<u>Vector spaces</u>: spaces of "vectors" that you can <u>add</u> and <u>multiply by scalars</u> (+8 rules for addition and mult.)

Vector spaces can be pretty abstract. For example, the "vectors" could be <u>functions</u> (you can add functions and multiply them by real number scalars.)

But for now, the vector spaces we mainly want to look at are <u>subspaces of $\boxed{\mathbb{R}^n}$</u> ← all vectors with n real number components.

Subspace = set of vectors in $\mathbb{R}^n$ such that:

1. Includes zero vector $\vec{0}$ (so subspace is <u>non-empty</u>)
2. Closed under addition: If $\vec{v}$ and $\vec{w}$ are in subspace, so is $\vec{v}+\vec{w}$
3. Closed under scalar multiplication: If $\vec{v}$ is in subspace, so is $c\vec{v}$ for <u>any</u> c.

Closed under linear combinations: If $\vec{v}, \vec{w}$ are in subspace, so are <u>all</u> $c\vec{v}+d\vec{w}$.

<u>First big example</u>: $C(A)$ = column space of $m \times n$ matrix A.

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

↑ ↑ ↑ columns of A

I.e., $C(A)$ = set of all linear combinations

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

A vector $\vec{b}$ in $\mathbb{R}^m$ is in $C(A)$ <u>precisely when</u> the system of equations $A\vec{x}=\vec{b}$ has at least one solution.

↑ each column of A has m components

This is because "$\vec{b} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$" means the <u>same thing</u> as

$$\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}.$$

<u>Example</u>: $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ Take $(-2)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ ← This is a vector in $C(A)$.

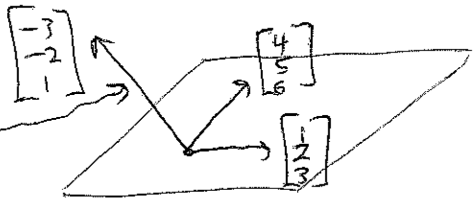
So the system $\begin{cases} x+4y=2 \\ 2x+5y=1 \\ 3x+6y=0 \end{cases}$ has a solution. It's $(x,y)=(-2,1)$.

what about $\begin{cases} x+4y=-3 \\ 2x+5y=-2 \\ 3x+6y=1 \end{cases}$?? Same question as: Is $\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$ in $C(A)$?

Check: $\left[\begin{array}{cc|c} 1 & 4 & -3 \\ 2 & 5 & -2 \\ 3 & 6 & 1 \end{array}\right] \xrightarrow[\text{Row 3}-3\text{ Row 1}]{\text{Row 2}-2\text{ Row 1}} \left[\begin{array}{cc|c} 1 & 4 & -3 \\ 0 & -3 & 4 \\ 0 & -6 & 10 \end{array}\right] \xrightarrow{\text{Row 3}-2\text{ Row 2}} \left[\begin{array}{cc|c} 1 & 4 & -3 \\ 0 & -3 & 4 \\ 0 & 0 & 2 \end{array}\right]$

$\rightsquigarrow \begin{cases} x+4y=-3 \\ -3y=4 \\ 0y=2 \end{cases}$ Inconsistent equation, No solution. So $\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$ is not in $C(A)$.

sticks out of the plane somehow



Comment: The set of all linear combinations of any subset in $\mathbb{R}^n$ is a subspace, called the "span."

In our example, $C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right)$

Important general problem: If S is a subspace, can you find a set of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in S such that $S = \text{span}\underbrace{(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m)}_{\text{called a "spanning set"}}$?

have to be vectors in the subspace

Example: $S = xy$-plane in $\mathbb{R}^3$, i.e. $S = $ all $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

Since $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $S = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

Also, $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \frac{x+y}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{x-y}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, so also $S = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$

different spanning sets, but same number of vectors in each.

But $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not a spanning set for S, even though $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - (x+y)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

← not vectors in S.

Second subspace from a $m \times n$ matrix A: <u>null</u> <u>space</u> $N(A)$

$$\vec{x} \xrightarrow{A} A\vec{x}$$ ← This vector is in $C(A)$

$\vec{x}$ in $\mathbb{R}^n$; $A\vec{x}$ in $\mathbb{R}^m$

This vector is in $N(A)$ <u>if</u> $A\vec{x} = \vec{0}$. I.e., $N(A) =$ <u>all</u> vectors $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x} = \vec{0}$.

