第二章习题解答

6（2）．错误。例如，$f(z) = |z|^2 = x^2 + y^2$, 则$f(z) = u(x,y) = x^2 + y^2, v(x,y) \equiv 0$, $\frac{\partial u}{\partial x}(0,0) = \frac{\partial v}{\partial y} = 0$, $\frac{\partial u}{\partial y}(0,0) = -\frac{\partial v}{\partial x} = 0$, 故$f(z)$在$z_0 = 0$处可导且$f'(0) = 0$，但易见$f(z)$在$z_0 \neq 0$处不可导，故$f(z)$处处不解析。

(3). 错误。例如，由上题的例子知$f(z) = |z|^2 = x^2 + y^2$在$z_0 = 0$可导且$f'(0) = 0$，但易见$f(z)$在$z_0 \neq 0$处不可导，故$f(z)$处处不解析，故$z_0 = 0$是$f(z)$的一个奇点，但$f(z)$在$z_0 = 0$可导。

(5). 错误。例如，$f(z) = \overline{z} = x - iy$, 则$u(x,y) = x, v(x,y) = -y \in C^1$, $\frac{\partial u}{\partial x}(= 1) \neq \frac{\partial v}{\partial y}(= -1)$。故$f(z)$处处不可导。

(6). 因$f(z) = u + iv$解析， 由Cauchy-Riemann 条件知当v为常数时$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$, 因而由$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \equiv 0$, 知$f(z) =$常数。

7．因$f(z)$解析， 知$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} =: u_x + iv_x$处处存在且有

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{0.1}$$

处处成立. 由$|f(z)|^2 = u^2 + v^2$, 得

$$\frac{\partial |f(z)|^2}{\partial x} = 2|f(z)|\frac{\partial |f(z)|}{\partial x} = 2\left[u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right],$$

$$\frac{\partial |f(z)|^2}{\partial y} = 2|f(z)|\frac{\partial |f(z)|}{\partial y} = 2\left[u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right].$$

以上两式除以 2，再平方后相加得

$$|f(z)|^2 \left[\left(\tfrac{\partial |f(z)|}{\partial x}\right)^2 + \left(\tfrac{\partial |f(z)|}{\partial y}\right)^2\right] =$$

$$\left(u^2u_x^2 + 2uvu_xv_x + v^2v_x^2 + u^2u_y^2 + 2uvu_yv_y + v^2v_y^2\right).$$

将等式(1)带入右边的等式中并利用$f'(z) = u_x + iv_x$, 化简后可得

$$|f(z)|^2 \left[\left(\frac{\partial |f(z)|}{\partial x}\right)^2 + \left(\frac{\partial |f(z)|}{\partial y}\right)^2\right] = |f(z)|^2|f'(z)|^2.$$

当$f(z)\equiv 0$时，所要证明的等式显然成立。当$f(z)\neq 0$时， 上式两边除以$|f(z)|^2$, 则得所要证的等式。

9. 由$x=r\cos\theta$, $y=r\sin\theta$, 知$u(x,y)=u(r\cos\theta,r\sin\theta)$, $v(x,y)=v(r\cos\theta,r\sin\theta)$, 因而有

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta,$$

及

$$\begin{aligned}\tfrac{\partial v}{\partial \theta} &= \tfrac{\partial v}{\partial x}\tfrac{\partial x}{\partial \theta}+\tfrac{\partial v}{\partial y}\tfrac{\partial y}{\partial \theta}=\tfrac{\partial v}{\partial x}(-r\sin\theta)+\tfrac{\partial v}{\partial y}r\cos\theta\\ &= r\left[\tfrac{\partial u}{\partial y}\sin\theta+\tfrac{\partial u}{\partial x}\cos\theta\right]=r\tfrac{\partial u}{\partial r}.\end{aligned}$$

这里用到了Cauchy-Riemann等式。同理可证第二个等式。

10(1). 这时$f(z)=u(x,y),v(x,y)\equiv 0$, 由$f(z)$解析，知$f'(z)=\frac{\partial u}{\partial x}$且由Cauchy-Riemann条件知$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\equiv 0$, 因而有$f'(z)\equiv 0$, 从而有$f(z)=$常数。

(2). 因$f(z)=u+iv$解析， 知$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, 又因$\overline{f}(z)=u-iv$解析， 知$\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y}$, 得$\frac{\partial u}{\partial x}=0$. 同理有$\frac{\partial v}{\partial x}=0$, 故有$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=0$, 因而有$f(z)=$常数。

(3), (4).

当$f(z)\equiv 0$时， (3)显然正确。 （这时(4)中的$argf(z)$不定义。）

当$f(z)\neq 0$时，令$h(z)=lnf(z)=ln|f(z)|+iargf(z)=u+iv$. 则易见$h(z)$解析， 且有$u=ln|f(z)|$, $v=argf(z)$. 由6(6)可知， 当$argf(z)=v$ 为常数时， $u=ln|f(z)|$ 也是常数， 从而有$h(z)=lnf(z)$为常数， 即$f(z)=e^{h(z)}$为常数。

同理可证当$u=ln|f(z)|$为常数时， $v=argf(z)$为常数， 从而有$h(z)$为常数，即$f(z)=e^{h(z)}$为常数。

15. $Ln(-i)=ln(-i)+2k\pi i=ln|-i|+iarg(-i)+2k\pi i=ln1-i\frac{\pi}{2}+2k\pi i=\frac{(4k-1)\pi}{2}i$, $k\in Z$. 主值为$-i\frac{\pi}{2}$.

