Calculus A(1)
12/15

Final exam

Chap 1 to 8 (included)

= everything until next thursday (included)

Last time:

$$\log : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$$

$$\mathbb{R}_{>0} = (0, +\infty)$$

$$\log(x) = \int_1^x \frac{1}{t}\, dt$$

$$\frac{d\log}{dx} = \frac{1}{x} \qquad \log(1) = 0$$

$\log$ is increasing on $(0, +\infty)$.

Properties of log :

$\log = \log_e$ = natural log

$\ln$ //

① Product rule

$$\forall x, y > 0, \quad \log(xy) = \log(x) + \log(y)$$

② Quotient rule

$$\forall x, y > 0, \quad \log\left(\frac{x}{y}\right) = \log(x) - \log(y)$$

Special case: $\log\left(\frac{1}{y}\right) = -\log(y)$

③ Power rule

$$\forall x > 0, \quad \forall \alpha \in \mathbb{Q}$$

$$\log(x^\alpha) = \alpha \cdot \log(x)$$

<u>Proof</u>: (1) Fix $y > 0$. Consider $x > 0$ as a variable.

Let $f(x) = \log(xy)$

$g(x) = \log(x) + \log(y)$

f and g are differentiable because $\log$ is.

$$f'(x) = y \cdot \log'(xy) = \frac{y}{xy}$$

$$= \frac{1}{x}$$

$$g'(x) = \frac{1}{x}$$

So $g'(x) = f'(x)$ $(\forall x > 0)$

So $\exists c$ s.t. $f = g + c$

$f(1) = g(1) = \log(y)$

So $C = 0 \implies f = g$.

(2) One only needs to prove the special case $\log\left(\frac{1}{x}\right) = -\log(x)$ because then write

$$\log\left(\frac{x}{y}\right) = \log\left(x \cdot \frac{1}{y}\right)$$
$$\overset{(1)}{=} \log(x) + \log\left(\frac{1}{y}\right)$$
$$= \log(x) - \log(y)$$

Proof: Let $f(x) = \log\left(\frac{1}{x}\right)$

$g(x) = -\log(x)$

$$f'(x) \overset{\text{chain rule}}{=} -\frac{1}{x^2} \cdot \log'\left(\frac{1}{x}\right)$$
$$= -\frac{1}{x^2} \cdot \frac{1}{1/x} = -\frac{1}{x}$$

$$g'(x) = -\frac{1}{x} = f'(x)$$

So $f = g + C$

$f(1) = g(1) = 0 \Rightarrow f = g.$

(3) Let $\alpha \in \mathbb{Q}$

$$f(x) = \log(x^\alpha)$$

$$g(x) = \alpha \log(x)$$

$$f'(x) = \alpha x^{\alpha-1} \log'(x^\alpha)$$

$$= \frac{\alpha x^{\alpha-1}}{x^\alpha} = \frac{\alpha}{x}$$

$$g'(x) = \frac{\alpha}{x} = f'(x)$$

$\Rightarrow f = g + C.$ $\quad f(1) = g(1) = 0$

So $f = g$. $\boxtimes$

<u>Moral</u>: log transforms multiplication into addition.

<u>Ex</u>:

- $\log(6) = \log(2 \cdot 3)$
  $= \log(2) + \log(3)$

- $\log(8) = \log(2^3) = 3\log(2)$
  $\| $
  $\log(2 \cdot 4) = \log(2) + \log(4)$
  $\log(4) = \log(2 \cdot 2) = 2\log(2)$
  $= 3\log(2)$

