Chapter 3 Vector Spaces and Subspaces

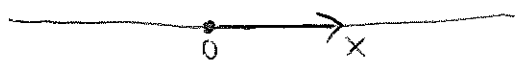
You can turn a set of <u>numbers</u> into a <u>vector</u>: $\{v_1, v_2, \dots, v_n\} \longrightarrow \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.

Now, we want to turn a set of <u>vectors</u> into a <u>vector space</u>:

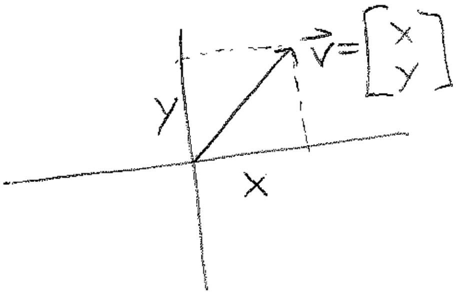
$\mathbb{R}^n$ = set of <u>all</u> vectors with n components, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.

That is, $\mathbb{R}^n$ = n-dimensional space.

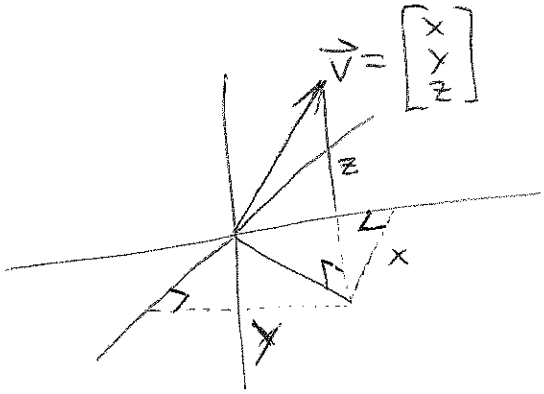
$\mathbb{R}^1$:



$\mathbb{R}^2$:

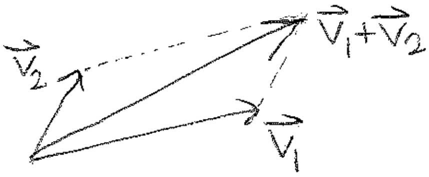


$\mathbb{R}^3$:



What can we do with vectors in $\mathbb{R}^n$?

We can add them:



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

We can multiply by scalars:

$$c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$$

Here are some properties of addition and scalar multiplication:

① $\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1$

② $\vec{v}_1 + (\vec{v}_2 + \vec{v}_3) = (\vec{v}_1 + \vec{v}_2) + \vec{v}_3$

③ There's a <u>zero vector</u>:

$$\vec{v} + \vec{0} = \vec{v}$$

④ There are <u>negative vectors</u>:

$$\vec{v} + (-\vec{v}) = \vec{0}$$

⑤ $1 \cdot \vec{v} = \vec{v}$

⑥ $(c_1 c_2) \cdot \vec{v} = c_1 \cdot (c_2 \cdot \vec{v})$

real number mult. Two scalar-vector multiplications

one scalar-vector multiplication

⑦ $c \cdot (\vec{v}_1 + \vec{v}_2) = c \cdot \vec{v}_1 + c \cdot \vec{v}_2$

⑧ $(c_1 + c_2) \cdot \vec{v} = c_1 \cdot \vec{v} + c_2 \cdot \vec{v}$

Now turn these properties of vectors in $\mathbb{R}^n$ into an abstract mathematical definition:

A <u>vector space</u> is a set of mathematical objects that we call "vectors." We call them vectors because they behave like vectors in $\mathbb{R}^n$:

1. We have a way to "add" them: vector + vector = another vector
2. We have a way to "multiply" them by scalars: (real number) · vector = another vector
3. Addition of vectors and scalar multiplication obey the 8 rules I wrote down for vectors in $\mathbb{R}^n$.

<u>Examples</u> (1) $\mathbb{R}^n$ is the basic example of a vector space.

(2) <u>Matrices</u>: you can add them <u>and</u> multiply by scalars:

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}+\begin{bmatrix}0&1\\2&3\end{bmatrix}=\begin{bmatrix}1&3\\5&7\end{bmatrix},\quad -2\cdot\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}-2&-4\\-6&-8\end{bmatrix}$$

Have to be the same size to add.

