Reminder TD_9

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- **Objective of the review:**
  1. Introduction to instrumental variables
  2. Overview of imperfect compliance in randomized experiments

(Warning: In this reminder we will look at the case of non-compliance. This is a sub-case of instrumental variables. These will be dealt with in more general terms in section 10).

1 Presentation of instrumental variables

We saw in the previous tutorials that, in order to estimate a causal effect using OLS estimators, it is necessary to be sure of the absence of selection bias (i.e. $Cov(D, Y(0)) = 0$, in other words D $\perp\!\!\!\perp$ (Y(0),Y(1)). But this is rarely the case in practice. Thus the theoretical regression of Y on D often fails to identify a causal effect.

Another approach to causal inference is the use of **instrumental variables**.

An instrument Z is valid if

- **relevance**: it has an effect on the explanatory variable (in our case treatment D)
  $Cov(Z, D) \neq 0$
- **exogeneity**: it has no direct effect on the potential outcome Y(d)
  $Cov(Z, Y(d)) = 0$

In other words, we need to find an instrument that has an effect on the potential outcome solely through its effect on treatment D.

2 Randomised trials: imperfect compliance framework

In randomized experiments, individuals are randomly selected to participate or not in the treatment. In practice, individuals may be assigned to the treatment but not receive it (for example, they are offered a treatment but ultimately decide not to take part) or they may be assigned to the control group but manage to receive the treatment all the same. This is a treatment with 'imperfect compliance': the people assigned to the treatment are not necessarily the individuals treated in the end. In this context, we need to distinguish between the random variable of treatment assignment Z (=1 if initially assigned to the treatment group, 0 otherwise), and the variable of treatment reception D (=1 if receives the treatment, 0 otherwise) (the latter is no longer randomly assigned and is correlated with the potential outcomes $Cov(D, Y(0)) \neq 0$).

We instrument the treatment by the assignment to the treatment. $Cov(D, Y(0)) \neq 0$ prevents us from obtaining a causal effect by a standard regression of Y on D (see Chapter 4). We therefore need to find a substitution

solution: instrumenting by Z. Assignment to treatment Z has an effect on the potential outcome only *via* its effect on the treatment. In other words, assignment to the treatment affects the treatment D but not directly the potential outcome Y(d). We therefore have both relevance and exogeneity.

2.1 Observed and potential variables

Observed variables

- Y = outcome
- D $\in \{0, 1\}$ = treatment actually received (allocated non-randomly. *Ex: in the case of job training D=0 are rather less motivated individuals who decide not to be treated*)
- Z $\in \{0, 1\}$ = initial treatment assignment (randomly assigned)

Potential variables

- Outcome : $Y(D) = DY(1) + (1 - D)Y(0)$
- Treatment : $D(Z) = ZD(1) + (1 - Z)D(0)$

There are 4 possible cases:

- D(1) = 1 means that the treatment has been allocated to the individual and that they have actually received it
- D(1) = 0 means that the treatment has been allocated to the individual but they have not actually received it
- D(0) = 1 means that the treatment has not been allocated to the individual but that they have actually received it
- D(0) = 0 means that the treatment has not been allocated to the individual and that they have not actually received it

2.2 Population partioning

Following the 4 possible cases described above, we can partition the population in the form of a table:

Table 1: Population breakdown

	D(1) = 0	D(1) = 1
D(0) = 0	**Never taker** (NT)	**Complier** (C)
D(0) = 1	**Defiers** (D)	**Always taker** (AT)

These four populations $\{\{NT\}, \{D\}, \{AT\}, \{C\}\}$ form a complete system of events.

2.3 Assumptions

In the case of non-compliance, it is difficult to estimate a causal effect since we are not in a classic case of random treatment. So, in order to estimate a causal effect, we need to make two assumptions:

- **Monotonicity** : D(1) $\geq$ D(0)
  $\rightarrow$ With this assumption, there are no defiers in the population (because for them D(0) = 1 > D(1) = 0). The treated are therefore either compliers who have been assigned to the treatment, or always takers.
- **Independency** : Z $\perp\!\!\!\perp$ (Y(0), Y(1), D(0), D(1))
  $\rightarrow$ This hypothesis is fairly credible because assignment to the treatment and control groups is random. (*Z is an instrumental variable! It only has an effect on Y through its effect on D, and* $cov(Z, \epsilon) = 0$)

2.4 Treatment effect estimator

Under the hypotheses of independence and monotonicity, we can estimate the **effect of the treatment on the compliers** δ^C (in a conventional framework, we estimate the causal effect on all the treated δ^T).

$$\begin{aligned}
\delta^C &= E(Y(1) - Y(0)|\overbrace{C}^{compliers}) \\
&= E(Y(1) - Y(0)|(C, Z = 1)) \quad \textit{(Under independency assumption)} \\
&= E(Y(1) - Y(0)|D(1) - D(0) = 1) \\
&= \frac{E(Y|Z = 1) - E(Y|Z = 0)}{E(D|Z = 1) - E(D|Z = 0)}
\end{aligned}$$

Note: the denominator is equal to the probability of being a complier.

We define δ^C as the **local average treatment effect** (LATE) because we are estimating the effect of the treatment on the sub-population of compliers. This is because the compliers comply with the assignment (unlike the NT who systematically refuse the treatment and the AT who systematically accept the treatment). In TD 9, for example, we look more closely at who these compliers are (i.e. what their characteristics are), since we want to know the population for which we have succeeded in estimating a causal effect.

The estimator is the following:

$$\hat{\delta}^C = \frac{\bar{Y}_1 - \bar{Y}_0}{\bar{D}_1 - \bar{D}_0}$$

with $\bar{Y}_z = \frac{1}{n_z} \sum_{i=1,i:Z_i=z}^{n} Y_i$ and $\bar{D}_z = \frac{1}{n_z} \sum_{i=1,i:Z_i=z}^{n} D_i$ and $n_z = \text{card}(\{i : Z_i = z\})$

- **Special case:** $\delta^T = \delta^C$

There is a case for which estimating the causal effect of the compliers (δ^C) is equivalent to estimating the causal effect on all the treaties (δ^T). When $E(D|Z = 1) > E(D|Z = 0) = 0$, then, as well as there being no defiers, there are also no 'always takers' (*P(AT)=0*). In other words, Never Takers and Compliers form a complete system of events. In this case, $\delta^T = \delta^C$. (Which is logical, since without always takers and defiers, only compliers are processed!)