H08

Ⓐ $\vec{r}(t) = t\vec{i} + (1-t)\vec{j} + \vec{k}$

16.1. 10. $\frac{d\vec{r}}{dt} = \vec{i} - \vec{j}$ $\quad x = t,\ y = 1-t,\ z = 1$

$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{2}$ $\quad \Downarrow$

$x - y + z - 2 = 2t - 2$

$\int_C f(x,y,z)\,ds = \int_0^1 (2t-2)\sqrt{2}\,dt$

$= \sqrt{2}\left[t^2 - 2t\right]_0^1 = -\sqrt{2}$

11. $\frac{d\vec{r}}{dt} = 2\vec{i} + \vec{j} - 2\vec{k}$ $\quad xy + y + z$

$= 2t \cdot t + t + (2-2t)$

$\left|\frac{d\vec{r}}{dt}\right| = 3$

$\int_C f(x,y,z)\,ds = \int_0^1 (2t^2 - t + 2)\,3\,dt$

$= 3\left[\frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t\right]_0^1 = \frac{13}{2}$

17. $\frac{d\vec{r}}{dt} = \vec{i} + \vec{j} + \vec{k}$ $\quad \frac{x+y+z}{x^2+y^2+z^2} = \frac{t+t+t}{t^2+t^2+t^2}$

$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{3}$ $\quad = \frac{1}{t}$

$\int_C f(x,y,z)\,ds = \int_a^b \left(\frac{1}{t}\right)\sqrt{3}\,dt$

$= \left[\sqrt{3}\ln|t|\right]_a^b = \sqrt{3}\ln\left(\frac{b}{a}\right)$ since $0 < a \le b$.

18. $\frac{d\vec{r}}{dt} = (-a\sin t)\vec{j} + (a\cos t)\vec{k}$ $\quad -\sqrt{x^2+z^2}$

$\left|\frac{d\vec{r}}{dt}\right| = |a|$

$= -\sqrt{0 + a^2\sin^2 t}$

$= \begin{cases} -|a|\sin t, & 0 \le t \le \pi \\ |a|\sin t, & \pi \le t \le 2\pi \end{cases}$

$\int_C f(x,y,z)\,ds = \int_0^\pi -|a|^2\sin t\,dt + \int_\pi^{2\pi} |a|^2 \sin t\,dt$

$= \left[a^2\cos t\right]_0^\pi - \left[a^2\cos t\right]_\pi^{2\pi}$

$= -4a^2$

Ⓑ 24. $\frac{d\vec{r}}{dt} = 2t\vec{j} + 2\vec{k}$

$\left|\frac{d\vec{r}}{dt}\right| = 2\sqrt{t^2+1}$

$M = \int_C \delta(x,y,z)\,ds$

$= \int_{-1}^1 \left(15\sqrt{(t^2-1)+2}\right)\left(2\sqrt{t^2+1}\right)dt$

$= \int_{-1}^1 30(t^2+1)\,dt$

$= 80.$

$M_{xy} = \int_C y\,\delta(x,y,z)\,ds$

$= \int_{-1}^1 (t^2-1)\left[30(t^2+1)\right]dt$

$= \int_{-1}^1 30(t^4-1)\,dt$

$= -48.$

$\bar{y} = -\frac{48}{80}$

$= -\frac{3}{5}$

$M_{yz} = 0. \Rightarrow \bar{x} = 0.$

$\bar{z} = 0$ by symmetry.

$(\bar{x}, \bar{y}, \bar{z}) = (0, -\frac{3}{5}, 0)$

27. Let $x = a\cos t$ and $y = a\sin t$, $0 \le t \le 2\pi$.

$\frac{dx}{dt} = -a\sin t,\ \frac{dy}{dt} = a\cos t,\ \frac{dz}{dt} = 0.$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = a\,dt.$