- $\log\left((x+1)^{1/5}\right) = \frac{1}{5}\log(x+1)$
  $\forall x > -1$

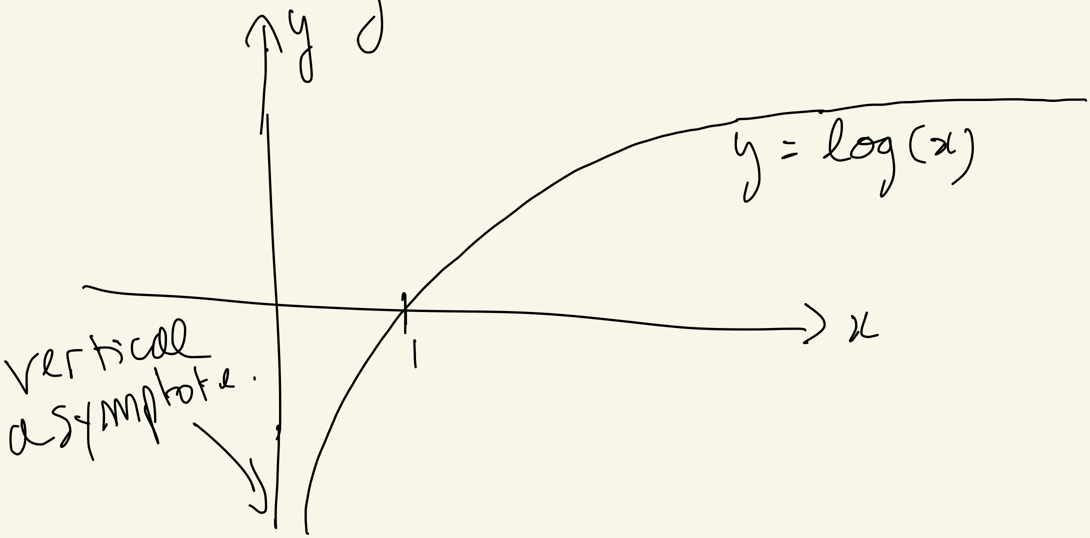
Graph of log

We have : log is increasing

$\log(1) = 0$

$$\log''(x) = \left(\frac{1}{x}\right)' = -\frac{1}{x^2} < 0$$

So log is concave.



Q : $\lim_{x \to +\infty} \log(x) = ?$

Fact: $\lim_{x \to +\infty} \log(x) = +\infty$

Proof: $\log(2^n) = n\log(2)$

$n \in \mathbb{N}$. $\log(2) > \underset{\overset{\|}{0}}{\log(1)}$

So $\lim_{n \to +\infty} n\log(2) = +\infty$

So $\lim_{n \to +\infty} \log(2^n) = +\infty$.

Similarly, $\lim_{x \to 0^+} \log(x) = -\infty$

$\left(\log(x) = -\log\left(\frac{1}{x}\right) \right)$

Applications of log to integrals

Prop: $\forall\, u : I \longrightarrow \mathbb{R}$ C^1 s.t. u does not vanish on I

We have $\displaystyle\int \frac{1}{u}\, du = \log(|u|) + C$

i.e. $\displaystyle\int \frac{u'(x)}{u(x)}\, dx = \log|u(x)| + C$

Proof: We just differentiate.

- if $\forall x \in I$, $u(x) > 0$ then

$$\frac{d}{dx} \log |u(x)| = \frac{d}{dx} \log(u(x))$$

$$= u'(x) \cdot \log'(u(x))$$

$$= \frac{u'(x)}{u(x)}$$

- if $\forall x \in I$, $u(x) < 0$,

then $\frac{d}{dx} \log |u(x)| = \frac{d}{dx} \log(-u(x))$

$$\overset{\text{Chain rule}}{=} \frac{-u'(x)}{-u(x)}$$

$$= \frac{u'(x)}{u(x)} \quad \boxtimes$$

Rem: We have seen before that $\forall \alpha \in \mathbb{Q}$, $\alpha \neq -1$

We have $\int u^{\alpha}\,du = \dfrac{u^{\alpha+1}}{\alpha+1} + C$

We now have a "formula" in the case $\alpha = -1$.

<u>Ex</u>: $\displaystyle\int_2^3 \frac{x^2}{x^3-2}\,dx = ?$

$u = x^3 - 2$

$du = 3x^2\,dx$

$x = 2,\ u = 6$

$x = 3,\ u = 25$

$$\int_2^3 \frac{x^2\,dx}{x^3-2} = \frac{1}{3}\int_6^{25} \frac{du}{u}$$

$$= \frac{1}{3}\Big[\log(u)\Big]_6^{25}$$

$$= \frac{1}{3}\left(\log(25) - \log(6)\right)$$

$$= \frac{1}{3}\log\left(\frac{25}{6}\right)$$

<u>Ex</u>: $\int \tan(x)\,dx$

(where we consider tan defined on some interval I, e.g

$$I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

or $I = \left(-\frac{\pi}{2}, 0\right)$)

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$$

$u = \cos(x) \qquad du = -\sin(x)\,dx$

$$\int \tan(x)\,dx = -\int \frac{du}{u}$$

$$= -\log|u(x)| + c$$

$$= -\log|\cos(x)| + c$$

$$\boxed{\sec(x) = \frac{1}{\cos(x)}}$$

$$= \log|\sec(x)| + c.$$

Logarithmic differentiation

Let I = interval

Let $u_1, \dots, u_n : I \longrightarrow \mathbb{R}$

s.t $\forall x \in I$, $\forall i = 1, \dots, n$,

$$u_i(x) \neq 0$$

ie. all the u_i's are non-vanishing

Let $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}$

Let $u = u_1^{\alpha_1} \cdot u_2^{\alpha_2} \cdots u_n^{\alpha_n}$

What is $u' = ?$

Technique :

$$\log|u| = \log\left(|u_1|^{\alpha_1} \cdots |u_n|^{\alpha_n}\right)$$

$$= \log\left(|u_1|^{\alpha_1}\right) + \cdots + \log\left(|u_n|^{\alpha_n}\right)$$

$$\log|u| = \alpha_1 \log|u_1| + \cdots + \alpha_n \log|u_n|$$

Differentiate both sides :

$$\boxed{\frac{u'(x)}{u(x)} = \sum_{i=1}^{n} \alpha_i \frac{u_i'(x)}{u_i(x)}}$$

Rem: $\frac{u'(x)}{u(x)}$ is called the logarithmic derivative of u.

