$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 6 & 8 & 9 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ 0 & 0 & 1 & 3 & -2b_1+b_2 \\ 0 & 0 & -1 & -3 & -3b_1+b_3 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ 0 & 0 & 1 & 3 & -2b_1+b_2 \\ 0 & 0 & 0 & 0 & -5b_1+b_2+b_3 \end{array}\right]$$

Keeps track of which linear combinations of the rows we are creating.

Tells us that $-5 \text{ Row } 1 + \text{Row } 2 + \text{Row } 3 = \vec{0}^T$

So $\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$ is in $N(A^T)$

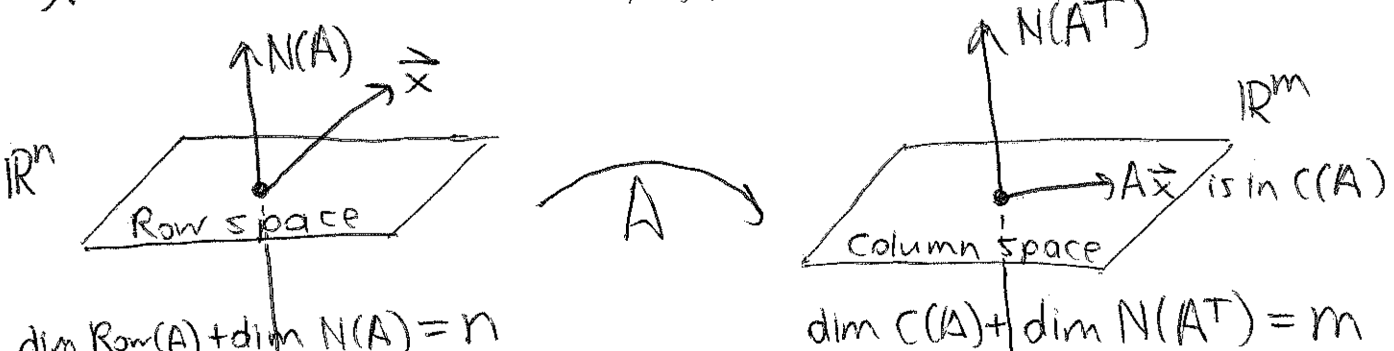
Since $\dim N(A^T) = m - r = 3 - 2 = 1$, we just need this one non-zero vector in $N(A^T)$ to get a basis.

If all 3 rows of A had been independent, we would have gotten $N(A^T) = \{\vec{0}\}$.

Chapter 4: Orthogonality

"Big picture" from last week:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, bend left, "m\times n \text{ matrix } A"] & \mathbb{R}^m \\ \vec{x} \arrow[r] & A\vec{x} \end{tikzcd}$$



$\dim \text{Row}(A) + \dim N(A) = n$

$\dim C(A) + \dim N(A^T) = m$

$\dim C(A)$ and $\dim \text{Row}(A)$ are rank of A, r.

I drew these subspaces perpendicular to each other.

Are they really? Yes!

Claim 1: Every vector in $N(A)$ is perpendicular (or orthogonal) to every vector in $R(A)$.

Why? $\vec{x}$ in $N(A)$ means $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, so

$$\underbrace{(\text{row}\,1)\cdot\vec{x}=0,\ (\text{row}\,2)\cdot\vec{x}=0,\ \ldots,\ (\text{row}\,m)\cdot\vec{x}=\overset{\cancel{\rightarrow}}{0}}_{\text{Entries of } A\vec{x}}$$

So $\vec{x}\perp$ every row $\longrightarrow$ $\vec{x}\perp$ every linear combination of the rows.

Claim 2: Same with column and left null spaces.

Why? Switch A with A^T in Claim 1: $N(A^T)\perp R(A^T)$

$\rightarrow N(A^T)\perp C(A)$.

Another proof: Vectors $\vec{y}$ in $N(A^T)$ satisfy $A^T\vec{y}=0$

Vectors in $C(A)$ look like $A\vec{x}$.

These two kinds of vectors are perpendicular:

$$(A\vec{x})\cdot\vec{y} \underset{\uparrow}{=} (A\vec{x})^T\vec{y} = (\vec{x}^T A^T)\vec{y} = \vec{x}^T(A^T\vec{y}) = \vec{x}^T\vec{0} = 0.$$

This formula is why we care about transposes.

So $A\vec{x}\perp\vec{y}$ if y is in $N(A^T)$.

