Chapter 3

Time Domain Analysis of Control Systems

Review

- **Stability**
  - Stability condition
  - Routh-Hurwitz criterion
- **Steady-state performance**
  - Definition of time response;
  - Typical test signals;
  - Performance specifications of steady-state response – steady-state error;
  - Impact of disturbance to steady-state error;
  - Impact of parameter variation to steady-state error.
- **Transient Response**
  - Performance Criteria
  - Transient Response of 1st-Order Systems
  - Transient Response of 2nd-Order Systems

Outline

- **Transient Response**
  - Approximation of High-Order Systems
- **Root loci**
  - Basic concept about root loci;
  - Rules for constructing root loci;

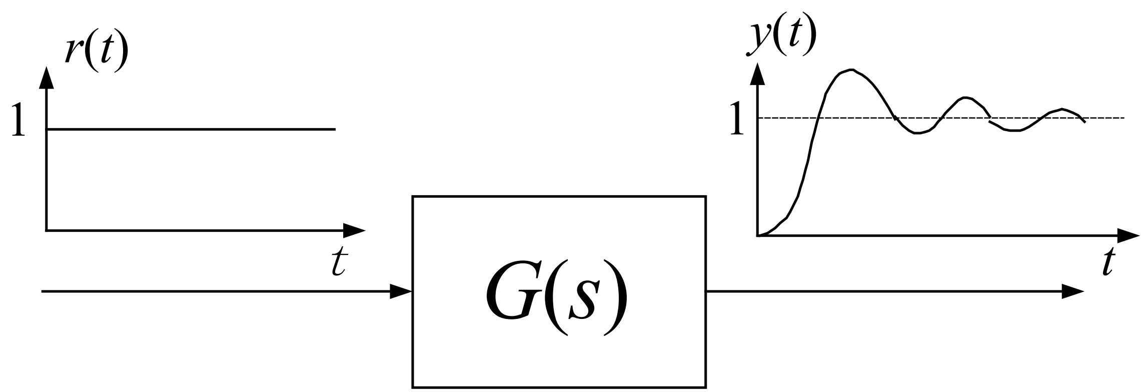
High-order Systems' Time Response

Laplace transform of the step response of a high-order system

$$Y(s) = G(s)R(s) = \frac{b_{m+1}s^m + b_m s^{m-1} + \cdots + b_2 s + b_1}{\prod_{j=1}^{q}(s+p_j)\prod_{k=1}^{r}(s^2 + 2\zeta_k\omega_k s + \omega_k^2)} \cdot \frac{1}{s}$$

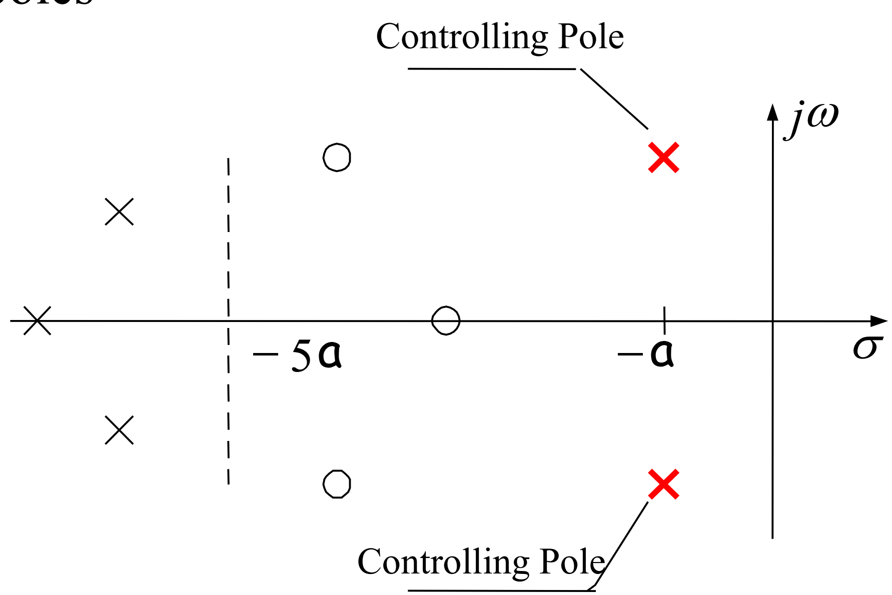
Time response of the system with a step-function as input

$$y(t) = G(0) + \sum_{j=1}^{q} A_j e^{-p_j t} + \sum_{k=1}^{r} B_k e^{-\zeta_k \omega_k t} \sin\left[\omega_{dk} t + \varphi_k\right]$$



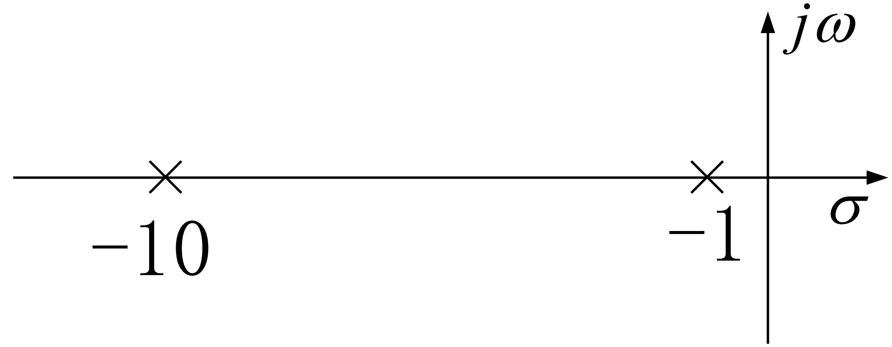
High-order Systems' Time Response

1. The time response of a high-order system is the linear combination of those of the 1st and 2nd order systems;
2. Poles of a high-order system which is far away from the imaginary axis has less impact on the system's transient response than those close to the imaginary axis. The farther the distance is, the smaller the impact is.
3. Zeros of a high-order system also has impact on the system's time response. They mainly affect the magnitude and phase of a dynamic mode.
4. The performance of a high-order system is mainly determined by its controlling poles



Approximation of High-order Systems by Low-Order Systems

Example: $$G(s) = \frac{10}{(s+1)(s+10)}$$



To keep the final value the same

$$G(s) = \frac{1}{s+1}$$

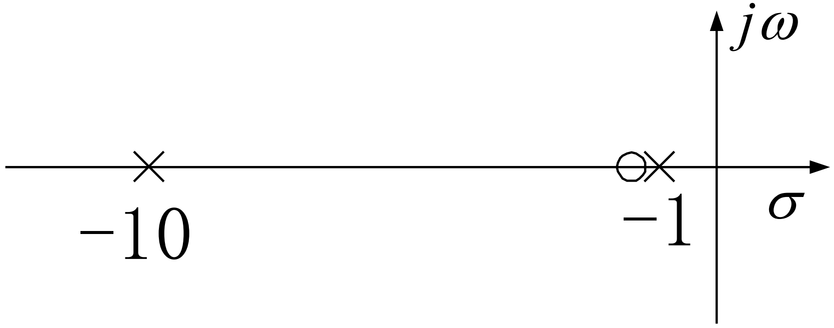
$$Y(s) = G(s)R(s) = \frac{10}{(s+1)(s+10)}\frac{1}{s}$$

$$y(t) = 1 - \frac{10}{9}e^{-t} + \frac{1}{9}e^{-10t} \approx 1 - \frac{10}{9}e^{-t} \approx 1 - e^{-t}$$

Approximation of High-order Systems by Low-Order Systems

Example: $$G(s)=\frac{10(\frac{9}{10}s+1)}{(s+1)(s+10)}$$

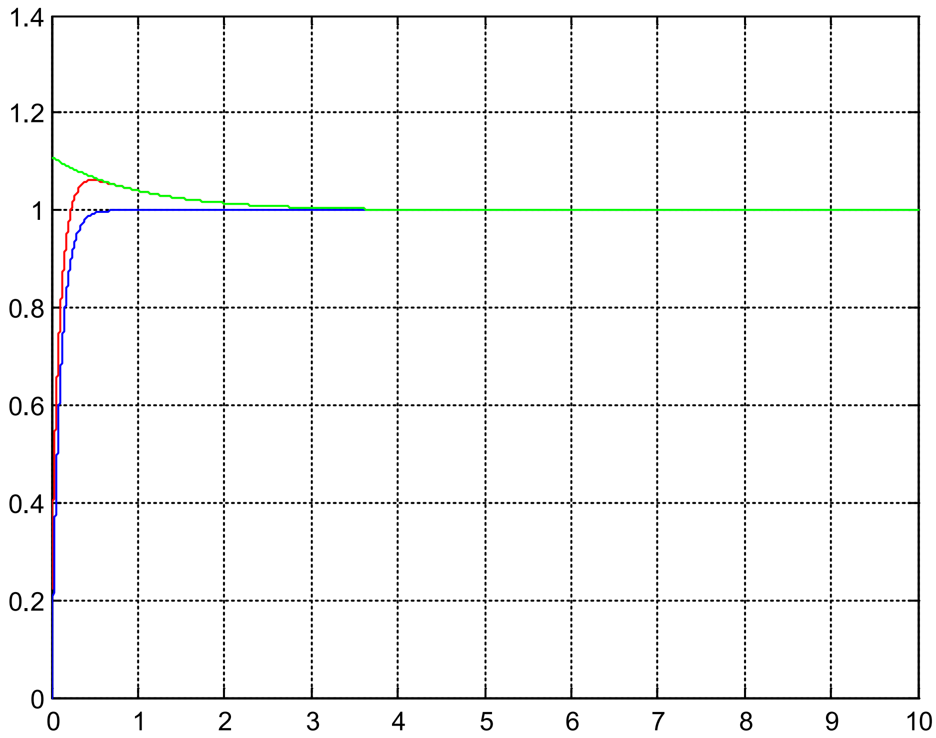
$$G(s)=\frac{10}{s+10}$$



$$Y(s)=G(s)R(s)=\frac{10(\frac{9}{10}s+1)}{(s+1)(s+10)}\frac{1}{s}$$

$$y(t)=1+\frac{1}{9}e^{-t}-\frac{8}{9}e^{-10t}\approx 1-e^{-10t}$$

Approximation of High-order Systems by Low-Order Systems



$$y(t) = 1 + \frac{1}{9}e^{-t} - \frac{8}{9}e^{-10t}$$

$$y(t) = 1 - e^{-10t}$$

$$y(t) = 1 + \frac{1}{9}e^{-t}$$

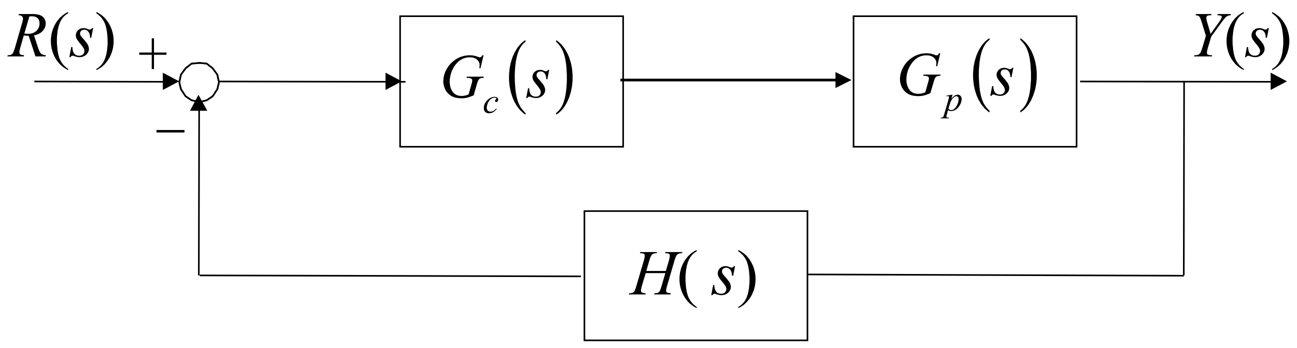
Approximation Criterion

The criterion of finding the low-order $M_L(s)$, given $M_H(s)$, is that the following relation should be satisfied as close as possible:

$$\frac{\left|M_H(j\omega)\right|^2}{\left|M_L(j\omega)\right|^2} = 1 \quad for \quad 0 \le \omega \le \infty$$

This condition implies that the amplitude characteristics of the two systems in the frequency domain ($s = j\omega$) are similar.

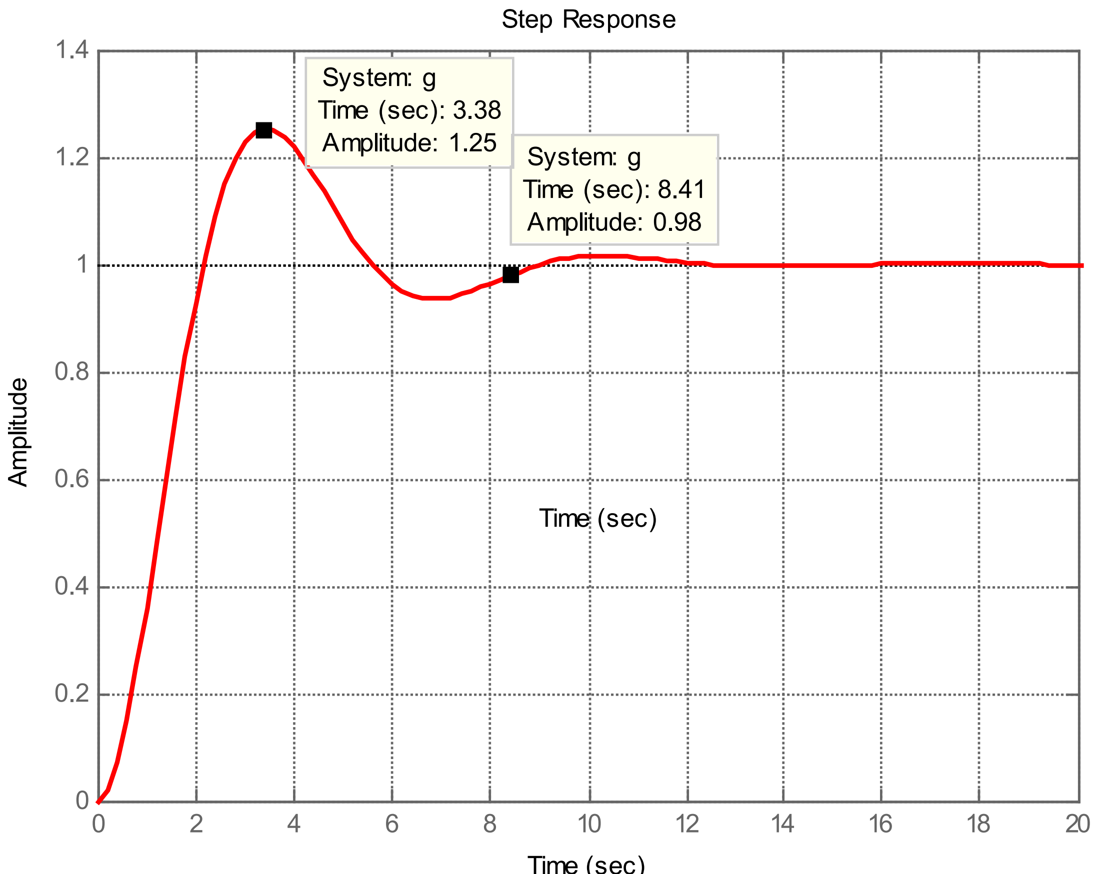
Impact of Additional Poles and Zeros



Adding a controller to a system actually is to add some poles and zeros to the original system. So, it is important to study the impact of adding additional poles and zeros.

