Chapter 2

Mathematical Models of Systems

Chapter 3

Time Domain Analysis of Control Systems

Third Lesson

Reviews

- Block diagram of a linear system
- Block diagram transformation
- Signal-flow graph
- Gain formula (Mason Formula)
- State space model
- State equation versus transfer function – from SE to TF

Outlines

- **Chapter 2**
- Transfer function versus state equation – from TF to SE
- Controllability canonical form (CCF)
- Observability canonical form (OCF)
- State diagram
- How to use TF to solve the dynamic response of a system
- How to solve a state equation
- **Chapter 3**
- Basics about time domain analysis
- Stability definitions
  - Liapunov stability
  - Asymptotical stability
  - BIBO stability

From TF to SE – An Example

Q: find a SE for the system described by the following TF

$$G(s)=\frac{Y(s)}{U(s)}=\frac{s^2+8s+15}{s^3+7s^2+14s+8}$$

A: introduce an intermediate variable V(s) and rewrite Y(s), U(s) as functions of V(s):

$$G(s)=\frac{Y(s)}{V(s)}\cdot\frac{V(s)}{U(s)}=\frac{s^2+8s+15}{s^3+7s^2+14s+8}$$

$$\frac{V(s)}{U(s)}=\frac{1}{s^3+7s^2+14s+8} \qquad \dddot{v}+7\ddot{v}+14\dot{v}+8v=u$$

$$\frac{Y(s)}{V(s)}=s^2+8s+15 \qquad \ddot{v}+8\dot{v}+15v=y$$

From TF to SE – An Example

If set:

$$x_1 = v \qquad \dot{x}_1 = x_2$$

$$x_2 = \dot{v} \qquad \dot{x}_2 = x_3$$

$$x_3 = \ddot{v} \qquad \dot{x}_3 = -8x_1 - 14x_2 - 7x_3 + u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -14 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$\ddot{v} + 8\dot{v} + 15v = y \qquad y = \begin{bmatrix} 15 & 8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Controllability Canonical Form (CCF)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{m+1}s^m + b_m s^{m-1} + \cdots + b_1}{s^n + a_n s^{n-1} + \cdots + a_1} \qquad (n > m)$$

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 & b_2 & \cdots & b_{m+1}, & 0 & \cdots & 0 \end{bmatrix}_{1\times n} X$$

Observability Canonical Form (OCF)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{m+1}s^{m} + b_{m}s^{m-1} + \cdots + b_1}{s^{n} + a_{n}s^{n-1} + \cdots + a_1} \qquad (n > m)$$

$$\dot{X} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_1 \\ 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 0 & \cdots & 1 & -a_n \end{bmatrix} X + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}_{1\times n} X$$

Remarks

- There exist other canonical forms, such as Diagonal Canonical Form and Jordan Canonical Form.
- There exist different ways to get those canonical forms.
- If n=m, it is a non inertial system. Do division first to get an expression comprised of an integer and a fraction.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s^3 + 8s^2 + 22s + 23}{s^3 + 7s^2 + 14s + 8} = 1 + \frac{s^2 + 8s + 15}{s^3 + 7s^2 + 14s + 8}$$

State Diagram

- State diagram is an extension of SFG to portray state equations and differential equations.
- The basic elements of a state diagram are similar to the conventional SFG, except for the integration operation.

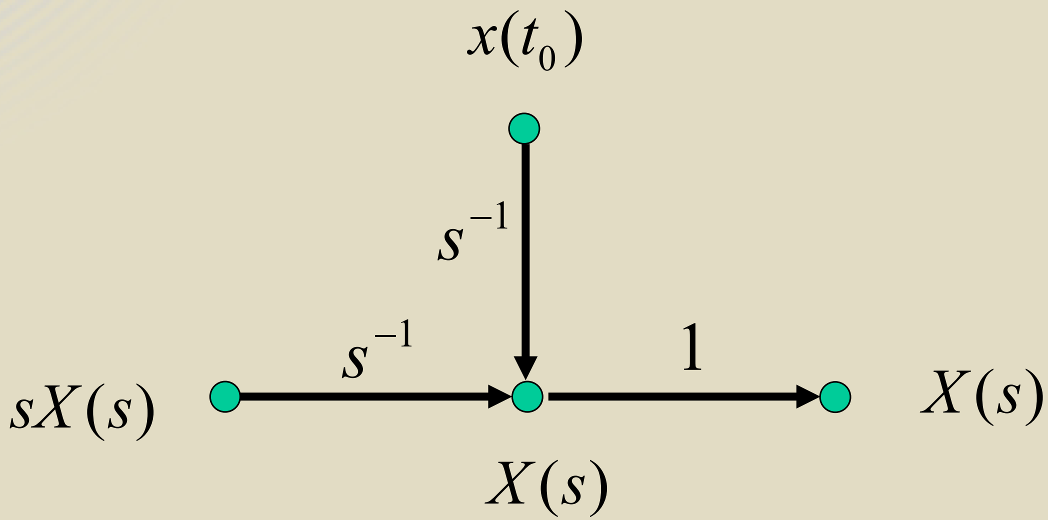
Differential Equation: $$\frac{dx}{dt} = \dot{x}$$

Integration: $$x(t) = \int_{t_0}^{t} \dot{x}(\tau)d\tau + x(t_0)$$

Laplace transform: $$X(s) = \frac{[sX(s)]}{s} + \frac{x(t_0)}{s} \qquad \tau \geq t_0$$

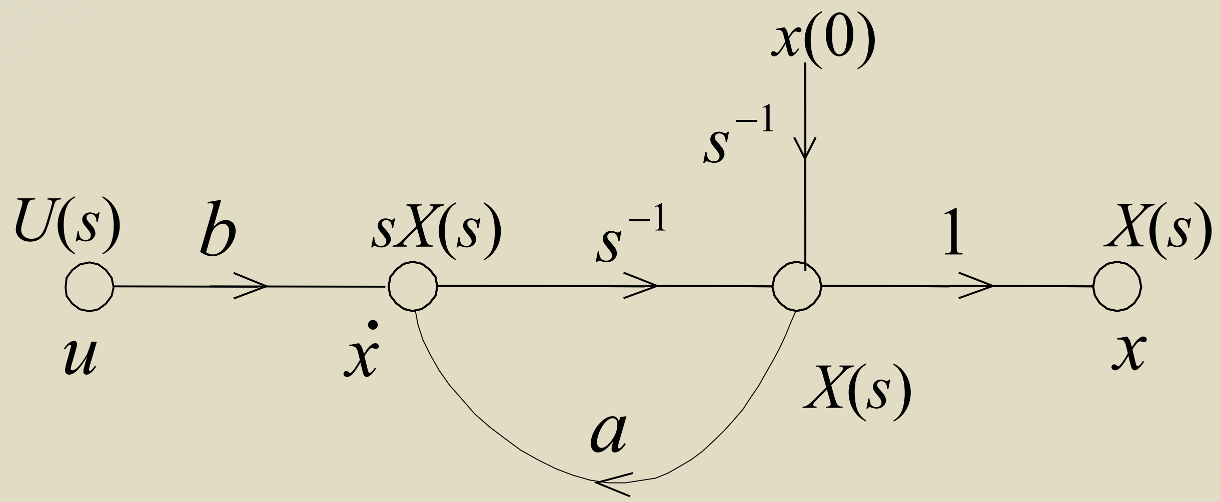
State Diagram of An Integration

$$X(s) = \frac{[sX(s)]}{s} + \frac{x(t_0)}{s} \qquad \tau \geq t_0$$



State Diagram of State Equations

$$\dot{x} = ax + bu$$



Advantages of State Diagrams

1. A state diagram can be constructed directly from the system' s differential equation. This allows the determination of the state variables and the state equations.
2. A state diagram can be constructed from the system' s transfer function.
3. The state diagram can be used for the programming of the system on an analog computer or simulation on a digital computer
4. The transfer functions of a system can be determined from the state diagram
5. The state equations and the output equations can be determined from the state diagram.
6. Other advantages.