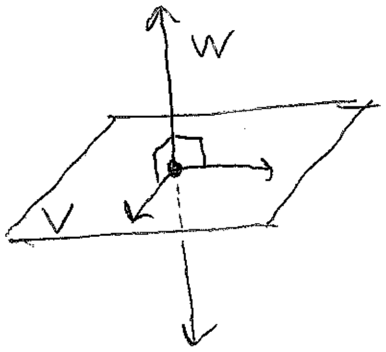
Definition: Two subspaces V and W in $\mathbb{R}^n$ are orthogonal if every vector in V is perpendicular to every vector in W.

Notation: $V\perp W$

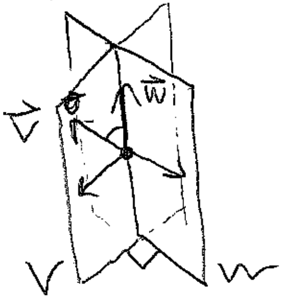
Examples: $N(A)\perp C(A^T)$ in $\mathbb{R}^n$, $N(A^T)\perp C(A)$ in $\mathbb{R}^m$.

Pictures:

Orthogonal subspaces:



Not quite orthogonal:



Some vectors in $\vec{v}$ are ⊥ to every vector in W, but not all

Important Fact: If $V \perp W$, then the only vector that is in both V and W is $\vec{0}$:

Why? If $\vec{v}$ is in both, it has to be ⊥ to itself:

$$\vec{v}^T\vec{v}=0, \text{ or } v_1^2+v_2^2+\dots+v_n^2=0$$

↗ only works if every $v_1, v_2, \dots, v_n = 0$, i.e., $\vec{v}=\vec{0}$.

Definition: Orthogonal complement of a subspace V:

$V^\perp$ = set of all vectors in $\mathbb{R}^n$ that are ⊥ to all vectors in V.

Is $V^\perp$ a subspace? Yes!

① $\vec{0}$ is in $V^\perp$: $\vec{0}\cdot(\text{every } \vec{v}) = 0$.

② If $\vec{x}, \vec{y}$ are in $V^\perp$, so is $\vec{x}+\vec{y}$: $(\vec{x}+\vec{y})\cdot(\text{every } \vec{v}) =$

$$\vec{x}\cdot\vec{v}+\vec{y}\cdot\vec{v}=0+0=0.$$

③ If $\vec{x}$ is in $V^\perp$, so is $c\vec{x}$: $(c\vec{x})\cdot(\text{every } \vec{v}) = c(\vec{x}\cdot\vec{v})$

$$=c(0)=0.$$

Example: $V=\text{span}\left\{\begin{bmatrix}1\\0\\2\\4\end{bmatrix}, \begin{bmatrix}2\\3\\3\\3\end{bmatrix}\right\}$ in $\mathbb{R}^4$. Find a basis for $V^\perp$.

(This is a basis for V.)

Note: If $\vec{x}\cdot$(every basis vector for V) $= 0$,

then $\vec{x}\cdot$(every linear combination of basis vectors) $= 0$ also,

so $\vec{x}\cdot$(every $\vec{v}$ in V) $= 0 \longrightarrow \vec{x}$ is in $V^\perp$.

So we just need to find all $\vec{x}$ such that:

$$\vec{x}\cdot\begin{bmatrix}1\\0\\2\\4\end{bmatrix}=0 \quad\text{and}\quad \vec{x}\cdot\begin{bmatrix}2\\3\\3\\3\end{bmatrix}=0$$

Or, $$\underbrace{\begin{bmatrix}1&0&2&4\\2&3&3&3\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

So, $V = \text{Row}(A)$ and $V^\perp = N(A)$!

Basis = special solutions of $A\vec{x}=\vec{0}$.

$$\begin{bmatrix}1&0&2&4\\2&3&3&3\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&2&4\\0&3&-1&-5\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&2&4\\0&1&-1/3&-5/3\end{bmatrix}$$

$x_1 = -2x_3 - 4x_4$

$x_2 = \frac{1}{3}x_3 + \frac{5}{3}x_4$

x_3, x_4 free

$$\longrightarrow \vec{x} = \begin{bmatrix}-2x_3-4x_4\\\frac{1}{3}x_3+\frac{5}{3}x_3\\x_3\\x_4\end{bmatrix} = x_3\begin{bmatrix}-2\\1/3\\1\\0\end{bmatrix} + x_4\begin{bmatrix}-4\\5/3\\0\\1\end{bmatrix}$$

These 2 vectors will form a <u>basis</u> for $V^\perp$.

Note that: $\dim V + \dim V^\perp = 2+2 = 4 = \dim$ of $\mathbb{R}^4$.

