# Homework 3 Solutions

**2.1.4** If $z=2$, then $(x,y)$ satisfy $\begin{cases} x+y+3(2)=6 \\ x-y+2=4 \end{cases}$

$$\rightarrow \begin{cases} x+y=0 \\ x-y=2 \end{cases} \quad \xrightarrow{\text{Add}} 2x=2 \rightarrow x=1, \qquad \xrightarrow{\text{subtract}} 2y=-2 \rightarrow y=-1 \qquad \rightarrow \boxed{(x,y,z)=(1,-1,2)}$$

If $z=0$, then $(x,y)$ satisfy $\begin{cases} x+y+3(0)=6 \\ x-y+0=4 \end{cases} \rightarrow \begin{cases} x+y=6 \\ x-y=4 \end{cases}$

$$\xrightarrow{\text{Add}} 2x=10 \rightarrow x=5, \qquad \xrightarrow{\text{Subtract}} 2y=2 \rightarrow y=1 \qquad \rightarrow \boxed{(x,y,z)=(5,1,0)}$$

Third point halfway in between should have average $(x,y,z)$ coordinates: $(x,y,z) = \dfrac{(1,-1,2)+(5,1,0)}{2} = \dfrac{(6,0,2)}{2} = \boxed{(3,0,1)}$

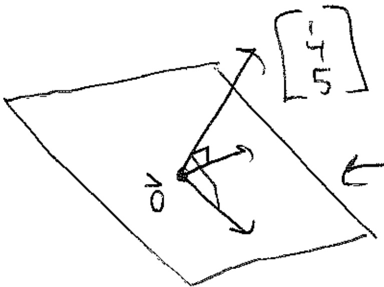
**2.1.17**
$$\begin{bmatrix} y \\ z \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{Q} \begin{bmatrix} y \\ z \\ x \end{bmatrix}$$

($P$ and $Q$ are inverses of each other)

**2.1.22** $(1,4,5)\cdot(x,y,z) = 1x+4y+5z = \underbrace{\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}}$

The columns of $A$ are vectors in only $\underline{1}$-dimensional space.

The solutions to $A\vec{x}=\vec{0}$ lie on a plane perpendicular to the vector $\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

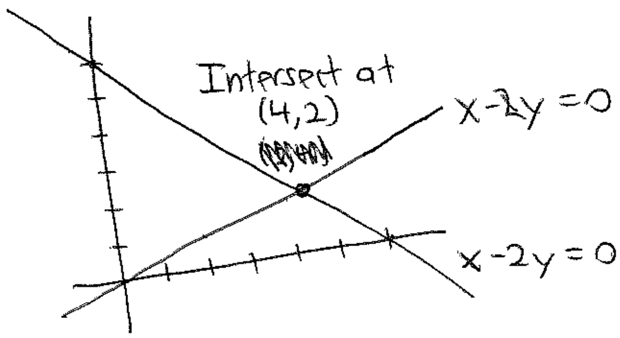



Plane of solutions $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, which satisfy $x+4y+5z=0$

2.1.26 $\begin{cases} x-2y=0 \\ x+y=6 \end{cases}$ $\begin{matrix} (y=\frac{1}{2}x) \\ (y=6-x \end{matrix}$ $\xrightarrow{\text{subtract}}$ $-3y=-6 \to y=2$

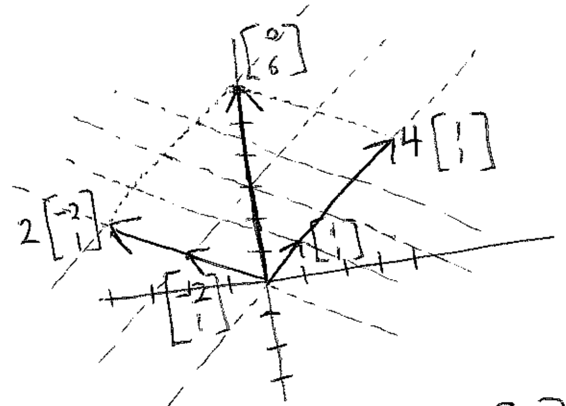
$x=2(2)=4$

Row Picture




Column picture:

$4\begin{bmatrix}1\\1\end{bmatrix}+2\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}0\\6\end{bmatrix}$




2.1.29 $\vec{u}_0=\begin{bmatrix}1\\0\end{bmatrix}$, $\vec{u}_1=\begin{bmatrix}.8 & .3\\.2 & .7\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}.8\\.2\end{bmatrix}$, $\vec{u}_2=\begin{bmatrix}.8 & .3\\.2 & .7\end{bmatrix}\begin{bmatrix}.8\\.2\end{bmatrix}=\begin{bmatrix}.7\\.3\end{bmatrix}$

$\vec{u}_3=\begin{bmatrix}.8 & .3\\.2 & .7\end{bmatrix}\begin{bmatrix}.7\\.3\end{bmatrix}=\begin{bmatrix}.65\\.35\end{bmatrix}$

For all 4 vectors, sum of x and y coordinates $=1$

(We can prove this will always work: suppose $x+y=1$. Then

$\begin{bmatrix}.8 & .3\\.2 & .7\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}.8x+.3y\\.2x+.7y\end{bmatrix}$, and we still get $(.8x+.3y)+(.2x+.7y)$

$=(.8+.2)x+(.3+.7)y=1x+1y=1.$)

2.2.6 $\begin{cases} 2x+by=16 \\ 4x+8y=g \end{cases} \longrightarrow \left[\begin{array}{cc|c} 2 & b & 16 \\ 4 & 8 & g \end{array}\right] \xrightarrow{\text{Row2-2Row1}} \left[\begin{array}{cc|c} 2 & b & 16 \\ 0 & 8-2b & g-32 \end{array}\right]$

Singular if $8-2b=0$, or $\boxed{b=4.}$

If system is singular, need $\boxed{g=32}$ to get solutions.