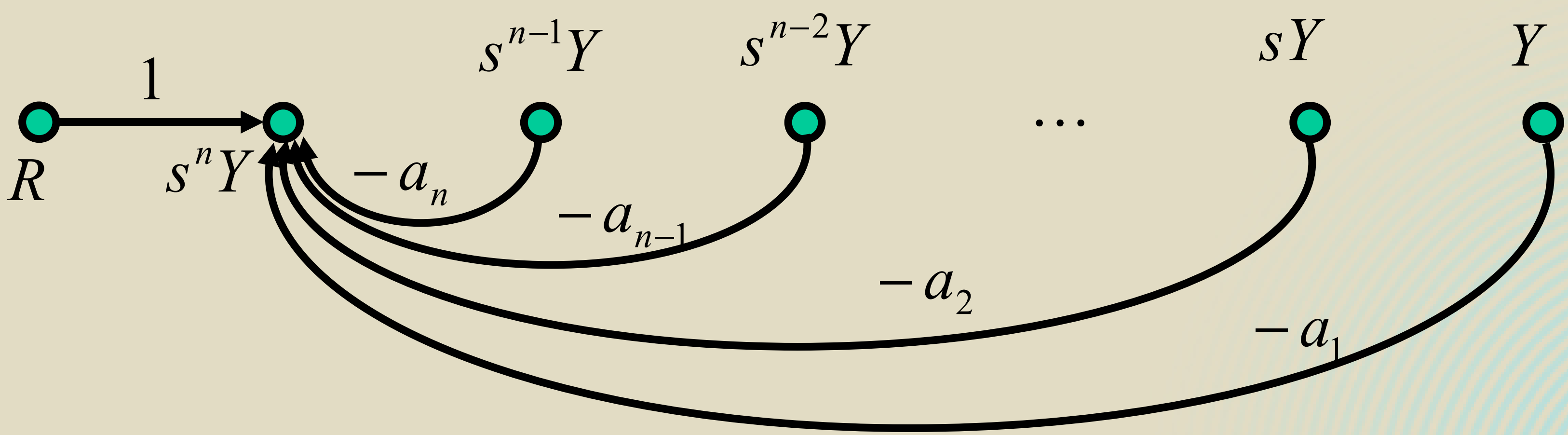
Example for Item 1

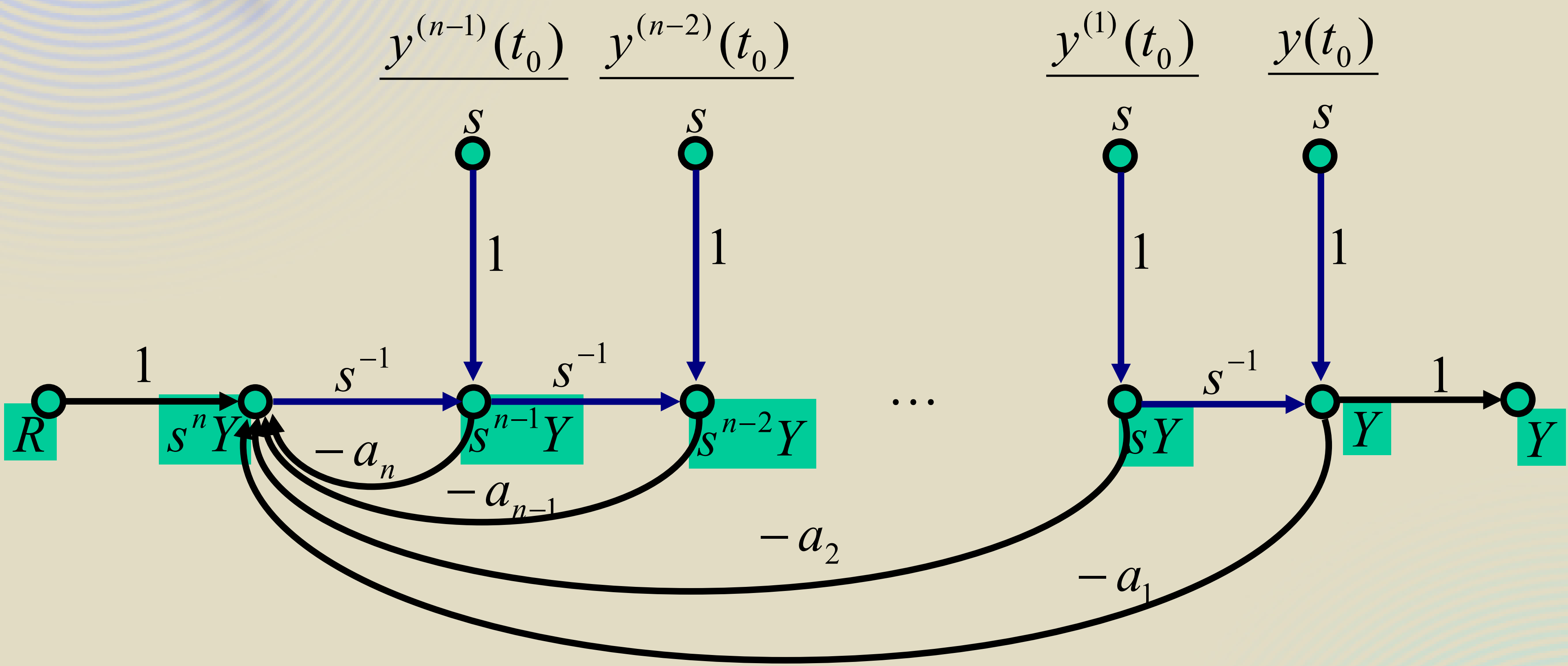
Q: Construct the SD of the following differential equation.

$$\frac{d^n y(t)}{dt} + a_n \frac{d^{n-1} y(t)}{dt} + \cdots + a_2 \frac{dy(t)}{dt} + a_1 y(t) = r(t)$$

A: rearrange the equation as

$$\frac{d^n y(t)}{dt} = -a_n \frac{d^{n-1} y(t)}{dt} - \cdots - a_2 \frac{dy(t)}{dt} - a_1 y(t) + r(t)$$





$y^{(n-1)}(t_0)$
$y^{(n-2)}(t_0)$
$y^{(1)}(t_0)$
$y(t_0)$
s
s
s
s
1
1
1
1
1
s^{-1}
s^{-1}
s^{-1}
1
$\cdots$
R
$s^n Y$
$s^{n-1}Y$
$s^{n-2}Y$
sY
Y
Y
$-a_n$
$-a_{n-1}$
$-a_2$
$-a_1$

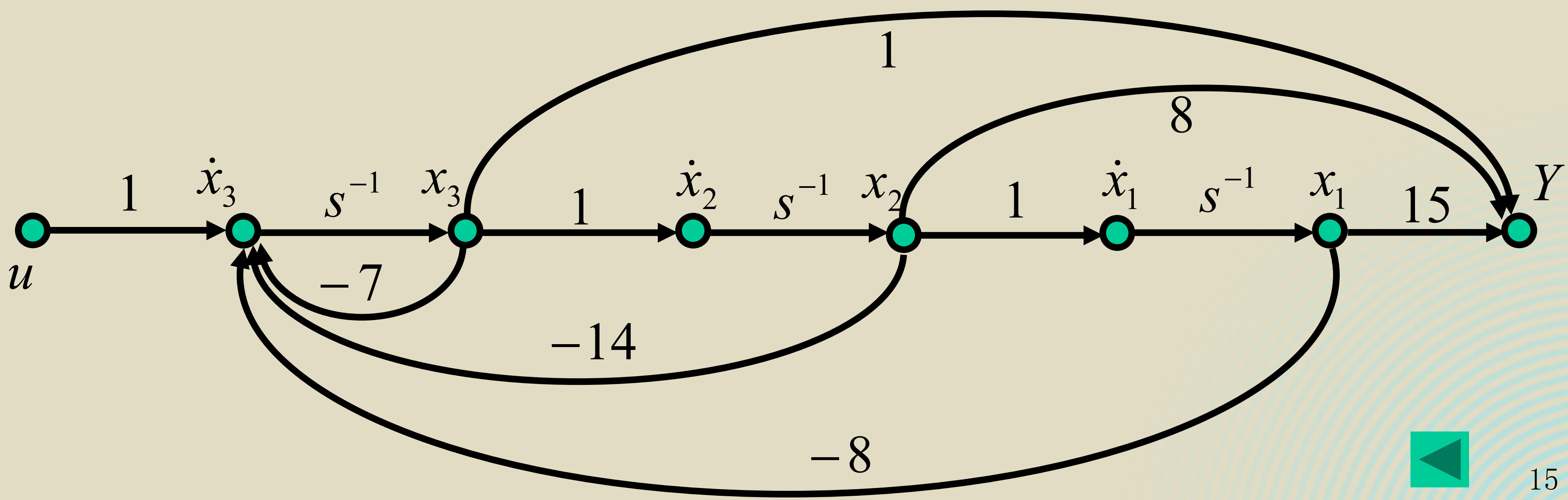
Example for Item 2

Q: Construct the SD from the following transfer function.

