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Homework 13
Solutions

6.2.7 Two linearly independent eigenvectors $\longrightarrow$ diagonalizable

$$A = X\Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{bmatrix}\left(-\frac{1}{2}\right)\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

any eigenvalues λ_1, λ_2 are possible

$$= \underbrace{\frac{1}{2}\begin{bmatrix} \lambda_1+\lambda_2 & \lambda_1-\lambda_2 \\ \lambda_1-\lambda_2 & \lambda_1+\lambda_2 \end{bmatrix}}_{\text{This is one correct answer.}}$$

Another correct answer:

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$ for any a, b real numbers.
(here $a = \frac{\lambda_1+\lambda_2}{2}$, $b = \frac{\lambda_1-\lambda_2}{2}$)

6.2.9 (a) $G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k$
$G_{k+1} = G_{k+1}$

$$\longrightarrow \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$$

$$\begin{vmatrix} \frac{1}{2}-\lambda & \frac{1}{2} \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0 \longrightarrow (\lambda-1)(\lambda+\tfrac{1}{2}) = 0 \longrightarrow \boxed{\lambda = 1, -\tfrac{1}{2}}$$

For $\lambda = 1$: $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow x_1 = x_2 \longrightarrow \boxed{\vec{x} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

For $\lambda = -\frac{1}{2}$: $\begin{bmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow x_1 = -\frac{1}{2}x_2 \longrightarrow \boxed{\vec{x} = x_2\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}}$

(b) $$A^n = X\Lambda^n X^{-1} = \begin{bmatrix} 1 & -1/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & (-1/2)^n \end{bmatrix}\begin{bmatrix} 1 & -1/2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & (-1/2)^{n+1} \\ 1 & (-1/2)^n \end{bmatrix}\frac{2}{3}\begin{bmatrix} 1 & 1/2 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{2}{3}\begin{bmatrix} 1-(-1/2)^{n+1} & 1/2+(-1/2)^{n+1} \\ 1-(-1/2)^n & 1/2+(-1/2)^n \end{bmatrix}.$$ so $$\lim_{n\to\infty} A^n = \frac{2}{3}\begin{bmatrix} 1 & 1/2 \\ 1 & 1/2 \end{bmatrix}.$$

(c) Note that $A^n \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} = \begin{bmatrix} G_{n+1} \\ G_n \end{bmatrix}$. So $\lim_{n\to\infty} G_n$ = 2nd component ②

of $\lim_{n\to\infty} A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 & 1/2 \\ 1 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix} \rightarrow \boxed{\lim_{n\to\infty} G_n = \frac{2}{3}}$

6.2.15 $X\Lambda^k X^{-1} = X\begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} X^{-1} \xrightarrow{k\to\infty} 0$ if $\lim_{k\to\infty} \lambda_i^k = 0$

for $i = 1, 2, \dots, n$. So every λ should have absolute value less than $\boxed{1}$.

For A_1: $\begin{vmatrix} 0.6-\lambda & 0.9 \\ 0.4 & 0.1-\lambda \end{vmatrix} = (0.6-\lambda)(0.1-\lambda) - 0.36 =$

$\lambda^2 - 0.7\lambda - 0.3 = (\lambda-1)(\lambda+0.3) = 0 \rightarrow \lambda = 1, -0.3$

$\lim_{k\to\infty} A_1^k \neq 0$.

For A_2: $\begin{vmatrix} 0.6-\lambda & 0.9 \\ 0.1 & 0.6-\lambda \end{vmatrix} = (0.6-\lambda)(0.6-\lambda) - 0.09 =$

$\lambda^2 - 0.12\lambda + 0.27 = (\lambda-0.9)(\lambda-0.3) \rightarrow \lambda = 0.9, 0.3$

$\lim_{k\to\infty} A_2^k = 0$.

