H05.

14.3

58. Find the value of $\frac{\partial x}{\partial z}$ at the point $(1,-1,-3)$: $xz+y\ln x - x^2+4=0$.

$$\left(\frac{\partial x}{\partial z}z + x\right) + \left(\frac{y}{x}\frac{\partial x}{\partial z}\right) - \left(2x\frac{\partial x}{\partial z}\right) = 0$$

$$\frac{\partial x}{\partial z}\left(z+\frac{y}{x}-2x\right) = -x$$

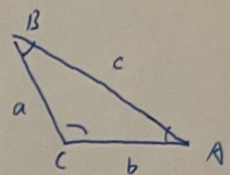
$$\frac{\partial x}{\partial z} = -\frac{x}{z+\frac{y}{x}-2x}$$

at $(1,-1,-3)$:

$$\frac{\partial x}{\partial z} = -\frac{1}{-3+\frac{-1}{1}-2}$$

$$\frac{\partial x}{\partial z} = \frac{1}{6}$$

59. Express A implicitly as a function of a, b, and c and calculate $\frac{\partial A}{\partial a}$ and $\frac{\partial A}{\partial b}$.



$a^2=b^2+c^2-2bc\cos A$ by cosine law.

$\frac{\partial A}{\partial a}$: $2a = 2bc\sin A\,\frac{\partial A}{\partial a}$

$$\frac{\partial A}{\partial a} = \frac{2a}{2bc\sin A}$$

$\frac{\partial A}{\partial b}$: $0 = 2b - 2c\cos A + 2bc\sin A\,\frac{\partial A}{\partial b}$

$$\frac{\partial A}{\partial b} = \frac{c\cos A - b}{bc\sin A}$$

60. Express a implicitly as a function of A, b, and B and calculate $\frac{\partial a}{\partial A}$ and $\frac{\partial a}{\partial B}$.

$\frac{a}{\sin A} = \frac{b}{\sin B}$ by sine law.

$\frac{\partial a}{\partial A}$:

$$\frac{\frac{\partial a}{\partial A}\sin A - a\cos A}{\sin^2 A} = 0$$

$$\frac{\partial a}{\partial A} = \frac{a\cos A}{\sin A}$$

$\frac{\partial a}{\partial B}$:

$$\frac{1}{\sin A}\frac{\partial a}{\partial B} = -b\frac{\cos B}{\sin^2 B}$$

$$\frac{\partial a}{\partial B} = -b\frac{\cos B\sin A}{\sin^2 B}$$

14.4

(a) express $\frac{dw}{dt}$ as a function of t. and (b) evaluate $\frac{dw}{dt}$ at the given value of t.

Express w in terms of t

1. $w=x^2+y^2$, $x=\cos t$, $y=\sin t$, $t=\pi$

$$\frac{dw}{dt} = 2x\cdot(-\sin t) + 2y\cdot\cos t$$

$$= -2\sin t\cos t + 2\sin t\cos t$$

$$= 0$$

$w = \cos^2 t + \sin^2 t$

$= 1$

$\frac{dw}{dt} = 0$

$\frac{dw}{dt}(\pi) = 0.$

2. $w=x^2+y^2$, $x=\cos t+\sin t$, $y=\cos t-\sin t$; $t=0$

$$\frac{dw}{dt} = 2x\cdot(-\sin t+\cos t) + 2y\cdot(-\sin t-\cos t)$$

$$= 2(\cos^2 t-\sin^2 t) + 2(\sin^2 t-\cos^2 t)$$

$$= 0$$

$w = (\cos t+\sin t)^2 + (\cos t-\sin t)^2$

$= 2\cos^2 t + 2\sin^2 t$

$= 2$

$\frac{dw}{dt} = 0$

$\frac{dw}{dt}(0) = 0$

3. $w = \frac{x}{z}+\frac{y}{z}$, $x=\cos^2 t$, $y=\sin^2 t$, $z=\frac{1}{t}$, $t=3$.

