Linear Independence: A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent if none is a linear combination of the others.

Same thing as: The only solution to $x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n = \vec{0}$ is $x_1 = x_2 = \ldots = x_n = 0$.

Put into a matrix: $A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$

Linearly independent if only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

$\longrightarrow N(A) = \{\vec{0}\}$ $\longleftarrow$ In general, $N(A) = \{\vec{0}\}$ means that $A\vec{x} = \vec{b}$ never has more than one solution (no free variables)

But if $A\vec{x} = \vec{0}$ has non-zero solutions, then the vectors are dependent. (In this case, there will be free variables.

Notice: If you have too many vectors, they will have to be dependent.

$$m \text{ rows} \left\{ \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ & & & \end{bmatrix}}_{n \text{ columns}} \right.$$

If $n > m$ $\rightarrow$ not every column can have a leading 1

$\rightarrow$ have to be free variables

$\rightarrow$ $A\vec{x} = \vec{0}$ has to have non-zero solutions.

Example $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix} \right\}$, 3 vectors in $\mathbb{R}^2$, $3 > 2$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & 6 \end{bmatrix} = R$$

pivot variables free variable

You can use R to read off a linear combination:

Notice that $\begin{bmatrix} -13 \\ 6 \end{bmatrix} = (-13)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Same relation works for original vectors: $\begin{bmatrix} -1 \\ 5 \end{bmatrix} = (-13)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 6\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, or $(13)\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 6\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Why does the same relation work?

$$\underbrace{\begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & 6 \end{bmatrix}\begin{bmatrix} 13 \\ -6 \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\text{Easy relation}} \underset{\text{Elimination}}{\overset{\text{Reverse elimination}}{\rightleftarrows}} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 13 \\ -6 \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Elimination doesn't change the solutions to a linear system:

So $A\vec{x}=\vec{0} \longleftrightarrow R\vec{x}=\vec{0}$.

So $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$ are independent, just like $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

But $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ is a linear combination of the other two.

Spanning Sets Remember: span $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ = set of all linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

Example: $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \rightsquigarrow C(A) = \text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$

$N(A) = \text{span}$ ("special solutions")

Examples Is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix}\right\}$ a spanning set for $\mathbb{R}^2$?

Translation: Can we write every $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ as a linear combination:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \overset{??}{=} x_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

Or: Does $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ have a solution?

Check: $\left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 1 & 3 & 5 & b_2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 0 & 1 & 6 & -b_1+b_2 \end{array}\right] \longrightarrow \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & -13 & 3b_1-2b_2 \\ 0 & 1 & 6 & -b_1+b_2 \end{array}\right]}$

No row of 0's means solution always exists (infinitely many, actually)

So yes, these vectors do span $\mathbb{R}^2$.

But do $\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\3\\5\end{bmatrix}$ span $\mathbb{R}^3$?

Translation: Is there always a solution for $\begin{bmatrix}1 & 1\\2 & 3\\-1 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$??

Check: $\left[\begin{array}{cc|c}1 & 1 & b_1\\2 & 3 & b_2\\-1 & 5 & b_3\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & 1 & b_1\\0 & 1 & -2b_1+b_2\\0 & 6 & b_1+b_3\end{array}\right] \xrightarrow[\text{Row3}-6\text{Row2}]{\text{Row1}-\text{Row2}} \underbrace{\left[\begin{array}{cc|c}1 & 0 & 3b_1-b_2\\0 & 1 & -2b_1+b_2\\0 & 0 & 13b_1-6b_2+b_3\end{array}\right]}$

solution exists only when $13b_1-6b_2+b_3=0$.

If $13b_1-6b_2+b_3 \neq 0$, then you cannot write $\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} = x_1\begin{bmatrix}1\\2\\-1\end{bmatrix} + x_2\begin{bmatrix}1\\3\\5\end{bmatrix}$

So these vectors do not span $\mathbb{R}^3$. This example illustrates a general rule:

$\boxed{\text{If } n<m, \text{ then } n \text{ vectors cannot span } \mathbb{R}^m.}$

We also saw:

$\boxed{\text{If } n>m, \text{ then } n \text{ vectors in } \mathbb{R}^m \text{ cannot be linearly independent.}}$

So: if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ spans $\mathbb{R}^m$ and is linearly independent, then we must have $m=n$.

Definition: A basis of a vector space (could be $\mathbb{R}^n$, or a subspace of $\mathbb{R}^n$, or something else) is a set of vectors that is a spanning set and is linearly independent.

Then, the dimension of a vector space is the number of vectors in a basis.

For $\mathbb{R}^n$: We saw that a basis (a linearly independent spanning set) has exactly n vectors. This means the dimension of $\mathbb{R}^n = n$ (as it should!)

