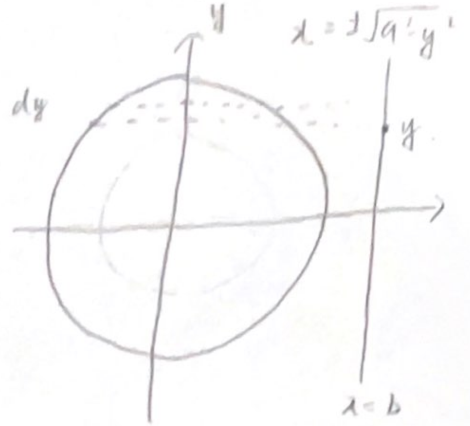
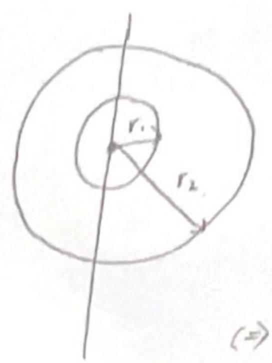
Calculus A(1) : Homework 8.

A1. Compute the volume of the soid of revolution obtained by rotating the disk $x^2+y^2 \le a^2$ about the line $x=b$, where $0<a<b$.





$r_1 = b - \sqrt{a^2-y^2}$

$r_2 = b + \sqrt{a^2-y^2}$

$S = \pi(r_2^2 - r_1^2)$

$\Leftrightarrow S = \pi 4b\sqrt{a^2-y^2}$

$$dv = S \cdot dy = 4\pi b\sqrt{a^2-y^2}\,dy \quad , y \in (-a,a)$$

$$\Leftrightarrow \; dv = 4\pi b \int_{-a}^{a} \sqrt{a^2-y^2}\,dy.$$

$$\Leftrightarrow \; dv = 4\pi b \cdot \frac{1}{2}\pi a^2$$

$$\Leftrightarrow \; dv = 2\pi^2 a^2 b.$$

A2 The shaded band shown here is cut from a sphere of radius R by parallel planes h units apart. Show that the surface area of the band is $2\pi R h$.

Sol:
$$A = \int_a^{a+h} 2\pi f(x)\sqrt{1+[f'(x)]^2}\, dx$$

$$f(x) = \sqrt{R^2 - x^2}$$

$$\Leftrightarrow f'(x) = \frac{1}{2}\sqrt{R^2-x^2}\,(-2x)$$

$$\Leftrightarrow f'(x) = \frac{-x}{\sqrt{R^2-x^2}}$$

$$\Leftrightarrow [f'(x)]^2 = \frac{x^2}{R^2-x^2}$$

Then, $A = \int_a^{a+h} 2\pi\sqrt{R^2-x^2}\sqrt{1+\frac{x^2}{R^2-x^2}}\, dx$

$$\Leftrightarrow A = \int_a^{a+h} 2\pi\sqrt{(R^2-x^2)+x^2}\, dx.$$

$$\Leftrightarrow A = \int_a^{a+h} 2\pi R\, dx.$$

$$\Leftrightarrow A = 2\pi R \; x\Big|_a^{a+h}$$

$$\Leftrightarrow A = 2\pi R\,(a+h-a)$$

$$\Leftrightarrow A = 2\pi R h$$

A3. a Show that the graph of e^x is concave up over every interval of x-values

Sol: $y = e^x \Rightarrow y' = e^x \Rightarrow y'' = e^x > 0$

which means the graph is concave up

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b. Show, by reference to the accompanying figure, that if $0 < a < b$ then

$$e^{(\ln a + \ln b)/2} \cdot (\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x \, dx < \frac{e^{\ln a} + e^{\ln b}}{2} \cdot (\ln b - \ln a)$$

Area of the trapezoid ABCD $< \int_{\ln a}^{\ln b} e^x \, dx <$ area of the trapezoid AEFD

$$\Rightarrow \frac{1}{2}(AB + CD)(\ln b - \ln a) < \int_{\ln a}^{\ln a} e^x \, dx < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a)$$

