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Homework 12 Solutions

5.3.1(b) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\quad A \quad\quad \vec{x} \quad\quad \vec{b}$

$$\det(A) = 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2(3) - 1(2) = 4$$

$$\det B_1 = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \longrightarrow \boxed{x_1 = \frac{3}{4}}$$

$$\det B_2 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix} \overset{\text{2nd}}{\underset{\text{col}}{=}} -1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2 \longrightarrow \boxed{x_2 = -\frac{1}{2}}$$

$$\det B_3 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix} \overset{\text{3rd}}{\underset{\text{col}}{=}} +1\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \longrightarrow \boxed{x_3 = \frac{1}{4}}$$

5.3.5 $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{a}_1$ $\quad x_1 = 1, x_2 = 0, x_3 = 0$ is a solution.

From Cramer's Rule: $x_1 = \frac{\det B_1}{\det A} = \frac{\det A}{\det A} = 1$ Because we are replacing the 1st col. of A with 1st col. of A $\rightarrow$ no change in A.

$x_2 = \frac{\det B_2}{\det A} = \frac{|\vec{a}_1 \vec{a}_1 \vec{a}_2|}{\det A} = 0$ Because two columns of B_1 are the same.

Similarly: $x_3 = \frac{|\vec{a}_1 \vec{a}_2 \vec{a}_1|}{\det A} = 0$

5.3.6 (b) Because A is symmetric, A^{-1} will be symmetric also.

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\det A = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 2(3)+(-2) = 4$$

$$C_{11} = +\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3, \quad C_{12} = -\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 2, \quad C_{13} = +\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1$$

$$C_{22} = +\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4, \quad C_{23} = -\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} = 2, \quad C_{33} = +\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

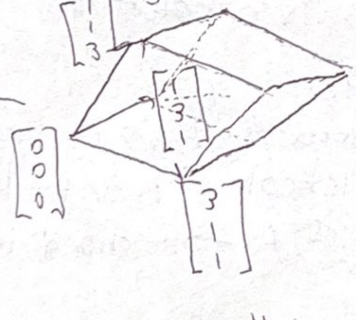
$$\text{So } A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}$$

5.3.15 For $n=5$, C contains $5 \cdot 5 = 25$ cofactors. Each cofactor contains $4! = 24$ terms, and each term need 3 multiplications.

$$a_{1,i_1}\, a_{2,i_2}\, a_{3,i_3}\, a_{4,i_4}$$

So computing C requires up to $25 \cdot 24 \cdot 3 = 1800$ multiplications.

5.3.17



$$\text{Volume} = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 3\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= 3(8) - 2 + (-2) = \boxed{20}$$

The box has 6 parallelogram faces, but only 3 distinct areas.

$$\text{Area } 1 = \left\| \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\| = \left\| \begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} \right\|$$

$$= \| -2i - 2j + 8k \| = \sqrt{(-2)^2 + (-2)^2 + 8^2} = \boxed{\sqrt{72}}$$

Area 2 $= \left\| \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\| = \left\| \begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 1 & 3 \end{vmatrix} \right\| = \| 2i - 8j + 2k \| = \sqrt{72}$ as before.

Area 3 $= \left\| \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\| = \left\| \begin{vmatrix} i & j & k \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} \right\| = \| 8i - 2j - 2k \| = \sqrt{72}$

So actually all faces have the same area $\sqrt{72}$.

5.3.23 $A^T A = \begin{bmatrix} \vec{a}^T \\ \vec{b}^T \\ \vec{c}^T \end{bmatrix} \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \begin{bmatrix} \vec{a}^T\vec{a} & \vec{a}^T\vec{b} & \vec{a}^T\vec{c} \\ \vec{b}^T\vec{a} & \vec{b}^T\vec{b} & \vec{b}^T\vec{c} \\ \vec{c}^T\vec{a} & \vec{c}^T\vec{b} & \vec{c}^T\vec{c} \end{bmatrix} = \begin{bmatrix} \|\vec{a}\|^2 & 0 & 0 \\ 0 & \|\vec{b}\|^2 & 0 \\ 0 & 0 & \|\vec{c}\|^2 \end{bmatrix}$

Because $\vec{a}, \vec{b}, \vec{c}$ are all $\perp$

So $\det(A^T A) = \|\vec{a}\|^2 \|\vec{b}\|^2 \|\vec{c}\|^2$

$\|$

$(\det A^T)(\det A) = (\det A)^2$

$\longrightarrow \det A = \pm \|\vec{a}\| \|\vec{b}\| \|\vec{c}\|$

$\downarrow$

$|\det A| = \|\vec{a}\| \|\vec{b}\| \|\vec{c}\|$

6.1.6 Eigenvalue of A: $\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0 \longrightarrow \boxed{\lambda = 1, 1}$

Eigenvalue of B: $\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0 \longrightarrow \boxed{\lambda = 1, 1}$

Eigenvalues of AB: $\begin{vmatrix} 1-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 2 = \lambda^2 - 4\lambda + 1 = 0$

$$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

(a) Eigenvalues of AB are <u>not</u> eigenvalues of A times eigenvalues of B.

