Homework 4 Solutions

2.3.9 (a) E_{21}: Subtract Row 1 from Row 2 on identity matrix: $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

P_{23}: Exchange Rows 2 and 3 of identity: $P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$M = P_{23}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

(b) $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

If you switch Rows 2 and 3 first, then subtract Row 1 from Row 3, that is the same as subtracting Row 1 from the original Row 2, and then switching Rows 2 and 3. So you get the same M: $P_{23}E_{21} = E_{31}P_{23}$, even though E's are different. ~~This shows~~

2.3.12

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{\begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix}$$

2.3.17 $(x,y) = (1,4)$: $4 = a + b(1) + c(1)^2$

$= (2,8)$: $8 = a + b(2) + c(2)^2$

$= (3,14)$: $14 = a + b(3) + c(3)^2$

$$\rightarrow \begin{cases} a + b + c = 4 \\ a + 2b + 4c = 8 \\ a + 3b + 9c = 14 \end{cases}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 14 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 14 \end{array}\right] \begin{array}{l} \text{Row 2} - \text{Row 1} \\ \text{Row 3} - \text{Row 1} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right] \xrightarrow{\text{Row 3} - 2\text{Row 2}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right] \xrightarrow{\frac{1}{2}\text{Row 3}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[\text{Row 2} - 3\text{Row 3}]{\text{Row 1} - \text{Row 3}}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{\text{Row 1} - \text{Row 2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \begin{array}{l} a = 2 \\ b = 1 \\ c = 1 \end{array}$$

$\rightsquigarrow$ $y = 2 + x + x^2$ is the correct parabola.

2.3.28 If $AB = I$ and $BC = I$:

$$A = AI = A(BC) = (AB)C = IC = C$$

associativity

2.4.6 $(A+B)^2 = \left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \right)^2 = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix}$

$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ Different

$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1+14+1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$

Correct rule: $(A+B)^2 = (A+B)(A+B) = A^2 + BA + AB + B^2$

Different, not $2AB$

2.4.15 (a) If A is $m\times n$, then A^2 is $(m\times n)(m \times n)$

have to be same, $m = n$

So true, A is square ($m=n$) if A^2 is defined.

(b) A $k\times \ell$, $\rightsquigarrow$ BA: $(m\times n)\cdot(k\times \ell)$ $\quad$ AB: $(k\times \ell)(m\times n)$

B $m\times n$

same, $n = k$ $\quad$ same, $\ell = m$

So A has to be $n\times m$ and B has to be $m\times n$ if AB, BA are both defined. So false, A and B don't have to be square ($m\neq n$ is okay).

(c) We saw that if AB, BA are both defined and B is $m \times n$, then A is $n \times m$. (3)

$\rightsquigarrow AB$ is $(n\times m)(m\times n) = n\times n$; square

$\rightsquigarrow BA$ is $(m\times n)(n\times m) = m\times m$, square

So true, AB and BA are both square.

(d) False, A doesn't need to be I since B might not be invertible.

For example, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then

$$AB = B \text{ if } \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\rightsquigarrow a = 1$, $c = 0$, no condition on b, d

So $\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} B = B$, $\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}$ might not be I.

Note: If B is invertible, then indeed A would have to be I.

Proof: $AB = B \longrightarrow (AB)B^{-1} = BB^{-1} = I$

$$(AB)B^{-1} = A(BB^{-1}) = AI = A$$

same if B^{-1} exists

2.4.18 (a) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $a_{ij} = $ min. of i and j $\rightarrow$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

(b) $a_{ij} = (-1)^{i+j}$, $A = \begin{bmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

(c) $a_{ij} = \frac{i}{j}$, $A = \begin{bmatrix} 1/1 & 1/2 & 1/3 \\ 2/1 & 2/2 & 2/3 \\ 3/1 & 3/2 & 3/3 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 2 & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix}$

2.4.21 $\underbrace{\begin{bmatrix}0&2&0&0\\0&0&2&0\\0&0&0&2\\0&0&0&0\end{bmatrix}\underbrace{\begin{bmatrix}0&2&0&0\\0&0&2&0\\0&0&0&2\\0&0&0&0\end{bmatrix}\underbrace{\begin{bmatrix}0&2&0&0\\0&0&2&0\\0&0&0&2\\0&0&0&0\end{bmatrix}\begin{bmatrix}0&2&0&0\\0&0&2&0\\0&0&0&2\\0&0&0&0\end{bmatrix}}_{A^2=\begin{bmatrix}0&0&4&0\\0&0&0&4\\0&0&0&0\\0&0&0&0\end{bmatrix}}}_{A^3=\begin{bmatrix}0&0&0&8\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}}}_{A^4=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}}$

So $A\vec{v}=\begin{bmatrix}0&2&0&0\\0&0&2&0\\0&0&0&2\\0&0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\\t\end{bmatrix}=\begin{bmatrix}2y\\2z\\2t\\0\end{bmatrix}$, $A^2\vec{v}=\begin{bmatrix}4z\\4t\\0\\0\end{bmatrix}$, $A^3\vec{v}=\begin{bmatrix}8t\\0\\0\\0\end{bmatrix}$,

$A^4\vec{v}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

2.4.26 $\begin{bmatrix}1&0\\2&4\\2&1\end{bmatrix}\begin{bmatrix}3&3&0\\1&2&1\end{bmatrix}=\begin{bmatrix}1\\2\\2\end{bmatrix}\begin{bmatrix}3&3&0\end{bmatrix}+\begin{bmatrix}0\\4\\1\end{bmatrix}\begin{bmatrix}1&2&1\end{bmatrix}$

$=\begin{bmatrix}3&3&0\\6&6&0\\6&6&0\end{bmatrix}+\begin{bmatrix}0&0&0\\4&8&4\\1&2&1\end{bmatrix}=\begin{bmatrix}3&3&0\\10&14&4\\7&8&1\end{bmatrix}$

2.4.32 $AX=A\begin{bmatrix}\vec{x}_1&\vec{x}_2&\vec{x}_3\end{bmatrix}=\begin{bmatrix}A\vec{x}_1&A\vec{x}_2&A\vec{x}_3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

So $AX=I$, the identity matrix.

Graded Problem 1 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$

commute if $\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \rightsquigarrow a = d$ and $c = 0$

So all matrices that commute with $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ look like $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$, a, b any real numbers

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \rightsquigarrow$ Need $\begin{cases} a = d \\ b = 0 \end{cases}$

All matrices commuting with $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ look like $\begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$, a, c any real numbers

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with both $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, then it has to look like both $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ and $\begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$. That is, $a = d$ and $b = c = 0$.

So $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, a any real number, are all such matrices.

Graded Problem 2.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 1 & 2 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 2 & -1 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1-2+1-1 & -1+2+1+0 \\ 1-4+3-4 & -1+4+3+0 \\ 1-8+9-16 & -1+8+9+0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -4 & 6 \\ -14 & 16 \end{bmatrix}$$