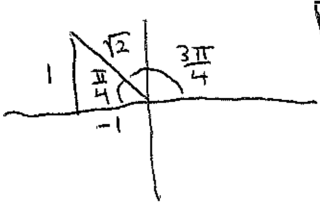
Homework 2 Solutions

1.2.7 $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$

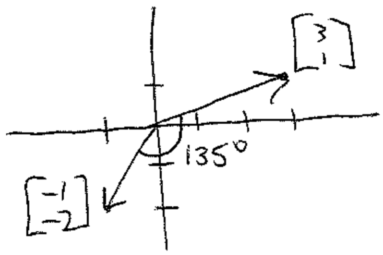
(b) $\cos\theta = \dfrac{\begin{bmatrix}2\\2\\-1\end{bmatrix}\cdot\begin{bmatrix}2\\-1\\2\end{bmatrix}}{\sqrt{2^2+2^2+(-1)^2}\sqrt{2^2+(-1)^2+2^2}} = \dfrac{2(2)+2(-1)+2(-1)}{\sqrt{9}\sqrt{9}} = 0$

$\rightarrow \boxed{\theta = \frac{\pi}{2} \text{ radians or } 90^\circ}$

(d) $\cos\theta = \dfrac{\begin{bmatrix}3\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\-2\end{bmatrix}}{\sqrt{3^2+1^2}\sqrt{(-1)^2+(-2)^2}} = \dfrac{3(-1)+1(-2)}{\sqrt{10}\sqrt{5}} = \dfrac{-5}{5\sqrt{2}} = -\dfrac{1}{\sqrt{2}}$



$\boxed{\theta = \frac{3\pi}{4} \text{ radians or } 135^\circ}$



1.2.16 $\|\vec{v}\| = \sqrt{\underbrace{1^2+1^2+\dots+1^2}_{9\text{ times}}} = \sqrt{9} = \boxed{3}$

To get $\vec{u}$, we need to divide $\vec{v}$ by its length: $\boxed{\begin{aligned}\vec{u} &= \tfrac{1}{3}\vec{v}\\ &= \left(\tfrac{1}{3},\tfrac{1}{3},\dots,\tfrac{1}{3}\right)\end{aligned}}$

To get $\vec{w}$, we could notice that $(1,-1,0,\dots,0)$ is perpendicular to $\vec{v}$ because $(1,-1,0,\dots,0)\cdot(1,1,1,\dots,1)=0$. But $(1,-1,0,\dots,0)$ isn't a unit vector, so need to divide by its length:

$\vec{w} = \dfrac{1}{\sqrt{1^2+(-1)^2+0^2+\dots+0^2}}(1,-1,0,\dots,0) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \dots, 0\right)$

(Note there are many possible solutions for $\vec{w}$.)

1.2.22 (a) $|\vec{v}\cdot\vec{w}|^2 = \left|\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\cdot\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right|^2 =$ ②

$|v_1w_1+v_2w_2|^2 = (v_1w_1+v_2w_2)^2 = v_1^2w_1^2+2v_1v_2w_1w_2+v_2^2w_2^2$

square means we can replace
$|\ |$ with $(\ )$

$\|\vec{v}\|^2\|\vec{w}\|^2 = \left(\sqrt{v_1^2+v_2^2}\right)^2\left(\sqrt{w_1^2+w_2^2}\right)^2 = (v_1^2+v_2^2)(w_1^2+w_2^2)$

$= v_1^2w_1^2+v_2^2w_1^2+v_1^2w_2^2+v_2^2w_2^2$

(b) Difference between two sides: $\|\vec{v}\|^2\|\vec{w}\|^2 - |\vec{v}\cdot\vec{w}|^2 =$

$(\cancel{v_1^2w_1^2}+v_1^2w_2^2+v_2^2w_1^2+\cancel{v_2^2w_2^2}) - (\cancel{v_1^2w_1^2}+2v_1v_2w_1w_2+\cancel{v_2^2w_2^2}) =$

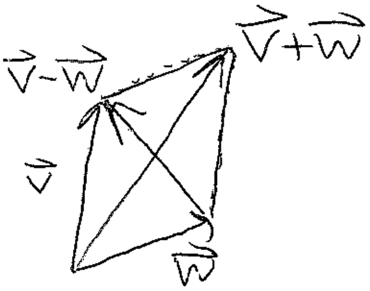
$v_1^2w_2^2-2v_1v_2w_1w_2+v_2^2w_1^2 = \underbrace{(v_1w_2-v_2w_1)^2}$

This is a square, so it's ≥ 0

Conclusion: $\|\vec{v}\|^2\|\vec{w}\|^2 - |\vec{v}\cdot\vec{w}|^2 = \text{square} \geq 0 \longrightarrow$