$I_z = \int_C (x^2+y^2)\,\delta\,ds$

$= \int_0^{2\pi} (a^2\sin^2 t + a^2\cos^2 t)\,a\,\delta\,dt.$

$= 2\pi\delta a^3.$

$M = \int_C \delta(x,y,z)\,ds$

$= \int_0^{2\pi} \delta\,a\,dt$

$= 2\pi\delta a.$

$R_z = \sqrt{\frac{I_z}{M}}$

$= \sqrt{\frac{2\pi a^3\delta}{2\pi a\delta}}$

$= a$

16.2 C.

17. $x = t$ and $y = x^2 = t^2$

$\Rightarrow \vec{r} = t\vec{i} + t^2\vec{j} \quad -1 \le t \le 2.$

$\frac{d\vec{r}}{dt} = \vec{i} + 2t\vec{j}$

$\vec{F} = xy\vec{i} + (x+y)\vec{j}$

$\Rightarrow \vec{F} = t^3\vec{i} + (t+t^2)\vec{j}$

$\vec{F}\cdot\frac{d\vec{r}}{dt} = t^3 + (2t^2 + 2t^3)$

$= 3t^3 + 2t^2$

$\int_C xy\,dx + (x+y)\,dy$

$= \int_{-1}^{2} (3t^3 + 2t^2)\,dt$

$= \frac{69}{4}$

18. Along $(0,0)$ to $(1,0)$:

$\vec{r} = t\vec{i} \quad 0 \le t \le 1$

$\vec{F} = (x-y)\vec{i} + (x+y)\vec{j}$

$\vec{F} = t\vec{i} + t\vec{j}$ $\quad \frac{d\vec{r}}{dt} = \vec{i}$ $\quad \vec{F}\cdot\frac{d\vec{r}}{dt} = t.$

Along $(1,0)$ to $(0,1)$.

$\vec{r} = (1-t)\vec{i} + t\vec{j}, \quad 0 \le t \le 1$

$\vec{F} = (x-y)\vec{i} + (x+y)\vec{j}$

$\vec{F} = (1-2t)\vec{i} + \vec{j}$

$\frac{d\vec{r}}{dt} = -\vec{i} + \vec{j}$ $\quad \vec{F}\cdot\frac{d\vec{r}}{dt} = 2t.$

Along $(0,1)$ to $(0,0)$

$\vec{r} = (1-t)\vec{j}, \quad 0 \le t \le 1$

$\vec{F} = (x-y)\vec{i} + (x+y)\vec{j}$

$\vec{F} = (t-1)\vec{i} + (1-t)\vec{j}$ $\quad \vec{F}\cdot\frac{d\vec{r}}{dt} = t-1$

$\frac{d\vec{r}}{dt} = -\vec{j}$

$\Rightarrow \int_0^1 t\,dt + \int_0^1 2t\,dt + \int_0^1 (t-1)\,dt$

$= 1$

19. $\vec{r} = x\vec{i} + y\vec{j} = y^2\vec{i} + y\vec{j}, \quad 2 \ge y \ge -1$

$\frac{d\vec{r}}{dy} = 2y\vec{i} + \vec{j}$

$\vec{F} = x^2\vec{i} - y\vec{j}$

$= y^4\vec{i} - y\vec{j}$

$\vec{F}\cdot\frac{d\vec{r}}{dy} = 2y^5 - y$

$\int_C \vec{F}\cdot\vec{T}\,ds = \int_2^{-1} \vec{F}\cdot\frac{d\vec{r}}{dy}\,dy$

$= \int_2^{-1} (2y^5 - y)\,dy$

$= -\frac{39}{2}$

20. $\vec{r} = (\cos t)\vec{i} + (\sin t)\vec{j} \quad 0 \le t \le \frac{\pi}{2}$

$\frac{d\vec{r}}{dy} = (-\sin t)\vec{i} + (\cos t)\vec{j}$

$\vec{F} = y\vec{i} - x\vec{j}$

$= (\sin t)\vec{i} - (\cos t)\vec{j}$

$\vec{F}\cdot\frac{d\vec{r}}{dt} = -\sin^2 t - \cos^2 t$

$= -1$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\frac{\pi}{2}} (-1)\,dt$

