$(\frac{3}{2}, \frac{3}{2})$

$(-\frac{1}{2}, \frac{1}{2})$. $(\frac{3}{2}, 3)$ (

$\overrightarrow{CA}$

$CA = CO + OA$

$CB = CO + OB$.

$(\overrightarrow{CO} + \overrightarrow{OA}) \cdot (\overrightarrow{CO} + \overrightarrow{OB})$.

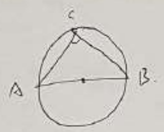
Calculus A(2) Final

23/06/12

$(\overrightarrow{CO}^2 + \overrightarrow{CO} \cdot \overrightarrow{OB} + \overrightarrow{OA} \cdot \overrightarrow{OB} + \overrightarrow{OA} \cdot \overrightarrow{OB})$

$\overrightarrow{OC}$

$\overrightarrow{CO}^2 + 2\overrightarrow{OC}$



A

5 marks

1. For $\underline{u} = (1, 2)$, $\underline{v} = (1, 1)$, $\underline{w} = (1, -1)$, find scalars a and b such that $\underline{u} = a\underline{v} + b\underline{w}$.
2. Find the angle between the planes
$$x + y = 0, \quad y + z = 0.$$
3. Find an equation for the plane through $P = (2, -1, 1)$ perpendicular to the vector from the origin to P.
4. For AB a diameter of a circle with center O, and C a point on the circle different from A and B. Show that $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.
5. Suppose L_1 and L_2 are non-parallel lines which do not meet. Does there exist a non-zero vector perpendicular to both L_1 and L_2?

B

2 marks

For vectors

$$\underline{u} = (1, 0, 0)$$
$$\underline{v} = (0, 2, 0)$$
$$\underline{w} = (0, 1, 1)$$

verify that the following holds.

$$(\underline{u} \times \underline{v}).\underline{w} = (\underline{v} \times \underline{w}).\underline{u}$$

C

3 marks

Consider the ellipse $R = \{x^2/a^2 + y^2/b^2 \leq 1\}$ for $a, b > 0$.

1. Make a sketch of R.
2. For the substitution $x = au$, $y = bv$, evaluate the Jacobian as follows.
$$J(u, v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

C (cont)

3. Find the area of R by evaluating the integral

$$\int_R dA$$

using the substitution given.

D

3 marks

For the function $f(x, y) = (4^2 - x^2 - y^2)^{-1/2}$ find

1. the domain,
2. the range,

and

3. sketch some level curves.

E

3 marks

A flat plate has shape $R = \{x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$. The density of the plate is

$$\delta(x, y) = 2 + x^2 - y^2.$$

Find the most and least dense points on the plate, including the boundary, and the density there.

F

4 marks

Find the direction in which the function increases most rapidly at P, and its derivative in this direction.

1. $f(x, y) = x^2 + xy + y^2$, $P = (-1, 1)$
2. $f(x, y, z) = \ln xy + \ln yz + \ln zx$, $P = (1, 1, 1)$

G

3 marks

For a closed curve C bounding a region R in the plane with unit normal $\underline{n}$, and vector field $\underline{F} = (M, N)$, Green's theorem states that

$$\int_C \underline{F}.\underline{n}\, ds = \iint_R \nabla.\underline{F}\, dA.$$

1. Use the theorem to prove that the area of R is given by the following formula.

$$\frac{1}{2}\int_C x\, dy - y\, dx$$

(1 marks)

2. Verify the conclusion of the theorem where

$$R = \{x^2 + y^2 \leq 1\}$$

and $\underline{F} = (-y, x)$.

(2 marks)

H

4 marks

Consider the function $f(x)$ with domain $[-\pi, \pi]$ defined as follows.

$$f(x) = \begin{cases} 0 & -\pi \leq x \leq 0 \\ 1 & 0 < x \leq \pi \end{cases}$$

1. Find a Fourier series for $f(x)$ of form

$$a_0 + \sum_{k=1}^{\infty} b_k \sin kx$$

using the following formulas.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx\, dx$$

(2 marks)

H (cont)

2. Does the series converge at $x = \pi$? Why?

(1 marks)

3. Show that your series converges at $x = \pi/2$ and find its value.

(1 marks)

I

3 marks

Recall that the centroid of a wire in the shape of a curve C is given by

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\int_C x\, ds, \int_C y\, ds, \int_C z\, ds\right) \bigg/ \int_C ds.$$

Sketch the curve

$$C = \{x = 0,\ y^2 + z^2 = 1,\ z \geq 0\}$$

and find the associated centroid.

$y = \sqrt{4 - x^2}$

$z = x^2 + y^2$

J

5 marks

1. A parabolic bowl given by the equation $z = x^2 + y^2 - 4$ in the region $z \leq 0$, is filled with water to 1 unit from the top. Find the volume of water in the bowl.

(2 marks)

2. Consider spherical coordinates where ρ is the distance from origin, and ϕ the angle from the positive z axis. Sketch the solid $\rho \leq 1$, $\phi \leq \pi/4$ and determine its volume. You may use $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

(3 marks)

K

2 marks

For a closed oriented surface S with normal $\underline{n}$ enclosing a region D, the domain of a vector field $\underline{F}$, the divergence theorem is as follows.

$$\iint_S \underline{F}.\underline{n}\, d\sigma = \iiint_D \nabla.\underline{F}\, dV$$

For $\underline{F} = (x, y, z)$ evaluate both sides for the region

$$D = \{x^2 + y^2 + z^2 \leq 1\}.$$

L

3 marks

For a function $f(x)$ the Taylor series with centre a is

$$f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \ldots$$

1. Use this to derive the following series for $\ln(1+y)$.

$$y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots$$

(2 marks)

2. Does the series converge for all $|y| < 1$? Why?

(1 marks)