Solving Linear Equations $A\vec{x} = \vec{b}$

$[A \mid \vec{b}] \xrightarrow{\text{elimination}} [R \mid \vec{d}]$ "reduced row echelon" form of R

Possibilities for R:

1. R has a leading 1 in every row and every column.
2. R has a leading 1 in every row, but not every column.
3. R has a leading 1 in every column, but not every row.
4. R has leading 1's missing from both rows and columns.

Saw examples of case 4 last time:

Because some rows don't have leading 1's: For most $\vec{b}$, $A\vec{x}=\vec{b}$ has no solution.

Because some columns don't have leading 1's (there are free variables): For some $\vec{b}$, $A\vec{x}=\vec{b}$ has infinitely many solutions. (For example, a plane of solutions if there are two free variables, i.e., two columns don't have leading 1's.)

Case 1: If R has a leading 1 in every row and column, it has n rows and n columns, and $R = n\times n$ identity matrix. If $R=I$, then every $A\vec{x}=\vec{b}$ has a unique solution, and A is invertible.

Example: Find the unique solution to $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

in terms of $\vec{b}$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & 2 & 3 & b_2 \\ 1 & 4 & 9 & b_3 \end{array}\right] \xrightarrow[\text{Row 3}-\text{Row 1}]{\text{Row 2}-\text{Row 1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & b_2-b_1 \\ 0 & 3 & 8 & b_3-b_1 \end{array}\right] \xrightarrow{\text{Row 3}-3\text{Row 2}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & -b_1+b_2 \\ 0 & 0 & 2 & 2b_1-3b_2+b_3 \end{array}\right]$$

$$\xrightarrow[\text{Row 2}-\text{Row 3}]{\text{Row 1}-\frac{1}{2}\text{Row 3}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \frac{3}{2}b_2-\frac{1}{2}b_3 \\ 0 & 1 & 0 & -3b_1+4b_2-b_3 \\ 0 & 0 & 2 & 2b_1-3b_2+b_3 \end{array}\right] \xrightarrow[\frac{1}{2}\text{Row 3}]{\text{Row 1}-\text{Row 2}} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3b_1-\frac{5}{2}b_2+\frac{1}{2}b_3 \\ 0 & 1 & 0 & -3b_1+4b_2-b_3 \\ 0 & 0 & 1 & b_1-\frac{3}{2}b_2+\frac{1}{2}b_3 \end{array}\right]}_{R=I}$$

Solution to $A\vec{x}=\vec{b}$ is:

$$\vec{x}=\begin{bmatrix}3b_1-\frac{5}{2}b_2+\frac{1}{2}b_3\\-3b_1+4b_2-b_3\\b_1-\frac{3}{2}b_2+\frac{1}{2}b_3\end{bmatrix}=\begin{bmatrix}3&-5/2&1/2\\-3&4&-1\\1&-\frac{3}{2}&1/2\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

This is A^{-1}.

only happens if $n>m$ (# columns > # rows)

Case 2: R has leading 1 in every row $\longrightarrow$ no row of 0's, so solutions to $A\vec{x}=\vec{b}$ always exist for all $\vec{b}$.

R has no leading 1 in some columns $\longrightarrow$ free variables, so $N(A)$ is bigger than $\{\vec{0}\}$ $\longrightarrow$ every system has infinitely many solutions.

Example Find all solutions to
$$\begin{bmatrix}-1&1&-1&1\\1&-2&3&-4\\-1&3&-6&10\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

