Section 5.3 Cramer's Rule, Inverses, Volumes

Today look at some applications of determinants. (The big application, eigenvalues, comes next chapter.)

First: Solving $n\times n$ linear systems

$$A\vec{x}=\vec{b} \xrightarrow{\text{arithmetic}} \text{Elimination algorithm}$$

$\downarrow$ algebra

Cramer's Rule (for solving $\vec{x}=A^{-1}\vec{b}$)

Idea: $$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} A\vec{x} & A\begin{bmatrix}0\\1\\ \vdots \\ 0\end{bmatrix} & \cdots & A\begin{bmatrix}0\\0\\ \vdots \\ 1\end{bmatrix}\end{bmatrix}$$

should equal $\vec{b}$

$$= \begin{bmatrix} | & | & & | \\ \vec{b} & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} \leftarrow$$ Call this matrix B_1 (you replace Col 1 of A with $\vec{b}$).

Now take det of both sides and use product rule:

$$(\det A)\underbrace{\begin{vmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & \cdots & 1 \end{vmatrix}}_{=x_1(\det \text{ of } (n-1)\times(n-1)\ I)=x_1} = \det B_1$$

So $x_1 = \dfrac{\det B_1}{\det A}$, <u>if</u> $\det A \neq 0$.

Same for $x_2, \ldots, x_n$:

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} 1 & & | & & 0 \\ & \ddots & \vec{x} & & \\ & & | & \ddots & \\ 0 & & | & & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | & & | \\ \vec{a}_1 & \cdots & \vec{b} & \cdots & \vec{a}_n \\ | & & | & & | \end{bmatrix}}_{B_i \text{ (replace Col } i \text{ of } A \text{ with } \vec{b}\text{)}.}$$

Put $\vec{x}$ in Col i

Expand across Row i:

$\det = x_i$

$$\boxed{x_i = \frac{\det B_i}{\det A}}$$

<u>Example</u>: Solve $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

We need to use <u>4</u> determinants:

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{vmatrix} = 2$$

$$\det B_1 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 4 & 9 \end{vmatrix} \underset{\text{col}}{\overset{\text{1st}}{=}} - \begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -5$$

This is the C_{21} cofactor.

$$\det B_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 1 & 0 & 9 \end{vmatrix} = 1 \underbrace{\begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix}} = 8$$

This is the C_{22} cofactor

$$\det B_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & 0 \end{vmatrix} = - \underbrace{\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}} = -3$$

This is the C_{23} cofactor

Now: $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} \det B_1 \\ \det B_2 \\ \det B_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -5 \\ 8 \\ -3 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 4 \\ -3/2 \end{bmatrix}$$

What did we really do here? We found the <u>2nd column</u> of A^{-1}!

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix} \overset{A^{-1}}{\begin{bmatrix} \\ \text{Col } 1 & \text{Col } 2 & \cdots & \text{Col } n \\ & \uparrow & & \\ \end{bmatrix}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$A \qquad \vec{x} \qquad = \qquad \vec{b}$$

The entries in <u>Column j</u> of A^{-1} come from the cofactors for <u>Row</u> j of A:

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det A}$$

Transpose!

cofactor matrix: $\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$

So we get an <u>algebraic formula</u> for A^{-1}:

$$A^{-1} = \frac{1}{\det A}\left[C_{ij}\right]^T \text{ important! } \quad (\text{if } \det A \neq 0).$$

Remember: $C_{ij} = \pm$ det of matrix you get after deleting Row i and column j from A.

Specifically, $(-1)^{i+j}$

Let's check how this works for 2×2:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} a_{11}C_{11}+a_{12}C_{12} & a_{11}C_{21}+a_{12}C_{22} \\ a_{21}C_{11}+a_{22}C_{12} & a_{21}C_{21}+a_{22}C_{22} \end{bmatrix}$$

Cofactor expansion for

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{21} & a_{22} \end{vmatrix} = 0$$

Cofactor expansions for det A (using Rows 1 and 2).

So matrix product is: $\begin{bmatrix} \det A & 0 \\ 0 & \det A \end{bmatrix}$

So just need to divide by det A to get I.

<u>Note</u>: Formula for A^{-1} is not too useful for <u>calculating</u> A^{-1} (elimination is usually faster)

But it can give <u>some</u> information about A^{-1} easily, if we don't care exactly what A is.

