Section 5.2 Permutations and cofactors ①

Last time: $A \xrightarrow{\det} |A|$

$n \times n$ matrix real number

$$\begin{cases} \det A \neq 0 \rightarrow A \text{ is invertible} \\ \det A = 0 \rightarrow A \text{ is } \underline{\text{not}} \text{ invertible} \end{cases}$$

We saw that determinants obey <u>3</u> <u>basic</u> properties:

① Identity matrix: $\det(I) = 1$

② Switch 2 rows → multiplies determinant by -1.

③ (a) Multiply <u>one</u> row by t → multiplies determinant by t.

(b) If <u>one</u> row is a sum of two rows → addition rule.

$$\begin{vmatrix} \vec{a}_1 \\ \vec{a}_2 + \vec{a}_2' \\ \vdots \\ \vec{a}_n \end{vmatrix} = \begin{vmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{vmatrix} + \begin{vmatrix} \vec{a}_1 \\ \vec{a}_2' \\ \vdots \\ \vec{a}_n \end{vmatrix}$$

Using only these 3 rules, we found many more properties of det, and a way to calculate them using row operations:

$$A \xrightarrow[\text{elimination}]{} U, \text{ upper triangular}$$

Keep track of row switches

$\det(U)$ = product of diagonal entries

To keep things simple, don't do the row operation where you multiply a row by a non-zero scalar.

Then $\det(A) = \underbrace{(-1)(-1)\cdots(-1)}_{\#\text{ of row switches}}\ \underbrace{(\det U)}_{\text{product of pivots}}$

Example

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -2 \end{vmatrix} =$$

switch Row 2 and Row 3

from row switch

$$-\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{vmatrix} = -(1)(-1)(-1) = \boxed{-1}$$

<u>But</u>: Is there a <u>formula</u> for det A that obeys the 3 rules? (We have an <u>algorithm</u>, but a formula is sometimes more useful if we need to do calculations with symbols instead of real numbers.)

Yes! There is a formula. (Actually, more than one.) Let's derive it for 2×2 matrices using the 3 rules:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \underset{\text{Trick}}{=} \begin{vmatrix} a+0 & 0+b \\ c & d \end{vmatrix} \underset{\text{Rule 3(b)}}{=} \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$\underset{\text{Trick}}{=} \begin{vmatrix} a & 0 \\ c+0 & 0+d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c+0 & 0+d \end{vmatrix} \underset{\text{Rule 3(b)}}{=} \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} \underset{\text{Rule 3(a)}}{=} ac \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

Rule 2 forces these to be 0.

$$+ bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + bd \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= ad\begin{vmatrix}1&0\\0&1\end{vmatrix} + bc\begin{vmatrix}0&1\\1&0\end{vmatrix}$$

Rule 1

$$\underset{\text{Rule 2}}{=} ad\begin{vmatrix}1&0\\0&1\end{vmatrix} - bc\begin{vmatrix}1&0\\0&1\end{vmatrix} = \boxed{ad-bc}$$

We have just proved that this is the <u>only</u> formula for 2×2 matrices that obeys Rules 1,2,3.

What about 3×3 matrices? Apply the "trick" to the first row:

$$\begin{vmatrix}a&b&c\\d&e&f\\g&h&i\end{vmatrix} = \begin{vmatrix}a+0+0&0+b+0&0+0+c\\d&e&f\\g&h&i\end{vmatrix}$$

$$\underset{\substack{\text{Rule}\\3(b)}}{=} \begin{vmatrix}a&0&0\\d&e&f\\g&h&i\end{vmatrix} + \begin{vmatrix}0&b&0\\d&e&f\\g&h&i\end{vmatrix} + \begin{vmatrix}0&0&c\\d&e&f\\g&h&i\end{vmatrix}$$

Now apply trick and Rule 3(b) to <u>2nd</u> row; some terms have to be 0 by Rule 2.

$$= \begin{vmatrix}a&0&0\\0&e&0\\g&h&i\end{vmatrix} + \begin{vmatrix}0&b&0\\d&0&0\\g&h&i\end{vmatrix} + \begin{vmatrix}0&b&0\\0&0&f\\g&h&i\end{vmatrix}$$

$$+ \begin{vmatrix}a&0&0\\0&0&f\\g&h&i\end{vmatrix} + \begin{vmatrix}0&0&c\\d&0&0\\g&h&i\end{vmatrix} + \begin{vmatrix}0&0&c\\0&e&0\\g&h&i\end{vmatrix}$$

$$= aei\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix} + bdi\begin{vmatrix}0&1&0\\1&0&0\\0&0&1\end{vmatrix} + bfg\begin{vmatrix}0&1&0\\0&0&1\\1&0&0\end{vmatrix}$$