Impact of Additional Poles and Zeros

Original system: $$G(s) = \frac{1}{s^2 + 0.8s + 1}$$



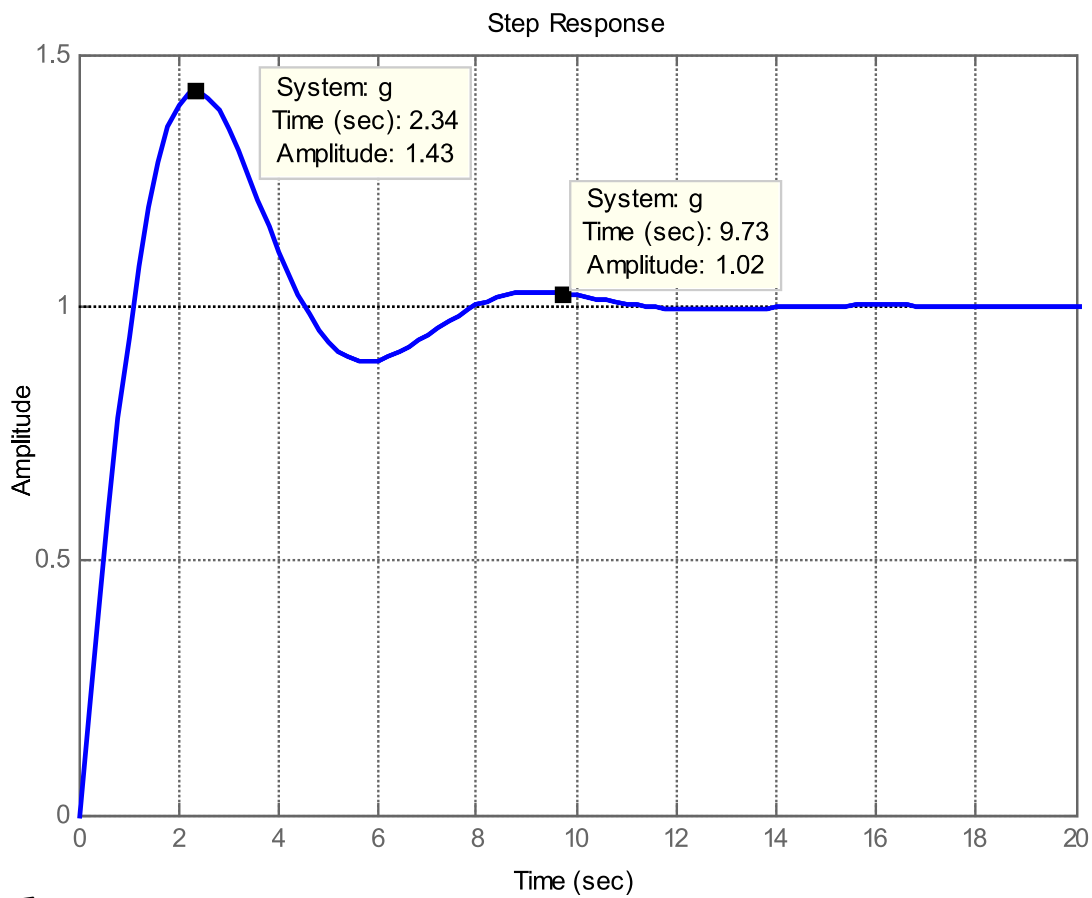
$$s_{1,2} = -0.4 \pm j0.917$$

$$\sigma\% = 25\%$$

$$t_s = 8.41s$$

Impact of Additional Zeros

With additional zero: $$G(s) = \frac{s+1}{s^2 + 0.8s + 1}$$



$$s_{1,2} = -0.4 \pm j0.917$$

$$\sigma\% = 43\%$$

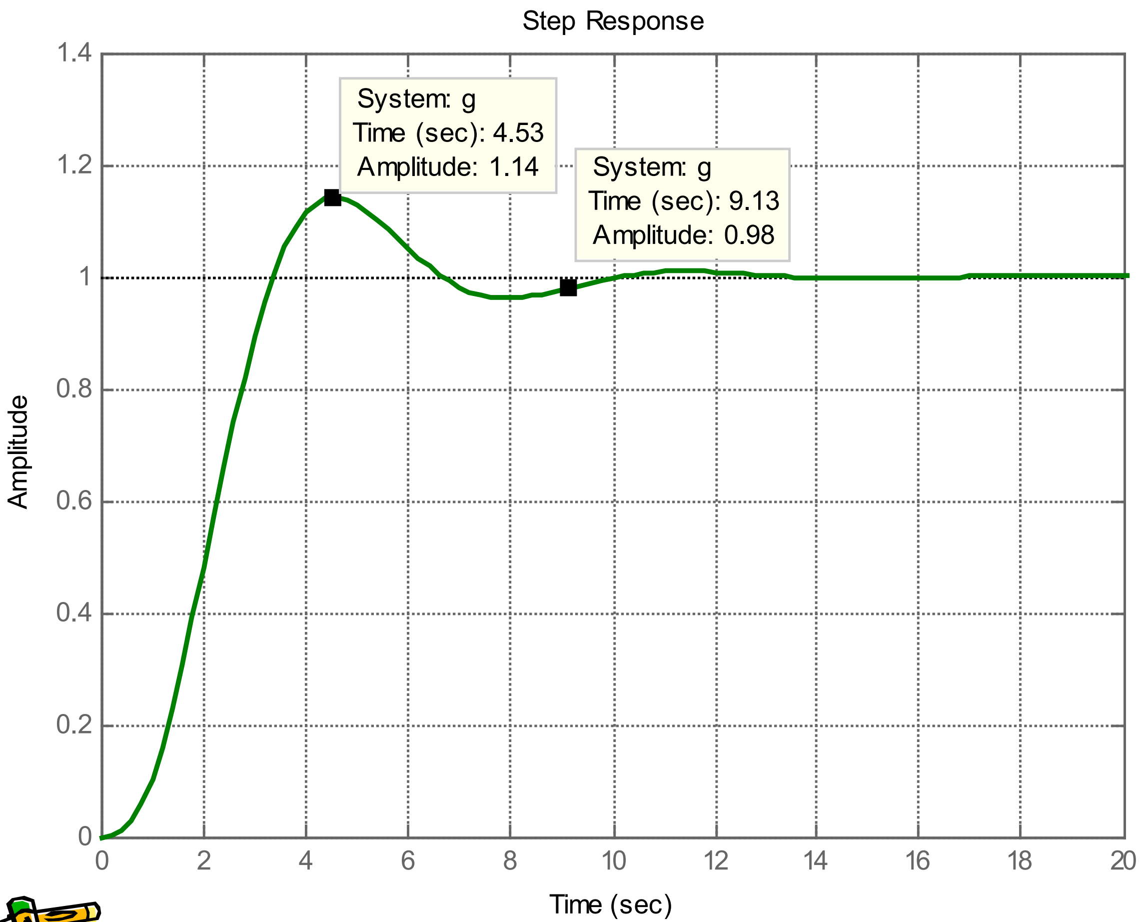
$$t_s = 9.73s$$

$$t_r \quad \downarrow$$

$$t_p \quad \downarrow$$

Impact of Additional Poles

With additional pole: $$G(s) = \frac{1}{(s^2 + 0.8s + 1)(s + 1)}$$



$$s_{1,2} = -0.4 \pm j0.917$$

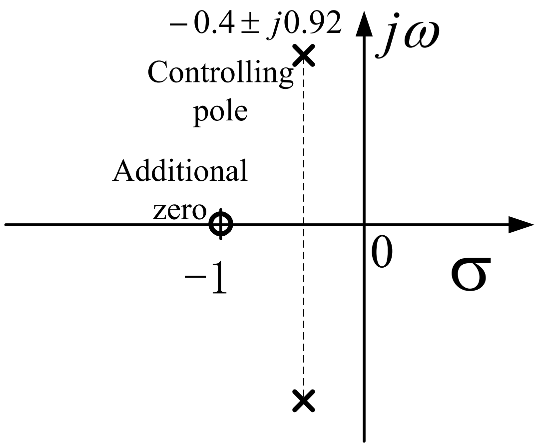
$$\sigma\% = 14\%$$

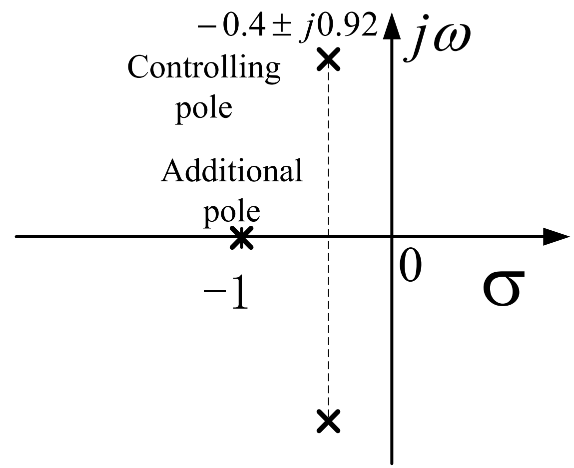
$$t_s = 9.13s$$

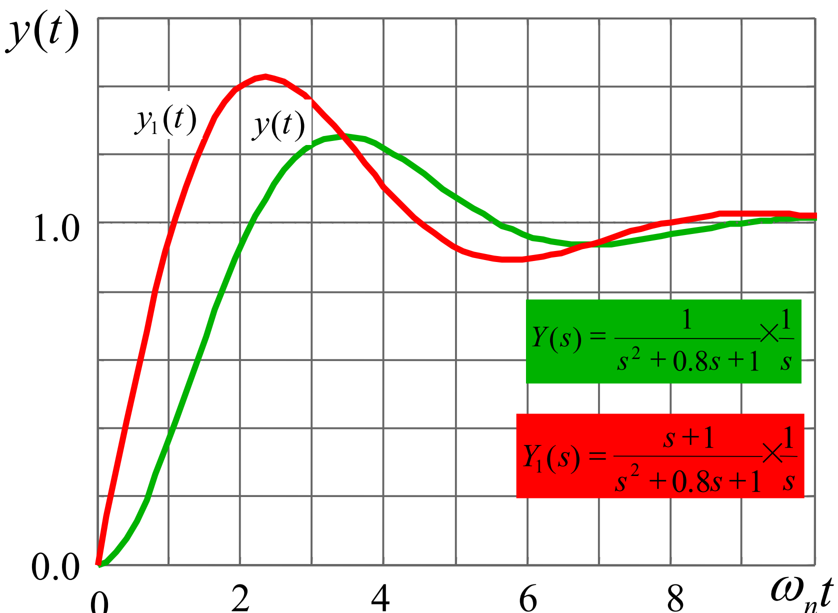
$$t_r \quad \uparrow$$

$$t_p \quad \uparrow$$

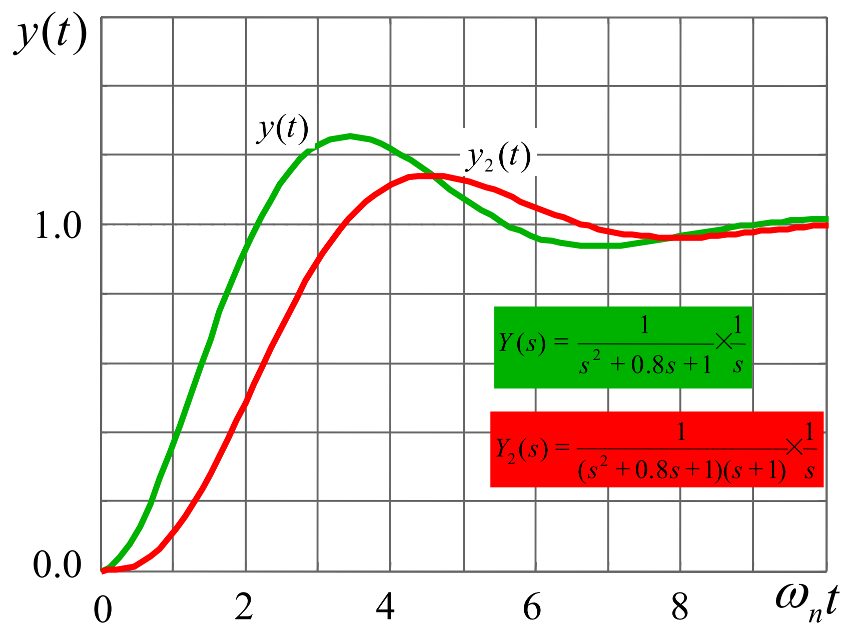
Conclusion on the Impact of Additional Poles and Zeros







Decrease Damping Ratio

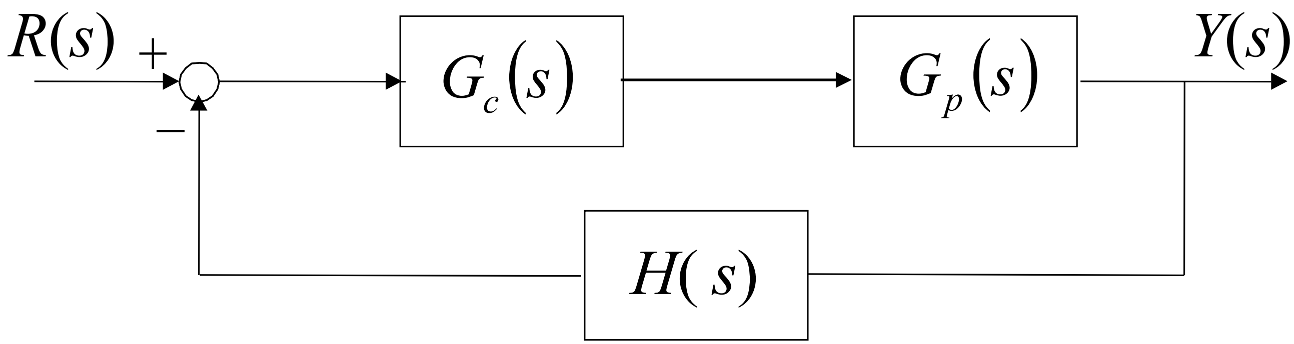


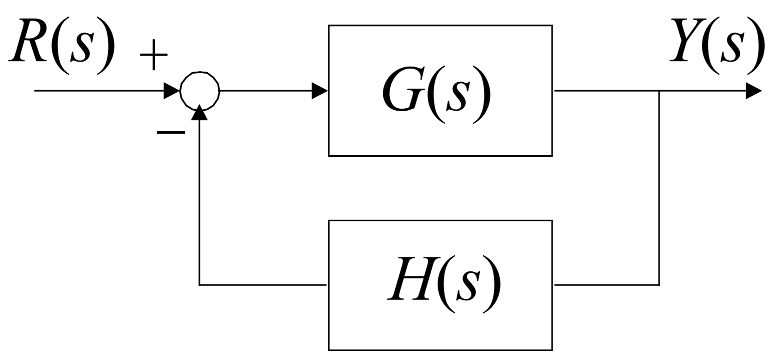
Increase Damping Ratio

Next Milestone

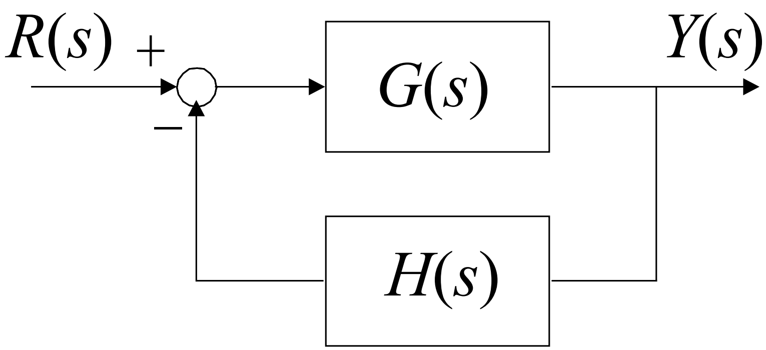
- **Root loci**
  - Basic concept about root loci;
  - Rules for constructing root loci;

Conditions for Being a Root of Characteristic Equation





Conditions for Being a Root of Characteristic Equation



Closed-loop TF of the system:

$$M(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{G(s)}{1 + G_0(s)}$$

The characteristic equation: $1 + G_0(s) = 0$ or $G_0(s) = -1$

Our goal is to comprehend the closed-loop system by analyzing the open loop system.

