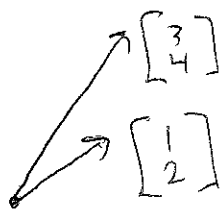


## Section 1.3 Matrices

(12)

So far: We have introduced vectors and some algebraic operations:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \quad 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = (1)(3) + (2)(4) = 11$$



For geometry with vectors:  
lengths and angles

Today: Introduce matrices and linear equations

Here is a  $2 \times 3$  matrix: It is a rectangular array of numbers. This one has 2 rows and 3 columns.

Here is a  $3 \times 2$  matrix:  $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$   $2 \times 2 = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$

$$1 \times 4 = [1 \ 2 \ 3 \ 4]$$

In general, an  $m \times n$  matrix has  $m$  rows and  $n$  columns

Typical abstract notation:  $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

Notice: A column vector in  $n$  dimensions is an  $n \times 1$  matrix:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

You can think of a matrix as a way to package information:  
the information of  $m \times n$  real numbers (scalars)

Why would we want to package information in rectangular form?

One reason: We have some algebraic operations with matrices (13)  
that let us manipulate the information in a matrix.

First operation: Matrix-vector multiplication

$$(m \times n \text{ matrix})(n\text{-dim vector}) = (m\text{-dim. vector})$$

$\nwarrow$   $n \times 1$  matrix

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{2 \times 3 \text{ matrix}} \underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}_{3 \times 1 \text{ column vector}} = \begin{bmatrix} 1(-1) + 2(-1) + 3(1) \\ 4(-1) + 5(-1) + 6(1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -3 \end{bmatrix}}_{2\text{-dim. column vector}} = A\vec{v}$$

"Row picture" of this operation:

The 1st component of  $A\vec{v} = (\text{1st row of } A) \cdot \vec{v}$   
2nd component of  $A\vec{v} = (\text{2nd row of } A) \cdot \vec{v}$   
3rd --- } However many  
4th --- } rows you have

Dot product

"Column picture" of this operation:

Think of the 3 columns of  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$  as 3 vectors in

2-dim. space:  $\vec{u} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ ,  $\vec{w} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$  (i.e., A contains the information of 3 vectors)

Now: the matrix-vector product  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$  calculates a

linear combination:  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 4 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$   
 $= \begin{bmatrix} -1 - 2 + 3 \\ -4 - 5 + 6 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$  Same result

Another example:  $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4-4 \\ 8-5 \\ 12-6 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$  (14)

2 column vectors in  
3-dim. space,  $\vec{u}, \vec{v}$

This vector is a linear  
combination of  $\vec{u}$  and  $\vec{v}$ .

We now have two ways of thinking about linear combinations:

1. Numbers multiplying vectors:  $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n$

2. Matrix multiplying a vector:  $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n$   
 1st column    2nd column

Problem 1.3.1  $\vec{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{s}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Use matrix-vector multiplication to calculate:

$$\begin{array}{ccc} 3\vec{s}_1 & + & 4\vec{s}_2 + 5\vec{s}_3 \\ \uparrow & & \uparrow \\ x_1 & & x_2 \end{array} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1(3)+0(4)+0(5) \\ 1(3)+1(4)+0(5) \\ 1(3)+1(4)+1(5) \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 12 \end{bmatrix}$$

$\vec{s}_1 \quad \vec{s}_2 \quad \vec{s}_3 \quad \vec{x}$ 
compute using dot products ("row picture")
 $\vec{b}$

Using matrix notation:

$$S\vec{x} = \vec{b}$$

Here, we knew  $S$  and  $\vec{x}$ , so we could calculate  $\vec{b} = \begin{bmatrix} 3 \\ 7 \\ 12 \end{bmatrix}$

Reverse problem: What if we knew  $S$  and  $\vec{b}$ ? Could we find  $\vec{x}$ ?

Problem 1.3.2: Solve  $S\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  for  $\vec{x}$ .

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \iff \begin{bmatrix} x_1 + 0x_2 + 0x_3 \\ x_1 + x_2 + 0x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Or,  $\begin{cases} x_1 = 1 \\ x_1 + x_2 = 1 \\ x_1 + x_2 + x_3 = 1 \end{cases}$

A system of  
linear equations!

