Example Approximate solution to

$$\begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad A = QR = \begin{bmatrix} -2/\sqrt{5} & -3/\sqrt{70} \\ 1/\sqrt{5} & -6/\sqrt{70} \\ 0 & 5/\sqrt{70} \end{bmatrix} \begin{bmatrix} \sqrt{5} & 6/\sqrt{5} \\ 0 & \sqrt{14/5} \end{bmatrix}$$

$$A \quad \vec{x} \quad \vec{b}$$

Just need to solve $R\hat{x} = Q^T \vec{b} = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -3/\sqrt{70} & -6/\sqrt{70} & 5/\sqrt{70} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ -3/\sqrt{70} \end{bmatrix}$

So $\sqrt{5}\, x + \frac{6}{\sqrt{5}} y = -\frac{2}{\sqrt{5}} \rightarrow x = \frac{1}{\sqrt{5}}\left(-\frac{2}{\sqrt{5}} - \frac{6}{\sqrt{5}} \cdot \frac{3}{5}\right) = -\frac{2}{5} - \frac{18}{25} = -\frac{28}{25}$

$$\sqrt{\frac{14}{5}}\, y = -\frac{3}{\sqrt{70}} \rightarrow y = -\frac{3\sqrt{5}}{\sqrt{14 \cdot 70}} = -\frac{3}{5}$$

Chapter 5 Determinants

The determinant is a function that takes a square matrix $(n \times n)$ as input, and outputs a real number:

$$\underset{n\times n}{A} \xrightarrow{\det} \underset{\text{in } \mathbb{R}}{\det(A), \text{ or } |A|}$$

It determines whether A is invertible:

$$\begin{cases} \det A \neq 0 \longrightarrow A \text{ has an inverse} \\ \det A = 0 \longrightarrow A \text{ has no inverse.} \end{cases}$$

But doesn't give us an efficient way of finding A^{-1}, so it's more useful as a theoretical tool.

Start with simplest examples:

1×1: $A = [a] \longrightarrow$ has inverse exactly when $a \neq 0$, because $[a]^{-1} = [a^{-1}]$.

So $\boxed{\det[a] = a}$

2x2: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Formula for A^{-1}: $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

So A^{-1} exists exactly when $ad-bc \neq 0$, and this is the determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

What about bigger matrices? There are several ways to define the determinant. To understand these formulas, it may be good first to look at what properties the determinant should have:

Properties: It turns out that det is the only possible function from nxn matrices to $\mathbb{R}$ that obeys 3 simple rules:

1. The determinant of the identity matrix is 1: $|I| = 1$.

   2x2 case: $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = (1)(1)-(0)(0) = 1$ ✓

2. If you switch two rows in A, then det changes by -1 factor:

   2x2 case:
$$\begin{array}{ccc} \begin{vmatrix} a & b \\ c & d \end{vmatrix} & \xrightarrow[\text{rows}]{\text{switch}} & \begin{vmatrix} c & d \\ a & b \end{vmatrix} \\ \downarrow \text{det} & & \downarrow \text{det} \\ ab-bc & \xrightarrow[\text{by } -1]{\text{Multiply}} & cb-da \end{array}$$

3. det is a "linear function" of each row separately. This means:

   (a) If you multiply one row by a scalar, det also changes by that scalar factor

   2x2 case:
$$\begin{array}{ccc} \begin{vmatrix} a & b \\ c & d \end{vmatrix} & \xrightarrow[t \text{ Row1}]{\text{Row1} \rightarrow} & \begin{vmatrix} ta & tb \\ c & d \end{vmatrix} \\ \downarrow \text{det} & & \downarrow \text{det} \\ ad-bc & \xrightarrow[\text{by } t]{\text{Multiply}} & tad-tbc = t(ad-bc) \end{array}$$

   (b) If you add one row of A to the same row of A', and all the other rows of A and A' are the same, then determinants also add.