<u>Example</u>: Show $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 7 & 14 & 25 \\ 4 & 11 & 25 & 50 \end{bmatrix}^{-1}$ has integer entries (whole numbers, no fractions)

First find $\det A$ by elimination:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 7 & 14 & 25 \\ 4 & 11 & 25 & 50 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 5 & 13 \\ 0 & 3 & 13 & 34 \end{vmatrix} \xrightarrow[\text{Then: Row 2} \leftrightarrow \text{Row 3}]{\text{Row 4} - 3\text{ Row 3}}$$

$$-\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 13 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -5 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 13 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -1$$

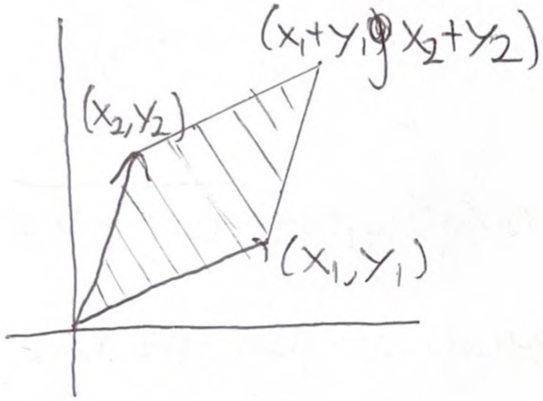
Now: entries of A^{-1} are $\dfrac{C_{ji}}{\det A} = -C_{ji}$

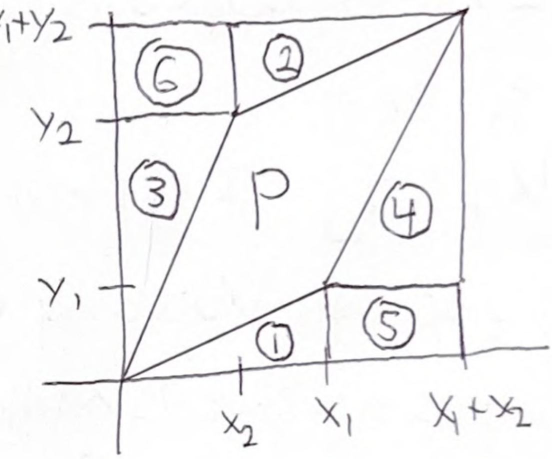
These are integers because $C_{ji} = \pm$ det of a submatrix of A = integer (since A has integer entries).

<u>In general</u>: If A has integer entries <u>and</u> $\det A = \pm 1$, then A^{-1} <u>also</u> has integer entries.

Areas and Volumes: Connect determinants to geometry (maybe this is the main reason determinants are natural.)

Let's find area of a parallelogram in the plane:





Area of $P = (x_1+x_2)(y_1+y_2) - (1+2+3+4+5+6)$

Area 1 = Area 2 $= \frac{1}{2}x_1y_1$ Area 5 $= x_2y_1$

Area 3 = Area 4 $= \frac{1}{2}x_2y_2$ Area 6 $= x_2y_1$

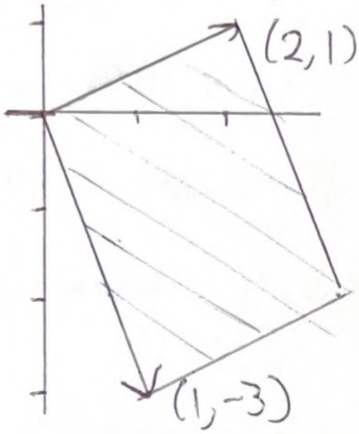
So Area of $P = (x_1+x_2)(y_1+y_2) - 2\left(\frac{1}{2}x_1y_1 + \frac{1}{2}x_2y_2 + x_2y_1\right)$

$= x_1y_1 + x_2y_1 + x_1y_2 + x_2y_2 - x_1y_1 - x_2x_2 - 2x_2y_1$

$= x_1y_2 - x_2y_1 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ ← Determinant is an <u>area</u>!

So if you know the coordinates of the vertices of a parallelogram, it's easy to find to area.

Example: $(x_1,y_1)=(2,1)$, $(x_2,y_2)=(1,-3)$



$$\text{Area} = \begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} = -6-1 = -7$$

?? The minus sign comes from the order we chose for the two vectors. Actual area is the absolute value, 7.

This works in n dimensions too: Volume of n-dim box determined by $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$

$$= \left| \det \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \right|$$

absolute value, not det!

or put $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ as the rows, get same det since $\det A = \det A^T$.

