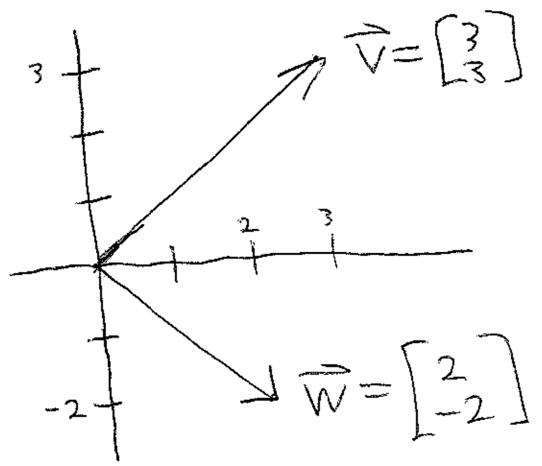
Homework 1 Solutions

1.1.3

$$\begin{array}{rl} & \vec{v}+\vec{w} = \begin{bmatrix}5\\1\end{bmatrix} \\ + & \vec{v}-\vec{w} = \begin{bmatrix}1\\5\end{bmatrix} \\ \hline & 2\vec{v} = \begin{bmatrix}6\\6\end{bmatrix}\end{array} \qquad \begin{array}{rl} & \vec{v}+\vec{w} = \begin{bmatrix}5\\1\end{bmatrix} \\ - & \vec{v}-\vec{w} = \begin{bmatrix}1\\5\end{bmatrix} \\ \hline & 2\vec{w} = \begin{bmatrix}4\\-4\end{bmatrix}\end{array}$$

$$\vec{v} = \frac{1}{2}\begin{bmatrix}6\\6\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix} \qquad \vec{w} = \frac{1}{2}\begin{bmatrix}4\\-4\end{bmatrix} = \begin{bmatrix}2\\-2\end{bmatrix}$$



1.1.6 Every linear combination looks like $c\begin{bmatrix}1\\-2\\1\end{bmatrix}+d\begin{bmatrix}0\\1\\-1\end{bmatrix} =$

$= \begin{bmatrix}c\\-2c+d\\c-d\end{bmatrix}$. components add up to $c+(-2c+d)+(c-d)$

$= (1-2+1)c + (1-1)d = \boxed{0}$

Write $\begin{bmatrix}3\\3\\-6\end{bmatrix}$ as a linear combination: $\begin{bmatrix}3\\3\\-6\end{bmatrix} = c\begin{bmatrix}1\\-2\\1\end{bmatrix}+d\begin{bmatrix}0\\1\\-1\end{bmatrix}$

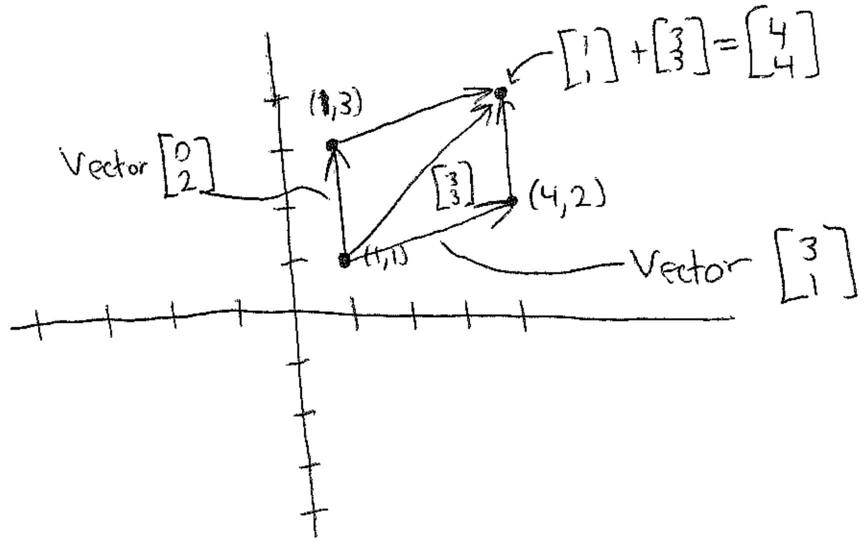
$$\rightarrow \begin{cases} 1c + 0d = 3 \rightarrow c = 3 \\ -2c + d = 3 \rightarrow -2(3)+d = 3 \rightarrow d = 9 \\ c - d = -6 \end{cases}$$

