Linear Algebra Homework 12

Problem 5.3.1 (b)

Solve this system of linear equations by Cramer's Rule, $x_j = \det B_j / \det A$:

$$\begin{aligned} 2x_1 + x_2 \quad\quad &= 1 \\ x_1 + 2x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 0. \end{aligned}$$

$$\det A = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 1 \\ 0 & -3 & -2 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix} = 4.$$

$$\det B_1 = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{vmatrix} = 3.$$

$$\det B_2 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix} = (-1)^{(3+3)} \times 2 \times \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 2 \times (-1) = -2.$$

$$\det B_3 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (-1)^{(1+3)} \times 1 \times \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} \det B_1 \\ \det B_2 \\ \det B_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$$

Problem 5.3.5.

If the right side $\vec{b}$ is the first column of A, solve the 3×3 system $A\vec{x} = \vec{b}$. How does each determinant in Cramer's Rule lead to the solution $\vec{x}$?

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix}$$

$\det A = \det B_1$ $\quad\quad$ $\det B_2 = 0$ $\quad\quad$ $\det B_3 = 0.$

$$x_1 = \frac{\det B_1}{\det A} = 1 \quad\quad x_2 = \frac{\det B_2}{\det A} = 0 \quad\quad x_3 = \frac{\det B_3}{\det A} = 0.$$

Problem 5.3.6(b)

Find A^{-1} from the cofactor formula $C^T/\det A$. You may use symmetry.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$C_{11} = (-1)^{(1+1)}\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$$C_{12} = (-1)^{(1+2)}\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = -1 \times (-2) = 2$$

$$C_{13} = (-1)^{(1+3)}\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1 - 0 = 1$$

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\det A = (8+0+0) - (0+2+2) = 4.$$

$$A^{-1} = \frac{C^T}{\det A} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Problem 5.3.15

cofactor

For $n = 5$, the cofactor matrix C contains $\underline{5\times5=25}$ cofactors. Each 4×4 contains $\underline{4\times3\times2\times1=24}$ terms and each term needs $\underline{3}$ ($1\times1\times1\times1$) multiplications. How many total multiplications to compute C? ($25\times24\times3=1800$) Compare with $5^3 = 125$ total multiplications for the Gauss-Jordan computation of A^{-1} in Section 2.4. 125.

Problem 5.3.17

設 $\vec{u}$ $\vec{v}$ $\vec{w}$

A box has edges from $(0,0,0)$ to $(3,1,1)$, to $(1,3,1)$, to $(1,1,3)$. Find its volume. Also find the area of each parallelogram face of the box using $\|\vec{u}\times\vec{v}\|$

$$V = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = (27+1+1) - (3+3+3) = 20$$

? area of face:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = (\vec{i} + \vec{j} + 9\vec{k}) - (\vec{k} + 3\vec{i} + 3\vec{j}) = -2\vec{i} - 2\vec{j} + 8\vec{k}$$

$(-2)^2 + (-2)^2 + 8^2$ area $= 6\sqrt{2}$

Problem 5.3.23.

When the edge vector $\vec{a}, \vec{b}, \vec{c}$ are perpendicular, the volume of the box should be $\|\vec{a}\|$ times $\|\vec{b}\|$ times $\|\vec{c}\|$. Check this formula using determinants: The matrix A^TA is ___ Then find $\det A^TA$ and $|\det A|$ $A = [\vec{a}\ \vec{b}\ \vec{c}]$

$$A^TA = \begin{bmatrix} \vec{a}^T \\ \vec{b}^T \\ \vec{c}^T \end{bmatrix} [\vec{a}\ \vec{b}\ \vec{c}] = \begin{bmatrix} \vec{a}^T\vec{a} & 0 & 0 \\ 0 & \vec{b}^T\vec{b} & 0 \\ 0 & 0 & \vec{c}^T\vec{c} \end{bmatrix}$$

$\because \vec{a} \perp \vec{b} \perp \vec{c}$

$\det A^TA = (\|\vec{a}\|\ \|\vec{b}\|\ \|\vec{c}\|)^2$

$\det A = \pm \|\vec{a}\|\ \|\vec{b}\|\ \|\vec{c}\|$

Problem 6.1.6

$\therefore A$ is orthernomal matrix $\therefore A^T = A^{-1} \Rightarrow \therefore A^TA = I$

Find the eigenvalues of A, B, AB and BA:

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix}$ $B - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix}$

$\det(A-\lambda I) = (1-\lambda)^2 - 0$ $\det(B-\lambda I) = (1-\lambda^2) - 0$

$\lambda = 1$ or $\lambda = 2$ $\lambda = 1$

$AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ $BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$AB - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}$ $BA - \lambda I = \begin{bmatrix} 3-\lambda & 2 \\ 1 & 1-\lambda \end{bmatrix}$

$\det(AB-\lambda I) = (1-\lambda)(3-\lambda) - 2$
$= 3 - 4\lambda + \lambda^2 - 2$
$= \lambda^2 - 4\lambda + 1$

$\det(BA-\lambda I) = (3-\lambda)(1-\lambda) - 2$
$= \lambda^2 - 4\lambda + 1$

$\lambda = 2+\sqrt{3}$ or $\lambda = 2-\sqrt{3}$. $\lambda = 2+\sqrt{3}$ or $\lambda = 2-\sqrt{3}$

(a) Are the eigenvalues of AB equal to eigenvalues of A times eigenvalues of B?