Ex: $u(x) = \frac{(x^2+1)(x+2)^{1/3}}{x-3}$

$x > 3$

$u'(x) = ?$

Write $u(x) = u_1(x)\, u_2(x)^{1/3}\, u_3(x)^{-1}$

where $u_1(x) = x^2+1 \qquad \alpha_1 = 1$
$u_2(x) = x+2 \qquad \alpha_2 = \frac{1}{3}$
$u_3(x) = x-3 \qquad \alpha_3 = -1$

$$\frac{u'(x)}{u(x)} = \alpha_1 \frac{u_1'(x)}{u_1(x)} + \alpha_2 \frac{u_2'(x)}{u_2(x)} + \alpha_3 \frac{u_3'(x)}{u_3(x)}$$

$$= \frac{2x}{x^2+1} + \frac{1}{3}\frac{1}{x+2} - \frac{1}{x-3}$$

So $$u'(x) = u(x)\left(\frac{2x}{x^2+1} + \frac{1}{3}\cdot\frac{1}{x+2} - \frac{1}{x-3}\right)$$

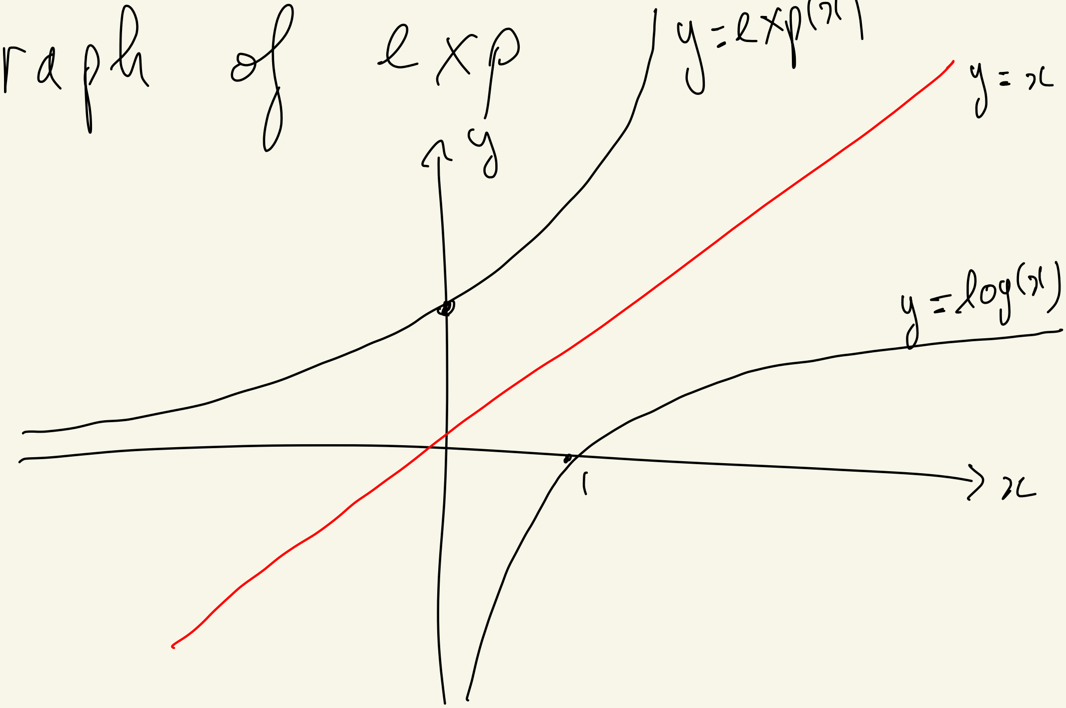
The exponential function

Def: Let exp be the inverse of log.

$$\exp : \mathbb{R} \longrightarrow \mathbb{R}_{>0} = (0, +\infty)$$

$$\exp = \log^{-1}$$

graph of exp



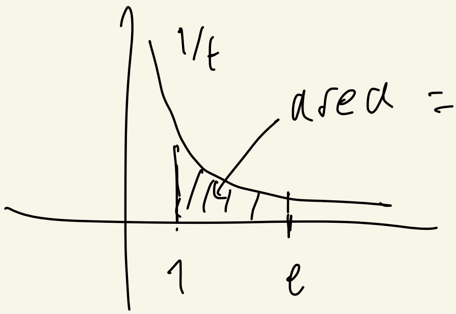
- exp is increasing on $\mathbb{R}$ since log is increasing.
- $\exp(\log(x)) = x$, $\forall x > 0$
- $\log(\exp(x)) = x$, $\forall x \in \mathbb{R}$

