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Homework 10 Solutions

4.3.1 Idea: find line $y = C + Dt$ which contains $(0,0), (1,8), (3,8), (4,20)$ (if possible):

$$\begin{aligned} 0 &= C + D(0) \\ 8 &= C + D(1) \\ 8 &= C + D(3) \\ 20 &= C + D(4) \end{aligned} \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}_{A} \begin{bmatrix} C \\ D \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}}_{\vec{b}}$$

(no solution)

Normal equations for best approximate solution:

$$A^T A \vec{x} = A^T \vec{b} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix} \xrightarrow[\frac{1}{2}\text{Row } 2]{\frac{1}{4}\text{Row } 1} \left[\begin{array}{cc|c} 1 & 2 & 9 \\ 4 & 13 & 56 \end{array}\right] \xrightarrow{\text{Row } 2 - 4\text{Row } 1}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 9 \\ 0 & \cancel{5}\ 1 & \cancel{20}\ 4 \end{array}\right] \xrightarrow{\text{Row } 1 - 2\text{Row } 2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 4 \end{array}\right] \longrightarrow \begin{aligned} C &= 1 \\ D &= 4 \end{aligned} \longrightarrow y = 1 + 4t$$

The four heights p_i:

$p_1 = 1 + 4(0) = 1$

$p_2 = 1 + 4(1) = 5$

$p_3 = 1 + 4(3) = 13$

$p_4 = 1 + 4(4) = 17$

Four errors e_i:

$e_1 = 1 - 0 = 1$

$e_2 = 5 - 8 = -3$

$e_3 = 13 - 8 = 5$

$e_4 = 17 - 20 = -3$

Total squared error:

$$E = 1^2 + (-3)^2 + (5)^2 + (-3)^2 = 1 + 9 + 25 + 9 = 44$$

4.3.10 Want to solve:

$$0 = C + D(0) + E(0)^2 + F(0)^3$$
$$8 = C + D(1) + E(1)^2 + F(1)^3$$
$$8 = C + D(3) + E(3)^2 + F(3)^3$$
$$20 = C + D(4) + E(4)^2 + F(4)^3$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$\xrightarrow[\text{Row4-Row1}]{\text{Row2-Row1, Row3-Row1}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 3 & 9 & 27 & 8 \\ 0 & 4 & 16 & 64 & 20 \end{array}\right] \xrightarrow[\text{Row 4-4 Row2}]{\text{Row3 - 3Row 2}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 24 & -16 \\ 0 & 0 & 12 & 60 & -12 \end{array}\right]$$

$$\xrightarrow{\text{Row 4-2Row3}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 24 & -16 \\ 0 & 0 & 0 & 12 & 20 \end{array}\right] \xrightarrow[\frac{1}{12}\text{Row 4}]{\frac{1}{6}\text{Row 3}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 1 & 4 & -8/3 \\ 0 & 0 & 0 & 1 & 5/3 \end{array}\right] \xrightarrow[\text{Row3 - 4Row4}]{\text{Row 2 - Row 4}}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 19/3 \\ 0 & 0 & 1 & 0 & -28/3 \\ 0 & 0 & 0 & 1 & 5/3 \end{array}\right] \xrightarrow[\text{-Row3}]{\text{Row 2}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 47/3 \\ 0 & 0 & 1 & 0 & -28/3 \\ 0 & 0 & 0 & 1 & 5/3 \end{array}\right] \rightarrow \begin{array}{l} C = 0 \\ D = 47/3 \\ E = -28/3 \\ F = 5/3 \end{array}$$

$$b = \frac{47}{3} t - \frac{28}{3} t^2 + \frac{5}{3} t^3$$

$$p_1 = \frac{47}{3}(0) - \frac{28}{3}(0)^2 + \frac{5}{3}(0)^3 = 0 = b_1$$

$$p_2 = \frac{47}{3}(1) - \frac{28}{3}(1)^2 + \frac{5}{3}(1)^3 = 8 = b_2$$

$$p_3 = \frac{47}{3}(3) - \frac{28}{3}(3)^2 + \frac{5}{3}(3)^3 =$$
$$= 47 - 84 + 45 = 8 = b_3$$

$$p_4 = \frac{47}{3}(4) - \frac{28}{3}(4)^2 + \frac{5}{3}(4)^3$$
$$= \frac{188 - 448 + 320}{3} = 20 = b_4$$

Since each $p_i = b_i$,

get error $e_i = 0$.

