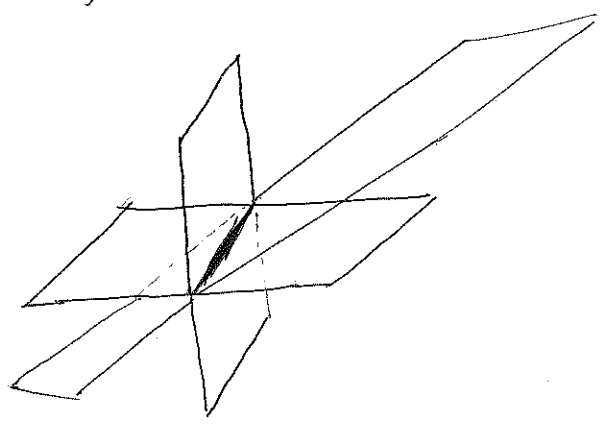
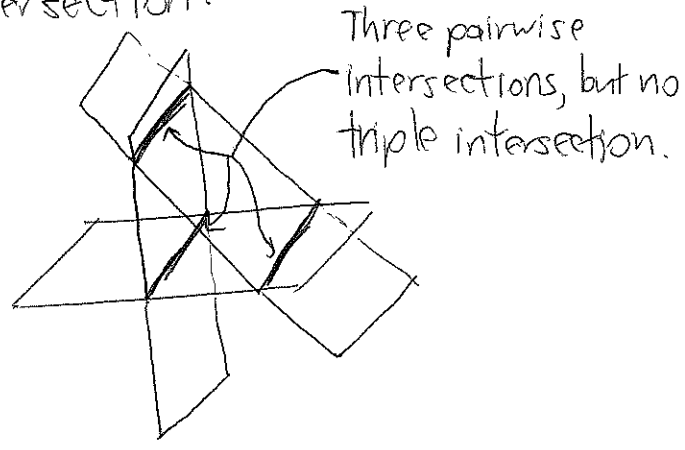


Important Warning: 3 planes in 3-dim. don't always intersect in a single point.

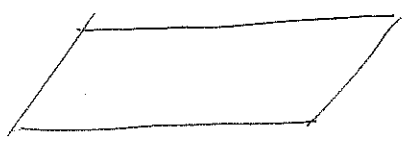
They might intersect in a line:



They might not have any common intersection:



They might even intersect in a plane, if all three are really the same plane in disguise:



$$\begin{cases} x+2y+3z=4 \\ 2x+4y+6z=8 \\ -x-2y-3z=-4 \end{cases}$$

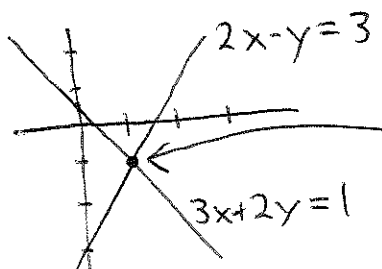
This means there are three possibilities when we solve a system of linear equations: No solution, exactly one solution, or infinitely many solutions (a whole, line, plane, etc.)

(It will never have just 2 or 3 solutions!)

Last time Pictures in 2 dimensions:

Row picture

$$\begin{cases} 2x-y=3 \\ 3x+2y=1 \end{cases}$$



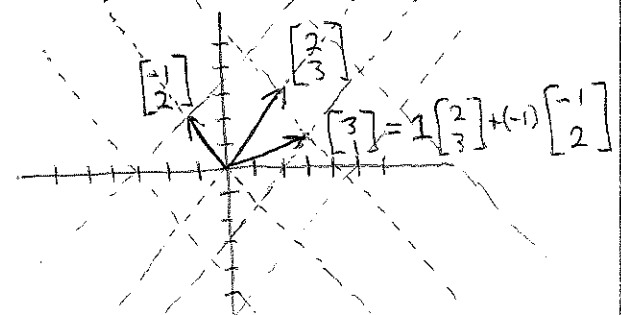
Matrix-vector equation

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

one intersection,
 $(x,y) = (1,-1)$

Column picture

$$x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



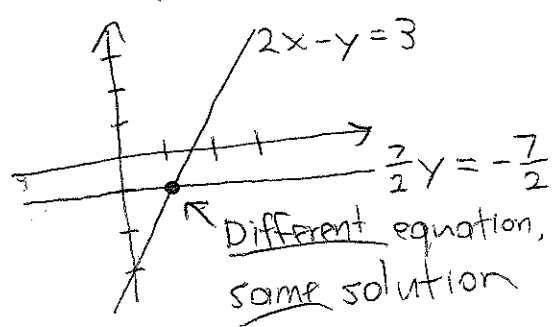
Solution using elimination:

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \xrightarrow[\text{Row 2} - \frac{3}{2}(\text{Row 1})]{\text{Row 2} \rightarrow} \begin{bmatrix} 2 & -1 \\ 0 & 7/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -7/2 \end{bmatrix}$$

First "pivot," the 1st non-zero entry in Row 1
a "multiplier"

Row 2 \rightarrow
 $\frac{2}{7}(\text{Row 2})$

New row picture:



$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

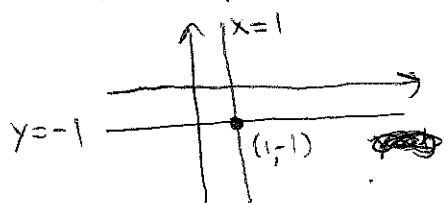
2nd "pivot," the 1st non-zero entry in the new Row 2

Row 1 \rightarrow
Row 1 + Row 2

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \xrightarrow[\frac{1}{2}(\text{Row 1})]{\text{Row 1} \rightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

we find the solution by doing a little arithmetic with the matrix-vector entries.

Final row picture:

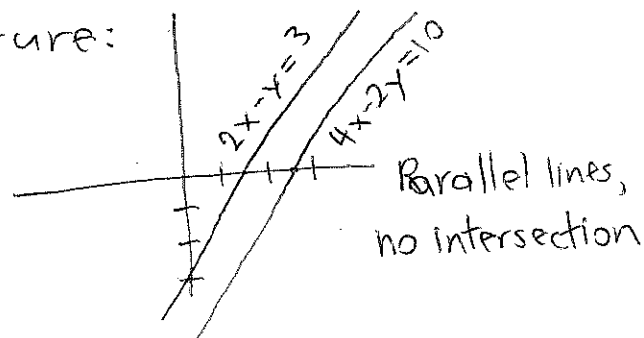


Row and column pictures with no solutions:

$$\begin{cases} 2x - y = 3 \\ 4x - 2y = 10 \end{cases} \xrightarrow[\text{Eqn. 2} - 2(\text{Eqn 1})]{\text{Eqn. 2} \rightarrow} \begin{cases} 2x - y = 3 \\ 0x + 0y = 4 \end{cases}$$

$0 = 4$ x no solution

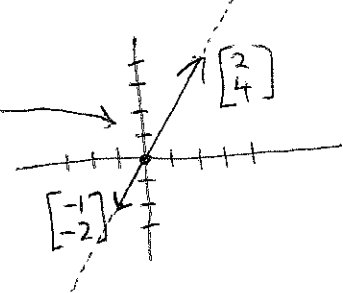
Row picture:



column picture:

$$x \begin{bmatrix} 2 \\ 4 \end{bmatrix} + y \begin{bmatrix} -1 \\ -2 \end{bmatrix} \stackrel{??}{=} \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

Parallel vectors, linear combinations only fill up a line, $\begin{bmatrix} 3 \\ 10 \end{bmatrix}$ is not on this line



Infinitely many solutions:

(25)

$$\begin{cases} 2x - y = 3 \\ -2x + y = -3 \end{cases}$$

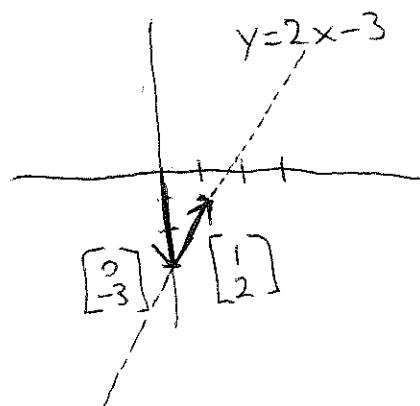
Egn. 2 + Egn. 1

$$\begin{cases} 2x - y = 3 \\ 0x + 0y = 0 \end{cases}$$

$0 = 0$, okay

All solutions are on the line $y = 2x - 3$.
x can be anything, say $x = c$. Then $y = 2c - 3$

Some line: $y = 2x - 3$



All solutions: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ 2c - 3 \end{bmatrix} = c \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

Direction of the line $y = 2x - 3$

$$c \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Point on the line

Now let's return to 3×3 systems in 3 dimensions

Row picture: Planes intersecting in space.

Column picture: Writing $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ as a linear combination of matrix columns.

We saw an example with a unique solution last time.

Today: $\begin{cases} x + y + z = b_1 \\ x + 2y + z = b_2 \\ 2x + y + 2z = b_3 \end{cases}$ Determine which values of b_1, b_2, b_3 give no solutions, and which give infinitely many.

