Example

$$\begin{cases} x_2 + x_3 = 4 \\ x_1 + x_2 + x_3 = 1 \\ x_1 + 2x_2 + 3x_3 = 2 \end{cases} \longleftrightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \end{array}\right]$$

First step: Eliminate these using Eqn. 1. But we can't!

Instead: Switch Rows 1 and 2 first.

$$\xrightarrow{\text{Row 1} \leftrightarrow \text{Row 2}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 4 \\ 1 & 2 & 3 & 2 \end{array}\right] \xrightarrow[\text{Row 3} - \text{Row 1}]{\text{Row 3} \to} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow[\text{Row 3} - \text{Row 2}]{\text{Row 3} \to}$$

Now eliminate this   Eliminate this

Eliminate   Eliminate

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow[\text{Row 2} - \text{Row 3}]{\text{Row 1} - \text{Row 3}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow{\text{Row 1} - \text{Row 2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

↓ or use back-substitution

Solution vector, $x_1 = -3,\ x_2 = 7,\ x_3 = -3$

$x_1 + x_2 + x_3 = 1$

$x_2 + x_3 = 4 \longrightarrow x_2 = 4 - x_3 = 4 - (-3) = \boxed{7}$

$\boxed{x_3 = -3}$

$x_1 = 1 - x_2 - x_3 = 1 - 7 - (-3) = \boxed{-3}$

Last time: Three operations for solving linear systems by elimination:

1. Add a multiple of one equation to another.
2. Multiply both sides of an equation by a non-zero scalar.
3. Switch the order of two equations.

↳ Sometimes need ③ to follow the systematic elimination method (eliminate lower left variables first)

Example (Problem 2.2.14)

$$\begin{cases} 2x + 5y + z = 0 \\ 4x + dy + z = 2 \\ y - z = 3 \end{cases}$$

What number d forces a row switch?

$$\underbrace{\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 4 & d & 1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right]}_{\text{Augmented matrix}} \xrightarrow{\text{Row 2}-2(\text{Row 1})} \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & d-10 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow[\text{Row 2} \leftrightarrow \text{Row 3}]{\text{If } d=10:}$$

Eliminate first

Eliminate using Eqn. 2:

Row 3 $\rightarrow$ Row 3 $- \frac{1}{d-10}$ Row 1

Doesn't work if $d=10$!

So $d=10$ forces a row switch.

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 2 \end{array}\right] \xrightarrow[\text{Row 2} - \text{Row 3}]{\text{continue eliminating: Row 1} + \text{Row 3}} \left[\begin{array}{ccc|c} 2 & 5 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

or back substitution

$$2x + 5y + z = 0$$
$$y - z = 3$$
$$-z = 2$$

$z = \boxed{-2}$ $\quad y = 3 + z = \boxed{1}$

$2x = -5y - z = -5 - (-2) = -3$

$\rightarrow x = \boxed{-3/2}$

Row 2 $-$ 5 Row 1

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

$\frac{1}{2}$ Row 1

$-$Row 3

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$(x = -3/2,\ y = 1,\ z = -2)$

We've now seen a number of examples of solving linear systems in practice using elimination.

Next: Look at some more theory behind linear systems and matrices (matrix multiplication, inverses, LU decomposition, ...)

First: A new algebraic operation, matrix multiplication

Motivate using elimination matrices: Let's look again at the first couple row operations in previous example.

$$\begin{bmatrix} 2 & 5 & 1 \\ 4 & 10 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \xrightarrow[\text{Row } 2-2(\text{Row }1)]{} \begin{bmatrix} 2 & 5 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2-2(2) \\ 3 \end{bmatrix}$$

Change this a bit

I can get this vector by multiplying $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ with an "elimination matrix."

$$\begin{bmatrix} 2 \\ 2-2(2) \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

How did I come up with this matrix? I applied "Row 2 → Row 2 − 2 (Row 1) to the identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Why does doing this work? ~~Start~~ with a true equation!

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \xrightarrow[\text{Row } 2-2(\text{Row }1)]{\text{Row } 2 \rightarrow} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2-2b_1 \\ b_3 \end{bmatrix}}$$

Still a true equation; this elimination matrix implements the row operation Row 2 → Row 2 − 2 (Row 1)

Next row operation:

$$\begin{bmatrix} 2 & 5 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} \xrightarrow{\text{Row } 2 \rightleftarrows \text{Row } 3} \begin{bmatrix} 2 & 5 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

Rows 2 and 3 of the identity matrix are switched. → $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$

Now how do these row operations combine?

$$\begin{bmatrix} 2 & 5 & 1 \\ 4 & 10 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \xrightarrow{\text{1st operation}} \begin{bmatrix} 2 & 5 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

2nd operation $\longrightarrow$ $$\begin{bmatrix} 2 & 5 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}\right)}$$

we need to perform 2 matrix-vector products to get right-side vector (Equivalently, 2 row operations.)

What if we try to "move the parentheses"?

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

a matrix multiplication

1 matrix-vector product

Idea: Let's define matrix multiplication so that we can move parentheses this way:

$$\underbrace{\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \underset{\text{equal}}{\overset{\text{Should}}{=}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\right)$$

can we figure out what this matrix should be?