If $b=4$, $g=32$, all solutions satisfy $2x+4y=16$, or $x=8-2y$

one solution: $\boxed{y=1,\ x=8-2(1)=6}$; another: $\boxed{y=2,\ x=8-2(2)=4}$

**2.2.13** $\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 4 & -5 & 1 & 7 \\ 2 & -1 & -3 & 5 \end{array}\right] \xrightarrow[\text{Subtract 1·Row 1 from Row 3}]{\text{Subtract 2·Row 1 from Row 2.}} \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & 2 \end{array}\right] \xrightarrow[\text{from Row 3}]{\text{Subtract 2·Row 2}}$

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 0 \end{array}\right] \longrightarrow \begin{array}{l} 2x-3y=3 \longrightarrow x=\frac{1}{2}(3+3y)=\frac{1}{2}(3+3)=\boxed{3} \\ y+z=1 \longrightarrow y=1-z=1-0=\boxed{1} \\ -5z=0 \longrightarrow z=\boxed{0} \end{array}$$

**2.2.18** (This problem has more than one solution.)

Rows 2 and 3 should be multiples of Row 1. For example:

$$\begin{array}{l} x+2y+3z=b_1 \\ -x-2y-3z=b_2 \\ 4x+8y+12z=b_3 \end{array} \quad \begin{array}{c} \xrightarrow{\text{Eqn. 2+Eqn. 1}} \\ \xrightarrow{\text{Eqn 3}-4\text{ Eqn. 1}} \end{array} \quad \begin{array}{l} x+2y+3z=b_1 \\ 0=b_1+b_2 \\ 0=-4b_1+b_3 \end{array}$$

This system has infinitely many solutions if $b_1+b_2=0$ and $-4b_1+b_3=0$.

It has no solutions if $b_1+b_2\neq 0$ or $-4b_1+b_3\neq 0$.

So to get solutions, we need $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b_1\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ satisfies this, but

$\begin{bmatrix} 1 \\ 10 \\ 100 \end{bmatrix}$ does not.

**2.2.21** $\left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right] \xrightarrow{\text{Row 2}-\frac{1}{2}\text{Row 1}} \left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right] \xrightarrow{\text{Row 3}-\frac{2}{3}\text{Row 2}}$

$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & 1 & 0 & 0 \\ 0 & 0 & 4/3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right] \xrightarrow{\text{Row 4}-\frac{3}{4}\text{Row 3}} \left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & 1 & 0 & 0 \\ 0 & 0 & 4/3 & 1 & 0 \\ 0 & 0 & 0 & 5/4 & 5 \end{array}\right]$

Now can use back substitution:

$\frac{5}{4}t=5 \longrightarrow \boxed{t=4}$

$\frac{4}{3}z+t=0 \rightarrow z=\frac{3}{4}(-4)=\boxed{-3}$, $\frac{3}{2}y+z=0 \rightarrow y=\frac{2}{3}(+3)=\boxed{2}$,

$2x+y=0 \rightarrow x=\frac{1}{2}(-2)=\boxed{-1}$, $\boxed{(x,y,z,t)=(-1,2,-3,4)}$

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right] \xrightarrow{\text{Row } 2 + \frac{1}{2}\text{Row } 1} \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right] \xrightarrow{\text{Row } 3 + \frac{2}{3}\text{Row } 2}$$

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right] \xrightarrow{\text{Row } 4 + \frac{3}{4}\text{Row } 3} \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & 0 & 5/4 & 5 \end{array}\right]$$

Continue elimination this time:

$$\xrightarrow{\text{Row } 3 + \frac{4}{5}\text{Row } 4}$$

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & 0 & 4/3 & 0 & 4 \\ 0 & 0 & 0 & 5/4 & 5 \end{array}\right] \xrightarrow{\text{Row } 2 + \frac{3}{4}\text{Row } 3} \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 & 3 \\ 0 & 0 & 4/3 & 0 & 4 \\ 0 & 0 & 0 & 5/4 & 5 \end{array}\right] \xrightarrow{\text{Row } 1 + \frac{2}{3}\text{Row } 2} \left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 2 \\ 0 & 3/2 & 0 & 0 & 3 \\ 0 & 0 & 4/3 & 0 & 4 \\ 0 & 0 & 0 & 5/4 & 5 \end{array}\right]$$

$\frac{1}{2}$ Row 1
$\frac{2}{3}$ Row 2
$\frac{3}{4}$ Row 3
$\frac{4}{5}$ Row 4

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \longrightarrow \boxed{(x,y,z,t) = (1,2,3,4)}$$

## Graded Problem

(a) Matrix-vector equation:

$$\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Vector equation:

$$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

(b)

$$\left[\begin{array}{cc|cc} 2 & -2 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\text{Row } 2 + \frac{1}{2}\text{Row } 1} \left[\begin{array}{cc|cc} 2 & -2 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 1 \end{array}\right] \xrightarrow{\text{Row } 1 + 2\text{ Row } 2} \left[\begin{array}{cc|cc} 2 & 0 & 2 & 2 \\ 0 & 1 & \frac{1}{2} & 1 \end{array}\right]$$

We can solve both cases for $\vec{b}$ at the same time this way

$$\xrightarrow{(1/2)\text{ Row } 1} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & 1 \end{array}\right]$$

If $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -2 \\ 2 \end{bmatrix}$

If $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 2 \end{bmatrix}$