"wide" matrix

$$\left[\begin{array}{cccc|c}-1&1&-1&1&1\\1&-2&3&-4&1\\-1&3&-6&10&1\end{array}\right]\xrightarrow[\text{Row 3}-\text{Row 1}]{\text{Row 2}+\text{Row 1}}\left[\begin{array}{cccc|c}-1&1&-1&1&1\\0&-1&2&-3&2\\0&2&-5&9&0\end{array}\right]\xrightarrow{\text{Row 3}+2\text{Row 2}}\left[\begin{array}{cccc|c}-1&1&-1&1&1\\0&-1&2&-3&2\\0&0&-1&3&4\end{array}\right]$$

$$\xrightarrow[\text{Row 2}+2\text{Row 3}]{\text{Row 1}-\text{Row 3}}\left[\begin{array}{cccc|c}-1&1&0&-2&-3\\0&-1&0&3&10\\0&0&-1&3&4\end{array}\right]\xrightarrow{\text{Row 1}+\text{Row 2}}\left[\begin{array}{cccc|c}-1&0&0&1&7\\0&-1&0&3&10\\0&0&-1&3&4\end{array}\right]\xrightarrow[\substack{-\text{Row 2}\\-\text{Row 3}}]{-\text{Row 1}}$$

$$\left[\begin{array}{cccc|c}1&0&0&-1&-7\\0&1&0&-3&-10\\0&0&1&-3&-4\end{array}\right]$$

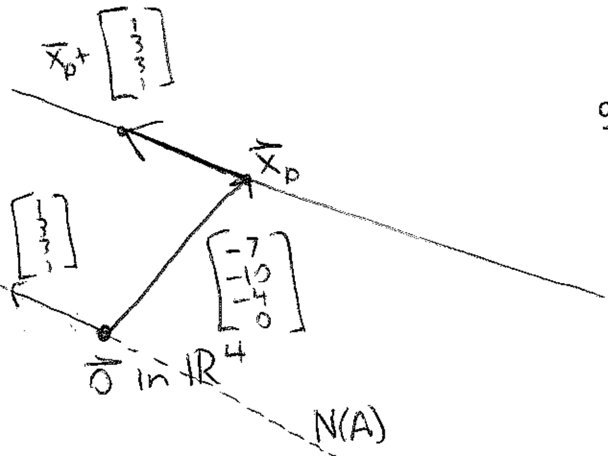
$R\vec{x}=\vec{d}$

Pivot variables x_1,x_2,x_3 — free variable x_4

$$\begin{aligned}x_1-x_4&=-7\\x_2-3x_4&=-10\\x_3-3x_4&=-4\end{aligned}\quad\longrightarrow\quad\begin{aligned}x_1&=-7+x_4\\x_2&=-10+3x_4\\x_3&=-4+3x_4\\x_4&=\text{anything (free)}\end{aligned}$$

All solutions look like:
$$\vec{x}=\begin{bmatrix}-7+x_4\\-10+3x_4\\-4+3x_4\\x_4\end{bmatrix}=\begin{bmatrix}-7\\-10\\-4\\0\end{bmatrix}+\underbrace{x_4\begin{bmatrix}1\\3\\3\\1\end{bmatrix}}_{\substack{\text{null space vectors}\\\text{(line in }\mathbb{R}^4)}}$$

one particular solution ($x_4=0$)



line of solutions, parallel to $N(A)$

Case 3: R has leading 1 in each column $\longrightarrow$ no free variables
$\longrightarrow N(A) = \{\vec{0}\}$ and $A\vec{x} = \vec{b}$ has at most one solution.

Only happens if $m > n$

Some rows missing leading 1's $\longrightarrow$ row of 0's in R
$\longrightarrow$ most $A\vec{x} = \vec{b}$ have no solutions (for most $\vec{b}$).

Example "skinny" matrix
$$\begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|c} 2 & 1 & b_1 \\ 3 & 1 & b_2 \\ 4 & 3 & b_3 \end{array}\right] \xrightarrow[\text{Row 3} - 2\text{Row 1}]{\text{Row 2} - \frac{3}{2}\text{Row 1}}$$

$$\left[\begin{array}{cc|c} 2 & 1 & b_1 \\ 0 & -\frac{1}{2} & -\frac{3}{2}b_1 + b_2 \\ 0 & 1 & -2b_1 + b_3 \end{array}\right] \xrightarrow[-2\text{Row 2}]{\frac{1}{2}\text{Row 1}} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2}b_1 \\ 0 & 1 & 3b_1 - 2b_2 \\ 0 & 1 & -2b_1 + b_3 \end{array}\right] \xrightarrow[\text{Row 3} - \text{Row 2}]{\text{Row 1} - \frac{1}{2}\text{Row 2}} \underbrace{\left[\begin{array}{cc|c} 1 & 0 & -b_1 + b_2 \\ 0 & 1 & 3b_1 - 2b_2 \\ 0 & 0 & -5b_1 + 2b_2 + b_3 \end{array}\right]}_{R}$$

