Chapter 6: Eigenvalues and Eigenvectors

$n \times n$ matrix A: the best way to understand A is to understand what it does to vectors in $\mathbb{R}^n$.

$\vec{x}$ in $\mathbb{R}^n$ $\xrightarrow{A}$ $A\vec{x}$, another vector in $\mathbb{R}^n$

Here's a matrix that's easy to understand:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

What does A do?
- It keeps x-axis vectors the same
- It stretches y-axis vectors by a factor of 2.
- It stretches z-axis vectors by a factor of 3.

Every other $\vec{x}$: $A\vec{x}$ = some linear combination of these stretchings

---

Idea of eigenvalues/vectors: Can we understand any $n \times n$ A this way, i.e., $A\vec{x}$ = linear combination of "stretchings"?

Problem: The axes that get stretched won't be the x-axis, y-axis, z-axis, ... if A isn't diagonal. We have to find these axes.

First, let's define "stretching axis" mathematically:

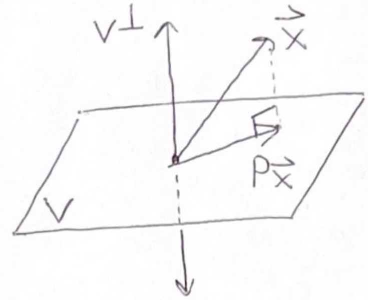
Definition: A non-zero vector $\vec{x}$ is called an eigenvector of A if $A\vec{x} = \lambda\vec{x}$ for some scalar λ.

A "stretches" the axis spanned by $\vec{x}$ by a factor of λ.

This scalar is called the eigenvalue.

Let's try to find eigenvalues of a few types of matrices geometrically, by asking: What non-zero $\vec{x}$ satisfy $A\vec{x} = \lambda\vec{x}$ for some λ (λ could be 0, or negative, or positive, ...).

Projection matrices:



What $\vec{x}$ has $P\vec{x} = \lambda\vec{x}$?

What if $\vec{x}$ is already in V? Then $P\vec{x}$ doesn't change anything:

$P\vec{x} = \vec{x} = (1)\vec{x}$

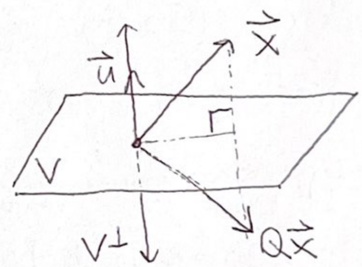
This is one of P's eigenvalues.

What if $\vec{x}$ is in $V^\perp$? Then $P\vec{x} = \vec{0} = 0 \cdot \vec{x}$

Another eigenvalue.

It turns out, 0 and 1 are always to two eigenvalues of P. The eigenvectors are the <u>non-zero</u> vectors in $V^\perp$ and V.

Reflection matrices:



$Q = I - 2\vec{u}\vec{u}^T$

What $\vec{x}$ have $Q\vec{x} = \lambda\vec{x}$?

If $\vec{x}$ is in V, reflection doesn't change it: $Q\vec{x} = \vec{x}$

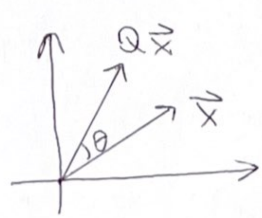
If $\vec{x}$ is in $V^\perp$, reflection flips it: $Q\vec{x} = -\vec{x}$.

$\longrightarrow$ The eigenvalues of Q are $+1$ and -1.

$\longrightarrow$ The eigenvectors are the <u>non-zero</u> vectors in V and $V^\perp$.

What does Q do to a general $\vec{x}$? It is a combination of "staying the same" and "totally flipping."

Rotation matrices:



$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

What <u>non-zero</u> $\vec{x}$ have $Q\vec{x} = \lambda\vec{x}$?

Usually none! For $\boxed{\text{most}}$ θ, $\vec{x}$ is always pointing in a different direction after you rotate it by θ!

$\longrightarrow$ Q has no real number eigenvalues. We will see that Q has complex number eigenvalues.

Two exceptions:

$\underline{\theta = 0}$ $\quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Eigenvalue $= 1$ (every vector stays the same if you rotate by 0)

$\underline{\theta = \pi}$ $\quad Q = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Eigenvalue $= -1$ (every vector gets flipped when you rotate by $180°$).

Big question: Given A, is there a systematic way to find the eigenvectors/values (the "stretching axes" with their stretching factors)?

We have to solve $A\vec{x} = \lambda\vec{x}$ for both $\vec{x}$ and λ.

We have to solve for λ first: Suppose λ is an eigenvalue of A. It must have an eigenvector: non-zero $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$.

Or: $A\vec{x} = \lambda I\vec{x} \longrightarrow A\vec{x} - \lambda I\vec{x} = \vec{0} \longrightarrow (A - \lambda I)\vec{x} = \vec{0}$

$\vec{x}$ is a non-zero vector in the null space of $A - \lambda I$!