Basic Concepts about Root Locus

Time-Constant form of $G_0(s)$: $$G_0(s) = \frac{k\prod_{i=1}^{m}(T_i s+1)}{s^r\prod_{j=1}^{v}(\tau_j s+1)}$$

Zero-pole form: $$G_0(s) = \frac{k'\prod_{i=1}^{m}(s+z_i)}{s^r\prod_{j=1}^{v}(s+p_j)} = \frac{k'\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$

$$k = k'\frac{\prod_{i=1}^{m} z_i}{\prod_{j=1}^{v} p_j}$$

Root locus gain $$k' = k\frac{\prod_{j=1}^{v} p_j}{\prod_{i=1}^{m} z_i}$$

Basic Concepts about Root Locus

What is root loci?

open-loop system

closed-loop system

$$G_0(s) = \frac{k' \prod_{i=1}^{m} (s + z_i)}{\prod_{j=1}^{n} (s + p_j)} \xrightarrow[\text{equation}]{\text{characteristic}} \frac{k' \prod_{i=1}^{m} (s + z_i)}{\prod_{j=1}^{n} (s + p_j)} + 1 = 0$$

zeros: z_i

poles: p_i

$\tilde{z}_i$ Roots of the above equation

Poles of the closed-loop system

when k' varies,

z_i and p_i keep the same

$\tilde{z}_i$ change

Root loci are the root trajectories of the closed-loop system when k' changes

Basic Concepts about Root Locus

An example:

open-loop system

$$G_0(s) = \frac{k'(s+1)}{(s+2)}$$

$\xrightarrow[\text{equation}]{\text{characteristic}}$

closed-loop system

$$\frac{k'(s+1)}{(s+2)} + 1 = 0$$

zeros: $z_i = -1$

poles: $p_i = -2$

when k' varies,

z_i and p_i keep the same

$\tilde{z}_i$: $s = -\dfrac{k'+2}{k'+1}$

$\tilde{z}_i$ change

Root loci are the root trajectories of the closed-loop system when k' changes

Basic Concepts about Root Locus

Condition on Magnitude:

$$\left|G_0(s)\right| = \left|\frac{k'\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}\right| = 1$$

$$k' = \frac{\prod_{j=1}^{n}\left|s+p_j\right|}{\prod_{i=1}^{m}\left|s+z_i\right|}$$

Basic Concepts about Root Locus

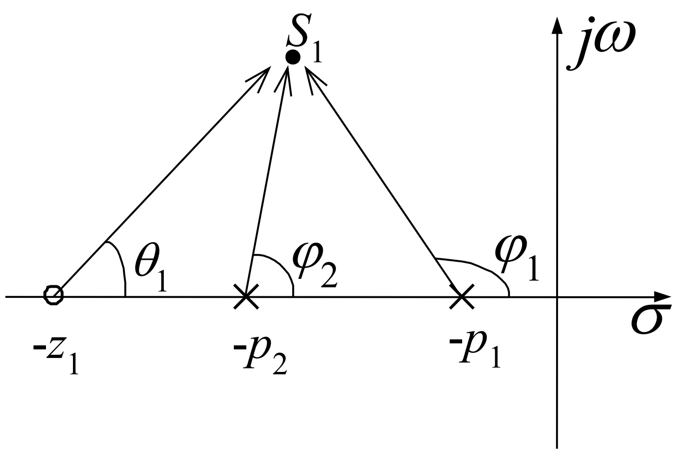
$$G_0(s)+1=0 \quad \longrightarrow \quad \frac{k'\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}=-1$$

Condition on Angle:

$$\angle G_0(s)=\sum_{i=1}^{m}\angle(s+z_i)-\sum_{i=1}^{n}\angle(s+p_j)=(2l+1)\pi$$

Basic Concepts about Root Locus

$$\angle G_0(s) = \sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_j) = (2l+1)\pi$$



$$\sum_{i=1}^{m} \theta_i - \sum_{j=1}^{n} \varphi_j = 180^0 \pm l \times 360^0$$

$$s + z_i = s - (-z_i)$$

$$s + p_j = s - (-p_j)$$

Remarks

- For an n-th order system, one k' is associated with n characteristic roots; one point on the root locus is associated with a value of k'.

- All the points on the root locus of a system must satisfy the angle condition; all the points that satisfy the angle condition must be on the root locus.

The angle condition is a sufficient and necessary condition for drawing the root loci.

Example 3.15

Q: Construct the root loci of the system with the following open-loop transfer function

$$G_0(s) = \frac{k}{s(2s+1)}$$

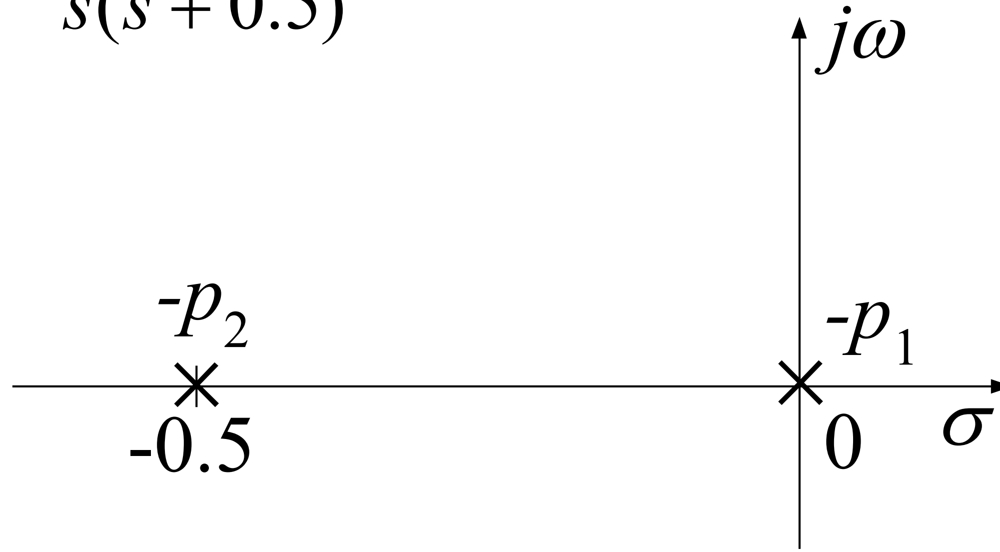
A: rewrite the open loop TF into zero-pole form

$$G_0(s) = \frac{0.5k}{s(s+0.5)} = \frac{k'}{s(s+0.5)}$$

$$k' = 0.5k$$

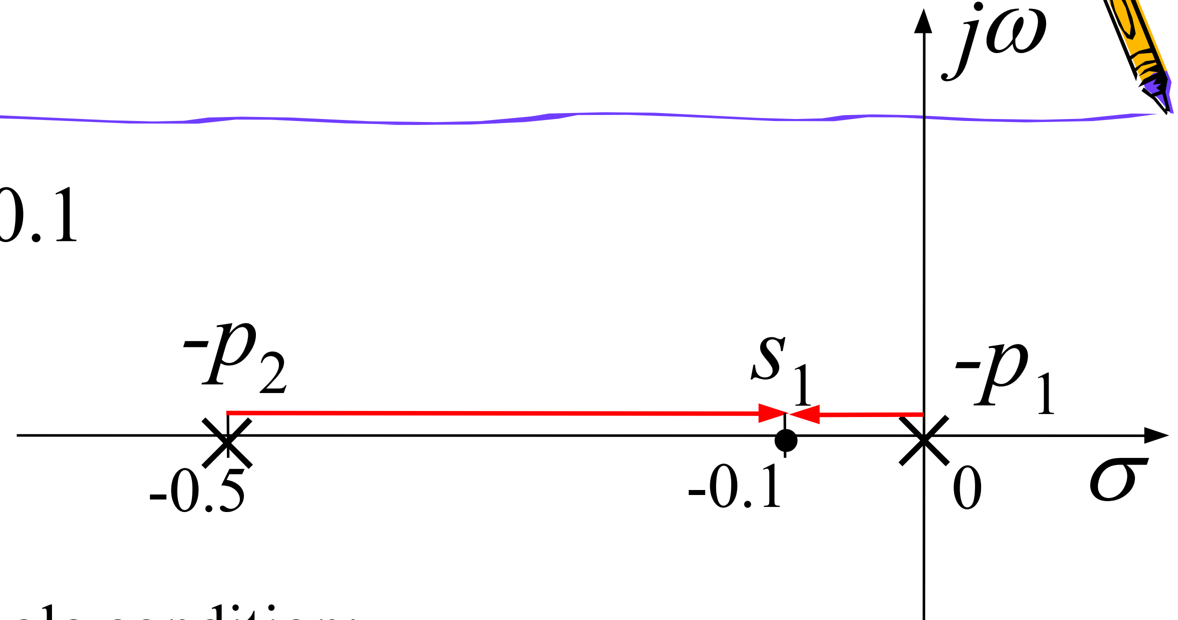
$$-p_1 = 0$$

$$-p_2 = -0.5$$



Trial and error:

Take a point $s_1 = -0.1$



Substitute into the angle condition:

$$-\angle(s_1 + p_1) - \angle(s_1 + p_2) = -180^\circ - 0^\circ = -180^\circ$$

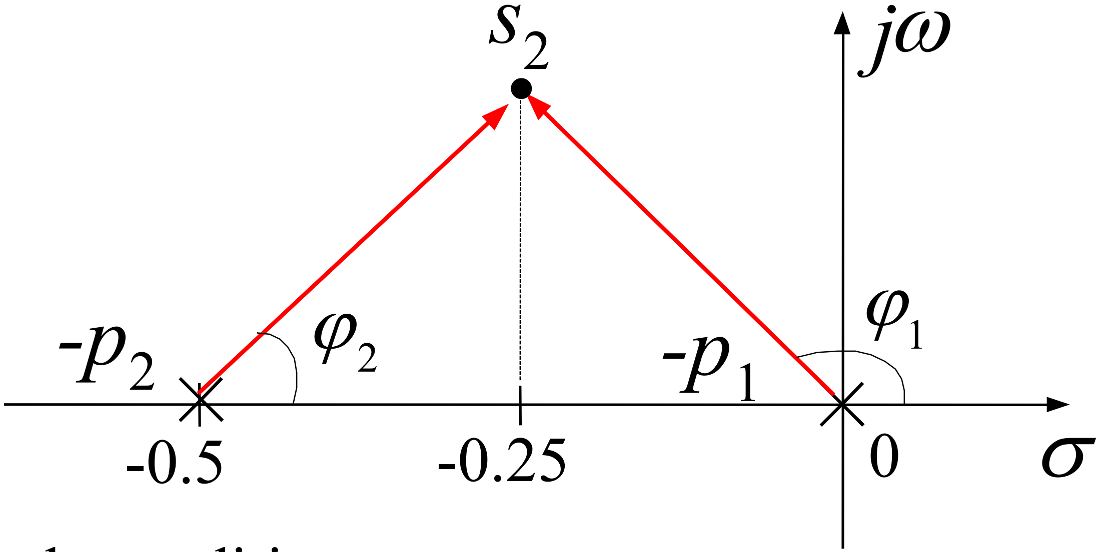
So, s_1 is on the root locus.

The root locus gain corresponding to s_1 is:

$$k' = |s_1 + p_1| \cdot |s_1 + p_2| = 0.1 \times 0.4 = 0.04$$

All the points between $-p_2 \sim -p_1$ are on the locus.

Take a point $s_2 = -0.25 + j0.25$



Substitute into the angle condition:

$$-\angle(s_1 + p_1) - \angle(s_1 + p_2) = -135^\circ - 45^\circ = -180^\circ$$

So, s_2 is on the root locus.

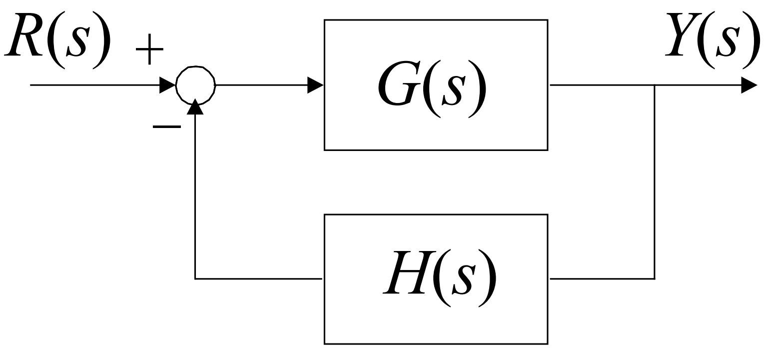
The root locus gain corresponding to s_2 is:

$$k' = |s_2 + p_1| \cdot |s_2 + p_2| = \left|0.25 \times \sqrt{2}\right|^2 = 0.125$$

All the points on the line $\sigma = -0.25$ are on the locus.

Properties and Construction of Root Loc

1. $k' = 0$ and $k' = \infty$ points



Closed-loop TF of the system:

$$M(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{G(s)}{1+G_0(s)}$$

The characteristic equation: $1+G_0(s)=0$ or $G_0(s)=-1$

Considering: $$G_0(s) = \frac{k' N_0(s)}{D_0(s)}$$ $$D_0(s) + k' N_0(s) = 0$$

$$k' = -\frac{D_0(s)}{N_0(s)}$$

$$k' = -\frac{D_0(s)}{N_0(s)}$$

$k' = 0$ ⇨ $D_0(s) = 0$ Poles of the open-loop TF

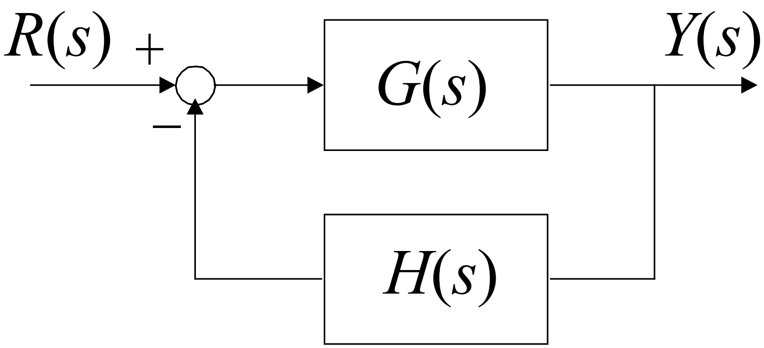
$k' = \infty$ ⇨ $N_0(s) = 0$ Zeros of the open-loop TF

When root locus gain varies from zero to infinity, the root locus starts from the poles of the open-loop TF and ends at the zeros of the open-loop TF

Q: how many branches of the root loci does a system have?

Properties and Construction of Root Loci

2. Numbers of Branches on the Root Loci



Closed-loop TF of the system:

$$M(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{G(s)}{1+G_0(s)}$$

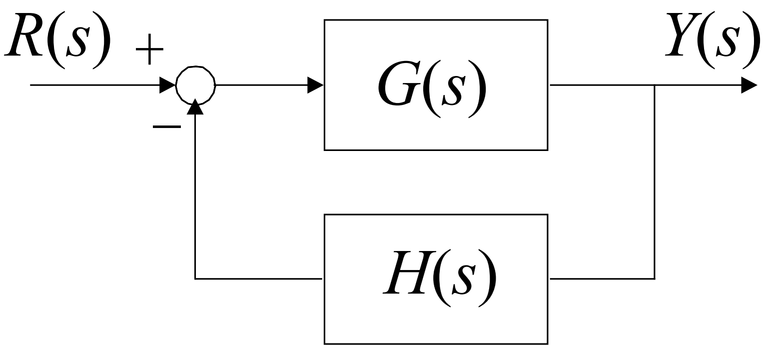
The characteristic equation: $1+G_0(s)=0$ or $G_0(s)=-1$

A branch of the root loci is the locus of one root when k' varies.

