



清华大学

①

Homework 9 Solutions

4.1.6 $\left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 2 & 2 & 3 & b_2 \\ 3 & 4 & 5 & b_3 \end{array} \right] \xrightarrow[\text{Row 3}-3\text{Row 1}]{\text{Row 2}-2\text{Row 1}} \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & -2 & -1 & b_2-2b_1 \\ 0 & -2 & -1 & b_3-3b_1 \end{array} \right] \xrightarrow{\text{Row 3}-\text{Row 2}} \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & -2 & -1 & b_2-2b_1 \\ 0 & 0 & 0 & -b_1-b_2+b_3 \end{array} \right]$

Look at $\underbrace{(-1)}_{\cancel{\text{Eqn 1}}}(\text{Eqn 1}) + \underbrace{(-1)}_{\cancel{\text{Eqn 2}}} \text{Eqn 2} + \underbrace{(1)}_{\cancel{\text{Eqn 3}}} \text{Eqn 3} =$

$$-(x_1 + 2y + 2z) = -5$$

$$-(2x + 2y + 3z) = -5$$

$$+(3x + 4y + 5z) = 9$$

$$0 = -1$$

So $(1)\text{Eqn 1} + (1)\text{Eqn 2} + (-1)\text{Eqn 3}$ gives $0 = 1$.

$$y_1 = 1, y_2 = 1, y_3 = -1$$

$\vec{y} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is in the left nullspace $N(A^T)$ since $\vec{y}^T A = \vec{0}$.

Recall: $C(A) \perp N(A^T)$, so if $\vec{b} = \begin{bmatrix} 5 \\ 5 \\ 9 \end{bmatrix}$ were in $C(A)$, it would be

\perp to \vec{y} . But since $\vec{y}^T \vec{b} = 1$, it is not in $C(A)$, meaning $A\vec{x} = \vec{b}$ has no solution.

4.1.11 $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

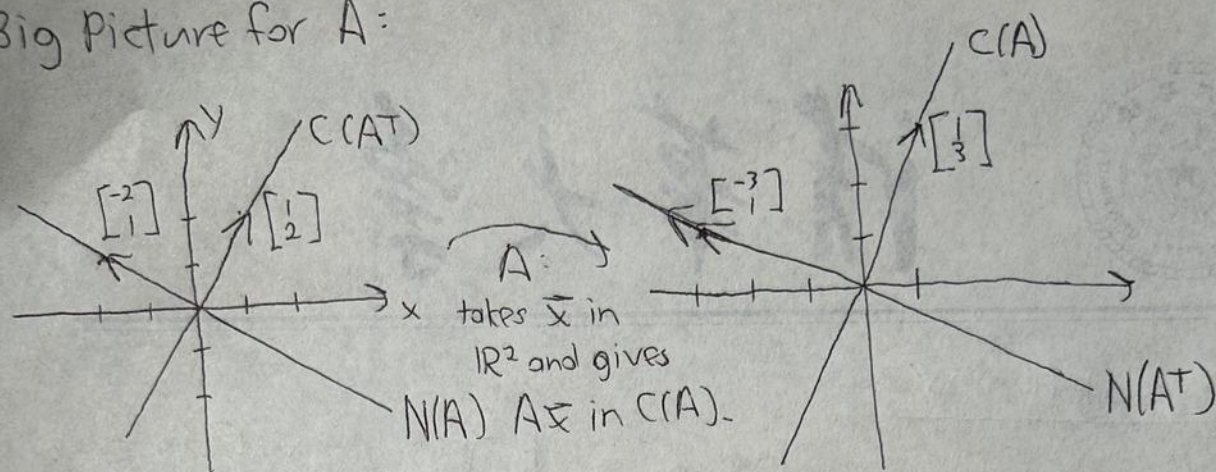
$$C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$$

$$N(A): x + 2y = 0 \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Row: $C(A^T) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$, $N(A^T): -3\text{Row 1} + \text{Row 2} = \vec{0}$
 $\rightarrow N(A^T) = \text{span}\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$

Big Picture for A:

(2)

