Last time: How do we find the dimension of a vectors space (like a subspace of $\mathbb{R}^n$)?

Answer: We find a basis: A set of vectors in the vector space that is linearly independent and a spanning set.

Then dimension = number of vectors in a basis

(Every vector space has a basis, and every basis has the same number of vectors, so this definition of dimension makes sense.)

Example: $\mathbb{R}^n$ has lots of different bases, but all of them have n vectors, so $\dim \mathbb{R}^n = n$.

How do you tell if a set of vectors in $\mathbb{R}^n$ is a basis?

$\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ in $\mathbb{R}^2$. Right number of vectors... but is it a basis?

Check independence: Does $x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ have non-zero solutions?

Check spanning: Does $x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ have a solution for all b_1, b_2?

Is every vector in $\mathbb{R}^2$ a linear combination of these two?

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 3 & 2 & b_2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -1 & -3b_1+b_2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -2b_1+b_2 \\ 0 & -1 & -3b_1+b_2 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & -2b_1+b_2 \\ 0 & 1 & 3b_1-b_2 \end{array}\right]$$

No row of 0's means solution always exists

Yes! Every $\vec{b}$ is a linear combination in a unique way:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = (-2b_1+b_2)\begin{bmatrix} 1 \\ 3 \end{bmatrix} + (3b_1-b_2)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Only one choice for x, y (given b_1, b_2)

For $\vec{b}=\vec{0}$, <u>only</u> solution is $x=0,\ y=0$.

$\longrightarrow$ means vectors are also <u>independent</u>, so they <u>are</u> a basis.

General test for deciding if $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$:

$$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \xrightarrow[\text{a matrix}]{\text{Put into}} A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \xrightarrow{\text{Elimination}} R$$

If $R = I$:
- <u>No</u> free variables $\longrightarrow A\vec{x}=\vec{0}$ has <u>unique</u> solution
  $\longrightarrow$ vectors are <u>independent</u>
- <u>No</u> row of 0's $\longrightarrow A\vec{x}=\vec{b}$ <u>always</u> has a solution
  $\longrightarrow$ vectors <u>span</u> $\mathbb{R}^n$.

$\longrightarrow$ So $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ <u>is</u> a basis.

If R is <u>not</u> I: Vectors are <u>not</u> a basis (<u>not</u> independent, <u>don't</u> span)

<u>Example:</u> $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}3\\4\\5\end{bmatrix} \right\}$ in $\mathbb{R}^3$

free variable

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R$$

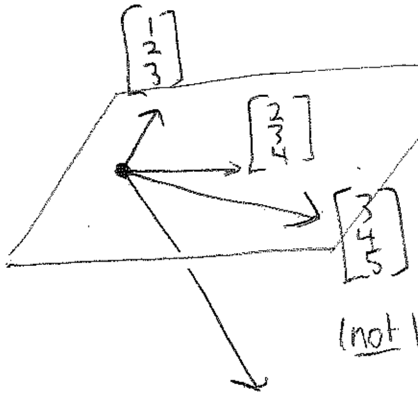
$R \neq I$, so vectors are dependent, <u>not</u> a basis

Null space: $x_1 - x_3 = 0$
$x_2 + 2x_3 = 0$

$$\rightarrow \vec{x} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

So $\begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}2\\3\\4\end{bmatrix} + \begin{bmatrix}3\\4\\5\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$,

or $\begin{bmatrix}3\\4\\5\end{bmatrix} = -\begin{bmatrix}1\\2\\3\end{bmatrix} + 2\begin{bmatrix}2\\3\\4\end{bmatrix}$



is in the same plane (<u>not</u> linearly independent)

Here's a vector that <u>isn't</u> a lin. comb. of the three, so the three <u>don't</u> span $\mathbb{R}^3$.

Now let's look at basis and dimension for subspaces of $\mathbb{R}^n$.

Important examples: $N(A)$ = all solutions to $A\vec{x}=\vec{0}$.

$C(A)$ = all $\vec{b}$ such that $A\vec{x}=\vec{b}$ has a solution.