So if λ is an eigenvalue: $N(A - \lambda I)$ has a non-zero vector

$\longrightarrow$ $A - \lambda I$ can't be invertible $\longrightarrow$ $\det(A - \lambda I) = 0$.

Result: λ is an eigenvalue of A exactly when $\det(A - \lambda I) = 0$.

So to find λ, we need to solve $\det(A-\lambda I)=0$ for λ.

You get this by multiplying entries of $A-\lambda I$ and adding/subtracting these terms ("Big Formula" for det) $\longrightarrow$ $\det(A-\lambda I)$ is a polynomial in λ, and we have to find the roots.

Called the characteristic polynomial of A.

Example Find eigenvalues of $A=\begin{bmatrix}1&3\\1&4\end{bmatrix}$

$$A-\lambda I=\begin{bmatrix}1&3\\1&4\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}=\begin{bmatrix}1-\lambda&3\\1&4-\lambda\end{bmatrix}$$

$$\det(A-\lambda I)=(1-\lambda)(4-\lambda)-3=4-5\lambda+\lambda^2-3=\lambda^2-5\lambda+1$$

When does this equal 0? Have to use quadratic formula.

$$\lambda=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(1)}}{2(1)}=\frac{5}{2}\pm\frac{\sqrt{21}}{2}$$

Square root shows we are doing a bit of nonlinear algebra.

For a 2x2 matrix: quadratic polynomial, up to 2 distinct roots $\longrightarrow$ at most 2 eigenvalues

---

Next let's do a bigger matrix but with nicer eigenvalues:

Example Find eigenvalues and eigenvectors of $A=\begin{bmatrix}-1&1&0\\0&-2&2\\3&-9&6\end{bmatrix}$

Eigenvalues: Solve $\det(A-\lambda I)=0$ for λ:

$$\det\left(\begin{bmatrix}-1&1&0\\0&-2&2\\3&-9&6\end{bmatrix}-\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}\right)=\begin{vmatrix}-1-\lambda&1&0\\0&-2-\lambda&2\\3&-9&6-\lambda\end{vmatrix}\overset{\text{1st col}}{=}(-1-\lambda)\begin{vmatrix}-2-\lambda&2\\-9&6-\lambda\end{vmatrix}$$

$$+3\begin{vmatrix}1&0\\-2-\lambda&2\end{vmatrix}=(-1-\lambda)\Big(\underbrace{(-2-\lambda)(6-\lambda)+18}_{\lambda^2-4\lambda+6}\Big)+3(2)=-(\lambda+1)(\lambda^2-4\lambda+6)+6$$

$$= -(\lambda^3 - 4\lambda^2 + 6\lambda + \lambda^2 - 4\lambda + \cancel{6}) + \cancel{6} = -(\lambda^3 - 3\lambda^2 + 2\lambda)$$

$$= -\lambda(\lambda^2 - 3\lambda + 2) = -\lambda(\lambda - 1)(\lambda - 2)$$

3×3 matrix $\rightarrow$ 3 roots: $\lambda = 0, 1, 2$

Now how do we find the eigenvectors?

For each λ, solve $A\vec{x} = \lambda\vec{x}$ for $\vec{x}$ $\rightarrow$ same as finding null space of $A - \lambda I$.

$\lambda = 0$: Solve $A\vec{x} = 0\vec{x} = \vec{0}$ (This is just $N(A)$!)

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix} \xrightarrow[-\text{Row }1]{\text{Row }3 + 3\text{Row }1,\ -\frac{1}{2}\text{Row }2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -6 & 6 \end{bmatrix} \xrightarrow[6\text{Row }3]{\text{Row }3 +} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[+\text{Row }2]{\text{Row }1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rightarrow$ $x_1 - x_3 = 0$, $x_2 - x_3 = 0$, x_3 free $\rightarrow \vec{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

All of these are eigenvectors, as long as $x_3 \neq 0$
(we don't count $\vec{0}$ as an eigenvector)

$\lambda = 1$: Solve $A\vec{x} = 1\vec{x} \rightarrow (A - I)\vec{x} = \vec{0} \rightarrow N(A - I)$

$$A - I = \begin{bmatrix} -1-1 & 1 & 0 \\ 0 & -2-1 & 2 \\ 3 & -9 & 6-1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 2 \\ 3 & -9 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 3 & -9 & 5 \end{bmatrix} \xrightarrow[3\text{ Row }1]{\text{Row }3 -}$$

$$\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & -15/2 & 5 \end{bmatrix} \xrightarrow[\text{Row }3 + \frac{15}{2}\text{Row }2]{\text{Row }1 + \frac{1}{2}\text{Row }2} \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 = \frac{1}{3}x_3 \\ x_2 = \frac{2}{3}x_3 \\ x_3 \text{ free} \end{matrix} \rightarrow \vec{x} = x_3 \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}$$

Eigenvectors for $\lambda = 1$: one-dimensional subspace minus $\vec{0}$.