Eigenvalues of BA: $\begin{vmatrix} 3-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda)-2 = \lambda^2-4\lambda+1=0$

So $\lambda = 2 \pm \sqrt{3}$ again.

(b) Eigenvalues of BA are not eigenvalues of A times eigenvalues of B.

6.1.12 Eigenvectors for $\lambda=1$: Solve $(P-I)\vec{x}=\vec{0}$

$$\begin{bmatrix} -0.8 & 0.4 & 0 \\ 0.4 & -0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = +\frac{1}{2}x_2$, $\longrightarrow \vec{x} = x_2\begin{bmatrix} +1/2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

x_2, x_3 free

For $\lambda=0$: Solve $P\vec{x}=\vec{0}$: $\begin{bmatrix} 0.2 & 0.4 & 0 \\ 0.4 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\rightarrow x_1 = -2x_2$, $x_3 = 0 \rightarrow \vec{x} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

To get an eigenvector with no 0 components we can take

$\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$, which has eigenvector 1.

6.1.15 First P: $\det(P-\lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & -\lambda \end{vmatrix}$

$= -\lambda^3 + 1 = -(\lambda^3-1) = -(\lambda-1)(\lambda^2+\lambda+1) = 0$

one eigenvalue is $\lambda=1$

$\lambda = \dfrac{-1 \pm \sqrt{1^2-4(1)(1)}}{2} = \dfrac{-1\pm\sqrt{-3}}{2}$

$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ ← two complex eigenvalues

Second P: $\det(P-\lambda I) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2-1)$

$= (1-\lambda)(\lambda-1)(\lambda+1) = 0 \longrightarrow \lambda = 1, 1, -1.$

<u>6.1.16</u> $\det(A-\lambda I)$ is a degree-n polynomial, so it can be factored into n linear factors, as long as you allow complex numbers (Fundamental Theorem of Algebra). So we can write:

$$\det(A-\lambda I) = \underbrace{C}_{\text{some constant factor}}\,\underbrace{(\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)}_{\text{eigenvalues}}$$

$\leftarrow$ If you multiply out, you get $\det(A-\lambda I) = C(-1)^n\lambda^n + \cdots$

Actually, $C=1$ because the only one of the $n!$ terms in the "Big Formula" for $\det(A-\lambda I)$ that has a λ^n is $(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$, so

$\det(A-\lambda I) = (-1)^n\lambda^n + \cdots$

Now plug in $\lambda=0$: $\det A = \det(A-0I) = (\lambda_1-0)(\lambda_2-0)\cdots(\lambda_n-0)$

$= \lambda_1\lambda_2\cdots\lambda_n$ (the product of all eigenvalues)

<u>6.1.27</u> $A = \begin{bmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1&1&1&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$, rank $=1$

Eigenvalues:

We can find eigenvalues without much work:

Since $\text{rank}(A)=1$, $\dim N(A)=4-1=3 \rightsquigarrow 0$ appears as an eigenvalue at least 3 times.

But we have one more eigenvalue since $\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}4\\4\\4\\4\end{bmatrix}=4\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

So the eigenvalues are $0,0,0,4$.

$$C=\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}\longrightarrow R=\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\longrightarrow \text{rank}(C)=2$$

Since $\text{rank}(C)=2$, 0 appears as an eigenvalue at least 2 times.

But another eigenvalue also appears at least 2 times:

$$\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}=\begin{bmatrix}2\\0\\2\\0\end{bmatrix}=2\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\quad \begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\2\\0\\2\end{bmatrix}=2\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$$

So the eigenvalues are $0,0,2,2$.

<u>6.1.32</u> (a) $N(A)=$ eigenvectors with $\lambda=0$ (plus $\vec{0}$) $=\text{span}(\vec{u})$

so $\{\vec{u}\}$ is a basis for $N(A)$.

$C(A)=$ All vectors $A\vec{x}$ for $\vec{x}$ in $\mathbb{R}^3 = $ All $A(c\vec{u}+d\vec{v}+e\vec{w})$

because $\vec{u},\vec{v},\vec{w}$ are independent so are a basis of $\mathbb{R}^3$

$=$ All $cA\vec{u}+dA\vec{v}+eA\vec{w} \overset{\text{All}}{=} c\vec{0}+d(3\vec{v})+e(5\vec{w})=$ All $(3d)\vec{v}+(5e)\vec{w}$

$=\text{span}(\vec{v},\vec{w})$. So $\{\vec{v},\vec{w}\}$ is a basis of $C(A)$.