$= -\frac{\pi}{2}$

D

25. $\vec{F}_1 = (a\cos t)\vec{i} + (a\sin t)\vec{j}$

$\frac{d\vec{r}_1}{dt} = (-a\sin t)\vec{i} + (a\cos t)\vec{j}$

$\vec{F}_1\cdot\frac{d\vec{r}}{dt} = 0$

$Circ_1 = 0.$

$M_1 = a\cos t.$

$N_1 = a\sin t,$

$dx = -a\sin t\,dt$

$dy = a\cos t\,dt.$

$Flux_1 = \int_C M_1\,dy - N_1\,dx.$

$= \int_0^{\pi} (a^2\cos^2 t + a^2\sin^2 t)\,dt$

$= \int_0^{\pi} a^2\,dt = a^2\pi.$

$\vec{F}_2 = t\vec{i},$

$\frac{d\vec{r}_2}{dt} = \vec{i}$

$Circ_2 = \int_{-a}^{a} t\,dt = 0.$

$M_2 = t$

$N_2 = 0$

$dx = dt$

$dy = 0.$

$Flux_2 = \int_C M_2\,dy - N_2\,dx$

$= \int_{-a}^{a} 0\,dt$

$= 0.$

$Circ = Circ_1 + Circ_2 = 0.$

$Flux = Flux_1 + Flux_2$

$= a^2\pi$

26. $F_1 = (a^2\cos^2 t)\vec{i} + (a^2\sin^2 t)\vec{j}$

$\frac{d\vec{r}_1}{dt} = (-a\sin t)\vec{i} + (a\cos t)\vec{j}$

$F_1 \cdot \frac{d\vec{r}_1}{dt} = -a^3\sin t\cos^2 t + a^3\cos t\sin^2 t$

$Circ_1 = \int_0^{\pi} (-a^3\sin t\cos^2 t + a^3\cos t\sin^2 t)\,dt$

$= -\frac{2a^3}{3}$

$M_1 = a^2\cos^2 t \quad dy = a\cos t\,dt$

$N_1 = a^2\sin^2 t \quad dx = -a\sin t\,dt$

$Flux_1 = \int_c M_1 dy - N_1 dx$

$= \int_0^{\pi} (a^3\cos^3 t + a^3\sin^3 t)\,dt$

$= \frac{4}{3}a^3$

$F_2 = t^2\vec{i}$

$\frac{d\vec{r}_2}{dt} = \vec{i}$

$\vec{F}_2 \cdot \frac{d\vec{r}}{dt} = t^2$

$Circ_2 = \int_{-a}^{a} t^2\,dt$

$= \frac{2a^3}{3}$

$M_2 = t^2 \quad dy = 0$

$N_2 = 0 \quad dx = dt$

$Flux_2 = \int_c M_2 dy - N_2 dx$

$= 0$

$Circ = Circ_1 + Circ_2 = 0$

$Flux = Flux_1 + Flux_2$

$= \frac{4}{3}a^3$

(E) 42.

$\vec{F} \cdot \frac{d\vec{r}}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}$

$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$

where $f(x,y,z) = \frac{1}{2}(x^2+y^2+z^2)$ by chain rule.

$\Downarrow$

$\vec{F} \cdot \frac{dr}{dt} = \frac{d}{dt}[f(\vec{r}(t))]$

$\int_c \vec{F} \cdot \frac{dr}{dt}\,dt$

$= \int_a^b \frac{d}{dt}[f(\vec{r}(t))]\,dt$

$= f[\vec{r}(b)] - f[\vec{r}(a)]$

Since C is an entir ellipse.

$\vec{r}(b) = \vec{r}(a)$

Circulation $= 0$.

43. Let $x = t$ be the parameter.

$\Downarrow$

$y = x^2 = t^2$

$z = x = t$

$\vec{r} = t\vec{i} + t^2\vec{j} + t\vec{k}, \ (0 \le t \le 1)$

$\frac{d\vec{r}}{dt} = \vec{i} + 2t\vec{j} + \vec{k}$

$\vec{F} = t^3\vec{i} + t^2\vec{j} - t^2\vec{k}$

$\vec{F} \cdot \frac{d\vec{r}}{dt} = t^3 + 2t^3 - t^3$

$= 2t^3$

$Flow = \int_0^1 2t^3\,dt = \frac{1}{2}$

16.3 (F) $\vec{F} = 2x\vec{i} + 3y\vec{j} + 4z\vec{k}$

7. $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x,y,z) = x^2 + g(y,z)$.

$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 3y. \Rightarrow g(y,z) = \frac{3y^2}{2} + h(z)$.

$\Downarrow$

$f(x,y,z) = x^2 + \frac{3y^2}{2} + h(z)$.

$\frac{\partial f}{\partial z} = 4z. = h(z) = 2z^2 + C$.

$f(x,y,z) = x^2 + \frac{3y^2}{2} + 2z^2 + C$.

