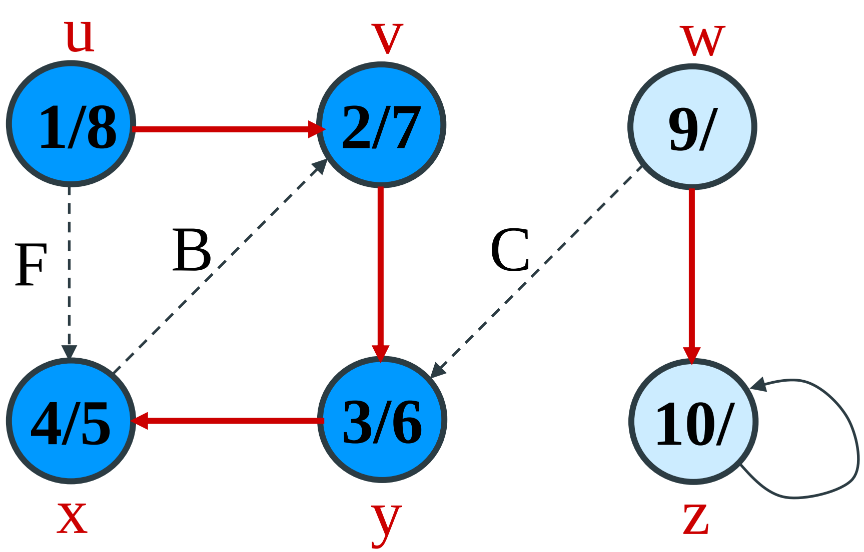
DFS Example



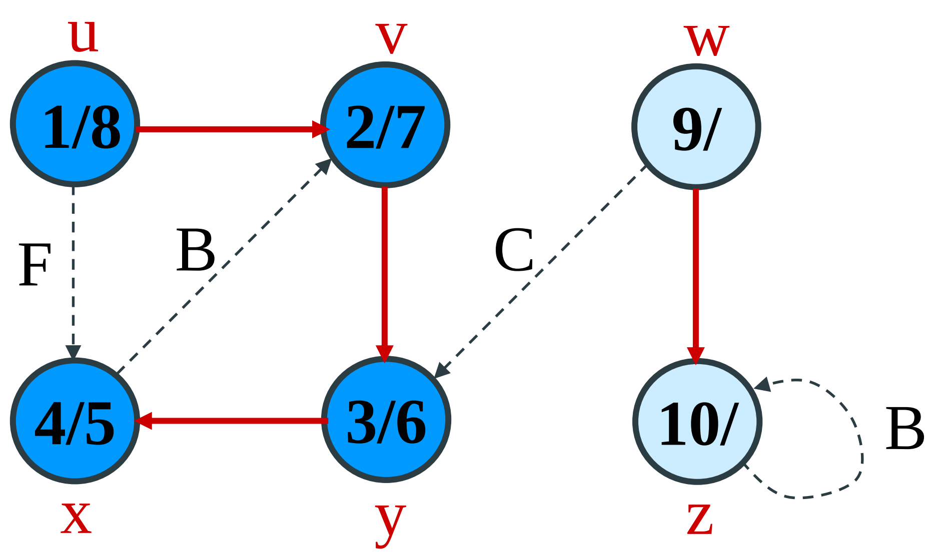
DFS Example



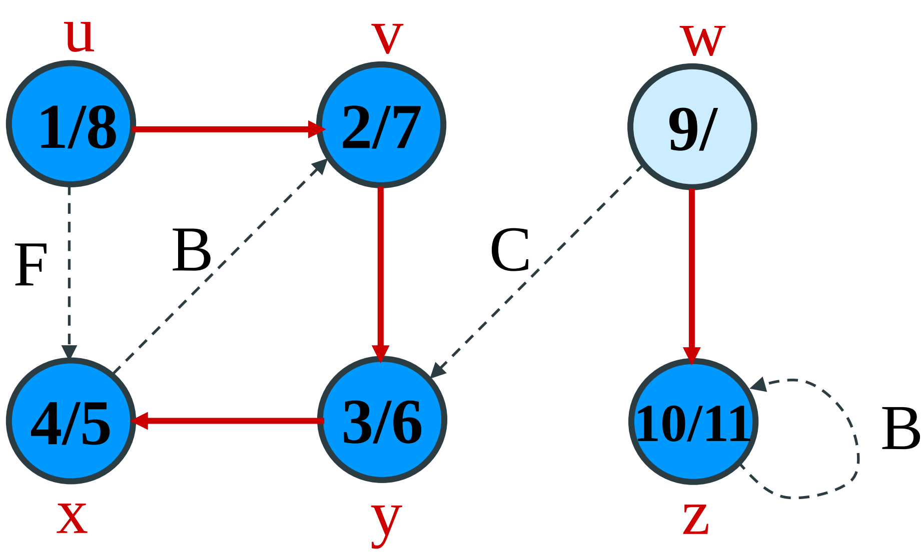
DFS Example



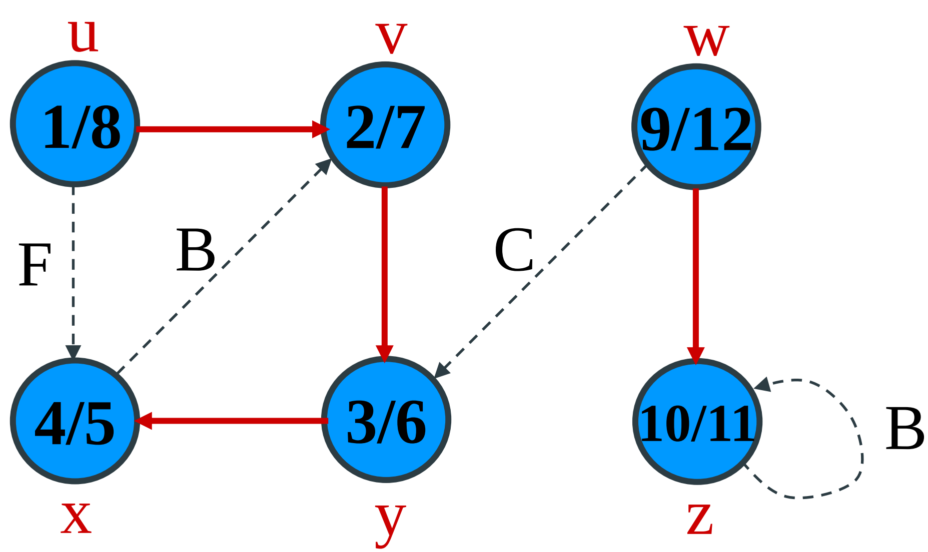
DFS Example



DFS Example



DFS Example



Analysis of DFS

- Loops on lines 1-2 & 5-7 take $\Theta(V)$ time, excluding time to execute DFS-Visit

- DFS-Visit is called once for each white vertex $v \in V$ when it's painted gray the first time. Lines 3-6 of DFS-Visit is executed $|\mathrm{Adj}[v]|$ times. The total cost of executing DFS-Visit is $\sum_{v \in V} |\mathrm{Adj}[v]| = \Theta(E)$

