Find the curve's unit tangent vector. Also find the length of the indicated portion of the curve

1. $\vec{r}(t) = (2\cos t)\vec{i} + (2\sin t)\vec{j} + \sqrt{5}\, t\vec{k}$, $0 \le t \le \pi$

$\vec{v} = (-2\sin t)\vec{i} + (2\cos t)\vec{j} + (\sqrt{5})\vec{k}$

$|\vec{v}| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (\sqrt{5})^2} = 3$

$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = (-\frac{2}{3}\sin t)\vec{i} + (\frac{2}{3}\cos t)\vec{j} + (\frac{\sqrt{5}}{3})\vec{k}$

$L = \int_0^{\pi} |\vec{v}|\, dt = \int_0^{\pi} 3\, dt = 3t\Big|_0^{\pi} = 3\pi$

2. $\vec{r}(t) = (6\sin 2t)\vec{i} + (6\cos 2t)\vec{j} + 5t\vec{k}$, $0 \le t \le \pi$

$\vec{v} = (12\cos 2t)\vec{i} - (12\sin 2t)\vec{j} + 5\vec{k}$

$|\vec{v}| = \sqrt{12^2 + 5^2} = 13$

$\vec{T} = (\frac{12}{13}\cos 2t)\vec{i} - (\frac{12}{13}\sin 2t)\vec{j} + \frac{5}{13}\vec{k}$

$L = \int_0^{\pi} |\vec{v}|\, dt = \int_0^{\pi} 13\, dt = 13\Big|_0^{\pi} = 13\pi$

3. $\vec{r}(t) = t\vec{i} + (\frac{2}{3})t^{\frac{3}{2}}\vec{k}$, $0 \le t \le 8$.

$\vec{v} = \vec{i} + t^{\frac{1}{2}}\vec{k}$

$|\vec{v}| = \sqrt{1 + (t^{\frac{1}{2}})^2} = \sqrt{1+t}$

$\vec{T} = \frac{1}{\sqrt{1+t}}\vec{i} + \frac{t^{\frac{1}{2}}}{\sqrt{1+t}}\vec{k}$

$L = \int_0^{8} \sqrt{1+t}\, dt = \frac{2}{3}(1+t)^{\frac{3}{2}}\Big|_0^{8} = \frac{2}{3} \times 27 = \frac{54}{3} - \frac{2}{3} = \frac{52}{3}$

Find the arc length parameter along the curve from the point where $t=0$ by $s = \int_0^t |\vec{v}(\tau)|\, d\tau$. Then find the length of the indicate portion of the curve.

11. $\vec{r}(t) = (4\cos t)\vec{i} + (4\sin t)\vec{j} + 3t\vec{k}$, $0 \le t \le \frac{\pi}{2}$

$\vec{v} = (-4\sin t)\vec{i} + (4\cos t)\vec{j} + 3\vec{k}$

$|\vec{v}| = \sqrt{4^2 + 3^2} = 5$

$s(t) = \int_0^t 5\, d\tau = 5\tau\Big|_0^t = 5t$

$L = s(\frac{\pi}{2}) - s(0) = \frac{5\pi}{2}$

12. $\vec{r}(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j}$, $\frac{\pi}{2} \le t \le \pi$

$\vec{v} = (-\sin t + \sin t + t\cos t)\vec{i} + (\cos t - \cos t + t\sin t)\vec{j}$

$= (t\cos t)\vec{i} + (t\sin t)\vec{j}$

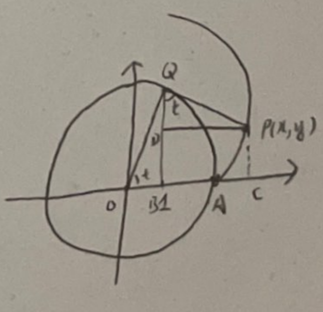
$|\vec{v}| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = t$

$s(t) = \int_0^t \tau\, d\tau = \frac{1}{2}\tau^2\Big|_0^t = \frac{t^2}{2}$

$L = s(\pi) - s(\frac{\pi}{2}) = \frac{\pi^2}{2} - \frac{(\frac{\pi}{2})^2}{2} = \frac{3\pi^2}{8}$

19. The involute of a circle.

$x = \cos t + t\sin t$, $y = \sin t - t\cos t$, $t > 0$



$\angle PQB = \angle QOB = t$.

$PQ = arc(AQ) = t$

$x = OB + BC = OB + DP = \cos t + t\sin t$

$y = PC = QB - QD = \sin t - t\cos t$

20. Find the unit tangent vector at point $P(x,y)$

$\vec{r} = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j}$

$\vec{v} = (-\sin t + \sin t + t\cos t)\vec{i} + (\cos t - \cos t + t\sin t)\vec{j}$

$= (t\cos t)\vec{i} + (t\sin t)\vec{j}$

$|\vec{v}| = t$

$\vec{T} = (\cos t)\vec{i} + (\sin t)\vec{j}$

13.2.