This always works (i.e. $\dim V + \dim V^\perp = n$ if V is in $\mathbb{R}^n$).

Also, this example illustrates the:

"<u>Fundamental Theorem of Linear Algebra, Part 2</u>":

For any $m\times n$ matrix A, $N(A) = C(A^T)^\perp$ (in $\mathbb{R}^n$)

$N(A^T) = C(A)^\perp$ (in $\mathbb{R}^m$)

rowspace,

Nice consequence: If $\vec{b}$ is in $C(A)$, then there is a unique solution to $A\vec{x}_r = \vec{b}$ such that $\vec{x}_r$ comes from the row space.

Why? If we have two solutions: $A\vec{x}_r = \vec{b}$, $A\vec{x}_r' = \vec{b}$, where $\vec{x}_r, \vec{x}_r'$ are both in $C(A^T)$, then $A(\vec{x}_r - \vec{x}_r') = \vec{0}$

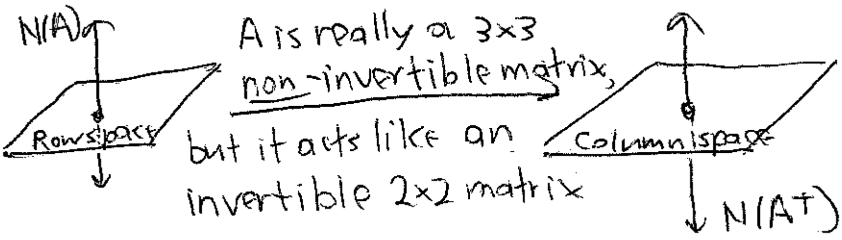
This vector is in both $C(A^T)$ and $N(A)$, so it's $\vec{0}$ by "important fact."

Since $N(A) \perp C(A^T)$, must have $\vec{x}_r - \vec{x}_r' = \vec{0}$, i.e., $\vec{x}_r = \vec{x}_r'$

solution is actually unique, if $\vec{x}_r$ comes from $C(A^T)$.

This means:

If you ignore $N(A)$ and $N(A^T)$, then A behaves like an invertible matrix: takes vectors from Row space to Column space in an ~~unique~~ invertible way:



here's an invertible 2x2 matrix inside A.

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

(not invertible)

Now to understand an $m \times n$ matrix A: we need a good basis for $\mathbb{R}^n$ that is related to A:

Claim: We can get a basis for $\mathbb{R}^n$ by combining bases for null and column spaces:

$$\text{Basis} = \{ \underbrace{\text{basis vectors for } C(A^T)}_{r \text{ of them}}, \underbrace{\text{basis vectors for } N(A)}_{n-r \text{ of them}} \}$$

get right number of vectors, n, but are they lin. ind?

Suppose (lin.comb. of row basis vectors) + (lin.comb. of N(A) basis) = $\vec{0}$.

call this $\vec{x}_r$

call this $\vec{x}_n$

Then $\vec{x}_r+\vec{x}_n=\vec{0} \rightsquigarrow \vec{x}_r=-\vec{x}_n$

↳ Shows $\vec{x}_r$ is in both $C(A^T)$ and $N(A)$

The only way to get lin.comb = 0 is to set all coefficients = 0

→ $\boxed{\vec{x}_r=\vec{0}}$ since $C(AT)\perp N(A)$

→ $\boxed{\vec{x}_n=\vec{0}}$ as well.

→ linearly independent and a basis (since we have right number of vectors for a basis).

This proves the claim, which implies:

spanning set property

Every $\vec{x}$ in $\mathbb{R}^n$ = lin. comb. of basis vectors

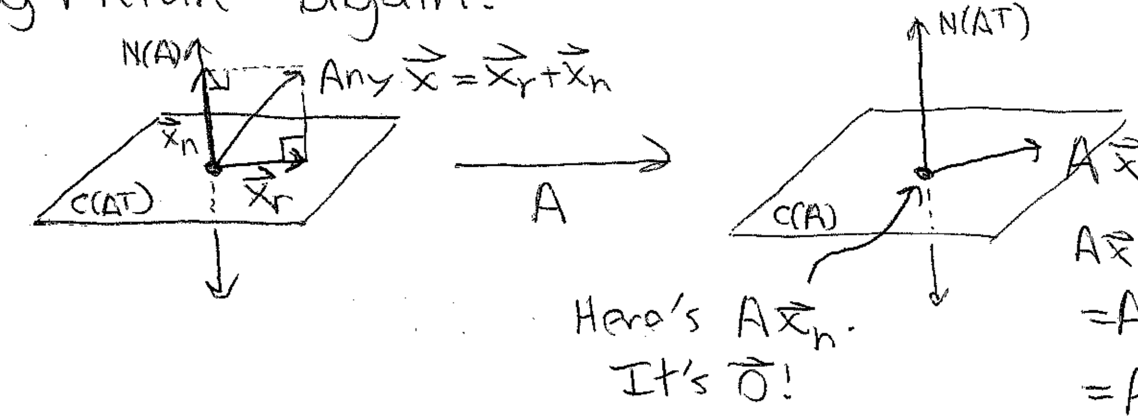
$$=\underbrace{\text{lin. comb. of row space vectors}}_{\vec{x}_r}+\underbrace{\text{lin comb. of null space vectors}}_{\vec{x}_n}$$

