# Homework 7 Solutions

3.1.27 (a) False: This subset never contains $\vec{0}$ (since $\vec{0}$ always is in $C(A)$), so it can't be a subspace.

(b) True: If $A$ is not the $0$ matrix, then it has a non-zero column which is in $C(A)$. So if $C(A) = \{\vec{0}\}$, then $A = 0$.

(c) True: If columns of $A$ are $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, then $C(A) = \text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$ and $C(2A) = \text{span}(2\vec{v}_1, 2\vec{v}_2, \dots, 2\vec{v}_n)$

$\vec{v}_i$ is in $C(2A)$ because $\vec{v}_i = \frac{1}{2}(2\vec{v}_i)$ and $C(2A)$ is closed under scalar multiplication.

in $C(A)$ because $C(A)$ is closed under scalar multiplication.

(d) False: For example, consider $A = 2\times 2$ identity.

Then $C(A) = C(I) = \text{span}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) = \mathbb{R}^2$

$C(A-I) = C(O) = \text{span}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right) = \{\vec{0}\}$

not the same

3.1.28 Column space should not be all of $\mathbb{R}^3$, so it should be only the plane spanned by $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$. We can take these two vectors as the 1st two columns, then take the 3rd to be a linear combination of the 1st two: $A = \begin{bmatrix}1 & 1 & 2\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix}$ is one example.

For column space to be a line, all columns should be multiples of one of them, for example $B = \begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\\1 & 2 & 3\end{bmatrix}$.

3.2.12 $\begin{bmatrix} 1 & -3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

$\uparrow$ $A$

Free variables $y, z$:

$x = 3y + z$,

$y, z$ free

$N(A)$: $\vec{x} = \begin{bmatrix} 3y+z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

The two special solutions, a basis for the plane $x - 3y - z = 0$.

3.2.20 Let's say we want $N(A) = \text{span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$, for example,

so $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{matrix} -2a + b = 0 \\ -2c + d = 0 \end{matrix} \rightsquigarrow \begin{matrix} b = 2a \\ d = 2c \end{matrix}$

We also need the columns to be multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$: $\begin{bmatrix} a \\ c \end{bmatrix} = x\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

and $\begin{bmatrix} b \\ d \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Choose $a = 2$: Then $b = 2(2) = 4$ and $a = -2x \to x = -1$

$b = -2y \to y = -2$

so $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \leftarrow C(A) = \text{span}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \end{bmatrix}\right)$

$= \text{span}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = N(A)$

3.2.31 (a) $\begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix} \xrightarrow[\text{Row3-Row1}]{\text{Row2-Row1}} \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{4}\text{Row1}} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{R}$

(b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} \xrightarrow[\text{Row3-3Row1}]{\text{Row2-2Row1}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{bmatrix} \xrightarrow[\text{Then: -Row2}]{\text{Row3-2Row2}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{\text{Row1-2Row2}} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow R$

(c) $\begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \xrightarrow[\text{Row 3} - \text{Row 1}]{\text{Row 2} - \text{Row 1}} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\text{Row 1}} \underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{R}$

3.2.32 Node 1: $y_3 = y_1 + y_4$ Node 2: $y_1 = y_2 + y_5$

Node 3: $y_2 = y_3 + y_6$ Node 4: $y_4 + y_5 + y_6 = 0$

$$\left\{\begin{array}{l} -y_1 + y_3 - y_4 = 0 \\ y_1 - y_2 - y_5 = 0 \\ y_2 - y_3 - y_6 = 0 \\ y_4 + y_5 + y_6 = 0 \end{array}\right. \longrightarrow \underbrace{\begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}}_{\vec{y}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[\text{Then: } -\text{Row 1}]{\text{Row 2} + \text{Row 1}}$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[\text{Then: } -\text{Row 2}]{\text{Row 3} + \text{Row 2}} \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[\text{Then: } -\text{Row 3}]{\text{Row 4} + \text{Row 3}}$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{Row 2} - \text{Row 3}]{\text{Row 1} - \text{Row 3}} \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{This is } R.$$

$N(A^T)$: $$\begin{array}{l} y_1 - y_3 - y_5 - y_6 = 0 \\ y_2 - y_3 - y_6 = 0 \\ y_4 + y_5 + y_6 = 0 \end{array} \longrightarrow \vec{y} = \begin{bmatrix} y_3 + y_5 + y_6 \\ y_3 + y_6 \\ y_3 \\ -y_5 - y_6 \\ y_5 \\ y_6 \end{bmatrix}$$

