

One more type of matrix where finding the inverse is easy:

Diagonal matrices: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ Only non-zero entries go on the diagonal

To find inverse, just invert diagonal entries:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$$

How about $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$?

↖ No inverse because we can't invert 0.

Last time: An $n \times n$ matrix is invertible if it has an inverse matrix:
 ↖ has to be square

$$AA^{-1} = I = A^{-1}A$$

↖ $n \times n$ identity matrix: $I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \end{bmatrix}$ satisfies $IB = B = BI$ for all $n \times n$ B

What does invertibility say about solutions to linear equations?

If A has an inverse, then for any vector \vec{b} , the system

$A\vec{x} = \vec{b}$ has a unique solution.

Why? Multiply on left by A^{-1} :

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b} \xrightarrow{\text{associativity}} (A^{-1}A)\vec{x} = A^{-1}\vec{b}$$

$$\rightsquigarrow I\vec{x} = A^{-1}\vec{b} \rightsquigarrow \vec{x} = A^{-1}\vec{b}$$

This is a vector. It's the only solution for \vec{x} .

In particular: Set $\vec{b} = \vec{0}$. If A has an inverse then $A\vec{x} = \vec{0}$ has only one solution. One solution is $\vec{x} = \vec{0}$, since $A\vec{0} = \vec{0}$. So $\vec{0}$ is the only solution to $A\vec{x} = \vec{0}$, if A is invertible.

Turn this logic around: If $A\vec{x} = \vec{0}$ has non-zero solutions, then A can't have an inverse.

Example Let's show $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ doesn't have an inverse (43)

by solving $A\vec{x} = \vec{0}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & | & 0 \\ 4 & 5 & 6 & | & 0 \\ 7 & 8 & 9 & | & 0 \end{bmatrix} \begin{array}{l} \text{Row 2} - 4\text{Row 1} \\ \text{Row 3} - 7\text{Row 1} \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 & | & 0 \\ 0 & -3 & -6 & | & 0 \\ 0 & -6 & -12 & | & 0 \end{bmatrix} \begin{array}{l} \text{Row 3} - 2\text{Row 2} \\ \text{Row 2} \rightarrow -\frac{1}{3}\text{Row 2} \end{array} \begin{bmatrix} 1 & 2 & 3 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{\text{Row 1} - 2\text{Row 2}} \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

We've reduced the system to: $x - z = 0$
 $y + 2z = 0$

We have no constraint on z , and then $x = z$, $y = -2z$

We have many solutions: $z = 0: (x, y, z) = (0, 0, 0)$

$$z = 1: (x, y, z) = (1, -2, 1)$$

$$z = -1: (x, y, z) = (-1, 2, -1)$$

So A^{-1} doesn't exist.

non-zero solutions

Some properties of inverse matrices:

① If A has an inverse, could it have more than one inverse?

No! To see why, what if B, C were both inverses of A ?

$$\text{Then } AB = I \xrightarrow[\text{C on left}]{\text{Multiply by C}} C(AB) = CI = C \xrightarrow{\text{associativity}} (CA)B = C$$

This is also I !

$$\longrightarrow IB = C \longrightarrow B = C$$

\sim So B and C have to be the same!

② If A and B both have inverses, what about AB ?

Turns out, inverse of AB is not $A^{-1}B^{-1}$!

Actually, it's $B^{-1}A^{-1}$. Why?

$$(AB)(\underbrace{B^{-1}A^{-1}}_I) = A(\underbrace{BB^{-1}}_I)A^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(\underbrace{A^{-1}A}_I)B = B^{-1}B = I. \quad \checkmark$$

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But, $(AB)(A^{-1}B^{-1}) = ??$

They don't cancel, and we can't switch order.

So: $(AB)^{-1} = B^{-1}A^{-1}$ (if A, B are both invertible).

Now the big question: Can we actually calculate inverse matrices in a practical way?