$$\frac{dw}{dt} = \frac{1}{z}(2\cos t\cdot(-\sin t)) + \frac{1}{z}(2\sin t\cos t) + \frac{-x-y}{z^2}\cdot\left(-\frac{1}{t^2}\right)$$

$$= -2t\sin t\cos t + 2t\sin t\cos t + \frac{-\cos^2 t+\sin^2 t}{\frac{1}{t^2}}\cdot\frac{1}{t^2}$$

$$= 1$$

$w = \frac{\cos^2 t}{\frac{1}{t}} + \frac{\sin^2 t}{\frac{1}{t}}$

$= t$

$\frac{dw}{dt} = 1$

$\frac{dw}{dt}(3) = 1$

4. $w=\ln(x^2+y^2+z^2)$, $x=\cos t$, $y=\sin t$, $z=4\sqrt{t}$, $t=3$

$$\frac{\partial w}{\partial t}=\frac{2x}{x^2+y^2+z^2}\cdot(-\sin t)+\frac{2y}{x^2+y^2+z^2}(\cos t)+\frac{2z}{x^2+y^2+z^2}\cdot 2t^{-\frac{1}{2}}$$

$$=\frac{(-2\cos t\sin t)+(2\sin t\cos t)+(2\times4\sqrt{t}\times2\times t^{-\frac{1}{2}})}{1+16t}$$

$$=\frac{16}{1+16t}$$

$w=\ln(\cos^2 t+\sin^2 t+(4\sqrt{t})^2)$

$w=\ln(1+16t)$ $\quad \frac{\partial w}{\partial t}=\frac{16}{1+16t}$

$$\frac{\partial w}{\partial t}(3)=\frac{16}{1+16\times3}=\frac{16}{49}$$

$\arctan' t=\frac{1}{1+t^2}$

5. $w=2ye^x-\ln z$, $x=\ln(t^2+1)$, $y=\tan^{-1}t$, $z=e^t$, $t=1$

$$\frac{\partial w}{\partial t}=2ye^x\cdot\frac{2t}{t^2+1}+2e^x\frac{1}{1+t^2}+(-\frac{1}{z})\cdot e^t$$

$$=\frac{4t\cdot\tan^{-1}t\cdot(t^2+1)}{t^2+1}+2(t^2+1)\cdot\frac{1}{1+t^2}-1$$

$$=4t\tan^{-1}t+1$$

$w=2\times\tan^{-1}t\cdot(t^2+1)-t.$

$w=2\tan^{-1}t(t^2+1)-t$

$$\frac{\partial w}{\partial t}=\frac{2}{1+t^2}(t^2+1)+2\tan^{-1}t\cdot(2t)-1$$

$$=4t\tan^{-1}t+1$$

$$\frac{\partial w}{\partial t}(1)=4\times\tan^{-1}1+1=4\times\frac{\pi}{4}+1=\pi+1$$

Show that the functions 69-75 are all solutions of the wave equation.

$$\frac{\partial^2 w}{\partial t^2}=c^2\frac{\partial^2 w}{\partial x^2}$$

69. $w=\sin(x+ct)$.

$\frac{\partial w}{\partial t}=\cos(x+ct)\cdot c.$ $\quad \frac{\partial w}{\partial x}=\cos(x+ct)$

$\frac{\partial^2 w}{\partial t^2}=-c\sin(x+ct)\cdot c.$ $\quad \frac{\partial w}{\partial x}=-ct\sin(x+ct)$

$=-c^2\sin(x+ct)$ $\quad =-\sin(x+ct)$

$$\frac{\partial^2 w}{\partial t^2}=c^2\frac{\partial^2 w}{\partial x^2}$$

70. $w=\cos(2x+2ct)$

$\frac{\partial w}{\partial t}=-\sin(2x+2ct)\cdot 2c$ $\quad \frac{\partial w}{\partial x}=-\sin(2x+2ct)\cdot 2$

$\frac{\partial^2 w}{\partial t^2}=-2c\cos(2x+2ct)\cdot 2c$ $\quad \frac{\partial^2 w}{\partial x^2}=-2\cos(2x+2ct)\cdot 2$