$\underbrace{c - d = -6}_{3-9=-6\ \checkmark}$

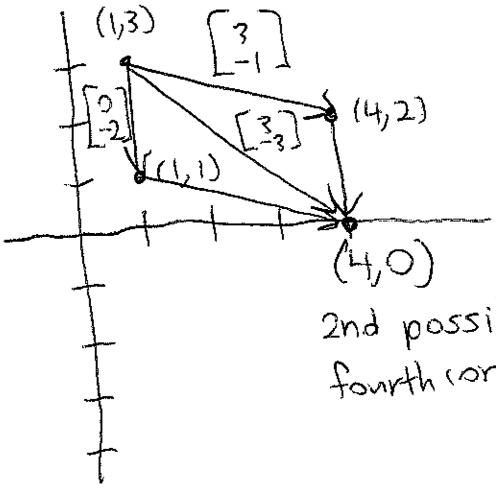
So $\begin{bmatrix}3\\3\\-6\end{bmatrix} = 3\begin{bmatrix}1\\-2\\1\end{bmatrix}+9\begin{bmatrix}0\\1\\-1\end{bmatrix}$

$\begin{bmatrix}3\\3\\6\end{bmatrix}$ is <u>not</u> a linear combination of $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\-1\end{bmatrix}$ because its components don't add up to 0: $3+3+6=12\neq 0$.

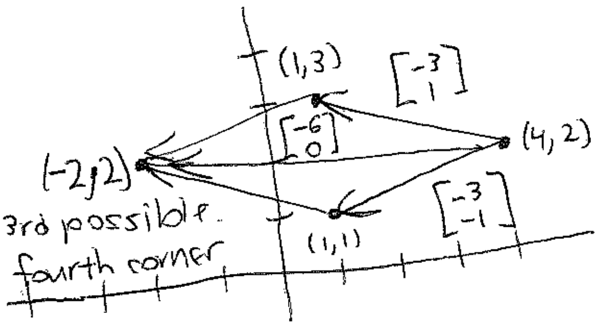
1.1.9



1st possible fourth corner: (4,4)

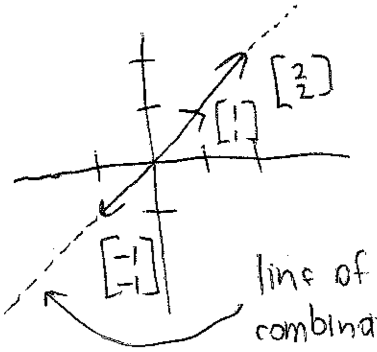


2nd possible fourth corner



3rd possible fourth corner

1.1.25 First, can take $\vec{u}=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}=\begin{bmatrix}2\\2\end{bmatrix}$, $\vec{w}=\begin{bmatrix}-1\\-1\end{bmatrix}$ in 2-dim. space:



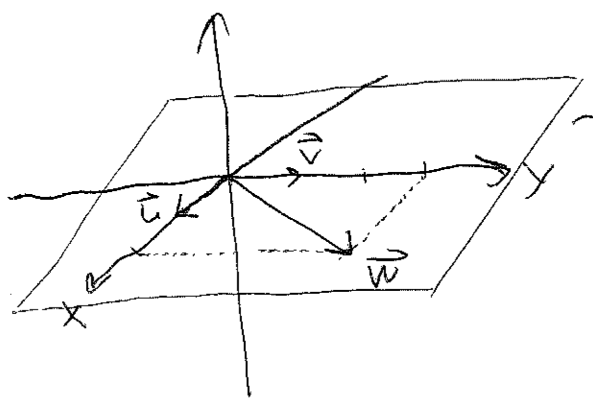
line of linear combinations = line where all vectors have same x and y coordinate

$$c\vec{u}+d\vec{v}+e\vec{w}=c\begin{bmatrix}1\\1\end{bmatrix}+d\begin{bmatrix}2\\2\end{bmatrix}+e\begin{bmatrix}-1\\-1\end{bmatrix}$$