If you fix m and n, then the set of <u>all</u> $m\times n$ matrices is a vector space (sometimes called $\mathbb{R}^{m\times n}$).

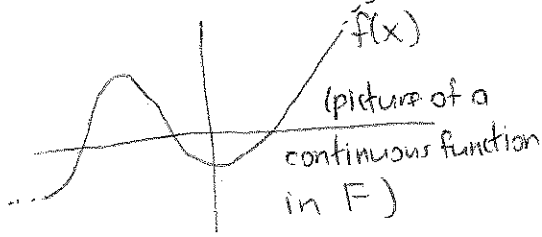
(3) The <u>zero vector space</u>: Has <u>one</u> vector in it, $\vec{0}$

$\vec{0}+\vec{0}=\vec{0}$, $c\cdot\vec{0}=\vec{0}$ (c any real number), $-\vec{0}=\vec{0}$

You could call this vector space $\mathbb{R}^0$.

Here is maybe the <u>most important</u> example, besides $\mathbb{R}^n$:

(4) Vector spaces of <u>functions</u>: F = set of all functions with domain $(-\infty,\infty)$



We can <u>add</u> functions:

$$(f+g)(x)=f(x)+g(x)$$

We can multiply by scalars: $(c\cdot f)(x)=c\,f(x)$

<u>Example</u> (Problem 3.1.6) <u>Linear combinations</u> of $f(x)=x^2$, $g(x)=5x$:

For example, $3\cdot f(x)-4\cdot g(x)=3x^2-4(5x)=3x^2-20x$.

You can think of a function $f(x)$ as being like a vector with infinitely many components: one component, $f(x)$, for <u>each</u> real number x.

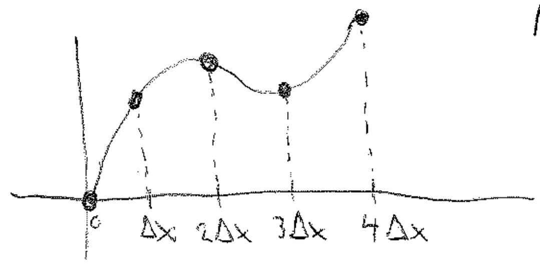
Thinking of functions as vectors in a vector space allows us to do calculus using linear algebra ideas!

In $\mathbb{R}^n$: $\vec{x} \xrightarrow{\text{matrix } A} A\vec{x}$ (another vector)

In F: $f(x) \xrightarrow[\text{derivative } d/dx]{} f'(x)$ (another function)

Calculus operations like derivatives behave like matrix-vector products ... but we won't do much of this now.

One example: "Discretizing" a function $f(x)$



Approximate $f(x)$ by its values at these points:

$$f(x) \approx \begin{bmatrix} f(0) \\ f(\Delta x) \\ f(2\Delta x) \\ f(3\Delta x) \\ f(4\Delta x) \end{bmatrix}$$

What about $f'(x)$?

$$f'(0) \approx \frac{f(\Delta x) - f(0)}{\Delta x}, \quad f'(\Delta x) \approx \frac{f(2\Delta x) - f(\Delta x)}{\Delta x}, \dots, f'(3\Delta x) \approx \frac{f(4\Delta x) - f(3\Delta x)}{\Delta x}$$

if Δx is small

(Let's not approximate $f'(4\Delta x)$ here)

$$\text{so} \begin{bmatrix} f'(0) \\ f'(\Delta x) \\ f'(2\Delta x) \\ f'(3\Delta x) \end{bmatrix} \approx \frac{1}{\Delta x} \begin{bmatrix} f(\Delta x) - f(0) \\ f(2\Delta x) - f(\Delta x) \\ f(3\Delta x) - f(2\Delta x) \\ f(4\Delta x) - f(3\Delta x) \end{bmatrix} = \frac{1}{\Delta x} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(\Delta x) \\ f(2\Delta x) \\ f(3\Delta x) \\ f(4\Delta x) \end{bmatrix}$$

Textbook calls this a "backward difference" matrix

Approximate $f'(x)$ by a matrix-vector multiplication

For now, we mainly want to understand matrices and linear equations $A\vec{x} = \vec{b}$. So the vectors we care about live in $\mathbb{R}^n$. But they may live in special <u>subsets</u> of $\mathbb{R}^n$ that we call <u>subspaces</u>:

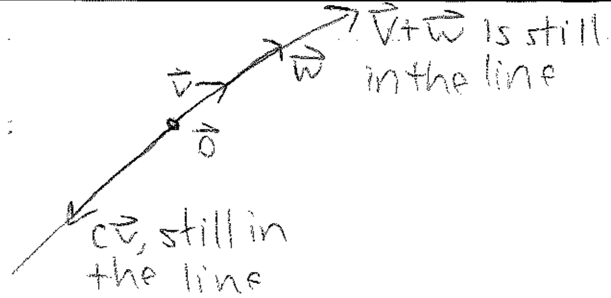
A <u>subspace</u> is a set of vectors in a vector space that satisfies:

1. The zero vector $\vec{0}$ is in the subspace.
2. If vectors $\vec{v}$ and $\vec{w}$ are in the subspace, so is $\vec{v} + \vec{w}$.
3. If $\vec{v}$ is in the subspace and c is a scalar, the $c\vec{v}$ is in the subspace

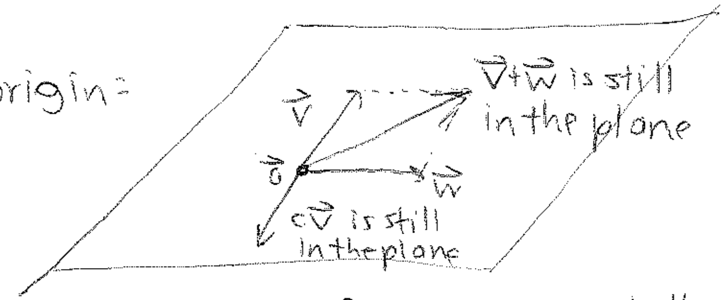
So a subspace is non-empty (it contains $\vec{0}$) and is "closed" under vector addition and scalar multiplication.

Subspaces in $\mathbb{R}^3$:

Lines through the origin:



Planes through the origin:

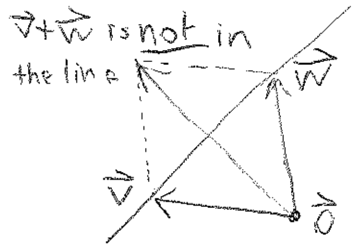


Two more subspaces that are kind of silly, but still work:

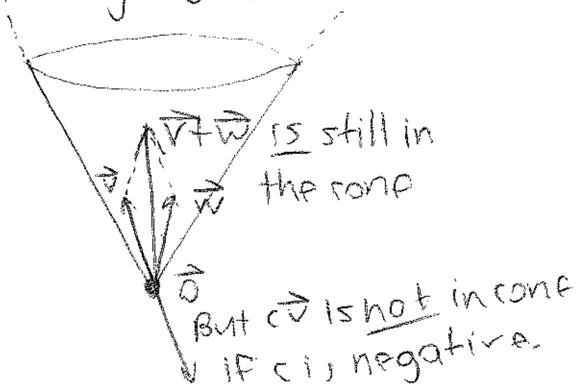
- All of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$
- The zero vector $\vec{0}$ by itself is a subspace.

Non-subspaces of $\mathbb{R}^3$

- A line that doesn't go through $\vec{0}$:



- A plane that doesn't go through $\vec{0}$ is also not a subspace.
- A cone of vectors:



Important comment: Subspaces of $\mathbb{R}^n$ are themselves vector spaces. They are sets of vectors that you can add and multiply by scalars, and the result is always another vector in the subspace.

This is the main reason we care about vector spaces right now: we need to understand subspaces that are related to a matrix A.

Important rule for subspaces: If $\vec{v}$ and $\vec{w}$ are both in a subspace, then so are all $\underbrace{\text{linear combinations}}_{c\vec{v}+d\vec{w}}$ of $\vec{v}$ and $\vec{w}$.