3rd row now, and Rule ②

$$+ afh\begin{vmatrix}1&0&0\\0&0&1\\0&1&0\end{vmatrix} + cdh\begin{vmatrix}0&0&1\\1&0&0\\0&1&0\end{vmatrix} + ceg\begin{vmatrix}0&0&1\\0&1&0\\1&0&0\end{vmatrix}$$

These are all 6 3×3 permutation matrices

Det of permutation matrix $= \underbrace{(-1)(-1)\cdots(-1)}_{\#\text{ of row switches to get } I}$

Rule 2

So $\begin{vmatrix}a&b&c\\d&e&f\\g&h&i\end{vmatrix} = aei - afh - bdi + bfg + cdh - ceg$

$$\begin{vmatrix}1&0&0\\0&0&1\\0&1&0\end{vmatrix} \quad \begin{vmatrix}0&1&0\\1&0&0\\0&0&1\end{vmatrix} \quad \begin{vmatrix}0&1&0\\0&0&1\\1&0&0\end{vmatrix} \quad \begin{vmatrix}0&0&1\\1&0&0\\0&1&0\end{vmatrix} \quad \begin{vmatrix}0&0&1\\0&1&0\\1&0&0\end{vmatrix}$$

1st 2nd 1st 2nd

What is the pattern here?

- det A has 6 terms, one for each 3×3 permutation matrix
- Each term has a product of 3 matrix entries, one from each row of A, and one from each

column. (which column it is for each row depends on the permutation).

- Each term has a ± 1 sign, which is det P. This is called the sign of the permutation : $(-1)^{\#\text{row switches}}$

The same trick and the same Rules 1, 2, 3 give us the same "Big Formula" for bigger $n \times n$ matrices:

$$\det A = \sum (\det P)\, a_{1,j_1} a_{2,j_2} \cdots a_{n,j_n}$$

± 1

sum over all $n \times n$ permutation matrices P

This is the product of matrix entries for:

$$P = \begin{array}{cc} \begin{bmatrix} 0 & \cdots & 1 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 1 & \cdots & 0 \\ & & & & & \\ 1 & \cdots & 0 & \cdots & 0 & \cdots\ 0 \end{bmatrix} & \begin{array}{l} \leftarrow \text{Row } 1 \\ \leftarrow \text{Row } 2 \\ \vdots \\ \leftarrow \text{Row } n \end{array} \end{array}$$

col j_n col j_1 col j_2

How many?

Row 1: n choices of column for the non-zero entry

Row 2: $n-1$ choices left

Row 3: $n-2$ choices left

$\vdots$

Row n: Only 1 choice remaining

So # of P's $=$ $n(n-1)(n-2)\cdots 1$ $= n!$

<u>Note</u>: This formula is an elegant mathematical formula, but it's <u>not</u> efficient for calculating because it has $n!$ terms.

↖ very large number!

<u>Unless</u>, a lot of the terms will be 0.

<u>Example</u> $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 \\ -1 & 1 & -1 & 2 \\ 0 & 4 & 0 & 1 \end{vmatrix} = ??$ ← $4(3)(2)(1) =$ 24 terms

Terms in the formula will be 0, <u>unless</u> the permutation chooses Col 2 for Row 2:

$$\begin{vmatrix} 1 & \cancel{2} & 3 & \cancel{4} \\ \cancel{0} & 2 & \cancel{0} & \cancel{0} \\ -1 & & -1 & \cancel{2} \\ \cancel{0} & \cancel{4} & \cancel{0} & 1 \end{vmatrix}$$

Only $3(2)(1) = 6$ terms choose Col 2 for Row 2. Even most of these will be 0, unless the permutation picks Col 4 for Row 4.

$$\text{So } \det A = (1)(2)(-1)(1)\underbrace{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}}_{+1} + (3)(-1)(2)(1)\underbrace{\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}}_{-1}$$

$$= -2 - (-6) = \boxed{4}$$

Cofactor Expansion: Let's revisit the 3×3 determinant formula.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - afh - bdi + bfg + cdh - ceg$$

$$= a(ei - fh) - b(di - \overset{f}{\cancel{b}}g) + c(dh - eg) \leftarrow$$ Separate out 1st-row terms

$$= -d(bi - ch) + e(ai - cg) - f(ah - bg) \leftarrow$$ 2nd row

$$= g(bf - ce) - h(af - cd) + i(ae - bd) \leftarrow$$ 3rd row

These expressions show that det A is indeed a linear function of the rows

Terms in parentheses are 2×2 determinants:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + b\left(-\begin{vmatrix} d & f \\ g & i \end{vmatrix}\right) + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

"Expanding across" the 1st row

These $\pm$ 2×2 determinants are called cofactors.