19. Firing from (x_0, y_0) Derive the equations $\begin{aligned} x &= x_0 + (v_0\cos\alpha)t \\ y &= y_0 + (v_0\sin\alpha)t - \frac{1}{2}gt^2 \end{aligned}$ by Differential equation: $\frac{d^2\vec{r}}{dt^2} = -g\vec{j}$

Initial conditions: $\vec{r}(0) = x_0\vec{i} + y_0\vec{j}$

$\frac{d\vec{r}}{dt}(0) = (v_0\cos\alpha)\vec{i} + (v_0\sin\alpha)\vec{j}$

$$\frac{d\vec{r}}{dt} = \int -g\vec{j}\,dt = -gt\vec{j} + C_1$$

$$\frac{d\vec{r}}{dt}(0) = -g\cdot 0\,\vec{j} + C_1 = (v_0\cos\alpha)\vec{i} + (v_0\sin\alpha)\vec{j}$$

$$C_1 = (v_0\cos\alpha)\vec{i} + (v_0\sin\alpha)\vec{j} \Rightarrow \frac{d\vec{r}}{dt} = (v_0\cos\alpha)\vec{i} + (v_0\sin\alpha - gt)\vec{j}$$

$$\vec{r} = \int\left[(v_0\cos\alpha)\vec{i} + (v_0\sin\alpha - gt)\vec{j}\right]dt = (v_0\cos\alpha\, t)\vec{i} + (v_0\sin\alpha\, t - \frac{1}{2}gt^2)\vec{j} + C_2.$$

$$\vec{r}(0) = C_2 = x_0\vec{i} + y_0\vec{j}.$$

$$\vec{r} = (x_0 + v_0\cos\alpha\, t)\vec{i} + (y_0 + v_0\sin\alpha\, t - \frac{1}{2}gt^2)\vec{j} \Rightarrow \begin{cases} x = x_0 + (v_0\cos\alpha)t \\ y = y_0 + (v_0\sin\alpha)t - \frac{1}{2}gt^2. \end{cases}$$

29. Linear drag Derive the equations $\begin{aligned} x &= \frac{v_0}{k}(1-e^{-kt})\cos\alpha \\ y &= \frac{v_0}{k}(1-e^{-kt})(\sin\alpha) + \frac{g}{k^2}(1-kt-e^{-kt}) \end{aligned}$ by Differential equation: $\frac{d^2\vec{r}}{dt^2} = -g\vec{j} - k\vec{v} = g\vec{j} - k\frac{d\vec{r}}{dt}$

$\vec{r}(0) = 0$

$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = v_0 = (v_0\cos\alpha)\vec{i} + (v_0\sin\alpha)\vec{j}$

$$\frac{d^2\vec{r}}{dt^2} = -g\vec{j} - k\frac{d\vec{r}}{dt}$$

$$\frac{d^2\vec{r}}{dt^2} + k\frac{d\vec{r}}{dt} = -g\vec{j}$$

Let $P(t) = k$ and $Q(t) = -g\vec{j}$

$$\int P(t)\,dt = kt$$

$$\Rightarrow v(t) = e^{\int P(t)dt} = e^{kt}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = \frac{1}{v(t)}\int v(t)Q(t)\,dt$$

$$= -ge^{-kt}\int e^{kt}\vec{j}\,dt$$

$$= -ge^{-kt}\left[\frac{e^{kt}}{k}\vec{j} + C_1\right]$$

$$= -\frac{g}{k}\vec{j} + Ce^{-kt}. \quad \text{where } C = -gC_1$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = (v_0\cos\alpha)\vec{i} + (v_0\sin\alpha)\vec{j} = -\frac{g}{k}\vec{j} + C$$

$$C = (v_0\cos\alpha)\vec{i} + (\frac{g}{k} + v_0\sin\alpha)\vec{j}$$

$$\frac{d\vec{r}}{dt} = (v_0e^{-kt}\cos\alpha)\vec{i} + (-\frac{g}{k} + e^{-kt}(\frac{g}{k} + v_0\sin\alpha))\vec{j}$$

$$\vec{r} = \int\frac{d\vec{r}}{dt}\,dt$$

$$= (-\frac{v_0}{k}e^{-kt}\cos\alpha)\vec{i} + (-\frac{gt}{k} - \frac{e^{-kt}}{k}(\frac{g}{k} + v_0\sin\alpha))\vec{j} + C_2.$$

$$\vec{r}(0) = 0 = (-\frac{v_0}{k}\cos\alpha)\vec{i} + (-\frac{g}{k^2} - \frac{v_0\sin\alpha}{k})\vec{j} + C_2$$