both columns are pivot

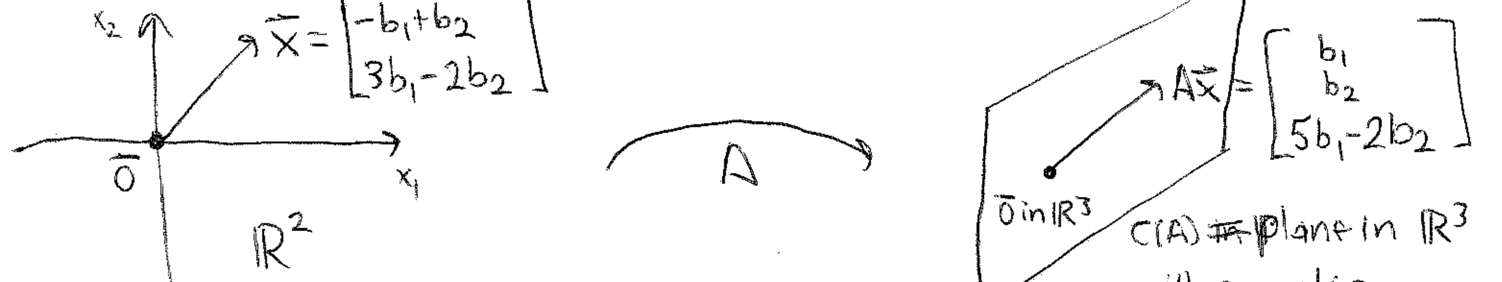
System $R\vec{x} = \vec{d}$:
$$\begin{cases} x_1 = -b_1 + b_2 \\ x_2 = 3b_1 - 2b_2 \\ 0 = -5b_1 + 2b_2 + b_3 \end{cases}$$

No solution if $-5b_1 + 2b_2 + b_3 \neq 0$.

One solution if $-5b_1 + 2b_2 + b_3 = 0$

It's $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -b_1 + b_2 \\ 3b_1 - 2b_2 \end{bmatrix}$

How to visualize this:



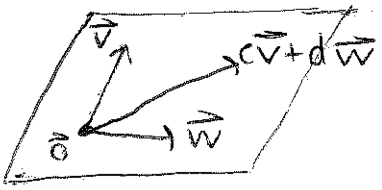
$C(A)$ ~~is~~ plane in $\mathbb{R}^3$ with equation $-5b_1 + 2b_2 + b_3 = 0$

Section 3.4 Linear Independence, Basis, Dimension

Note: These are maybe the most important ideas in linear algebra, but warning: they are a little abstract.

Motivation: What do we mean by dimension?

Here's a plane in $\mathbb{R}^3$:



we think it is 2-dimensional, even though it is living in $\mathbb{R}^3$.

Why is dimension = 2 for a plane?

Answer attempt: Every vector in the plane is a linear combination of 2 vectors, $\vec{v}$ and $\vec{w}$

But: Every vector in plane is also a linear combination of 3 vectors $\vec{v}$, $\vec{w}$, and $\vec{v}-\vec{w}$, because:

$$c\vec{v}+d\vec{w} = c\vec{v}+d\vec{w}+0(\vec{v}-\vec{w}), \text{ or } c\vec{v}+d\vec{w} = (c-1)\vec{v}+(d+1)\vec{w} + 1(\vec{v}-\vec{w}), \ldots$$

So we should say that the dimension is the minimum number of vectors required to span the plane, which is 2, not 3.

Question: How can we tell if a spanning set is minimal.

Every vector = lin. comb. of these ones

Answer: The vectors in the ~~...~~ spanning set should be independent: None of them is a linear combination of the others.

