15.3.

Change the cartesian integral into polar integral.

1. $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\,dx$

$= \int_0^{\pi}\int_0^{1} r\,dr\,d\theta$

$= \frac{1}{2}\int_0^{\pi} d\theta$

$= \frac{\pi}{2}$

3. $\int_0^1\int_0^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy$

$= \int_0^{\frac{\pi}{2}}\int_0^1 r^3\,dr\,d\theta$

$= \frac{1}{4}\int_0^{\frac{\pi}{2}} d\theta$

$= \frac{\pi}{8}$

5. $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$

$= \int_0^{2\pi}\int_0^{a} r\,dr\,d\theta$

$= \frac{a^2}{2}\int_0^{2\pi} d\theta$

$= \pi a^2$

13. $\int_0^2\int_0^{\sqrt{1-(x-1)^2}} \frac{x+y}{x^2+y^2}\,dy\,dx$

$= \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} \frac{r(\cos\theta+\sin\theta)}{r^2}\,r\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} (2\cos^2\theta + 2\sin\theta\cos\theta)\,d\theta.$

$= \left[\theta + \frac{\sin 2\theta}{2} + \sin^2\theta\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} + 1$

18. Find the area of the region the lies insides the cardioid $r = 1+\cos\theta$ and outside the circle $r=1$.

$A = 2\int_0^{\frac{\pi}{2}}\int_1^{1+\cos\theta} r\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} (2\cos\theta + \cos^2\theta)\,d\theta$

$= \frac{8+\pi}{4}$

19. Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$.

$A = 2\int_0^{\frac{\pi}{6}}\int_0^{12\cos 3\theta} r\,dr\,d\theta$

$= 144\int_0^{\frac{\pi}{6}} \cos^2 3\theta\,d\theta.$

$= 12\pi$

39. Integrate the function $f(x,y) = \frac{1}{(1-x^2-y^2)}$ over the disk $x^2+y^2 \le \frac{3}{4}$. Does the integral of $f(x,y)$ over the disk $x^2+y^2 \le 1$ exist?

① Over the disk $x^2+y^2 \le \frac{3}{4}$

$\iint_R \frac{1}{1-x^2-y^2}\,dA$

$= \int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}} \frac{r}{1-r^2}\,dr\,d\theta$

$= \int_0^{2\pi} (-\frac{1}{2}\ln\frac{1}{4})\,d\theta$

$= \pi\ln 4.$

② Over the disk $x^2+y^2 \le 1$

$\iint_R \frac{1}{1-x^2-y^2}\,dA$

$= \int_0^{2\pi}\int_0^1 \frac{r}{1-r^2}\,dr\,d\theta.$

$= \int_0^{2\pi} \lim_{a\to 1^-}\left[-\frac{1}{2}\ln(1-a^2)\right]d\theta.$

$= 2\pi\cdot\lim_{a\to 1^-}\left[-\frac{1}{2}\ln(1-a^2)\right]$

$= 2\pi\cdot\infty$

Does not exist ove $x^2+y^2 \le 1$

40. Use the double integral in polar coordinates to derive the formula. $A = \int_\alpha^\beta \frac{1}{2}r^2\,d\theta.$

$A = \int_\alpha^\beta\int_0^{f(\theta)} r\,dr\,d\theta$

$= \int_\alpha^\beta \left[\frac{r^2}{2}\right]_0^{f(\theta)}$

$= \frac{1}{2}\int_\alpha^\beta f^2(\theta)\,d\theta.$

$= \int_\alpha^\beta \frac{1}{2}r^2\,d\theta$ where $r = f(\theta)$

15.4

23. $V = \int_0^1 \int_{-1}^1 \int_0^{y^2} dz\,dy\,dx$

$= \int_0^1 \int_{-1}^1 y^2\,dy\,dx$

$= \frac{2}{3} \int_0^1 dx$

$= \frac{2}{3}$

27. $V = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-\frac{3y}{2}} dz\,dy\,dx$

$= \int_0^1 \int_0^{2-2x} (3-3x-\frac{3}{2}y)\,dy\,dx$

$= \int_0^1 3(1-x)^2\,dx$

$= 1$

29. $V = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\,dy\,dx$

$= 8\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy\,dx$

$= 8\int_0^1 (1-x^2)\,dx$

$= \frac{16}{3}$

48. what domain D in space maximises the value $\iiint_D (1-x^2-y^2-z^2)\,dV$?

$1-x^2-y^2-z^2 \geq 0$, which is a solid sphere of radius 1 centered at the origin

15.6

15. $\int_0^{\pi} \int_0^{2\sin\theta} \int_0^{4-r\sin\theta} f(r,\theta,z)\,dz\,r\,dr\,d\theta$

17. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^{1+\cos\theta} \int_0^4 f(r,\theta,z)\,dz\,r\,dr\,d\theta$

82.

Let the base radius of the cone be a and the height h, and place the cone's axis of symmetry along the z-axis with the vertex at the origin.

$M = \frac{\pi a^2 h}{3}$ and $M_{xy} = \int_0^{2\pi} \int_0^a \int_{(\frac{h}{a})r}^{h} z\,dz\,r\,dr\,d\theta$

$= \frac{3}{4}h$

and $\bar{x} = \bar{y} = 0$, by symmetry.

$\Downarrow$

the centroid is one fourth of the way from the base to the vertex

15.7.

6. $V = 8\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{2\sin\varphi} \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$.

