Linear Algebra - Homework 11

Problem 5.1.2

If a 3x3 matrix has $\det A = -1$, find $\det(\frac{1}{2}A)$, $\det(-A)$, $\det(A^2)$ and $\det(A^{-1})$

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, then $\det A = 1 \times 1 \times (-1) = -1$

$\det(\frac{1}{2}A)$:

$\frac{1}{2}A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$, $\det(\frac{1}{2}A) = \frac{1}{2} \times \frac{1}{2} \times -\frac{1}{2} = -\frac{1}{8}$

$\det(-A)$:

$-A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\det(-A) = -1 \times (-1) \times 1 = 1$

$\det(A^2)$:

$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\det(A^2) = 1 \times 1 \times 1 = 1$.

$\det(A^{-1})$:

$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $\det(A^{-1}) = 1 \times 1 \times (-1) = -1$

Problem 5.1.7.

Find the determinants of rotations and reflections:

$Q_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\det Q_1 = \cos^2\theta + \sin^2\theta. = 1.$

$Q_2 = \begin{bmatrix} 1-2\cos^2\theta & -2\cos\theta\sin\theta. \\ -2\cos\theta\sin\theta & 1-2\sin^2\theta \end{bmatrix}$

$\det Q_2 = (1-2\cos^2\theta)(1-2\sin^2\theta) - (-2\cos\theta\sin\theta)(-2\cos\theta\sin\theta)$

$= (1 - 2\sin^2\theta - 2\cos^2\theta + 4\sin^2\theta\cos^2\theta) - 4\sin^2\theta\cos^2\theta.$

$= 1 - 2(\sin^2\theta + \cos^2\theta)$

$= -1$

Problem 5.1.13.

Reduce A to U and find $\det A$ = product of the pivots

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}. \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \quad \det A_1 = 1 \times 1 \times 1 = 1$$

$$A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & -3 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & -\frac{3}{2} \end{bmatrix} \quad \det A_2 = 1 \times (-2) \times (-\tfrac{3}{2}) = 3.$$

Problem 5.1.18.

Use row operations to show that the 3×3 "Vandermonde determinant" is

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$$

$$A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} \xrightarrow{-\frac{c-a}{b-a}} \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-a)(c-b) \end{bmatrix}, \quad \det A = 1 \times (b-a) \times (c-a)(c-b) = (b-a)(c-a)(c-b)$$

Problem 5.1.30.

Show that the partial derivatives of $\ln(\det A)$ give A^{-1}!

$$\boxed{\ln x' = \frac{1}{x}}$$

$f(a,b,c,d) = \ln(ad-bc)$ Leads to $\begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial c} \\ \frac{\partial f}{\partial b} & \frac{\partial f}{\partial d} \end{bmatrix} = A^{-1}$.

Derivatives of $f = \ln(ad-bc)$:

$$\begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial c} \\ \frac{\partial f}{\partial b} & \frac{\partial f}{\partial d} \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}$$

Problem 5.2.1

Compute the determinants of A, B, C from 6 terms, Are their $\boxed{\text{row independent}}$?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \quad \det A = (1+12+18) - (9+4+6) = 31 - 19 = 12$$

$$\text{Yes} \quad \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & \\ 0 & -4 & \end{bmatrix} \qquad \text{Yes!}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{bmatrix} \quad \det B = (28 + 40 + 72) - (60 + 56 + 24) = 140 - 80 = 60$$

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 4 & 6 \\ 3 & 4 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 5 \\ 0 & -4 & -4 \\ 0 & -8 & -8 \end{bmatrix} \qquad \text{No!}$$

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \det C = (0 + 0 + 0) - (1 + 0 + 0) = -1$$

$$\text{Yes} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{Yes!}$$

Problem 5.2.15.

The tridiagonal 1, 1, 1 matrix of order n has determinant E_n :

$$E_1 = |1| \quad E_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \quad E_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \quad E_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}.$$

(a) By cofactors show that $E_n = E_{n-1} - E_{n-2}$

$$E_n \underbrace{\begin{vmatrix} 1 & 1 & & \\ 1 & 1 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 1 \end{vmatrix}}_{n \times n} = \overbrace{\begin{vmatrix} 1 & 1 & & \\ 1 & 1 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 1 \end{vmatrix}}^{(1,1)\leftrightarrow \ E_{n-1}} - \overbrace{\begin{vmatrix} \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & & 1 & & \\ 0 & & & 1 & 1 \\ 0 & & & 1 & 1 \end{vmatrix}}^{(1,1)\updownarrow} = E_{n-1} - \overbrace{\begin{vmatrix} 1 & 1 & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & 1 \\ & & & 1 & 1 \end{vmatrix}}^{E_{n-2}}$$