(b) One solution is $\vec{x}=\frac{1}{3}\vec{v}+\frac{1}{5}\vec{w}$ because $A\left(\frac{1}{3}\vec{v}+\frac{1}{5}\vec{w}\right)=\frac{1}{3}(3\vec{v})+\frac{1}{5}(5\vec{w})$

$=\vec{v}+\vec{w}$.

All solutions look like $\frac{1}{3}\vec{v}+\frac{1}{5}\vec{w}$ + a vector in $N(A)$, so all solutions are $\frac{1}{3}\vec{v}+\frac{1}{5}\vec{w}+c\vec{u}$ for c in $\mathbb{R}$.

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(c) If $A\vec{x}=\vec{u}$ had a solution, then $\vec{u}$ would be in the column space $C(A)=\text{span}(\vec{v},\vec{w})$. So $\vec{u}$ would be a linear combination of $\vec{v}$ and $\vec{w}$. But $\{\vec{u},\vec{v},\vec{w}\}$ is linearly independent, so in fact $A\vec{x}=\vec{u}$ has <u>no</u> solution.

<u>Graded Problem 1:</u> (a) Volume = absolute value of

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \\ 1 & -2 & 4 & -8 \end{vmatrix} \xlongequal[\substack{\text{Row3}-\text{Row1} \\ \text{Row4}-\text{Row1}}]{\text{Row2}-\text{Row1}} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & -3 & 3 & -9 \end{vmatrix} \xlongequal[\text{Row4}-\frac{3}{2}\text{Row2}]{\text{Row3}+\frac{1}{2}\text{Row2}} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & -6 \end{vmatrix}$$

$$\xlongequal{\text{Row4}-\text{Row3}} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & -12 \end{vmatrix} = 1(-2)(3)(-12) = \boxed{72}$$

(b) Volume is absolute value of $\begin{vmatrix} Q\vec{x}_1 & Q\vec{x}_2 & Q\vec{x}_3 & Q\vec{x}_4 \end{vmatrix} =$

$$\det\left(Q\begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \vec{x}_4 \end{bmatrix}\right) = \det(Q)\begin{vmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \vec{x}_4 \end{vmatrix} = 72\det Q$$

Now $\det Q = \pm 1$ since $\det(Q^TQ)=\det I \longrightarrow (\det Q^T)(\det Q)=1 \longrightarrow (\det Q)^2 = 1 \longrightarrow \det Q = \pm 1$.

So Volume $= |\pm 72| = 72$ (<u>no</u> change in volume)

Graded Problem 2: Eigenvalues: $\begin{vmatrix} 1-\lambda & -2 & 2 \\ 2 & -3-\lambda & 2 \\ 2 & -4 & 3-\lambda \end{vmatrix} =$

$$(1-\lambda)\begin{vmatrix} -3-\lambda & 2 \\ -4 & 3-\lambda \end{vmatrix} - (-2)\begin{vmatrix} 2 & 2 \\ 2 & 3-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & -3-\lambda \\ 2 & -4 \end{vmatrix} =$$

$$(1-\lambda)(\lambda^2-1) + 2(-2\lambda+2) + 2(-2+2\lambda) =$$

$$(1-\lambda)(\lambda^2-1+2(2)-2(2)) = (1-\lambda)(\lambda-1)(\lambda+1) = 0 \rightarrow \boxed{\lambda = 1, 1, -1}$$

Eigenvectors for $\lambda=1$: Solve $(A-I)\vec{x}=\vec{0}$

$$\begin{bmatrix} 0 & -2 & 2 \\ 2 & -4 & 2 \\ 2 & -4 & 2 \end{bmatrix} \xrightarrow{\text{Row3-Row2}} \begin{bmatrix} 0 & -2 & 2 \\ 2 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row1+2Row2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = x_3$
$x_2 = x_3$ $\rightarrow$ $\boxed{\vec{x} = x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (x_3 \neq 0 \text{ for eigenvectors})}$
x_3 free

Eigenvectors for $\lambda=3$: Solve $(A+I)\vec{x}=\vec{0}$

$$\begin{bmatrix} 2 & -2 & 2 \\ 2 & -2 & 2 \\ 2 & -4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$
$\rightarrow$ $x_2 = x_3$ $\rightarrow$ $\boxed{\vec{x} = x_3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$
x_3 free

Only two linearly independent eigenvectors, so $\mathbb{R}^3$ does <u>not</u> have a basis of eigenvectors for A.