$$G(s)=\frac{s^2+8s+15}{s^3+7s^2+14s+8}$$

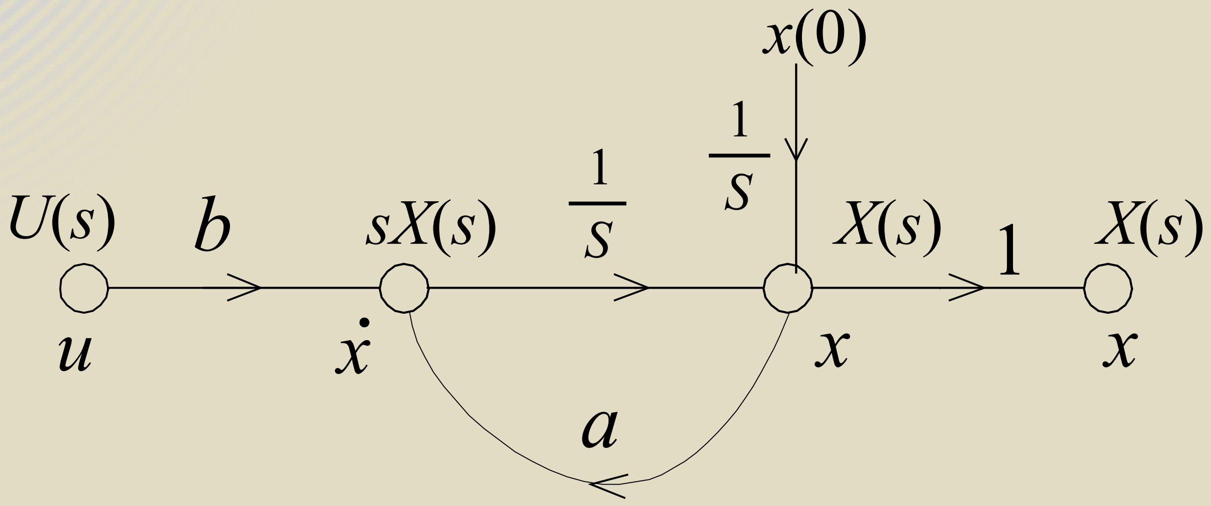
A: convert the TF to state equations

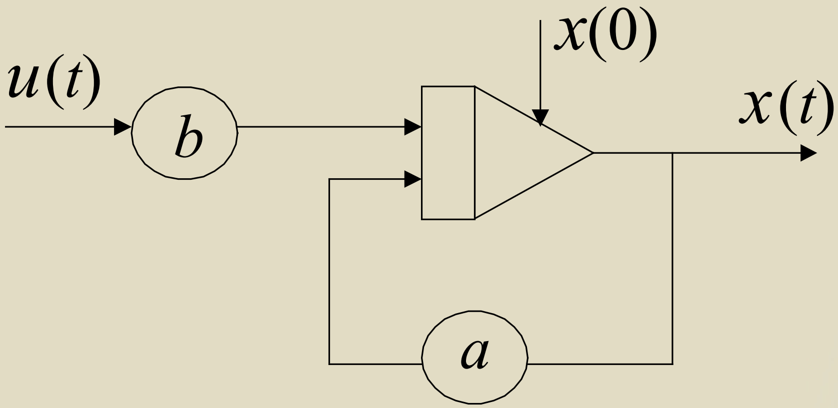
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -8 & -14 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u \qquad y=\begin{bmatrix}15 & 8 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$



Example for Item 3

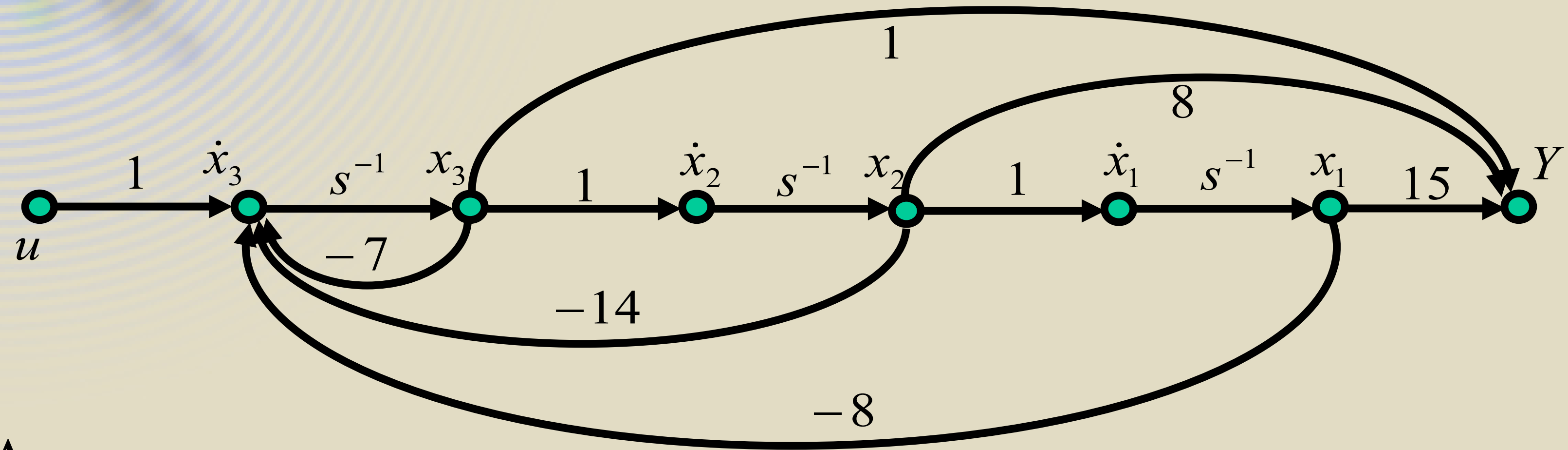
$$\dot{x} = ax + bu$$





Example for Item 4

Q: Get the transfer function from the following SD.



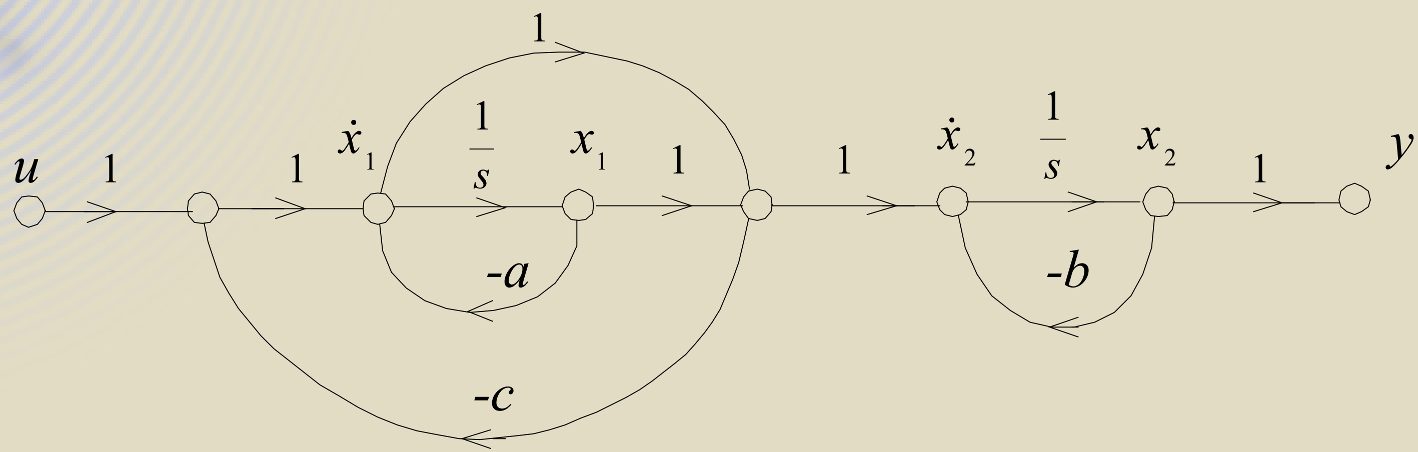
A:

loops: $\dfrac{-7}{s}, \dfrac{-14}{s^2}, \dfrac{-8}{s^3}$ forward paths: $\dfrac{1}{s}, \dfrac{8}{s^2}, \dfrac{15}{s^3}$

$$G(s) = \frac{\frac{1}{s} + \frac{8}{s^2} + \frac{15}{s^3}}{1 + \frac{7}{s} + \frac{14}{s^2} + \frac{8}{s^3}} = \frac{s^2 + 8s + 15}{s^3 + 7s^2 + 14s + 8}$$

Example for Item 5

Q: Get the state equations from the following SD.