$y_3, y_5, y_6$ free

$$= y_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + y_6 \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

3 special solutions

**3.3.4**

$$\begin{bmatrix} 1 & 3 & 1 & 2 & | & 1 \\ 2 & 6 & 4 & 8 & | & 3 \\ 0 & 0 & 2 & 4 & | & 1 \end{bmatrix} \xrightarrow{\text{Row 1}-2\text{ Row 2}} \begin{bmatrix} 1 & 3 & 1 & 2 & | & 1 \\ 0 & 0 & 2 & 4 & | & 1 \\ 0 & 0 & 2 & 4 & | & 1 \end{bmatrix} \xrightarrow[\text{Then: } \frac{1}{2}\text{Row 2}]{\text{Row 3}-\text{Row 2}}$$

$$\begin{bmatrix} 1 & 3 & 1 & 2 & | & 1 \\ 0 & 0 & 1 & 2 & | & 1/2 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{\text{Row 1}-\text{Row 2}} \begin{bmatrix} 1 & 3 & 0 & 0 & | & 1/2 \\ 0 & 0 & 1 & 2 & | & 1/2 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \longrightarrow \begin{cases} x+3y = \frac{1}{2} \\ z+2t = \frac{1}{2} \\ y, t \text{ free} \end{cases}$$

$$\vec{x} = \begin{bmatrix} 1/2 - 3y \\ y \\ 1/2 - 2t \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}}_{\vec{x}_p} + \underbrace{y \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\vec{x}_n}$$

special null space solutions

**3.3.7**

$$A = \begin{bmatrix} 1 & 3 & 1 & | & b_1 \\ 3 & 8 & 2 & | & b_2 \\ 2 & 4 & 0 & | & b_3 \end{bmatrix} \xrightarrow[\text{Row 3}-2\text{ Row 1}]{\text{Row 2}-3\text{ Row 1}} \begin{bmatrix} 1 & 3 & 1 & | & b_1 \\ 0 & -1 & -1 & | & b_2 - 3b_1 \\ 0 & -2 & -2 & | & b_3 - 2b_1 \end{bmatrix} \xrightarrow[-2\text{ Row 2}]{\text{Row 3}}$$

$$\begin{bmatrix} 1 & 3 & 1 & | & b_1 \\ 0 & -1 & -1 & | & b_2 - 3b_1 \\ 0 & 0 & 0 & | & \underbrace{(b_3 - 2b_1) - 2(b_2 - 3b_1)} \end{bmatrix}$$

Need $4b_1 - 2b_2 + b_3 = 0$ for $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ to be in $C(A)$.

Some combination of rows gives 0: 4 Row 1 − 2 Row 2 + Row 3 = 0

**3.3.13** (a) Not true unless $\vec{b} = \vec{0}$. For example, could $2\vec{x}_p + 3\vec{x}_n$ be a solution to $A\vec{x} = \vec{b}$? Then $A(2\vec{x}_p + 3\vec{x}_n) = \vec{b}$, ~~but~~

so $\vec{b} = A(2\vec{x}_p + 3\vec{x}_n) = 2\underbrace{A\vec{x}_p}_{\text{also } \vec{b}} + 3\underbrace{A\vec{x}_n}_{\vec{0}} = 2\vec{b} \longrightarrow$

$2\vec{b} - \vec{b} = 0 \longrightarrow \vec{b} = \vec{0}$. Doesn't work unless $\vec{b} = \vec{0}$.

(b) Not true if $N(A)$ is bigger than $\{\vec{0}\}$, because then ~~any~~ if $\vec{x}_p$ works as a particular solution, so does $\vec{x}_p + \vec{x}_n$

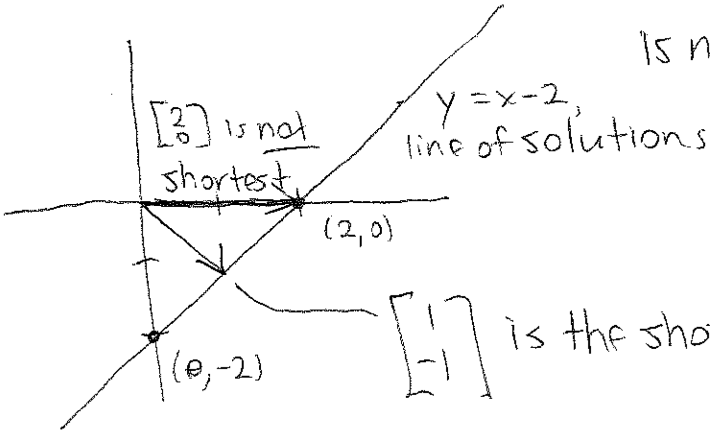
any non-zero vector in $N(A)$.