Two more subspaces coming from $m\times n$ matrix A:

Row space = all linear combinations of the rows of A (a subspace of $\mathbb{R}^n$)

= all linear combinations of the columns of A^T

$= C(A^T)$

If $A=\begin{bmatrix}\vec{v}_1^T\\ \vec{v}_2^T\\ \vdots\\ \vec{v}_m^T\end{bmatrix}$, then $R(A)=\text{span}(\vec{v}_1,\vec{v}_2,\dots,\vec{v}_m)$

Left nullspace $= N(A^T)$ (subspace of $\mathbb{R}^m$)

= all solutions to $A^T\vec{y}=\vec{0}$

= all solutions to $(A^T\vec{y})^T=\vec{0}^T$

= all solutions to $\boxed{\vec{y}^TA=\vec{0}^T}$

Let's look at these 4 subspaces for $A=\begin{bmatrix}1&2&3&4\\2&4&7&11\\3&6&8&9\end{bmatrix}$

Best way is to do elimination and find R:

$$\begin{bmatrix}1&2&3&4\\2&4&7&11\\3&6&8&9\end{bmatrix}\xrightarrow[\text{Row 3}-3\text{Row 1}]{\text{Row 2}-2\text{Row 1}}\begin{bmatrix}1&2&\boxed{3}&4\\0&0&1&3\\0&0&\boxed{-1}&-3\end{bmatrix}\xrightarrow[\text{Row 3}+\text{Row 2}]{\text{Row 1}-3\text{Row 2}}\begin{bmatrix}1&2&0&-5\\0&0&1&3\\0&0&0&0\end{bmatrix}=R$$

Nullspace $N(A)$

$x_1+2x_2-5x_4=0$

$x_3+3x_4=0$

$x_1=-2x_2+5x_4$

$x_3=-3x_4$

x_2, x_4 are free

$$N(A)=\text{all}\begin{bmatrix}-2x_2+5x_4\\x_2\\-3x_4\\x_4\end{bmatrix}=x_2\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}5\\0\\-3\\1\end{bmatrix}$$

Two special solutions, one for each free variable

The special solutions are a spanning set for $N(A)$, and they are also linearly independent. $\longrightarrow$ they are a basis for $N(A)$

$\longrightarrow \dim N(A) = 2$

General fact: $\dim N(A) = \#$ of special solutions

$$= \# \text{ of free variables}$$

$$= \underset{\substack{\uparrow \\ \text{columns}}}{n} - \underset{\substack{\nwarrow \\ \text{rank}}}{r}$$

Column space $C(A)$

By definition, columns of A are a spanning set for $C(A)$:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 11 \\ 9 \end{bmatrix} \right\}$$

But they might not be independent. We have to get rid of dependent columns to get a basis. How to tell which columns are dependent?

Key fact: Row operations don't change relations between the columns. So columns of R are dependent exactly when the corresponding columns of A are dependent.

$$R = \begin{bmatrix} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Col 2 = 2 Col 1

Col 4 = (−5) Col 1 + 3 Col 3

{Col 1, Col 3} are independent

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 6 & 8 & 9 \end{bmatrix}$$

Col 2 = 2 Col 1

Col 4 = (−5) Col 1 + 3 Col 3

{Col 1, Col 3} are independent $\leftarrow$ basis

So one basis for $C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} \right\}$,

and $\dim C(A) = 2$.

<u>Warning</u>: $C(R)$ is <u>not</u> the same subspace as $C(A)$, though they have the same dimension.

Basis of $C(R) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. This is <u>not</u> a basis for $C(A)$.

<u>General fact</u>: Basis for $C(A)$ = pivot columns <u>of A</u> (the columns that will have a leading 1 in R).

So $\dim C(A) = \#$ leading 1's $= r$, rank of A.

<u>Note</u>: $\dim C(A) + \dim N(A) = r + (n-r) = \boxed{n}$

<u>Row space $R(A)$</u> One option: do elimination on A^T and find the pivot columns of A^T (since $R(A) = C(A^T)$).

<u>Or</u>: Use another <u>Key Fact</u>: Elimination on A <u>doesn't change</u> row space (so row space of A <u>is the same</u> as row space of R).

$$R = \begin{bmatrix} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} \cdots \end{bmatrix}$$

↳ Basis for Row(R) is just the two <u>non-zero</u> rows of R:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\} \longleftarrow \text{This is } \underline{\text{also}} \text{ a basis for Row}(A)$$

Why does this work? Elimination involves taking reversible linear combinations of the rows, so it doesn't change the <u>set of all</u> linear combinations.