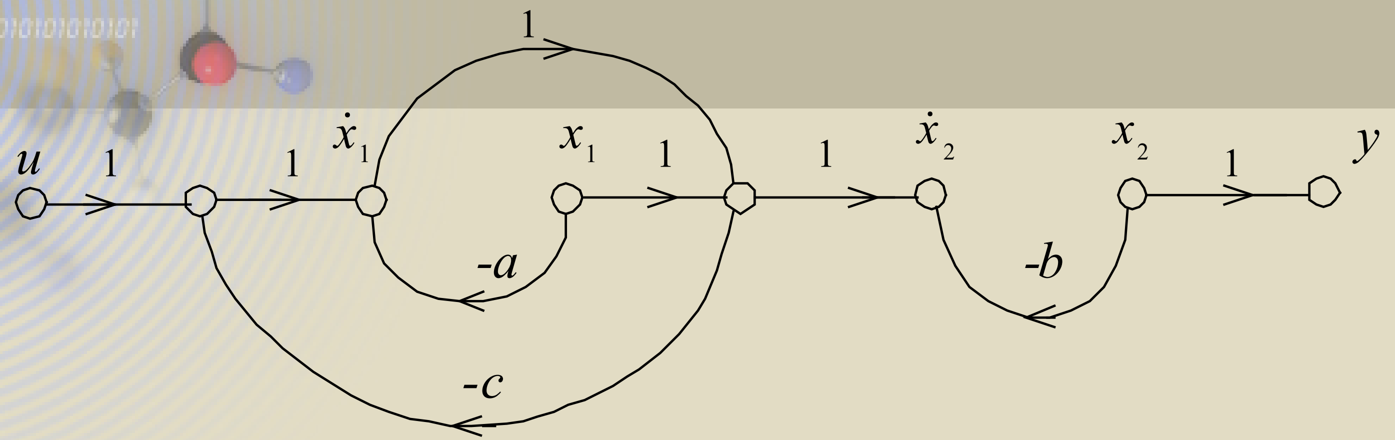


A: the general form of state equations is:

$$\dot{X} = AX + BU$$

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots b_{1m}u_m$$

How to get the parameters? Mason formula



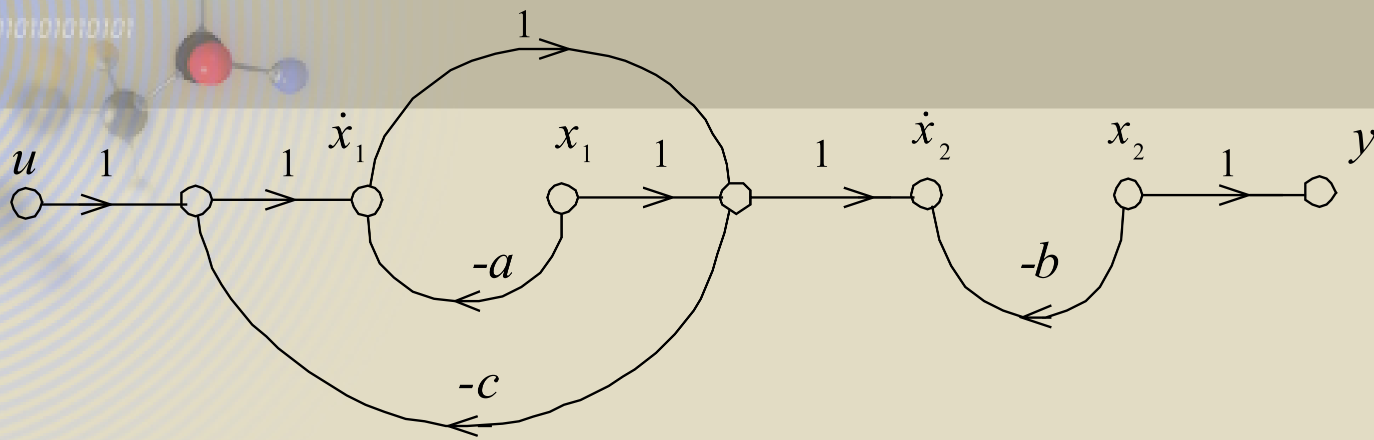
General form of state equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

loops: $-c$

forward paths: $-a, -c$

$$a_{11} = \frac{-a-c}{1+c}$$



General form of state equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{-a-c}{1+c} & 0 \\ \dfrac{1-a}{1+c} & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{1+c} \\ \dfrac{1}{1+c} \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Milestone 1

- Modeling of a LTI system
  1. ODE
  2. TF
  3. SE
- Graphical description of a LIT system
  1. Block diagrams
  2. Signal flow graph
  3. State diagram

Next Milestone

- Solving a LTI system
  1. What does it mean to solve a LTI system
  2. How to solve a LTI system
     - Solve ODEs
     - Solve TF
     - Solve SE

Application of TF to the Solution of Response

What does it mean to solve a LTI system ?

For a LTI system which is expressed by a TF

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_{m+1}s^{m}+b_{m}s^{m-1}+\cdots+b_{1}}{s^{n}+a_{n}s^{n-1}+\cdots+a_{1}}\quad (n\geqslant m)$$

$$Y(s)=G(s)U(s)$$

$$y(\mathrm{t})=L^{-1}[G(s)U(s)]$$

Find the time-domain response of the system subject to an input.

Application of TF to the Solution of Response

Basic definitions:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_{m+1}s^{m}+b_{m}s^{m-1}+\cdots+b_{1}}{s^{n}+a_{n}s^{n-1}+\cdots+a_{1}} \qquad (n \geqslant m)$$

Perform partial-fraction expansion:

$$G(s)=\frac{N(s)}{D(s)}=\frac{k(s+z_{1})(s+z_{2})\cdots(s+z_{m})}{(s+p_{1})(s+p_{2})\cdots(s+p_{n})} \qquad (n \geqslant m)$$

$-z_{1},-z_{2},\cdots,-z_{m}$ → Zeros, roots of N(s)=0

$-p_{1},-p_{2},\cdots,-p_{n}$ → Poles, roots of D(s)=0

D(s)=0: characteristic equation

Application of TF to the Solution of Response

Suppose we have a response subject to an input R(s)

$$Y(s) = G(s)\cdot R(s) = \sum_{i=1}^{r}\left[\frac{k_{i1}}{s+p_i} + \frac{k_{i2}}{(s+p_i)^2} + \cdots + \frac{k_{in_i}}{(s+p_i)^{n_i}}\right]$$

where p_i are poles; r is the number of poles, including simple poles and multi-order poles; n_i is the order of p_i . k_{in_i} is the residue of p_i .

Application of TF to the Solution of Response

If there only exist simple poles,

$$Y(s) = \frac{N(s)}{D(s)} = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \cdots + \frac{k_r}{s+p_r}$$

$$k_i = \left[(s+s_i)\frac{N(s)}{D(s)} \right] \Bigg|_{s=-s_i}$$

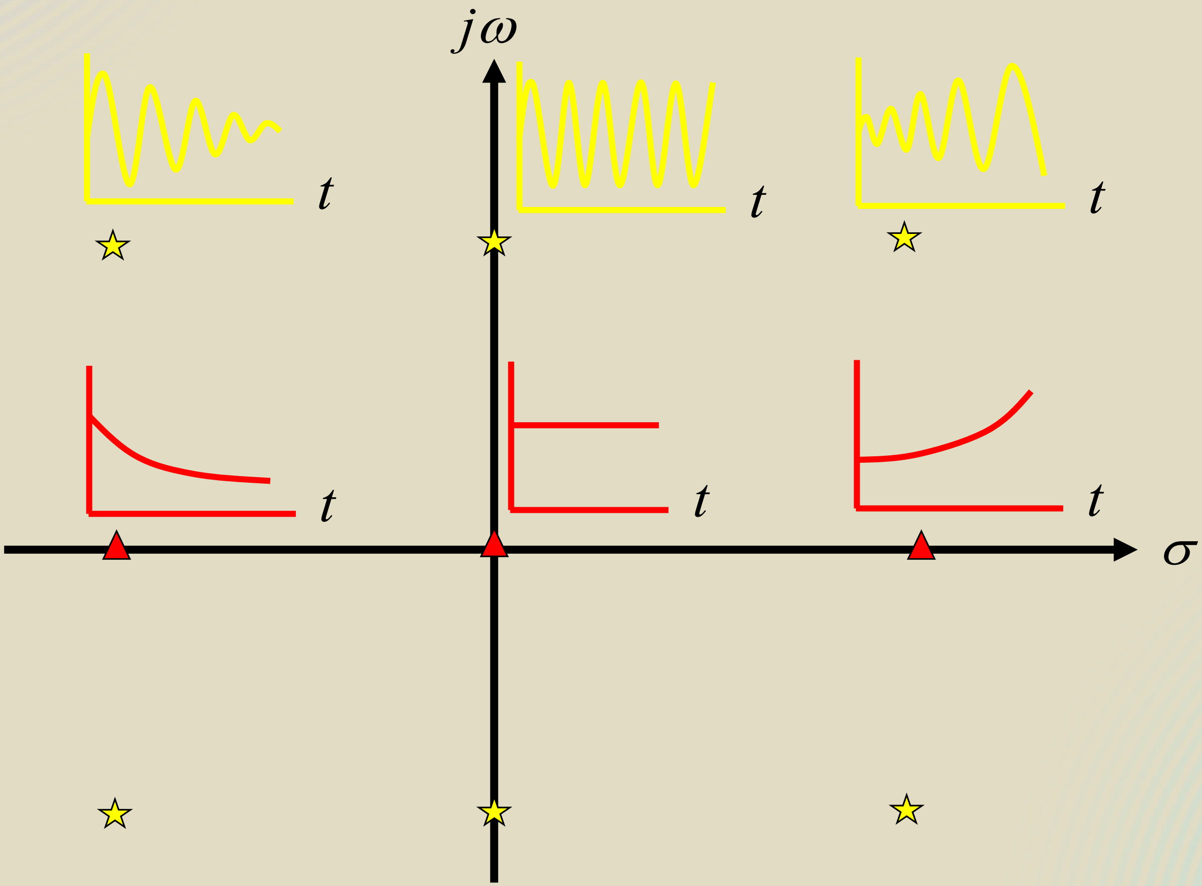
Time-domain response is:

$$y(t) = \sum_{i=1}^{r} \left[k_i e^{-p_i t} \right] = k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots + k_r e^{-p_r t}$$

Application of TF to the Solution of Response

(1) Dynamic modes are determined by poles;

(2) Magnitude and phase are affected by zeros.