8. $\vec{F} = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$

$\frac{\partial f}{\partial x} = y+z \Rightarrow f(x,y,z) = (y+z)x + g(y,z)$.

$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x+z \rightarrow \frac{\partial g}{\partial y} = z. \Rightarrow f(x,y,z) = (y+z)x + yz + h(z)$.

$\frac{\partial f}{\partial z} = x+y+\frac{\partial h}{\partial z} = x+y = \frac{\partial h}{\partial z} = 0 \Rightarrow f(x,y,z) = (y+z)x + zy + C$.

16. $\int_{(0,0,0)}^{(3,3,1)} 2x\,dx - y^2dy - \frac{4}{1+z^2}dz.$

Let $\vec{F}(x,y,z) = 2x - y^2\vec{j} - (\frac{4}{1+z^2})\vec{k}$

$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y}$.
$\Downarrow$ exact.

$\frac{\partial f}{\partial x} = 2x \Rightarrow f(x,y,z) = x^2 + g(y,z).$

$\frac{\partial f}{\partial y} = -y^2 \Rightarrow f(x,y,z) = x^2 - \frac{y^2}{3} + h(x).$

$\frac{\partial f}{\partial z} = -\frac{4}{1+z^2} \Rightarrow f(x,y,z) = x^2 - \frac{y^2}{3} - 4\tan^{-1}z + C$

$\int_{(0,0,0)}^{(3,3,1)} 2x\,dx - y^2dy - \frac{4}{1-z^2}dz$

$= f(3,3,1) - f(0,0,0)$

$= -\pi$

17. $\int_{(1,0,0)}^{(0,1,1)} \sin y\cos x\,dx + \cos y\sin x\,dy + dz.$

Let $\vec{F}(x,y,z) = (\sin y\cos x)\vec{i} + (\cos y\sin x)\vec{j} + \vec{k}$

$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = \cos y\cos x = \frac{\partial M}{\partial y}$. $\Rightarrow$ exact

~~$\Rightarrow M\,dx + N\,dy + P\,dz$ exact.~~

$\frac{\partial f}{\partial x} = \sin y\cos x \Rightarrow f(x,y,z) = \sin y\sin x + g(y,z)$

$\frac{\partial f}{\partial y} = \cos y\sin x + \frac{\partial g}{\partial y} = \cos y\sin x. \Rightarrow \frac{\partial g}{\partial y} = 0.$

$f(x,y,z) = \sin y\sin x + h(z)$

$\frac{\partial f}{\partial z} = 1 \Rightarrow f(x,y,z) = \sin y\sin x + z + C.$

$\int_{(1,0,0)}^{(0,1,1)} \sin y\cos x\,dx + \cos y\sin x\,dy + dz$

$= f(0,1,1) - f(1,0,0)$

$= 1$

G

34. $\vec{F} = \nabla f \Rightarrow g(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} \vec{F}\cdot d\vec{r} = \int_{(0,0,0)}^{(x,y,z)} \nabla f\cdot d\vec{r} = f(x,y,z) - f(0,0,0)$

$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} - 0$, $\frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} - 0$, and $\frac{\partial g}{\partial z} = \frac{\partial f}{\partial z} - 0 \Rightarrow \nabla g = \nabla f = \vec{F}$,

35. The path will not matter, the work along any path will be the same because the field is conservati[ve]

H. 16.4

1. $\vec{F} = -y\vec{i} + x\vec{j}$

$M = -y = -a\sin t$, $N = x = a\cos t.$

$dx = -a\sin t\,dt$, $dy = a\cos t\,dt.$

$\frac{\partial M}{\partial x} = 0$, $\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$, and $\frac{\partial N}{\partial y} = 0$

$\oint_C M\,dy - N\,dx = \int_0^{2\pi} [(-a\sin t)(a\cos t) - (a\cos t)(-a\sin t)]dt.$

$= \int_0^{2\pi} 0\,dt$

$= 0.$

$\iint_R (\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y})dx\,dy = \iint_R 0\,dx\,dy = 0.$

$\oint_C M\,dx + N\,dy = \int_0^{2\pi} [(-a\sin t)(-a\sin t) - (a\cos t)(a\cos t)]dt$