The number of branches of the root loci should be equal to the number of the roots of the characteristic equation, or the order of the characteristic equation, or the number of the poles of $G_0(s)$.

Properties and Construction of Root Loci

3. Asymptotes of the Root Loci



Closed-loop TF of the system:

$$M(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{G(s)}{1+G_0(s)}$$

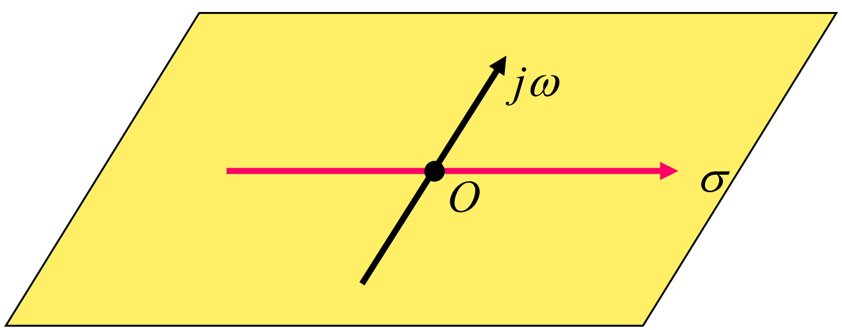
The characteristic equation: $1+G_0(s)=0$ or $G_0(s)=-1$

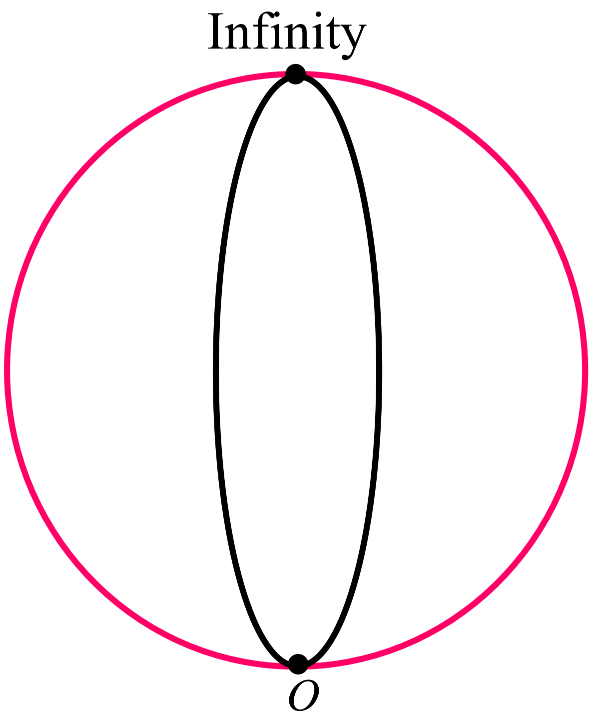
$$G_0(s) = \frac{k'N_0(s)}{D_0(s)} \qquad D_0(s)+k'N_0(s)=0$$

When n, the order of $D_0(s)$, is not equal to m, the order of $N_0(s)$, some of The loci will approach infinity in the s-plane. The properties of the root near infinity in the s-plane are described by the asymptotes of the loci when $|s| \to \infty$

More specifically, if $G_0(s)$ has n poles and m zeros, the root loci of the system will have (n-m) asymptotes. At the infinite point, the (n-m) root loci branches approach the asymptotes unlimitedly.

But where are the asymptotes and where do the asymptotes go? Before we answer this question, we need to understand a concept first – the infinity point.





Look at the characteristic equation:

$$1+G_0(s)=0 \qquad \text{or} \qquad G_0(s)=-1$$

Because $$G_0(s)=\frac{k'\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} \qquad -k'=\frac{\prod_{j=1}^{n}(s+p_j)}{\prod_{i=1}^{m}(s+z_i)}$$

$$-k'=\frac{(s+p_1)(s+p_2)\cdots(s+p_n)}{(s+z_1)(s+z_2)\cdots(s+z_m)}=\frac{s^n+(p_1+p_2+\cdots+p_n)s^{n-1}+\cdots}{s^m+(z_1+z_2+\cdots+z_m)s^{m-1}+\cdots}$$

$$-k'\approx\frac{s^n+\sum_{j=1}^{n}p_js^{n-1}}{s^m+\sum_{i=1}^{m}z_is^{m-1}}\approx s^{n-m}+(\sum_{j=1}^{n}p_j-\sum_{i=1}^{m}z_i)s^{n-m-1}$$

$$-k' = \frac{\prod_{j=1}^{n}(s+p_j)}{\prod_{i=1}^{m}(s+z_i)}$$

At the infinity point, root loci and asymptotes superpose each other. If observing from the infinity point, it seems that the (n-m) asymptotes or root loci branches are from a same point - F.

$$-k' = \frac{\prod_{j=1}^{n}(s+p_j)}{\prod_{i=1}^{m}(s+z_i)} \quad \Longrightarrow \quad -k' = \frac{\prod_{j=1}^{n}(s+F)}{\prod_{i=1}^{m}(s+F)}$$

$$-k' = (s+F)^{n-m} \approx s^{n-m} + (n-m)Fs^{n-m-1}$$

Comparing the following two equations:

$$-k' \approx s^{n-m} + (\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i) s^{n-m-1}$$

$$-k' \approx s^{n-m} + (n-m) F s^{n-m-1}$$

The following relationship holds:

$$F = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n-m}$$

The coordinate of F point, which is the intersect of the asymptotes, is:

$$\sigma_F = \frac{\sum_{j=1}^{n} (-p_j) - \sum_{i=1}^{m} (-z_i)}{n-m}$$

⇨ on the real axis

At the infinity point, the point on the root loci should also satisfy the angle condition:

General expression of AC: $$\sum_{i=1}^{m}\theta_i - \sum_{j=1}^{n}\varphi_j = 180^0 \pm l \times 360^0$$

For point on the asymptotes in the infinity, there are (n-m) angles that start from F and end at the infinity point. These angles are identical. If we denote the angle by α, then we can get the following equation:

$$\alpha(n-m) = (2l+1)180^0$$

$$\alpha = \frac{(2l+1)\times 180^0}{(n-m)}$$

4. Symmetry of the Root Loci

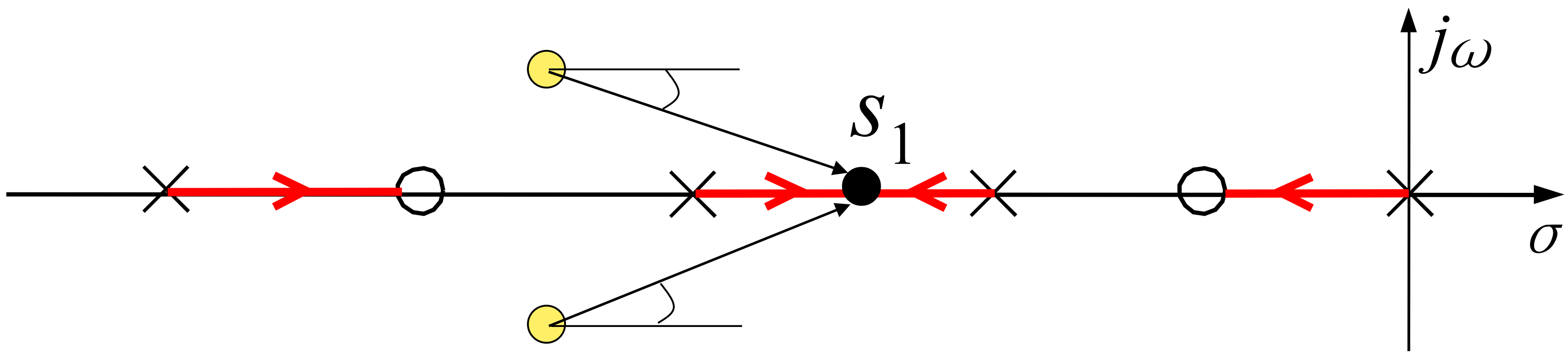
The root loci are symmetrical with respect to the real axis of the s-plane.

Why ?

The characteristic equation has real coefficients. The roots must be real or in complex-conjugate pairs.

5. Root Loci on the Real Axis

On a given section of the real axis, root loci are found in the section only if the total number of poles and zeros of the open-loop TF to the right of the section is ***odd*.**



What do the complex-conjugate poles and zeros contribute to the angular relation?

What do the real poles and zeros that lie to the right of S_1 contribute?

What do the real poles and zeros that lie to the left of S_1 contribute?

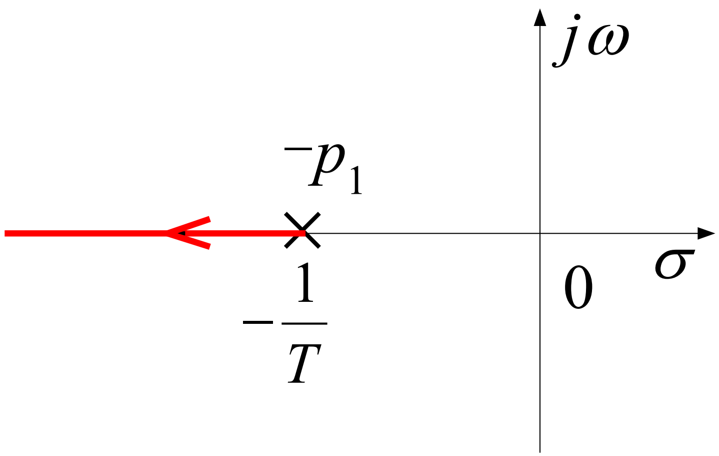
Example 3.16

Q: Construct the root loci of the system with the following open-loop TF.

$$G_0(s) = \frac{k'}{Ts+1}$$

A: the pole of the open-loop TF is

$$-p_1 = -\frac{1}{T}$$



The number of poles of $G_0(s)$

$$n = 1$$

The number of zeros of $G_0(s)$

$$m = 0$$

There is n-m=1 asymptote

$$\sigma_F = \frac{-1/T - 0}{1-0}$$

$$\alpha = \frac{(2l+1)}{1-0} \times 180^\circ = 180^\circ$$

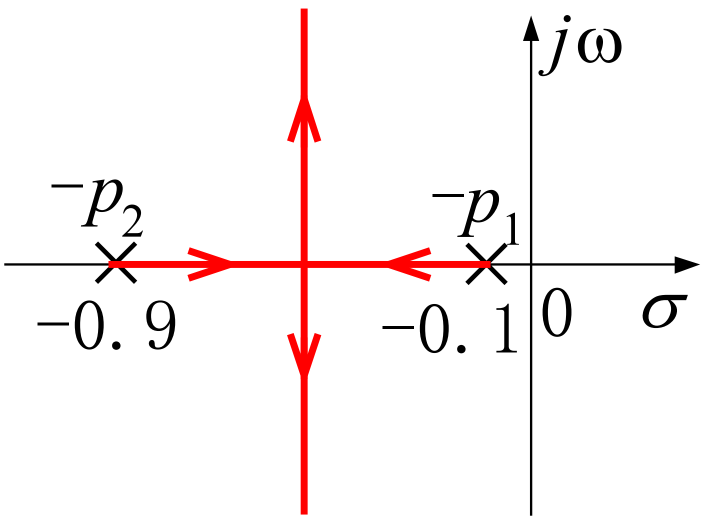
Example 3.17

Q: Construct the root loci of the system with the following open-loop TF.

$$G_0(s) = \frac{k'}{(s+0.1)(s+0.9)}$$

A: the pole of the open-loop TF is

$$p_1 = -0.1 \qquad p_2 = -0.9$$



The number of poles of $G_0(s)$

$$n = 2$$

The number of zeros of $G_0(s)$

$$m = 0$$

There are n-m=2 asymptotes

$$\sigma_F = \frac{-0.1-0.9}{2} = -0.5$$

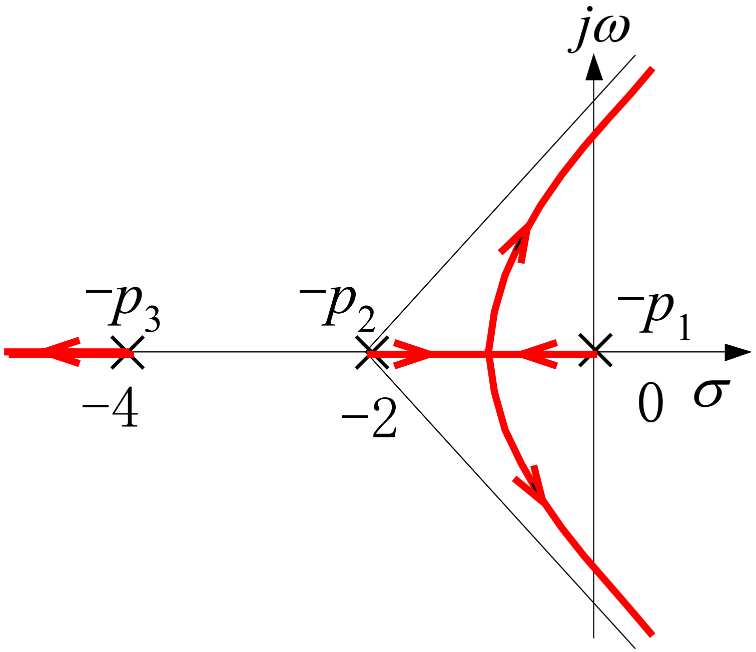
$$\alpha = \frac{(2l+1)}{2} \times 180^\circ = 90^\circ、270^\circ$$

Example 3.18

Q: Construct the root loci of the system with the following open-loop TF.

$$G_0(s)=\frac{k'}{s(s+2)(s+4)}$$

A: the poles of the open-loop TF are



$$p_1=0 \quad p_2=-2 \quad p_3=-4$$

The number of poles of $G_0(s)$

$$n=3$$

The number of zeros of $G_0(s)$

$$m=0$$

There are n-m=3 asymptotes

$$\sigma_F=\frac{-2-4}{3}=-2$$

$$\alpha=\frac{(2l+1)}{3}\times 180^\circ=60^\circ、180^\circ、300^\circ$$

Example 3.19

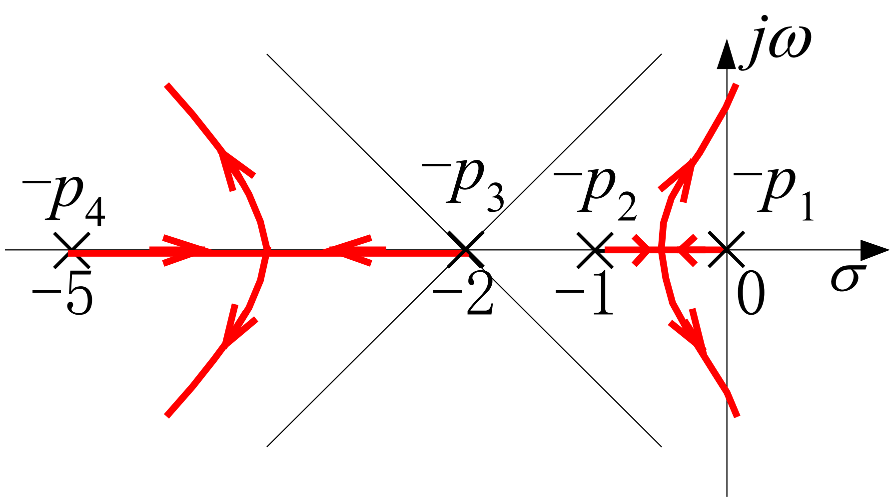
Q: Construct the root loci of the system with the following open-loop TF.