In our plane: $\vec{v} \neq c\vec{w}$, $\vec{w} \neq d\vec{v}$ $\longrightarrow$ $\{\vec{v},\vec{w}\}$ is an independent set of vectors

But: $\vec{v}-\vec{w} = 1\vec{v}+(-1)\vec{w}$

Also: $\vec{v} = 1\vec{w}+1(\vec{v}-\vec{w})$

$\vec{w} = 1\vec{v}+(-1)(\vec{v}-\vec{w})$

$\longrightarrow$ $\{\vec{v},\vec{w},\vec{v}-\vec{w}\}$ is a dependent set of vectors

Definition: A set of vectors $\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_m\}$ in $\mathbb{R}^m$ is dependent if one of the vectors is a linear combination of the others. If none is a linear combination of the others, the set of vectors is independent.

Quick example: Is $\{\vec{v},\vec{w},\vec{0}\}$ dependent?

Yes! For example, $\vec{0} = 0\vec{v}+0\vec{w}$.

Another formulation: $\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_m\}$ dependent means something like

$$\vec{v}_1 = c_2\vec{v}_2+\cdots+c_m\vec{v}_m$$

Or: $1\vec{v}_1+(-c_2)\vec{v}_2+\dots+(-c_m)\vec{v}_m=\vec{0}$

Definitely not $\vec{0}$ Could be 0's.

So if $\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_m\}$ is <u>dependent</u>, then there is a non-zero linear combination adding up to $\vec{0}$:

$$x_1\vec{v}_1+x_2\vec{v}_2+\dots+x_m\vec{v}_m=\vec{0}$$

some of these could be 0, but <u>not all</u> are 0

Turn the logic around: If the <u>only</u> way to get $x_1\vec{v}_1+x_2\vec{v}_2+\dots+x_m\vec{v}_m=\vec{0}$ is to set $x_1=x_2=\dots=x_m=0$, then the vectors must be <u>independent</u>:

$\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_m\}$ independent $\Longleftrightarrow$ Only solution to $x_1\vec{v}_1+x_2\vec{v}_2+\dots+x_m\vec{v}_m=\vec{0}$ is $\vec{0}$

$\Longleftrightarrow$ Only solution to $\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}=\vec{0}$ is $\vec{x}=\vec{0}$.

$\Longleftrightarrow$ For $A=\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}$, $N(A)=\{\vec{0}\}$

$\Longleftrightarrow$ R for A has <u>no free variables</u> (there's a leading 1 in <u>every column</u>.

means R look like $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hline & & & \end{bmatrix}$ could be rows of 0's here if # vectors < # components in each vector

So how to tell if $\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_m\}$ is independent?

1. Put $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_m$ as the columns of a matrix A
2. Do elimination on A to find R.
3. If R has $\longrightarrow$ free variables $\longrightarrow$ dependent
   $\searrow$ <u>no</u> free variables $\longrightarrow$ independent

Let's do a 3x3 example for both cases.

Check if $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is independent:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \xrightarrow[\text{Row 2}]{\text{Row 1} \leftrightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \xrightarrow[\text{Row 3+Row 2}]{\text{Row 3+Row 1, then}} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{} \xrightarrow[\text{elimination}]{\text{more}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

can already see: <u>no</u> free variables (vectors are <u>independent</u>)

So the <u>only</u> way to get

$$x_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is to set $x_1 = x_2 = x_3 = 0$.

Check if $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right\}$ is dependent or not.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[\text{Row 3}-3\text{Row 1}]{\text{Row 2}-2\text{Row 1}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow{\text{Row 3}-2\text{Row 2}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

free variable; (vectors are <u>dependent</u>)

$$\xrightarrow[\text{Then: }-\text{Row 2}]{\text{Row 1}+2\text{ Row 2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Null space equations: $x_1 = x_3$, $x_2 = -2x_3$, $x_3 = \text{anything}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Special solution tells us <u>how</u> to write the vectors as linear combinations of each other.

$$1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-2) \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

<u>Dependent</u> because we can write at least one of the vectors as a linear combination of the other two.