Example: $A = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1(3) & 1(1) & 1(4) \\ -1(3) & -1(1) & -1(4) \\ 2(3) & 2(1) & 2(4) \end{bmatrix}$

Every row is a multiple of the 1st row.

$$A \xrightarrow[\text{Row3}-2\text{Row1}]{\text{Row2}+\text{Row1}} \begin{bmatrix} 3 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}\text{Row 1}} \begin{bmatrix} 1 & 1/3 & 4/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R$$

one pivot column, one leading 1

Fun fact: You can write any A as a linear combination of "outer products": $A = \vec{u}_1 \vec{v}_1^T + \vec{u}_2 \vec{v}_2^T + \cdots + \vec{u}_r \vec{v}_r^T$

The rank r is the minimum number of outer products required to add up to r.

Let's see how to do this using LU decomposition:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[\text{Row 3}-\boxed{7}\text{Row1}]{\text{Row 2}-\boxed{4}\text{Row 1}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \xrightarrow{\text{Row 3}-\boxed{2}\text{Row 2}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ +4 & 1 & 0 \\ +7 & 2 & 1 \end{bmatrix}}_{L} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

U has rank$=2 \rightarrow A$ has rank $2 \rightarrow A \stackrel{?}{=} \vec{u}_1 \vec{v}_1^T + \vec{u}_2 \vec{v}_2^T$?

In fact, can write $U = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -3 & -6 \end{bmatrix}$ (no need for $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$(something) since 3rd row is all 0)

$$\rightarrow A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -3 & -6 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \left(\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 0 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & -3 & -6 \end{bmatrix}$$

Why care about this? It might take less computer memory to store $\{\underbrace{\vec{u}_1}_{m \text{ entries}}, \underbrace{\vec{v}_1}_{n \text{ entries}}, \vec{u}_2, \vec{v}_2, \ldots, \vec{u}_r, \vec{v}_r\}$ than A itself

$$\underbrace{\{\vec{u}_1, \vec{v}_1, \ldots, \vec{u}_r, \vec{v}_r\}}_{r(m+n) \text{ entries}} \quad \text{vs.} \quad \underbrace{A}_{mn \text{ entries}}$$

smaller if r is small and m,n are big

Example: Store a photograph as a matrix (1 pixel = 1 matrix entry, different numerical values for matrix entries represent different colors)

red	red	red	red	red	red
red	red	red	red	red	red
red	red	blue	blue	red	red
red	red	blue	blue	red	red
red	red	red	red	red	red
red	red	red	red	red	red

$$\rightsquigarrow \quad A = \underbrace{\begin{bmatrix} 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \\ 1&1&2&2&1&1 \\ 1&1&2&2&1&1 \\ 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \end{bmatrix}}_{36 \text{ matrix entries}} \begin{array}{l} \text{Row 2} - \text{Row 1} \\ \text{Row 3} - \text{Row 1} \\ \text{Row 4} - \text{Row 1} \\ \text{Row 5} - \text{Row 1} \\ \text{Row 6} - \text{Row 1} \end{array}$$

$$\begin{bmatrix} 1&1&1&1&1&1 \\ 0&0&0&0&0&0 \\ 0&0&1&1&0&0 \\ 0&0&1&1&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix} \xrightarrow[\text{-Row 3}]{\text{Row 4}} \underbrace{\begin{bmatrix} 1&1&1&1&1&1 \\ 0&0&0&0&0&0 \\ 0&0&1&1&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}}_{U} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1&1&1&1&1&1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0&0&1&1&0&0 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1&0&0&0&0&0 \\ 1&1&0&0&0&0 \\ 1&0&1&0&0&0 \\ 1&0&\mathbf{1}&1&0&0 \\ 1&0&0&0&1&0 \\ 1&0&0&0&0&1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1&1&1&1&1&1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0&0&1&1&0&0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1&1&1&1&1&1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0&0&1&1&0&0 \end{bmatrix}$$

← 24 total vector entries

This idea of writing $A = \vec{u}_1\vec{v}_1^T + \cdots + \vec{u}_r\vec{v}_r^T$ returns in Chapter 7 on singular value decomposition.

Last time Looked at column and null spaces of $m \times n$ A:

$C(A)$ = linear combinations of the columns (subspace of $\mathbb{R}^m$)

$N(A)$ = all $\vec{x}$ such that $A\vec{x} = \vec{0}$ (subspace of $\mathbb{R}^n$)

Why do we care?

$C(A)$ tells us if $A\vec{x} = \vec{b}$ has any solution (it does if $\vec{b}$ is a vector in $C(A)$)

$N(A)$ tells us how many solutions $A\vec{x} = \vec{b}$ can have.