How would you prove this volume formula? Show that the function Vol:

$$\begin{bmatrix} - & \vec{a}_1 & - \\ - & \vec{a}_2 & - \\ & \vdots & \\ - & \vec{a}_n & - \end{bmatrix} \xrightarrow{\text{Vol}} \text{"Signed" volume of box determined by } \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$$

obeys the 3 rules of determinants.

Since det is the only function that obeys all 3 rules, the "signed" volume function has to be the same as det.

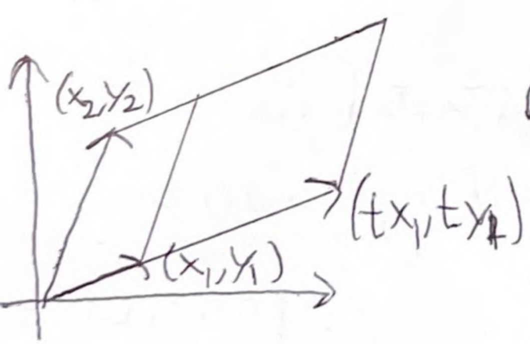
Check these rules:

Rule 1: $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \xrightarrow{\text{Vol}} 1\ ??$

← This gives the unit "hypercube" in n dimensions. Volume $= \underbrace{(1)(1)\cdots(1)}_{n \text{ times}} = 1$ ✓

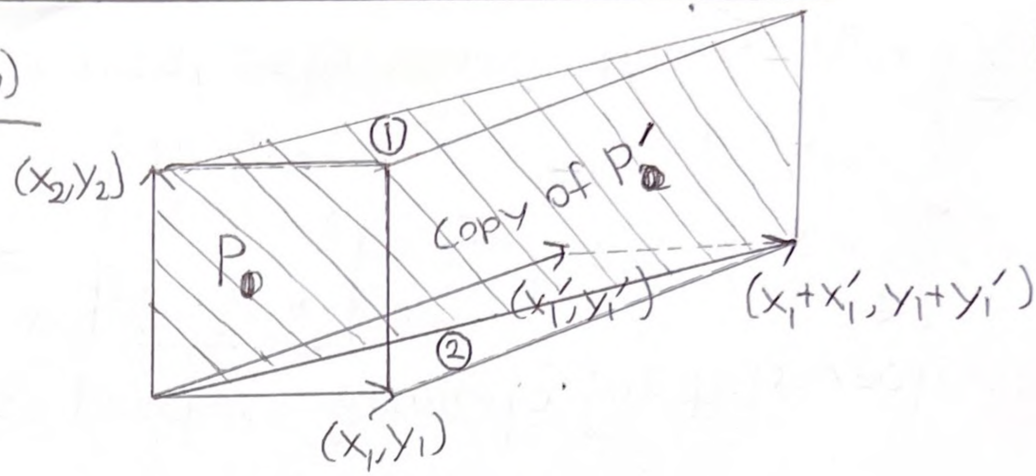
Rule 2: How does volume change if we switch two rows? Volume doesn't change if we reorrange the vectors since the box they span is the same. But "signed" volume changes by a sign, like det.

Rule 3(a) Multiply one row by t: $\begin{bmatrix} tx_1 & ty_1 \\ x_2 & y_2 \end{bmatrix}$



One side stretches by t while other dimensions of box are the same → Volume is multiplied by t (just like det).

Rule 3(b)



$$\text{Area}\left(\begin{bmatrix} x_1+x_1' & y_1+y_1' \\ x_2 & y_2 \end{bmatrix}\right) = \text{shaded area} =$$

$$= \text{Area } P + \text{Area } P' + \underbrace{\text{Area ①} - \text{Area ②}}$$

same, both = area of triangle spanned by (x_1,y_1) and (x_1+x_1', y_1+y_1')

$$= \text{Area}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\right) + \text{Area}\left(\begin{bmatrix} x_1' & y_1' \\ x_2 & y_2 \end{bmatrix}\right)$$

Areas add, so Area function obeys Rule 3(b). ✓

<u>Note</u>: For n-dimensional space, you might wonder, Do we really know what "n-dimensional volume" really means? One way to fix this problem is to use determinants to <u>define</u> volume.

