Last week: Eigenvalues and Eigenvectors

$$A\vec{x} = \lambda\vec{x}$$ ← non-zero eigenvector

A: n×n matrix

λ: eigenvalue ← Solutions of $\underbrace{\det(A-\lambda I)}_{\text{characteristic polynomial of } A} = 0$

General comment: When you find eigenvectors for a matrix in the homework problems, your answers might be different from the answer key's.

Why? Probably because your ~~answer~~ eigenvectors are multiples or linear combinations of the answer key's.

This is okay! General fact: Suppose λ is an eigenvalue of A. Then the set of all eigenvectors + zero vector is a subspace.

called the eigenspace for λ.

Closed under addition: $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \lambda\vec{x} + \lambda\vec{y} = \lambda(\vec{x}+\vec{y})$

Scalar multiplication: $A(c\vec{x}) = c(A\vec{x}) = c(\lambda\vec{x}) = \lambda(c\vec{x})$

Linear combinations of eigenvectors are still eigenvectors (if they are non-zero)

Note: Eigenspace for λ is just null space of $A-\lambda I$.

So if your homework answer is "$\vec{x}$" and answer key's is "$c\vec{x}$", both are correct!

Today: Solving differential equations with linear algebra.

1×1 system of ordinary differential equations

$$\begin{cases} \dfrac{du}{dt} = \lambda u \\ u(0) = u_0 \end{cases}$$

← General solution is $u(t) = Ce^{\lambda t}$, C a scalar,

because $\frac{d}{dt} Ce^{\lambda t} = C\lambda e^{\lambda t} = \lambda(Ce^{\lambda t})$

Since also $u(0) = u_0$:

$u_0 = C e^{\lambda(0)} = C \longrightarrow u(t) = e^{\lambda t} u_0$

Change to nxn system: $\vec{u} = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}$

$u_1(t)$ and $u_2(t)$ are "coupled"

$$\frac{d\vec{u}}{dt} = A\vec{u}(t)$$

nxn matrix

Example: $\begin{cases} u_1'(t) = 2u_1(t) + u_2(t) \\ u_2'(t) = u_1(t) + 2u_2(t) \end{cases}$

Or, $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

Guess an exponential solution: $\vec{u}(t) = e^{\lambda t} \vec{x}$

constant vector

Then we need: $\vec{u}'(t) = \lambda \cancel{e^{\lambda t}} \vec{x}$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cancel{e^{\lambda t}} \vec{x}$$

$$\longrightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{x} = \lambda \vec{x}$$

That is: We need λ to be an eigenvalue and $\vec{x}$ to be a corresponding eigenvetor (to get a non-zero solution)

Eigenvalues in this example: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3$$

$\longrightarrow (\lambda - 1)(\lambda - 3) = 0 \longrightarrow \lambda = 1, 3$

Eigenvetors for $\lambda = 1$: Solve $(A - I)\vec{x} = \vec{0} \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So we need $x_1 = -x_2 \to \vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Eigenvectors for $\lambda = 3$: Solve $(A - 3I)\vec{x} = \vec{0}$, or

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So $x_1 = x_2 \longrightarrow \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$

Two eigenvalue/eigenvector pairs $\longrightarrow$ two different non-zero solutions of the differential equations:

$\vec{u}(t) = e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -e^t \\ e^t \end{bmatrix}$ and $\vec{u}(t) = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$

General solution of the system of differential equations:

All linear combinations of the two basic solutions

because it's a 2×2 system

$$\vec{u}(t) = C e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + D e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -Ce^t + De^{3t} \\ Ce^t + De^{3t} \end{bmatrix} \begin{matrix} \leftarrow u_1(t) \\ \leftarrow u_2(t) \end{matrix}$$

We can pick out one particular solution by choosing initial values for $u_1(t)$ and $u_2(t)$.