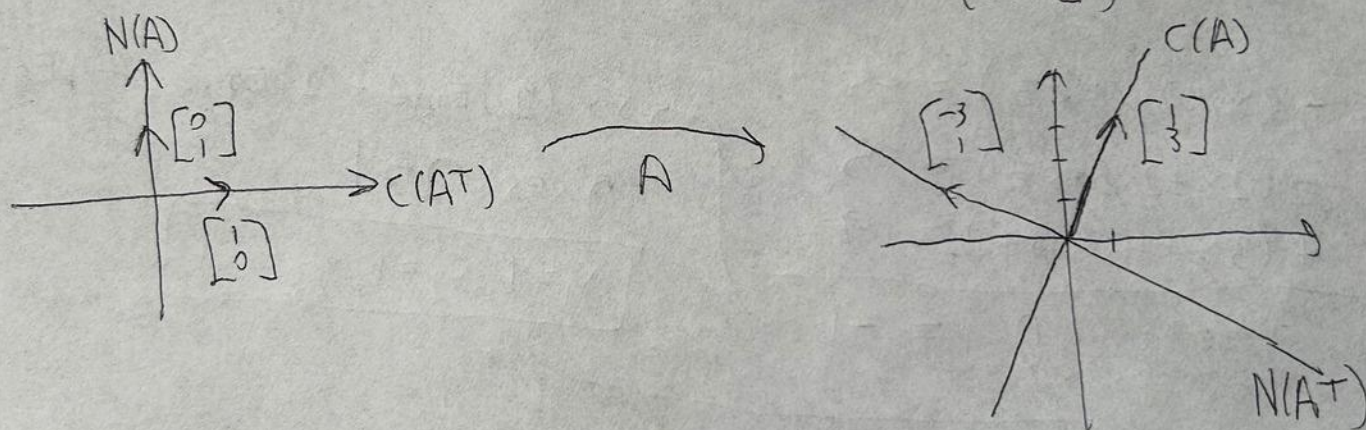


$$B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \xrightarrow[-3\text{Row } 1]{\text{Row } 2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C(A) = \text{span} \left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) \quad N(A) = x=0 \rightsquigarrow \text{span} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$\text{Row: } C(A^T) = \text{span} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), \quad N(A^T) = \text{Row } 3 - 3\text{Row } 1 = 0$$

$$\text{So } N(A^T) = \text{span} \left(\begin{bmatrix} -3 \\ 1 \end{bmatrix} \right)$$



4.1.17 If $S = \{\vec{0}\}$: $S^\perp = \text{all } \vec{x} \text{ such that } \vec{x} \cdot \vec{0} = 0$
 $= \text{all } \vec{x} \text{ in } \mathbb{R}^3 \rightarrow \boxed{S^\perp = \mathbb{R}^3}$

If $S = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$: $S^\perp = \text{all } \vec{x} \text{ such that } \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow$

$$x = -y - z \rightarrow \text{all } \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{So } S^\perp = \text{span} \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Basis for S^\perp



清华大学

③

If $S = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) : S^\perp = \text{all } \vec{x} \text{ such that } \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \vec{x} = 0 \text{ and}$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \vec{x} = 0. \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0} \quad \text{Null space of } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} :$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow[\text{Row 2}]{\text{Row 1} -} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[-\text{Row 2}]{\text{Row 1} + \text{Row 2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{matrix} x+y=0 \\ z=0 \end{matrix}$$

$$\text{So } S^\perp = \text{all } \vec{x} \text{ ~~such that~~ } = \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} = \text{all } y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

So $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is a basis for S^\perp in this case.

4.1.30 If \vec{b} is a vector in $C(B)$, then $\vec{b} = B\vec{x}$ for some \vec{x} .

So $A\vec{b} = A(B\vec{x}) = (AB)\vec{x} = \vec{0}\vec{x} = \vec{0}$, since $AB = \vec{0}$. This means \vec{b} is in $N(A)$.