<u>General fact</u>: Basis for $\overset{\text{Row}(A)}{C(A^T)}$ = Non-zero rows of R

$\dim C(A^T) = \#$ non-zero rows in R

$= \#$ leading 1's

$= r$, rank of A.

$C(A)$ and $C(A^T)$ have <u>same dimension</u>, but they are usually <u>different</u> subspaces.

More bases for $C(A^T)$: One you know that $\dim C(A^T) = r$, you can just pick r linearly independent rows of A itself (but you have to check that they are linearly independent.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 6 & 8 & 9 \end{bmatrix}, \quad r = 2 \rightsquigarrow 3 \text{ possible bases: } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \\ 11 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 8 \\ 9 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 2 \\ 4 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 8 \\ 9 \end{bmatrix} \right\}$$

Left null space $N(A^T)$ - we know its dimension:

Remember: $\dim N(A) + \dim C(A) = \#$ columns in A

Same for A^T: $\dim N(A^T) + \underbrace{\dim C(A^T)}_{r} = \underbrace{\# \text{ columns in } A^T}_{m}$

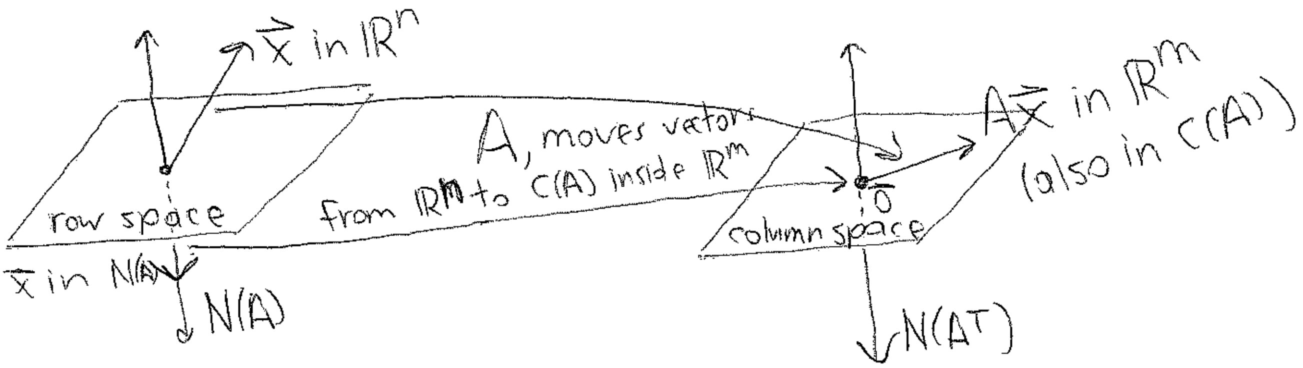
So $\dim N(A^T) = m - r$

Summary: "Fundamental Theorem of Linear Algebra, Part 1":

There are 3 important numbers for A: m, n, and r.

- r = dim of column space and dim of row space
- $N(A)$, $N(A^T)$ have dimensions $n-r$, $m-r$.

The "big picture" when $m = n = 3$, $r = 2$



Now: Can we actually find a basis for $N(A^T)$.

One option: Do elimination on A^T and find the null space as usual.

Different approach: Left null space = all $\vec{y}$ such that $\underbrace{\vec{y}^T A}_{\text{Linear combination of the rows of } A.} = \vec{0}^T$

We can find $N(A^T)$ by finding which linear combinations of the rows of A give $\vec{0}$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 6 & 8 & 9 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ 0 & 0 & 1 & 3 & -2b_1+b_2 \\ 0 & 0 & -1 & -3 & -3b_1+b_3 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ 0 & 0 & 1 & 3 & -2b_1+b_2 \\ 0 & 0 & 0 & 0 & -5b_1+b_2+b_3 \end{array}\right]$$

Keeps track of which linear combinations of the rows we are creating.

Tells us that $-5 \text{ Row } 1 + \text{Row } 2 + \text{Row } 3 = \vec{0}^T$

So $\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$ is in $N(A^T)$

Since $\dim N(A^T) = m - r = 3 - 2 = 1$, we just need this <u>one</u> non-zero vector in $N(A^T)$ to get a basis.

If all 3 rows of A had been independent, we would have gotten $N(A^T) = \{\vec{0}\}$.