Example (of an initial value problem): Suppose $u_1(0) = 2$, $u_2(0) = 3$, or $\vec{u}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Then $\begin{cases} -Ce^0 + De^{3(0)} = 2 \\ Ce^0 + De^{3(0)} = 3 \end{cases} \longrightarrow \begin{cases} -C + D = 2 \\ C + D = 3 \end{cases}$

$2C = 1 \to C = \frac{1}{2}$

$2D = 5 \to D = \frac{5}{2}$

Get $\vec{u}(t) = \begin{bmatrix} -\frac{1}{2}e^t + \frac{5}{2}e^{3t} \\ \frac{1}{2}e^t + \frac{5}{2}e^{3t} \end{bmatrix}$

Another perspective for solving differential equations:

"Matrix exponential" of an $n \times n$ matrix A.

$$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{6} + \frac{A^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

Technically, this is $\lim_{N \to \infty} \underbrace{\sum_{n=0}^{N} \frac{A^n}{n!}}_{\text{"partial sum" } n\times n \text{ matrix, } S_N}$, means <u>every entry</u> of S_N approaches the corresponding entry of e^A as $N \to \infty$.

Derivatives: $$\frac{d}{dt} e^{tA} = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \overset{(?)}{=} \sum_{n=0}^{\infty} \frac{d}{dt} \frac{t^n}{n!} A^n$$

$$= \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} A^n = A \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = A e^{tA}.$$

Shows that solutions to $\vec{u}'(t) = A\vec{u}(t)$ <u>should</u> be

$\vec{u}(t) = e^{tA} \vec{u}(0)$ — the initial value, can be any constant vector

But can we actually <u>calculate</u> this matrix exponential?

Yes if A is diagonalized! $A = X \Lambda X^{-1}$ — eigenvalue matrix: $\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

Because then $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \underbrace{(X\Lambda X^{-1})^n}_{X\Lambda^n X^{-1}} = X \underbrace{\left(\sum_{m=0}^{\infty} \frac{t^m}{m!} \begin{bmatrix} \lambda_1^m & & & 0 \\ & \lambda_2^m & & \\ & & \ddots & \\ 0 & & & \lambda_n^m \end{bmatrix} \right)}_{\text{Diagonal matrix with entries}} X^{-1}$$

Diagonal matrix with entries $\sum_{m=0}^{\infty} \frac{t^m \lambda_1^m}{m!}, \sum_{m=0}^{\infty} \frac{t^m \lambda_2^m}{m!}, \ldots, \sum_{m=0}^{\infty} \frac{t^m}{m!} \lambda_n^m$

So $e^{tA} = X \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \quad e^{\lambda_n t} \end{bmatrix} X^{-1}$

So here are the solutions to $\vec{u}'(t) = A\vec{u}(t)$ when $A = X \Lambda X^{-1}$:

$$\vec{u}(t) = X \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \quad e^{\lambda_n t} \end{bmatrix} X^{-1} \vec{u}(0)$$

Let's check this with our previous example, $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$:

$\lambda_1 = 1,\ \lambda_2 = 3,\quad \vec{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $A = X \Lambda X^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$

$\frac{1}{-2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

The solutions to $\vec{u}'(t) = A\vec{u}(t)$ are:

$$\vec{u}(t) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}$$

$$= \begin{bmatrix} -e^t & e^{3t} \\ e^t & e^{3t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2}u_1(0) + \frac{1}{2}u_2(0) \\ \frac{1}{2}u_1(0) + \frac{1}{2}u_2(0) \end{bmatrix} = \begin{bmatrix} \underbrace{\tfrac{1}{2}(u_1(0) - u_2(0))}e^t + \underbrace{\tfrac{1}{2}(u_1(0) + u_2(0))}e^{3t} \\ -\tfrac{1}{2}(u_1(0) - u_2(0))e^t + \tfrac{1}{2}(u_1(0) + u_2(0))e^{3t} \end{bmatrix}$$

These are the C and D from before.