I.e., $\vec{p} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} = \vec{b}$, $\vec{e} = \vec{0}$

Cubic goes through all 4 points exactly.

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4.3.12 (a) $\vec{a}^{T}\vec{a}\,\hat{x}=\vec{b} \rightsquigarrow [1\ 1 \cdots 1]\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}\hat{x}=[1\ 1\cdots 1]\begin{bmatrix}b_1\\b_2\\\vdots\\b_m\end{bmatrix}$

$\rightsquigarrow m\hat{x}=b_1+b_2+\cdots+b_m \longrightarrow \hat{x}=\dfrac{b_1+b_2+\cdots+b_m}{m}$ (the mean)

(b) $\vec{e}=\vec{b}-\vec{a}\hat{x}=\begin{bmatrix}b_1\\b_2\\\vdots\\b_m\end{bmatrix}-\dfrac{b_1+b_2+\cdots+b_m}{m}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}=\dfrac{1}{m}\begin{bmatrix}(m-1)b_1-b_2-\cdots-b_m\\-b_1+(m-1)b_2-\cdots-b_m\\\vdots\\-b_1-b_2-\cdots+(m-1)b_m\end{bmatrix}$

So $\|\vec{e}\|^2=\dfrac{1}{m^2}\Big(\big((m-1)b_1-b_2-\cdots-b_m\big)^2+\big(-b_1+(m-1)b_2-\cdots-b_m\big)^2+\cdots\Big)$

For $i=1,2,\ldots,m$, we have an $(m-1)^2 b_i^2$ term and $m-1$ other b_i^2 terms

$\rightsquigarrow \|\vec{e}\|^2$ contains $\big((m-1)^2+(m-1)\big)b_i^2=m(m-1)b_i^2$

For $i,j=1,2,\ldots,m$, with $i\neq j$ have $-2(m-1)b_ib_j$ coming from ith and jth terms, also $2b_ib_j$ coming from other $m-2$ terms

$\rightsquigarrow \|\vec{e}\|^2$ contains $\big(2(m-2)-4(m-1)\big)b_ib_j=-2m\,b_ib_j$

So $\|\vec{e}\|^2=\dfrac{1}{m^2}\left(\sum_{i=1}^{m} m(m-1)b_i^2-2m\sum_{i\neq j}b_ib_j\right)=\dfrac{1}{m}\left(\sum_{i=1}^{m}(m-1)b_i^2-2\sum_{i\neq j}b_ib_j\right)$

$\|\vec{e}\|$ is the square root of this

(c) If $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$, then $\hat{x} = \dfrac{1+2+6}{3} = 3$

So $\vec{e} = \vec{b} - \vec{a}\hat{x} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$

So $\vec{e} \perp \vec{p}$ because $\begin{bmatrix} -2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = -6-3+9 = 0.$

3x3 P: $P = \dfrac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}} = \dfrac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \dfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4.3.17 $\begin{cases} 7 = C + D(-1) \\ 7 = C + D(1) \\ 21 = C + D(2) \end{cases} \longrightarrow \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A}\begin{bmatrix} C \\ D \end{bmatrix} = \underbrace{\begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}}_{\vec{b}}$

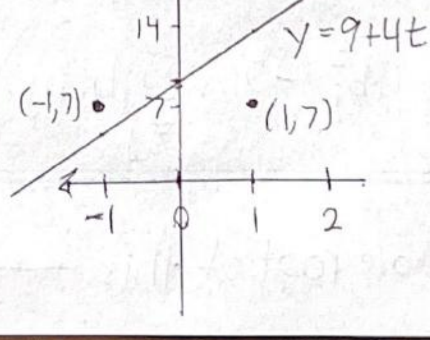
Least squares: $A^T A \hat{x} = A^T \vec{b}$

$\longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix} \longrightarrow \begin{bmatrix} C \\ D \end{bmatrix} = \dfrac{1}{18-4}\begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 35 \\ 42 \end{bmatrix}$

$= \dfrac{1}{14}\begin{bmatrix} 126 \\ 56 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$

$\boxed{y = 9 + 4t}$



4.4.2 $\left\|\begin{bmatrix}2\\2\\-1\end{bmatrix}\right\| = \sqrt{2^2+2^2+(-1)^2} = \sqrt{9} = 3$, $\left\|\begin{bmatrix}-1\\2\\2\end{bmatrix}\right\| = 3$ also.