(c) consider $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \rightsquigarrow \begin{cases} x = 2+y \\ y \text{ free} \end{cases}$

or, $y = x-2$

$\rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2+y \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + y\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{x}_n}$

This $\vec{x}_p$ has free variable $y=0$, but it isn't the shortest solution




$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the shortest solution $\left(\left\|\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\| = \sqrt{2}\right)$

(d) The null space always has at least one vector, $\vec{0}$.

3.3.34 (a) One special solution means $\underbrace{\#\text{columns}}_{4} = \text{rank} + 1$

So rank of $A = 3$.

Complete solution to $A\vec{x} = \vec{0}$ is just all linear combinations of special solutions, so $\vec{x} = x_3 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

the free variable.

(b) Free variable $x_3$ means pivot columns are 1, 2, and 4.

$R = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ To get $\begin{cases} x_1 = 2x_3 \\ x_2 = 3x_3 \end{cases}$ in null space equations.

(c) R has no row of 0's, so no condition on $b_1, b_2, b_3$ is necessary to guarantee $A\vec{x} = \vec{b}$ has a solution.

## Graded Problem 1

$$\begin{bmatrix} -3 & -1 & 0 & 1 \\ -4 & -1 & 1 & 1 \\ -1 & 1 & 4 & -1 \end{bmatrix} \xrightarrow[\text{Row 1} \leftrightarrow \text{Row 3}]{-\text{Row 3, then}} \begin{bmatrix} 1 & -1 & -4 & 1 \\ -4 & -1 & 1 & 1 \\ -3 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\text{Row 3}+3\text{Row 1}]{\text{Row 2}+4\text{Row 1}} \begin{bmatrix} 1 & -1 & -4 & 1 \\ 0 & -5 & -15 & 5 \\ 0 & -4 & -12 & 4 \end{bmatrix} \xrightarrow[\text{Then Row 3}-\text{Row 2}]{-\frac{1}{5}\text{Row 2},\ -\frac{1}{4}\text{Row 3}} \begin{bmatrix} 1 & -1 & -4 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 1}+\text{Row 2}}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{R} \longrightarrow \begin{array}{l} x_1 - x_3 = 0 \\ x_2 + 3x_3 - x_4 = 0 \\ x_3, x_4 \text{ free} \end{array} \rightarrow \vec{x} = \begin{bmatrix} x_3 \\ -3x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

special solutions

## Graded Problem 2

$$A: \left[\begin{array}{ccc|c} 2 & -1 & -1 & b_1 \\ -1 & 2 & -1 & b_2 \\ -1 & -1 & 2 & b_3 \end{array}\right] \xrightarrow[\text{Row 3}+\frac{1}{2}\text{Row 1}]{\text{Row 2}+\frac{1}{2}\text{Row 1}} \left[\begin{array}{ccc|c} 2 & -1 & -1 & b_1 \\ 0 & 3/2 & -3/2 & b_2 + \frac{1}{2}b_1 \\ 0 & -3/2 & 3/2 & b_3 + \frac{1}{2}b_1 \end{array}\right]$$

$$\xrightarrow{\text{Row 3}+\text{Row 2}} \left[\begin{array}{ccc|c} 2 & -1 & -1 & b_1 \\ 0 & 3/2 & -3/2 & b_2 + \frac{1}{2}b_1 \\ 0 & 0 & 0 & \underbrace{b_1 + b_2 + b_3} \end{array}\right] \xrightarrow[b_3 = -2]{b_1 = 1,\ b_2 = 1} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 1 \\ 0 & 3/2 & -3/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The condition $b_1 + b_2 + b_3 = 0$ guarantees $\vec{b}$ is in $C(A)$

$$\xrightarrow{\frac{2}{3}\text{Row 2}} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{Row 1}+\text{Row 2}} \left[\begin{array}{ccc|c} 2 & 0 & -2 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}\text{Row 1}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \begin{array}{l} x_1 = x_3 + 1 \\ x_2 = x_3 + 1 \\ x_3 \text{ free} \end{array} \longrightarrow \boxed{\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$