<u>Example</u> $A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix}$ We want to find <u>all</u> <u>solutions</u> for $A\vec{x} = \vec{0}$

Put into augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 4 & 0 \\ 1 & 2 & 2 & 5 & 0 \\ 1 & 3 & 2 & 6 & 0 \end{array}\right] \xrightarrow[\text{Row 3-Row 1}]{\text{Row 2-Row 1}} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 & \\ 0 & 2 & 0 & 2 & \end{array}\right] \xrightarrow{\text{Row 3-2 Row 2}} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

one more variable we can eliminate

$$\xrightarrow{\text{Row 1-Row 2}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

→ System is reduced to <u>two</u> "independent" equations: $\begin{cases} x_1 + 2x_3 + 2x_4 = 0 \\ x_2 + x_4 = 0 \end{cases}$

can solve for x_1 and x_2 in terms of x_3 and x_4:

$$x_1 = -2x_3 - 2x_4$$
$$x_2 = -x_4$$

x_3, x_4: "free variables," they are free to take <u>any</u> real number value

So vectors in $N(A)$ look like: $\vec{x} = \begin{bmatrix} -2x_3 - 2x_4 \\ -x_4 \\ x_3 \\ x_4 \end{bmatrix}$ ← We have found <u>all</u> solutions.

Or: $\vec{x} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ ⇝ So $N(A) =$ <u>all</u> linear combinations of the 2 "special solutions"

two "special solutions"

$$= \operatorname{span}\left(\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

A <u>plane</u> in $\mathbb{R}^4$.

Now that we can find $N(A)$ for a matrix, let's check that it <u>really</u> is a subspace of $\mathbb{R}^n$:

① Is $\vec{0}$ in $N(A)$? $A\vec{0} = \vec{0}$ ✓ Yes, it is.

② If $\vec{x}$ and $\vec{y}$ are in $N(A)$, what about $\vec{x}+\vec{y}$?

$$A(\vec{x}+\vec{y}) = \underbrace{A\vec{x}}_{\vec{0}} + \underbrace{A\vec{y}}_{\vec{0}} = \vec{0}+\vec{0} = \vec{0} \checkmark$$ So $\vec{x}+\vec{y}$ is in $N(A)$

③ If $\vec{x}$ is in $N(A)$, what about $c\vec{x}$?

$$A(c\vec{x}) = c\underbrace{(A\vec{x})}_{\vec{0}} = c\vec{0} = \vec{0} \checkmark$$ So $c\vec{x}$ is in $N(A)$.

So $N(A)$ is a subspace!

Some more examples:

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ Find all vectors in $N(A)$.

Need to solve $A\vec{x} = \vec{0}$: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] \xrightarrow[\text{Row 3} - 7\,\text{Row 1}]{\text{Row 2} - 4\,\text{Row 1}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \xrightarrow[-2\,\text{Row 2}]{\text{Row 3}}$

"free variable" column

$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

"pivot columns"

We can now solve for variables in the "pivot columns" in terms of the variable in the "free" column.

But first let's do a little more elimination:

$\xrightarrow{-\frac{1}{3}\text{Row 2}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{Row 1} - 2\,\text{Row 2}} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}$

This matrix is called a "reduced row echelon form." It is the simplest form for reading off solutions.

$$x_1 - x_3 = 0$$
$$x_2 + 2x_3 = 0$$

$\rightarrow$ $x_1 = x_3$, $x_2 = -2x_3$ $\quad$ x_3 = free variable, can take any value

All solutions: $\vec{x} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ ← one "special solution"

$N(A)$ = all multiples (or, linear combinations) of the "special solution"

$= \operatorname{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$ = a line in $\mathbb{R}^3$.

Let's review the full elimination procedure:

$$A \xrightarrow[\text{left variables}]{\text{eliminate lower}} U \xrightarrow[\text{"leading 1's"}]{\text{eliminate upper right variables, create}} R$$

U: upper triangular

R: "reduced row echelon form"

Means: all variables above leading 1's are eliminated

Means: non-zero entries in each row begin with a "leading 1," and leading 1's go down from upper left corner towards lower right.
(If a row has all 0's put it at the bottom.)