- Total running time of DFS is $\Theta(V+E)$

Depth-First Trees

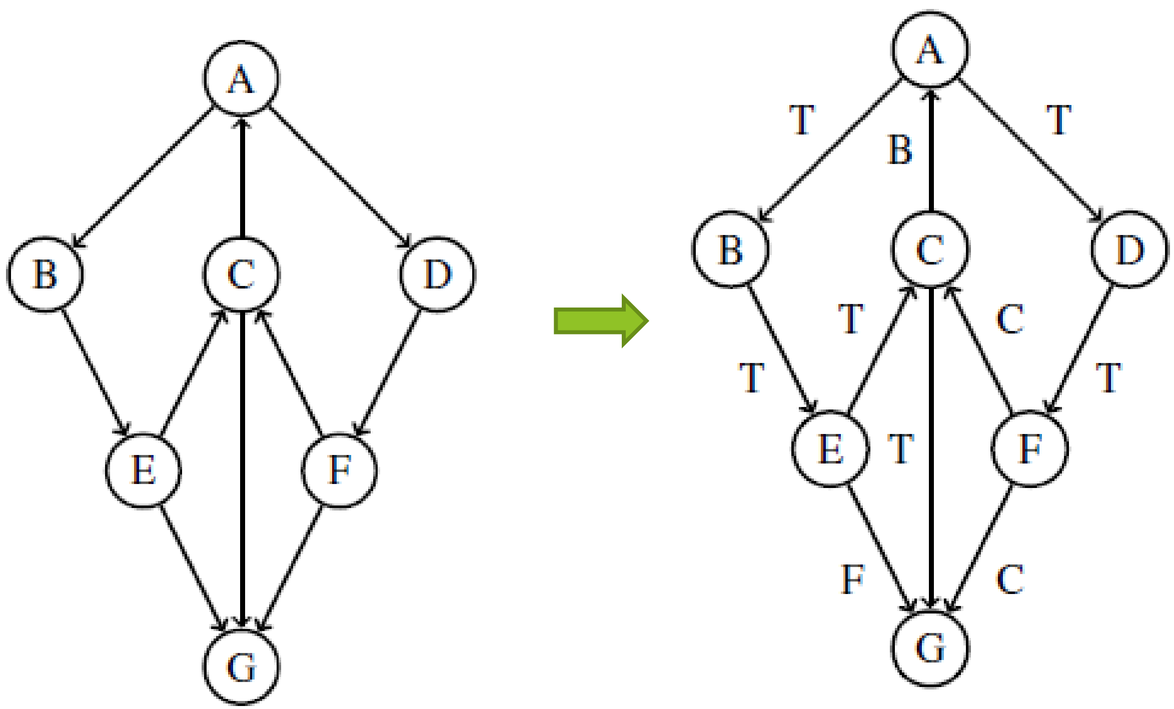
- Predecessor subgraph defined slightly different from that of BFS
- The predecessor subgraph of DFS is $G_\pi = (V, E_\pi)$ where $E_\pi = \{(\pi[v], v) : v \in V \text{ and } \pi[v] \neq \text{NIL}\}$.
  - How does it differ from that of BFS?
  - The predecessor subgraph G_π forms a *depth-first forest* composed of several *depth-first trees*. The edges in E_π are called *tree edges*.

Classification of Edges

- Tree edge: in the depth-first forest.
  Found by exploring (u, v)
- Back edge: (u, v), where u is a descendant of v
  (in the depth-first tree)
- Forward edge: (u, v), where v is a descendant of u, but not a tree edge
- Cross edge: any other edge
  Can go between vertices in same depth-first tree or in different depth-first trees

Practice Problems

- Perform a depth-first search on the following graph starting at A. Break all ties by picking the vertices in alphabetical order (i.e A before Z).