$$G_0(s)=\frac{k'}{s(s+1)(s+2)(s+5)}$$

A: the poles of the open-loop TF are

$$p_1=0 \qquad p_3=-2$$

$$p_2=-1 \qquad p_4=-5$$



The number of poles of $G_0(s)$

$$n=4$$

The number of zeros of $G_0(s)$

$$m=0$$

There are n-m=4 asymptotes

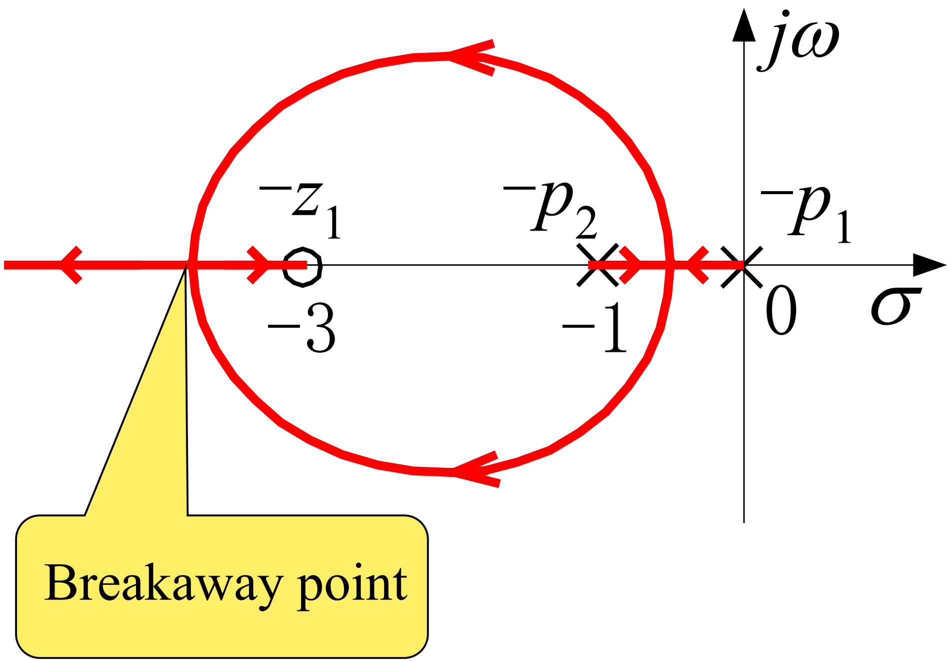
$$\sigma_F=\frac{-1-2-5}{4}=-2$$

$$\alpha=\frac{(2l+1)}{4}\times180^\circ=45^\circ、135^\circ、225^\circ、315^\circ$$

Example 3.20

Q: Construct the root loci of the system with the following open-loop TF.

$$G_0(s) = \frac{k'(s+3)}{s(s+1)}$$



A: the poles and zeros of the open-loop TF

$$p_1 = 0 \qquad p_2 = -1$$

$$z_1 = -3$$

The number of poles of $G_0(s)$

$$n = 2$$

The number of zeros of $G_0(s)$

$$m = 1$$

There is n-m=1 asymptote

$$\sigma_F = \frac{0-1+3}{1} = 2$$

$$\alpha = \frac{(2l+1)}{1} \times 180^\circ = 180^\circ$$

6. Breakaway Points on the Root Loci

In the previous several cases, we saw two branches of the root loci meet at a breakaway point on the real axis and they depart from the axis in opposite direction.

The breakaway point represents a double root of the characteristic equation.

The breakaway points on the root loci of $1+G_0(s)=0$ must satisfy:

$$\frac{dk'}{ds}=0$$

Where $G_0(s)=\frac{k' N_0(s)}{D_0(s)}$, $\quad k'=-\frac{D_0(s)}{N_0(s)}$

The above condition is a necessary condition.

Proof : calculate $\dfrac{dk'}{ds}$

Because $\quad \dfrac{dk'}{ds} = \lim\limits_{\Delta s \to 0} \dfrac{\Delta k'}{\Delta s}$

According to the characteristic equation:

$$1 + G_0(s) = 1 + \frac{k' N_0(s)}{D_0(s)} = \frac{D_0(s) + k' N_0(s)}{D_0(s)} = 0$$

So $\quad D_0(s) + k' N_0(s) = 0$

Take an increase of k', $\quad D_0(s) + (k' + \Delta k')N_0(s) = 0$

Both sides divide $D_0(s) + k' N_0(s)$,

We get

$$1 + \frac{\Delta k' N_0(s)}{D_0(s) + k' N_0(s)} = 0$$

If S_i is an n-th order root

$$\frac{N_0(s)}{D_0(s)+k'N_0(s)}=\frac{N_0(s)}{(s-s_i)^n\cdot O(s)}=\frac{T(s)}{(\Delta s)^n}$$

Where $T(s)=\dfrac{N_0(s)}{O(s)}$

$$1+\frac{\Delta k' T(s)}{(\Delta s)^n}=0 \qquad \frac{\Delta k'}{\Delta s}=-\frac{(\Delta s)^{n-1}}{T(s)}$$

$$s\to s_i \qquad \Delta s\to 0$$

$$\frac{\Delta k'}{\Delta s}\to\frac{dk'}{ds} \qquad \frac{(\Delta s)^{n-1}}{T(s)}\to 0$$

so $\dfrac{dk'}{ds}=0$

More words about breakaway points

- Another formula used to calculate the coordinate of breakaway points

$$\sum_{i=1}^{n} \frac{1}{d + p_i} = \sum_{j=1}^{m} \frac{1}{d + z_j}$$

- The angles between branches leaving a breakaway point are determined by

$$\alpha = \frac{(2k+1)\pi}{l}$$

- Proof

$$\sum_{i=1}^{n}\frac{1}{d+p_i}=\sum_{j=1}^{m}\frac{1}{d+z_j}$$

z_j、p_i are coordinates of zeros and poles of the open loof TF

（1）

$$1+G_0(s)=1+\frac{k'\prod_{j=1}^{m}(s+z_j)}{\prod_{i=1}^{n}(s+p_i)}=\frac{\prod_{i=1}^{n}(s+p_i)+k'\prod_{j=1}^{m}(s+z_j)}{\prod_{i=1}^{n}(s+p_i)}=0$$

$$\Rightarrow D(s)=\prod_{i=1}^{n}(s+p_i)+k'\prod_{j=1}^{m}(s+z_j)=0$$

（2） The breakaway points are multiple roots, thus

$$\frac{d}{ds}D(s)=0$$

$$\Rightarrow \frac{d}{ds}\left[\prod_{i=1}^{n}(s+p_i)+k'\prod_{j=1}^{m}(s+z_j)\right]=0$$

$$\Rightarrow \frac{d}{ds}\left[\prod_{i=1}^{n}(s+p_i)\right]+\frac{d}{ds}\left[k'\prod_{j=1}^{m}(s+z_j)\right]=0$$

（3） Rewrite (1) and (2)

$$\prod_{i=1}^{n}\left(s+p_i\right)=-k'\prod_{j=1}^{m}\left(s+z_j\right) \qquad \text{(a)}$$

$$\frac{d}{ds}\prod_{i=1}^{n}\left(s+p_i\right)=-\frac{d}{ds}k'\prod_{j=1}^{m}\left(s+z_j\right) \qquad \text{(b)}$$

（4） $\left(\mathrm{b}\right)\div\left(\mathrm{a}\right)$

$$\frac{\dfrac{d}{ds}\prod_{i=1}^{n}\left(s+p_i\right)}{\prod_{i=1}^{n}\left(s+p_i\right)}=\frac{\dfrac{d}{ds}\prod_{j=1}^{m}\left(s+z_j\right)}{\prod_{j=1}^{m}\left(s+z_j\right)} \Longrightarrow \frac{d}{ds}\ln\left[\prod_{i=1}^{n}\left(s+p_i\right)\right]=\frac{d}{ds}\ln\left[\prod_{j=1}^{m}\left(s+z_j\right)\right]$$

P.S. $$\frac{d}{dx}\ln f\left(x\right)=\frac{1}{f\left(x\right)}\frac{d}{dx}f\left(x\right)$$

（5） $$\frac{d}{ds}\ln\left[\prod_{i=1}^{n}(s+p_i)\right]=\frac{d}{ds}\ln\left[\prod_{j=1}^{m}(s+z_j)\right]$$

P.S. $\ln f_1 f_2 = \ln f_1 + \ln f_2$

$$\Rightarrow \quad \frac{d}{ds}\sum_{i=1}^{n}\ln(s+p_i)=\frac{d}{ds}\sum_{j=1}^{m}\ln(s+z_j)$$

$$\Rightarrow \quad \sum_{i=1}^{n}\frac{d}{ds}\ln(s+p_i)=\sum_{j=1}^{m}\frac{d}{ds}\ln(s+z_j)$$

$$\Rightarrow \quad \sum_{i=1}^{n}\frac{1}{(s+p_i)}=\sum_{j=1}^{m}\frac{1}{(s+z_j)}$$

QED

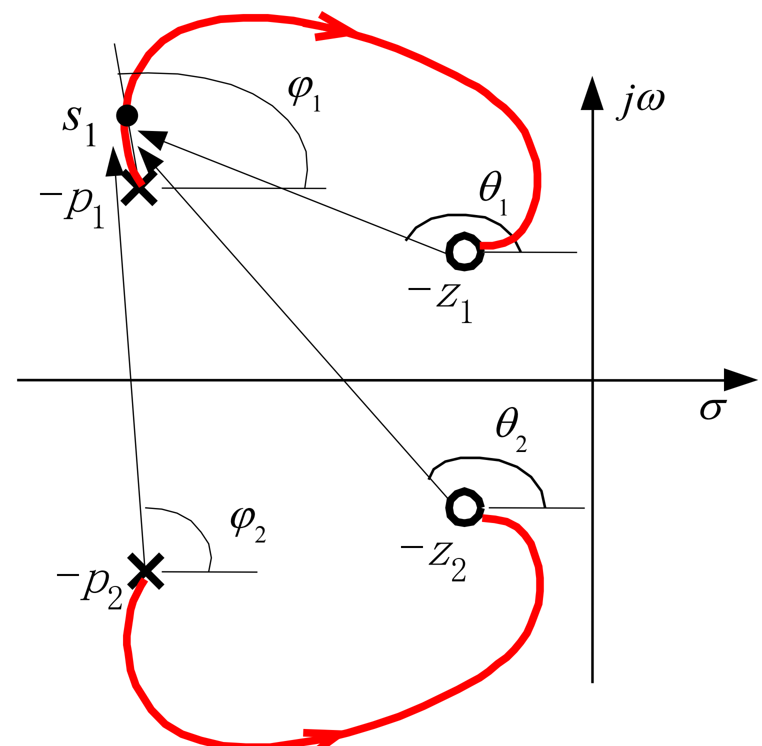
Properties and Construction of Root Loci

7. Angles of Departure and Arrival

The angle of departure or arrival of a root locus at a pole or zero, respectively, of $G_0(s)$ denotes the angle of the tangent to the locus near the point.

The angle of departure or arrival of a root locus at a pole or zero of $G_0(s)$ can be determined by assuming a point s_1 that is very close to the pole, or zero, and applying the angular condition.

$$\theta_1 + \theta_2 - \varphi_1 - \varphi_2 = 180^\circ$$



Properties and Construction of Root Loci

8. Intersection of the Root Loci with the Imaginary Axis

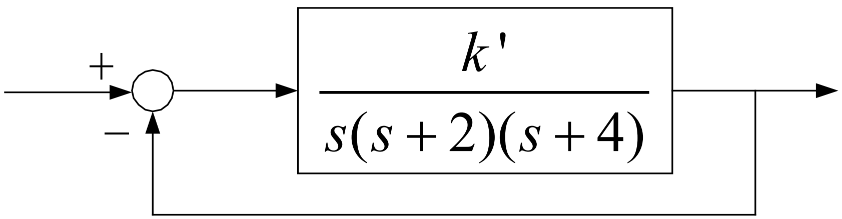
The points where the root loci intersect the imaginary axis of the s-plane, if any, and the corresponding values of k', may be determined by means of the Routh-Hurwitz criterion.

That elements of a row are all zeros indicates that one or more of the following conditions may exist:

1. one pair of real roots with equal magnitude but opposite signs;
2. one or more pairs of imaginary roots;
3. pairs of complex-conjugate roots forming symmetry about the origin of the s-plane.

Example 3.21

Q: Construct the root loci of the following system.



A: the poles of the open-loop TF are

$$p_1 = 0 \qquad p_2 = -2 \qquad p_3 = -4$$

The number of poles of $G_0(s)$

$$n = 3$$

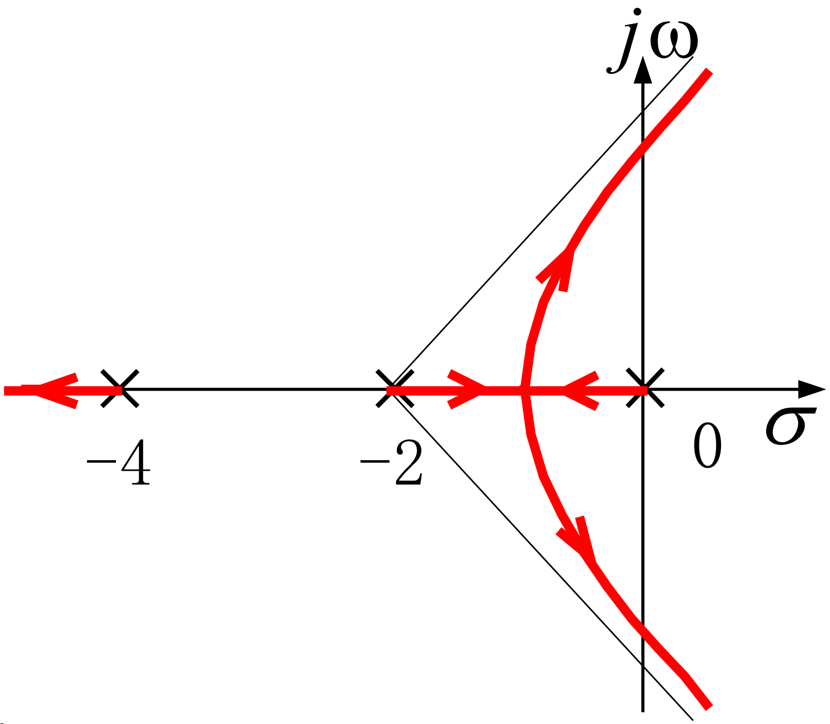
The number of zeros of $G_0(s)$

$$m = 0$$

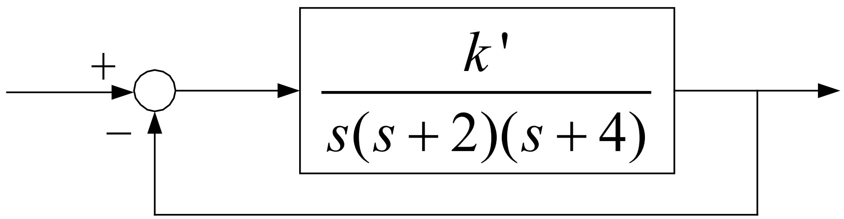
There are n-m=3 asymptotes

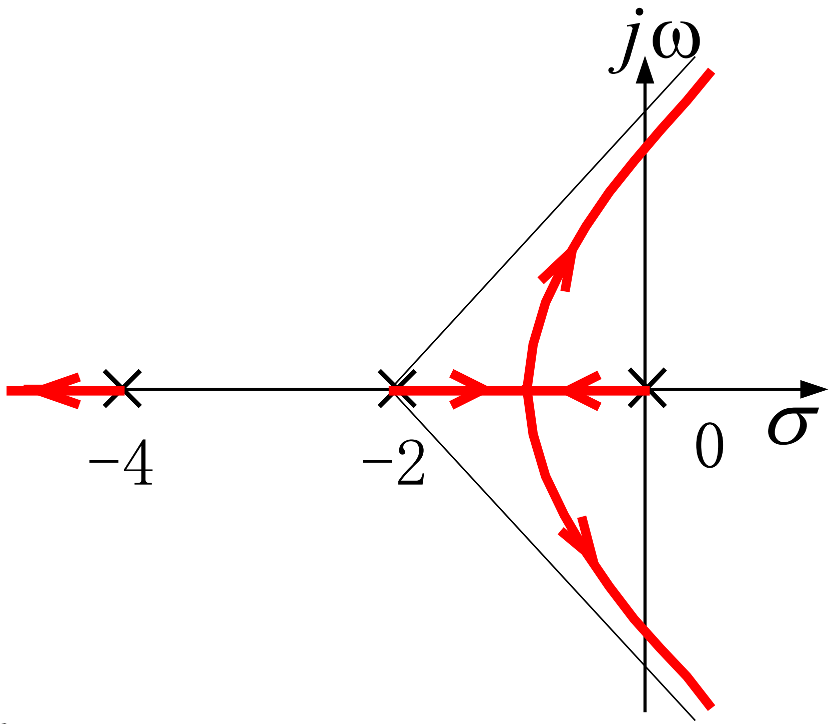
$$\sigma_F = \frac{0-2-4}{3} = -2$$

$$\alpha = \frac{(2l+1)}{3} \times 180^\circ = 60^\circ、180^\circ、300^\circ$$



Example 3.21 (continue)





