This shows x_3 is allowed to be <u>any</u> real number. But once we've picked x_3, we must take $x_1 = -x_3$ and $x_2 = -x_3$.

So all solutions look like $\begin{bmatrix} -c \\ -c \\ c \end{bmatrix} = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ (c can be any real number)

For example, if we pick $c=1$: tells us that

$$(-1)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>Note</u>: Today we <u>briefly</u> looked at some of the major ideas in solving linear equations. We will look at these ideas in more detail later, so don't worry if you didn't completely understand everything today.

<u>Chapter 2</u>: Solving Linear Equations

Today, begin a <u>systematic</u> study of systems of linear equations.

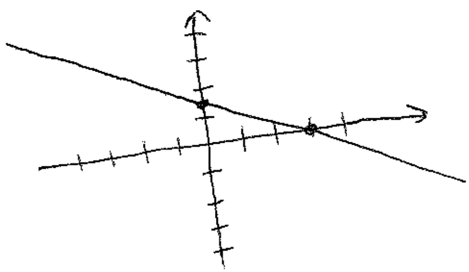
<u>Algebraically</u>: Linear equations involve <u>1st</u> powers of the variables, as well as constants:

$x + 2y + 3z = 4$ is linear, $\sin x + xy + y^3 = 1$ is <u>not</u>

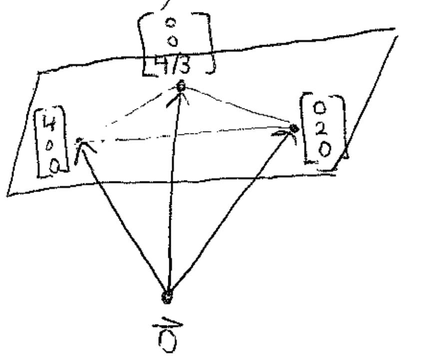
<u>Geometrically</u>: Linear equations describe lines in the plane, planes in space, "hyperplanes" in higher-dimensional space.

$x + 2y = 3$

$\left(y = -\frac{1}{2}x + \frac{3}{2}\right)$

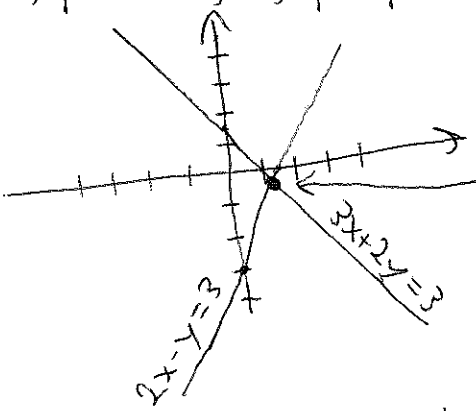


$x + 2y + 3z = 4$



Linear systems can involve more than one linear equation. They describe how lines, planes, hyperplanes intersect.

$$\begin{cases} 2x - y = 3 \\ 3x + 2y = 3 \end{cases}$$



This point has coordinates (x,y) that satisfy both equations.

What's the best way to find the point of intersection?

We will use elimination:

$$2x - y = 3$$
$$3x + 2y = 3$$

(optional first step) Divide Eqn. 1 by 2

$$x - \tfrac{1}{2}y = \tfrac{3}{2}$$
$$3x + 2y = 3$$

Eliminate x from Eqn. 2: Eqn. 2 $\rightarrow$ Eqn. 2 $-$ 3 (Eqn. 1)

$$x - \tfrac{1}{2}y = \tfrac{3}{2}$$
$$0x + \tfrac{7}{2}y = -\tfrac{3}{2}$$

Solve for y

$$x - \tfrac{1}{2}y = \tfrac{3}{2}$$
$$y = -\tfrac{3}{7}$$

Eliminate y in Eqn. 1: Eqn. 1 $\rightarrow$ Eqn 1 + $\tfrac{1}{2}$(Eqn. 2)

$$x + 0y = \tfrac{3}{2} + \tfrac{1}{2}\left(-\tfrac{3}{7}\right) = \tfrac{21-3}{14} = \tfrac{9}{7}$$
$$y = -\tfrac{3}{7}$$

Matrix form: $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

The coefficient matrix

Divide Row 1 by 2

$$\begin{bmatrix} 1 & -\tfrac{1}{2} \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \tfrac{3}{2} \\ 3 \end{bmatrix}$$

Row 2 $\rightarrow$ Row 2 $-$ 3 (Row 1)

$$\begin{bmatrix} 1 & -\tfrac{1}{2} \\ 0 & \tfrac{7}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \tfrac{3}{2} \\ -\tfrac{3}{2} \end{bmatrix}$$

Row 2 $\rightarrow$ $\tfrac{2}{7}$(Row 2)

$$\begin{bmatrix} 1 & -\tfrac{1}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \tfrac{3}{2} \\ -\tfrac{3}{7} \end{bmatrix}$$

Row 1 $\rightarrow$ Row 1 + $\tfrac{1}{2}$(Row 2)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9/7 \\ -3/7 \end{bmatrix}$$

So the solution is $(x,y) = \left(\frac{9}{7}, -\frac{3}{7}\right)$

The final matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the 2×2 identity matrix. It has the property $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

Doesn't change the vector, just like $1 \cdot x = x$ for scalars.