Example 2.18

Q: Give out the form of the response of the following system subject to the given input signal.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2(3s+1)}{(s+1)(s+2)} \qquad u(t) = u_1 + u_2 e^{-5t}$$

A:

Laplace transform of the input: $$U(s) = \frac{u_1}{s} + \frac{u_2}{s+5}$$

Frequency domain response:

$$Y(s) = \frac{2(3s+1)}{(s+1)(s+2)} \cdot \left(\frac{u_1}{s} + \frac{u_2}{s+5}\right)$$

Time domain response:

$$y(t) = v_1 + v_2 e^{-5t} + v_3 e^{-t} + v_4 e^{-2t}$$

Example 2.19

Q: Compare the responses of the following two systems to the same unit-step signal.

$$G_1(s) = \frac{4s+2}{(s+1)(s+2)} \qquad G_2(s) = \frac{1.5s+2}{(s+1)(s+2)}$$

A:

Laplace transform of the input: $u(t) = 1(t) \qquad U(s) = 1/s$

Zero of G_1: $-1/2$ Zero of G_2: $-4/3$

Time domain response:

$$y_1(t) = 1 + 2e^{-t} - 3e^{-2t}$$

$$y_2(t) = 1 - 0.5e^{-t} - 0.5e^{-2t}$$

Example 2.20

Q: find the response of the following system to the given input signal.

$$G(s) = \frac{s+1}{s+2} \qquad u(t) = e^{-t}$$

A: Laplace transform of the input: $U(s) = \dfrac{1}{s+1}$

Frequency domain response:

$$Y(s) = G(s)U(s) = \frac{(s+1)}{(s+2)(s+1)} = \frac{1}{s+2}$$

Time domain response: $y(t) = ke^{-2t}$

Zero-pole cancellation blocks the cancelled mode from showing up in the output.

Application of TF to the Solution of Response

What does it mean to solve a LTI system ?

For a LTI system which is expressed by a state space model

SISO, first-order ODE, zero-input

Q: $\dot{x} = ax \qquad x(0) = x_0$

A: $x(t) = Ce^{at} \quad \Rightarrow \quad x(t) = x_0 e^{at} = x(0)e^{at}$

Find the time-domain trajectory of the state variable subject to the initial condition.

The Solution of State Equations

SISO, first-order ODE

Q: $\dot{x} = ax + bu \qquad x(0) = x_0$

A: $sX(s) - x(0) = aX(s) + bU(s)$

$$X(s) = \frac{1}{s-a}x(0) + \frac{1}{s-a}bU(s)$$

Real Convolution

$$F_1(s)F_2(s) = L[f_1(t) * f_2(t)] = L[\int_0^t f_2(\tau)f_1(t-\tau)d\tau]$$

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

Find the time-domain trajectory of the state variable subject to both the initial condition and the input.

The Solution of State Equations

MIMO, nth-order ODE

$$\dot{X} = AX + BU$$

$$X(s) = (sI - A)^{-1}X(0) + (sI - A)^{-1}BU(s)$$

$$e^{At} = L^{-1}[(sI - A)^{-1}]$$

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-\tau)}BU(\tau)d\tau$$

State-Transition Matrix $\Phi(t) = e^{At}$

$$e^{At} = 1 + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!}A^kt^k$$

The Solution of State Equations

How to prove $$\sum_{k=0}^{\infty}\frac{1}{k!}A^k t^k = L^{-1}[(sI-A)^{-1}]$$

The Solution of State Equations

How to prove $$\sum_{k=0}^{\infty}\frac{1}{k!}A^k t^k = L^{-1}[(sI-A)^{-1}]$$

$$L[e^{At}] = L[\sum_{k=0}^{\infty}\frac{A^k t^k}{k!}]$$

$$= \sum_{k=0}^{\infty}\frac{A^k}{k!}L[t^k]$$

$$= \sum_{k=0}^{\infty}\frac{A^k}{k!}\bullet\frac{k!}{s^{k+1}}$$

$$= \sum_{k=0}^{\infty}\frac{A^k}{s^{k+1}}$$

The Solution of State Equations

Both sides are multiplied with $(sI - A)$

$$
\begin{aligned}
(sI - A)L[e^{At}] &= (sI - A)\sum_{k=0}^{\infty}\frac{A^k}{s^{k+1}} \\
&= \sum_{k=0}^{\infty}\frac{A^k}{s^k} - \sum_{k=0}^{\infty}\frac{A^{k+1}}{s^{k+1}} \\
&= \sum_{k=0}^{\infty}\frac{A^k}{s^k} - \sum_{k=1}^{\infty}\frac{A^k}{s^k} \\
&= \left.\frac{A^k}{s^k}\right|_{k=0} \\
&= I
\end{aligned}
$$

The Solution of State Equations

$$
\begin{aligned}
(sI - A)L[e^{At}] &= I \\
L[e^{At}] &= (sI - A)^{-1} \\
e^{At} &= L^{-1}[(sI - A)^{-1}] \\
\sum_{k=0}^{\infty} \frac{A^k t^k}{k!} &= L^{-1}[(sI - A)^{-1}]
\end{aligned}
$$

The Solution of State Equations

Important feature of e^{At}

$$\frac{de^{at}}{dt} = ae^{at} \quad \Rightarrow \quad ? \quad \frac{de^{At}}{dt} = Ae^{At}$$

$$e^{At} = 1 + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!}A^kt^k$$

$$\frac{de^{At}}{dt} = 0 + A + A^2t + \cdots + \frac{1}{(k-1)!}A^kt^{k-1} + \cdots$$

$$= A(1 + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots)$$

$$= A(\sum_{k=0}^{\infty}\frac{1}{k!}A^kt^k) = Ae^{At}$$

Example 2.21

Q: find the response of the following system to the unit-step input

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

A: Laplace transform of the state equation

$$X(s) = (sI - A)^{-1} X(0) + (sI - A)^{-1} bU(s)$$

First item $\quad (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$

$$= \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ -\frac{2}{s+1} + \frac{2}{s+2} & -\frac{1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

Example 2.21

the state-transition matrix

$$\Phi(t) = e^{At} = L^{-1}[(sI - A)^{-1}] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

the second item

$$(sI - A)^{-1} bU(s) = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ -\frac{2}{s+1} - \frac{2}{s+2} & -\frac{1}{s+1} + \frac{2}{s+2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s}$$

$$= \begin{bmatrix} \frac{1}{s(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2} \\ \frac{1}{s+1} - \frac{1}{s+2} \end{bmatrix}$$

Example 2.21

$$L^{-1}[(sI-A)^{-1}bU(s)] = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

the response to the unit-step input:

$$X(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Chapter 3

Time Domain Analysis
of
Control Systems

Outlines

- **Chapter 2**
- Transfer function versus state equation – from TF to SE
- Controllability canonical form (CCF)
- Observability canonical form (OCF)
- State diagram
- How to use TF to solve the dynamic response of a system
- How to solve a state equation
- **Chapter 3**
- Basics about time domain analysis
- Stability definitions
  - Liapunov stability
  - Asymptotical stability
  - BIBO stability

Basics in Control System Analysis

- Among the many forms of performance specifications used in design, the most important requirement is that the system be stable.
- When all types of systems are considered – linear, nonlinear, time-invariant, and time-varying – the definition of stability can be given in many forms.
- For analysis and design purposes, we can clarify stability as absolute stability and relative stability.
- Absolute stability refers to the condition of whether the system is stable or unstable; it is a yes or no answer.
- Relative stability refers to the degree of stability; it defines how stable a stable system is.