2x2 case: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c & d \end{bmatrix} \xrightarrow[\text{Keep Row 2 the same}]{\text{Add Row 1's}} \begin{bmatrix} a+a' & b+b' \\ c & d \end{bmatrix}$

Two matrices with the same Row 2

$\downarrow$ Take both determinants $\qquad\qquad$ $\downarrow$ det

$ad-bc,\ a'd-b'c \xrightarrow[\text{determinants}]{\text{Add}}$ $(a+a')d-(b+b')c = (ad-bc)+(a'd-b'c)$

Warning: Rule ③ does not mean: $\begin{cases} \det(tA) = t\det A \\ \det(A+B) = \det A + \det B \end{cases}$

For tA, you multiply all n rows of A by t, so Rule ③ means you multiply $\det A$ by t n times:

Correct relation: $\det(tA) = t^n \det A$ if A is $n\times n$.

But there is no relation between $\det(A+B)$ and $\det A$, $\det B$ in general.

Using Rules 1-3, we can find many more properties of determinants and also find ways to calculate them for big matrices:

④ If 2 rows of A are the same, then $\det A = 0$.

2x2 case: $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0$ ✓

General case:

$$\begin{tikzcd}
A \arrow[rr, "\text{switch two rows}"', "\text{that are same}"] \arrow[d, "\det"] & & \text{same } A \arrow[d, "\det"] \\
{|A|} \arrow[rr, bend right, "\text{Rule ②: multiply}"', "\text{det by } -1"] & & {|A| = -|A|}
\end{tikzcd}$$

$|A| = -|A|$ means $2|A| = 0$, or $\boxed{|A| = 0}$

⑤ The big row operation: If we add a multiple of one row to another row, the determinant doesn't change.

2x2 case:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[\text{Row 2+t Row 1}]{\text{Row 2} \rightarrow} \begin{bmatrix} a & b \\ c+ta & d+tb \end{bmatrix}$$

det ↓ ↓ det

$$ad-bc = a(d+tb)-b(c+ta) = ad-bc+(\cancel{atb}-\cancel{bta})$$

General case: det of new A $= \begin{vmatrix} \text{Row 1} \\ \text{Row 2} + t\,\text{Row 1} \\ \text{Row 3} \\ \vdots \\ \text{Row } n \end{vmatrix} \overset{\text{Rule}}{\underset{3b}{=}} \underbrace{\begin{vmatrix} \text{Row 1} \\ \text{Row 2} \\ \vdots \\ \text{Row } n \end{vmatrix}}_{\text{old A}} + \begin{vmatrix} \text{Row 1} \\ t\,\text{Row 1} \\ \vdots \\ \text{Row } n \end{vmatrix}$

$\overset{\text{Rule}}{\underset{3a}{=}} \det(\text{old A}) + t \begin{vmatrix} \text{Row 1} \\ \text{Row 1} \\ \vdots \\ \text{Row } n \end{vmatrix}$ same rows $\overset{\text{Rule}}{\underset{4}{=}} \det(\text{old A}) + t(0)$

So det (new A) = det (old A)

Rules 2, 3, and 5 tell us how row operations affect determinants.
This means we can calculate determinants using elimination!

Example Find determinant of $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix} \xrightarrow[\text{det changes by } (-1)]{\text{Row 1} \leftrightarrow \text{Row 2}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 3 & 2 \end{bmatrix} \xrightarrow[\text{in det}]{\text{No change}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -4 \end{bmatrix} \xrightarrow[\text{in det}]{\text{No change}}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow[\text{in det}]{\text{No change}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow[\text{in det}]{\text{No change}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

This shows:

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 2 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{vmatrix} \overset{\text{Rule}\ 3a}{=} (-1)(-2) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\overset{\text{Rule}}{\underset{1}{=}} (-1)(-2)(1) = \boxed{2}$$

Since $\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 2 \end{vmatrix} \neq 0$, this matrix should be invertible.

But we don't need determinants to see that! We saw that the reduced row echelon form is I, which shows that A is invertible.

