Reminder: We have a class from 19:20 to 21:45 on saturday.

Calculus A(1)
12/8

Last time: FTC

Version 1: $f : [a,b] \to \mathbb{R}$ $\overset{\circ}{\subseteq}$

$$F(x) = \int_a^x f(t)\,dt$$

$$F' = f$$

Version 2: $$\int_a^b f(t)\,dt = F(b) - F(a)$$

for any antiderivative F of f.

Notation: $f : I \to \mathbb{R}$

We say that f is of class C^n for some $n \geq 1$ $(n \in \mathbb{N})$ if $\frac{d^n f}{dx^n}$ exists and is C^0.

e.g. f is (of class) C^1 if f' exists and is C^0.

Theorem (The substitution rule for indefinite integrals).

Let $f : I \to \mathbb{R}$ C^0

Let $g : J \to \mathbb{R}$ C^1

s.t $g(J) \subset I$.

We have:

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$$

where $u = g(x)$

where $\int f(u)\,du = F(u) + C$
$= F(g(x)) + C$

where F is an antiderivative of f.

Last time we saw the motivation behind this formula:

$du = g'(x)\,dx$ as differentials

(by definition)

<u>Proof</u>: We want:

$F(g(x))$ is an antiderivative

of $f(g(x)) \cdot g'(x)$.

One can check that directly:

$$\frac{d}{dx} F(g(x)) \overset{\text{chain rule}}{=} g'(x) \cdot F'(g(x))$$

$$= g'(x) \cdot f(g(x))$$

$\boxed{X}$

<u>Ex</u>: ① $\int \cos(2\theta + 3)\, d\theta = ?$

$u = 2\theta + 3$

$$du = 2\,d\theta$$

So $$\int \cos(2\theta+3)\,d\theta$$

$$= \int \cos(u)\cdot\frac{1}{2}\,du$$

$$= \frac{1}{2}\sin(u) + c$$

$$= \frac{1}{2}\sin(2\theta+3) + C.$$

(2) $$\int x^4\cos(x^5)\,dx = ?$$

let $u = x^5$ $\qquad du = 5x^4\,dx$

i.e. $x^4\,dx = \frac{1}{5}\,du$

So: $$\int x^4\cos(x^5)\,dx = \frac{1}{5}\int \cos(u)\,du$$

$$= \frac{1}{5}\sin(u) + C.$$

③ $$\int \frac{1}{\cos(3x)^2}\,dx = ?$$

Let $u = 3x$ $\quad du = 3dx$

So $$\int \frac{1}{\cos(3x)^2}\,dx = \frac{1}{3}\int \frac{1}{\cos(u)^2}\,du$$

$$= \frac{1}{3}\tan(u) + C$$

$$= \frac{1}{3}\tan(3x) + C$$

<u>Rem</u>: Sometimes there are several good change of variables which allow us to compute

the integral. Sometimes one may need to do several consecutive change of variables.

Ex: $\displaystyle\int \frac{x\,dx}{(x^2+1)^{1/3}} = ?$

First way: $u = x^2+1$

$du = 2x\,dx$

So: $\displaystyle\int \frac{x\,dx}{(x^2+1)^{1/3}} = \frac{1}{2}\int \frac{du}{u^{1/3}}$

$\displaystyle = \frac{1}{2}\int u^{-1/3}\,du.$

$\boxed{\displaystyle\int u^{\alpha}\,d\alpha = \frac{1}{\alpha+1}u^{\alpha+1}+C \quad \alpha \neq -1}$

$\displaystyle = \frac{1}{2}\left(\frac{1}{1-1/3}u^{1-\frac{1}{3}} + C\right) \quad (\alpha = -\tfrac{1}{3})$

$$= \frac{3}{4} u^{2/3} + C.$$

$$= \frac{3}{4}(x^2+1)^{2/3} + C$$

<u>Other way</u>: Let

$$u = (x^2+1)^{1/3}$$

$$du = \left(2x \cdot \frac{1}{3}(x^2+1)^{1/3-1}\right) dx$$

$$= \frac{2}{3} x (x^2+1)^{-2/3} \quad dx$$

$$du = \frac{2}{3} \cdot u^{-2} \cdot x\,dx$$

So: $$\int \frac{x\,dx}{(1+x^2)^{1/3}} = \int \frac{\frac{3}{2} u^2\,du}{u}$$

$$= \frac{3}{2} \int u\,du$$

$$= \frac{3}{4} u^2 + c.$$

$$= \frac{3}{4}\left(x^2+1\right)^{2/3} + c$$

Substitutions for definite integrals

Question:

$$\int_0^{\pi/12} \frac{1}{\left(\cos(3x)\right)^2}\, dx = ?$$

One solution:

$$u = 3x$$

$$dx = \frac{1}{3}\, du$$

if x ranges from 0 to $\pi/12$
then u ranges from 0 to $\pi/4$

So $$\int_0^{\pi/12} \frac{1}{\cos(3x)^2}\,dx = \frac{1}{3}\int_0^{\pi/4} \frac{1}{\cos(u)^2}\,du$$

$$\overset{FTC}{=} \frac{1}{3}\left(\tan(\pi/4) - \tan(0)\right)$$

$$= \frac{1}{3}$$

<u>Other solution</u>:

We have seen in the examples above that $\int \frac{1}{\cos(3x)^2}\,dx$

is $\frac{1}{3}\tan(3x) + C$

So by the FTC, we get

$$\int_0^{\pi/12} \frac{1}{\cos(3x)^2}\,dx = \frac{1}{3}\tan\left(3\cdot\frac{\pi}{12}\right)$$

$$- \frac{1}{3} \tan(3 \cdot 0))$$

$$= \frac{1}{3}$$

To justify the 1^{st} way, we have the following result:

<u>Theorem</u> (substitution rule for definite integrals)

Let $f : [c,d] \to \mathbb{R}$ C^0

and $g : [a,b] \to \mathbb{R}$ C^1

s.t $g([a,b]) \subset [c,d]$

Then $$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

<u>Proof</u>: let F = antiderivative of f. We have seen that $F(g(x))$ is antiderivative of $f(g(x)) \cdot g'(x)$.

So by the FTC:

$$\int_a^b f(g(x)) \cdot g'(x)\,dx \overset{FTC}{=} \Big[F(g(x))\Big]_a^b$$

$$= F(g(b)) - F(g(a))$$

$$\overset{FTC}{=} \int_{g(a)}^{g(b)} f(u)\,du$$

$\boxtimes$

Examples :

1. $$\int_{-1}^{1} x^4 \sqrt{x^5+2}\, dx = ?$$

$$u = x^5 + 2$$
$$du = 5x^4\, dx$$

So $$\int_{-1}^{1} x^4 \sqrt{x^5+2}\, dx$$

$$= \frac{1}{5} \int_{1}^{3} \sqrt{u}\, du$$

When $x = -1, \dots, 1$
$u = 1, \dots, 3$

$$= \frac{1}{5} \left[\frac{1}{3/2} u^{3/2} \right]_1^3$$

$$= \frac{2}{15} \cdot \left(3^{3/2} - 1 \right).$$

Rem: Sometimes it is easier to directly apply the substitution rule in the definite integral rather than doing it for the indefinite integral and applying the FTC.

Theorem: Let $f : [-a, a] \to \mathbb{R}$ C^0 where $a > 0$

(1) if f is even then

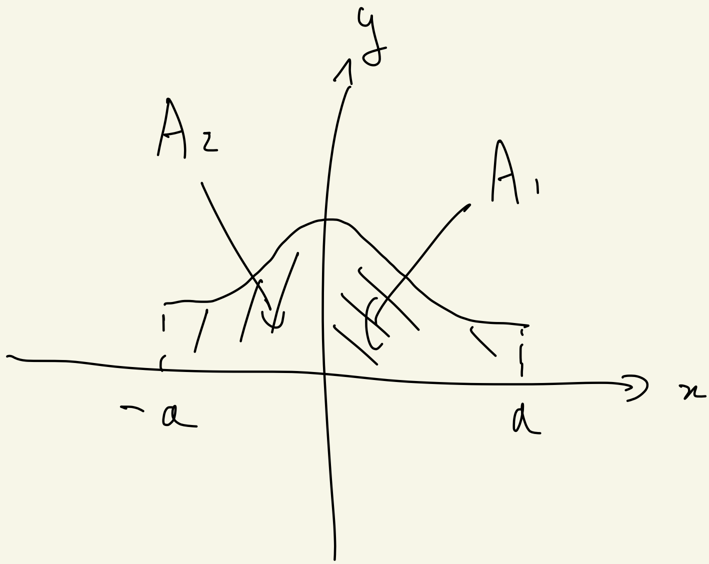
$$\int_{-a}^{a} f(x)\,dx = 2 \int_{0}^{a} f(x)\,dx$$

(2) if f is odd,

$$\int_{-a}^{a} f(x)\,dx = 0$$

Pictures:

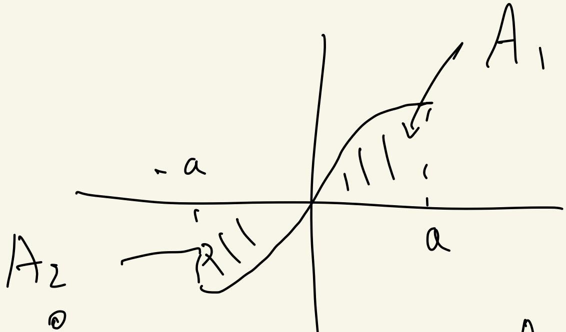
(1)