(b) Stating $E_1 = 1$ and $E_2 = 0$, find $E_3, E_4, \dots, E_8$

$E_3 = E_2 - E_1 = 0 - 1 = -1$

$E_4 = E_3 - E_2 = -1 - 0 = -1$

$E_5 = E_4 - E_3 = -1 - (-1) = 0$

$E_6 = E_5 - E_4 = 0 - (-1) = 1$

$E_7 = E_6 - E_5 = 1 - 0 = 1$

$E_8 = E_7 - E_6 = 1 - 1 = 0.$

(C) Find E_{100}

the cycle of the value of E is (1, 0, -1, -1, 0, 1)

$100 \bmod 6 \equiv 4$

$E_{100} = E_4 = -1$

Problem 5.2.19

The goal of this problem is to find the 4x4 Vandermonde determinant

$$V_4 = \det \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & x & x^2 & x^3 \end{bmatrix}$$

(a) Explain why V_4 is a cubic polynomial in the variable x.

Since x, x^2, x^3 are all in the same row, they never multiply each other in det V_4.

(b) Find three possible values r_1, r_2, r_3 for x that make V_4 equal to 0. These are the roots of V_4 as a polynomial in x

$r_1 = a$, $r_2 = b$, $r_3 = c$.

because when a determinat has a some rows , the determinant will be 0

(c) Explain why $V_4 = A(x-r_1)(x-r_2)(x-r_3)$ for some value A, and show that the value of A is the 3x3 Vandermonde determinant from Problem 5.1.18. ?

$$V_4 = \overset{(4,4)}{x^3} \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \_\_\_\_$$

(d) Write down a formula for V_4 in terms of a, b, c, x

$$V_4 = (b-a)(c-a)(c-b)(x-a)(x-b)(x-c)$$

Problem 5.2.31 Find the det of this cyclic P by cofactors of row 1 and then the 'big formul'
How many exchanges reorder 4,1,2,3 into 1,2,3,4? Is $|P^2| = 1$ or -1?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad P^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{bmatrix} \qquad \det P = -1^{(1+4)} \begin{vmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{vmatrix} = -1$$

$$\begin{vmatrix} 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{vmatrix} \overset{①}{=} - \begin{vmatrix} 0&0&1&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{vmatrix} \overset{②}{=} \begin{vmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{vmatrix} \overset{③}{=} - \begin{vmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{vmatrix} \quad \text{3 time!}$$

$$P^2 \begin{bmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{bmatrix} \qquad \det P^2 = \begin{vmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{vmatrix} \Rightarrow - \begin{vmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{vmatrix} = \begin{vmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{vmatrix} = 1$$

Problem 5.2.34

This problem shows in two ways that $\det A = 0$ (the x's are any numbers; they don't have to all be the same):

$$A = \begin{bmatrix} x&x&x&x&x \\ x&x&x&x&x \\ 0&0&0&x&x \\ 0&0&0&x&x \\ 0&0&0&x&x \end{bmatrix} \qquad \begin{bmatrix} 0&0&0&x& \\ 0&0&0&0&x \\ 0&0&0&0&0 \end{bmatrix}$$

(a) How do you know that the rows are linearly dependent?

The last three rows must be dependet becaus only 2 colums are nonzero.

(b) Explan why all 120 terms are zero in the big for formula for $\det A$.

The term of BIG FORMULA must contains 0 because

$$\det A = \sum (\det A)\, a_{1\alpha} a_{2\beta} a_{3\gamma} a_{4w} a_{5\ldots}$$

Graded Problems.

Problem 1

Use row operations to calculate the determinant:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 3 & 9 & 19 & 34 \\ 0 & 4 & 14 & 34 & 69 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 3 & 10 & 22 \\ 0 & 0 & 6 & 22 & 53 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 4 & 17 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \times 1 \times 1 \times 1 \times 1 = 1$$

Problem 2

Use cofactors to calculate the determinant

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & 0 & 0 & 0 \\ 4 & 0 & -2 & 0 & 1 \\ 0 & -2 & 0 & 2 & 0 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & 0 & 0 & 0 \\ 4 & 0 & -2 & 0 & 1 \\ 0 & -2 & 0 & 2 & 0 \end{vmatrix} = -1^{(3+2)} \begin{vmatrix} 1 & 1 & -1 & 1 \\ 1 & 3 & 4 & 5 \\ 4 & -2 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{vmatrix} = -1 \times (-1)^{(4+3)}\, 2 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 4 & -2 & 1 \end{vmatrix}$$

$$= 2 \times \left[(3+20-2) - (12-10+1) \right] = 2\times(21-3) = -2\times 18 = -36.$$