Homework 6 Solutions

2.7.5: $\vec{x}^T A \vec{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{A\vec{y} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 5$

(a)

(b) $\vec{x}^T A = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$

(c) $A\vec{y} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

2.7.11 $A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$ Row 1 → Row 3, Row 3 → Row 2, Row 2 → Row 1 $\rightsquigarrow P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Multiplying A on the right by P_2 will exchange the columns of A.

$$\begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} \xleftrightarrow[\leftrightarrow \text{Row } 3]{\text{Row } 2} \begin{bmatrix} 0 & 0 & 6 \\ 0 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow[\text{Col } 3]{\text{Col } 1 \leftrightarrow} \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

So
$$\begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \underset{P_1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}} \underset{P_2}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$$

P_1 (switch Rows 2 and 3 of I) P_2 (switch cols. 1 and 3 of I)

2.7.16 (a) $(A^2 - B^2)^T = (AA)^T - (BB)^T = A^T A^T - B^T B^T$
$= A^2 - B^2 \rightsquigarrow$ certainly symmetric

(b) $[(A+B)(A-B)]^T = (A-B)^T(A+B)^T = (A^T - B^T)(A^T+B^T)$

$= (A-B)(A+B)$

Note: $\left.\begin{array}{l}(A+B)(A-B) = A^2 + BA - AB + B^2 \\ (A-B)(A+B) = A^2 - BA + AB + B^2\end{array}\right\}$ not the same unless $AB = BA$

$\rightsquigarrow$ <u>not</u> necessarily symmetric

(c) $(ABA)^T = A^T B^T A^T = ABA$ $\rightsquigarrow$ certainly symmetric

(d) $(ABAB)^T = B^T A^T B^T A^T = BABA$ $\rightsquigarrow$ <u>not</u> necessarily symmetric.

For example: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ (both symmetric)

$$ABAB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\neq$$

$$BABA = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

<u>2.7.22</u> $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{\text{Row 1} \leftrightarrow \text{Row 2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow[-\boxed{2}\text{Row 1}]{\text{Row 3}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}$

This is PA, with $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\downarrow$ Row 3 $-\boxed{2}$ Row 2

$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$$\underset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}\underset{A}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}}$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[\text{Row 3} - \text{Row 1}]{\text{Row 2} - 2\,\text{Row 1}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$ ← Need to switch Rows 2 and 3

$$\underset{P}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow[\text{Row 3} - \boxed{2}\,\text{Row 1}]{\text{Row 2} - \boxed{1}\,\text{Row 1}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$= \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

<u>2.7.39</u> $Q^TQ = I \rightsquigarrow$

$$\underset{Q^T}{\begin{bmatrix} \vec{q}_1^{\,T} \\ \vec{q}_2^{\,T} \\ \vdots \\ \vec{q}_n^{\,T} \end{bmatrix}} \underset{Q}{\begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_n \end{bmatrix}} = \underset{I}{\begin{bmatrix} q_1^Tq_1 & q_1^Tq_2 & \cdots & q_1^Tq_n \\ q_2^Tq_1 & q_2^Tq_2 & \cdots & q_2^Tq_n \\ \vdots & \vdots & \ddots & \vdots \\ q_n^Tq_1 & q_n^Tq_2 & \cdots & q_n^Tq_n \end{bmatrix}}$$

(a) I has 1's on diagonal: $q_1^Tq_1 = 1, q_2^Tq_2 = 1, \ldots, q_n^Tq_n = 1$

$\rightsquigarrow \|\vec{q}_i\|^2 = 1$ for $i = 1, 2, \ldots, n$

(b) I has 0's off diagonal: $\vec{q}_i^{\,T}\vec{q}_j = 0$ if $i \neq j$.

(c) $Q = \begin{bmatrix} \cos\theta & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ $\quad \cos^2\theta + q_{21}^2 = 1 \rightsquigarrow q_{21} = \pm\sqrt{1-\cos^2\theta}$

$= \pm\sin\theta$

~~Also~~ Then $q_{12}\cos\theta \pm q_{22}\sin\theta = 0$

can take $q_{12} = \sin\theta$, $q_{22} = \mp\cos\theta$

One example: $Q = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ Another: $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

3.1.4 zero vector: $0=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$\frac{1}{2}A=\frac{1}{2}\begin{bmatrix}2&-2\\2&-2\end{bmatrix}=\begin{bmatrix}1&-1\\1&-1\end{bmatrix}$, $-A=\begin{bmatrix}-2&2\\-2&2\end{bmatrix}$

smallest subspace $=\text{span}\left(\begin{bmatrix}2&-2\\2&-2\end{bmatrix}\right)=$ set of all $c\begin{bmatrix}2&-2\\2&-2\end{bmatrix}=\begin{bmatrix}2c&-2c\\2c&-2c\end{bmatrix}$

3.1.10 (a) Subspace: Contains $\vec{0}=(0,0,0)$ (since $0=0$)

closed under addition: $(b_1,b_1,b_3)+(c_1,c_1,c_3)=(b_1+c_1,b_1+c_1,b_3+c_3)$

still equal

closed under scalar multiplication:

$c(b_1,b_1,b_3)=(cb_1,cb_1,cb_3)$

still equal

(b) Not a subspace: $\vec{b}=(1,0,0)$ is in the set, but $2\cdot(1,0,0)=(2,0,0)$ is not. (not closed under scalar multiplication, for example)

(c) Not a subspace: $(1,0,1)$ and $(0,1,0)$ are in the set, but $(1,0,1)+(0,1,0)=(1,1,1)$ is not (not closed under addition)

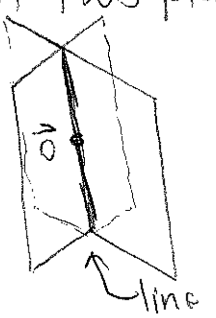
(d) Subspace: This is span of $\vec{v},\vec{w}$, which is always a subspace.

(e) Subspace: We could check the 3 conditions, or we could notice this set $=$ all $\begin{bmatrix}b_1\\b_2\\-b_1-b_2\end{bmatrix}=$ all $b_1\begin{bmatrix}1\\0\\-1\end{bmatrix}+b_2\begin{bmatrix}0\\1\\-1\end{bmatrix}=$

$=\text{span}\left(\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}0\\1\\-1\end{bmatrix}\right)$, a subspace.

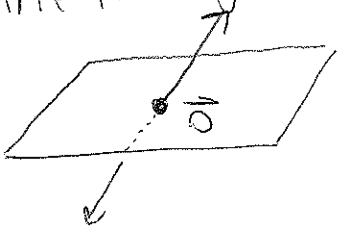
(f) Not a subspace: $(1,2,3)$ is in the set, but $-1\cdot(1,2,3)=$
$=(-1,-2,-3)$ is not (not closed under scalar multiplication)

3.1.15 (a) Intersection of two planes through $(0,0,0)$ is probably a line:

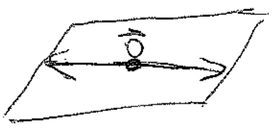


but it could be a plane (if the two planes are really the same)

(b) The intersection of a plane and line through $(0,0,0)$ is probably a point (just $(0,0,0)$):



but it could be a line:



(c) (1) $S \cap T$ contains $\vec{0}$ since both S and T do.

(2) If $\vec{x}, \vec{y}$ in $S \cap T$, what about $\vec{x}+\vec{y}$?

$\vec{x}, \vec{y}$ both in $S \rightarrow \vec{x}+\vec{y}$ in S because S is a subspace.

$\vec{x}, \vec{y}$ both in $T \rightarrow \vec{x}+\vec{y}$ in T also

shows $\vec{x}+\vec{y}$ in $S \cap T$ ✓

(3) If $\vec{x}$ in $S \cap T$, what about $c\vec{x}$?

$\vec{x}$ in $S \rightarrow c\vec{x}$ is in S

$\vec{x}$ in T also $\rightarrow c\vec{x}$ is in T also $\rightarrow c\vec{x}$ is in $S \cap T$ ✓

3.1.20 (a) $\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right] \xrightarrow[\text{Row 3 + Row 1}]{\text{Row 2} - 2\text{Row 1}} \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 0 & 0 & b_1+b_3 \end{array}\right]$

So $A\vec{x}=\vec{b}$ has a solution if both $b_2=2b_1$ and $b_1+b_3=0$

(two conditions are necessary)

(b) $\left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array}\right] \xrightarrow[\text{Row 3 + Row 1}]{\text{Row 2} - 2\text{Row 1}} \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2-2b_1 \\ 0 & 0 & b_3+b_1 \end{array}\right]$ Solution exists if $b_3+b_1=0$.

3.1.25 $\vec{z} = \vec{x} + \vec{y}$ since $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$ (6)

$= \vec{b} + \vec{b}^*$

Graded Problem 1: S = all $n \times n$ A such that $A^T = A$

(1) Does S contain 0 matrix? $0^T = 0$ ✓

(2) Closed under addition: If A, B are symmetric, then $(A+B)^T = A^T + B^T = A + B$, so $A+B$ is symmetric also. ✓

(3) If A is in S, then $(cA)^T = cA^T = cA$, so cA is in S as well $\longrightarrow$ closed under scalor multiplication.

Graded Problem 2: (1) P_n contains the zero function:

$0 = 0 + 0x + \dots + 0x^n$ ✓

(2) Addition:
$$p(x) = a_0 + a_1 x + \dots + a_n x^n$$
$$+ \; q(x) = b_0 + b_1 x + \dots + b_2 x^n$$
$$(p+q)(x) = (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_n + b_n)x^n$$

$\rightsquigarrow$ still a polynomial with degree $\leq n$. ✓

(3) Scalar multiplication:
$$c\,p(x) = c\,(a_0 + a_1 x + \dots + a_n x^n)$$
$$= (ca_0) + (ca_1)x + \dots + (ca_n)x^n$$

$\rightsquigarrow$ still a polynomial with degree $\leq n$ ✓