Calculus A (1): Homework 7

Assigned exercises

A 1.

(a) ① $U = max_1 \Delta x + max_2 \Delta x + \dots + max_n \Delta x$, where $max_1 = f(x_1)$, $max_2 = f(x_2)$ ... $max_n = f(x_n)$

Since f is increasing on $[a,b]$

② $L = min_1 \Delta x + min_2 \Delta x + \dots min_n \Delta x$, where $min_1 = f(x_0)$, $min_2 = f(x_1)$ ... $min_n = f(x_{n-1})$

Since f is increasing on $[a,b]$

$$U - L = (max_1 - min_1)\Delta x + (max_2 - min_2)\Delta x + \dots (max_n - min_n)\Delta x.$$

$$= [\cancel{f(x_1)} - f(x_0)]\Delta x + [\cancel{f(x_2)} - \cancel{f(x_1)}]\Delta x + \dots [f(x_n) - \cancel{f(x_{n-1})}]\Delta x.$$

$$= [f(x_n) - f(x_0)]\Delta x$$

$$= [f(b) - f(a)]\Delta x.$$

(b) ① $U = max_1 \Delta x_1 + max_2 \Delta x_2 + \dots max_n \Delta x_n$ where $max_1 = f(x_1)$, $max_2 = f(x_2)$, ... $max_n = f(x_n)$

Since f is increasing on $[a,b]$.

② $L = min_1 \Delta x_1 + min_2 \Delta x_2 + \dots + min_n \Delta x_n$ where $min_1 = f(x_0)$, $min_2 = f(x_1)$ ... $min_n = f(x_{n-1})$

Since f is increasing on $[a,b]$

$$U - L = (max_1 - min_1)\Delta x_1 + (max_2 - min_2)\Delta x_2 + \dots + (max_n - min_n)\Delta x_n.$$

$$= [f(x_1) - f(x_0)]\Delta x_1 + [f(x_2) - f(x_1)]\Delta x_2 \dots + [f(x_n) - f(x_{n-1})]\Delta x_n.$$

$$\leq [f(x_1) - f(x_0)]\Delta x_{max} + [f(x_2) - f(x_1)]\Delta x_{max} + [f(x_n) - f(x_{n-1})]\Delta x_{max}.$$

$\Leftrightarrow$ $U - L \leq [f(x_n) - f(x_0)]\Delta max$. (by question (a))

$\Leftrightarrow$ $U - L \leq [f(b) - f(a)]\Delta x_{max}$ (by question (a))

$\Leftrightarrow$ $U - L \leq |f(b) - f(a)|\Delta x_{max}$ ($f(b) \geq f(a)$)

Therefore,

$$\lim_{\|P\| \to 0} (U - L) = \lim_{\|P\| \to 0} [f(b) - f(a)]\Delta x_{max} = 0, \text{ sin } \Delta x_{max} = \|P\|$$

A2.

a. True by FTC, $h'(x) = f(x)$ Since f is differentiable for all x, h has a second derivative for all x.

b. True Since they are differentiable, they're continous.

c. True. $h'(x) = f(x) \Rightarrow h'(1) = f(1) = 0$

d. True f has a negative derivative for all x. Since $h'(1) = 0$ and $h''(1) = f'(1) < 0$

e False $h'(1) = 0$, $h''(1) = f'(1) < 0$.

f False $h''(x) = f'(x) < 0$ never changes sign

g. True since $h'(1) = f(1) = 0$ and $h'(x) = f(x)$ is a decreasing function of x (For $f'(x) < 0$)

A3.

$$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\cos^3\sqrt{\theta}}\,d\theta$$