$1 \times 1 \neq 2+\sqrt{3}$ or $2-\sqrt{3}$

(b) Are the eigenvalues of AB equal to the eigenvalues of BA?

Yes!

3

Problem 6.1.12

Find three eigenvectors for this projection matrix P (you may assume that the eigenvalues of P are 1 and 0):

$$P = \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0.4 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P - \lambda I = \begin{bmatrix} 0.2-\lambda & 0.4 & 0 \\ 0.4 & 0.8-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \qquad \det(P-\lambda I) = \sim \qquad \lambda = 1 \text{ or } \lambda = 0$$

① $\lambda = 0$.

$P\vec{x} = \lambda\vec{x}$

$P\vec{x} = \vec{0}$

$$P = \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0.4 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 is free variable

$$\begin{cases} x_1 + 2x_2 = 0 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_2 \\ x_3 = 0 \end{cases} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

eigenvector.

② $\lambda = 1$.

$P\vec{x} = \lambda\vec{x}$

$P\vec{x} = \vec{x}$

$\vec{x}(P-I) = 0.$

$$P - I = \begin{bmatrix} -0.8 & 0.4 & 0 \\ 0.4 & -0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 4 & -2 & 0 \\ -8 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 4 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 x_3 are free variables.

$$x_1 = \frac{1}{2}x_2, \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

eigenvectors

If two eigenvectors share the same λ, so do all their linear combinations. Find an eigenvector of P with no zero components.

$$\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix} \text{ has } \lambda = 1$$

Problem 6.1.15.

Every permutation matrix leave $\vec{x} = (1,1,1 \dots, 1)$ unchanged, so one eigenvalue is $\lambda = 1$.

Find two more λ's (possibly complex) for these permutations, from $\det(P - \lambda I) = 0$:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix}$$

$$\det(P - \lambda I) = (-\lambda^3 + 1) + 0 = 1 - \lambda^3.$$

$$\lambda = 1, \lambda = \tfrac{1}{2}(-1 + i\sqrt{3}), \lambda = \tfrac{1}{2}(-1 - i\sqrt{3})$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P - \lambda I = \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix}.$$

$$\det(P - \lambda I) = (1-\lambda)\lambda^2 - (1-\lambda)$$
$$= \lambda^2 - \lambda^3 - 1 + \lambda.$$

$\lambda = 1$ or $\lambda = -1$

Problem 6.1.16.

Show that the det of A equal the product of eigenvalues $\lambda_1 \lambda_2 \dots \lambda_n$: Start with the polynomial $\det(A - \lambda I) = 0$ separated into its n factors. (always possible as long as you allow the λ's to be complex numbers). Then set $\lambda = 0$.

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda) \text{ so } \det A = \underline{\lambda_1 \lambda_2 \cdots \lambda_n}$$

Problem 6.1.27

Find the rank and all eigenvalues of A and C.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$\text{rank}(A) = 1.$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 4-\lambda & 4-\lambda & 4-\lambda \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} = (4-\lambda) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -\lambda & 0 & 1 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & -\lambda \end{vmatrix}$$

$$= (4-\lambda) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} = -(4-\lambda) \cdot (-\lambda^3)$$

$$= \lambda^3(\lambda - 4).$$

$\lambda = 0$ or $\lambda = 4$.

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{rank}(C) = 2.$$

$$C - I\lambda = \begin{bmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{bmatrix}$$

$$\det(C - I\lambda) = \begin{vmatrix} 2-\lambda & 2-\lambda & 2-\lambda & 2-\lambda \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1-\lambda & 0 & 1 \\ 0 & -1 & -\lambda & -1 \\ 0 & 1 & 0 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda) \times (-1)^{1+1}\, 1 \times \begin{vmatrix} 1-\lambda & 0 & 1 \\ -1 & -\lambda & -1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)\left[(-\lambda)(1-\lambda)^2 - (-\lambda)\right]$$

$$= (2-\lambda)(2\lambda^2 - \lambda^3)$$

$$= \lambda^2 (2-\lambda)(2-\lambda)$$

$$= \lambda^2 (2-\lambda)^2.$$

$\lambda = 0,\ \lambda = 2$

eigenvalue.

Problem 6.1.32.