$$C_2 = (\frac{v_0}{k}\cos\alpha)\vec{i} + (\frac{g}{k^2} + \frac{v_0\sin\alpha}{k})\vec{j}$$

$$\Rightarrow \vec{r}(t) = \frac{v_0}{k}(1-e^{-kt})\cos\alpha + (\frac{v_0}{k}(1-e^{-kt})\sin\alpha + \frac{g}{k^2}(1-kt-e^{-kt}))\vec{j}$$

$$\Rightarrow \begin{cases} x = \frac{v_0}{k}(1-e^{-kt})\cos\alpha \\ y = \frac{v_0}{k}(1-e^{-kt})(\sin\alpha) + \frac{g}{k^2}(1-kt-e^{-kt}) \end{cases}$$

13.4

Find $\vec{T}$, $\vec{N}$ and κ

1 $\vec{r}(t) = t\vec{i} + (\ln\cos t)\vec{j}$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

$\vec{v} = \vec{i} + (\frac{1}{\cos t}\sin t)\vec{j}$

$|\vec{v}| = \frac{1}{\cos t}$

$\vec{T} = \cos t\,\vec{i} + \sin t\,\vec{j}$

$\frac{d\vec{T}}{dt} = -\sin t\,\vec{i} + \cos t\,\vec{j}$

$\left|\frac{d\vec{T}}{dt}\right| = 1$

$\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{\frac{1}{\cos t}} \cdot 1 = \cos t$

$\vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|} = -\sin t\,\vec{i} + \cos t\,\vec{j}$

2. $\vec{r}(t) = (\ln\sec t)\vec{i} + t\vec{j}$ $-\frac{\pi}{2} < t < \frac{\pi}{2}$

$\vec{v} = (\frac{1}{\sec t} \cdot \sec t\tan t)\vec{i} + \vec{j}$

$|\vec{v}| = \sqrt{\tan^2 t + 1} = \sqrt{\sec^2 t} = \sec t$

$\vec{T} = \frac{\tan t}{\sec t}\vec{i} + \frac{1}{\sec t}\vec{j} = \sin t\,\vec{i} + \cos t\,\vec{j}$

$\frac{d\vec{T}}{dt} = \cos t\,\vec{i} - \sin t\,\vec{j}$ $\kappa = \frac{1}{\sec t} \cdot 1 = \cos t$.

$\left|\frac{d\vec{T}}{dt}\right| = 1$ $\vec{N} = \cos t\,\vec{i} - \sin t\,\vec{j}$

3. $\vec{r}(t) = (2t+3)\vec{i} + (5-t^2)\vec{j}$

$\vec{v} = 2\vec{i} - 2t\vec{j}$

$|\vec{v}| = 2\sqrt{1+t^2}$

$\vec{T} = \frac{1}{\sqrt{1+t^2}}\vec{i} - \frac{t}{\sqrt{1+t^2}}\vec{j}$

$\frac{d\vec{T}}{dt} = -\frac{t}{\sqrt{1+t^2}^3}\vec{i} - \frac{1}{\sqrt{1+t^2}^3}\vec{j}$

$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{(1+t^2)^2}} = \frac{1}{1+t^2}$

$\kappa = \frac{1}{2\sqrt{1+t^2}} \cdot \frac{1}{1+t^2} = \frac{1}{2(1+t^2)^{\frac{3}{2}}}$

$\vec{N} = \frac{-t}{\sqrt{1+t^2}}\vec{i} - \frac{1}{\sqrt{1+t^2}}\vec{j}$

Find $\vec{T}$, $\vec{N}$ and κ

9 $\vec{r}(t) = (3\sin t)\vec{i} + (3\cos t)\vec{j} + 4t\vec{k}$

$\vec{v} = (3\cos t)\vec{i} - (3\sin t)\vec{j} + 4\vec{k}$

$|\vec{v}| = 5$

$\vec{T} = (\frac{3}{5}\cos t)\vec{i} - (\frac{3}{5}\sin t)\vec{j} + \frac{4}{5}\vec{k}$

$\frac{d\vec{T}}{dt} = (-\frac{3}{5}\sin t)\vec{i} - (\frac{3}{5}\cos t)\vec{j}$

$\left|\frac{d\vec{T}}{dt}\right| = \frac{3}{5}$

$\kappa = \frac{1}{5} \cdot \frac{3}{5} = \frac{3}{25}$

$\vec{N} = (-\sin t)\vec{i} - (\cos t)\vec{j}$

10. $r(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j} + 3\vec{k}$

$\vec{v} = (-\sin t + \sin t + t\cos t)\vec{i} + (\cos t - \cos t + t\sin t)\vec{j}$