$=-4c^2\cos(2x+2ct)$ $\quad =-4\cos(2x+2ct)$

$$\frac{\partial^2 w}{\partial t^2}=c^2\frac{\partial^2 w}{\partial x^2}$$

71. $w=\sin(x+ct)+\cos(2x+2ct)$

$\frac{\partial w}{\partial t}=c\cos(x+ct)-2c\sin(2x+2ct)$ $\quad \frac{\partial w}{\partial x}=\cos(x+ct)-2\sin(2x+2ct)$

$\frac{\partial^2 w}{\partial t^2}=-c^2\sin(x+ct)-4c^2\cos(2x+2ct)$ $\quad \frac{\partial^2 w}{\partial x^2}=-\sin(x+ct)-4\cos(2x+2ct)$

$$\frac{\partial^2 w}{\partial t^2}=c^2\frac{\partial^2 w}{\partial x^2}$$

72. $w = \ln(2x+2ct)$

$$\frac{\partial w}{\partial t} = \frac{2c}{2x+2ct} = \frac{c}{x+ct} \qquad \frac{\partial w}{\partial x} = \frac{2}{2x+2ct} = \frac{1}{x+ct}$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{-c^2}{(x+ct)^2} \qquad \frac{\partial^2 w}{\partial x^2} = \frac{-1}{(x+ct)^2}$$

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

73. $w = \tan(2x-2ct)$

$$\frac{\partial w}{\partial t} = \sec^2(2x-2ct)\cdot(-2c) \qquad \frac{\partial w}{\partial x} = \sec^2(2x-2ct)\cdot 2$$

$$\frac{\partial^2 w}{\partial t^2} = 8c^2\sec^2(2x-2ct)\tan(2x-2ct) \qquad \frac{\partial^2 w}{\partial x^2} = 8\sec^2(2x-2ct)\tan(2x-2c$$

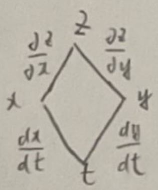
$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

14.4

draw a tree diagram and write a Chain Rule formula for each derivative.

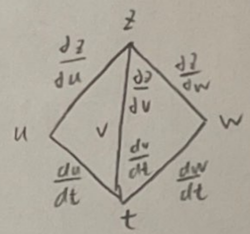
13. $\frac{dz}{dt}$ for $z = f(x,y)$, $x = g(t)$, $y = h(t)$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}.$$



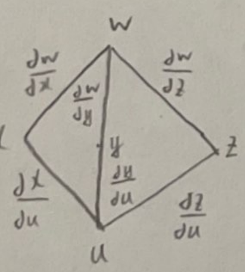
14. $\frac{dz}{dt}$ for $z = f(u,v,w)$, $u = g(t)$, $v = h(t)$, $w = k(t)$

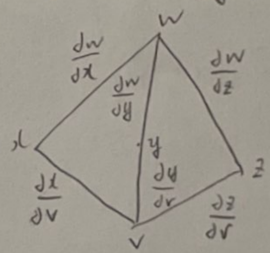
$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\cdot\frac{du}{dt} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dt} + \frac{\partial z}{\partial w}\frac{dw}{dt}$$



15. $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ for $w = h(x,y,z)$, $x = f(u,v)$, $y = g(u,v)$, $z = k(u,v)$

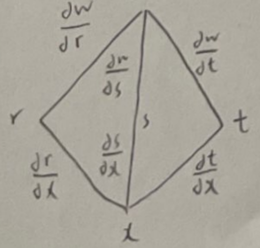
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} \qquad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

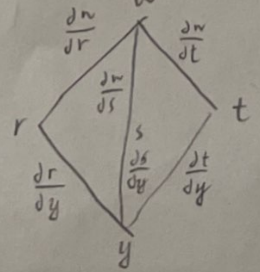




16. $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ for $w = f(r,s,t)$, $r = g(x,y)$, $s = h(x,y)$, $t = k(x,y)$.

