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Homework 11 Solutions

5.1.2 A is a 3×3 matrix, $\det(A) = -1$

$$\det\left(\tfrac{1}{2}A\right) = \left(\tfrac{1}{2}\right)^3 \det(A) = \tfrac{1}{8}(-1) = -\tfrac{1}{8}$$

$$\det(-A) = (-1)^3 \det(A) = (-1)(-1) = 1$$

$$\det(A^2) = \det(A)\det(A) = (-1)(-1) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{-1} = -1$$

5.1.7 Rotation: $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta - (-\sin\theta)(\sin\theta)$

$$= \cos^2\theta + \sin^2\theta = 1$$

Reflection: $\begin{vmatrix} 1-2\cos^2\theta & -2\sin\theta\cos\theta \\ -2\cos\theta\sin\theta & 1-2\sin^2\theta \end{vmatrix} =$

$$(1-2\cos^2\theta)(1-2\sin^2\theta) - (-2\cos\theta\sin\theta)^2 =$$

$$1-2\cos^2\theta - 2\sin^2\theta + 4\cos^2\theta\sin^2\theta - 4\cos^2\theta\sin^2\theta =$$

$$1-2(\cos^2\theta+\sin^2\theta) = 1-2 = -1$$

5.1.13 $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} \underset{\text{Row2-Row1}}{\overset{\text{Row3-Row1}}{=}} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} \underset{\text{-Row2}}{\overset{\text{Row3}}{=}} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1(1)(1) = 1$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{vmatrix} \underset{\text{Row 3}-3\text{Row 1}}{\overset{\text{Row 2}-2\text{Row 1}}{=}} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & -3 & -6 \end{vmatrix} \overset{\text{Row 3}-\frac{3}{2}\text{Row 2}}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & -3/2 \end{vmatrix}$$

$$= 1(-2)\left(-\frac{3}{2}\right) = 3$$

5.1.18

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \underset{\text{Row 3}-\text{Row 1}}{\overset{\text{Row 2}-\text{Row 1}}{=}} \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \underset{\text{If } b \neq a}{\overset{\text{Row 3} - \frac{c-a}{b-a}\text{Row 2}}{=}}$$

$$\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & c^2-a^2-\frac{(c-a)(b^2-a^2)}{b-a} \end{vmatrix} = 1(b-a)\left(c^2-b^2-\frac{(c-a)(b^2-a^2)}{b-a}\right)$$

$$= (c^2-a^2)(b-a) - (c-a)\underbrace{(b^2-a^2)}_{(b-a)(b+a)} = (b-a)\left(c^2-b^2-(c-a)(b+a)\right)$$

$$= (b-a)\left(c^2-a^2-bc+ab-ac+a^2\right) = (b-a)\left(c(c-a)-b(c-a)\right)$$

$$= (b-a)(c-a)(c-b)$$

5.1.30 $\frac{\partial f}{\partial a} = \frac{\partial}{\partial a}\ln(ad-bc) = \frac{1}{ad-bc}\cdot d = \frac{d}{ad-bc}$

$\frac{\partial f}{\partial b} = \frac{1}{ad-bc}(-c)$ $\frac{\partial f}{\partial c} = \frac{1}{ad-bc}(-b)$, $\frac{\partial f}{\partial d} = \frac{a}{ad-bc}$

So $\begin{bmatrix} \partial f/\partial a & \partial f/\partial c \\ \partial f/\partial b & \partial f/\partial d \end{bmatrix} = \begin{bmatrix} d/(ad-bc) & -b/(ad-bc) \\ -c/(ad-bc & a/(ad-bc) \end{bmatrix} =$

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}.$$

5.2.1

$$\det(A) = 1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 1(1)(1) - 1(2)(2) - 2(3)(1) + 2(2)(3) + 3(3)(2) - 3(1)(3)$$

$$= 1 - 4 - 6 + 12 + 18 - 9 = 12 \neq 0 \rightarrow \underline{\text{independent}} \text{ rows}$$

$$\det(B) = 1\begin{vmatrix} 4 & 4 \\ 6 & 7 \end{vmatrix} - 2\begin{vmatrix} 4 & 4 \\ 5 & 7 \end{vmatrix} + 3\begin{vmatrix} 4 & 4 \\ 5 & 6 \end{vmatrix}$$

$$= 1(4)(7) - 1(4)(6) - 2(4)(7) + 2(4)(5) + 3(4)(6) - 3(4)(5)$$

$$= 28 - 24 - 56 + 40 + 72 - 60 = 0 \rightarrow \underline{\text{dependent}} \text{ rows}$$

$$\det(C) = 1\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 1(1)(0) - 1(0)(0) - 1(1)(0) + 1(0)(1) + 1(1)(0) - 1(1)(1)$$

$$= 0 - 0 - 0 + 0 + 0 - 1 = -1 \rightarrow \underline{\text{independent}} \text{ rows}$$

5.2.15

(a)

$$E_n = \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & & & \vdots \\ 0 & 1 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{vmatrix} \underset{\text{row}}{\overset{\text{1st}}{=}} 1 \underbrace{\begin{vmatrix} 1 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 1 \end{vmatrix}}_{E_{n-1}}$$

$$-1 \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & & 0 & \\ 0 & 1 & 1 & \ddots & & \\ \vdots & & & \ddots & 1 & 1 \\ 0 & 0 & & & 1 & 1 \end{vmatrix} = E_{n-1} - \underset{\substack{\uparrow \\ \text{1st} \\ \text{column}}}{1}(1) E_{n-2} = E_{n-1} - E_{n-2}$$

E_{n-2}

(b) $E_1 = 1,\ E_2 = 0,\ E_3 = E_2 - E_1 = -1,\ E_4 = E_3 - E_2 = -1$

$E_5 = E_4 - E_3 = 0,\ E_6 = E_5 - E_4 = 1,\ E_7 = E_6 - E_5 = 1$

$E_8 = E_7 - E_6 = 0$

~~NOA~~ (c) Notice $E_7 = E_1$ and $E_8 = E_2$

So the values will repeat in cycles of 6:

1, 0, −1, −1, 0, 1, $\leftarrow E_n = E_{n-6}$ for any n.