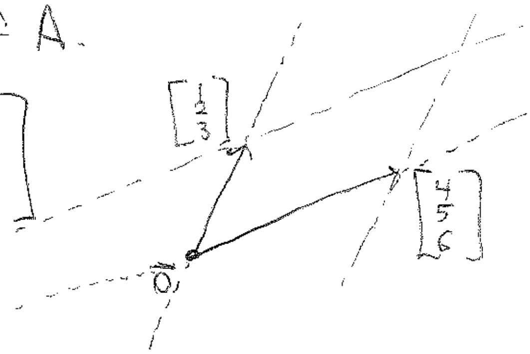
$$\begin{array}{ccccc} \vec{v} & \xrightarrow{\text{Rule } 3} & c\vec{v} & & \\ \text{and} & & \text{and} & \xrightarrow{\text{Rule } 2} & c\vec{v}+d\vec{w} \text{ also in subspace.} \\ \vec{w} & \xrightarrow{\text{Rule } 3} & d\vec{w} & & \\ \text{in subspace} & & \text{also in subspace} & & \end{array}$$

We can actually use linear combinations to create subspaces.

Here's an important example: Column space of a matrix.

If A is a $m \times n$ matrix, its column space $C(A)$ = set of all linear combinations of the columns of A.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \quad C(A) = \text{all } c\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$



$C(A)$ = plane sitting inside $\mathbb{R}^3$ ← # rows of A

plane ← 2 independent columns

Why is this a subspace?

① Contains $\vec{0}$: $\vec{0} = 0\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

② Closed under addition: $\left(c_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d_1\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right) + \left(c_2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d_2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right)$

$$= (c_1+c_2)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (d_1+d_2)\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \leftarrow \text{still a linear combination}$$

③ Closed under scalar multiplication:

$$e\left(c\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right) = (ec)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (ed)\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \leftarrow \text{still a linear combination.}$$

Why is $C(A)$ important? It tells us when we can find solutions to linear equations $A\vec{x} = \vec{b}$

A: $m \times n$ matrix; $\vec{x}$: in $\mathbb{R}^n$; $\vec{b}$: in $\mathbb{R}^m$

Vector form of the equation: $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}$

columns of A

$$\downarrow$$

$$\underbrace{x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n}_{\text{linear combination, in } C(A)} = \vec{b}$$

If there's a solution, the $\vec{b}$ has to be a vector in $C(A)$. If there's no solution, then $\vec{b}$ isn't in $C(A)$:

So $A\vec{x} = \vec{b}$ has a solution $\Leftrightarrow$ $\vec{b}$ is a vector in $C(A)$, the column space.

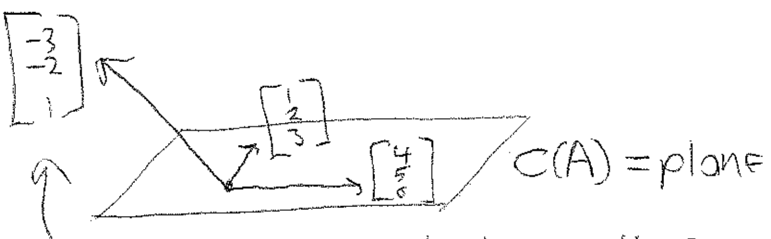
Example $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ again. $(-2)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ ← This is a vector in $C(A)$

So the system $\begin{cases} x + 4y = 2 \\ 2x + 5y = 1 \\ 3x + 6y = 0 \end{cases}$ has a solution. It's $(x, y) = (-2, 1)$

What about $\begin{cases} x + 4y = -3 \\ 2x + 5y = -2 \\ 3x + 6y = 1 \end{cases}$?? $\left[\begin{array}{cc|c} 1 & 4 & -3 \\ 2 & 5 & -2 \\ 3 & 6 & 1 \end{array}\right] \xrightarrow[\text{Row 3} - 3\text{Row 1}]{\text{Row 2} - 2\text{Row 1}}$

$$\left[\begin{array}{cc|c} 1 & 4 & -3 \\ 0 & -3 & 4 \\ 0 & -6 & 10 \end{array}\right] \xrightarrow{\text{Row 3} - 2\text{Row 2}} \left[\begin{array}{cc|c} 1 & 4 & -3 \\ 0 & -3 & 4 \\ 0 & 0 & 2 \end{array}\right] \rightsquigarrow \begin{cases} x + 4y = -3 \\ -3y = 4 \\ \underbrace{0y = 2}_{\text{inconsistent equation}} \end{cases}$$

No solution $\longrightarrow$ $\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$ is not a vector in $C(A)$



This vector "sticks out" of the plane somehow.