We can solve it:  $x_1 = 1 \checkmark$   
 $x_2 = 1 - x_1 = 1 - 1 = 0 \checkmark$   
 $x_3 = 1 - x_1 - x_2 = 1 - 1 - 0 = 0 \checkmark$

So  $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ , and indeed  $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Or:  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

↖ This is a linear combination of the 3 vectors (actually, it is one of the 3 vectors)

Harder question: Is every vector in 3-dim. space a linear combination of  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ?

Solution: Every vector in 3-dim space looks like this:  $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

( $b_1, b_2, b_3$  can be any real numbers)

The question is: can we find  $x_1, x_2, x_3$  so that

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} ??$$

They would depend on  $b_1, b_2, b_3$ .

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

we need to solve a matrix-vector equation for  $x_1, x_2, x_3$

$$\begin{bmatrix} x_1 + 0x_2 + 0x_3 \\ x_1 + x_2 + 0x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \rightarrow \begin{cases} x_1 = b_1 \\ x_1 + x_2 = b_2 \\ x_1 + x_2 + x_3 = b_3 \end{cases}$$

we can solve for  $x_1, x_2, x_3$  in terms of the general parameters  $b_1, b_2, b_3$ :

$$x_1 = b_1 \checkmark$$

$$x_2 = b_2 - x_1 = b_2 - b_1 \checkmark$$

$$x_3 = b_3 - x_1 - x_2$$

$$= b_3 - b_1 - (b_2 - b_1) = b_3 - b_2 \checkmark$$

This tells us:

$$b_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (b_2 - b_1) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (b_3 - b_2) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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No matter what I give you for  $\vec{b}$ , you can tell me  $\vec{x}$ :

If  $\vec{b} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$ , then  $\vec{x} = \begin{bmatrix} b_1 \\ b_2 - b_1 \\ b_3 - b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 - 1 \\ 9 - 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ .

Let's look at this formula more carefully:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 + 0b_2 + 0b_3 \\ -b_1 + b_2 + 0b_3 \\ 0b_1 - b_2 + b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The original problem was: Solve  $S\vec{x} = \vec{b}$  for  $\vec{x}$ .

The answer turns out to be:  $\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$S\vec{x} = \vec{b} \rightarrow \vec{x} = S^{-1}\vec{b}$$

If you can find an "inverse" matrix, you can use it to solve linear equations

This matrix is called the inverse of  $S$ , or  $S^{-1}$ .

Easy question: What's the inverse of a  $1 \times 1$  matrix?

$$A = [a] \text{ (really just a number)}$$

$$A\vec{x} = \vec{b} \text{ is really just } ax = b \rightsquigarrow x = \underbrace{a^{-1}}_{\text{This is the inverse.}} b$$

$$\text{So } [a]^{-1} = [a^{-1}]$$

But this only works if  $a \neq 0!!$

It's possible that a matrix doesn't have an inverse.

Example: 3 vectors that lie in a plane

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ (the vector sum)}$$

Can we write  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  as a linear combination of  $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ ? (17)

Try to solve:  $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightsquigarrow$

$$\begin{cases} x_1 + x_3 = 1 \\ x_1 + x_2 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \rightsquigarrow \begin{cases} x_1 - x_2 = 1 \\ x_1 - x_2 = 0 \end{cases}$$

These equations are inconsistent!

There is no solution to this system of equations.

The problem is that  $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  is a linear combination of  $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ .

It is "dependent."

This means the matrix  $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$  has no inverse (just like  $0^{-1}$  doesn't exist)

When the columns of the matrix are dependent (no inverse exists) we might also get infinitely many solutions to linear equations.

Example How many ways can we write the zero vector  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  as a linear combination of  $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ .

One way:  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Other ways: Solve  $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{cases} x_1 + x_3 = 0 \\ x_1 + x_2 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$

$\begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \end{cases}$   
 $x_1 + x_2 + 2x_3 = -x_3 - x_3 + 2x_3 = 0 \checkmark$  (No problem. Also, no new information about  $x_1, x_2, x_3$ )

This shows  $x_3$  is allowed to be any real number. But once we've picked  $x_3$ , we must take  $x_1 = -x_3$  and  $x_2 = -x_3$ . (18)

So all solutions look like  $\begin{bmatrix} -c \\ -c \\ c \end{bmatrix} = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$  ( $c$  can be any real number)

For example, if we pick  $c=1$ : tells us that

$$(-1) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Note: Today we briefly looked at some of the major ideas in solving linear equations. We will look at these ideas in more detail later, so don't worry if you didn't completely understand everything today.