Solve when $\vec{u}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$:

$u_1(t) = -\frac{1}{2}e^t + \frac{5}{2}e^{3t},\quad u_2(t) = \frac{1}{2}e^t + \frac{5}{2}e^{3t}$, like before.

Matrix exponential is still useful when eigenvalues don't behave well (repeated roots)

<u>Example</u> $y'' - 2y' + y = 0$

2nd-order equation for $y(t)$

Trick to turn into a <u>1st-order system</u>: write $u_1(t) = y(t)$, $u_2(t) = y'(t)$.

Then: $u_1'(t) = u_2(t)$ (by definition)

$$u_2'(t) = y''(t) = -y(t) + 2y'(t) = -u_1(t) + 2u_2(t)$$

That is: $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, or $\begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$

Eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \longrightarrow \lambda = 1, 1$

Eigenvectors: solve $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow x_1 = x_2 \rightarrow \vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

only one independent eigenvector
(matrix is not diagonalizable)

One solution is $\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = Ce^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} Ce^t \\ Ce^t \end{bmatrix}$

But since we have a 2×2 system, we should have a second independent solution.

Good news: Matrix exponential does give us all solutions, if only we can calculate e^{tA}!

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = e^{tA} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} \overset{\text{Trick}}{=} e^{tI + t(A-I)} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

Warning: $e^{A+B} = e^A e^B$ only if $AB = BA$.

$$e^{tI} e^{t(A-I)}$$

$$\begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}$$

$$I + t(A-I) + \frac{t^2}{2}(A-I)^2 + \dots =$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \frac{t^2}{2} \underbrace{\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} !!} + \dots =$$

$$= \begin{bmatrix} +1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \frac{t^2}{2} 0 + 0 + 0 + \cdots$$

Conclusion: $\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}}_{e^{It}} \underbrace{\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right)}_{e^{t(A-I)}} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$

$$= \underbrace{\begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}}_{e^{At}} \underbrace{\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}}_{\vec{u}(0)}$$

Example: Initial value problem with $y(0) = 0$, $y'(0) = 1$

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}}_{e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \underbrace{\begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\begin{bmatrix} t \\ 1+t \end{bmatrix}} = \begin{bmatrix} te^t \\ (1+t)e^t \end{bmatrix}$$

$\boxed{\text{So } y(t) = te^t}$

The 2nd basic solution to $y'' - 2y' + y = 0$ (besides $y(t) = e^t$)

Matrix exponential is <u>also</u> useful when you have complex eigenvalues:

Example $y'' = -y$ $\to u_1(t) = y(t)$, $u_2(t) = y'(t)$ $\to$ $u_1' = u_2$, $u_2' = y'' = -y = -u_1$

So $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ Eigenvalues: $\begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = 0 \to$

$\lambda^2 + 1 = 0 \to \lambda = \pm i$

So if you don't mind working with complex numbers, the solutions are: $\vec{u}(t) = Ce^{it}\vec{x}_1 + De^{-it}\vec{x}_2$

complex eigenvectors

If you <u>don't</u> like complex exponentials, you could try matrix exponential instead:

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \exp\left(t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

$$\hookrightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} + t\underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} + \frac{t^2}{2}\underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{A^2} + \frac{t^3}{3!}\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A^3} +$$

$$\frac{t^4}{4!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{t^5}{5!}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \cdots$$

$A^4 = I$! Back to A!

$$= \begin{bmatrix} 1 - t^2/2! + t^4/4! - \cdots & t - t^3/3! + t^5/5! - \cdots \\ -t + t^3/3! - t^5/5! + \cdots & 1 - t^2/2! + t^4/4! - \cdots \end{bmatrix}$$

Power series for $\sin t$ and $\cos t$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Conclusion: $\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} y(0)\cos t + y'(0)\sin t \\ -y(0)\sin t + y'(0)\cos t \end{bmatrix}$