Linear Algebra - Homework 13

Problem 6.2.7

Write down all 2x2 matrices that have eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$

$$X = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \quad , \quad X^{-1} = \frac{1}{-1-1}\begin{bmatrix}-1 & -1\\-1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$A = X\Lambda X^{-1}$$

$$A = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\cdot\frac{1}{2}$$

$$A = \begin{bmatrix}\lambda_1 & \lambda_2\\\lambda_1 & -\lambda_2\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\frac{1}{2}$$

$$A = \begin{bmatrix}\lambda_1+\lambda_2 & \lambda_1-\lambda_2\\\lambda_1-\lambda_2 & \lambda_1+\lambda_2\end{bmatrix}\frac{1}{2}$$

These are the matrices $\begin{bmatrix}a & b\\b & a\end{bmatrix}$

Problem 6.2.9

Suppose G_{k+2} is the average of the two previous number G_{k+1} and G_k: ?

$$G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k \qquad \rightarrow \begin{bmatrix}G_{k+2}\\G_{k+1}\end{bmatrix} = A\begin{bmatrix}G_{k+1}\\G_k\end{bmatrix}$$

$$G_{k+1} = G_{k+1}$$

(a) Find the eigenvalues and eigenvectors of A.

$$\begin{bmatrix}G_{k+2}\\G_{k+1}\end{bmatrix} = A\begin{bmatrix}G_{k+1}\\G_k\end{bmatrix}$$

$$A = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\1 & 0\end{bmatrix} \;,\; \lambda_1 = 1, \lambda_2 = -\frac{1}{2} \;,\; x_1 = \begin{bmatrix}1\\1\end{bmatrix}, x_2 = \begin{bmatrix}1\\-2\end{bmatrix}$$

(b) Find the limit as $n \to \infty$ of the matrices $A^n = X \Lambda^n X^{-1}$

$$A^n = X\Lambda^n X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & \frac{1}{2^n} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \to A^\infty = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

(c) If $G_0 = 0$ and $G_1 = 1$, show that $\lim_{n\to\infty} G_n = \frac{2}{3}$

Problem 6.2.15.

$A^k = X\Lambda^k X^{-1}$ approaches the 0 matrix as $k \to \infty$ if and only if every λ has absolute value less than $\underline{1}$ $|\lambda|<1$. Which of these matrices has $A^k \to 0$?

B

$$A_1 = \begin{bmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0.6 & 0.9 \\ 0.1 & 0.6 \end{bmatrix}$$

$\lambda = 1$, $\lambda = 0.2$ $\qquad$ $\lambda = 0.6 \pm 0.3$

Problem 6.2.16

Find Λ and X to diagonalize A_1 in Problem 6.2.15. What is the limit of Λ^k as $k \to \infty$? What is the lim of $X\Lambda^k X^{-1}$? In the columns of this limiting matrix you see the

$A\vec{x} = \lambda \vec{x}$

$\vec{x}(A_1 - \lambda I) = 0.$

$$A_1 - \lambda I = \begin{bmatrix} 0.6-\lambda & 0.9 \\ 0.4 & 0.1-\lambda \end{bmatrix}$$

$\det(A_1 - \lambda I) = (0.6-\lambda)(0.1-\lambda) - 0.36$

$\det(A_1 - \lambda I) = (0.06 - 0.6\lambda - 0.1\lambda + \lambda^2) - 0.36$

$\det(A_1 - \lambda I) = \lambda^2 - 0.7\lambda - 0.3$

$\det(A_1 - \lambda I) = (\lambda - 1)(\lambda - 0.3)$

$\therefore \lambda_1 = 1$, $\lambda_2 = 0.3$

① $\lambda = 1$

$A - I = 0.$

$$\begin{bmatrix} -0.4 & 0.9 \\ 0.4 & -0.9 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{9}{4} \\ 0 & 0 \end{bmatrix} \quad x_1 = -\frac{9}{4} x_2 \quad ?$$

$$x_2 \begin{bmatrix} -\frac{9}{4} \\ 1 \end{bmatrix}$$

② $\lambda = 0.3$

$A - 0.3I =$

$$\begin{bmatrix} 0.3 & 0.9 \\ 0.4 & 0.2 \end{bmatrix}$$

Ans

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Lambda^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_1{}^k = X\Lambda^k X^{-1} \to \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Problem 6.2.30

The "Cayley-Hamilton Theorem" states that if $p(\lambda)$ is the characteristic polynomial of an $n\times n$ matrix A, then the $n\times n$ matrix $p(A)$ is the zero matrix.