Let $u = \cos\sqrt{\theta}$

$$du = -\sin\sqrt{\theta}\cdot\frac{1}{2\sqrt{\theta}}\,d\theta$$

$$du = -\frac{\sin\sqrt{\theta}}{2\sqrt{\theta}}\,d\theta$$

$$d\theta = -\frac{du\,2\sqrt{\theta}}{\sin\sqrt{\theta}}$$

$$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\cos^3\sqrt{\theta}}\,d\theta$$

$$= \int \frac{\cancel{\sin\sqrt{\theta}}}{\cancel{\sqrt{\theta}}\cos^3\sqrt{\theta}}\cdot\left(-\frac{du\,2\cancel{\sqrt{\theta}}}{\cancel{\sin\sqrt{\theta}}}\right) \quad \text{(by Substitution Rule)}$$

$$= \int -\frac{2}{\sqrt{u^3}}\,du$$

$$= \int -2u^{-\frac{3}{2}}\,du$$

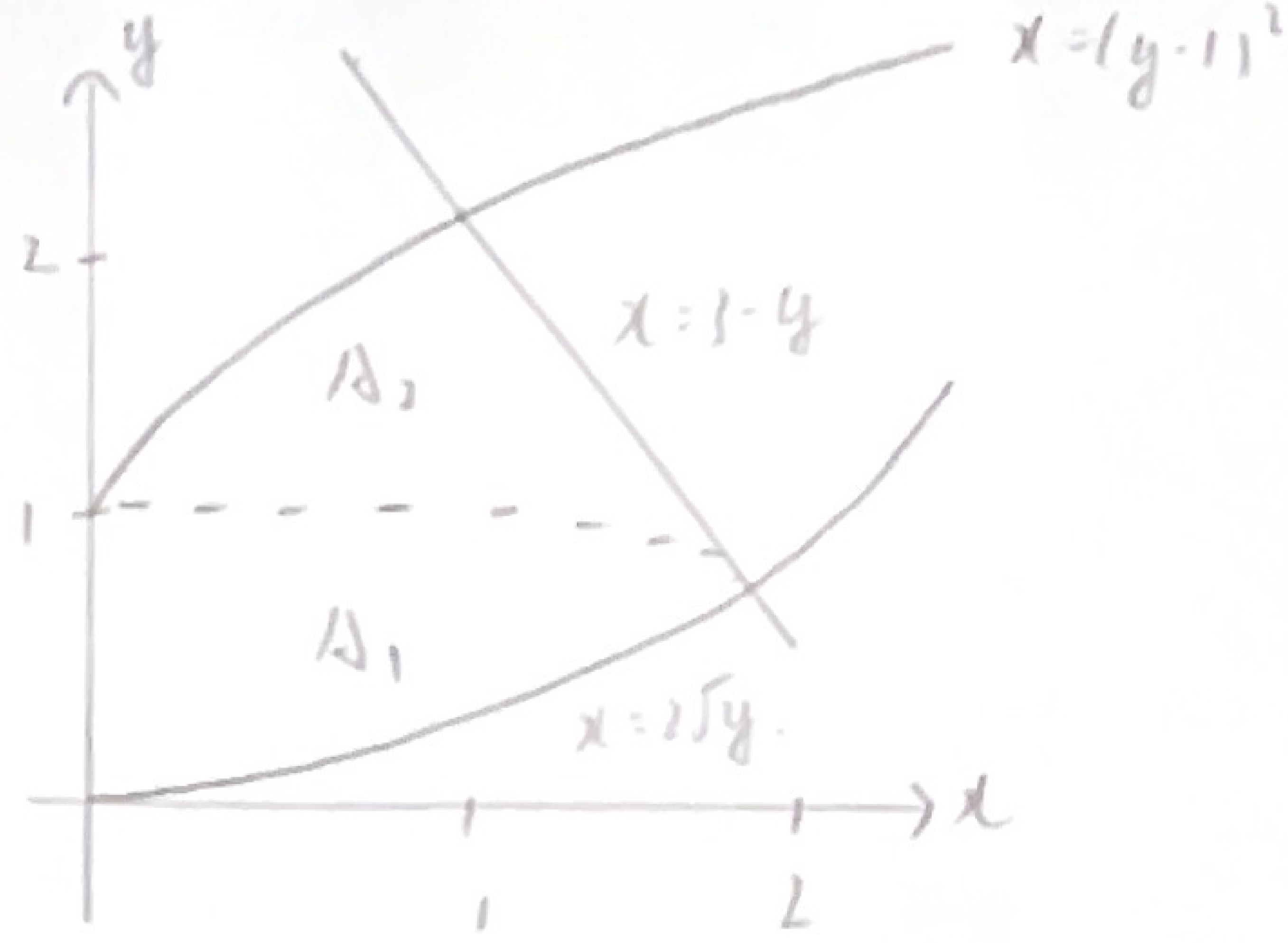
$$= -2\cdot\frac{1}{1-\frac{3}{2}}u^{-\frac{1}{2}} + C$$

$$= 4u^{-\frac{1}{2}} + C$$

$$= \frac{4}{\sqrt{\cos\sqrt{\theta}}} + C$$

A4.