→ $\vec{x}=\vec{x}_r+\vec{x}_n$

Called "orthogonal projection" of $\vec{x}$ onto N(A).

called "orthogonal projection" of $\vec{x}$ onto C(AT)

"Big Picture" again:



$A\vec{x}$, also $A\vec{x}_r$:

$$A\vec{x}=A(\vec{x}_r+\vec{x}_n)$$
$$=A\vec{x}_r+A\vec{x}_n$$
$$=A\vec{x}_r+\vec{0}=A\vec{x}_r$$

Problem 4.1.12 Draw this picture for $A=\begin{bmatrix}1&-1\\0&0\\0&0\end{bmatrix}$ and $\vec{x}=\begin{bmatrix}2\\0\end{bmatrix}$

$C(A) = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\end{bmatrix}\right) = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$

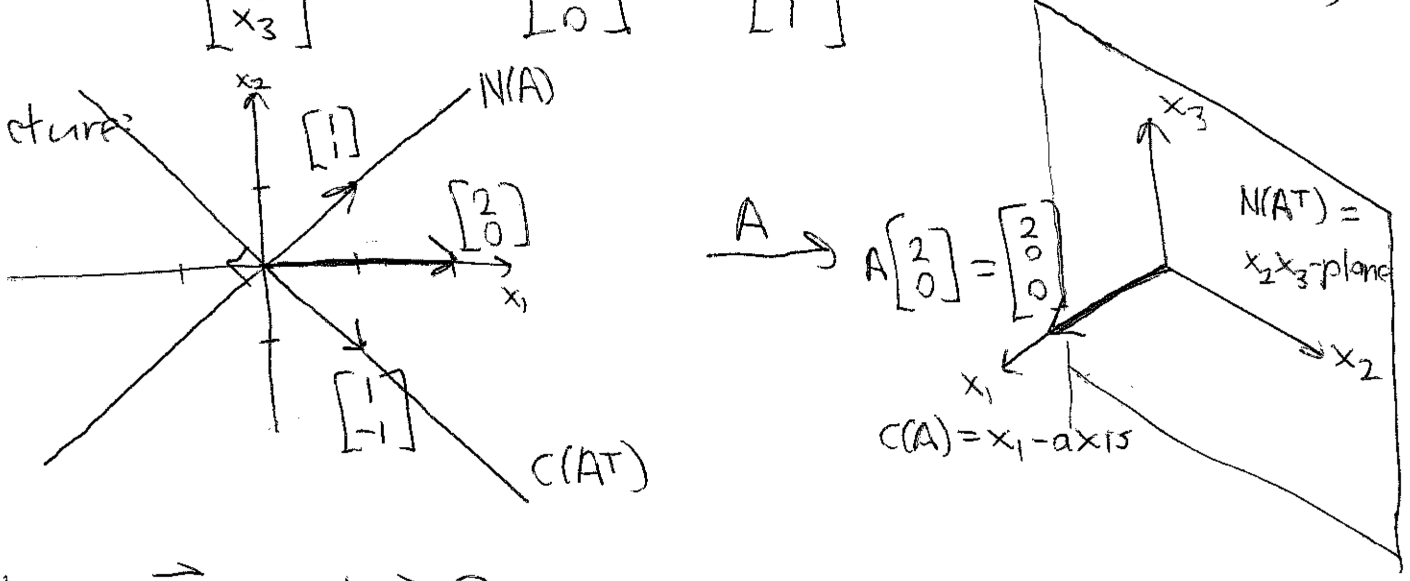
$N(A)$: $\begin{bmatrix}1&-1\\0&0\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ means $x_1 = x_2 \rightsquigarrow N(A) = \text{all } x_2\begin{bmatrix}1\\1\end{bmatrix}$

$C(A^T) = \text{span}\left(\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right) = \text{span}\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)$ perpendicular!