$= \frac{64}{3} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin^4\varphi\,d\varphi\,d\theta$.

$= \frac{64}{3} \int_0^{\frac{\pi}{2}} \left[-\frac{\sin^3\varphi\cos\varphi}{4} \Big|_0^{\frac{\pi}{2}} + \frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2\varphi\,d\varphi \right] d\theta$

$= 16\int_0^{\frac{\pi}{2}} \left[\frac{\varphi}{2} - \frac{\sin 2\varphi}{4} \right]_0^{\frac{\pi}{2}} d\theta$

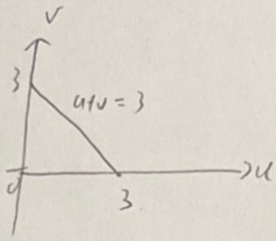
$= 4\pi \int_0^{\frac{\pi}{2}} d\theta$

$= 2\pi^2$.

15.7

1. Solve the system $u = x - y$, $v = 2x + y$., for x and y in terms of u and v, Then find the value of the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$

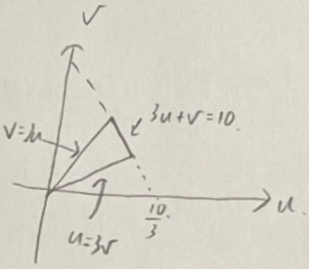
(a) $\begin{cases} u = x - y \\ v = 2x + y \end{cases} \Rightarrow \begin{cases} 3x = u + v \\ y = x - u. \end{cases} \Rightarrow \begin{cases} x = \frac{1}{3}(u+v) \\ y = \frac{1}{3}(-2u+v). \end{cases}$



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

3. $u = 3x + 2y$, $v = x + 4y$.

$\begin{cases} 3x + 2y = u \\ x + 4y = v \end{cases} \Rightarrow \begin{cases} -5x = -2u + v \\ y = \frac{1}{2}(u - 3x) \end{cases} \Rightarrow \begin{cases} x = \frac{2u - v}{5} \\ y = \frac{3v - u}{10} \end{cases}$



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{vmatrix} = \frac{6}{50} - \frac{1}{50} = \frac{1}{10}$$

6. $\iint_R (2x^2 - xy - y^2)\,dx\,dy$. $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$, and $y = x + 1$.

$$\iint_R (2x^2 - xy - y^2)\,dx\,dy.$$
$$= \iint_R (x-y)(2x+y)\,dx\,dy$$
$$= \iint_G uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$
$$= \frac{1}{3}\iint_G uv\,du\,dv.$$

$$\Rightarrow \frac{1}{3}\iint_G uv\,du\,dv$$
$$= \frac{1}{3}\int_{-1}^{2}\int_{4}^{7} uv\,dv\,du$$
$$= \frac{1}{3}\int_{-1}^{2} u\left[\frac{v^2}{2}\right]_4^7 du.$$
$$= \frac{11}{2}\int_{-1}^{2} u\,du$$
$$= \frac{33}{4}$$

7. $\iint_R (3x^2 + 14xy + 8y^2)\,dx\,dy$. , $y = -\frac{3}{2}x + 1$, $y = -\frac{3}{2}x + 3$, $y = -\frac{1}{4}x$, and $y = -\frac{1}{4}x + 1$.

$$\iint_R (3x^2 + 14xy + 8y^2)\,dx\,dy$$
$$= \iint_R (3x + 2y)(x + 4y)\,dx\,dy$$
$$= \iint_G uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$
$$= \frac{1}{10}\iint_G uv\,du\,dv.$$

$$\Rightarrow \frac{1}{10}\iint_G uv\,du\,dv$$
$$= \frac{1}{10}\int_2^6\int_0^4 uv\,dv\,du$$
$$= \frac{1}{10}\int_2^6 u\left[\frac{v^2}{2}\right]_0^4 du$$
$$= (\frac{4}{5})\int_2^6 u\,du$$
$$= \frac{64}{5}$$

12. The area of an ellipse.

$$\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

$$A = \iint_R dy\,dx$$

$$= \iint_G ab\,du\,dv$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab\,dv\,du$$

$$= 2ab\int_{-1}^{1}\sqrt{1-u^2}\,du$$

$$= 2ab\left[\frac{u}{2}\sqrt{1-u^2}+\frac{1}{2}\sin^{-1}u\right]_{-1}^{1}$$

$$= ab\pi$$

18. Centroid of boomerang. $y^2=-4(x-1)$ and $y^2=-2(x-2)$ in the xy-plane.

$$M = \int_{-2}^{2}\int_{1-\frac{y^2}{4}}^{2-\frac{y^2}{2}} dx\,dy = \int_{-2}^{2}\left(1-\frac{y^2}{4}\right)dy = \frac{8}{3}$$

$$M_y = \int_{-2}^{2}\int_{1-\frac{y^2}{4}}^{2-\frac{y^2}{2}} x\,dx\,dy = \int_{-2}^{2}\left[\frac{x^2}{2}\right]_{1-\frac{y^2}{4}}^{2-\frac{y^2}{2}} = \int_{-2}^{2}\frac{3}{32}(4-y^2)dy = \frac{3}{32}\int_{-2}^{2}(16-8y^2+y^4)dy = \frac{48}{15}$$

$$\bar{x} = \frac{M_y}{M} = \frac{48}{15}\cdot\frac{3}{8} = \frac{6}{5}, \text{ and } \bar{y}=0 \text{ by symmetry.}$$