1, 0, −1, −1, 0, 1, ...

So $E_{100} = E_{94} = E_{88} = E_{82} = \cdots = E_{10} = E_4 = -1$

5.2.19 (a) Expand across 4th row:

$$V_4 = -1 \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + x \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} - x^2 \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} + x^3 \underbrace{\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}}_{V_3}$$

The 3x3 determinants are just numbers, so V_4 is a cubic polynomial in x.

(b) $V_4 = 0$ when $x = a, b, c$ because then two rows of the matrix will be the same, so determinant will be 0.

So $r_1 = a,\ r_2 = b,\ r_3 = c$.

(c) Any cubic polynomial can be factored as $A(x-r_1)(x-r_2)(x-r_3)$ where A is the coefficient of x^3 and r_1, r_2, r_3 are the three roots (which could be complex numbers in general). For V_4,

$V_4 = V_3(x-a)(x-b)(x-c)$ from parts (a) and (b)

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(d) Using formula for V_3 from 5.1.18,

$V_4 = (b-a)(c-a)(c-b)(x-a)(x-b)(x-c)$

5.2.3 $\det(P) = 0\underbrace{\begin{vmatrix} \cdots \end{vmatrix}}_{\text{doesn't matter}} - 0\begin{vmatrix} \cdots \end{vmatrix} + 0\begin{vmatrix} \cdots \end{vmatrix} - 1\underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}}_{I}$

$= -1(1) = -1$

For "big formula," $\det P =$ sum of 24 terms $\pm p_{1i_1} p_{2i_2} p_{3i_3} p_{4i_4}$

different column for each row

The only non-zero term chooses Col 4 for Row 1, Col 1 for Row 2, Col 2 for Row 3, and Col 3 for Row 4.

Then $\det P = \pm(1)(1)(1)(1) +$ a bunch of 0's

$(-1)^{\#\text{row switches to reorder } 4,1,2,3 \text{ to } 1,2,3,4}$

we can do $4,1,2,3 \to 1,4,2,3 \to 1,2,4,3 \to 1,2,3,4$

(3 row switches.

So $\det P = (-1)^3(1)(1)(1)(1) = -1$, same as before.

Then $\det(P^2) = (\det P)(\det P) = (-1)(-1) = +1$.

5.2.34 $A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$

(a) The last 3 rows are all contained in span$([0,0,0,1,0],[0,0,0,0,1])$, which is 2 dimensional, So one of the last three rows has to be a linear combination of the other 2.

(b) Each of the 120 terms in the big formula looks like $\pm a_{1,i_1} a_{2,i_2} a_{3,i_3} a_{4,i_4} a_{5,i_5}$, where Columns i_1, i_2, i_3, i_4, i_5 are all different. ~~So in each term~~ For this to be non-zero, we'd have to choose either Col 4 or 5 for Row 3, and then the other one ~~of~~ Col 4 or 5 for Row 4. But then we have to choose one of Cols 1,2,3 for Row 5, so $a_{5,i_5} = 0 \rightsquigarrow$ the whole term is 0.

Graded Problem 1.

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{vmatrix} \underset{\text{Row 5}-\text{Row 1}}{\overset{\text{Row 2}-\text{Row 1},\ \text{Row 3}-\text{Row 1}}{=}} \text{(Row 4}-\text{Row 1)} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 3 & 9 & 19 & 34 \\ 0 & 4 & 14 & 34 & 69 \end{vmatrix}$$

$$\underset{\text{Row 4}-3\text{Row 2},\ \text{Row 5}-4\text{Row 2}}{\overset{\text{Row 3}-2\text{Row 2}}{=}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 3 & 10 & 22 \\ 0 & 0 & 6 & 22 & 53 \end{vmatrix} \underset{\text{Row 5}-6\text{Row 3}}{\overset{\text{Row 4}-3\text{Row 3}}{=}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 4 & 17 \end{vmatrix}$$

$$\overset{\text{Row 5}-4\text{Row 4}}{=} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1(1)(1)(1)(1) = 5$$

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Graded Problem 2

$$\begin{vmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & 0 & 0 & 0 \\ 4 & 0 & -2 & 0 & 1 \\ 0 & -2 & 0 & 2 & 0 \end{vmatrix} \overset{\text{Row3}}{=} -(-1) \begin{vmatrix} 1 & 1 & -1 & 1 \\ 1 & 3 & 4 & 5 \\ 4 & -2 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{vmatrix}$$

$$\overset{\text{Row 4}}{=} 1(-2) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 4 & -2 & 1 \end{vmatrix} \overset{\text{Row1}}{=} -2\left(1 \begin{vmatrix} 3 & 5 \\ -2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 5 \\ 4 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix}\right)$$

$$= -2(13 + 19 - 14) = -36$$