$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} U$$

(Reverse ~~order~~ the row operations to get inverses)

$$\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & -1 & 1 \end{bmatrix} = L$$

So
$$\underset{A}{\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & -1 & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}}$$

(Check.)

Last time: Started LU decomposition

$n \times n$ matrix $\to A = LU$

lower triangular (L), upper triangular (U)

How? $A \xrightarrow[\text{Same as: multiply A by lower triangular elimination matrices}]{\text{Eliminate lower left variables}} U$ (upper triangular)

$$A \to L_1 A \to L_2 L_1 A \to \cdots \to \underbrace{L_m \cdots L_2 L_1}_{\text{not } L \text{ yet}} A = U$$

Then: $A = \underbrace{L_1^{-1} L_2^{-1} \cdots L_m^{-1}}_{\text{This is } L!} U$

4x4 Example:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \xrightarrow[\text{Row 2} - \text{Row 1},\ \text{Row 3} - \text{Row 1},\ \text{Row 4} - \text{Row 1}]{\text{Eliminate lower left:}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{bmatrix}$$

Or: multiply by
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\text{Row 4} - 3\text{Row 1}]{\text{Row 3} - 2\text{Row 1}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{bmatrix} \xrightarrow{\text{Row} - 3\text{Row 3}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or: Multiply by
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}$$

Or: multiply by
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

Upper triangular; this is U

$$A \to \begin{bmatrix}1&0&0&0\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{bmatrix}A \longrightarrow \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-2&1&0\\0&-3&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{bmatrix}A \longrightarrow$$

$$\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-3&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-2&1&0\\0&-3&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{bmatrix}}_{\text{Not } L \text{ yet; this is } L^{-1}}A = U$$

Solve for A: $$A = \underbrace{\left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-3&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-2&1&0\\0&-3&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{bmatrix}\right)^{-1}}\, U$$

Switch order when you invert a product $\to$ $$\underbrace{\begin{bmatrix}1&0&0&0\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-2&1&0\\0&-3&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-3&1\end{bmatrix}^{-1}}$$

Reverse the row operations $\to$ $$\underbrace{\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\1&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\0&3&0&1\end{bmatrix}}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&3&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&0&1\end{bmatrix} \qquad \begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}$$

This is L!

Conclusion: $$\underbrace{\begin{bmatrix}1&1&1&1\\1&2&3&4\\1&3&6&10\\1&4&10&20\end{bmatrix}}_{A} = \underbrace{\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&0&1\end{bmatrix}}_{U}$$

LU decomposition works whenever you can do elimination $A \longrightarrow U$ <u>without</u> switching rows. How to find the entries of L more efficiently?

Diagonal entries of L: all 1's

Below-diagonal entries: Put ℓ_{ij} in row i, column j if you did the operation $\text{Row } i \to \text{Row } i - \ell_{ij} \text{ Row } j$

Example: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \xrightarrow[\text{Row 3} - \underbrace{\boxed{1}}_{\ell_{31}} \text{Row 1}]{\text{Row 2} - \overbrace{\boxed{1}}^{\ell_{21}} \text{Row 1}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{bmatrix} \xrightarrow[\ell_{32}]{\text{Row 3} - \boxed{3} \text{Row 2}}$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = U \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

Check that this is correct: $LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} = A$ ✓

Summarize LU decomposition algorithm:

$A \xrightarrow[\text{elimination}]{\text{lower left}} U \rightsquigarrow A = LU$

L: lower triangular, U: upper triangular

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & 1 & \cdots & 0 \\ & \boxed{\ell_{ij}} & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

Comes from operation
$\text{Row } i \to \text{Row } i - \ell_{ij} \text{ Row } j$
note the minus sign

Warning 1: Formula for L only works if you do the elimination operations in the right order:

Eliminate Column 1 variables using Row 1 first,
Eliminate Column 2 variables using Row 2 next,
⋮

Warning 2: LU decomposition only works if you can go from A to U without switching rows.

Now: What is the use of LU decomposition?

It turns one hard system of equations $A\vec{x}=\vec{b}$ into two easy triangular systems:

$$A\vec{x}=\vec{b} \rightsquigarrow L\underbrace{U\vec{x}}_{\text{call this vector } \vec{y}}=\vec{b}$$

First solve $L\vec{y}=\vec{b}$ for $\vec{y}$, then solve $U\vec{x}=\vec{y}$ for $\vec{x}$

same $\vec{y}$

Check that this works: Does $A\vec{x}$ really equal $\vec{b}$?

$$A\vec{x}=(LU)\vec{x}=L\underbrace{(U\vec{x})}_{\text{This is } \vec{y}}=\underbrace{L\vec{y}}_{\text{This is } \vec{b}}=\vec{b} \quad \checkmark$$

Example Solve
$$\begin{cases} x_1+x_2+x_3+x_4=1 \\ x_1+2x_2+3x_3+4x_4=0 \\ x_1+3x_2+6x_3+10x_4=-1 \\ x_1+4x_2+10x_3+20x_4=0 \end{cases}$$

Already know: $A=\underbrace{\begin{bmatrix} 1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&0&1 \end{bmatrix}}_{U}$