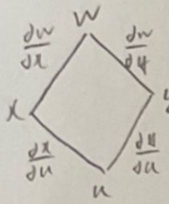
$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x} \qquad \frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$$

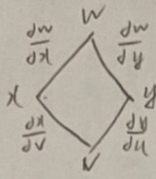




17. $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ for $w = g(x,y)$, $x = h(u,v)$, $y = k(u,v)$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} \qquad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$$





Use $\frac{dy}{dx} = \frac{-F_x}{F_y}$ to find $\frac{dy}{dx}$ at the given point.

25. $x^3 - 2y^2 + xy = 0$ $(1,1)$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2+y}{-4y+x}$$

$$\frac{dy}{dx}(1,1) = -\frac{3+1}{-4+1} = \frac{4}{3}$$

26. $xy + y^2 - 3x - 3 = 0$ $(-1,1)$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y-3}{x+2y}$$

$$\frac{dy}{dx}(-1,1) = -\frac{1-3}{-1+2} = 2.$$

27. $x^2 + xy + y^2 - 7 = 0$ $(1,2)$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+y}{x+2y}$$

$$\frac{dy}{dx}(1,2) = -\frac{2+2}{1+4} = -\frac{4}{5}$$

28. $xe^y + \sin xy + y - \ln 2 = 0$ $(0, \ln 2)$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{e^y + y\cos xy}{xe^y + x\cos xy + 1}$$

$$\frac{dy}{dx}(0,\ln 2) = -\frac{2+\ln 2}{0+0+1} = -2-\ln 2.$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

29. $z^3 - xy + yz + y^3 - 2 = 0$ $(1,1,1)$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-y}{3z^2+y} \qquad \frac{\partial z}{\partial x}(1,1,1) = -\frac{-1}{3+1} = \frac{1}{4}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-x+z+3y^2}{3z^2+y} \qquad \frac{\partial z}{\partial y}(1,1,1) = -\frac{-1+1+3}{3+1} = -\frac{3}{4}$$

14.5

Find the derivative of the function at P_0 in the direct of $\vec{A}$.

9. $f(x,y) = 2xy - 3y^2$, $P_0(5,5)$, $\vec{A} = 4\vec{i} + 3\vec{j}$

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$$

$$f_x(5,5) = 2y\,|_{(5,5)} = 10.$$

$$f_y(5,5) = (2x-6y)|_{(5,5)} = -20.$$

$$\nabla f = 10\vec{i} - 20\vec{j}$$

$$(D_u f)_{P_0} = \nabla f \cdot \vec{u} = 10\times\frac{4}{5} - 20\times\frac{3}{5} = -4$$

10. $f(x,y) = 2x^2 + y^2$. $P_0(-1,1)$, $\vec{A} = 3\vec{i} - 4\vec{j}$

$$\vec{u} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$$

$$f_x(-1,1) = 4x\,|_{(-1,1)} = -4$$

$$f_y(-1,1) = 2y\,|_{(-1,1)} = 2$$

$$\nabla f = -4\vec{i} + 2\vec{j}$$

$$(D_u f)_{P_0} = \nabla f \cdot \vec{u} = -\frac{12}{5} - \frac{8}{5} = -4$$

11. $g(x,y) = x - \frac{y^2}{x} + \sqrt{3}\sec^{-1}(2xy)$, $P_0(1,1)$ $\vec{A} = 12\vec{i} + 5\vec{j}$

$$\vec{u} = \frac{12}{13}\vec{i} + \frac{5}{13}\vec{j} \qquad \operatorname{arcsec} x = \frac{1}{x\sqrt{x^2-1}}$$

$$g_x(1,1) = \left(1 + \frac{y^2}{x^2} + \frac{2y\sqrt{3}}{2xy\sqrt{4x^2y^2-1}}\right)\Big|_{(1,1)} = 1+1+1 = 3$$

$$g_y(1,1) = \left(-\frac{2y}{x} + \frac{2x\sqrt{3}}{2xy\sqrt{4x^2y^2-1}}\right)\Big|_{(1,1)} = -2+1 = -1$$

$$\nabla g = 3\vec{i} - 1\vec{j}$$

$$(D_u g)_{P_0} = \nabla g\,\vec{u} = 3\times\frac{12}{13} - 1\,\frac{5}{13} = \frac{31}{13}$$