To see why, let's review the full elimination method for solving $A\vec{x} = \vec{b}$:

$$\left[A \mid \vec{b}\right] \longrightarrow \left[R \mid \vec{d}\right]$$

Augmented matrix "reduced row echelon form" of A.

(1) Eliminate lower left variables in A first.

(2) Identify pivot columns (they contain the first non-zero entries for the rows.)

(3) Turn first non-zero entries into leading 1's.

(4) Eliminate all variables above the leading 1's.

This is easier if $\vec{b} = \vec{0}$ (solving $A\vec{x} = \vec{0}$, i.e., finding $N(A)$):

Example Find R for $A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & -2 & 1 & 2 \\ 3 & 1 & 2 & -1 \end{bmatrix}$

Step (2): These are the pivot columns

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & -2 & 1 & 2 \\ 3 & 1 & 2 & -1 \end{bmatrix} \xrightarrow[\text{Row 3} - 3\,\text{Row 1}]{\substack{\text{Step (1):} \\ \text{Row 2} - 2\,\text{Row 1}}} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & -8 & -1 & 0 \\ 0 & -8 & -1 & -4 \end{bmatrix} \xrightarrow{\text{Row 3} - \text{Row 2}} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & -8 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \xrightarrow[-\frac{1}{4}\,\text{Row 3}]{\substack{\text{Step (3)} \\ -\frac{1}{8}\,\text{Row 2}}}$$

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1/8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{Row 1} - \text{Row 3}]{\substack{\text{Step (4):} \\ \text{Eliminate these}}} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1/8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Row 1} - 3\,\text{Row 2}} \begin{bmatrix} 1 & 0 & 5/8 & 0 \\ 0 & 1 & 1/8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = R.$$

We can use R to find $N(A)$ easily: Solve $A\vec{x} = \vec{0}$

$$\left[A \mid \vec{0}\right] \xrightarrow{\text{elimination}} \left[R \mid \vec{0}\right] \longrightarrow \begin{cases} x_1 + \frac{5}{8}x_3 = 0 \\ x_2 + \frac{1}{8}x_3 = 0 \\ x_4 = 0 \end{cases}$$

"free variable"

pivot column variables

All solutions look like: $\vec{x} = \begin{bmatrix} -5/8\, x_3 \\ -1/8\, x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -5/8 \\ -1/8 \\ 1 \\ 0 \end{bmatrix}$ ← "special solution" Provides a spanning set for $N(A)$.

In general: All solutions to $A\vec{x}=\vec{0}$ is a linear combination of "special solutions," and there is one special solution for each free variable.

What about $A\vec{x}=\vec{b}$ with $\vec{b}\neq\vec{0}$? $[A \mid \vec{b}] \xrightarrow{\text{elimination}} [R \mid \vec{d}]$

not $\vec{0}$ any more

still easy to solve for pivot column variables in terms of free variables.

Example $\begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \rightsquigarrow [A \mid \vec{b}] = \left[\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 1 \\ 1 & 2 & 3 & 6 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right]$

$$\xrightarrow{\text{Row 2}-\text{Row 1}} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow[-\text{Row 2}]{\text{Row 3}} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{Row 1}-2\text{Row 2}} \begin{array}{c} \quad R \qquad\qquad \vec{d} \\ \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Eliminate 2

pivot columns

consistent equation $0=0$, so solutions exist

Equations $R\vec{x}=\vec{d}$ look like: $\begin{cases} x_1 + 2x_2 = -1 \\ x_3 + 2x_4 = 1 \end{cases}$

pivot column variables ← free variables

We can get one particular solution by setting all free variables $=0$.

So $x_1=-1$, $x_2=0$, $x_3=0$, $x_4=0$ $\rightsquigarrow$ One solution is $\vec{x}_p = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

What if $\vec{y}$ is another solution, $A\vec{y}=\vec{b}$? Then

$A(\underbrace{\vec{y}-\vec{x}_p}) = A\vec{y} - A\vec{x}_p = \vec{b}-\vec{b} = \vec{0}$

This is a solution to $A\vec{x}=\vec{0}$, so $\vec{y}-\vec{x}_p$ is in $N(A)$

So if $\vec{y}$ is any solution, we can write $\vec{y} = \vec{x}_p + \underbrace{(\vec{y}-\vec{x}_p)}_{\vec{x}_n}$

it's a sum of a particular solution and a null space vector.