6.2.16 $\lambda = 1$: $\begin{bmatrix} -0.4 & 0.9 \\ 0.4 & -0.9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow 4x = 9y \rightarrow \vec{x} = c\begin{bmatrix} 9 \\ 4 \end{bmatrix}$

$\lambda = -0.3$: $\begin{bmatrix} 0.9 & 0.9 \\ 0.4 & 0.4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x + y = 0 \rightarrow \vec{x} = c\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

eigenvector

So $A_1 = \begin{bmatrix} 9 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -0.3 \end{bmatrix}\begin{bmatrix} 9 & -1 \\ 4 & 1 \end{bmatrix}^{-1}$, $\lim_{k\to\infty} \Lambda^k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,

$\lim_{k\to\infty} A_1^k = \begin{bmatrix} 9 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{13}\begin{bmatrix} 1 & 1 \\ -4 & 9 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 9 & 0 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -4 & 9 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 9 & 9 \\ 4 & 4 \end{bmatrix}$

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6.2.30

(a) $(A-aI)(A-dI) = \begin{bmatrix} a-a & b \\ 0 & d-a \end{bmatrix}\begin{bmatrix} a-d & b \\ 0 & d-d \end{bmatrix}$

$= \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix}\begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓

(b) $A^2 - A - I = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓

6.3.4 $\frac{d}{dt}(v(t)+w(t)) = \frac{dv}{dt} + \frac{dw}{dt} = (w-v)+(v-w) = 0,$

so $v(t)+w(t) = \text{constant} = v(0)+w(0) = 40.$

If $\vec{u}(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$, then $\frac{d\vec{u}}{dt} = \begin{bmatrix} -v+w \\ v-w \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}}_{A}\vec{u}(t)$

Eigenvalues: $\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+1)^2 - 1 = \lambda^2 + 2\lambda = 0 \longrightarrow \lambda = 0, -2.$

Eigenvectors for $\lambda=0$: $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \vec{x} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda=-2$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \vec{x} = x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So $\vec{u}(t) = Ce^{0t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{-2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} C - De^{-2t} \\ C + De^{-2t} \end{bmatrix}$

Initial conditions: $\begin{cases} C - D = v(0) = 30 & \rightarrow 2C = 40 \rightarrow C = 20 \\ C + D = w(0) = 10 & \rightarrow 2D = -20 \rightarrow D = -10 \end{cases}$

So $\begin{cases} v(t) = 20 + 10e^{-2t} \\ w(t) = 20 - 10e^{-2t} \end{cases}$

At $t=1$: $v(1) = 20 + 10e^{-1} \approx 24$ At $t=\infty$: $v = 20$

$w(1) = 20 - 10e^{-1} \approx 16$ $w = 20$

6.3.21 Eigenvalues: $\begin{vmatrix} 1-\lambda & 4 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 - \lambda = \lambda(\lambda-1) = 0 \rightarrow \lambda = 0, 1$

For $\lambda = 0$: $\begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 = -4x_2 \rightarrow \vec{x} = x_2\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

For $\lambda = 1$: $\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_2 = 0 \rightarrow \vec{x} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So $A = \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \longrightarrow$

$e^{At} = \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e^{0t} & 0 \\ 0 & e^{1t} \end{bmatrix}\left(-\begin{bmatrix} 0 & -1 \\ -1 & -4 \end{bmatrix}\right) = \begin{bmatrix} -4 & e^t \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}$

$= \begin{bmatrix} e^t & e^{4t} \\ 0 & 1 \end{bmatrix}$

6.4.8 $S = Q\Lambda Q^T$

Eigenvalues: $\begin{vmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{vmatrix} = 144 - 25\lambda + \lambda^2 - 144 = \lambda(\lambda-25) = 0 \rightarrow \lambda = 0, 25$

For $\lambda = 0$: $\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow 3x_1 = -4x_2 \rightarrow \vec{x} = c\begin{bmatrix} -4 \\ 3 \end{bmatrix}$

Two unit eigenvectors: $\vec{x} = \pm\frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$

For $\lambda = 25$: $\begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow 4x_1 = 3x_2 \rightarrow \vec{x} = c\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Two unit eigenvectors: $\vec{x} = \pm\frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

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To make Q, we have two choices of unit eigenvector for each of two eigenvalues, and we can put the eigenvectors in either order $\longrightarrow 2\cdot 2\cdot 2 = 8$ possibles Q's:

$$Q = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}, \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix}, \begin{bmatrix} -4/5 & -3/5 \\ 3/5 & -4/5 \end{bmatrix}, \begin{bmatrix} 4/5 & -3/5 \\ -3/5 & -4/5 \end{bmatrix},$$

$$\begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}, \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}, \begin{bmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{bmatrix}, \begin{bmatrix} -3/5 & 4/5 \\ -4/5 & -3/5 \end{bmatrix}$$