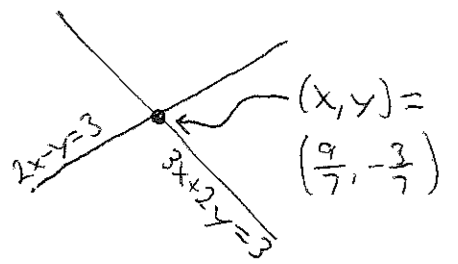
We have two different ways of thinking about linear equations:

"Row picture"
(lines/planes/etc. intersecting in space)

1st row of a matrix

$$\begin{cases} 2x - y = 3 \\ 3x + 2y = 3 \end{cases}$$

2nd row of a matrix

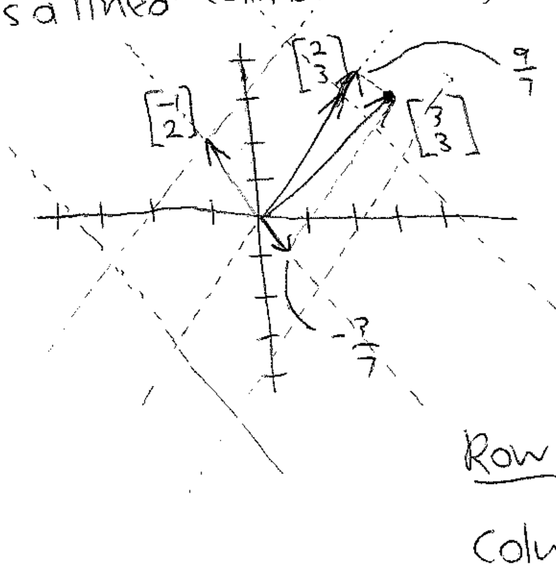


$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

"Column picture"
(writing a vector as a linear combination)

$$x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

columns of a matrix



$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \frac{9}{7}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \left(-\frac{3}{7}\right)\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Row picture advantage: Geometrically natural

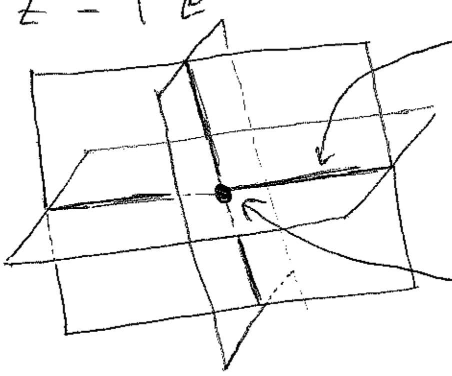
Column picture advantage: Easier to imagine in higher dimensions.

In some sense, the matrix-vector product equation unifies these two pictures.

Now for a 3x3 example:

Row picture: $\begin{cases} 2x - y + z = 1 \\ -x + 2y - z = 1 \\ x - y + 2z = 1 \end{cases}$ ← These are equations of planes in 3-dim. space

Harder to visualize:



Two planes in 3-dim. space (usually) intersect in a line.

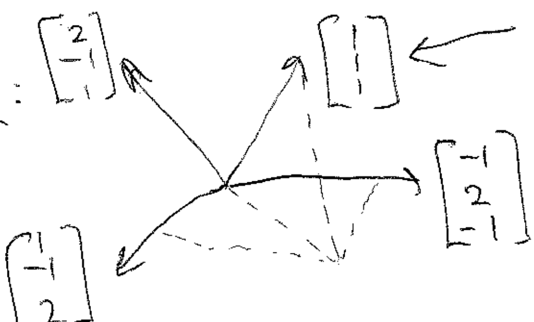
Three planes in 3-dim. space usually intersect in a single point (the solution to the system of linear equations.)

Matrix form of the equations:

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Coefficient matrix

Column picture:



Can we write this as a linear combination of the other 3?

