Last time: We started to look at ~~matrix~~ matrix multiplication:

(matrix)(matrix) = another matrix

Key property of matrix multiplication: $A(B\vec{x}) = (AB)\vec{x}$

Two matrix-vector multiplications — matrix-matrix multiplication — matrix-vector

If we want this property, there's only one way to define AB:

$$AB = A\underbrace{\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}}_{\text{columns of } B} = \underbrace{\begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{bmatrix}}_{\text{Columns of } AB}$$

For this to work, the matrix-vector products $A\vec{b}_1, \dots, A\vec{b}_n$ have to make sense: # columns of A = $\underbrace{\#\text{ components in } \vec{b}_1, \dots, \vec{b}_n}_{\text{same as \# rows in } B}$

Also: # rows in AB = # components in $A\vec{b}_1, \dots, A\vec{b}_n$ = # rows of A

So: ($k \times m$ matrix)($m \times n$ matrix) = $k \times n$ matrix

same

Also last time: Matrix multiplication is not "commutative":

$AB \neq BA$ usually, even if both products exist and have same size.

Sometimes $AB = BA$, but not always.

Example (Problem 2.4.34) Find all matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that satisfy $A\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}A$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} b \\ d \end{bmatrix}$$

When are these two the same? We need:

$$\left.\begin{aligned} a+b &= a+c \\ a+b &= b+d \\ c+d &= a+c \\ c+d &= b+d \end{aligned}\right\} \begin{array}{l}\text{4 linear equations} \\ \text{in 4 variables !}\end{array} \longrightarrow \begin{aligned} b&=c \\ a&=d \\ d&=a \\ c&=b \end{aligned} \longrightarrow A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \text{ where}$$

a, b can be <u>any</u> real numbers

This matrix will "commute" with $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

For example: $a=2, b=-1$:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \left(\text{both equal } \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} !\right)$$

But:

$$\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}} \neq \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}_{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}$$

$a \neq d$

<u>Also last time</u>: Matrix multiplication <u>is</u> associative:

$A(BC) = (AB)C$ (the Key Property $A(B\vec{x}) = (AB)\vec{x}$ is the special case where C is a column vector $\vec{x}$)

So far, I've shown you how to calculate AB one column at a time:

$$A\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{bmatrix}$$

because this is related to the Key Property $(AB)\vec{x} = A(B\vec{x})$

Strang calls this the "Second Way" of matrix multiplication.

There are other ways to do the calculation:

<u>"1st Way"</u> Think: How do I get the (i,j)-entry of AB?

$$\begin{bmatrix} & \vdots & \\ \cdots & (AB)_{ij} & \cdots \\ & \vdots & \end{bmatrix} \leftarrow i\text{th row}$$

jth column

We know the jth column of AB is $A\vec{b}_j$.

What is the ith component of $A\vec{b}_j$?

It's a dot product: (ith <u>row</u> of A) $\cdot\ \vec{b}_j$

call it $\vec{a}_i$

So here's the matrix multiplication formula: the (i,j)-entry of AB is the dot product (ith row of A)$\cdot$(jth column of B):

$$\begin{bmatrix} \text{---} \vec{a}_1 \text{---} \\ \text{---} \vec{a}_2 \text{---} \\ \vdots \\ \text{---} \vec{a}_m \text{---} \end{bmatrix} \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix} = \begin{bmatrix} \vec{a}_1\cdot\vec{b}_1 & \vec{a}_1\cdot\vec{b}_2 & \cdots & \vec{a}_1\cdot\vec{b}_n \\ \vec{a}_2\cdot\vec{b}_1 & \vec{a}_2\cdot\vec{b}_2 & \cdots & \vec{a}_2\cdot\vec{b}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{a}_m\cdot\vec{b}_1 & \vec{a}_m\cdot\vec{b}_1 & \cdots & \vec{a}_m\cdot\vec{b}_n \end{bmatrix}$$

"3rd Way": Calculate AB one row at a time:

$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix} B = \begin{bmatrix} \vec{a}_1 B \\ a_2 B \\ \vdots \\ \vec{a}_m B \end{bmatrix}$$

← Rows of AB
(row vector × matrix = another row vector)

"4th Way": Multiply columns of A by rows of B, then add.

Example: $$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \end{bmatrix}$$

(col 1)(row 1) + (col 2)(row 2)

$$= \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} -3 & -4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 5 & 6 \end{bmatrix}$$

Compare "1st way": $$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2(1)-1(3) & 2(2)-1(4) \\ -1(1)+2(3) & -1(2)+2(4) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 5 & 6 \end{bmatrix} \checkmark$$

This method involves another operation: matrix addition.