You can also find det A by "expanding across" Rows 2 and 3, but you have to be careful about $\pm$:

Row 2: $\det A = -d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - f\begin{vmatrix} a & b \\ g & h \end{vmatrix}$.

General formula for $n \times n$ A

The "cofactors": they are $\pm$ [$(n-1)\times(n-1)$ det] you get by deleting Row 1 and Col j

Expanding across 1st row: $\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$

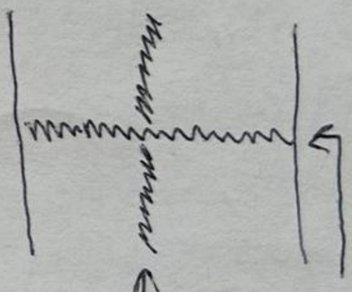
permutations that pick Col 2 for Row 2.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{12} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

This collects all $(n-1)!$ terms in the "Big Formula" that pick Col 1 for Row 1

Expanding across Row i: $\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

C_{ij} cofactor $= (-1)^{i+j}$



Delete Col j

Delete Row i

Take det of what is left.

This is ± 1. Gives alternating sign pattern:

$$\begin{array}{cc} & \\ \text{Row } i \rightarrow & \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \\ & \uparrow \\ & \text{col } j \end{array}$$

sign for C_{ij} is -1 here.

The term $a_{ij}C_{ij}$ collects all $(n-1)!$ in the "Big Formula" that come from permutations which pick Col j for Row i.

Previous Example

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 \\ -1 & 1 & -1 & 2 \\ 0 & 4 & 0 & 1 \end{vmatrix} \overset{\text{1st}}{\underset{\text{row}}{=}} 1\begin{vmatrix} 2 & 0 & 0 \\ 1 & -1 & 2 \\ 4 & 0 & 1 \end{vmatrix}$$

$$-2\underbrace{\begin{vmatrix} 0 & 0 & 0 \\ -1 & -1 & 2 \\ 0 & 0 & 1 \end{vmatrix}}_{\substack{\text{Will be } 0 \\ (\text{Row } 1 = 0)}} + 3\begin{vmatrix} 0 & 2 & 0 \\ -1 & 1 & 2 \\ 0 & 4 & 1 \end{vmatrix} - 4\underbrace{\begin{vmatrix} 0 & 2 & 0 \\ -1 & 1 & -1 \\ 0 & 4 & 0 \end{vmatrix}}_{\substack{\text{Will be } 0 \\ (\text{Row } 3 = 2\,\text{Row} 1)}}$$

$$= 1\left(2\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix}\right)$$

$$+3\left(0\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} -1 & 1 \\ 0 & 4 \end{vmatrix}\right)$$

$$= 1\Big(2\big((-1)(1) - 2(0)\big)\Big) + 3\Big(-2\big((-1)(1) - 2(0)\big)\Big)$$

$$= -2 + 6 = 4 \text{ again}$$

It would be more efficient to take advantage of the 0's:

same sign pattern

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 \\ -1 & 1 & -1 & 2 \\ 0 & 4 & 0 & 1 \end{vmatrix} \overset{\text{2nd}}{\underset{\text{row}}{=}} -0\begin{vmatrix} \cdots \end{vmatrix} + 2\begin{vmatrix} 1 & 3 & 4 \\ -1 & -1 & 2 \\ 0 & 0 & 1 \end{vmatrix} - 0\begin{vmatrix} \cdots \end{vmatrix} + 0\begin{vmatrix} \cdots \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 3 & 4 \\ -1 & -1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \overset{\text{3rd}}{\underset{\text{row}}{=}} 2\left(+1\begin{vmatrix} 1 & 3 \\ -1 & -1 \end{vmatrix}\right)$$

$$= 2\left(1(-1) - 3(-1)\right) = 4 \quad \checkmark$$

We can <u>also</u> do cofactor expansion <u>down</u> <u>columns</u>:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 \\ -1 & 1 & -1 & 2 \\ 0 & 4 & 0 & 1 \end{vmatrix} \overset{\text{1st}}{\underset{\text{col}}{=}} 1\begin{vmatrix} 2 & 0 & 0 \\ 1 & -1 & 2 \\ 4 & 0 & 1 \end{vmatrix} - 0\begin{vmatrix} \cdots \end{vmatrix}$$

