## Section 6.2 Diagonalizing a matrix

Idea: If you want to understand $n\times n$ $A$, it is best to use a basis of $\mathbb{R}^n$ that is well-suited to $A$.

Maybe the best basis would be: a basis of eigenvectors for $A$.

Example $A=\begin{bmatrix}7 & 6\\ -8 & -7\end{bmatrix}$ Eigenvalues/vectors satisfy:

$$A\vec{x}=\lambda\vec{x}$$

$\lambda$: eigenvalue

$\vec{x}$: eigenvector, non-zero

$(A-\lambda I)\vec{x}=\vec{0}$ has non-zero null space $\rightarrow$ not invertible $\longrightarrow$ $\boxed{\det(A-\lambda I)=0}$

$$\begin{vmatrix}7-\lambda & 6\\ -8 & -7-\lambda\end{vmatrix}=(7-\lambda)(-7-\lambda)+48=\lambda^2-7\lambda+7\lambda-49+48$$

$$=\lambda^2-1=0\longrightarrow\lambda=\pm1$$

Eigenvectors for $\lambda=1$: Solve $(A-I)\vec{x}=\vec{0}\longrightarrow$

$$\begin{bmatrix}6 & 6\\ -8 & -8\end{bmatrix}\longrightarrow\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\longrightarrow x_1=-x_2\longrightarrow\vec{x}=x_2\begin{bmatrix}-1\\ 1\end{bmatrix}$$

every eigenvector for $\lambda=1$ is a non-zero multiple of this one

For $\lambda=-1$: Solve $(A+I)\vec{x}=\vec{0}\longrightarrow$

$$\begin{bmatrix}8 & 6\\ -8 & -6\end{bmatrix}\longrightarrow\begin{bmatrix}4 & 3\\ 0 & 0\end{bmatrix}\longrightarrow 4x_1=-3x_2\longrightarrow\vec{x}=x_2\begin{bmatrix}-3/4\\ 1\end{bmatrix}$$

What can you do with this?

Put two special eigenvectors into a matrix: $X=\begin{bmatrix}-1 & -3/4\\ 1 & 1\end{bmatrix}$

Columns are eigenvectors, so $AX$ is nice:

$$AX = \underset{A}{\begin{bmatrix} 7 & 6 \\ -8 & -7 \end{bmatrix}} \underset{X}{\begin{bmatrix} -1 & -3/4 \\ 1 & 1 \end{bmatrix}} = \begin{bmatrix} -1 & 3/4 \\ 1 & -1 \end{bmatrix} = \underset{X}{\begin{bmatrix} -1 & -3/4 \\ 1 & 1 \end{bmatrix}} \underset{\Lambda}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

diagonal eigenvalue matrix

Multiply on right by $X^{-1}$: $(AX)X^{-1} = (X\Lambda)X^{-1} \longrightarrow$

$A = X \underset{\substack{\uparrow \\ \text{diagonal}}}{\Lambda} X^{-1}$ We have "diagonalized" $A$.

One thing we can do with this: Find <u>all</u> matrix powers $A^n$.

$$A^n = \underbrace{(X\Lambda X^{-1})(\cancel{X}\Lambda X^{-1}) \cdots (X\Lambda X^{-1})}_{n \text{ times}} = X\Lambda^n X^{-1}$$

cancel  cancel  This is easy: $\begin{bmatrix} 1^n & 0 \\ 0 & (-1)^n \end{bmatrix}$

$$= \underset{X}{\begin{bmatrix} -1 & -3/4 \\ 1 & 1 \end{bmatrix}} \underset{\Lambda^n}{\begin{bmatrix} 1 & 0 \\ 0 & (-1)^n \end{bmatrix}} \underset{X^{-1}}{\frac{1}{-1/4}\begin{bmatrix} 1 & 3/4 \\ -1 & -1 \end{bmatrix}} = \begin{bmatrix} -1 & -\frac{3}{4}(-1)^n \\ 1 & (-1)^n \end{bmatrix} \begin{bmatrix} -4 & -3 \\ 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3(-1)^n & 3-3(-1)^n \\ -4+4(-1)^n & -3+4(-1)^n \end{bmatrix}$$

If $n$ is odd, $(-1)^n = -1$ $\begin{bmatrix} 7 & 6 \\ -8 & -7 \end{bmatrix}$

If $n$ is even, $(-1)^n = 1$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So $A^n = \begin{cases} A & \text{if } n \text{ is odd} \\ I & \text{if } n \text{ is even} \end{cases}$

---

<u>In general</u>: A $n \times n$ matrix is "diagonalizable" if we can write $A = X\Lambda X^{-1}$ with ~~Anda~~ $\Lambda$ <u>diagonal</u>.

This works if $\mathbb{R}^n$ has a <u>basis</u> of eigenvectors for $A$.

Eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_n$

Eigenvectors: $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$

Then we create $\Lambda$ with the eigenvalues: $\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

$X$ comes from the eigenvectors: $X = \begin{bmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{bmatrix}$ ← invertible because $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are a basis for $\mathbb{R}^n$.