1st "pivot": the first non-zero entry
 $1x + y + z = b_1$ in 1st row

$$1x + 2y + z = b_2$$

$$2x + y + 2z = b_3$$

2nd eqn \rightarrow 2nd - 1st
3rd eqn \rightarrow 3rd - 2(1st)

$$x + y + z = b_1$$

$$= b_2 - b_1$$

$$= b_3 - 2b_1$$

use 2nd "pivot" to eliminate y from 3rd equation

3rd \rightarrow 3rd + 2nd

$$x + y + z = b_1$$

$$y = b_2 - b_1$$

$$0 = -3b_1 + b_2 + b_3$$

2nd "pivot" - y

From last equation: We get solutions only if $-3b_1 + b_2 + b_3 = 0$, (26)
or $b_3 = 3b_1 - b_2$

If $b_3 = 3b_1 - b_2$, we get infinitely many solutions:

$$\begin{cases} X + y + z = b_1 \\ y = b_2 - b_1 \\ 0 = 0 \checkmark \end{cases} \xrightarrow{\text{Eliminate } y \text{ from 1st equation}} \begin{cases} X + z = 2b_1 - b_2 \\ y = b_2 - b_1 \end{cases}$$

(no more simplification is possible)

We can choose z to be anything, let's say $z = c$

Then: $x = 2b_1 - b_2 - c$
 $y = b_2 - b_1$
 $z = c$

← Infinitely many choices for c ,
 infinitely many solutions

Specific examples: $b_1 = 1, b_2 = 1, b_3 = 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ has no solutions because } 3(1) - 1 \neq 1$$

How about $b_1 = 1, b_2 = 1, b_3 = 2$? $3(1) - 1 = 2 \checkmark$

Solutions are: $x = 2b_1 - b_2 - c = 1 - c$
 $y = b_2 - b_1 = 0$
 $z = c$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - c \\ 0 \\ c \end{bmatrix} \xrightarrow{\text{Rewrite}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

one choice of solution, $(1, 0, 0)$
 All solutions form a whole line.

Now let's do a 4x4 example:

$$\begin{cases} 2x_1 - x_2 = 2 \\ -x_1 + 2x_2 - x_3 - x_4 = -2 \\ -x_2 + 2x_3 = -2 \\ -x_2 + 2x_4 = 2 \end{cases}$$

← Equations for 4 "hyperplanes" intersecting in 4-dimensional space

Systematic elimination: Eliminate lower left variables first, from left to right. Then eliminate upper right variables from right to left.

1st "pivot"

$$\begin{cases} 2x_1 - x_2 = 2 \\ -x_1 + 2x_2 - x_3 - x_4 = -2 \\ -x_2 + 2x_3 = -2 \\ -x_2 + 2x_4 = 2 \end{cases}$$

Eliminate first

Eqn. 2 →
Eqn. 2 + 1/2 (Eqn. 1)

$$\begin{cases} 2x_1 - x_2 = 2 \\ \frac{3}{2}x_2 - x_3 - x_4 = -1 \\ -x_2 + 2x_3 = -2 \\ -x_2 + 2x_4 = 2 \end{cases}$$

Eliminate next, use

2nd "pivot," 3/2

Row 3 →
Row 3 + 2/3 (Row 2)
Row 4 →
Row 4 + 2/3 (Row 2)

$$\begin{cases} 2x_1 - x_2 + 0x_3 + 0x_4 = 2 \\ 0x_1 + \frac{3}{2}x_2 - x_3 - x_4 = -1 \\ 0x_1 + 0x_2 + \frac{4}{3}x_3 - \frac{2}{3}x_4 = -\frac{8}{3} \\ 0x_1 + 0x_2 - \frac{2}{3}x_3 + \frac{4}{3}x_4 = \frac{4}{3} \end{cases}$$

3rd "pivot"

Eliminate using Eqn. 3

Eqn. 4 → Eqn. 4 + 1/2 Eqn. 3

$$\begin{cases} 2x_1 - x_2 = 2 \\ \frac{3}{2}x_2 - x_3 - x_4 = -1 \\ \frac{4}{3}x_3 - \frac{2}{3}x_4 = -\frac{8}{3} \\ x_4 = 0 \end{cases}$$

Row 1 → 1/2 Row 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & -2 \\ 0 & 0 & 1 & 0 & | & -2 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Coefficient matrix

$$\begin{bmatrix} 2 & -1 & 0 & 0 & | & 2 \\ -1 & 2 & -1 & -1 & | & -2 \\ 0 & -1 & 2 & 0 & | & -2 \\ 0 & -1 & 0 & 2 & | & 2 \end{bmatrix}$$

Augmented matrix

Row 2 →
Row 2 + 1/2 Row 1

$$\begin{bmatrix} 2 & -1 & 0 & 0 & | & 2 \\ 0 & 3/2 & -1 & -1 & | & -1 \\ 0 & -1 & 2 & 0 & | & -2 \\ 0 & -1 & 0 & 2 & | & 2 \end{bmatrix}$$