$\|\vec{v}\|^2\|\vec{w}\|^2 \geq |\vec{v}\cdot\vec{w}| \xrightarrow[\text{roots}]{\text{take square}} \|\vec{v}\|\|\vec{w}\| \geq |\vec{v}\cdot\vec{w}|$

1.2.27



$\|\vec{v}+\vec{w}\|^2+\|\vec{v}-\vec{w}\|^2 =$

$(\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w})+(\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w}) =$

$\underbrace{\vec{v}\cdot\vec{v}}_{\|\vec{v}\|^2}+\cancel{\vec{w}\cdot\vec{v}}+\cancel{\vec{v}\cdot\vec{w}}+\underbrace{\vec{w}\cdot\vec{w}}_{\|\vec{w}\|^2}+\vec{v}\cdot\vec{v}-\cancel{\vec{w}\cdot\vec{v}}-\cancel{\vec{v}\cdot\vec{w}}+\vec{w}\cdot\vec{w}$

$= 2\|\vec{v}\|^2+2\|\vec{w}\|^2$

1.2.33 (There is more than one solution to this problem).

We could start $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Then all three other vectors need to have two $+\frac{1}{2}$'s and two $-\frac{1}{2}$'s to get a dot product of 0.

We could pick $(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$, $(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$, and $(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$.

Check that the last three are perpendicular to each other:

$$(\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})\cdot(\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})=\tfrac{1}{4}-\tfrac{1}{4}-\tfrac{1}{4}+\tfrac{1}{4}=0 \quad \checkmark$$

$$(\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})\cdot(\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})=\tfrac{1}{4}-\tfrac{1}{4}+\tfrac{1}{4}-\tfrac{1}{4}=0 \quad \checkmark$$

$$(\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})\cdot(\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})=\tfrac{1}{4}+\tfrac{1}{4}-\tfrac{1}{4}-\tfrac{1}{4}=0 \quad \checkmark$$

1.3.4 We want to find x_2, x_3 so that

$$1\begin{bmatrix}1\\2\\3\end{bmatrix}+x_2\begin{bmatrix}4\\5\\6\end{bmatrix}+x_3\begin{bmatrix}7\\8\\9\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}, \quad \text{or} \quad \begin{cases}4x_2+7x_3=-1\\5x_2+8x_3=-2\\6x_2+9x_3=-3\end{cases}$$

$$4\text{ Eqn. }2-5\text{ Eqn. }1: \quad \begin{aligned}&20x_2+32x_3=-8\\ -&(20x_2+35x_3=-5)\\ \hline &-3x_3=-3 \rightarrow \boxed{x_3=1}\end{aligned}$$

Then $4x_2=-1-7x_3=-1-7=-8 \rightarrow \boxed{x_2=-2}$

Check: $1\begin{bmatrix}1\\2\\3\end{bmatrix}-2\begin{bmatrix}4\\5\\6\end{bmatrix}+1\begin{bmatrix}7\\8\\9\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ $\checkmark$

These vectors are dependent because we can write one as a linear combination of the other two (for example, $\vec{w}_3=-\vec{w}_1+2\vec{w}_2$).

The three vectors lie in a plane in 3-dimensional space.

1.3.5 We need to solve

$$y_1\begin{bmatrix}1\\4\\7\end{bmatrix}+y_2\begin{bmatrix}2\\5\\8\end{bmatrix}+y_3\begin{bmatrix}3\\6\\9\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}, \quad \text{or} \quad \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c}1&2&3&0\\4&5&6&0\\7&8&9&0\end{array}\right]\xrightarrow[\text{Row }3-7\text{ Row}1]{\text{Row }2-4\text{ Row }1}\left[\begin{array}{ccc|c}1&2&3&0\\0&-3&-6&0\\0&-6&-12&0\end{array}\right]\xrightarrow{\text{Row }3-2\text{ Row }2}$$

$$\left[\begin{array}{ccc|c}1&2&3&0\\0&-3&-6&0\\0&0&0&0\end{array}\right]\xrightarrow{-\frac{1}{3}\text{Row }2}\left[\begin{array}{ccc|c}1&2&3&0\\0&1&2&0\\0&0&0&0\end{array}\right]\xrightarrow{\text{Row }1-2\text{Row }2}\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&2&0\\0&0&0&0\end{array}\right]\xrightarrow{\text{Back to equations}}$$

$$\begin{cases}y_1-y_3=0\\ y_2+2y_3=0\\ y_3 \text{ can be anything}\end{cases} \longrightarrow \begin{array}{l}y_1=y_3\\ y_2=-2y_3\end{array}$$

Two choices: $y_3=1 \rightarrow y_1=1,\ y_2=-2$

$y_3=-2 \rightarrow y_1=-2,\ y_2=4$

Check: $1\begin{bmatrix}1\\4\\7\end{bmatrix}-2\begin{bmatrix}2\\5\\8\end{bmatrix}+1\begin{bmatrix}3\\6\\9\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ ✓ $\quad -2\begin{bmatrix}1\\4\\7\end{bmatrix}+4\begin{bmatrix}2\\5\\8\end{bmatrix}-2\begin{bmatrix}3\\6\\9\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ ✓