In fact, the elimination method for determinants gives us a way to prove the:

Big Determinant Theorem: $\begin{cases} \text{If } A \text{ is invertible, then } \det(A) \neq 0 \\ \text{If } A \text{ is not invertible, then } \det(A) = 0. \end{cases}$

Prove this: $A \xrightarrow{\text{elimination}} R$, reduced row echelon form

The three eliminations change $|A|$ by:

1. Factor of -1 (row switches)
2. Factor of t (multiply a row by non-zero t)
3. Nothing (factor of 1) (add a multiple of one row to another)

Whatever elimination operations we have to do, they change $|A|$ by a non-zero scalar: So $|R| = (\text{non-zero scalar})|A|$,

or $|A| = (\text{non-zero})^{-1}|R|$

Now: If A is invertible: then $R = I$, so

$$|A| = (\text{non-zero scalar})|I| = (\text{non-zero})(1) \neq 0 \quad \checkmark$$

Rule 1

But if A is not invertible: Then R has a row of 0's.

$$|A| = (\text{non-zero scalar})\begin{vmatrix} \text{stuff} \\ 0 \; 0 \cdots 0 \end{vmatrix} = (\text{non-zero})(0)\begin{vmatrix} \text{stuff} \\ 1 \; 1 \cdots 1 \end{vmatrix} = 0 \quad \checkmark$$

Rule 3a

These don't matter.

Another big determinant theorem:

$\det(AB) = (\det A)(\det B)$, or $|AB| = |A||B|$.

Remember this kind of rule doesn't work for addition!

Check 2×2: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}$

$(\det A)(\det B) \downarrow$ $\qquad\qquad \downarrow \det(AB)$

$$(ad-bc)(ps-qr) \overset{??}{=} (ap+br)(cq+ds) - (aq+bs)(cp+dr)$$

① ② ③ ④ $\qquad$ ① ② ③ ④

other terms cancel

General case: First possibility: A is not invertible. Then AB is not invertible either (if $(AB)^{-1}$ existed, then we'd get $A^{-1} = B(AB)^{-1}$)

So $\det(AB)$ and $(\det A)(\det B)$ are both 0.

0

2nd possibility: A is invertible, so $R = I$

$A \xrightarrow{\text{elimination}} I$ $\qquad$ actually, this is $|A|$.

$|A| \longrightarrow$ (non-zero scalar) $|I|$

Now do the same elimination operations to AB. This is the same as multiplying by elimination matrices on the left, so:

$AB \xrightarrow[\text{elimination}]{\text{same}} IB = B$, $\quad$ Determinants: $|AB| \longrightarrow$ (same non-zero) $|B|$

So $|AB| = |A||B|$ ✓

Some more properties:

- If A is invertible, then $|A^{-1}| = |A|^{-1}$. $\quad$ Non-zero, so $|A|^{-1}$ exists!

  Why? $|A||A^{-1}| \underset{\text{product rule}}{=} |AA^{-1}| = |I| = 1 \longrightarrow |A^{-1}| = \dfrac{1}{|A|}$.

- Det of <u>triangular</u> matrix is easy: product of diagonal entries

$$\begin{vmatrix} a & \text{stuff} & \\ 0 & b & \\ 0 & 0 & c \end{vmatrix} \xrightarrow[\text{no change in det}]{\text{elimination}} \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \underset{\text{Rule 3a}}{=} (a)(b)(c) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = abc$$

If one of a, b, c is 0, matrix isn't invertible, so $\det = 0$, still the same as abc.

Same for <u>lower</u> triangular matrices.

<u>Example</u> Find det of $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

Just need to get to <u>triangular</u> form.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow[\text{in det}]{\text{No change}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow[\text{change}]{\text{No}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow[\text{change}]{\text{No}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

Determinant is $2\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right) = \boxed{5}$

- Transposes: $\det(A^T) = \det(A)$ (no difference)

Why? Use LU decomposition: $PA = LU \rightarrow |A| = |L||U| / |P|$

permutation matrix in case of row switches

from product rule

Also: $(PA)^T = (LU)^T \longrightarrow A^T P^T = U^T L^T \longrightarrow |A^T| = |U^T||L^T| / |P^T|$

Now: $|U^T| = |U|$ (both are triangular, one upper and one lower, with <u>same</u> diagonal entries)

$|L^T| = |L|$ (same reason

$|P^T| = |P|$ (You can row-switch P to get I, so $|P| = \pm 1$. Also, P is orthogonal: $P^T P = I$, so $|P||P^T| = 1 \longrightarrow$ either both are $+1$ or both are -1)

so $\boxed{|A^T| = |L||U| / |P| = |A|}$