So $\vec{q}_1 = \begin{bmatrix}2/3\\2/3\\-1/3\end{bmatrix}$, $\vec{q}_2 = \begin{bmatrix}-1/3\\2/3\\2/3\end{bmatrix} \rightsquigarrow Q = \begin{bmatrix}2/3 & -1/3\\2/3 & 2/3\\-1/3 & 2/3\end{bmatrix}$

$QQ^T = \begin{bmatrix}2/3 & -1/3\\2/3 & 2/3\\-1/3 & 2/3\end{bmatrix}\begin{bmatrix}2/3 & 2/3 & -1/3\\-1/3 & 2/3 & 2/3\end{bmatrix} = \begin{bmatrix}5/9 & 2/9 & -4/9\\2/9 & 8/9 & 2/9\\-4/9 & 2/9 & 5/9\end{bmatrix}$ ← Projection matrix for projection onto span$(\vec{q}_1, \vec{q}_2)$

$Q^TQ = \begin{bmatrix}2/3 & 2/3 & -1/3\\-1/3 & 2/3 & 2/3\end{bmatrix}\begin{bmatrix}2/3 & -1/3\\2/3 & 2/3\\-1/3 & 2/3\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

4.4.10 (a) Suppose $c_1\vec{q}_1 + c_2\vec{q}_2 + c_3\vec{q}_3 = \vec{0}$

Dot with $\vec{q}_1$: $c_1\underbrace{(\vec{q}_1\cdot\vec{q}_1)}_{1} + c_2\underbrace{(\vec{q}_1\cdot\vec{q}_2)}_{0} + c_3\underbrace{(\vec{q}_1\cdot\vec{q}_3)}_{0} = \underbrace{\vec{q}_1\cdot\vec{0}}_{0} \rightarrow c_1 = 0$

Similary: dot with $\vec{q}_2 \rightarrow c_2 = 0$, dot with $\vec{q}_3 \rightarrow c_3 = 0$.

Only solution is $c_1 = c_2 = c_3 = 0$, so $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$ is independent.

(b) Write $Q = \begin{bmatrix}\vec{q}_1 & \vec{q}_2 & \vec{q}_3\end{bmatrix}$, so $c_1\vec{q}_1 + c_2\vec{q}_2 + c_3\vec{q}_3 = \vec{0}$ means $Q\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \vec{0}$

For orthogonal matrices, $Q^TQ = I$ (even if Q isn't square)

So $Q^TQ\vec{c} = Q^T\vec{0} \rightarrow I\vec{c} = \vec{0} \rightarrow \vec{c} = 0 \rightarrow$ columns of Q are independent

<u>4.4.18</u> $A = \frac{1}{\|\vec a\|}\vec a = \frac{1}{\sqrt 2}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$

$\vec B$: project $\vec b$ onto span($\vec a$), take difference $\vec e$:

$$\vec e = \vec b - \frac{\vec a \vec a^T \vec b}{\vec a^T \vec a} = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\underbrace{\begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}}_{-1} = \begin{bmatrix}1/2\\1/2\\-1\\0\end{bmatrix}$$

Normalize: $\vec B = \frac{1}{\|\vec e\|}\vec e = \frac{1}{\sqrt{\frac14+\frac14+1+0}}\begin{bmatrix}1/2\\1/2\\-1\\0\end{bmatrix} = \sqrt{\frac{2}{3}}\begin{bmatrix}1/2\\1/2\\-1\\0\end{bmatrix} = \frac{1}{\sqrt 6}\begin{bmatrix}1\\1\\-2\\0\end{bmatrix}$

$\vec C$: project $\vec c$ onto span $(\vec A, \vec B)$, take difference $\vec e$:

$$\vec e = \vec c - \underbrace{\underbrace{\begin{bmatrix}1/\sqrt2 & 1/\sqrt6\\ -1/\sqrt2 & 1/\sqrt6\\ 0 & -2/\sqrt6\\ 0 & 0\end{bmatrix}}_{Q}\begin{bmatrix}1/\sqrt2 & -1/\sqrt2 & 0 & 0\\ 1/\sqrt6 & 1/\sqrt6 & -2/\sqrt6 & 0\end{bmatrix}}_{\text{Projection matrix}}\underbrace{\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}}_{\vec c}$$

$$= \begin{bmatrix}0\\0\\1\\-1\end{bmatrix} - \begin{bmatrix}1/\sqrt2 & 1/\sqrt6\\ -1/\sqrt2 & 1/\sqrt6\\ 0 & -2/\sqrt6\\ 0 & 0\end{bmatrix}\begin{bmatrix}0\\-2/\sqrt6\end{bmatrix} = \begin{bmatrix}0\\0\\1\\-1\end{bmatrix} - \begin{bmatrix}-1/3\\-1/3\\2/3\\0\end{bmatrix} = \begin{bmatrix}1/3\\1/3\\1/3\\-1\end{bmatrix}$$