If two matrices A and B have the same size ($m\times n$), you can add them just by adding corresponding entries:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+1 & -1+1 \\ -1+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Matrix multiplication is commutative: $A+B = B+A$

and associative: $A+(B+C) = (A+B)+C$

One final operation: scalar multiplication with matrices:

$$(-3)\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -3(2) & -3(-1) \\ -3(-1) & -3(2) \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}.$$

We have a <u>distributive law</u> for scalar multiplication:

$$c(A+B) = cA + cB$$

scalar $m \times n$ matrices

But we have <u>two</u> distributive laws for <u>matrix</u> multiplication:

$$A(B+C) = AB + AC$$

multiply A on left or on right

These are not usually the same, so we have to write out both distributive laws

$$(B+C)A = BA + CA$$

<u>Problem 2.4.3</u>: Let's check these rules with $A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$

$$A(B+C) \overset{??}{=} AB + AC = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}\left(\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1(3)+5(0) & 1(3)+5(1) \\ 2(3)+3(0) & 2(3)+3(1) \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 1(0)+5(0) & 1(2)+5(1) \\ 2(0)+3(0) & 2(2)+3(1) \end{bmatrix} + \begin{bmatrix} 1(3)+5(0) & 1(1)+5(0) \\ 2(3)+3(0) & 2(1)+3(0) \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 0 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix} \quad \checkmark$$

Other way:

$$(B+C)A = \begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 24 \\ 2 & 3 \end{bmatrix} \quad \checkmark$$

$$BA + CA = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 18 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 24 \\ 2 & 3 \end{bmatrix}$$

But notice $\begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix} \neq \begin{bmatrix} 9 & 24 \\ 2 & 3 \end{bmatrix}$

So far: We can add matrices: $A+B$ } if sizes are compatible
subtract: $A-B = A+(-1)B$ ← scalar multiplication
multiply: AB

But can we divide by a matrix? $B/A = ??$ (or: $A^{-1}B$)

For real numbers: $\frac{b}{a}$ is the same thing as $\underbrace{a^{-1}}_{\text{"inverse" of } a} b$

The inverse a^{-1} satisfies $aa^{-1} = 1$ (also $a^{-1}a = 1$),
and the number 1 satisfies $1b = b$ (also $b1 = b$).

Now go from numbers (1×1 matrices) to $n \times n$ matrices ← square

$a \rightsquigarrow A$

$1 \rightsquigarrow$ Identity matrix $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

Identity matrix satisfies $IA = A$ and $AI = A$

Check for 2×2: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1(a)+0(c) & 1(b)+0(c) \\ 0(a)+1(c) & 0(b)+1(d) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ✓

Other way: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a(1)+b(0) & a(0)+b(1) \\ c(1)+d(0) & c(0)+d(1) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ✓

The identity matrix allows us to define inverse matrices:
An $n\times n$ matrix A is invertible if it has an inverse A^{-1} such that

$$\underbrace{AA^{-1} = I \text{ and } A^{-1}A = I}_{\text{"two-sided inverse"}}$$

Warning: Not all $n \times n$ matrices have inverses.

Simplest cases: 1×1 matrix $[a]$.

$[a][a^{-1}] = [aa^{-1}] = [1] = I$ and $[a^{-1}][a] = [a^{-1}a] = [1] = I$

Inverse matrix $[a^{-1}]$ exists exactly when $a \neq 0$, so a 1×1 matrix $[a]$ is invertible exactly when $a \neq 0$.

What about 2×2? $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

There's a simple formula for the inverse (if it exists):

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

switch

negate

divide by "determinant"

Let's test this formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & -ab+ba \\ cd-dc & -cb+da \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

Not quite the identity I, but it will be if we divide by $ad-bc$.
But A^{-1} won't exist if $ad-bc=0$! So A is invertible exactly when $ad-bc \neq 0$.

Formula also works the other way:

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} da-bc & \overbrace{db-bd}^{0} \\ \underbrace{-ca+ac}_{0} & -cb+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Notice: In the 2×2 case non-zero matrices can be non-invertible:

Example: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $(1)(4)-(2)(2)=0 \rightarrow$ no inverse

not the zero matrix

On the other hand: $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ $(1)(4)-(3)(1)=1 \rightarrow$ yes inverse

Inverse is: $\dfrac{1}{1(4)-3(1)}\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}.$

One more type of matrix where finding the inverse is easy:

Diagonal matrices: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ Only non-zero entries go on the diagonal

To find inverse, just invert diagonal entries:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$$

How about $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$?

No inverse because we can't invert 0.