$S = A_1 + A_2.$



① $A_1 = \int_0^1 2\sqrt{y}\, dy$

$A_1 = 2 \frac{1}{1+\frac{1}{2}} y^{\frac{3}{2}} \Big|_0^1$

$A_1 = \frac{4}{3} y^{\frac{3}{2}} \Big|_0^1$

$A_1 = \frac{4}{3}$

② $A_2 = \int_1^2 [3-y-(y-1)^2]\, dy.$

$A_2 = \int_1^2 (3-y-y^2+2y-1)\, dy$

$A_2 = \int_1^2 (-y^2+y+2)\, dy$

$A_2 = \left(-\frac{1}{3}y^3+\frac{1}{2}y^2+2y\right)\Big|_1^2$

$A_2 = \left(-\frac{8}{3}+2+4\right)-\left(-\frac{1}{3}+\frac{1}{2}+2\right)$

$A_2 = \frac{7}{6}$

by ①, ②.

$S = A_1 + A_2$

$S = \frac{4}{3} + \frac{7}{6}$

$S = \frac{5}{2}$

A5. If f is a continuous function, find the value of the integral

$$I = \int_0^a \frac{f(x)\,dx}{f(x)+f(a-x)}$$

by making the substitution $u = a - x$ and adding the resulting integral to I.

Let $u = a - x$, then we have.

$$du = -dx \qquad u = a - x \qquad ① x = a \Rightarrow u = 0$$

$$dx = -du \quad \text{and} \quad x = a - u \Rightarrow ② x = 0 \Rightarrow u = a.$$

$$I = \int_0^a \frac{f(x)\,dx}{f(x)+f(a-x)}$$

$$I = \int_a^0 \frac{f(a-u)\cdot(-du)}{f(a-u)+f[a-(a-u)]} \quad \text{(by Substitution Rule)}$$

$$I = \int_0^a \frac{f(a-u)}{f(a-u)+f(u)}\,du$$

$$I = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}\,dx.$$

Therefore,

$$I + I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx + \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}\,dx.$$

$$2I = \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}\,dx$$

$$2I = \int_0^a 1\,dx$$

$$2I = x\Big|_0^a$$

$$2I = a$$

$$I = \frac{a}{2}$$

B1.

Prove that

$$\int_0^x \left(\int_0^u f(t)\,dt \right) du = \int_0^x f(u)(x-u)\,du.$$

The derivative of LHS:

$$\frac{d}{dx}\left[\int_0^x \left[\int_0^u f(t)\,dt\right] du \right]$$

$$= \int_0^x f(t).dt.$$

The derivative of RHS:

$$\frac{d}{dx}\left[\int_0^x f(u)(x-u)\,du \right]$$

$$= \frac{d}{dx}\int_0^x f(u)x\,du - \frac{d}{dx}\int_0^x uf(u)\,du.$$

$$= \frac{d}{dx}\left[x\int_0^x f(u)\,du\right] - \frac{d}{dx}\int_0^x uf(u)\,du.$$

$$= \int_0^x f(u)\,du + x\left[\frac{d}{dx}\int_0^x f(u)\,du\right] - xf(x)$$

$$= \int_0^x f(u)\,du + \cancel{xf(x)} - \cancel{xf(x)}$$

$$= \int_0^x f(u)\,du.$$

Since each side has the same derivative, they differ by a constant, and since both sides equal 0 when $x=0$, the constant must be 0.

Therefor. $\int_0^x \left[\int_0^u f(t)\,dt \right] du = \int_0^x f(u)(x-u)\,du$