$N(A^T)$: $\begin{bmatrix}1&0&0\\-1&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ means $x_1 = 0 \rightsquigarrow$

$N(A^T) = \text{all}\begin{bmatrix}0\\x_2\\x_3\end{bmatrix} = \text{all } x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$ ($= x_2x_3$-plane)

The picture:



What are $\vec{x}_r$ and $\vec{x}_n$? Need to write

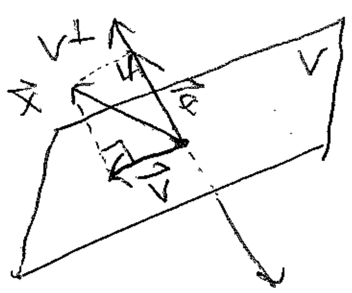
$\begin{bmatrix}2\\0\end{bmatrix} = c\underbrace{\begin{bmatrix}1\\-1\end{bmatrix}}_{\vec{x}_r} + d\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\vec{x}_n}$ Looks like $c = d = 1$ will work.

Section 4.2 Projections

Big linear algebra problem: Figure out how to write $\vec{x} = \vec{x}_r + \vec{x}_n$

To say another way: Figure out how to project $\vec{x}$ onto a subspace (such as N(A)).

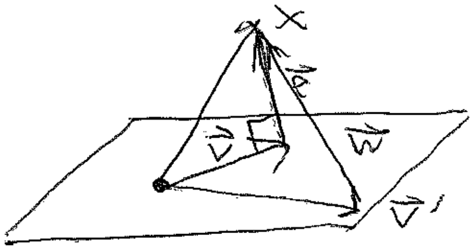
This means: Write $\vec{x} = \underset{\substack{\nearrow \\ \text{in } V}}{\vec{v}} + \underset{\substack{\nwarrow \\ \text{in } V^\perp}}{\vec{e}}$



Why call it $\vec{e}$? We are trying to approximate $\vec{x}$ by a vector in V.

$\vec{v}$ is the approximation

$\vec{e}$ is the error of the approximation.

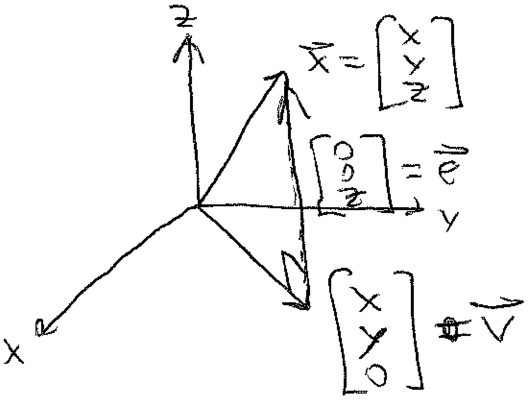


If we did it differently:

We could also write $\vec{x} = \vec{v}' + \vec{w}$ where $\vec{w}$ is not in $V^\perp$, but then $\|\vec{w}\| > \|\vec{e}\|$, worse error.

So what we want to do is orthogonal projection (minimize error).

Very simple example: Project $\vec{x}$ in $\mathbb{R}^3$ onto the xy-plane.



Error vector $\vec{e}$ is just $\begin{bmatrix}0\\0\\z\end{bmatrix}$, which is $\perp$ to $\begin{bmatrix}x\\y\\0\end{bmatrix}$ in the xy-plane.

Size of the error is $\left\|\begin{bmatrix}0\\0\\z\end{bmatrix}\right\| = |z|$.

Turns out, there is a matrix that does this projection!

$$\begin{matrix} \vec{x} & = & \vec{v} & + & \vec{e} \\ \nearrow & & \uparrow & & \nwarrow \\ \begin{bmatrix}x\\y\\z\end{bmatrix} & & \begin{bmatrix}x\\y\\0\end{bmatrix} & & \begin{bmatrix}0\\0\\z\end{bmatrix} \end{matrix}$$

$$\begin{bmatrix}x\\y\\0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\vec{v} = P\vec{x}$$

To project $\vec{x}$ onto $V^\perp$ instead (z-axis) use $I-P$:

$$\begin{bmatrix}0\\0\\z\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\vec{e} = (I-P)\vec{x}$$

P and $I-P$ have two special properties:

① Symmetric: $P = P^T$

② $P^2 = P$ (projecting twice is the same as projecting once)

Any matrix with these two properties projects vectors onto some subspace!