$$=\begin{bmatrix}c+2d-e\\c+2d-e\end{bmatrix}$$

x and y coordinates always have to be equal

Second: can take $\vec{u}=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v}=\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w}$ = some linear combination of $\vec{u}$ and $\vec{v}$, for example $\vec{w}=\begin{bmatrix}2\\3\\0\end{bmatrix}=2\vec{u}+3\vec{v}$. Then all linear combinations look like $c\begin{bmatrix}1\\0\\0\end{bmatrix}+d\begin{bmatrix}0\\1\\0\end{bmatrix}+e\begin{bmatrix}2\\3\\0\end{bmatrix}=\begin{bmatrix}c+2e\\d+3e\\0\end{bmatrix}$ z-coordinate is always 0, so linear combinations fill up the xy-plane



plane of linear combinations = xy-plane (all vectors with z-coordinate = 0)

1.1.26 $c\begin{bmatrix}1\\2\end{bmatrix}+d\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}14\\8\end{bmatrix}$

$\begin{bmatrix}c+3d\\2c+d\end{bmatrix} \longrightarrow \begin{cases}c+3d=14\\2c+d=8\end{cases}$

Solve: Eqn 2 − 2 Eqn 1: $-5d=-20$, $d=4$

So $c=14-3(4)=2$

$\begin{bmatrix}14\\8\end{bmatrix}=2\begin{bmatrix}1\\2\end{bmatrix}+4\begin{bmatrix}3\\1\end{bmatrix}$

1.1.29 We need $c\begin{bmatrix}1\\3\end{bmatrix}+d\begin{bmatrix}2\\7\end{bmatrix}+e\begin{bmatrix}1\\5\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$, or

$\begin{cases}c+2d+e=0\\3c+7d+5e=1\end{cases} \xrightarrow{\text{Eqn 2}-3\text{ Eqn 1}} \begin{cases}c+2d+e=0\\d+2e=1\end{cases}$

$\xrightarrow{\text{Eqn. 1}-2\text{ Eqn. 2}} \begin{cases}c-3e=-2\\d+2e=1\end{cases} \longrightarrow \begin{array}{l}c=3e-2\\d=1-2e\\e=\text{anything}\end{array}$

For example, we get one solution if we pick $e=1$: $c=1, d=-1, e=1$

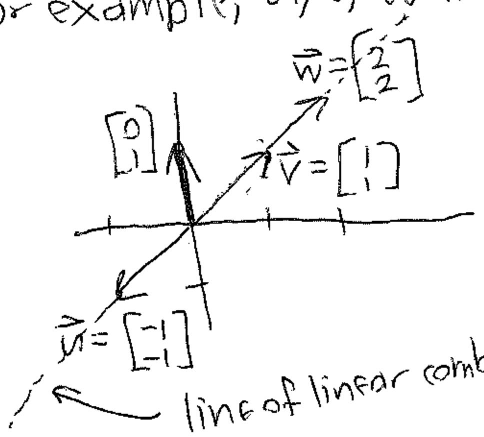
So $\begin{bmatrix}0\\1\end{bmatrix}=1\begin{bmatrix}1\\3\end{bmatrix}+(-1)\begin{bmatrix}2\\7\end{bmatrix}+1\begin{bmatrix}1\\5\end{bmatrix}$

We get another solution if we pick $e=-1$: $c=-5, d=3, e=-1$

So $\begin{bmatrix}0\\1\end{bmatrix}=(-5)\begin{bmatrix}1\\3\end{bmatrix}+3\begin{bmatrix}2\\7\end{bmatrix}+(-1)\begin{bmatrix}1\\5\end{bmatrix}$

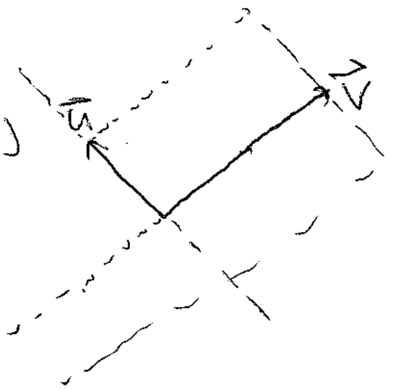
If $\vec{u}, \vec{v}, \vec{w}$ are <u>any</u> 3 vectors in 2-dim. space, we might <u>not</u> be able to write $\begin{bmatrix}0\\1\end{bmatrix}$ as a linear combination of $\vec{u}, \vec{v}, \vec{w}$ in two different ways.