12. $h(x,y)=\tan^{-1}\left(\frac{y}{x}\right)+\sqrt{3}\sin^{-1}\left(\frac{xy}{2}\right)$, $P_0(1,1)$ $\vec{A}=3\vec{i}-2\vec{j}$

$\vec{u}=\frac{3}{\sqrt{13}}\vec{i}-\frac{2}{\sqrt{13}}\vec{j}$

$\nabla h=\frac{1}{2}\vec{i}+\frac{3}{2}\vec{j}$.

$h_x(1,1)=\left(\frac{-\frac{y}{x^2}}{(\frac{y}{x})^2+1}+\frac{\frac{y}{2}\sqrt{3}}{\sqrt{1-(\frac{xy}{2})^2}}\right)\Big|_{(1,1)}=-\frac{1}{2}+1=\frac{1}{2}$

$h_y(1,1)=\left(\frac{\frac{1}{x}}{(\frac{y}{x})^2+1}+\frac{\frac{x}{2}\sqrt{3}}{\sqrt{1-(\frac{xy}{2})^2}}\right)\Big|_{(1,1)}=\frac{1}{2}+1=\frac{3}{2}$

$(D_u h)_{P_0}=\nabla g\cdot\vec{u}$

$=\frac{3}{2\sqrt{13}}-\frac{6}{2\sqrt{13}}$

$=-\frac{3}{2\sqrt{13}}$

13. $f(x,y,z)=xy+yz+zx$, $P_0(1,-1,2)$, $\vec{A}=3\vec{i}+6\vec{j}-2\vec{k}$

$\vec{u}=\frac{3}{7}\vec{i}+\frac{6}{7}\vec{j}-\frac{2}{7}\vec{k}$

$\nabla f=\vec{i}+3\vec{j}$

$(D_u f)_{P_0}=\nabla f\cdot\vec{u}$

$=\frac{3}{7}+\frac{18}{7}$

$=3$

$f_x(1,-1,2)=(y+z)\big|_{(1,-1,2)}=1$

$f_y(1,-1,2)=(x+z)\big|_{(1,-1,2)}=3.$

$f_z(1,-1,2)=(x+y)\big|_{(1,-1,2)}=0.$

14.4

回路

Changing voltage in a circuit.

39. $V=IR$, $R=600$, $I=0.04$, $\frac{dR}{dt}=0.5$, $\frac{dV}{dt}=-0.01$

$\frac{dV}{dt}=\frac{\partial V}{\partial I}\frac{dI}{dt}+\frac{\partial V}{\partial R}\frac{dR}{dt}$

$V=IR$

$\Rightarrow \frac{\partial V}{\partial I}=R$, $\frac{\partial V}{\partial R}=I$

$\frac{dV}{dt}=\frac{\partial V}{\partial I}\frac{dI}{dt}+\frac{\partial V}{\partial R}\frac{dR}{dt}$

$=R\frac{dI}{dt}+I\frac{dR}{dt}$

$-0.01=600\frac{dI}{dt}+0.04\times0.5$

$\frac{dI}{dt}=-0.00005\ (A/s)$

40. Changing dimensions in a box.

$a=1m$, $b=2m$, $c=3m$, $\frac{da}{dt}=\frac{db}{dt}=1\ m/sec$, $\frac{dc}{dt}=-3\ m/sec$. At what rates are the box's volume V and surface area S changing at that instant? Are the box's interior diagonals increasing in length or decreasing?

$V=abc.$

$\frac{dV}{dt}=\frac{\partial V}{\partial a}\frac{da}{dt}+\frac{\partial V}{\partial b}\frac{db}{dt}+\frac{\partial V}{\partial c}\frac{dc}{dt}$

$=bc\frac{da}{dt}+ac\frac{db}{dt}+ab\frac{dc}{dt}$

$=2\times3\times1+1\times3\times1+1\times2\times(-3)$

$=3\ (m^3/sec)>0$

the volume is increasing.