6.4.21 Eigenvectors for $\lambda = 1$:

$$S: \begin{bmatrix} -1 & d & 0 \\ d & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

one solution is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$B: \begin{bmatrix} -d-1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

one solution is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Eigenvectors for $\lambda = d$:

$$S: \begin{bmatrix} -d & d & 0 \\ d & -d & 0 \\ 0 & 0 & 1-d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

one solution is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$B: \begin{bmatrix} -2d & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

one solution is $\begin{bmatrix} 1 \\ 0 \\ 2d \end{bmatrix}$

Eigenvectors for $\lambda = -d$:

$$S: \begin{bmatrix} d & d & 0 \\ d & d & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

one solution: $(-1, 1, 0)$

$$B: \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

one solution: $(1, 0, 0)$

So Q for S is: $\begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix}$, X for B is: $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 2d & 0 \end{bmatrix}$

make these unit vectors

$\det = -1 \begin{vmatrix} 1 & 1 \\ 2d & 0 \end{vmatrix} = 2d$

But columns of X are <u>not</u> perpendicular. $\longrightarrow$ Invertible if $d \neq 0$

<u>Graded Problem</u>

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & -2 \\ 1 & -2 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ 1 & 3-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 3-\lambda \\ 1 & -2 \end{vmatrix}$$

$$= (2-\lambda)\underbrace{(\lambda^2 - 6\lambda + 5)}_{(\lambda-5)(\lambda-1)} \underbrace{- (3-\lambda+2) + (-2-3+\lambda)}_{+2(\lambda-5)}$$

$$= (\lambda-5)\left((2-\lambda)(\lambda-1) + 2\right) = (\lambda-5)(-\lambda^2+3\lambda) = 0 \longrightarrow$$

$\lambda = 0, 3, 5$

<u>$\lambda = 0$</u>: $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 \\ 2 & 1 & 1 \\ 1 & -2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & -5 & 5 \\ 0 & -5 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{matrix} x_1 = -x_3 \\ x_2 = x_3 \end{matrix} \longrightarrow \vec{x} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \longrightarrow \vec{q}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

<u>$\lambda = 3$</u>: $\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{matrix} x_1 = 2x_3 \\ x_2 = x_3 \end{matrix}$

$\longrightarrow \vec{x} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \longrightarrow \vec{q}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

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$\lambda=5$: $\begin{bmatrix} -3 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & -5 & -5 \\ 1 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\longrightarrow \begin{matrix} x_1 = 0 \\ x_2 = -x_3 \end{matrix} \longrightarrow \vec{x} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \longrightarrow \vec{q}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Basis: $\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

Then $A^N = Q \Lambda^N Q^T = \begin{bmatrix} -1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3^N & 0 \\ 0 & 0 & 5^N \end{bmatrix} Q^T$

$= \begin{bmatrix} 0 & 2\cdot 3^N/\sqrt{6} & 0 \\ 0 & 3^N/\sqrt{6} & -5^N/\sqrt{2} \\ 0 & 3^N/\sqrt{6} & 5^N/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 4\cdot 3^N/6 & 2\cdot 3^N/6 & 2\cdot 3^N/6 \\ 2\cdot 3^N/6 & 3^N/6 + 5^N/2 & 3^N/6 - 5^N/2 \\ 2\cdot 3^N/6 & 3^N/6 - 5^N/2 & 3^N/6 + 5^N/2 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 4\cdot 3^{N-1} & 2\cdot 3^{N-1} & 2\cdot 3^{N-1} \\ 2\cdot 3^{N-1} & 3^{N-1}+5^N & 3^{N-1}-5^N \\ 2\cdot 3^{N-1} & 3^{N-1}-5^N & 3^{N-1}+5^N \end{bmatrix} = \begin{bmatrix} 2\cdot 3^{N-1} & 3^{N-1} & 3^{N-1} \\ 3^{N-1} & \frac{3^{N-1}+5^N}{2} & \frac{3^{N-1}-5^N}{2} \\ 3^{N-1} & \frac{3^{N-1}-5^N}{2} & \frac{3^{N-1}+5^N}{2} \end{bmatrix}$