Normalize: $\vec C = \frac{1}{\sqrt{\frac19+\frac19+\frac19+1}}\begin{bmatrix}1/3\\1/3\\1/3\\-1\end{bmatrix} = \frac{\sqrt3}{2}\begin{bmatrix}1/3\\1/3\\1/3\\-1\end{bmatrix} = \begin{bmatrix}\sqrt3/6\\\sqrt3/6\\\sqrt3/6\\-\sqrt3/2\end{bmatrix}$

Note $\vec A, \vec B, \vec C$ are all $\perp$ to $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, just like $\vec a, \vec b, \vec c$

4.4.22: $\vec{A} = \frac{1}{\|\vec{a}\|}\begin{bmatrix}1\\1\\2\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\2\end{bmatrix}$

$\vec{B}$: project $\vec{b}$ onto span($\vec{a}$), take error $\vec{e}$:

$$\vec{e} = \vec{b} - \underset{\substack{\uparrow\\ \text{projection onto span}(\vec{a})}}{\frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}}}\vec{b} = \begin{bmatrix}1\\-1\\0\end{bmatrix} - \frac{1}{6}\begin{bmatrix}1\\1\\2\end{bmatrix}\underbrace{\begin{bmatrix}1&1&2\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix}}_{0,\ \vec{a}\perp\vec{b}\text{ already}} = \begin{bmatrix}1\\-1\\0\end{bmatrix}$$

So $\vec{b} = \frac{1}{\|\vec{e}\|}\vec{e} = \frac{1}{\|\vec{b}\|}\vec{b} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$

$\vec{C}$: project $\vec{c}$ onto span($\vec{A},\vec{B}$), take error $\vec{e}$:

$$\vec{e} = \vec{c} - \begin{bmatrix}1/\sqrt{6} & 1/\sqrt{2}\\ 1/\sqrt{6} & -1/\sqrt{2}\\ 2/\sqrt{6} & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6}\\ 1/\sqrt{2} & -1/\sqrt{2} & 0\end{bmatrix}\begin{bmatrix}1\\0\\4\end{bmatrix} = \begin{bmatrix}1\\0\\4\end{bmatrix} - \begin{bmatrix}1/\sqrt{6} & 1/\sqrt{2}\\ 1/\sqrt{6} & -1/\sqrt{2}\\ 2/\sqrt{6} & 0\end{bmatrix}\begin{bmatrix}9/\sqrt{6}\\ 1/\sqrt{2}\end{bmatrix}$$

$$= \begin{bmatrix}1\\0\\4\end{bmatrix} - \begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}-1\\-1\\1\end{bmatrix} \longrightarrow \vec{C} = \frac{1}{\|\vec{e}\|}\vec{e} = \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

4.4.31 (a) All columns of Q have length $\sqrt{1+1+1+1} = 2$, so $\boxed{c = \frac{1}{2}}$

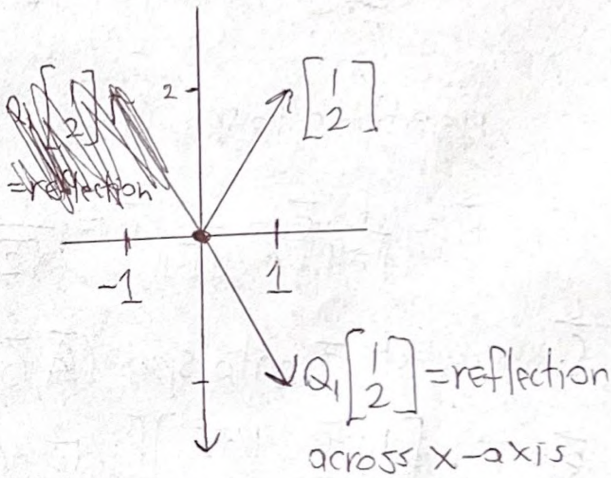
(b) $$\vec{p} = P\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}1&-1&-1&-1\end{bmatrix}}{\begin{bmatrix}1&-1&-1&-1\end{bmatrix}\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1&-1&-1&-1\\-1&1&1&1\\-1&1&1&1\\-1&1&1&1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\1/2\\1/2\end{bmatrix}$$