We need to solve the matrix equation $AX = I$ for the unknown X .

$$A \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \dots & \vec{x}_n \end{bmatrix} = I \longrightarrow \begin{bmatrix} A\vec{x}_1 & A\vec{x}_2 & \dots & A\vec{x}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

We need to solve n systems of linear equations at the same time:

$$A\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad A\vec{x}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

same coefficient matrix for each system, can write down a big augmented matrix to solve all systems by elimination at the same time:

$$\left[\begin{array}{c|cccc} A & 1 & 0 & \dots & 0 \\ & 0 & 1 & \dots & 0 \\ & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \dots & 1 \end{array} \right] \xrightarrow{\text{Apply elimination}} \left[\begin{array}{c|cccc} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{array} \right] A^{-1}$$

solutions for $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ on the right side will be the columns of the inverse matrix.

Example Find inverse of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}$

Solution: Apply elimination on a big augmented matrix.

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\text{Row 2} + \frac{1}{2}\text{Row 1}} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -2 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array} \right]$$

Put identity matrix on the right.

$$\xrightarrow{\text{Row 3} + \frac{2}{3}\text{Row 2}} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -2 & 1/2 & 1 & 0 \\ 0 & 0 & 2/3 & 1/3 & 2/3 & 1 \end{array} \right] \xrightarrow{\text{Row 2} \times 3/2}$$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 3/2 & 3 & 3 \\ 0 & 0 & 2/3 & 1/3 & 2/3 & 1 \end{array} \right] \xrightarrow{\text{Row 1} + \frac{2}{3}\text{Row 2}} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 2 & 2 & 2 \\ 0 & 3/2 & 0 & 3/2 & 3 & 3 \\ 0 & 0 & 2/3 & 1/3 & 2/3 & 1 \end{array} \right]$$

$$\xrightarrow{\substack{\frac{1}{2}\text{Row 1}, \frac{2}{3}\text{Row 2} \\ \frac{3}{2}\text{Row 3}}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1/2 & 1 & 3/2 \end{array} \right] \quad \text{This is } A^{-1}$$

Good to check: $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1/2 & 1 & 3/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A \quad A^{-1} \quad I$

We have to divide by 2 in some entries; suggests that the "determinant" of A is the non-zero number 2.

Another example: Upper triangular

$$A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & -1 & -2 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Do elimination on:}} \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & -4 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} \text{Row 1} + 4\text{Row 4} \\ \text{Row 2} - 3\text{Row 4} \\ \text{Row 3} - 2\text{Row 4} \end{array} \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 0 & 1 & 0 & 0 & 4 \\ 0 & -1 & -2 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \text{Row 1} + 3\text{Row 3} \\ \text{Row 2} - 2\text{Row 3} \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & -2 & -2 \\ 0 & -1 & 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow[\text{Then: } \begin{array}{l} -\text{Row 2} \\ -\text{Row 3} \end{array}]{\text{Row 1} + 2\text{Row 2}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

This is A^{-1} . It's still upper triangular.