$\vec{v} = (t\cos t)\vec{i} + (t\sin t)\vec{j}$

$|\vec{v}| = t$

$\vec{T} = \cos t\,\vec{i} + \sin t\,\vec{j}$

$\frac{d\vec{T}}{dt} = -\sin t\,\vec{i} + \cos t\,\vec{j}$

$\left|\frac{d\vec{T}}{dt}\right| = 1$

$\kappa = \frac{1}{t} \cdot 1 = \frac{1}{t}$

$\vec{N} = -\sin t\,\vec{i} + \cos t\,\vec{j}$

17. Show that $y = ax^2$, $a \neq 0$, has its largest curvature at its vertex has no minimum curvature.

$y = ax^2$

$y' = 2ax$

$y'' = 2a$

$\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{\frac{3}{2}}}$

$= \frac{|2a|}{[1+(2ax)^2]^{\frac{3}{2}}}$

$= |2a|(1+4a^2x^2)^{-\frac{3}{2}}$

$\kappa'(x) = -\frac{3}{2}|2a|(1+4a^2x^2)^{-\frac{5}{2}}(8a^2x)$.

$\kappa'(x) = 0$

$x = 0$

$\kappa'(x) > 0$ for $x < 0$ and $\kappa'(x) < 0$ for $x > 0$,

which means $\kappa(x)$ has an absolute maximum at $x = 0$.

which is the vertex of the parabola.

Since $x = 0$ is the only critical point for $\kappa(x)$, the curvature has no minimum value

18. Show that $x = a\cos t, y = b\sin t, a > b > 0$ has its largest curvature on its major axis and its smallest curvature on its minor axis.

$\vec{r} = (a\cos t)\vec{i} + (b\sin t)\vec{j}$

$\vec{v} = (-a\sin t)\vec{i} + (b\cos t)\vec{j}$

$\vec{a} = (-a\cos t)\vec{i} - (b\sin t)\vec{j}$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & b\cos t & 0 \\ -a\cos t & -b\sin t & 0 \end{vmatrix} = ab\vec{k}$$

$|\vec{v} \times \vec{a}| = |ab| = ab$

$K(t) = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$

$= ab(a^2\sin^2 t + b^2\cos^2 t)^{-\frac{3}{2}}$

$K'(t) = -\frac{3}{2}(ab)(a^2-b^2)(\sin 2t)(a^2\sin^2 t + b^2\cos^2 t)^{-\frac{5}{2}}$

Therefore $K'(t) = 0$

$\sin 2t = 0$

$\Rightarrow t = 0$ or $t = \pi$

or $t = \frac{\pi}{2}, \frac{3\pi}{2}$

$K'(t) < 0$ for $0 < t < \frac{\pi}{2}$, $\pi < t < \frac{3\pi}{2}$

$K'(t) > 0$ for $\frac{\pi}{2} < t < \pi$, $\frac{3\pi}{2} < t < 2\pi$

Therefore, the points associated with $t = 0$ and $t = \pi$ on the major axis give absolute maximum curvature and the points associated with $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ on the minor axis give axis give absolute minmum curvature.

14.1

(a) Find the function's domain

(b) Find the function's range

(c) describe the function's level curves.

(d) Find the boundary of the function's domain

(e) Determine if the domain is an open region, a closed region, or neither, and.

(f) decide if the domain is bounded or unbounded

1. $f(x,y) = y - x$.

(a) $\mathbb{R}$

(b) $\mathbb{R}$.

(c) straight lines $y - x = c$ parallel to the line $y = x$.

(d) no.

(e) both open and closed

(f) unbounded

2. $f(x,y) = \sqrt{y-x}$.

(a) $y \geq x$

(b) $z \geq 0$.

(c) straight lines of the form $y - x = c$, where $c \geq 0$

(d) $\sqrt{y-x} = 0 \Rightarrow y = x$

(e) closed

(f) unbounded

3. $f(x,y) = 4x^2 + 9y^2$

(a) $\mathbb{R}$

(b) $z \geq 0$.

(c) for $f(x,y) = 0$, the origin; for $f(x,y) = c > 0$, ellipses with center $(0,0)$ and major and minor axes along the x- and y-axes, respective.

(d) no

(e) both open and closed

(f) unbounded

4. $f(x,y) = x^2 - y^2$

(a) $\mathbb{R}$

(b) $\mathbb{R}$

(c). for $(x,y) = 0$, the union of the lines $y = \pm x$ for $f(x,y) = c \neq 0$, hyperbolas centered at $(0,0)$ with foci on the x-axis if $c > 0$ and on the y-axis if $c < 0$.

(d) no

(e) both open and closed

(f) unbounded

5. $f(x,y) = xy$

(a) $\mathbb{R}$

(b) $\mathbb{R}$

(c) are hyperbolas with the x-and y-axes as asymptotes when $f(x,y) \neq 0$, and the x and y-axes when $f(x,y) = 0$

(d) no

(e) both open and closed (f) unbounded