Projection for plane of 1st two columns: can use orthogonal columns: $P = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \\ -1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix}$

$= \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$ Multiply by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$: get $\vec{p} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

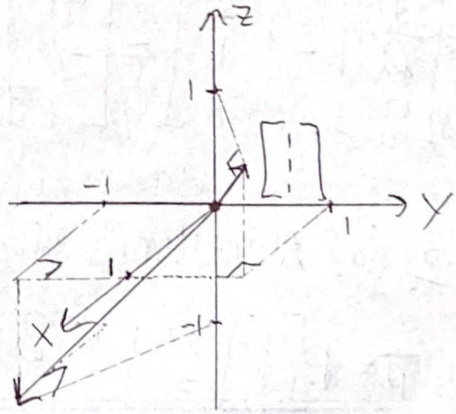
4.4.32 $Q_1 = I - 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$Q_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$



$Q_2 = I - 2\begin{bmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$, $Q_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$



Reflection in plane $\perp$ to $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ (spanned by x-axis and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$) $\rightarrow \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

Graded Problem 1

$t = 0, 1, 2, 3$
$h = 50, 44, 32, 6$ ⟶ Try to solve

$C + D(0) + E(0)^2 = 50$
$C + D(1) + E(1)^2 = 44$
$C + D(2) + E(2)^2 = 32$
$C + D(3) + E(3)^2 = 6$

$$\rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}}_{A} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \underbrace{\begin{bmatrix} 50 \\ 44 \\ 32 \\ 6 \end{bmatrix}}_{\vec{b}}$$

No solution, solve normal equations $A^T A \hat{x} = A^T \vec{b}$ instead.

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}}_{\begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix}} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 50 \\ 44 \\ 32 \\ 6 \end{bmatrix}}_{\begin{bmatrix} 132 \\ 126 \\ 226 \end{bmatrix}}$$

$$\boxed{\begin{gathered}\text{Best fit parabola:} \\ h(t) = \frac{248}{5} + \frac{3}{5}t - 5t^2 \\ \rightarrow g \approx 2(5) = 10\end{gathered}}$$

Divide all rows by 2

$$\left[\begin{array}{ccc|c} 2 & 3 & 7 & 66 \\ 3 & 7 & 18 & 63 \\ 7 & 18 & 49 & 113 \end{array}\right] \xrightarrow[\text{Row3} - \frac{7}{2}\text{Row1}]{\text{Row2} - \frac{3}{2}\text{Row1}} \left[\begin{array}{ccc|c} 2 & 3 & 7 & 66 \\ 0 & 5/2 & 15/2 & -36 \\ 0 & 15/2 & 49/2 & -118 \end{array}\right]$$

$$\xrightarrow{\text{Row3} - 3\text{Row2}} \left[\begin{array}{ccc|c} 2 & 3 & 7 & 66 \\ 0 & 5/2 & 15/2 & -36 \\ 0 & 0 & 2 & -10 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 3/2 & 7/2 & 33 \\ 0 & 1 & 3 & -72/5 \\ 0 & 0 & 1 & -5 \end{array}\right] \xrightarrow[\text{Row 2} - 3\text{Row3}]{\text{Row 1} - \frac{7}{2}\text{Row3}} \left[\begin{array}{ccc|c} 1 & 3/2 & 0 & 101/2 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

$$\xrightarrow{\text{Row 1} - \frac{3}{2}\text{Row2}} \cancel{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2788/95 \\ 0 & 1 & 0 & -573/95 \\ 0 & 0 & 1 & -31/19 \end{array}\right]}$$

~~(Warning: there~~

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 248/5 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

Graded Problem 2

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow[\text{Row3}-\text{Row1}]{\text{Row2}-\frac{1}{2}\text{Row1}} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[2\,\text{Row2}]{\frac{1}{2}\text{Row1}} \begin{bmatrix} 1 & 1/2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[-\frac{1}{2}\text{Row2}]{\text{Row1}}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{Row space} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

Null space: $x_1 + x_3 = 0$, $x_2 = 0$ $\longrightarrow \vec{x} = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix}$

$\longrightarrow$ basis vector: $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Orthonormal basis for Rowspace: basis vectors are already $\perp$, just need to rescale first one:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Orthonormal basis for null space:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

These 3 vectors together form an orthonormal basis for $\mathbb{R}^3$.