$A_1 = A_2$

$A = A_1 + A_2 = 2A_1$

(2)



$A_2 = \int_{-a}^{0} f(x)\,dx$ $A_1 = -A_2$

$$A_1 = \int_0^a f(x)\,dx$$

$$\Rightarrow \int_{-a}^{a} = A_1 + A_2 = 0$$

Proof :

$$\int_{-a}^{a} f(x)\,dx$$

$$= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$u = -x \iff x = -u$$

$$du = -dx$$

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-u)\,du$$

$$= \int_0^a f(-u)\,du.$$

So $$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx$$

① if $f(-x) = f(x)$ $(\forall x)$

i.e. f even, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

② if f odd, $f(-x) = -f(x)$

So $$\int_{-a}^{a} f(x)\,dx = 0. \quad \boxed{X}$$

Ex:
$$\int_{-\pi/2}^{\pi/2} \left(\sin(x)\right)^{11} dx$$

$$= 0$$ by the theorem.

But it is not so easy to find an antiderivative of $\sin(x)^{11}$

i.e find $\int \sin(x)^{11} dx$.

It is still possible to

find it.

<u>Trick</u>: $\left(\sin(x)\right)^{11} = \left(\sin(x)\right)^{10} \cdot \sin(x)$

So: $\displaystyle\int \left(\sin(x)\right)^{11} dx = -\int \left(\sin(x)\right)^{10} \cdot d\left(\cos(x)\right)$

$\left(d\left(\cos(x)\right) = -\sin(x)\, dx \right)$

$$\left(\sin(x)\right)^{10} = \left(\sin(x)^2\right)^5$$
$$= \left(1 - \cos(x)^2\right)^5$$

So if $u = \cos(x)$, we get

$$\int \left(\sin(x)\right)^{11} dx = -\int \underbrace{\left(1 - u^2\right)^5}_{\text{polynomial}} du$$

Expand : $(1-u^2)^5 = \underbrace{---}_{\text{expand}}$

take the antiderivative of each term.

Then replace u by $\cos(x)$

Applications of definite integrals to volumes, lengths and areas

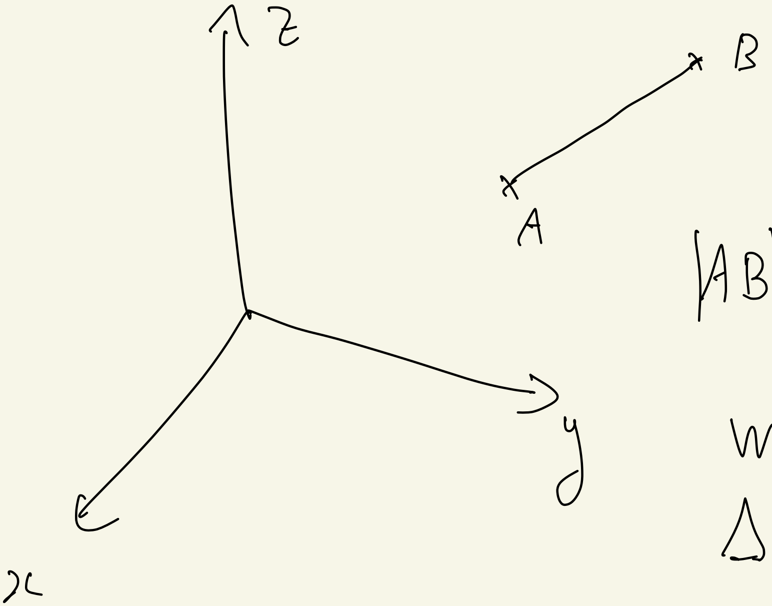
(Chap. 6 of our book)

We have already seen the link between definite integrals and the area below a graph.

In this chapter, we consider more complicated "shapes" in the 3-dimensional Euclidean space $\mathbb{R}^3$

and their volume, area and length.