Since $\lim_{x \to +\infty} \log(x) = +\infty$, $\exists e > 0$

s.t $\log(e) = 1$

This e is unique, $e > 1$

because $\log(1) = 0$



By definition, we have

$$\exp(1) = e.$$

More generally, for $\alpha \in \mathbb{Q}$.

we have $\log(e^{\alpha}) \overset{\text{Prop of log}}{=} \alpha \log(e)$

$$= \alpha$$

$$\Rightarrow \boxed{\exp(\alpha) = e^{\alpha}}$$

This identity right now only makes sense if $\alpha \in \mathbb{Q}$.
But we take this as a definition if $\alpha \in \mathbb{R}$ in general.

Def: Let $\alpha \in \mathbb{R}$.
We define $e^{\alpha} := \exp(\alpha)$

This is our <u>first definition</u> of a power to a general real exponent.

What about a^{α} for any $a > 0$ and $\alpha \in \mathbb{R}$?

Write $a = \exp(\log(a))$

$$= e^{\log(a)}$$

whatever definition we take

motivation

$$a^\alpha = \left(e^{\log(a)}\right)^\alpha$$

$$= e^{\log(a)\cdot\alpha}$$

$$= \exp(\alpha \log(a))$$

<u>Def</u>: $\forall a > 0, \quad \forall \alpha \in \mathbb{R}$

we let $a^\alpha = e^{\alpha \log(a)}$

$$\overset{\text{def}}{=} \exp(\alpha \log(a))$$

e.g. $2^{\sqrt{3}} = e^{\sqrt{3}\log(2)}$

$$\sqrt{5}^{\pi} = e^{\pi \log(\sqrt{5})}$$

$$= e^{\pi \frac{\log(5)}{2}}$$

Properties of exp

① Derivative : exp is differentiable and

$$\boxed{\exp' = \exp}$$

ie. $$\frac{d}{dx} e^{x} = e^{x}$$

② $\forall x, y \in \mathbb{R}$, $\boxed{e^{x+y} = e^{x} \cdot e^{y}}$

$$\exp(x+y) = \exp(x) \cdot \exp(y)$$

③ $$e^{-x} = \frac{1}{e^{x}}$$

and more generally:

$$\forall x, y \in \mathbb{R}, \quad e^{x-y} = \frac{e^{x}}{e^{y}}$$

④ $$\forall x, y \in \mathbb{R}, \quad \boxed{(e^{x})^{y} = e^{xy}}$$

<u>Proof of ①</u> : $\exp = \log^{-1}$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$\exp'(x) = \frac{1}{\log'(\exp(x))}$$

$$= \frac{1}{1/\exp(x)} = \exp(x)$$

(2) Let $x, y \in \mathbb{R}$

We want to show:

$$(*)\ \exp(x+y) = \exp(x) \cdot \exp(y)$$

$$x = \log(a) \qquad a > 0 \quad a = e^{x}$$

$$y = \log(b) \qquad b > 0 \quad b = e^{y}$$

$$\log(a \cdot b) = \log(a) + \log(b)$$

$$= x + y$$

We apply exp on both sides:

$$\exp(\log(a \cdot b)) = \exp(x+y)$$

$$\begin{array}{ccc} \| & & // \\ a \cdot b & & \\ & = \exp(x) \cdot \exp(y) & \end{array}$$

$\boxtimes$ -

Corollary of ①:

$$\int e^u \, du = e^u + C$$

i.e. $$\int e^{u(x)} u'(x) \, dx = e^{u(x)} + C$$

Pf: the derivative of the right-hand side is

$$\left(e^{u(x)}\right)' = u'(x) \exp'(u(x))$$
$$= u'(x) \exp(u(x))$$
$$= u'(x) e^{u(x)}$$

$\boxtimes$.

Ex : $\int_0^{\log(2)} e^{3x}\,dx = ?$

$u = 3x \qquad du = 3\,dx$

$$\int_0^{\log(2)} e^{3x}\,dx = \frac{1}{3}\int_0^{3\log(2)} e^{u}\,du$$

$$= \frac{1}{3}\left[e^{u}\right]_0^{3\log(2)}$$

$$= \frac{1}{3}\left(e^{3\log(2)} - e^{0}\right)$$

$$= \frac{1}{3}\left(e^{\log(8)} - e^{0}\right)$$

$$= \frac{1}{3}(8 - 1)$$

$$= 7/3.$$