Basics in Control System Analysis

- Why analyze?
- Analyze what?
- How to analyze?

Basics in Control System Analysis

- In preparation of the definition of stability, we define the following two types of responses for LTI systems:

  - Zero-state response: the zero-state response is due to the input only; all the initial conditions of the system are zero.

  - Zero-input response: the zero-input response is due to the initial conditions only; all the inputs are zero.

  - From the principle of superposition, when a system is subject to both inputs and initial conditions, the total response is written

total response = zero-state response + zero-input response

Zero-input Stability (Liapunov Stability)

- Zero-input stability refers to the stability condition when the input is zero, and the system is driven by its initial conditions.
- A LTI dynamic system without input is depicted by a homogeneous ODE

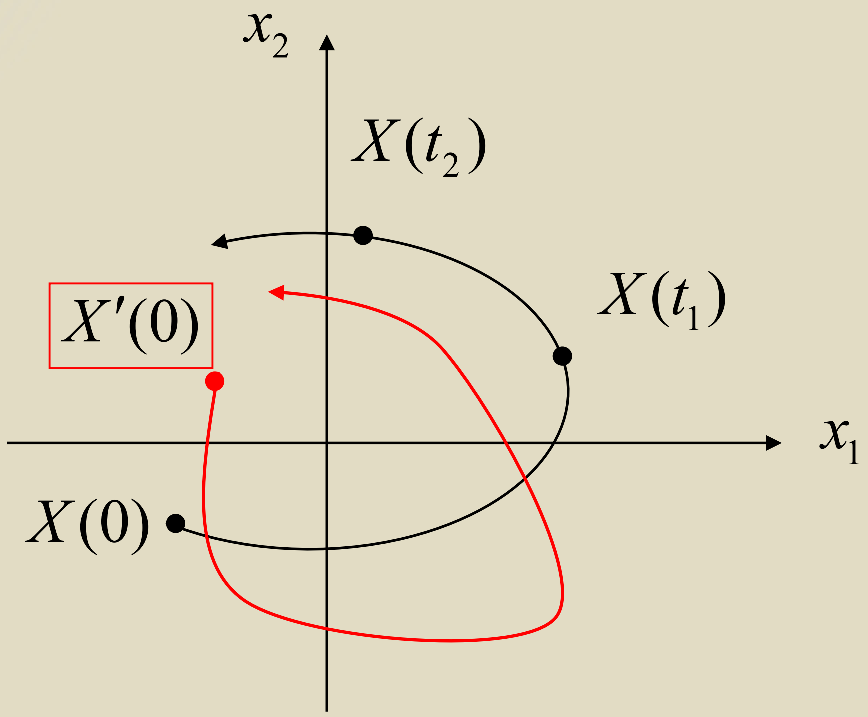
$$\dot{X}(t) = AX(t)$$

Solution: $$X(t) = e^{At}X(0)$$

Modes: $$\begin{bmatrix} e^{-\lambda_1 t} & e^{-\lambda_2 t} & \cdots & e^{-\lambda_n t} \end{bmatrix}^{\mathrm{T}}$$

Definition of Zero-input Stability

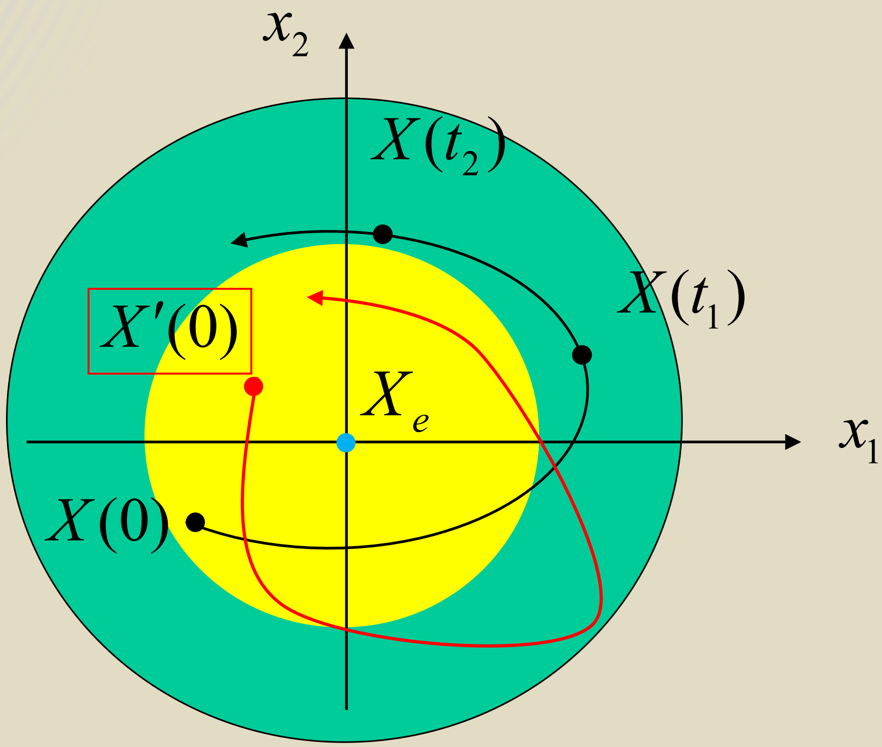
System trajectory can be sketched in the phase space:



Different initial points are corresponding to different system trajectories.

Definition of Zero-input Stability

Assume a system has an equilibrium point: X_e



I want the system trajectories to be within a certain range of X_e

Can you tell me what range the initial point should be located in?