$\mathbb{R}^3 = \{ (x, y, z), \ x, y, z \in \mathbb{R} \}$



$|AB|^2 = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$

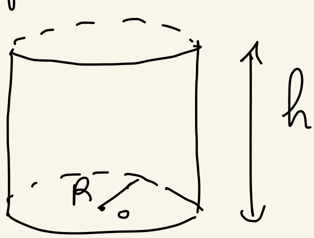
where

$\Delta x = x_B - x_A$

$\vdots$

Volumes: Slicing

Simplest volume: cylinders



$V = \pi R^2 h$

R = radius of the

base

h = height

$$V = (\text{Area of base}) \cdot \text{height}.$$

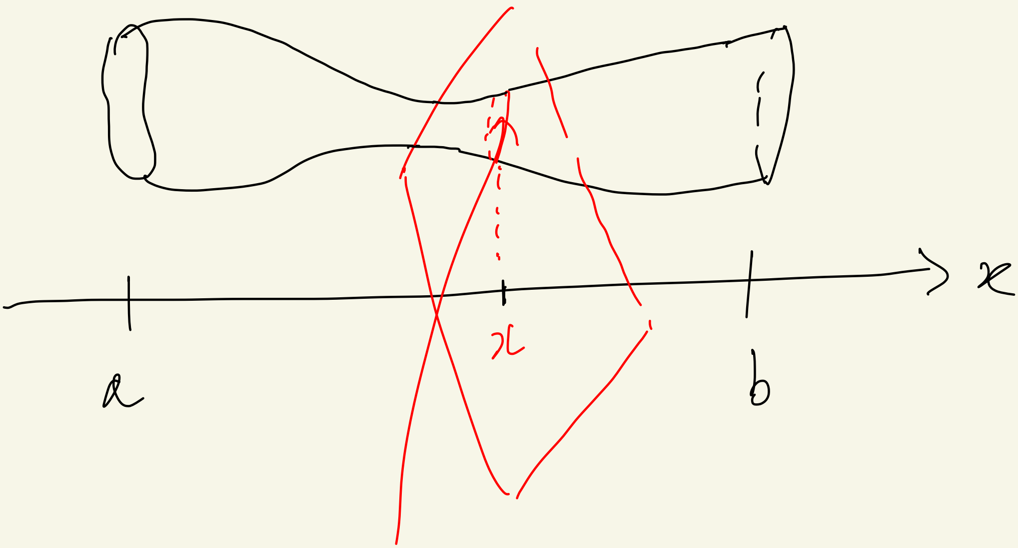
More general cylinders

h = height

$A :=$ area of the base

$$V = A \cdot h.$$

What is the volume of a general solid like



$R(x) =$  $=$ cross-section

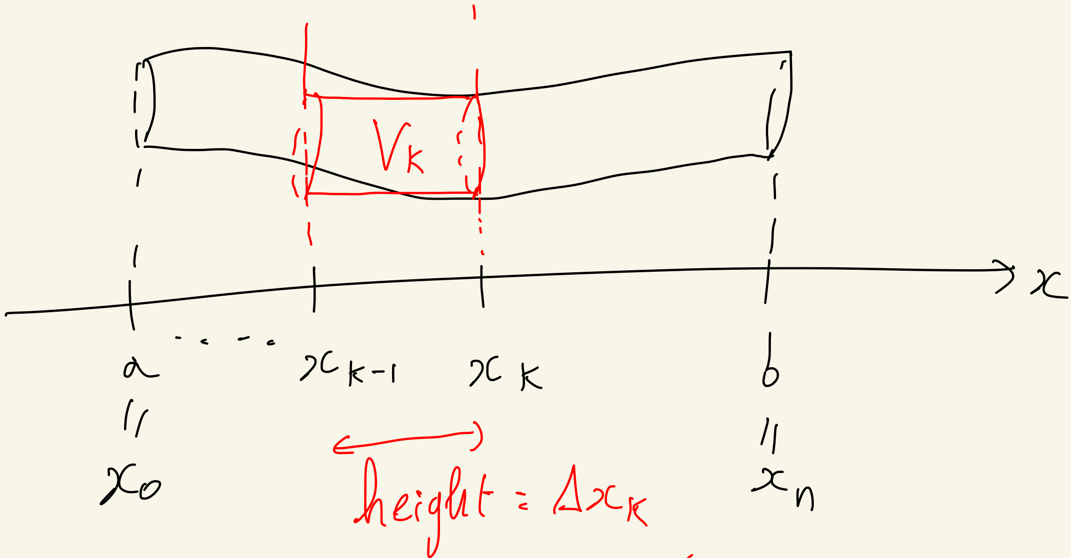
$=$ 2 dimensional region.

A cross-section = intersection of the solid with a plane (here perpendicular to the x-axis)

Let $A(x)$ = area of $R(x)$

= depends on x in general.

Idea: Slice into small cylinders



$$V_k = A(x_k) \cdot \Delta x_k .$$

$V :=$ Volume of our solid

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k) \cdot \Delta x_k$$

where $P = \{ x_0 < x_1 < \cdots < x_n$

$\overset{\shortparallel}{a}$ $\qquad$ b

is a partition of $[a, b]$.

$$V \overset{\text{def}}{=} \int_a^b A(x)\,dx$$