$= \int_0^{2\pi} a^2\,dt$

$= 2\pi a^2.$

$\iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dx\,dy$

$= \int_{-a}^{a}\int_{-c}^{\sqrt{a^2-x^2}} 2\,dy\,dx.$

$= \int_{-a}^{a} 4\sqrt{a^2-x^2}\,dx.$

$= 4\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{-a}^{a}$

$= 2a^2\pi$

3. $\vec{F} = 2x\vec{i} - 3y\vec{j}$

$M = 2x = 2a,\ N = -3y = -3a\sin t.$

$dx = -a\sin t\,dt,\ dy = a\cos t\,dt$

$\frac{\partial M}{\partial x} = 2,\ \frac{\partial M}{\partial y} = 0,\ \frac{\partial N}{\partial x} = 0,\ \frac{\partial N}{\partial y} = -3.$

$\oint_C M\,dy - N\,dx = \int_0^{2\pi} [(2a\cos t)(a\cos t) + (3a\sin t)(-a\sin t)]\,dt.$

$= -\pi a^2$

$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) = \iint_R -1\,dx\,dy = \int_0^{2\pi}\int_0^a -r\,dr\,d\theta$

$= -\pi a^2.$

$\oint_C M\,dx + N\,dy = \int_0^{2\pi} [(2a\cos t)(-a\sin t) + (-3a\sin t)(a\cos$

$= 0.$

$\iint_R 0\,dx\,dy = 0.$

I

5. $F = (x-y)\vec{i} + (y-x)\vec{j}$ C: The square bounded by $x=0, x=1, y=0, y=1.$

$M = x - y,\ N = y - x.$

$\frac{\partial M}{\partial x} = 1,\ \frac{\partial M}{\partial y} = -1,\ \frac{\partial N}{\partial x} = -1,\ \frac{\partial N}{\partial y} = 1.$

$\text{Flux} = \int_0^1\int_0^1 2\,dx\,dy = 2.$

$\text{Circ} = \iint_R [-1-(-1)]\,dx\,dy = 0$

9. $F = (x + e^x\sin y)\vec{i} + (x + e^x\cos y)\vec{j}$ $C: r^2 = \cos 2\theta.$

$M = x + e^x\sin y,\ N = x + e^x\cos y$

$\frac{\partial M}{\partial x} = 1 + e^x\sin y,\ \frac{\partial M}{\partial y} = e^x\cos y,\ \frac{\partial N}{\partial x} = 1 + e^x\cos y,\ \frac{\partial N}{\partial y} = -e^x\sin y.$

$\text{Flux} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}} r\,dr\,d\theta.$

$= \frac{1}{2}$

$\text{Circ} = \iint_R (1 + e^x\cos y - e^x\cos y)\,dx\,dy$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}} r\,dr\,d\theta = \frac{1}{2}$

J. 21. $M = x = a\cos t,\ N = y = b\sin t.$

$dx = -a\sin t\,dt,\ dy = b\cos t\,dt.$

$A = \frac{1}{2}\int_0^{2\pi} (ab\cos^2 t + ab\sin^2 t)\,dt$

$= \frac{1}{2}\int_0^{2\pi} ab\,dt$

$= \pi ab.$

22. $M = x = a\cos t,\ N = y = b\sin t$

$dx = -a\sin t\,dt,\ dy = b\cos t\,dt$

$A = \frac{1}{2}\int_0^{2\pi} (ab\cos^2 t + ab\sin^2 t)\,dt$

$= \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab$

23. $M = x = a\cos^3 t,\ N = y = \sin^3 t$

$dx = -3\cos^2 t\sin t\,dt,\ dy = 3\sin^2 t\cos t\,dt$

$A = \frac{1}{2}\int_0^{2\pi} (3\sin^2 t\cos^2 t)(\cos^2 t + \sin^2 t)\,dt.$

$= \frac{1}{2}\int_0^{2\pi} (3\sin^2 t\cos^2 t)\,dt.$

$= \frac{3}{8}\int_0^{2\pi} \sin^2 2t\,dt.$

$= \frac{3}{8}\pi$

24. $M = x = t^2,\ N = y = \frac{t^3}{3} - t$

$dx = 2t\,dt,\ dy = (t^2 - 1)\,dt$

$A = \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} (\frac{1}{3}t^4 + t^2)\,dt.$

$= \frac{8}{5}\sqrt{3}.$

(t)