Example: What is R for

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} ??$$

$$\xrightarrow[\text{Row 3}-\text{Row 1}]{\text{Row 2}-\text{Row 1}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{bmatrix} \xrightarrow{\text{Row 3}-3\text{ Row 2}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

This is U

two "leading 1's" already

$$\xrightarrow[\text{to get a "leading 1" in Row 3}]{\frac{1}{2}\text{Row 3}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{Row 2}-2\text{ Row 3}]{\text{Row 1}-\text{Row 3}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Row 1}-\text{Row 2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is R; leading 1's go down diagonal, all variables above leading 1's are eliminated.

In this case, $R = I$! We can now easily determine $N(A)$:

Solve $A\vec{x} = \vec{0}$: $\left[A \,\middle|\, \vec{0}\right] \xrightarrow{\text{elimination}} \left[R \,\middle|\, \vec{0}\right] = \left[I \,\middle|\, \vec{0}\right]$

Elimination operations don't change $\vec{0}$.

$x_1 = 0, x_2 = 0, x_3 = 0$

There's only one solution: $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, So $N(A)$ = the zero vector subspace.

This always works if $R = I$: $A \xrightarrow{\text{elimination}} I$ means $N(A)$ = zero subspace. Another way of saying this: the columns of A are "independent."

Problem 3.2.1(a) Find R for $A=\begin{bmatrix}1&2&2&4&6\\1&2&3&6&9\\0&0&1&2&3\end{bmatrix}$

$$\begin{bmatrix}1&2&2&4&6\\1&2&3&6&9\\0&0&1&2&3\end{bmatrix}\xrightarrow{\text{Row 2-Row 1}}\begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&1&2&3\end{bmatrix}\xrightarrow{\text{Row 3-Row 2}}\begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}$$

leading 1's

This is U; lower-left variables are eliminated.

$$\xrightarrow{\text{Row 1-2 Row2}}\begin{bmatrix}1&2&0&0&0\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}$$ This is R.

leading 1's in Cols 1 and 3 → these are the "pivot columns"
columns 2,4,5 give free variables.

We can easily solve $A\vec{x}=\vec{0}$ using R: $[A \mid \vec{0}] \longrightarrow [R \mid \vec{0}]$

$$\begin{cases}x_1+2x_2=0\\x_3+2x_4+3x_5=0\end{cases}\xrightarrow[\text{in terms of free variables}]{\text{solve for } x_1, x_3}\begin{aligned}x_1&=-2x_2\\x_3&=-2x_4-3x_5\end{aligned}$$

from R

All solutions: $\vec{x}=\begin{bmatrix}-2x_2\\x_2\\-2x_4-3x_5\\x_4\\x_5\end{bmatrix}=\underbrace{x_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\-2\\1\\0\end{bmatrix}+x_5\begin{bmatrix}0\\0\\-3\\0\\1\end{bmatrix}}$

Vectors in N(A): Three free variables, three special solutions

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For an $m\times n$ A, the number of leading 1's in R is important. It is called the rank of the matrix. (Use r for rand of A)
Since A has n total columns, the number of free variables is $n-r$ (also the number of special solutions in N(A)).

Interesting question: What matrices have $r=1$?
Answer: "Outer products" of vectors, $\underset{m\times n}{A}=\underset{m\times 1}{\vec{u}}\,\underset{1\times n}{\vec{v}^T}$

Example: $A = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1(3) & 1(1) & 1(4) \\ -1(3) & -1(1) & -1(4) \\ 2(3) & 2(1) & 2(4) \end{bmatrix}$

Every row is a multiple of the 1st row.

$A \xrightarrow[\text{Row3}-2\text{Row1}]{\text{Row2}+\text{Row1}} \begin{bmatrix} 3 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}\text{Row 1}} \begin{bmatrix} 1 & 1/3 & 4/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R$

one pivot column, one leading 1

Fun fact: You can write any A as a linear combination of "outer products": $A = \vec{u}_1 \vec{v}_1^T + \vec{u}_2 \vec{v}_2^T + \cdots + \vec{u}_r \vec{v}_r^T$

The rank r is the minimum number of outer products required to add up to r.