I.e., define Vol(n-dim box) to equal

$$\left| \det \begin{bmatrix} \vec{a_1} \\ \vec{a_2} \\ \vdots \\ \vec{a_n} \end{bmatrix} \right|$$

. But we would still want this function to behave like volume. For example:

Problem 5.3.20 Find volume of 4-dim "hypercube" spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

means the vectors are orthogonal

Since this is a hypercube, volume should equal product of side lengths:

$$\left\| \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\| = \sqrt{1^2+1^2+1^2+1^2} = 2 \rightarrow$$ same for other 3 vectors

$$\longrightarrow \text{Volume} \overset{??}{=} (2)(2)(2)(2) = 16$$

But let's check this using the determinant volume formula.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} \longrightarrow \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & -2 \end{vmatrix} \overset{\text{1st}}{\underset{\text{col}}{=}} 1 \begin{vmatrix} 0 & -2 & -2 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{vmatrix}$$

$$= -(-2)\begin{vmatrix} -2 & -2 \\ 0 & -2 \end{vmatrix} + (-2)\begin{vmatrix} -2 & -2 \\ -2 & 0 \end{vmatrix} = 2(4) + (-2)(-4)$$

$= \boxed{16}$ ✓ Determinant produces correct volume

<u>Cross Product in $\mathbb{R}^3$</u> Given $\vec{u}, \vec{v}$ in $\mathbb{R}^3$, find a vector $\perp$ to <u>both</u> of them (i.e., a vector in $\operatorname{span}(\vec{u}, \vec{v})^\perp$).

The cross product formula for this vector:

$$\vec{u} \times \vec{v} = \text{"}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\text{"} = (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$$

(not a real determinant)

Here: $\vec{i} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \vec{j} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \vec{k} = \begin{bmatrix}0\\0\\1\end{bmatrix}$. components of $\vec{u}\times\vec{v}$ are C_{11}, C_{12}, C_{13} cofactors.

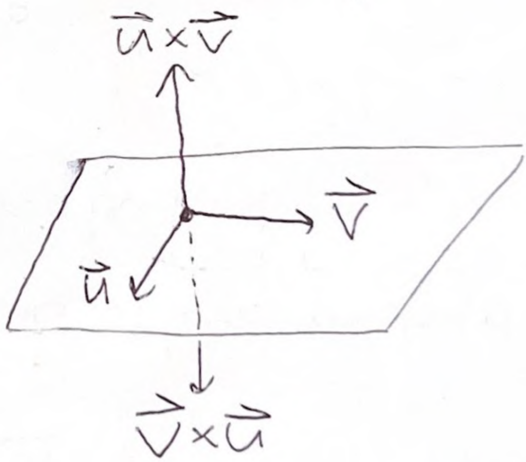
Why is $\vec{u}\times\vec{v} \perp \vec{u}, \vec{v}$?

$$\vec{u}\cdot(\vec{u}\times\vec{v}) = u_1 C_{11} + u_2 C_{12} + u_3 C_{13}$$
$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0. \checkmark$$

Same for $\vec{v}\cdot(\vec{u}\times\vec{v})$.

From Rule 2 for determinants: If you switch $\vec{u}$ and $\vec{v}$, cross product changes sign.

$$\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$$



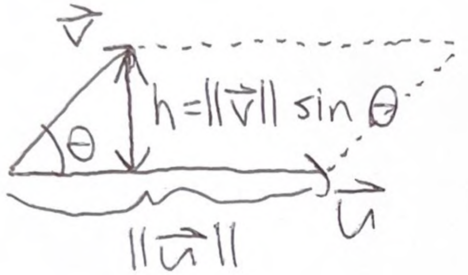
How <u>long</u> is $\vec{u}\times\vec{v}$? Check when $\vec{u}, \vec{v}$ are in xy-plane:

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix}$$

So $\|\vec{u}\times\vec{v}\|$ = absolute value of $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$

= Area of parallelogram spanned by $\vec{u}$ and $\vec{v}$.

Can express this area using angle between $\vec{u},\vec{v}$:



So $\|\vec{u}\times\vec{v}\|$ = area

$= \|\vec{u}\|\,\|\vec{v}\|\,|\sin\theta|$

Similar to dot product:

$$|\vec{u}\cdot\vec{v}| = \|\vec{u}\|\,\|\vec{v}\|\,|\cos\theta|$$

<u>Triple Product Formula</u> for volume of a box in $\mathbb{R}^3$:

Vol of box spanned by $\vec{u},\vec{v},\vec{w}$ $= \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

$$= w_1\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$= \vec{w}\cdot(\vec{u}\times\vec{v})$ (Take absolute value to get actual volume.)