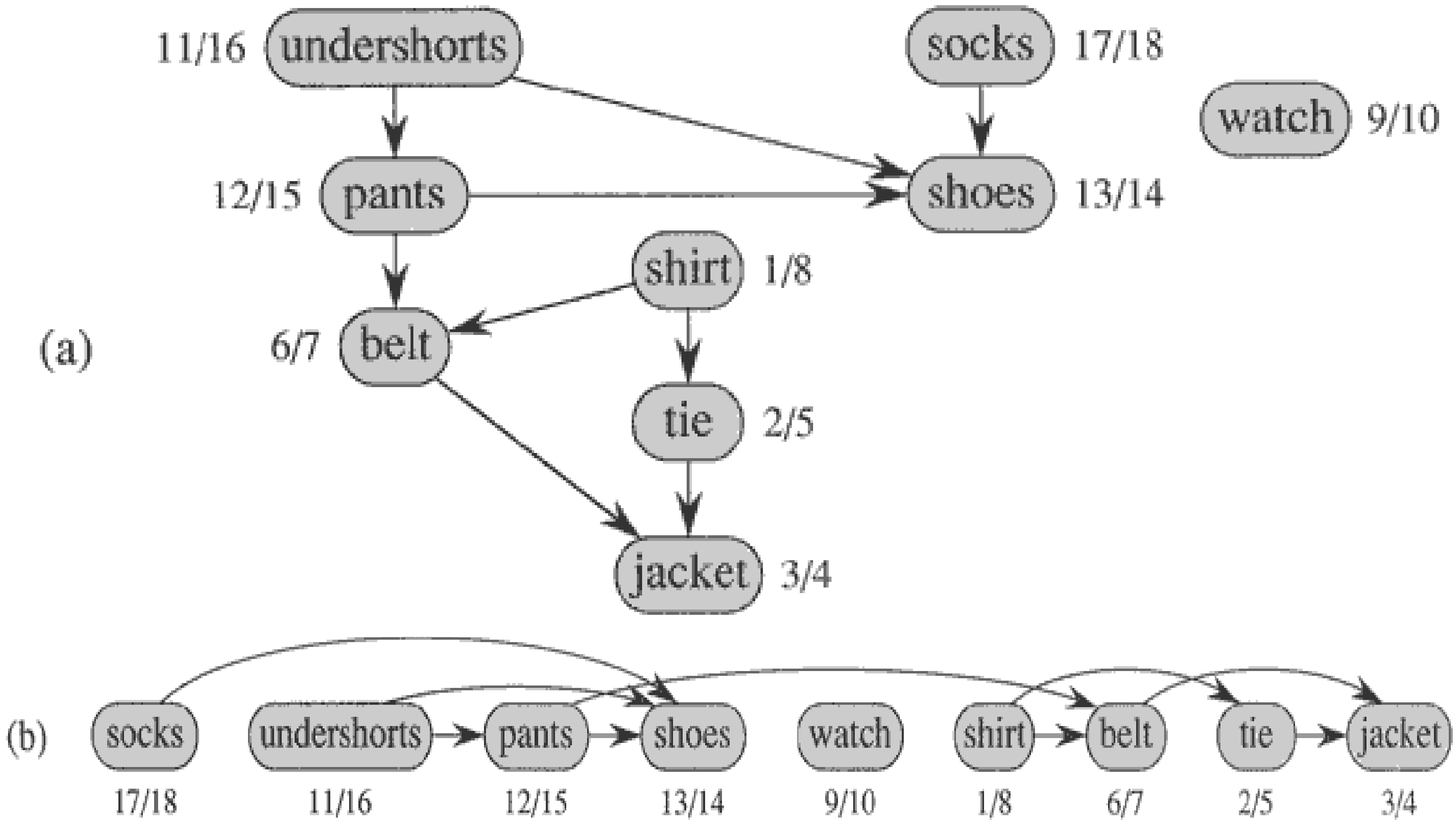


Topological Sort

- A topological sort of a Directed Acyclic Graph (DAG) is a linear order of all its vertices s.t. if G contains an edge (u, v), then u appears before v in the ordering
  - If the graph is not acyclic, then no linear ordering is possible.
  - A topological sort can be viewed as an ordering of its vertices along a horizontal line so that all directed edges go from left to right
- TOPOLOGICAL-SORT(G) $\Theta(V+E)$

```
call DFS(G) to compute finishing times f[v] for each vertex v
as each vertex is finished, insert it onto the front of a linked list
return the linked list of vertices
```

Topological Sort



Thanks to contributors

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Prof. Hyunseung Choo (2001 – 2022)

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