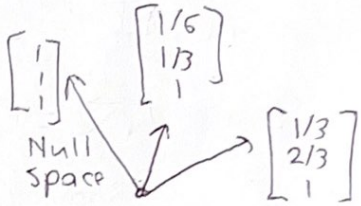
$\lambda = 2$: Solve $A\vec{x} = 2\vec{x} \rightarrow (A - 2I)\vec{x} = \vec{0} \rightarrow N(A - 2I)$

$$A-2I=\begin{bmatrix}-1-2 & 1 & 0\\ 0 & -2-2 & 2\\ 3 & -9 & 6-2\end{bmatrix}=\begin{bmatrix}-3 & 1 & 0\\ 0 & -4 & 2\\ 3 & -9 & 4\end{bmatrix}\xrightarrow[\text{Row 1}]{\text{Row 3 +}}\begin{bmatrix}-3 & 1 & 0\\ 0 & -4 & 2\\ 0 & -8 & 4\end{bmatrix}\xrightarrow[\text{2Row 2}]{\text{Row 3 −}}$$

$$\begin{bmatrix}-3 & 1 & 0\\ 0 & -4 & 2\\ 0 & 0 & 0\end{bmatrix}\longrightarrow\begin{bmatrix}1 & -1/3 & 0\\ 0 & 1 & -1/2\\ 0 & 0 & 0\end{bmatrix}\xrightarrow[\frac{1}{3}\text{Row 2}]{\text{Row 1 +}}\begin{bmatrix}1 & 0 & -1/6\\ 0 & 1 & -1/2\\ 0 & 0 & 0\end{bmatrix}\rightarrow \begin{matrix}x_1=\frac{1}{6}x_3\\ x_2=\frac{1}{2}x_3\end{matrix} \quad x_3 \text{ free}$$

$\longrightarrow$ Eigenvectors are $\vec{x}=x_3\begin{bmatrix}1/6\\1/2\\1\end{bmatrix}$ (if $x_3 \neq 0$)

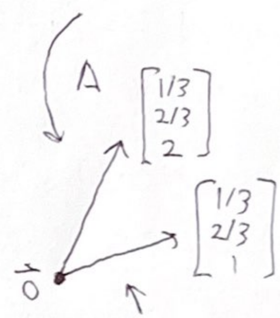
What does all of this show about A?



A collapses the $\begin{bmatrix}1\\1\\1\end{bmatrix}$-axis to $\vec{0}$.

A doesn't affect the $\begin{bmatrix}1/3\\2/3\\1\end{bmatrix}$-axis

A stretches the $\begin{bmatrix}1/6\\1/3\\1\end{bmatrix}$-axis by factor of 2.



Column space of A is the plane spanned by these two vectors.

To find eigenvalues, you have to solve a polynomial equation.

$\longrightarrow$ roots of a polynomial might not be real.

$\longrightarrow$ A might have <u>complex</u> eigenvalues.

<u>Example</u> Rotation matrix $Q=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

$$\det(A-\lambda I)=\begin{vmatrix}\cos\theta-\lambda & -\sin\theta\\ \sin\theta & \cos\theta-\lambda\end{vmatrix}=(\cos\theta-\lambda)^2+\sin^2\theta$$

$$=\lambda^2-(2\cos\theta)\lambda+\underbrace{\cos^2\theta+\sin^2\theta}_{1}=\lambda^2-(2\cos\theta)\lambda+1\overset{?}{=}0$$

Quadratic formula: $\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{-\sin^2\theta}$

$\sqrt{-1}$

$= \cos\theta \pm i\sin\theta$ Not real, unless $\sin\theta = 0 \rightarrow \theta = 0, \pi$

You can find eigenvectors for these eigenvalues, but they will have complex number components (i.e., no eigenvectors in $\mathbb{R}^2$)

real components

Characteristic polynomial could also have repeated roots
$\longrightarrow$ might lead to fewer eigenvalues than you expect.

Example $A = \begin{bmatrix} -3 & 16 \\ -1 & 5 \end{bmatrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 16 \\ -1 & 5-\lambda \end{vmatrix}$

$= (-3-\lambda)(5-\lambda) + 16 = \lambda^2 - 5\lambda + 3\lambda - 15 + 16 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$

Only root is $\lambda = 1$.

Eigenvectors: Solve $A\vec{x} = \vec{x} \rightarrow N(A - I)$

$A - I = \begin{bmatrix} -4 & 16 \\ -1 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \rightarrow x_1 = 4x_2 \rightarrow \vec{x} = x_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

What this tells us about A: $\mathbb{R}^2$ doesn't have a basis of eigenvectors for $A \rightarrow$ There is only one axis that A stretches/compresses/flips/keeps the same (It keeps the $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$-axis the same)

What it does to the rest of $\mathbb{R}^2$ is more complicated...