Let's check that indeed $A = X\Lambda X^{-1}$, or $AX = X\Lambda$:

$$A\cancel{X} = A\begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} = \begin{bmatrix} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 & \cdots & \lambda_n\vec{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = X\Lambda \checkmark$$

↑

Multiplying a diagonal matrix on the right multiplies the columns of $X$ by the scalar diagonal entries.

Notice: $A = X\Lambda X^{-1}$ does not mean $A = \Lambda$, because $X\Lambda \neq \Lambda X$

Also: to create $X$ and $\Lambda$, you need to order the columns of $X$ and entries of $\Lambda$ consistently:

$\vec{x}_1$ is an eigenvector for $\lambda_1$, $\vec{x}_2$ is an eigenvector for $\lambda_2$, etc.

So you can choose a different order for the $\vec{x}$'s, but then you should adjust the order of $\lambda$'s.

---

When does $\mathbb{R}^n$ have a basis of eigenvectors for $A$.

1. What if all $n$ eigenvalues are real and different? Then each of $\lambda_1, \lambda_2, \ldots, \lambda_n$ has an eigenvector: $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$. Eigenvectors for different eigenvalues are independent, so $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ is a basis → $A$ is diagonalizable.

(2) What if A has complex number eigenvalues? Then eigenvectors will also be complex, so $\mathbb{R}^n$ won't have a basis of eigenvectors. But A might still be diagonalizable if you don't mind working with complex numbers. (11)

(3) What if A has a repeated eigenvalue? Each distinct eigenvalue will give you <u>at least one</u> eigenvector, but this <u>might not</u> be enough for a basis $\longrightarrow$ A <u>might</u> or <u>might not</u> be diagonalizable.

<u>Example of Case (2)</u> $A = \begin{bmatrix} -5 & 13 \\ -2 & 5 \end{bmatrix}$ Eigenvalues: Solve $\det(A-\lambda I)=0$

$$\begin{vmatrix} -5-\lambda & 13 \\ -2 & 5-\lambda \end{vmatrix} = (-5-\lambda)(5-\lambda)-13(-2) = \lambda^2+5\lambda-5\lambda-25+26 = \lambda^2+1=0$$

Eigenvalues are complex numbers: $\lambda = \pm\sqrt{-1}$ (or, $\pm i$)

But since they are <u>different</u>, there are two independent <u>complex</u> eigenvectors, so $A = X \Lambda X^{-1}$

complex eigenvector matrix $\qquad \Lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

This still gives us information about $A^n$. For example, $A^4$.

$$A^4 = (X\Lambda \cancel{X^{-1}})(\cancel{X}\Lambda X^{-1})(X\Lambda X^{-1})(X\Lambda X^{-1}) = X\Lambda^4 X^{-1} = XIX^{-1} = I$$

$$\begin{bmatrix} i^4 & 0 \\ 0 & (-i)^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So $A^4 = I$ (or $A^{-1} = A^3$), not obvious from A itself, but we can check:

$$\underbrace{\begin{bmatrix} -5 & 13 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -5 & 13 \\ -2 & 5 \end{bmatrix}}_{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}\underbrace{\begin{bmatrix} -5 & 13 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -5 & 13 \\ -2 & 5 \end{bmatrix}}_{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Example of non-diagonalizable A with repeated eigenvalues:
See end of notes from last time.

Even if A has repeated eigenvalues, it *might* still be diagonalizable:

Example: $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ ← not invertible, non-zero null space, so *one* eigenvalue will be 0.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & -1-\lambda & 1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & -1-\lambda \\ 1 & -1 \end{vmatrix}$$

$$= (1-\lambda)\big(\underbrace{(-1-\lambda)(1-\lambda)+1}_{\lambda^2-\lambda+\lambda-1+1}\big) + \cancel{(1-\lambda-1) + (-1+1+\lambda)} = (1-\lambda)\lambda^2 = 0$$

Eigenvalues are: $\lambda = 1, 0$ ← repeated root

Eigenvectors for $\lambda=1$: Solve $(A-I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow[\text{Row 3}]{\text{Row 1} \leftrightarrow} \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[\text{-Row 1}]{\text{Row 2}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[\text{Then: -Row 2}]{\text{Row 3-Row 2}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{Row 1+Row 2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{matrix} \rightarrow \vec{x} = x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

one basis (?) eigenvector

For $\lambda=0$: Solve $A\vec{x} = \vec{0}$ (null space) $\rightarrow x_1 - x_2 + x_3 = 0$

two free variables

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Two linearly independent eigenvectors for the same eigenvalue, 0.

Only two different eigenvalues, but one of them has two linearly independent eigenvectors, so we still get a basis.

$$\vec{x_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x_3} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So $A = X\Lambda X^{-1}$:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$

$\Lambda$ is a projection matrix, so $\Lambda^2 = \Lambda$ (also $\Lambda^n = \Lambda$). This means $A^n = A$ for any $n$ as well. But $A$ is not an orthogonal projection matrix because $A$ isn't symmetric.