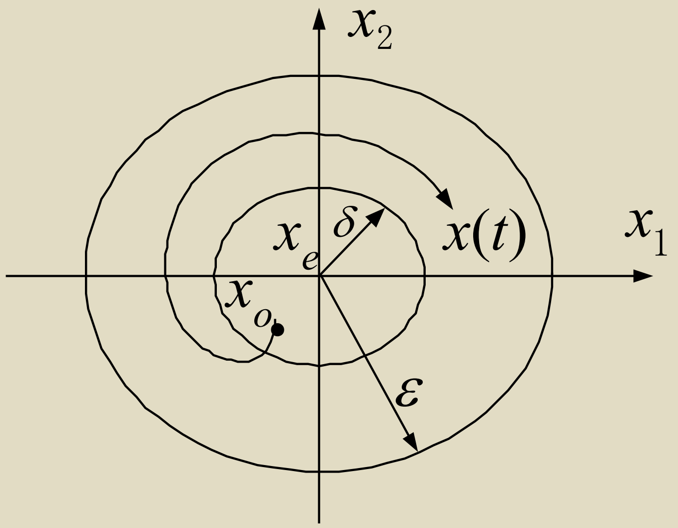
Definition of Zero-input Stability

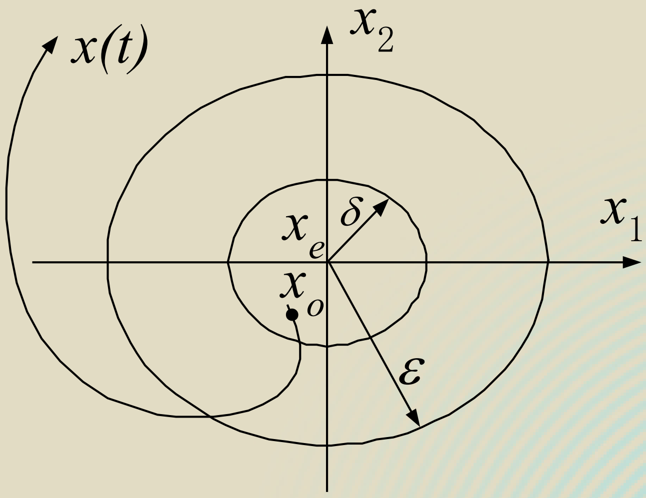
A system: $\dot{X}(t) = f[X(t), t]$

Equilibrium point: X_e

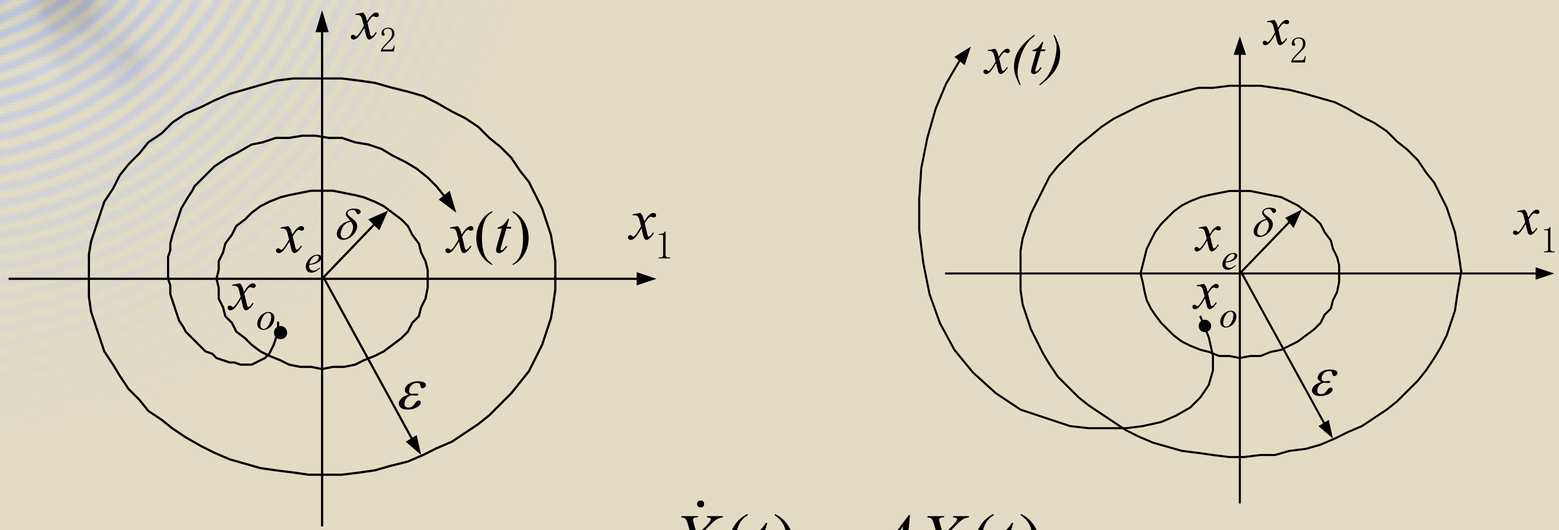
Given finite trajectory: $\|X_t - X_e\| \le \varepsilon$

Initial condition range: $\|X_0 - X_e\| \le \delta$





Definition of Zero-input Stability



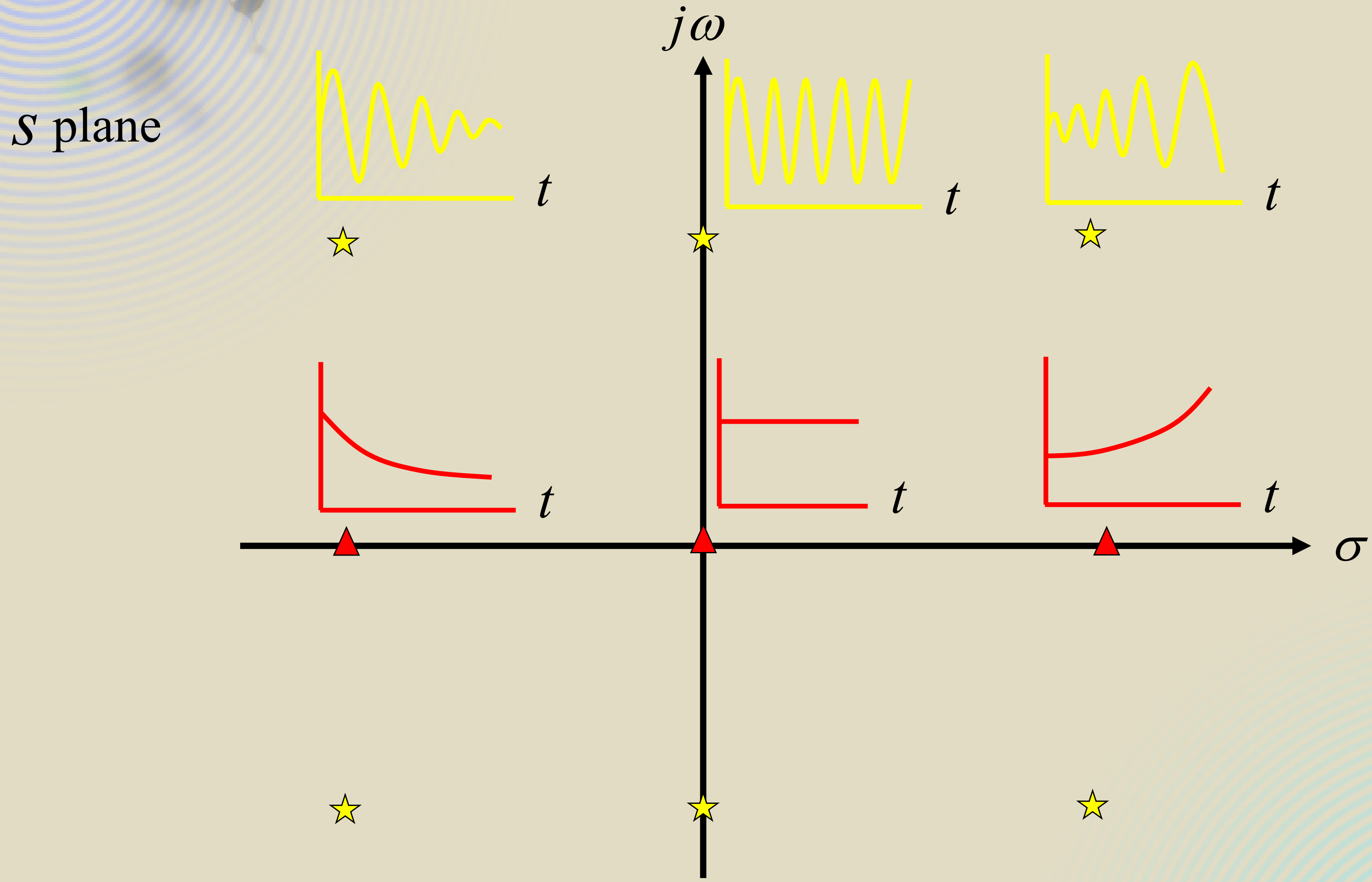
$$\dot{X}(t) = AX(t)$$

Solution: $$X(t) = e^{At}X(0)$$

Modes: $$[e^{-p_1 t}, e^{-p_2 t}, \cdots, e^{-p_r t}]$$

$$x_1(t) = \sum_{i=1}^{r}\left[k_i e^{-p_i t}\right] = k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots + k_r e^{-p_r t}$$

Stability Region in S-plane



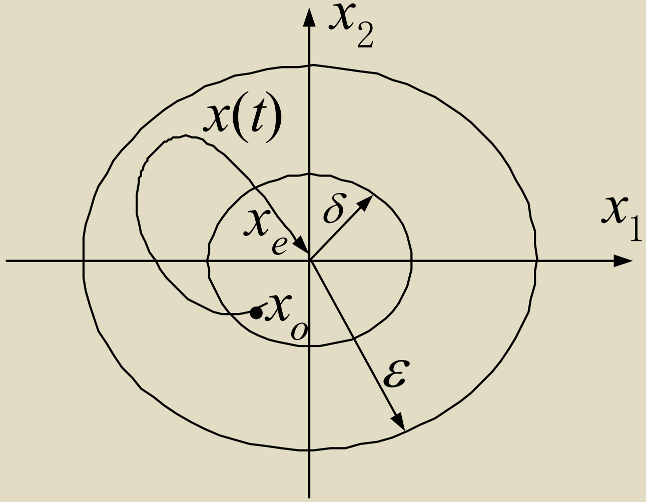
Asymptotic Stability

A system: $\dot{X}(t) = f[X(t), t]$

Equilibrium point: X_e

Finite initial condition: $\left\| X_0 - X_e \right\| \le \delta$

Finite response: $\lim\limits_{t\to\infty} X(t) = \lim\limits_{t\to\infty} e^{At} X(t_0) = X_e$



Definition of Zero-State Stability

Bounded-Input Bounded-Output Stability

Definition: let $u(t)$, $y(t)$, and $g(t)$ be the input, output, and impulse response of a linear time-invariant system, respectively. With zero initial conditions, the system is said to be bounded-input bounded-output (BIBO) stable, or simply stable, if its output $y(t)$ is bounded to a bounded input $u(t)$

What's the sufficient and necessary condition for BIBO stable?

Definition of Zero-State Stability

The convolution integral: $$y(t) = \int_0^\infty u(t-\tau)g(\tau)d\tau$$

The absolute value:

$$|y(t)| = \left|\int_0^\infty u(t-\tau)g(\tau)d\tau\right|$$

$$|y(t)| \le \int_0^\infty |u(t-\tau)||g(\tau)|d\tau$$

If u(t) is bounded: $|u(t)| \le M$ M is a finite positive number

$$|y(t)| \le M\int_0^\infty |g(\tau)|d\tau$$

If y(t) is bounded: $$|y(t)| \le M\int_0^\infty |g(\tau)|d\tau \le N < \infty$$

The condition: $$\boxed{\int_0^\infty |g(\tau)|d\tau \le Q < \infty}$$

Zero-State Stability Criterion

Study the stability condition in s-plane:

Laplace transform of $g(t)$ $$G(s) = \int_0^\infty g(t)e^{-st}dt$$

$$|G(s)| = \left|\int_0^\infty g(t)e^{-st}dt\right| \le \int_0^\infty |g(t)||e^{-st}|dt$$

Where s is a complex number, $s = \sigma + j\omega$, σ, ω are real numbers.