On the other hand, any nullspace vector is a linear combination of "special solutions":

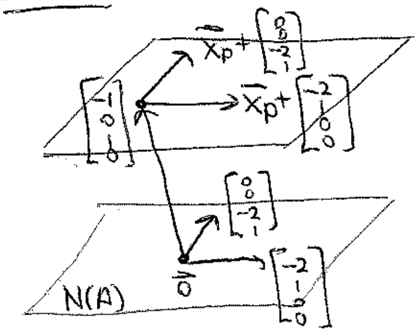
$$\begin{cases} x_1+2x_2=0 \\ x_3+2x_4=0 \end{cases} \rightsquigarrow \begin{matrix} x_1=-2x_2 \\ x_3=-2x_4 \end{matrix} \rightsquigarrow \vec{x}=\begin{bmatrix} -2x_2 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

So here is the complete solution to $A\vec{x}=\vec{b}$: All solutions look like

$$\vec{x}=\vec{x}_p+\vec{x}_n=\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$\vec{x}_p \neq \vec{0}$, so plane of solutions doesn't contain $\vec{0}$ (not a subspace)

two free variables means solutions form a plane in $\mathbb{R}^4$.



The solution plane is parallel to N(A).

For this matrix A, R has a row of 0's, means $A\vec{x}=\vec{b}$ has no solution for most $\vec{b}$'s.

In general: If rank of A < #rows of A, then $A\vec{x}=\vec{b}$ usually has no solutions (i.e., most $\vec{b}$'s are not in C(A), so C(A) is smaller than $\mathbb{R}^m$).

Problem 3.3.1 $A=\begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}$, $\vec{b}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ Find a condition on b_1, b_2, b_3 for solutions of $A\vec{x}=\vec{b}$ to exist

$$A\vec{x}=\vec{b} \to \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & b_1 \\ 2 & 5 & 7 & 6 & b_2 \\ 2 & 3 & 5 & 2 & b_3 \end{array}\right] \xrightarrow[\text{Row 3}-\text{Row 1}]{\text{Row 2}-\text{Row 1}} \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2-b_1 \\ 0 & -1 & -1 & -2 & b_3-b_1 \end{array}\right] \xrightarrow{\text{Row 3}+\text{Row 2}}$$

$$\left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2-b_1 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_3 \end{array}\right] \xrightarrow{\frac{1}{2}\text{Row 1}} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & \frac{1}{2}b_1 \\ 0 & 1 & 1 & 2 & b_2-b_1 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_3 \end{array}\right] \xrightarrow{\text{Row 1}-2\text{Row 2}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & \frac{5}{2}b_1-2b_2 \\ 0 & 1 & 1 & 2 & b_2-b_1 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_3 \end{array}\right]$$

Solutions exist only if $\boxed{-2b_1+b_2+b_3=0}$. This is the equation of a plane in $\mathbb{R}^3$, and this plane is exactly the column space C(A).

If $\vec{b}$ is in $C(A)$ (which means $-2b_1+b_2+b_3=0$), then there are many solutions.

Null space: Take $b_1=b_2=b_3=0$ (note $-2(0)+0+0=0$)

$$\begin{cases} x_1 + x_3 - 2x_4 = 0 \\ x_2 + x_3 + 2x_4 = 0 \end{cases} \longrightarrow \begin{matrix} x_1 = -x_3 + 2x_4 \\ x_2 = -x_3 - 2x_4 \end{matrix}$$ (2 free variables, x_3 and x_4)

$N(A)$ = all vectors like $\begin{bmatrix} -x_3+2x_4 \\ -x_3-2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

special solutions

Now take $\vec{b}=\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$. Is $\vec{b}$ in $C(A)$? Check: $-2(4)+3+5=0$ ✓

In this case, get equations $\begin{cases} x_1+x_3-2x_4 = \frac{5}{2}(4)-2(3) = 4 \\ x_2+x_3+2x_4 = 3-4 = -1 \end{cases}$

One particular solution: set $x_3=x_4=0$, get $x_1=4$, $x_2=-1$

So $\vec{x}_p=\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. All solutions: $\vec{x}=\vec{x}_p+\vec{x}_n=\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}+x_3\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}+x_4\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

plane of solutions parallel to $N(A)$ again.

In these examples, $N(A)$ was a plane (non-zero) so we got a whole plane of solutions for $A\vec{x}=\vec{b}$ (or no solution if $\vec{b}$ is not in $C(A)$). I.e., $N(A)$ tells you the max number of solutions $A\vec{x}=\vec{b}$ can have.

Question: When we do $[A \mid \vec{b}] \xrightarrow{\text{elimination}} [R \mid \vec{d}]$, what are the basic possibilities for R?

(1) R has a leading 1 in every row and every column.

(2) R has a leading 1 in every row, but not every column.

(3) R has a leading 1 in every column, but not every row.

(4) R has leading 1's missing from both rows and columns

We just saw examples of Case (4): For most $\vec{b}$, $A\vec{x}=\vec{b}$ has no solution, but if $\vec{b}$ is in $C(A)$, then $A\vec{x}=\vec{b}$ has infinitely many solutions,