Calculate the breakaway point:

$$1+\frac{k'}{s(s+2)(s+4)}=0$$

$$k'=-s^3-6s^2-8s$$

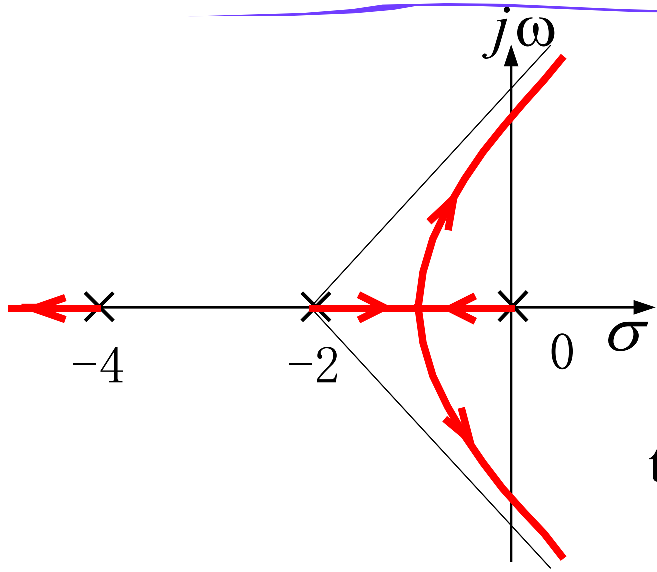
$$\frac{dk'}{ds}=-3s^2-12s-8=0$$

$$s_{1,2}=-2\pm\frac{2\sqrt{3}}{3}$$

$$s_1=-2+\frac{2\sqrt{3}}{3}=-0.85$$

s_1 is the breakaway point.

Example 3.21 (continue)



Calculate the intersection of the root loci with the imaginary axis

$$s^3 + 6s^2 + 8s + k' = 0$$

When $k' > 0$ and $8 - \frac{k'}{6} > 0$

that is when $0 < k' \le 48$, the system is stable.

When $k' = 48$, the coefficients of s^1 row are all zero. The auxiliary equation is:

$$6s^2 + 48 = 0$$

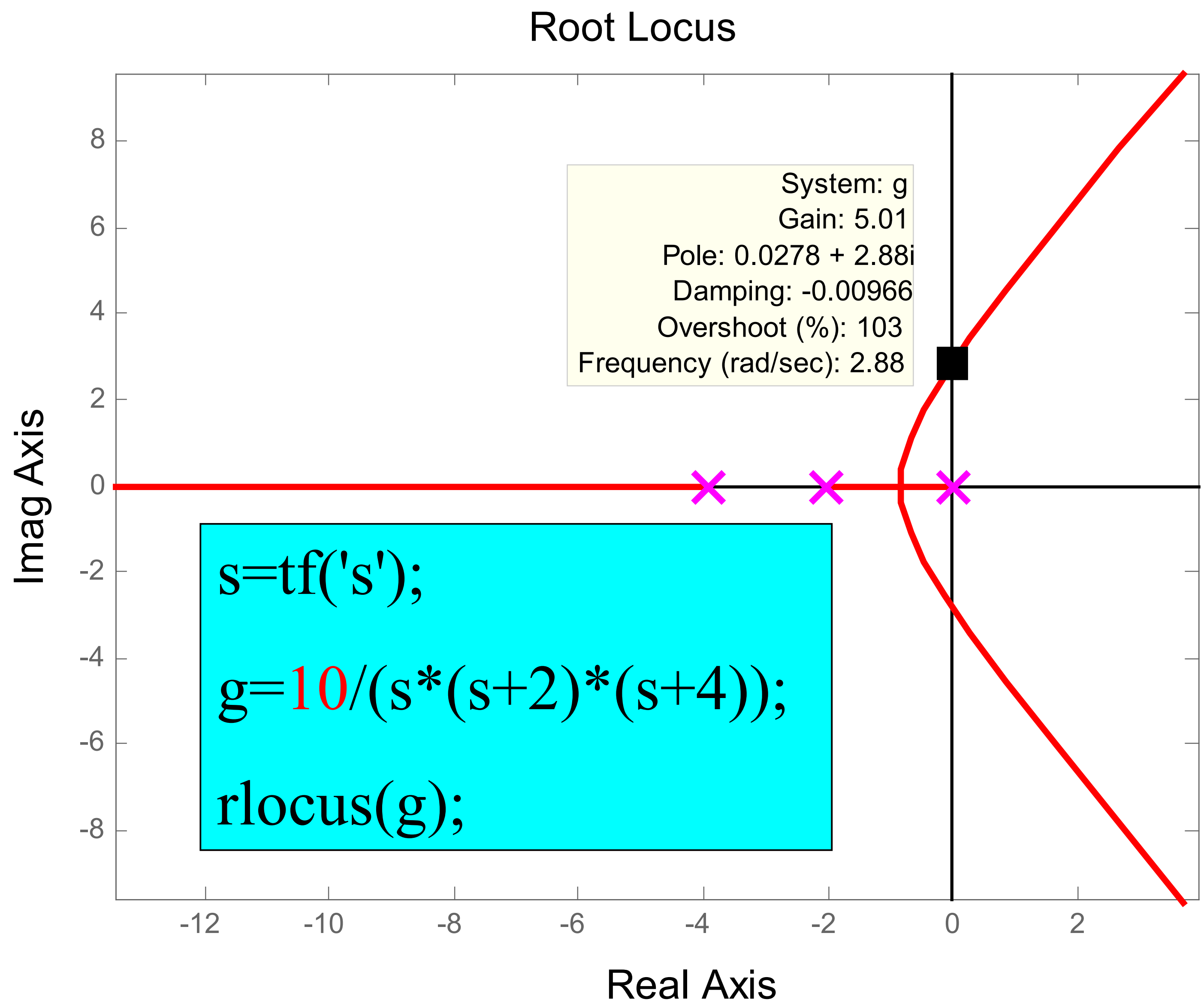
Solve the auxiliary equation:

$$s = \pm j2\sqrt{2}$$

These are the intersection of the root loci with the imaginary axis.

s^3	1	8
s^2	6	k'
s^1	$8 - \frac{k'}{6}$	0
s^0	k'	0

How to Use Matlab to Construct the Root Loci




```
s=tf('s');
g=10/(s*(s+2)*(s+4));
rlocus(g);
```

Example 3.22

Q: Construct the root loci of the open loop transfer function

$$G(s)H(s)=\frac{K^*}{s(s+3)(s^2+2s+2)}$$

A: the open loop transfer function is

$$G(s)H(s)=\frac{K^*}{s(s+3)(s+1+j)(s+1-j)}$$

So， the poles are

$$p_1=0 \qquad p_2=-3 \qquad p_{3,4}=-1\pm j$$

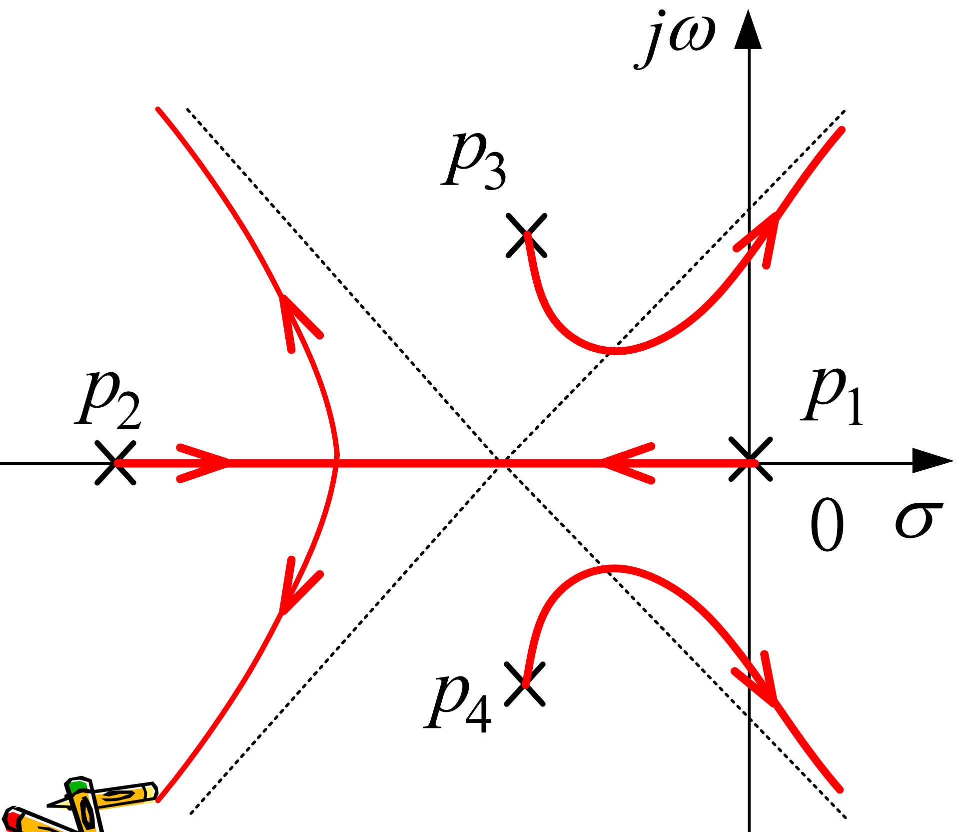
The number of poles $n=4$ The number of zeros $m=0$

There are n-m=4 asymptotes

$$\sigma_F=\frac{0-3-1+j-1-j}{4}=-1.25 \qquad \alpha=\frac{2l+1}{4}\times 180^\circ=\pm 45^\circ,\pm 135^\circ$$

Example 3.22 (continue)

$$G(s)H(s)=\frac{K^*}{s(s+3)\left(s^2+2s+2\right)}$$



Calculate the breakaway point:

$$1+\frac{K^*}{s(s+3)\left(s^2+2s+2\right)}=0$$

$$K^*=-s^4-5s^3-8s^2-6s$$

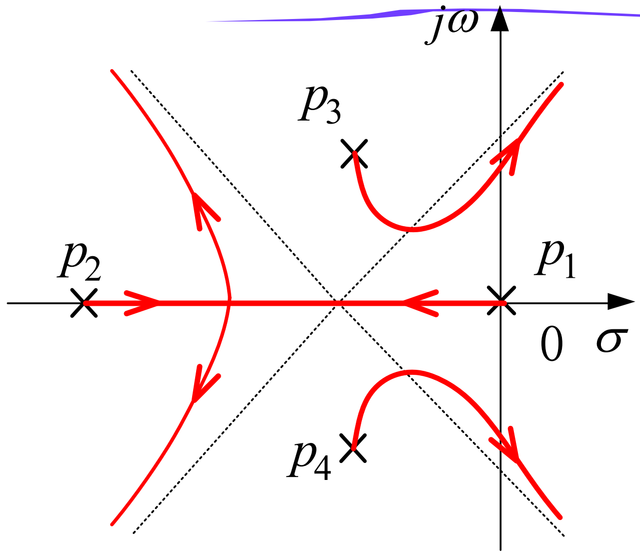
$$\frac{dK^*}{ds}=-4s^3-15s^2-16s-6=0$$

$$s_1=-2.29$$

$$s_{2,3}=-0.731\pm j0.35$$

s_1 is the breakaway point.

Example 3.22 (continue)



Calculate the intersection of the root loci with the imaginary axis

$$s^4+5s^3+8s^2+6s+K^*=0$$

s^4	1	8	K^*
s^3	5	6	
s^2	$\frac{34}{5}$	K^*	
s^1	$\frac{204-25K^*}{34}$		
s^0	K^*		

When $K^*=8.16$, the coefficients of s^1 row are all zero. The auxiliary equation is:

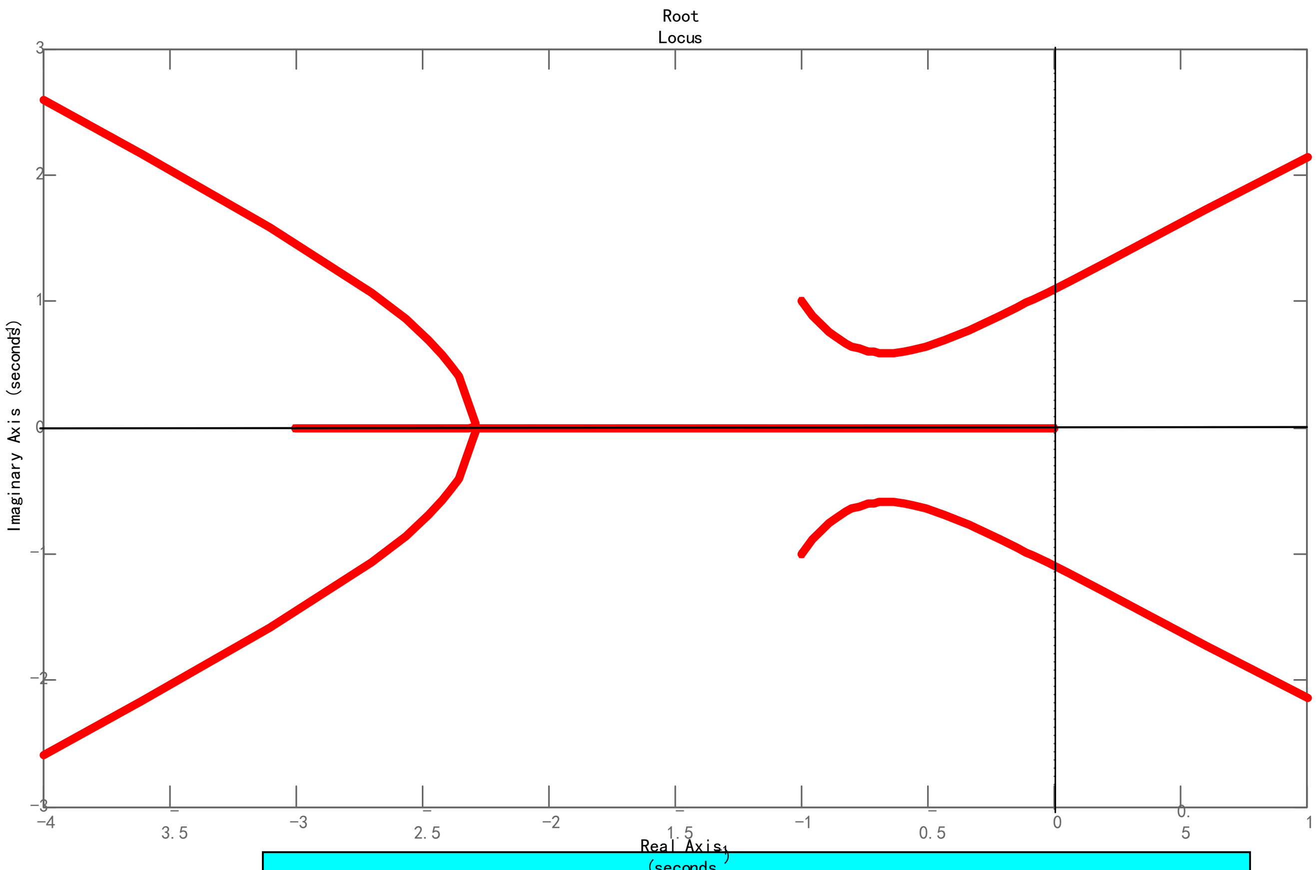
$$\frac{34}{5}s^2+8.16=0$$

Solve the auxiliary equation:

$$s=\pm 1.1j$$

These are the intersection of the root loci with the imaginary axis.