Suppose A has eigenvalues 0, 3, 5 with independent eigenvector $\vec{u}, \vec{v}, \vec{w}$

(a) Give a basis for the nullspace and a basis for the column space.

$A\vec{x} = 0 = \vec{u}$ — basis for the null space.

$A\vec{x} = 3\vec{x} = \vec{v}$ basis for the column space.

$A\vec{x} = 5\vec{x} = \vec{w}$

?

(b) Find a particular solution to $A\vec{x} = \vec{v} + \vec{w}$. Find all solutions.

$A\vec{x} = A\left(\frac{\vec{v}}{3} + \frac{\vec{w}}{5}\right) = \frac{3\vec{v}}{3} + \frac{5\vec{w}}{5} = \vec{v} + \vec{w}$ So. $\vec{x} = \frac{\vec{v}}{3} + \frac{\vec{w}}{5}$ is a particular solutions to $A\vec{x} = \vec{v} + \vec{w}$

(c) $A\vec{x} = \vec{u}$ has no solution: If it did, then $\underline{\vec{u}}$ would be in the column space

Graded Problems.

Problem 1

(a) Find the volume the box in $\mathbb{R}^4$ determined by the vectors

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \\ 4 \end{bmatrix}, \quad \vec{x}_4 = \begin{bmatrix} 1 \\ -1 \\ 8 \\ -8 \end{bmatrix}.$$

Volume of the box:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -2 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 8 & -8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 7 & -9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & -6 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -12 \end{vmatrix} = 1 \times (-2) \times 3 \times (-12) = 72.$$

(b) If Q is any 4×4 orthogonal matrix, what is the volume of the box determined by $Q\vec{x}_1, Q\vec{x}_2, Q\vec{x}_3, Q\vec{x}_4$? Hint: What does $Q^TQ = I$ tell you about $\det Q$?

let $B = [Qx_1\ Qx_2\ Qx_3\ Q_4]$, then we have.

$B = Q[x_1\ x_2\ x_3\ x_4]$.

$\det B = \det Q \det([x_1, x_2, x_3, x_4])$.

Since $Q^TQ = I.$, $Q^T = Q$

$\therefore \det Q^T \det Q = 1.$

$\det Q^2 = 1$

$\therefore \det Q = \pm 1.$

because Volume must be a possitive numbe,

$|\det B| = |\det Q \det([x_1, x_2, x_3, x_4])|$

$\det B = 1 \times 72 = 72.$

Problem 2

Find all eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 2 \\ 2 & -4 & 3 \end{bmatrix}.$$

$$A - I\lambda = \begin{bmatrix} 1-\lambda & -2 & 2 \\ 2 & -3-\lambda & 2 \\ 2 & -4 & 3-\lambda \end{bmatrix}$$

$$\det(A - I\lambda) = \begin{vmatrix} 1-\lambda & -2 & 2 \\ 2 & -3-\lambda & 2 \\ 0 & \lambda-1 & 1-\lambda \end{vmatrix}$$

$$= \left[(1-\lambda)^2(-3-\lambda) + (4\lambda-4)\right] - \left[2(1-\lambda)(\lambda-1) + (4\lambda-4)\right]$$

$$= \left[4\lambda - 4 - (1-2\lambda+\lambda^2)(3+\lambda)\right] - \left[(-2\lambda^2+4\lambda-2) + (4\lambda-4)\right]$$

$$= -3 - \lambda + 6\lambda + 2\lambda^2 - 3\lambda^2 - \lambda^3 + 2\lambda^2 - 4\lambda + 2$$

$$= -\lambda^3 + \lambda^2 + \lambda - 1$$

$$= \lambda - 1 - (\lambda^3 - \lambda^2)$$

$$= \lambda - 1 - \lambda^2(\lambda - 1)$$

$$= (\lambda - 1)(1 - \lambda^2)$$

$$\underbrace{\lambda = 1 \text{ or } \lambda = -1.}_{\text{eigenvalues}}$$

① $\lambda = 1$

$A\vec{x} = \vec{x}$

$\vec{x}(A - I) = 0.$

$$A - I = \begin{bmatrix} 0 & -2 & 2 \\ 2 & -4 & 2 \\ 2 & -4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$ x_3 is free variable.

$$\therefore \begin{cases} x_1 - x_3 = 0. \\ x_2 - x_3 = 0. \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ eigenvector

② $\lambda = -1$

$A\vec{x} = -\vec{x}$

$\vec{x}(A+I) = 0.$

$$A+I = \begin{bmatrix} 2 & -2 & 2 \\ 2 & -2 & 2 \\ 2 & -4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$ x_3 is free variable.

$$\therefore \begin{cases} x_1 = 0 \\ x_2 - x_3 = 0. \end{cases} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$ — eigenvector.

Does $\mathbb{R}^3$ have a basis consisting of eigenvector for A?

$\mathbb{R}^3$ doesn't have a basis consisting of eigenvector for A, because there are only two bases which are $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$