Now $\frac{1}{2}(AB + CD)$ is the height of the midpoint. $M = e^{(\ln a + \ln b)/2}$

Since the curve containing the points B and C is linear

Therefore, $$e^{(\ln a + \ln b)/2}(\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x \, dx < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a)$$

(c) Use the inequality in part (b) to conclude that

$$\sqrt{ab} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2}$$

by part (b), we have

$$e^{(\ln a + \ln b)/2}(\ln b - \ln a) < b - a < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a)$$

$$\Leftrightarrow \quad e^{(\ln a + \ln b)/2} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2}$$

$$\Leftrightarrow \quad e^{\frac{\ln a}{2}} \cdot e^{\frac{\ln b}{2}} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2}$$

$$\Leftrightarrow \quad \sqrt{e^{\ln a}} \cdot \sqrt{e^{\ln b}} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2}$$

$$\Leftrightarrow \sqrt{ab} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2}$$

A4. Compute $\lim_{x\to 0^+} x^x$ (if it exists). You should justify your answer.

Sol: let $x = e^{\ln x}$

$\lim_{x\to 0^+} x^x$

$\Leftrightarrow \; = \lim_{x\to 0^+} (e^{\ln x})^x.$

$\Leftrightarrow \; = \lim_{x\to 0} e^{x\ln x}.$

$\Leftrightarrow \; = e^{\lim_{x\to 0} x\ln x}.$

$\Leftrightarrow \; = e^{\lim_{x\to 0} \frac{\ln x}{\frac{1}{x}}}$ — Since. $\lim_{x\to 0^+} \ln x = -\infty$, $\lim_{x\to 0^+} \frac{1}{x} = +\infty$.

we use L'hospital rule.

$\Leftrightarrow \; = e^{\lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}}}$

$\Leftrightarrow \; = e^{\lim_{x\to 0} (-x)}$

$\Leftrightarrow \; = e^0$

$\Leftrightarrow \; = 1$

A5. Prove that for all $x \in \mathbb{R}$ with $|x| \geq 1$, we have $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$

Proof: Let $y = \sec^{-1}(-x)$, $x \in \mathbb{R}$, $|x| \geq 1$

Therefore, $\sec(y) = -x$

$\Leftrightarrow \quad -\sec(y) = x$

$\Leftrightarrow \quad \sec(\pi - y) = x$

$\Leftrightarrow \quad \pi - y = \sec^{-1}(x)$

$\Leftrightarrow \quad y = \pi - \sec^{-1}(x)$

$\Leftrightarrow \quad \sec^{-1}(-x) = \pi - \sec^{-1}(x)$

B2. Let $f: [a,b] \to \mathbb{R}$ be a continuous function which is one to one.

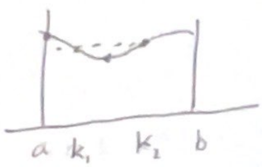
Show that f is monotonic.

Proof: Suppose that f is not monotonic on $[a,b]$.

Since f is continous on $[a,b]$, there must be a global maximun on

a or b or C, $C \in [a,b]$

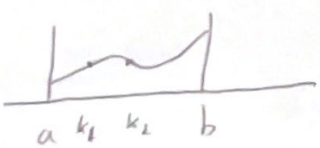
① when the global maximum is on a, and f is not monotonic



There $\exists x_1, x_2$, s.t. $f(k_1) = f(k_2)$

Contradicts with f is one to one!

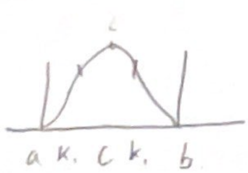
② when the global maximun is on b, and f is not monotonic.



There $\exists x_1 x$, st $f(k_1) = f(k_2)$.

Contradicts with f is one to one!

③. when the global maximum is on C, $C \in (a,b]$



There $\exists x_1 x_2$, st $f(k_1) = f(k_2)$.

Contradicts with f is one to one!

by ①.②.③. if $f: [a,b] \to \mathbb{R}$ is continous function which is one to one, it is monotonic