First solve $L\vec{y}=\vec{b}$:
$$\begin{aligned} y_1 &= 1 \\ y_1+y_2 &= 0 \to y_2=-1 \\ y_1+2y_2+y_3 &= -1 \to y_3=-1-1-2(-1)=0 \\ y_1+3y_2+3y_3+y_4 &= 0 \to y_4=-1-3(-1)=2 \end{aligned}$$

Now solve $U\vec{x}=\vec{y}$:
$$\begin{aligned} x_1+x_2+x_3+x_4 &= 1 \to 1-5-(-6)-2=0 \\ x_2+2x_3+3x_4 &= -1 \to x_2=-1-2(-6)-3(2)=5 \\ x_3+3x_4 &= 0 \to x_3=-3(2)=-6 \\ x_4 &= 2 \end{aligned}$$

so $\vec{x}=\begin{bmatrix} 0\\5\\-6\\2 \end{bmatrix}$

Note: The two triangular systems are easy to solve, but we still have to work hard to find L and U.

Solving systems using LU works best when you have a lot of systems with the same A:

$$A\vec{x}_1 = \vec{b}_1,\ A\vec{x}_2 = \vec{b}_2,\ A\vec{x}_3 = \vec{b}_3, \ldots, A\vec{x}_{1000} = \vec{b}_{1000}$$

Even though you have 1000 systems, you only need to find LU once.

Some statistics from the textbook: (if A is $n\times n$)

To find LU, it usually takes about $\frac{2}{3}n^3$ arithmetic operations

multiplication and addition

To solve a triangular system usually takes about $2n^2$ arithmetic operations

So what if you solve 1000 systems individually?

$$\underbrace{A \xrightarrow[\frac{2}{3}n^3]{} U \xrightarrow[2n^2]{\text{solve triangular}} \text{solution}}_{1000 \text{ times}} \longrightarrow 2000\left(\frac{n^3}{3} + n^2\right) \text{ operations}$$

Using LU: $A \to U$ once $\rightsquigarrow \frac{2}{3}n^3$

$$\underbrace{\text{Two triangular systems} \rightsquigarrow 4n^2}_{1000 \text{ times}} \to \frac{2}{3}n^3 + 4000n^2 \text{ operations}$$

If n is big, then n^2 is much less than n^3, so solving by LU is almost 1000 times faster:

$$\frac{2000\left(\frac{n^3}{3}+n^2\right)}{\frac{2}{3}n^3+4000n^2} = \frac{\frac{2000n^3}{3}\left(1+\frac{3}{n}\right)}{\frac{2}{3}n^3\left(1+\frac{6000}{n}\right)} = 1000\cdot \underbrace{\frac{1+3/n}{1+6000/n}}_{\substack{\text{Goes to } 1 \\ \text{as } n\to\infty}}$$

Transposes and Permutation

One final matrix operation: transpose

↑ switch rows and columns of a matrix

$m \times n$ matrix $A \xrightarrow[\text{into columns}]{\text{turn rows}} n \times m$ matrix A^T

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

(i,j)-entry of A^T $= (j,i)$-entry of A

$$\begin{array}{c} \\ j\text{th row} \rightarrow \\ \\ \end{array} \begin{bmatrix} & | & \\ — & A_{ji} & — \\ & | & \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} & | & \\ — & (A^T)_{ij} & — \\ & | & \end{bmatrix} \leftarrow i\text{th row now}$$

↑ ith column $\qquad\qquad$ jth column now

Another viewpoint: "Flip A over the diagonal" to get A^T:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Rules $(A+B)^T = A^T + B^T$

$(AB)^T = ??$

$(A^{-1})^T = ??$

$$\begin{aligned} (A+B)^T_{ij} &= (A+B)_{ji} \\ &= A_{ji} + B_{ji} \\ &= A^T_{ij} + B^T_{ij} = (A^T + B^T)_{ij} \end{aligned}$$

(i,j)-entry of $(AB)^T$ $= (j,i)$-entry of AB

$= (j\text{th row of } A) \cdot (i\text{th column of } B)$

$= (j\text{th column of } A^T) \cdot (i\text{th row of } B^T)$

$= (i\text{th row of } B^T) \cdot (j\text{th column of } A^T)$

$= (i,j)$-entry of $B^T A^T$.

so $\boxed{(AB)^T = B^T A^T}$ (switch order, just like for inverse)

Now we can find $(A^{-1})^T$:

$$AA^{-1} = I \longrightarrow (AA^{-1})^T = I^T \longrightarrow (A^{-1})^T A^T = I$$

$$A^{-1}A = I \longrightarrow (A^{-1}A)^T = I^T \longrightarrow \underbrace{A^T (A^{-1})^T = I}$$

$(A^{-1})^T$ looks like inverse of A^T.

So : $\boxed{(A^{-1})^T = (A^T)^{-1}}$ This means that A^T is invertible if A is, and A^T isn't invertible if A isn't.

Example

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \xrightarrow{\text{Inverse}} A^{-1} = \frac{1}{\underbrace{(1)(4)-(1)(3)}_{1}} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$\downarrow$ Transpose $\qquad\qquad$ $\downarrow$ Transpose

$$A^T = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \xrightarrow{\text{Inverse}} (A^T)^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} = (A^{-1})^T$$