$S=2ab+2ac+2bc$

$\frac{dS}{dt}=\frac{\partial S}{\partial a}\frac{da}{dt}+\frac{\partial S}{\partial b}\frac{db}{dt}+\frac{\partial S}{\partial c}\frac{dc}{dt}$

$=2(b+c)\frac{da}{dt}+2(a+c)\frac{db}{dt}+2(a+b)\frac{dc}{dt}$

$=\overset{10}{2\times(2+3)\times1}+\overset{6}{2\times(1+3)\times1}+\overset{-18}{2(1+2)\times(-3)}$

$=0.$

the surface are is not changing

$D = \sqrt{a^2+b^2+c^2}$

$$\frac{dD}{dt} = \frac{\partial D}{\partial a}\frac{da}{dt} + \frac{\partial D}{\partial b}\frac{db}{dt} + \frac{\partial D}{\partial c}\frac{dc}{dt}$$

$$= \frac{a}{\sqrt{a^2+b^2+c^2}}\frac{da}{dt} + \frac{b}{\sqrt{a^2+b^2+c^2}}\frac{db}{dt} + \frac{c}{\sqrt{a^2+b^2+c^2}}\frac{db}{dt}$$

$$= \frac{1}{\sqrt{14}}\times 1 + \frac{2}{\sqrt{14}}\times 1 + \frac{3}{\sqrt{14}}\times(-3)$$

$$= -\frac{6}{\sqrt{14}}\ (m/sec) < 0.$$

the diagonals are decreasing in length.

41. If $f(u,v,w)$ is differentiable and $u = x-y$, $v = y-z$, and $w = z-x$ show that $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} + 0 - \frac{\partial f}{\partial w}$$

$$\frac{\partial f}{\partial y} = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

$$\frac{\partial f}{\partial z} = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}\right) + \left(-\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) + \left(-\frac{\partial f}{\partial v} + \frac{\partial r}{\partial w}\right)$$

$$= 0.$$

14.5

Find the directions in which the functions increas and derease most rapidly at P_0. Then find the derivatives of the functions in these directions.

17. $f(x,y) = x^2+xy+y^2$, $P_0(-1,1)$

$f_x(-1,1) = (2x+y)|_{(-1,1)} = -1$

$f_y(-1,1) = (x+2y)_{(-1,1)} = 1$

$\nabla f = -\vec{i}+\vec{j}$

$\vec{u} = \frac{\nabla f}{|\nabla f|} = -\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$

increases most rapidly in $\vec{u} = -\frac{1}{\sqrt{2}}\vec{i}+\frac{1}{\sqrt{2}}\vec{j}$

decreases most rapidly in $-\vec{u} = \frac{1}{\sqrt{2}}\vec{i}-\frac{1}{\sqrt{2}}\vec{j}$

$(D_u f)_{P_0} = \nabla f\cdot\vec{u} = \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}} = \sqrt{2}$

$(D_{-u} f)_{P_0} = \nabla f\cdot -\vec{u} = -\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}} = -\sqrt{2}.$

18. $f(x,y) = x^2y + e^{xy}\sin y$, $P_0(1,0)$

$f_x(1,0) = (2xy + ye^{xy}\sin y)|_{(1,0)} = 0.$

$f_y(1,0) = (x^2 + xe^{xy}\sin y + e^{xy}\cos y) = 1+1 = 2.$

$\nabla f = 2\vec{j}$

$\vec{u} = \vec{j}$

increases most rapidly in $\vec{u} = \vec{j}$

decreases most rapidly in $-\vec{u} = -\vec{j}$

$(D_u f)_{P_0} = \nabla f\cdot\vec{u} = 2$

$(D_{-u} f)_{P_0} = \nabla f\cdot-\vec{u} = -2.$

19. $f(x,y,z)=\frac{x}{y}-yz$, $P_0(4,1,1)$

$f_x(4,1,1)=\frac{1}{y}\Big|_{(4,1,1)}=1$

$f_y(4,1,1)=(-\frac{x}{y^2}-z)\Big|_{(4,1,1)}=-4-1=-5$

$f_z(4,1,1)=-y\Big|_{(4,1,1)}=-1$

$\nabla f=\vec{i}-5\vec{j}-\vec{k}$

$\vec{u}=\frac{1}{3\sqrt{3}}\vec{i}-\frac{5}{3\sqrt{3}}\vec{j}-\frac{1}{3\sqrt{3}}\vec{k}$