$$+(-1)\begin{vmatrix} 2 & 3 & 4 \\ 2 & 0 & 0 \\ 4 & 0 & 1 \end{vmatrix} - 0\begin{vmatrix} \cdots \end{vmatrix}$$

same cofactors, same sign pattern

$$\overset{\text{2nd}}{\underset{\text{col}}{=}} 1\left(-1\begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix}\right) + (-1)\left(-3\begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix}\right)$$

$$= 1(-1)(2)(1) - 1(-3)(2)(1) = 4$$

That is: $$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

collects all terms for permutations that choose Row n for Col j.

<u>Why does this work?</u> Because $\det A = \det A^T$!

So expanding down Col j = expand across Row j of A^T $\longrightarrow$ gives <u>same</u> determinant.

We now have <u>two</u> ways to calculate determinants: Row operations and cofactor expansion.

<u>Usually</u>, row operations are more efficient, but cofactor expansion is good when A has <u>many</u> <u>0's</u>.

<u>Problem 5.2.16</u>

$$F_n = \underbrace{\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 1 & -1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}}_{n \times n}$$

<u>Row operations</u>: need to n of them (lot of work)

<u>cofactor</u> <u>expan.</u> using 1st row: reveals a pattern.

$$\det F_n \overset{\text{1st row}}{=} 1 \underbrace{\begin{vmatrix} 1 & -1 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \\ 0 & 0 & \cdots & 1 \end{vmatrix}}_{\text{This is } \det F_{n-1}} - (-1) \underbrace{\begin{vmatrix} 1 & -1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \ddots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 1 \end{vmatrix}}_{\underline{\text{not}}\ F_{n-1}!}$$

$$= \det F_{n-1} + 1 \cdot 1 \underbrace{\begin{vmatrix} 1 & -1 & & 0 \\ 1 & 1 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & 1 & 1 \end{vmatrix}}_{\text{This is } \det F_{n-2}}$$

1st column

So $\det F_n = \det F_{n-1} + \det F_{n-2}$.

Also: $\det F_1 = |1| = 1$

$\det F_2 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$

These determinants are Fibonacci numbers.

$$\begin{array}{ccccccccc} F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & \\ 1, & 2, & 3, & 5, & 8, & 13, & 21, & 34, & \ldots \end{array}$$

Let's do F_4 explicitly:

$$\begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 2 + 1 + 2 = 5. \checkmark$$

With row operations:

$$\begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix} \underset{\text{Row 3} - \frac{1}{2}\text{Row 2}}{=} \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 3/2 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$\underset{\text{Row 4} - \frac{2}{3}\text{Row 3}}{=} \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 3/2 & -1 \\ 0 & 0 & 0 & 5/3 \end{vmatrix} = 1(2)\left(\frac{3}{2}\right)\left(\frac{5}{3}\right) = 5 \checkmark$$

<u>A similar example:</u>

$$A_n = \underbrace{\begin{bmatrix} 2 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{bmatrix}}_{n \times n \text{ matrix}}$$

$\det A_1 = |2| = 2$

$\det A_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$

In general:

$$\begin{vmatrix} 2 & -1 & & & 0 \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ 0 & & & -1 & 2 \end{vmatrix} = 2 \underbrace{\begin{vmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ 0 & & -1 & 2 \end{vmatrix}}_{\det A_{n-1}}$$

$$-(-1) \begin{vmatrix} -1 & -1 & \cdots & 0 & 0 \\ 0 & 2 & -1 \cdots & 0 & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ & & & 2 & -1 \\ 0 & \cdots & & -1 & 2 \end{vmatrix}$$

A_{n-2}

$$= 2 \det A_{n-1} + 1 \left(-1 \det A_{n-2} \right)$$

$$= 2 \det A_{n-1} - \det A_{n-2}$$ ← "Recursive formula"

$\det A_3 = 2 \det A_2 - \det A_1 = 2(3) - 2 = 4$

$\det A_4 = 2 \det A_3 - \det A_2 = 2(4) - 3) = 5$

$\vdots$

$\det A_n = n+1$ in general

$\det A_{n+1} = 2 \det A_n - \det A_{n-1}$

$= 2(n+1) - (n) = n+2 = (n+1)+1$ ✓