Now: A is 3×4 $\dim N(A) = 4 - \text{rank}(A)$

B is 4×5 $\dim C(B) = \text{rank}(B)$

we also know $C(B)$ is inside $N(A)$, so

$$\dim C(B) \leq \dim N(A)$$

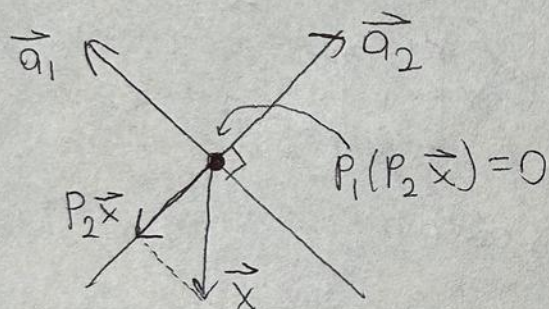
$$\text{so } \text{rank } B \leq 4 - \text{rank}(A) \rightarrow \boxed{\text{rank } A + \text{rank } B \leq 4}$$

$$4.2.5 \quad \vec{a}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, P_1 = \frac{\vec{a}_1 \vec{a}_1^T}{\vec{a}_1^T \vec{a}_1} = \frac{\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}}{\begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}} = \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \quad (4)$$

$$\vec{a}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, P_2 = \frac{\vec{a}_2 \vec{a}_2^T}{\vec{a}_2^T \vec{a}_2} = \frac{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \end{bmatrix}}{\begin{bmatrix} 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}} = \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$\text{So } P_1 P_2 = \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} = \frac{1}{81} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Why is $P_1 P_2 = 0$? Because $\vec{a}_1 \perp \vec{a}_2$:



$P_1 P_2$ means project to $\text{span}(\vec{a}_2)$ first, then project the result to $\text{span}(\vec{a}_1)$. But since $\vec{a}_1 \perp \vec{a}_2$, the result in $\text{span}(\vec{a}_2)$ is \perp to \vec{a}_1 , and gets projected to $\vec{0}$ by P_1 . I.e., $P_1 P_2 \vec{x} = \vec{0}$ for all \vec{x} , which means $P_1 P_2 = 0$.

$$4.2.6 \text{ Project onto } \text{span}(\vec{a}_1): P_1 \vec{b} = \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/9 \\ -2/9 \\ -2/9 \end{bmatrix} = \vec{p}_1$$

$$\text{Project onto } \text{span}(\vec{a}_2): P_2 \vec{b} = \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/9 \\ 4/9 \\ -2/9 \end{bmatrix} = \vec{p}_2$$

$$\text{Project onto } \text{span}(\vec{a}_3): \vec{p}_3 = \frac{\vec{a}_3 \vec{a}_3^T}{\vec{a}_3^T \vec{a}_3} \vec{b} = \frac{1}{9} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/9 \\ -2/9 \\ 4/9 \end{bmatrix}$$

$$\text{So } \vec{p}_1 + \vec{p}_2 + \vec{p}_3 =$$

$$\frac{1}{9} \left(\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{For 4.2.7: } P_3 = \frac{1}{9} \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}$$



清华大学

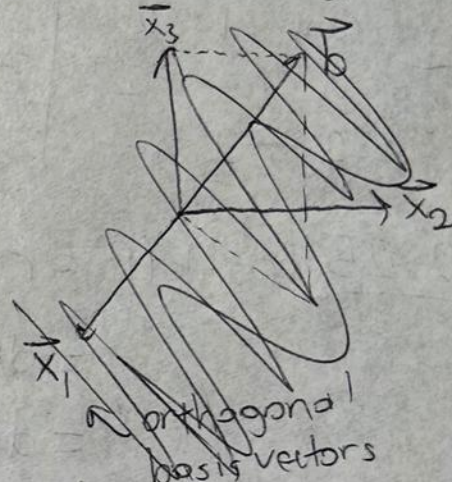
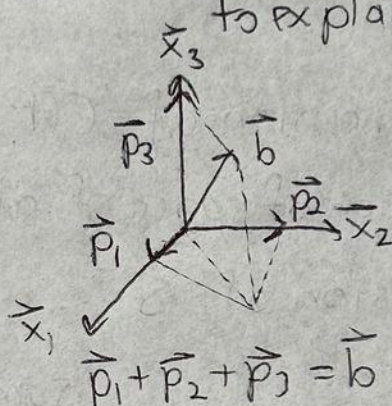
5

4.2.7 $P_1 + P_2 + P_3 = \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}$

$$= \frac{1}{9} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = I$$

General picture to explain:

Orthogonal basis



4.2.19 Plane $x - y - 2z = 0 \rightsquigarrow x = y + 2z$

Plane = all $\begin{bmatrix} y+2z \\ y \\ z \end{bmatrix} = \text{all } y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, so $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, and

$$P = A(A^T A)^{-1} A^T = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\frac{1}{6} \begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

4.2.20 $\vec{e} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ is \perp to the plane \rightarrow

(6)

$$Q = \frac{\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \end{bmatrix}}{\begin{bmatrix} 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}} = \frac{1}{6} \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix}, \text{ Then } P = I - Q =$$

$$\frac{1}{6} \begin{bmatrix} 6-1 & 1 & 2 \\ 1 & 6-1 & -2 \\ 2 & -2 & 6-4 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix} \quad (\text{same as before})$$

4.2.25 $\text{rank}(P) = r = \text{dimension of } C(P)$

$= \text{dimension of space of all } P\vec{b}$

$= \text{dimension of } S$

$= r$

\rightarrow Why are these the same? Well, $P\vec{b}$ is in S for any \vec{b} , and on the other hand, if \vec{b} is already in S , then $P\vec{b} = \vec{b}$, so S is also contained in the set of all vectors $P\vec{b}$.

4.2.34 P is a projection matrix if and only if $P = P^T$ and $P^2 = P$.

What about $P_1 P_2$? First, suppose $P_1 P_2 = P_2 P_1$. Then

$$(P_1 P_2)^T = P_2^T P_1^T = P_2 P_1 = P_1 P_2 \quad \checkmark$$

P_1 and P_2 are projections

$$\text{and } (P_1 P_2)^2 = P_1 P_2 P_1 P_2 = P_1 P_1 P_2 P_2 = P_1 P_2 \quad \checkmark$$

} So $P_1 P_2$ is a projection.

on the other hand, suppose $P_1 P_2$ is a projection matrix. Then

$$(P_1 P_2) = (P_1 P_2)^T = P_2^T P_1^T = P_2 P_1. \text{ So } P_1 \text{ and } P_2 \text{ commute. } \checkmark$$



清华大学

7

Graded Problem 1 If $V = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, then $V^\perp =$ all \vec{x} such that

$$\begin{cases} \vec{v}_1^T \vec{x} = 0 \\ \vec{v}_2^T \vec{x} = 0 \\ \vec{v}_3^T \vec{x} = 0 \end{cases} = \text{all } \vec{x} \text{ such that } \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vec{v}_3^T \end{bmatrix} \vec{x} = \vec{0} = \text{null space of}$$

$$\begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vec{v}_3^T \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 6 \\ 1 & 2 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{\substack{\text{Row 2} - \text{Row 1} \\ \text{Row 3} - \text{Row 1}}} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \end{bmatrix} \xrightarrow{\text{Row 3} + 2\text{Row 2}}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 1} - 3\text{Row 2}} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} x_1 + 2x_2 + x_4 + 2x_5 = 0 \\ x_3 + x_4 + x_5 = 0 \\ x_2, x_4, x_5 \text{ free} \end{cases}$$

$$V^\perp = \text{all } \begin{bmatrix} -2x_2 - x_4 - 2x_5 \\ x_2 \\ -x_4 - x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

↑ ↑ ↑
Basis for V^\perp

Graded Problem 2 $P = A(A^T A)^{-1} A^T$ where $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}$

$$\left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 2 \\ 0 & -1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 2 & 4 \\ 4 & 10 \end{bmatrix}^{-1} = \frac{1}{4} \begin{bmatrix} 10 & -4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 5/2 & -1 \\ -1 & 1/2 \end{bmatrix}$$

$$\text{So } P = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 2 \\ 0 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} 5/2 & -1 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 2 & -1 \end{bmatrix}}_{\begin{bmatrix} 1/2 & 1 & 1/2 & 1 \\ 0 & -1/2 & 0 & -1/2 \end{bmatrix}} = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix} \quad (8)$$

$$\text{Project } \vec{x}: P\vec{x} = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix} = \vec{p}$$

$$\text{Error: } \vec{e} = \vec{x} - \vec{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix},$$

$$\|\vec{e}\| = \sqrt{(-1)^2 + (-1)^2 + 1^2 + 1^2} = 2$$