Elimination Procedure:

$$2x - y + z = 1$$
$$-x + 2y - z = +1$$
$$x - y + 2z = 1$$

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Eliminate x from 2nd and 3rd eqns:

2nd eqn → 2nd + $\frac{1}{2}$(1st)

3rd eqn → 3rd − $\frac{1}{2}$(1st)

Row 2 → Row 2 + $\frac{1}{2}$(Row 1)

Row 3 → Row 3 − $\frac{1}{2}$(Row 1)

$$2x - y + z = 1$$
$$0x + \tfrac{3}{2}y - \tfrac{1}{2}z = \tfrac{3}{2}$$
$$0x - \tfrac{1}{2}y + \tfrac{3}{2}z = \tfrac{1}{2}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Eliminate y from 3rd eqn (use 2nd eqn, not 1st)

3rd eqn $\rightarrow$ 3rd + $\frac{1}{3}$(2nd)

Row 3 $\rightarrow$ Row 3 + $\frac{1}{3}$(Row 2)

$$2x - y + z = 1$$
$$0x + \tfrac{3}{2}y - \tfrac{1}{2}y = \tfrac{3}{2}$$
$$0x + 0y + \tfrac{4}{3}z = 1$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & -1/2 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 1 \end{bmatrix}$$

An upper triangular matrix.

Solve for z

$$2x - y + z = 1$$
$$\tfrac{3}{2}y - \tfrac{1}{2}z = \tfrac{3}{2}$$
$$\boxed{z = \tfrac{3}{4}}$$

We could continue eliminating z from 2nd eqn., and z and y from 1st eqn. Or, we can solve for y and x using substitution:

$$\frac{3}{2}y = \frac{3}{2} + \frac{1}{2}z = \frac{3}{2} + \frac{1}{2}\left(\frac{3}{4}\right) = \frac{15}{8}$$

$$\rightarrow \boxed{y = \frac{2}{3}\left(\frac{15}{8}\right) = \frac{5}{4}}$$

$$\rightarrow 2x = 1 + y - z = 1 + \frac{5}{4} - \frac{3}{4} = \frac{3}{2} \longrightarrow \boxed{x = \frac{3}{4}}$$

So the 3 planes intersect at the single point $\boxed{(x,y,z) = \left(\frac{3}{4}, \frac{5}{4}, \frac{3}{4}\right)}$

Check: $2\left(\frac{3}{4}\right) - \frac{5}{4} + \frac{3}{4} = \frac{3}{2} - \frac{1}{2} = 1$ ✓

$-\frac{3}{4} + 2\left(\frac{5}{4}\right) - \frac{3}{4} = \frac{10-6}{4} = 1$ ✓

$\frac{3}{4} - \frac{5}{4} + 2\left(\frac{3}{4}\right) = -\frac{1}{2} + \frac{3}{2} = 1$ ✓

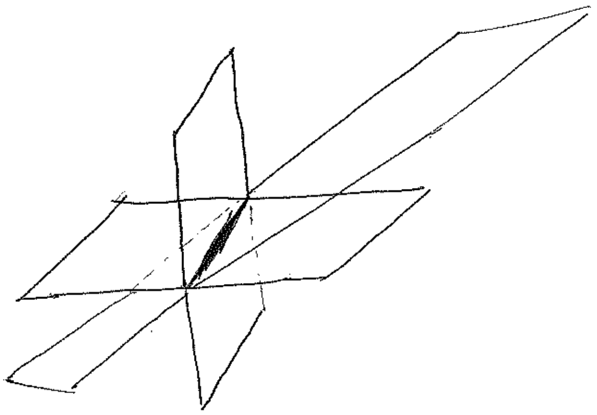
Column interpretation: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a linear combination of

$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$:

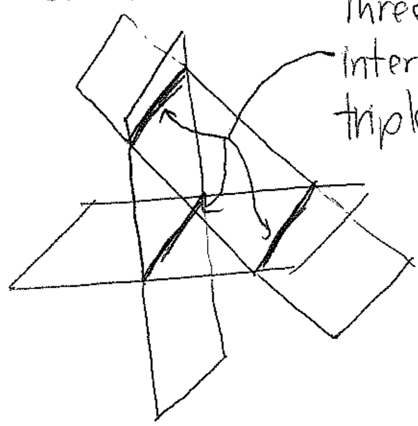
$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{3}{4}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \frac{5}{4}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Important Warning: 3 planes in 3-dim. don't always intersect in a single point.

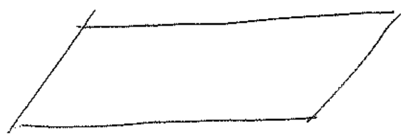
They might intersect in a line:



They might not have any common intersection:



They might even intersect in a plane, if all three are really the same plane in disguise:



$$\begin{cases} x+2y+3z=4 \\ 2x+4y+6z=8 \\ -x-2y-3z=-4 \end{cases}$$

This means there are three possibilities when we solve a system of linear equations: No solution, exactly one solution, or infinitely many solutions (a whole line, plane, etc.)

(It will never have just 2 or 3 solutions!)