(a) If $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, then the determinant of $A - \lambda I$ is $(\lambda - a)(\lambda - d)$. Check that $(A - aI)(A - dI) = 0$ matrix as predicted by the Cayley-Hamilton Theorem.

① $\lambda = a$ ② $\lambda = d$.

$$A - aI = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix} \qquad A - dI = \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix}$$

$$(A - aI)(A - dI) = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix}\begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) Test in matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ The theorem predicts that $A^2 - A - I = 0$, since the polynomial $\det(A - \lambda I)$ is $\lambda^2 - \lambda - 1$.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^2 - A - I = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 6.3.4

A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \quad \text{and} \quad \frac{dw}{dt} = v - w$$

Show that the total $v(t) + w(t)$ is constant (40 people). Find the matrix in $\frac{d\vec{u}}{dt} = A\vec{u}$ and its eigenvalues and eigenvectors. What are v and w at $t = 1$ and $t = \infty$?

$$\frac{d(v+w)}{dt} = (w - v) + (v - w) = 0$$

So the total $v(t) + w(t)$ is constant

?

$$\begin{bmatrix} v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix} = 20\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\Downarrow$

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \lambda_1 = 0 \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = 2 \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$v(1) = 20 + 10e^{-1}$ $w(\infty) = 20$

$w(1) = 20 - 10e^{-1}$ $w(\infty) = 20$

Problem 6.3.21

write $A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$ in the form $X \Lambda X^{-1}$. Find e^{At} from $Xe^{\Lambda t}X^{-1}$

$$A = X\Lambda X^{-1}$$

$$A = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e^{At} = Xe^{\Lambda t}X^{-1} = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} e^t & 4e^t - 4 \\ 0 & 1 \end{bmatrix}$$

Problem 6.4.8.

Find all orthogonal matrices that diagonalize $S = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$

$$S - \lambda I = \begin{bmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{bmatrix}$$

$$\det(S - \lambda I) = (9-\lambda)(16-\lambda) - 144$$

$$= 144 - 25\lambda + \lambda^2 - 144$$

$$= \lambda(\lambda - 25).$$

$$\lambda_1 = 0, \ \lambda_2 = 25.$$

① $\lambda = 0$:

$$S - I = \begin{bmatrix} 8 & 12 \\ 12 & 15 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

② $\lambda = 25$.

$$S - 25I = \begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & 0 \end{bmatrix}.$$

$$x_1 = \frac{3}{4}x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}.$$

$$Q = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix} \quad ?$$

Problem 6.4.21

Find the eigenvector matrices Q for S and X for B. Show that X is still invertible at $d=1$, even though $\lambda = 1$ is repeated. Are these eigenvectors perpendicular?

$$S = \begin{bmatrix} 0 & d & 0 \\ d & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -d & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{bmatrix} \text{ have } \lambda = 1, d, -d.$$

① $\lambda = 1$

$$S - \lambda I = \begin{bmatrix} -1 & d & 0 \\ d & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & d & 0 \\ 0 & d^2-1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x_3 \text{ is free variable}$$

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ – eigenvector}$$

② $\lambda = d$

$$S - dI = \begin{bmatrix} -d & d & 0 \\ d & -d & 0 \\ 0 & 0 & 1-d \end{bmatrix} \rightarrow \begin{bmatrix} -d & d & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-d \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad x_2 \text{ is free variable}$$

$$\begin{cases} x_1 - x_2 = 0 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_2 \\ x_3 = 0 \end{cases} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ – eigenvector}$$

③ $\lambda = -d$

$$S + dI = \begin{bmatrix} d & d & 0 \\ d & d & 0 \\ 0 & 0 & 1+d \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad x_2 \text{ is free variable.}$$

$$\begin{cases} x_1 + x_2 = 0 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -x_2 \\ x_3 = 0 \end{cases} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ – eigenvector}$$

$$Q = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -d & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{bmatrix}$$

① $\lambda = 1$

$$B - I = \begin{bmatrix} -d-1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & d-1 \end{bmatrix} \to \begin{bmatrix} d+1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ x_2 is free variable

$$\begin{cases} x_1 = 0 \\ x_3 = 0 \end{cases} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ — eigenvector.