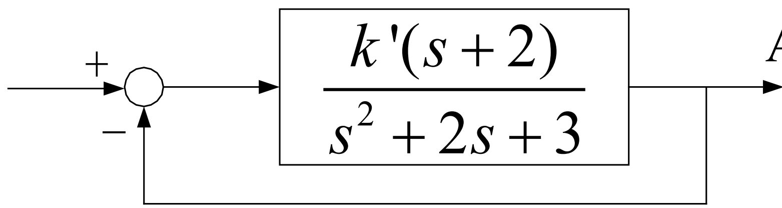
How to Use Matlab to Construct the Root Loci




```
s=tf('s');
m=10/(s*(s+3)*(s^2+2*s+2));
rlocus(m);
```

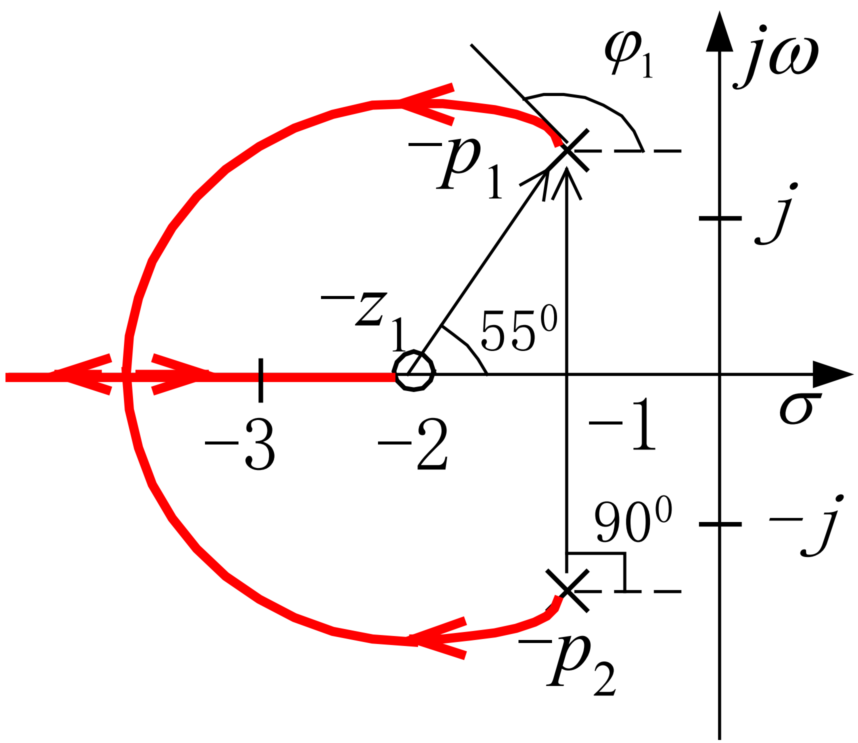
Example 3.23

Q: Construct the root loci of the following system.



A: the poles and zero of the open-loop TF are:

$$p_{1,2} = -1 \pm j\sqrt{2} \qquad z_1 = -2$$



The number of poles of $G_0(s)$

$$n = 2$$

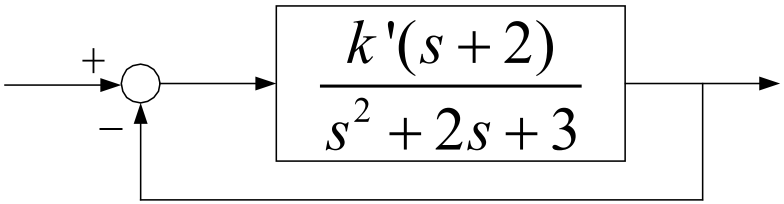
The number of zeros of $G_0(s)$

$$m = 1$$

There is n-m=1 asymptotes

$$\alpha = \frac{(2l+1)}{1} \times 180^\circ = 180^\circ$$

Example 3.23 (Continue)



The characteristic equation is:

$$1+\frac{k'(s+2)}{s^2+2s+3}=0$$

$$s^2+2s+3+k'(s+2)=0$$

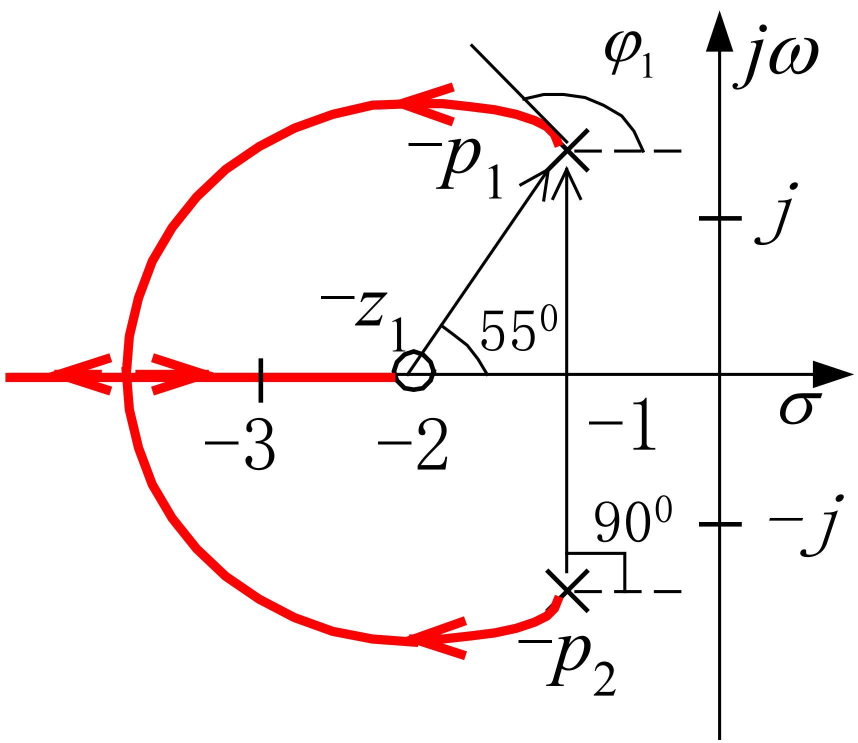
$$k'=-\frac{s^2+2s+3}{s+2}=0$$

$$\frac{dk'}{ds}=-\frac{s^2+4s+1}{(s+2)^2}=0$$

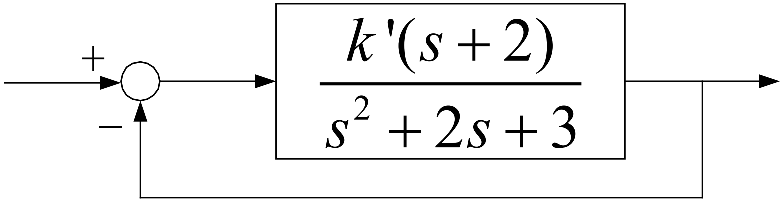
$$s_{1,2}=-2\pm\sqrt{3}$$

$$s_1=-2-\sqrt{3}\approx-3.73$$

$$k'=5.47$$



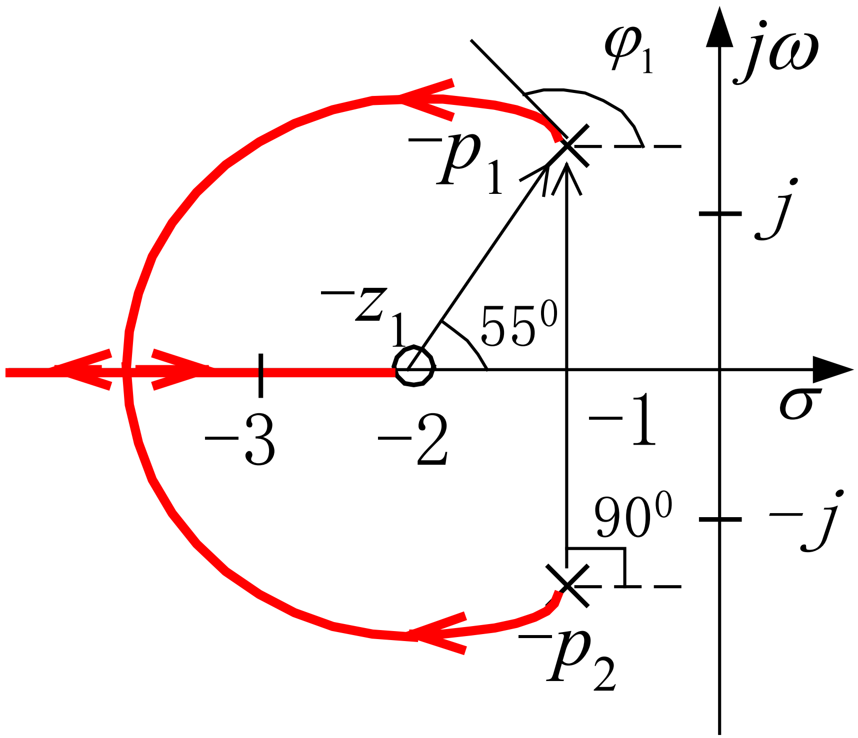
Example 3.23 (Continue)



$$\frac{k'(s+2)}{s^2+2s+3}$$

Calculate the angle of departure

$$-\varphi_1 - 90^0 + 55^0 = 180^0$$



$$\varphi_1 = -180^0 + 55^0 - 90^0$$

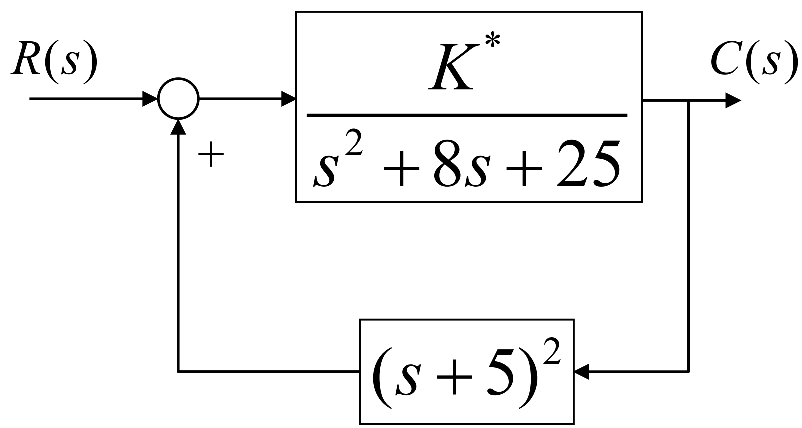
$$= -215^0 (145^0)$$

$$\varphi_2 = -145^0$$

More Examples: Example A

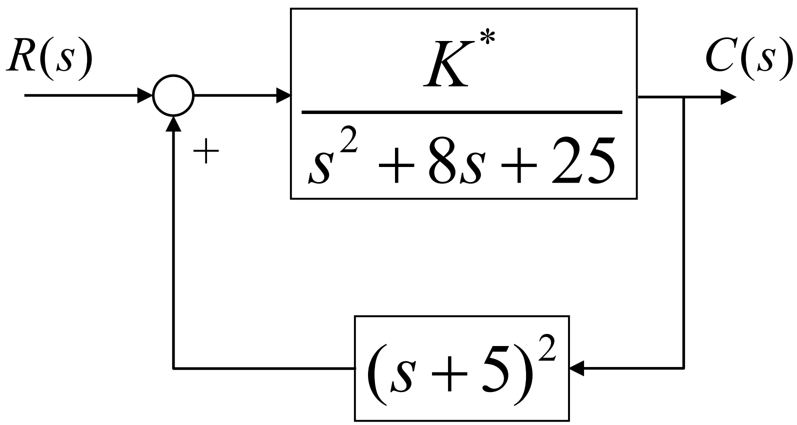
Q: A positive feedback system structure is shown as follows.

1. Construct the root locus of the system.
2. Determine the range of the open loop gain K^* to make the system stable and stay overdamped.
3. Determine the open loop gain K^* and the coordinate of closed-loop pole to make $\zeta = 0.707$, and calculate the dynamic performance of the system. $(\sigma\%, t_s)$



Example A

Q: 1. Construct the root locus of the system.



$$G_K(s) = \frac{K^*(s+5)^2}{s^2+8s+25}$$

$$p_{1,2} = -4 \pm j3$$

$$z_1 = z_2 = -5$$

$$\sum_{i=1}^{n} \frac{1}{d-p_i} = \sum_{j=1}^{m} \frac{1}{d-z_j}$$

$$\frac{1}{d+4+j3} + \frac{1}{d+4-j3} = \frac{1}{d+5} + \frac{1}{d+5}$$

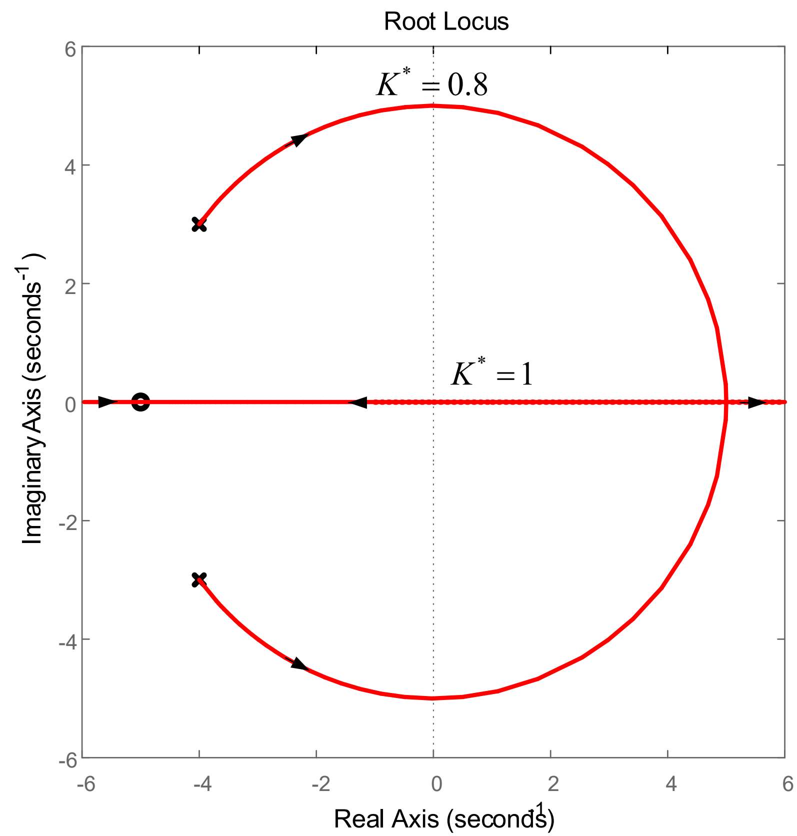
$$d = 5 \qquad K^* = \frac{\left|d^2+8d+25\right|}{\left|d+5\right|^2} = 0.9$$