Row 3 → Row 3 + 2/3 (Row 2)
Row 4 → Row 4 + 2/3 (Row 2)

$$\begin{bmatrix} 2 & -1 & 0 & 0 & | & 2 \\ 0 & 3/2 & -1 & -1 & | & -1 \\ 0 & 0 & 4/3 & -2/3 & | & -8/3 \\ 0 & 0 & -2/3 & 4/3 & | & 4/3 \end{bmatrix}$$

Row 4 → Row 4 + 1/2 (Row 3)

Eliminate using Row 4

$$\begin{bmatrix} 2 & -1 & 0 & 0 & | & 2 \\ 0 & 3/2 & -1 & -1 & | & -1 \\ 0 & 0 & 4/3 & -2/3 & | & -8/3 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Row 2 → Row 2 + Row 4
Row 3 → Row 3 + 3/2 Row 4

Eliminate using Row 3

$$\begin{bmatrix} 2 & -1 & 0 & 0 & | & 2 \\ 0 & 3/2 & -1 & -1 & | & -1 \\ 0 & 0 & 4/3 & 0 & | & -8/3 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Eliminate

Row 2 → Row 2 + 3/4 (Row 3)
(Also Row 3 → 1/2 Row 3)

Row 1 →
Row 1 + 2/3 (Row 2)
(Row 2 → 2/3 Row 2)

$$\begin{bmatrix} 2 & -1 & 0 & 0 & | & 2 \\ 0 & 3/2 & 0 & 0 & | & -3 \\ 0 & 0 & 1 & 0 & | & -2 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Solution = final column of final augmented matrix:

(28)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -2 \\ 0 \end{bmatrix}$$

column picture: $\begin{bmatrix} 2 \\ -2 \\ -2 \\ 2 \end{bmatrix} = (-2) \begin{bmatrix} 2^{\text{nd}} \\ \text{col} \end{bmatrix} + (-2) \begin{bmatrix} 3^{\text{rd}} \\ \text{col} \end{bmatrix}$

↑
right: last column of
first augmented matrix

↑
columns of coefficient
matrix

$$= (-2) \begin{bmatrix} -1 \\ 2 \\ -1 \\ -1 \end{bmatrix} + (-2) \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \checkmark$$

So far: We have applied two operations for solving linear systems by elimination:

- ① Add a multiple of one equation ~~to~~ another.
- ② Multiply both sides of an equation by a non-zero scalar.

These change the equations, but not the solutions.

Why? Because these operations are reversible:

- If I add $2(\text{Eqn. 1})$ to Eqn. 2, I can reverse this by subtracting $2(\text{Eqn. 1})$ back off.
- If I multiply Eqn. 3 by C , I can reverse this by dividing Eqn. 3 by C (as long as $C \neq 0$!)

There's one more reversible operation we can perform, though it might seem a bit silly:

- ③ Switch the order of two equations.

Why would we want to switch the order of two equations?

Might need to if we want to follow our systematic elimination method (i.e., eliminate lower left variables first)

Example

(29)

$$\begin{cases} x_2 + x_3 = 4 \\ x_1 + x_2 + x_3 = 1 \\ x_1 + 2x_2 + 3x_3 = 2 \end{cases} \longleftrightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \end{bmatrix}$$

First step: Eliminate these using Eqn. 1. But we can't!

Instead: Switch Rows 1 and 2 first.

$$\text{Row 1} \leftrightarrow \text{Row 2} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 4 \\ \textcircled{1} & 2 & 3 & 2 \end{bmatrix} \xrightarrow[\text{Row 3 - Row 1}]{\text{Row 3} \rightarrow} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & \textcircled{1} & 2 & 1 \end{bmatrix} \xrightarrow[\text{Row 3 - Row 2}]{\text{Row 3} \rightarrow}$$

Now eliminate this

Eliminate this

$$\begin{bmatrix} 1 & 1 & \textcircled{1} & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix} \xrightarrow[\text{Row 2 - Row 3}]{\text{Row 1 - Row 3}} \begin{bmatrix} 1 & \textcircled{1} & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -3 \end{bmatrix} \xrightarrow{\text{Row 1 - Row 2}} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Eliminate

Eliminate

or use back
-substitution

Solution vector,
 $x_1 = -3, x_2 = 7, x_3 = -3$

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_2 + x_3 &= 4 \end{aligned} \rightarrow \begin{aligned} x_2 &= 4 - x_3 \\ x_3 &= -3 \end{aligned} \rightarrow \begin{aligned} x_2 &= 4 - (-3) = 7 \\ x_1 &= 1 - x_2 - x_3 \\ &= 1 - 7 - (-3) = -3 \end{aligned}$$