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ -2b_1+b_2 \\ b_3 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 \\ b_3 \\ -2b_1+b_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\longrightarrow$ Yes! This is the matrix product.

Now let's try to multiply 2×2 matrices:

Question: What should $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ equal?

Answer: We need to make sure: $\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\right)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} =$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \underbrace{\left(\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right)}_{\text{This is a vector} = \begin{bmatrix} b_1c_1 + b_2c_2 \\ b_3c_1 + b_4c_2 \end{bmatrix}} = \begin{bmatrix} a_1(b_1c_1+b_2c_2) + a_2(b_3c_1+b_4c_2) \\ a_3(b_1c_1+b_2c_2) + a_4(b_3c_1+b_4c_2) \end{bmatrix}$$

$$\underset{\text{Rearrange}}{=} \begin{bmatrix} (a_1b_1 + a_2b_3)c_1 + (a_1b_2 + a_2b_4)c_2 \\ (a_3b_1 + a_4b_3)c_1 + (a_3b_2 + a_4b_4)c_2 \end{bmatrix}$$

$$\underset{\substack{\text{Write as}\\ \text{matrix-vector}\\ \text{product}}}{=} \begin{bmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

This should be the matrix product $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

1st column: $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \underbrace{\begin{bmatrix} b_1 \\ b_3 \end{bmatrix}}_{\text{1st column of } B}$ 2nd column: $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \underbrace{\begin{bmatrix} b_2 \\ b_4 \end{bmatrix}}_{\text{2nd column of } B}$

Conclusion: For 2×2 matrices A, B:

$$AB = A \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 \end{bmatrix}$$

Columns of B Columns of AB

We compute matrix products by doing matrix-vector multiplication to each column of B. This works for $n \times n$ matrices too!

Definition of matrix multiplication:

$$AB = A \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{bmatrix}$$

columns of B columns of AB

Actually, A, B don't have to be square matrices, $n \times n$. We just need to make sure $A\vec{b}_1, A\vec{b}_2, \ldots, A\vec{b}_n$ are defined.

If B is $m \times n$, then each of $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n$ has m components.

$\uparrow$ # of rows $\quad$ $\uparrow$ # of columns

Then A needs to have m columns for $A\vec{b}_1, A\vec{b}_2, \dots, A\vec{b}_n$ to be defined.

If A is $k \times m$, then $A\vec{b}_1, \dots,$ have k components $\rightarrow$ AB has k rows.

$\uparrow$ # of rows

In general: $(k \times m \text{ matrix})(m \times n \text{ matrix}) = k \times n$ matrix

need to match; if they don't, then AB isn't defined.

Example $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & -2 \end{bmatrix}$

1st column of AB: $A\vec{b}_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(1)+2(2)+3(1) \\ 4(1)+5(2)+6(1) \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \end{bmatrix}$

2nd column of AB: $A\vec{b}_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(2)+2(-1)+3(-2) \\ 4(2)+5(-1)+6(-2) \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \end{bmatrix}$

So $AB = \begin{bmatrix} 8 & -6 \\ 20 & -9 \end{bmatrix}$

What about BA?

1st column: $\begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \\ -7 \end{bmatrix}$ $\quad$ 2nd column: $\begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 12 \\ -1 \\ -8 \end{bmatrix}$

3rd column: $\begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \\ -9 \end{bmatrix}$ $\quad$ So $BA = \begin{bmatrix} 9 & 12 & 15 \\ -2 & -1 & 0 \\ -7 & -8 & -9 \end{bmatrix}$

In this example, AB and BA are <u>not</u> the same. They don't even have the same size!

Even if A, B are square matrices, we <u>usually</u> get $AB \neq BA$.

<u>Example</u> $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

But: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ <u>Not</u> the same.

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

<u>Conclusion</u>: Matrix multiplication is <u>not</u> "commutative.":

$AB \neq BA$ usually (even if both products exist).

<u>But</u>, matrix multiplication <u>is</u> "associative":

$A(BC) = (AB)C$

You can move parentheses
(which means you can ignore them)

the columns of BC

Why? $A(BC) = A\left(B\left[\vec{c}_1\ \vec{c}_2 \cdots \vec{c}_n\right]\right) = A\left[B\vec{c}_1\ B\vec{c}_2 \cdots B\vec{c}_n\right]$

the columns of $A(BC)$

$= \left[A(B\vec{c}_1)\ A(B\vec{c}_2) \cdots A(B\vec{c}_n)\right] = \left[(AB)\vec{c}_1\ (AB)\vec{c}_2 \cdots (AB)\vec{c}_n\right]$

the columns of $(AB)C$

Definition of matrix multiplication was motivated by the goal of having $(AB)\vec{c} = A(B\vec{c})$

Technically, we only proved this when A, B are 2×2, but this property is still true for A, B any <u>compatible</u> sizes.

$= (AB)\left[\vec{c}_1\ \vec{c}_2 \cdots \vec{c}_n\right]$

$= (AB)C$ ✓