Important Problem: A vector space usually has a lot of different bases. How do we know that every basis of any vector space has the same number of vectors. If they could have different numbers, definition of dimension doesn't make sense.

Important Fact: Every basis of a vector space does have the same number of vectors.

For $\mathbb{R}^m$, this comes from matrix shapes: $A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$

If n is too big: A is wide and short $\longrightarrow$ free variables in R

$\longrightarrow \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ not independent.

If n is too small: A is narrow and tall $\longrightarrow$ row of 0's in R

$\longrightarrow \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ don't span.

Same idea works in general, but we need to create the right matrix A:

For example, what if $\{\vec{v}_1, \vec{v}_2\}$ and $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ are two bases of the same vectorspace (not $\mathbb{R}^n$!) Let's prove that this isn't possible.

In fact, since $\{\vec{v}_1, \vec{v}_2\}$ is a spanning set, we can write:

$$\vec{w}_1 = a_{11}\vec{v}_1 + a_{21}\vec{v}_2$$
$$\vec{w}_2 = a_{12}\vec{v}_1 + a_{22}\vec{v}_2 \quad \rightsquigarrow \quad A = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}}_{\text{Too wide, has to have a non-zero nullspace.}} \quad \rightsquigarrow \quad A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$
$$\vec{w}_3 = a_{13}\vec{v}_1 + a_{23}\vec{v}_2$$

a non-zero vector in $N(A)$

Then: $x_1\vec{w}_1 + x_2\vec{w}_2 + x_3\vec{w}_3 = x_1(a_{11}\vec{v}_1 + a_{21}\vec{v}_2) + x_2(a_{12}\vec{v}_1 + a_{22}\vec{v}_2) + x_3(a_{13}\vec{v}_1 + a_{23}\vec{v}_2)$

$$= (\underbrace{a_{11}x_1 + a_{12}x_2 + a_{13}x_3}_{\text{1st component of } A\vec{x}})\vec{v}_1 + (\underbrace{a_{21}x_1 + a_{22}x_2 + a_{23}x_3}_{\text{2nd component}})\vec{v}_2$$

$= 0\vec{v}_1 + 0\vec{v}_2 = \vec{0} \longleftarrow$ Means $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ are dependent!

Conclusion: If $\{\vec{v}_1, \vec{v}_2\}$ is a spanning set, then $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ can't be independent! They can't both be bases if they have different numbers of vectors!

In general: If $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ is a spanning set of a vector space and $n > m$, then any set $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ is dependent

use same kind of proof

Problem 3.4.16 Basis for subspaces of $\mathbb{R}^4$.

(a) Subspace = all vectors with all 4 components equal.

$$= \text{all vectors like } \begin{bmatrix} x \\ x \\ x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Every vector in S is a multiple of this one, so it's a spanning set for S. It's also linearly independent since a single non-zero vector always forms an independent set.

Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ $\left(\left\{ \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix} \right\} \text{ also works, so does } \left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \right\} \right)$

(b) S = all vectors whose components add to 0.

$$= \text{all } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ with } x_1 + x_2 + x_3 + x_4 = 0$$

$$= \text{all } \begin{bmatrix} -x_2 - x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Every vector in S is a linear combination of these 3 (so it's a spanning set. Check independence:

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

no free variables so $N(A) = \{\vec{0}\}$ ✓

Basis: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) S = all vectors perpendicular to $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\1\\1\end{bmatrix}$

= all $\vec{x}$ such that $\begin{cases}1x_1+1x_2+0x_3+0x_4=0\\1x_1+0x_2+1x_3+1x_4=0\end{cases}$

= Null space of $A=\begin{bmatrix}1&1&0&0\\1&0&1&1\end{bmatrix}$

x_3 and x_4 are free variables

Elimination: $\begin{bmatrix}1&1&0&0\\1&0&1&1\end{bmatrix}\longrightarrow\begin{bmatrix}1&1&0&0\\0&-1&1&1\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&+1&+1\\0&1&-1&-1\end{bmatrix}$

$\begin{cases}x_1+x_3+x_4=0\\x_2-x_3-x_4=0\end{cases}\rightsquigarrow\begin{array}{l}x_1=-x_3-x_4\\x_2=x_3+x_4\\x_3,x_4=\text{anything}\end{array}$

So $S=N(A)=$ all $\vec{x}=\begin{bmatrix}-x_3-x_4\\x_3+x_4\\x_3\\x_4\end{bmatrix}=x_3\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-1\\1\\0\\1\end{bmatrix}$

The special solutions will be a <u>basis</u> for $N(A)$.