1.3.12

$$\begin{bmatrix}0&1&0&0\\-1&0&1&0\\0&-1&0&1\\0&0&-1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}\longrightarrow\left[\begin{array}{cccc|c}0&1&0&0&b_1\\-1&0&1&0&b_2\\0&-1&0&1&b_3\\0&0&-1&0&b_4\end{array}\right]\xrightarrow{\text{Row }1\leftrightarrow\text{Row }2}$$

$$\left[\begin{array}{cccc|c}-1&0&1&0&b_2\\0&1&0&0&b_1\\0&-1&0&1&b_3\\0&0&-1&0&b_4\end{array}\right]\xrightarrow{\text{Row }3+\text{Row }2}\left[\begin{array}{cccc|c}-1&0&1&0&b_2\\0&1&0&0&b_1\\0&0&0&1&b_1+b_3\\0&0&-1&0&b_4\end{array}\right]\xrightarrow{\text{Row }3\leftrightarrow\text{Row }4}$$

$$\left[\begin{array}{cccc|c}-1&0&1&0&b_2\\0&1&0&0&b_1\\0&0&-1&0&b_4\\0&0&0&1&b_1+b_3\end{array}\right]\xrightarrow{\text{Row }1+\text{Row }3}\left[\begin{array}{cccc|c}-1&0&0&0&b_2+b_4\\0&1&0&0&b_1\\0&0&-1&0&b_4\\0&0&0&1&b_1+b_3\end{array}\right]\xrightarrow[-\text{Row }3]{-\text{Row }1}\left[\begin{array}{cccc|c}1&0&0&0&-b_2-b_4\\0&1&0&0&b_1\\0&0&1&0&-b_4\\0&0&0&1&b_1+b_3\end{array}\right]$$

$$\text{So}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-b_2-b_4\\b_1\\-b_4\\b_1+b_3\end{bmatrix}=\underbrace{\begin{bmatrix}0&-1&0&-1\\1&0&0&0\\0&0&0&-1\\1&0&1&0\end{bmatrix}}_{\text{This is } C^{-1}.}\begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$$

Graded Problem 1 (a) Divide $\vec{u}$ by $\|\vec{u}\|$:

$$\|\vec{u}\| = \sqrt{1^2+2^2+1^2} = \sqrt{6}, \quad \text{unit vector} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}$$

(b) There are many solutions. For $\vec{v}$, we could pick

$$\vec{v} = \begin{bmatrix}1\\0\\-1\end{bmatrix}, \text{ since } \vec{u}\cdot\vec{v} = 1(1)+2(0)+1(-1) = 0.$$

For $\vec{w} = \begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$, we need to solve $\begin{cases}\vec{v}\cdot\vec{w}=0\\ \vec{u}\cdot\vec{w}=0\end{cases} \rightarrow \begin{matrix}w_1 \quad -w_3 = 0\\ w_1+2w_2+w_3=0\end{matrix}$

Eqn. 2 − Eqn. 1: $2w_2+2w_3=0$. So: $\begin{cases}w_1-w_3=0\\ w_2+w_3=0\end{cases} \rightarrow \begin{matrix}w_1=w_3\\ w_2=-w_3\end{matrix}$

Pick $w_3=1$: then $w_1=1, w_2=-1$ $\rightarrow \vec{w} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$

Graded Problem 2

(a) $\begin{bmatrix}4&7\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$

$$\left[\begin{array}{cc|c}4&7&b_1\\1&2&b_2\end{array}\right] \xrightarrow[\text{(for convenience)}]{\text{Row1}\leftrightarrow\text{Row2}} \left[\begin{array}{cc|c}1&2&b_2\\4&7&b_1\end{array}\right] \xrightarrow[\text{4 Row 1}]{\text{Row 2 −}} \left[\begin{array}{cc|c}1&2&b_2\\0&-1&b_1-4b_2\end{array}\right] \xrightarrow[\text{2 Row 2}]{\text{Row 1 +}}$$

$$\left[\begin{array}{cc|c}1&0&2b_1-7b_2\\0&-1&b_1-4b_2\end{array}\right] \xrightarrow{-\text{Row 2}} \left[\begin{array}{cc|c}1&0&2b_1-7b_2\\0&1&-b_1+4b_2\end{array}\right]$$

$$\boxed{\vec{x} = \begin{bmatrix}2b_1-7b_2\\-b_1+4b_2\end{bmatrix}}$$

(b) Since $\begin{bmatrix}2b_1-7b_2\\-b_1+4b_2\end{bmatrix} = \begin{bmatrix}2&-7\\-1&4\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}$, $\boxed{A^{-1} = \begin{bmatrix}2&-7\\-1&4\end{bmatrix}}$