$Ln(-3+4i)=ln(-3+4i)+2k\pi i=ln|-3+4i|+iarg(-3+4i)+2k\pi i=ln5+i(\pi-arctan\frac{4}{3})+2k\pi i$, $k\in Z$. 主值为$ln5+iarg(-3+4i)=\ln 5+i(\pi-\arctan\frac{4}{3})$.

18. $e^{1-\frac{\pi i}{2}}=ee^{-\frac{\pi i}{2}}=e[\cos\left(-\frac{\pi}{2}\right)-i\sin\left(\frac{\pi}{2}\right)]=-ie.$

$exp[\frac{1+i\pi}{4}]=e^{\frac{1}{4}}e^{\frac{\pi i}{4}}=e^{\frac{1}{4}}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)=e^{\frac{1}{4}}\frac{(1+i)}{\sqrt{2}}.$

$3^i=e^{iLn3}=e^{i(\ln 3+2k\pi i)}=e^{iln3-2k\pi}=e^{-2k\pi}\left(\cos ln3+i\sin ln3\right),\quad k\in Z.$

$$
\begin{aligned}
(1+i)^i &= e^{iLn(1+i)} = e^{i[ln(1+i)+2k\pi i]} \\
&= e^{-2k\pi + i[ln|1+i| + iarg(1+i)]} \\
&= e^{-2k\pi + i[ln\sqrt{2} + i\frac{\pi}{4}]} \\
&= e^{-(2k\pi+\frac{\pi}{4})} e^{i\frac{ln2}{2}} \\
&= e^{-(2k+\frac{1}{4})\pi}\left[\cos(\tfrac{ln2}{2}) + i\sin(\tfrac{ln2}{2})\right], \quad k\in Z.
\end{aligned}
$$

补充题：由

$$
\begin{aligned}
\sin z &= \tfrac{e^{iz}-e^{-iz}}{2i} = \tfrac{e^{i(x+iy)}-e^{-i(x+iy)}}{2i} \\
&= \tfrac{e^{ix}e^{-y}-e^{-ix}e^{y}}{2i} = \tfrac{(\cos x+i\sin x)e^{-y}-(\cos x-i\sin x)e^{y}}{2i} \\
&= \tfrac{e^{y}+e^{-y}}{2}\sin x + i\tfrac{(e^{y}-e^{-y})\cos x}{2}.
\end{aligned}
$$

由此可得

$$
Re\{\sin z\} = \frac{e^y+e^{-y}}{2}\sin x, \qquad Im\{\sin z\} = \frac{(e^y-e^{-y})\cos x}{2}.
$$

令

$$
\frac{e^y+e^{-y}}{2}\sin x = A, \qquad \frac{(e^y-e^{-y})\cos x}{2} = B. \tag{0.2}
$$

由方程(0.2), 可知，

(I). 当$B=0$, $|A|\le 1$时，可取$y=0$, $\sin x=A$, 这时方程$\sin z=\sin x=A$ 有无穷多组解：$z=x_k=2k\pi+\arcsin A,\ y=0,\ k\in Z.$

(II). 当$B=0$, $|A|>1$时, 可取$\cos x=0$, 这时有$|\sin x|=1$. 由方程(0.2)的第一式可知，$\sin x=1$,若$A>1$; $\sin x=-1$, 若$A<-1$. 这时，方程(0.2)变为求方程

$$
\frac{e^y+e^{-y}}{2} = |A| > 1. \tag{0.3}
$$

令$g(y)=\frac{e^y+e^{-y}}{2},\ y\in(-\infty,+\infty)$,因为$g(y)$是偶函数，只需讨论$y\ge 0$即可. 因为$g'(y)>0, \forall y>0$. $g(0)=1<|A|, \lim_{y\to+\infty} g(y)=+\infty$, 由单调递增函数理论，可知存在唯一$y_A>0$, 使得$g(\pm y_A)=|A|>1$.

这时方程(0.2) 有无穷多组解：$z=z_k=(2k\pm\frac{1}{2})\pi\pm iy_A, k\in Z.$

(III). 当$B \neq 0$时.由方程(0.2), 消去x可得，

$$\frac{4A^2}{(e^y+e^{-y})^2}+\frac{4B^2}{(e^y-e^{-y})^2}=1. \tag{0.4}$$

令$h(y)=\frac{4A^2}{(e^y+e^{-y})^2}+\frac{4B^2}{(e^y-e^{-y})^2}$, 则$h(y)$是偶函数，故只需讨论当$y>0$时即可. (由(0.2), $y \neq 0$,) 因$y \to 0^+$时, $h(y) \to +\infty$, 而$y \to +\infty$时，$h(y) \to 0$. 由$h(y)$的连续性，可知存在$y_B>0$, 使得$h(\pm y_B)=1$.

将$y=y_B$代入方程(0.2)中的任意一式，再利用(0.4), 可得$|\sin x| \leq 1$, $|\cos x| \leq 1$, $x=x_k$, $k \in Z$, $y=y_B$, $z=z_k=x_k \pm i y_B$, $k \in Z$有无穷多解.□