The dynamics of a control system is determined by its poles, so we need to study the above equation when s is a pole.

Zero-State Stability Criterion

If s is a pole of G(s), and considering $\left|e^{-st}\right| = \left|e^{-\sigma t}\right|$,

$$\left|G(s)\right| = \left|\int_0^\infty g(t)e^{-st}dt\right| \le \int_0^\infty \left|g(t)\right|\left|e^{-st}\right|dt$$

$$\infty = \left|G(s)\right| \le \int_0^\infty \left|g(t)\right|\left|e^{-\sigma t}\right|dt$$

If one or more roots of the characteristic equation are in the right-half s-plane or on the imaginary axis, $\sigma \ge 0$

Then: $\left|e^{-\sigma t}\right| \le K = 1$

So: $\infty \le \int_0^\infty K\left|g(t)\right|dt \le \int_0^\infty \left|g(t)\right|dt$

This conflicts with the stability condition!

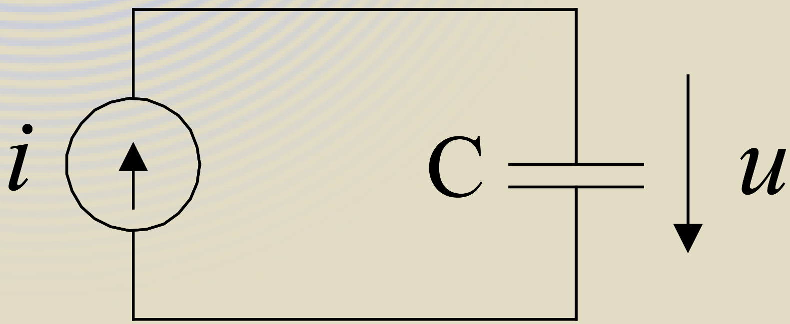
Zero-State Stability Criterion

Thus, for BIBO stability, the roots of the characteristic equation, or the poles of G(s), cannot be located in the right-half s-plane or on the imaginary axis, or they must lie in the left-half s-plane.

A system is said to be unstable if it is not BIBO stable. When a system has roots on the imaginary axis, say, at $s = j\omega_0$ and $s = -j\omega_0$, if the input is a sinusoid, $\sin\omega_0 t$, the output will be of the form of $t\sin\omega_0 t$, which is unbounded, and the system is unstable.

Example – 3.1

Q: find the response of the following circuit to given inputs, and analyze its BIBO stability.



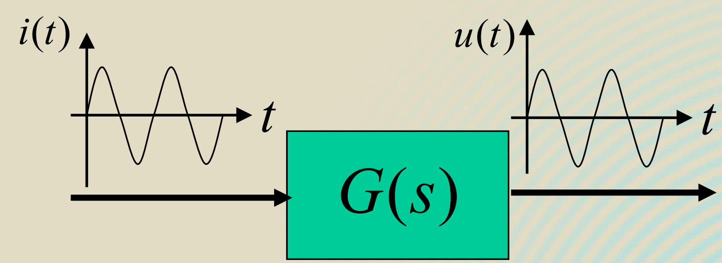
$$i = c\frac{du}{dt} \qquad I(s) = sCU(s)$$

$$G(s) = {U(s)}\big/{I(s)} = 1\big/{sC}$$

(1) When the input is $i(t) = \sin\omega_0 t$

$$\dot{U} = G(j\omega)\dot{I} = \frac{1}{j\omega C}\dot{I} = \frac{I}{\omega C}\angle -90^0$$

$$u(t) = \frac{1}{\omega C}\sin(\omega t - 90^0)$$

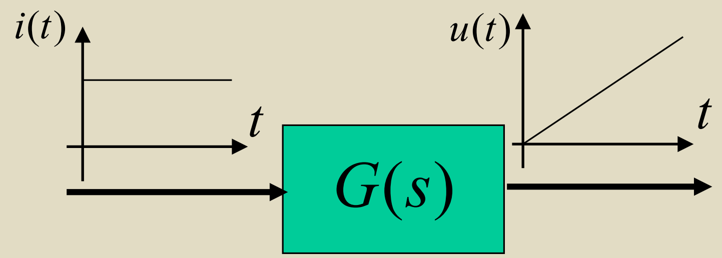


Example – 3.1

(2) When the input is $i(t) = 1(t)$

$$G(s) = \frac{1}{sC} \qquad I(s) = \frac{1}{s}$$

$$U(s) = G(s)I(s) = \frac{1}{s^2 C} \Rightarrow u(t) = \frac{1}{C}t$$

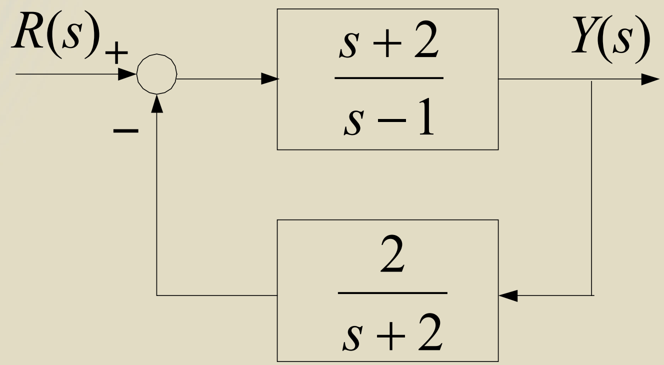


Is the system BIBO stable?

What can we get from this example?

Example – 3.2

Q: find if the following system BIBO stable, asymptotic stable.



A: the transfer function of the above system is:

$$G(s) = \frac{\dfrac{s+2}{s-1}}{1+\dfrac{s+2}{s-1}\dfrac{2}{s+2}} = \frac{(s+2)^2}{(s+1)(s+2)} = \frac{s+2}{s+1}$$

One pole is at -1, a cancelled pole is at -2. Both are located at the left-half s-plane. The system is both BIBO and asymptotic stable.

Wrap-up

- Transfer function versus state equation – from TF to SE
- Controllability canonical form (CCF)
- Observability canonical form (OCF)
- State diagram
- How to use TF to solve the dynamic response of a system
- How to solve a state equation
- Basics about time domain analysis
- Stability
  - Zero-State & Zero-Input Response
  - Liapunov stability
  - Asymptotical stability
  - BIBO stability

Assignment

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1，(2)

DISCUSSION

Is a time delay block a linear block?

$$y(t) = f[u(t)] = u(t-\tau)$$

DISCUSSION

Basic idea:

According to the definition, if a system is linear, the system must satisfy the following two requirements:

- **Principle of superposition**

$$f[u_1(t)+u_2(t)] = f[u_1(t)]+f[u_2(t)]$$

- **Property of homogeneity**

$$f[\beta u_1(t)]=\beta f[u_1(t)]$$

DISCUSSION

Proof: **Principle of superposition**

There is a time delay block whose parameter is τ .

With two inputs $u_1(t)$ and $u_2(t)$, corresponding outputs are

$$y_1(t) = f[u_1(t)] = u_1(t-\tau)$$

$$y_2(t) = f[u_2(t)] = u_2(t-\tau)$$

Let $u_3(t)$ denote the sum of $u_1(t)$ and $u_2(t)$

$$u_3(t) = u_1(t) + u_2(t)$$

The output of the time delay block with respect to $u_3(t)$ is

$$f[u_3(t)] = u_3(t-\tau) = u_1(t-\tau) + u_2(t-\tau)$$

$$f[u_1(t)+u_2(t)] \qquad = \qquad f[u_1(t)] + f[u_2(t)]$$

Thus, the principle of superposition holds

DISCUSSION

Proof: Property of homogeneity

$u_4(t)$ **is denoted as :**

$$u_4(t) = \beta u_1(t)$$

The output of the time delay block with respect to $u_4(t)$ **is**

$$\underline{f[u_4(t)]} = u_4(t-\tau) = \underline{\beta u_1(t-\tau)}$$

$$f[\beta u_1(t)] \qquad = \qquad \beta f[u_1(t)]$$

Thus, the property of homogeneity holds

Now, we have proved that a time delay block is a linear block.

DISCUSSION

Further information：

In time domain, the time delay block is not a linear function with respect to time t , but it is a linear block with respect to the input $u(t)$.

In s domain, the Laplace transform of the time delay block $e^{-\tau s}$ is not a linear function with respect to the parameter s , but it is still a linear block with respect to the input.