Good to check: $A \cdot A^{-1} = I$

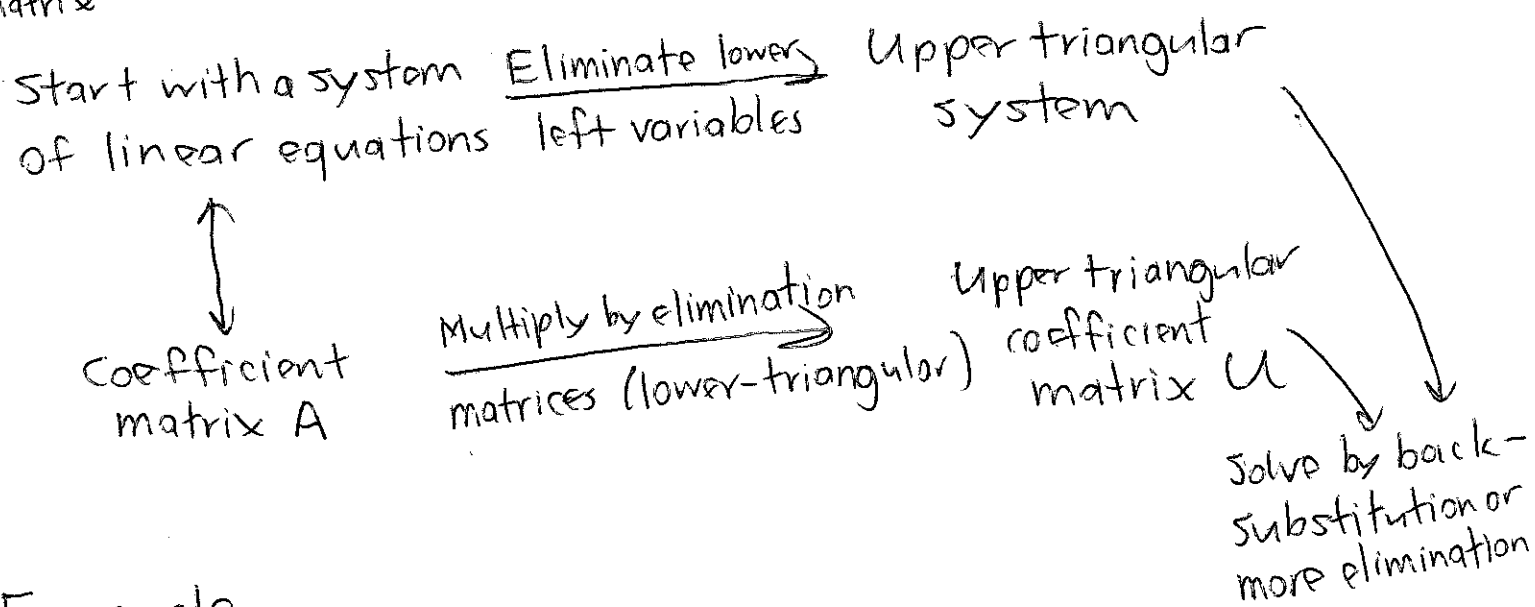
$$\begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & -1 & -2 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Section 2.6 LU Decomposition

Factorizing a matrix as a product of simpler factors.

$A = L U$
 $\nwarrow \nearrow$
 $\begin{matrix} \text{nxn} \\ \text{matrix} \end{matrix} \quad \begin{matrix} \text{lower} \\ \text{triangular} \end{matrix} \quad \begin{matrix} \text{upper} \\ \text{triangular} \end{matrix}$

This comes from linear systems / elimination.



Example

$$\begin{array}{rcl} 2x - 2y & = & 2 \\ -2x - 3y & = & 3 \end{array} \xrightarrow{\text{Eqn. 2} + \text{Eqn. 1}} \begin{array}{rcl} 2x - 2y & = & 2 \\ -5y & = & 5 \end{array}$$

\leftrightarrow

$$\begin{bmatrix} 2 & -2 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

\nwarrow This is A

Multiply both sides by $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -2 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

That is, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} A = U \xrightarrow[\text{for } A]{\text{Solve}} A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} U$

Elimination matrix $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ does the "Row 2 \rightarrow Row 2 + Row 1"

operation, so $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$ does the inverse row operation that undoes

it: "Row 2 \rightarrow Row 2 - Row 1" $\leadsto \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \leftarrow$ This is L.

So the LU decomposition (or factorization) of A is:

$$\begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 0 & -5 \end{bmatrix}$$

A L U

Basic idea: Elimination from A to an upper triangular system is the same as factoring A ~~into~~ as a product of lower and upper triangular matrices.

3x3 example; Let's eliminate $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ to upper triangular

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow[\text{or: multiply by } \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}]{\begin{array}{l} \text{Row 2} + \frac{1}{2} \text{Row 1} \\ \text{Row 3} + \frac{1}{2} \text{Row 1} \end{array}} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{bmatrix} \xrightarrow[\text{or: multiply by } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}]{\begin{array}{l} \text{Row 3} + \text{Row 2} \end{array}} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Upper triangular now; this is U

$$A \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} A \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} A = U$$

$$\text{So } A = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \right)^{-1} U$$

This is L.

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} U$$

(Switch order when you invert a product of matrices)

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} U$$

(Reverse ~~order~~ the row operations to get inverses)

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$$\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & -1 & 1 \end{bmatrix} = L$$

$$\text{So } \underbrace{\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}}_A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & -1 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}}_U \quad (\text{check.})$$