For example, $\vec{u}, \vec{v}, \vec{w}$ might <u>all</u> be on a single line that doesn't include $\begin{bmatrix}0\\1\end{bmatrix}$:



<u>No</u> way to write $\begin{bmatrix}0\\1\end{bmatrix}$ as a linear combination of $\vec{u}, \vec{v}, \vec{w}$.

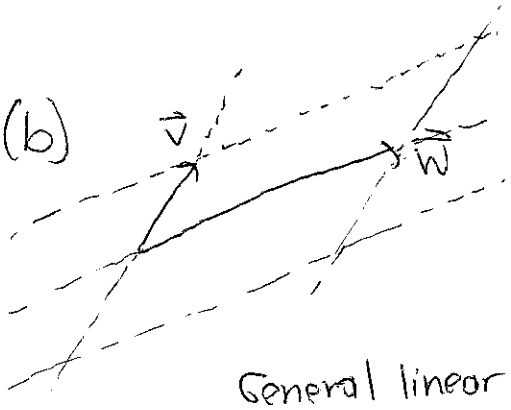
<u>Graded problem</u> (a)



$\vec{u}$ and $\vec{v}$ are not multiples of each other (they are "independent") so because there are <u>two</u> vectors, their linear combinations fill up a <u>plane</u>.

General linear combination: $c\vec{u} + d\vec{v} = c\begin{bmatrix}2\\0\\0\end{bmatrix} + d\begin{bmatrix}0\\2\\2\end{bmatrix} = \begin{bmatrix}2c\\2d\\2d\end{bmatrix}$

(b)



$\vec{v}$ and $\vec{w}$ are not multiples of each other, so their linear combinations also fill up a <u>plane</u>.

General linear combination: $e\vec{v} + f\vec{w} = e\begin{bmatrix}0\\2\\2\end{bmatrix} + f\begin{bmatrix}2\\2\\3\end{bmatrix} = \begin{bmatrix}2f\\2e+2f\\2e+3f\end{bmatrix}$

(e) Question: What vectors can we write as <u>both</u> $\begin{bmatrix}2c\\2d\\2d\end{bmatrix}$ and $\begin{bmatrix}2f\\2e+2f\\2e+3f\end{bmatrix}$?

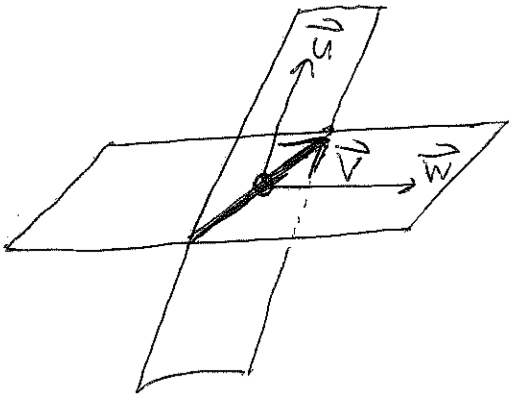
Need $\begin{cases}2c = 2f\\2d = 2e+2f\\2d = 2e+3f\end{cases} \rightarrow \begin{cases}c = f\\d = e+f\\d = e+\frac{3}{2}f\end{cases}$ $f = d-e = \frac{3}{2}f$ $c = 0$ also

only works if $f=0$

Now $d = e+f \xrightarrow[f=0]{} d = e$. So these vectors look like $\begin{bmatrix}2(0)\\2(d)\\2(d)\end{bmatrix}$, or $\begin{bmatrix}2(0)\\2(d)+2(0)\\2(d)+3(0)\end{bmatrix}$ (same)

$\longrightarrow d \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ (all multiples of $\vec{v}$)

So if a vector is a linear comb. of $\vec{u}, \vec{v}$ and a linear comb. of $\vec{v}, \vec{w}$, then it is a multiple of $\vec{v}$. These multiples fill up a line:



Intersection of these two planes is a line.