---

Fun application of diagonalizing to Fibonacci numbers

$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ 5, 8, 13, 21, 34, 55, 89, \ldots$

In general, $F_k = F_{k-1} + F_{k-2}$ (recursion formula)

Use eigenvalues to find a "closed-form" formula for $F_k$:

Idea: $\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} F_1 + F_0 \\ F_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$

In general: $\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} F_k + F_{k-1} \\ F_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k-1} + F_{k-2} \\ F_{k-1} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} F_{k-1} \\ F_{k-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 \begin{bmatrix} F_{k-2} \\ F_{k-3} \end{bmatrix} = \cdots = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

$F_{k-2} + F_{k-3}$ same $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

We can use eigenvalues to compute matrix powers: $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$F_k$ is the second component of this vector.

Eigenvalues: $\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \rightarrow \lambda^2 - \lambda - 1 = 0 \rightarrow \lambda = \dfrac{1 \pm \sqrt{(-1)^2 + 4(1)(-1)}}{2(1)}$

$$= \frac{1 \pm \sqrt{5}}{2}$$

"+" eigenvalue is called the "Golden Ratio". Typical notation is $\phi$: $\phi = \frac{1+\sqrt{5}}{2}$

"−" eigenvalue is $1 - \left(\frac{1+\sqrt{5}}{2}\right) = 1 - \phi$

Eigenvectors: $\lambda = \phi$, Solve $\begin{bmatrix} 1-\phi & 1 \\ 1 & -\phi \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2nd equation tells us that $x_1 - \phi x_2 = 0$, so $\vec{x} = \begin{bmatrix} \phi \\ 1 \end{bmatrix}$ is an eigenvector.

For $\lambda = 1-\phi$, Solve $\begin{bmatrix} 1-(1-\phi) & 1 \\ 1 & -(1-\phi) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

~~1st equation tells us that $\phi x_1 + x_2 = 0$, so $\vec{x} =$~~

2nd equation says $x_1 - (1-\phi)x_2 = 0$, so $\vec{x} = \begin{bmatrix} 1-\phi \\ 1 \end{bmatrix}$ is an eigenvector.

So
$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \phi & 1-\phi \\ 1 & 1 \end{bmatrix}}_{X}\underbrace{\begin{bmatrix} \phi & 0 \\ 0 & 1-\phi \end{bmatrix}}_{\Lambda}\underbrace{\frac{1}{\phi-(1-\phi)}\begin{bmatrix} 1 & -(1-\phi) \\ -1 & \phi \end{bmatrix}}_{X^{-1}}$$

Finally:
$$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \phi & 1-\phi \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \phi^k & 0 \\ 0 & (1-\phi)^k \end{bmatrix}\frac{1}{2\phi-1}\begin{bmatrix} 1 & -(1-\phi) \\ -1 & \phi \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \phi^{k+1} & (1-\phi)^{k+1} \\ \phi^k & (1-\phi)^k \end{bmatrix}\frac{1}{2\phi-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2\phi-1}\begin{bmatrix} \phi^{k+1} - (1-\phi)^{k+1} \\ \phi^k - (1-\phi)^k \end{bmatrix}$$

Closed-form formula: $F_k$ = 2nd component =

$$\frac{1}{2\left(\frac{1+\sqrt{5}}{2}\right)-1}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

It's a little remarkable that this expression is a positive integer, since there are so many $\sqrt{5}$'s in here. (But note: formula stays same when you change $\sqrt{5} \to -\sqrt{5}$, means $\sqrt{5}$'s have to cancel out.)

What does this formula tell us about $F_k$?

When k gets large: $\phi > 1 \rightsquigarrow \phi^k$ <u>grows</u> exponentially

$-1+\phi<1 \rightsquigarrow (\phi-1)^k$ <u>decays</u> exponentially

So if k is large, $F_k \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k$ ← Shows $F_k$ grows approximately exponentially, with base $\phi \approx 1.61803398\ldots$

Ratio of consecutive Fibonacci numbers:

$$F_{k+1}/F_k \approx \frac{\frac{1}{\sqrt{5}}\phi^{k+1}}{\frac{1}{\sqrt{5}}\phi^k} = \phi \quad \text{(if } k \text{ is large)}$$

<u>Examples</u>: $\phi = 1.61803398\ldots$

$F_6/F_5 = 8/5 = 1.60000000$

$F_7/F_6 = 13/8 = 1.62500000$

$F_8/F_7 = 21/13 = 1.61538462\ldots$

$F_9/F_8 = 34/21 = 1.61904762\ldots$

$F_{10}/F_9 = 55/34 = 1.61764706\ldots$

$F_{11}/F_{10} = 89/55 = 1.61818182\ldots$

$F_{12}/F_{11} = 144/89 = 1.61797753\ldots$