② $\lambda = d$

$$B - dI = \begin{bmatrix} -2d & 0 & 1 \\ 0 & 1-d & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2d} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ x_3 is free variable

$$\begin{cases} x_1 - \frac{1}{2d} x_3 = 0 \\ x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2d} x_3 \\ x_2 = 0 \end{cases} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2d} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2d} \\ 0 \\ 1 \end{bmatrix}$$ — eigenvector

③ $\lambda = -d$

$$B + dI = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1+d & 0 \\ 0 & 0 & 2d \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ x_1 is free variable

$$\begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ — eigenvector

$$X = \begin{bmatrix} 0 & \frac{1}{2d} & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

when $d = 1$

$$X = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\det X = (0+1+0) - (0+0+0)$

$\det X = 1$

X is invertible when $d = 1$.

Perpendicular for A

Not perpendicular ... since $B^T \neq B$.

Graded Problem

Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors for the symmetric matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

Then compute the matrix power A^N for any positive integer N.

Eigenvalues:

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & -2 \\ 1 & -2 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & -2 \\ 0 & \lambda-5 & 5-\lambda \end{vmatrix} = (-1)^{3+2}(\lambda-5)\begin{vmatrix} 2-\lambda & 1 \\ 1 & -2 \end{vmatrix} +$$

$$(-1)^{3+3}(5-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (5-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & -2 \end{vmatrix} + (5-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (5-\lambda)\left(\begin{vmatrix} 2-\lambda & 1 \\ 1 & -2 \end{vmatrix} + \begin{vmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}\right)$$

$$= (5-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= (5-\lambda)\left[(2-\lambda)(1-\lambda)-2\right]$$

$$= (5-\lambda)(2-2\lambda-\lambda+\lambda^2-2)$$

$$= (5-\lambda)(\lambda^2-3\lambda)$$

$$= \lambda(\lambda-3)(5-\lambda)$$

$\lambda_1 = 0,\ \lambda_2 = 3,\ \lambda_3 = 5.$

① $\lambda = 0$

$A\vec{x} = 0$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -2 \\ 1 & -2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 \\ 0 & -5 & 5 \\ 0 & -5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$ x_3 is freevariable

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -x_3 \\ x_2 = x_3 \end{cases} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$ – eigenvector. $\vec{x}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

② $\lambda = 3$.

$$A\vec{x} = 3\vec{x}$$

$$\vec{x}(A-3I)=0.$$

$$A-3I=0.$$

$$A-3I = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$ x_3 is free variable.

$$\begin{cases} x_1 - 2x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 2x_3 \\ x_2 = x_3 \end{cases} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$ – eigenvector.

$$\vec{x}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

③ $\lambda = 5$.

$$A\vec{x} = 5\vec{x}$$

$$\vec{x}(A-5I)=0.$$

$$A-5I = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -2 \\ -3 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -2 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ x_3 is free var

$$\begin{cases} x_1 = 0 \\ x_2 + x_1 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = -x_3 \end{cases} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$ – eigen vector.

Orthonormal Basis of R^3: $X = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$\vec{x}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Since A is symmetric, $X^{-1} = X^T$

$$A = X \Lambda X^T$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A^N = X \Lambda^N X^T$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}^N = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3^N & 0 \\ 0 & 0 & 5^N \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}^N = \begin{bmatrix} 0 & \frac{2\times 3^N}{\sqrt{6}} & 0 \\ 0 & \frac{3^N}{\sqrt{6}} & -\frac{5^N}{\sqrt{2}} \\ 0 & \frac{3^N}{\sqrt{6}} & \frac{5^N}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 6 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}^N = \begin{bmatrix} 2\times 3^{N-1} & 3^{N-1} & 3^{N-1} \\ 3^{N-1} & \frac{3^N}{6} + \frac{5^N}{2} & \frac{3^N}{6} - \frac{5^N}{2} \\ 3^{N-1} & \frac{3^N}{6} - \frac{5^N}{2} & \frac{3^N}{6} + \frac{5^N}{2} \end{bmatrix}$$