increases most rapidly in $\vec{u}=\frac{1}{3\sqrt{3}}\vec{i}-\frac{5}{3\sqrt{3}}\vec{j}-\frac{1}{3\sqrt{3}}\vec{k}$

decreases most rapidly in $-\vec{u}=-\frac{1}{3\sqrt{3}}\vec{i}+\frac{5}{3\sqrt{3}}\vec{j}+\frac{1}{3\sqrt{3}}\vec{k}$

$(D_u f)_{P_0}=\nabla f\,\vec{u}$

$=\frac{1}{3\sqrt{3}}+\frac{25}{3\sqrt{3}}+\frac{1}{3\sqrt{3}}$

$=3\sqrt{3}$

$(D_{-u} f)_{P_0}=-3\sqrt{3}$

20. $g(x,y,z)=xe^y+z^2$, $P_0(1,\ln 2,\frac{1}{2})$

$g_x(1,\ln 2,\frac{1}{2})=e^y\Big|_{(1,\ln 2,\frac{1}{2})}=2.$

$g_y(1,\ln 2,\frac{1}{2})=xe^y\Big|_{(1,\ln 2,\frac{1}{2})}=2.$

$g_z(1,\ln 2,\frac{1}{2})=2z\Big|_{(1,\ln 2,\frac{1}{2})}=1$

$\nabla g=2\vec{i}+2\vec{j}+1\vec{k}$

$\vec{u}=\frac{2}{3}\vec{i}+\frac{2}{3}\vec{j}+\frac{1}{3}\vec{k}$

increases most rapidly in $\vec{u}=\frac{2}{3}\vec{i}+\frac{2}{3}\vec{j}+\frac{1}{3}\vec{k}$

decreases most rapidly in $-\vec{u}=-\frac{2}{3}\vec{i}-\frac{2}{3}\vec{j}-\frac{1}{3}\vec{k}$

$(D_u g)_{P_0}=\nabla f\,\vec{u}$

$=\frac{4}{3}+\frac{4}{3}+\frac{1}{3}$

$=3$

$(D_{-u} g)_{P_0}=\nabla f\cdot(-\vec{u})$

$=-3$

21. $f(x,y,z)=\ln xy+\ln yz+\ln xz$, $P_0(1,1,1)$

$f_x(1,1,1)=(\frac{y}{xy}+\frac{z}{xz})\Big|_{(1,1,1)}=2$

$f_y(1,1,1)=(\frac{x}{xy}+\frac{z}{yz})\Big|_{(1,1,1)}=2.$

$f_z(1,1,1)=(\frac{y}{yz}+\frac{x}{xz})\Big|_{(1,1,1)}=2.$

$\nabla f=2\vec{i}+2\vec{j}+2\vec{k}$

$\vec{u}=\frac{1}{\sqrt{3}}\vec{i}+\frac{1}{\sqrt{3}}\vec{j}+\frac{1}{\sqrt{3}}\vec{k}$

increases most rapidly in $\vec{u}=\frac{1}{\sqrt{3}}\vec{i}+\frac{1}{\sqrt{3}}\vec{j}+\frac{1}{\sqrt{3}}\vec{k}$

decreases most rapidly in $-\vec{u}=-\frac{1}{\sqrt{3}}\vec{i}-\frac{1}{\sqrt{3}}\vec{j}-\frac{1}{\sqrt{3}}\vec{k}$

$(D_u f)_{P_0}=\nabla f\,\vec{u}$

$=\frac{2}{\sqrt{3}}\times 3$

$=2\sqrt{3}.$

$(D_{-u} f)_{P_0}=-2\sqrt{3}.$