Intersection point with the imaginary axis

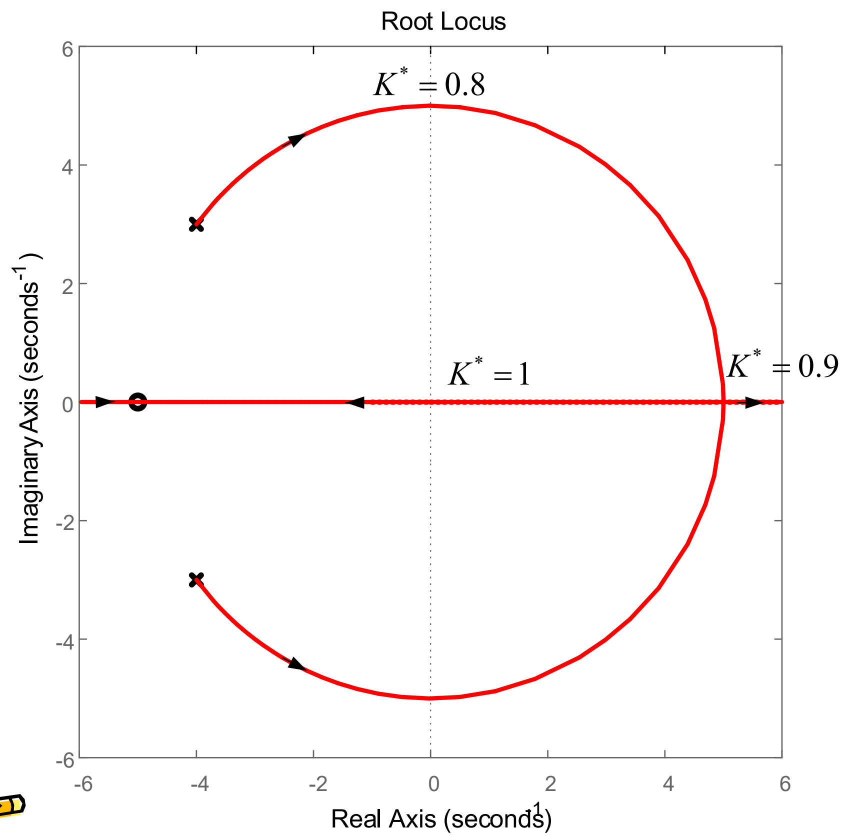
$$(1-K^*)s^2+(8-10K^*)s+25(1-K^*)=0 \qquad s=j\omega$$

$$K^*=0.8, s=\pm j5$$

$$K^*=1, s=0$$



Q2: Determine the range of the open loop gain K^* to make the system stable and stay overdamped.



Q2. Observed from the root locus

Stable:

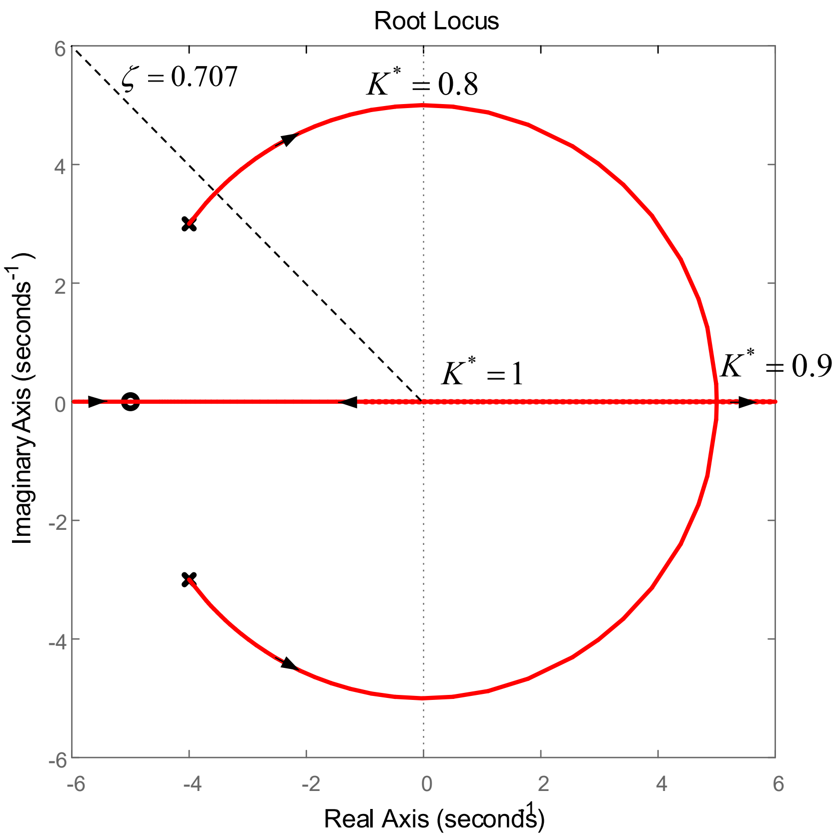
$0 < K^* < 0.8 \quad or \quad K^* > 1$

Overdamped :

$K^* > 1$

Q3: Determine the open loop gain K^* and the coordinate of closed-loop pole to make $\zeta = 0.707$, and calculate the dynamic performance of the system.

A: Draw the line of $\zeta = 0.707$ from the origin, intersecting the root locus at s_1. Let $s_1 = -\omega + j\omega$, then the closed-loop characteristic polynomial is



$$D(s) = (s+\omega+j\omega)(s+\omega+j\omega) = s^2 + 2\omega s + 2\omega^2$$

The closed-loop transfer function is

$$\Phi(s) = \frac{K^*}{s^2 + \dfrac{8-10K^*}{1-K^*}s + 25}$$

Compare D(s) and Φ(s)

$$2\omega^2 = 25, 2\zeta\omega_n = 2\omega = \frac{8-10K^*}{1-K^*}$$

$$\omega = 3.535, K^* = 0.317, s_1 = -3.535 + j3.535$$

$$\sigma\% = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\% = 4.33\% \qquad t_s = \frac{3.5}{\zeta\omega_n} = 0.99s$$

Example B

If the open loop transfer function of a unit feedback system is

$$G(s)=\frac{K^*(s+2)}{s(s+1)}$$

Try to draw its closed-loop system root locus, and prove: The complex part of the root locus is a circle whose center is $(-2, j0)$ and radius is $\sqrt{2}$.

$$p_{1,2}=0,-1$$

$$z_1=-2$$

$$n=2$$

$$m=1$$

There is n-m=1 asymptotes

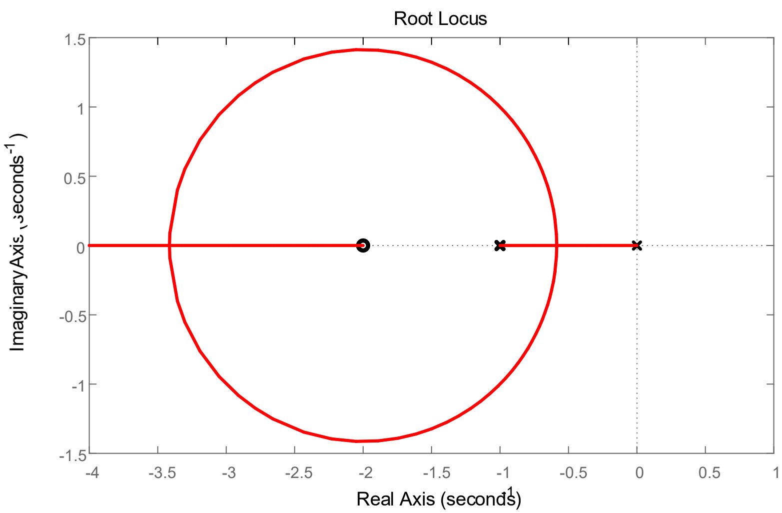
$$\alpha=\frac{(2l+1)}{1}\times 180^\circ=180^\circ$$

$$\sum_{i=1}^{n}\frac{1}{d-p_i}=\sum_{j=1}^{m}\frac{1}{d-z_j}$$

$$\frac{1}{d}+\frac{1}{d+1}=\frac{1}{d+2}$$

$$d^2+4d+2=0$$

$$d_1=-0.59, d_2=-3.41$$



Proof:

Take a point s_1 on the root locus

$$s_1 = \sigma + j\omega$$

s_1 should satisfy the closed-loop characteristic equation:

$$s^2 + (K^* + 1)s + 2K^* = 0$$

Then we get

$$(\sigma + j\omega)^2 + (K^* + 1)(\sigma + j\omega) + 2K^* = 0$$

$$\sigma^2 - \omega^2 + (K^* + 1)\sigma + 2K^* + j[2\sigma\omega + (K^* + 1)\omega] = 0$$

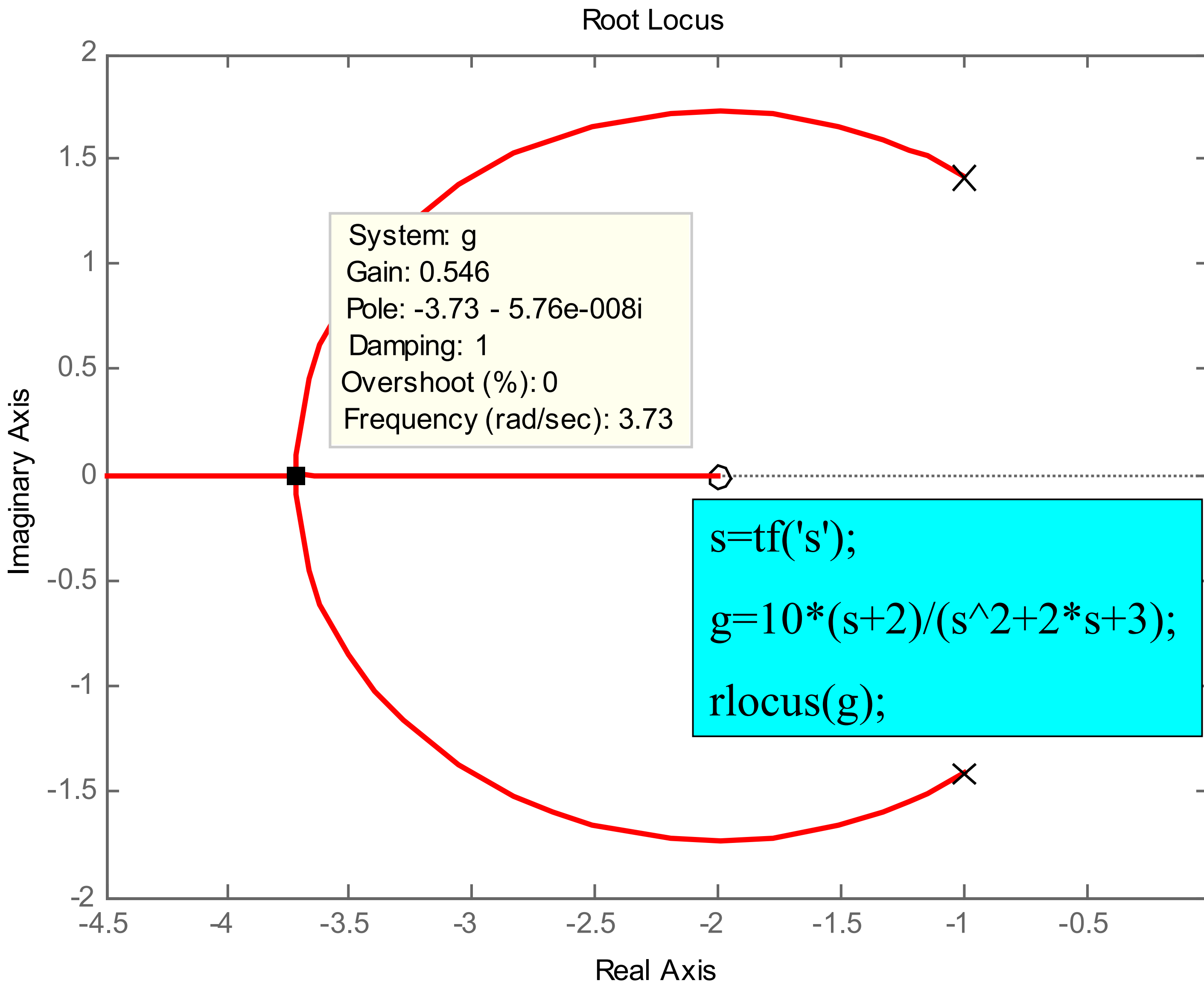
So

$$\begin{cases} \sigma^2 - \omega^2 + (K^* + 1)\sigma + 2K^* = 0 \\ 2\sigma\omega + (K^* + 1)\omega = 0 \end{cases}$$

Eliminate K^*

$$(\sigma + 2)^2 + \omega^2 = (\sqrt{2})^2$$

How to Use Matlab to Construct the Root Loci

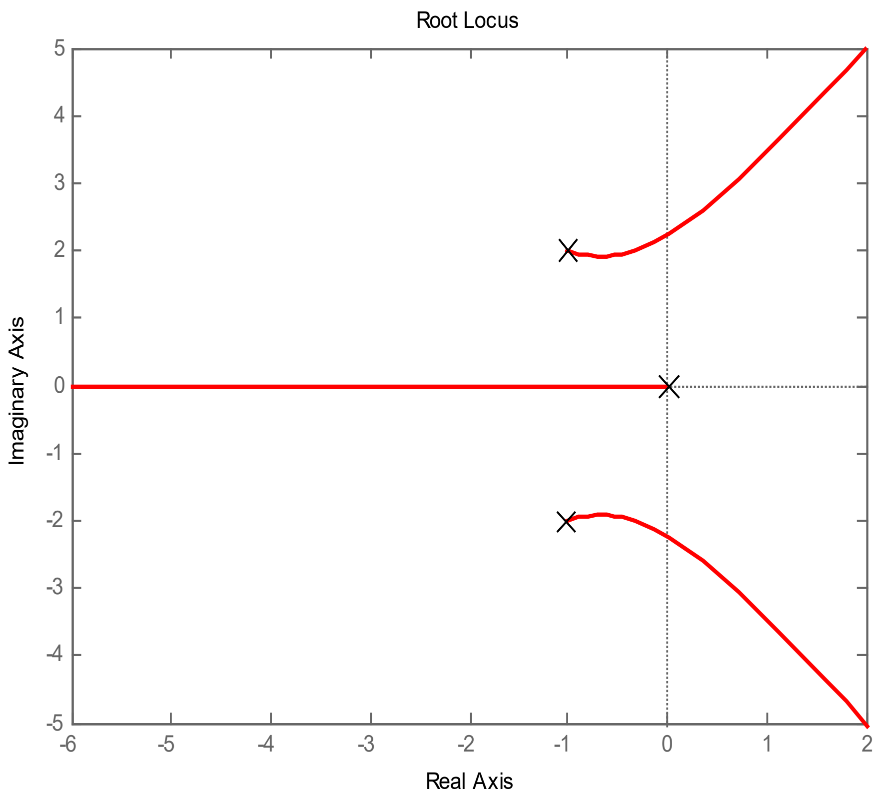


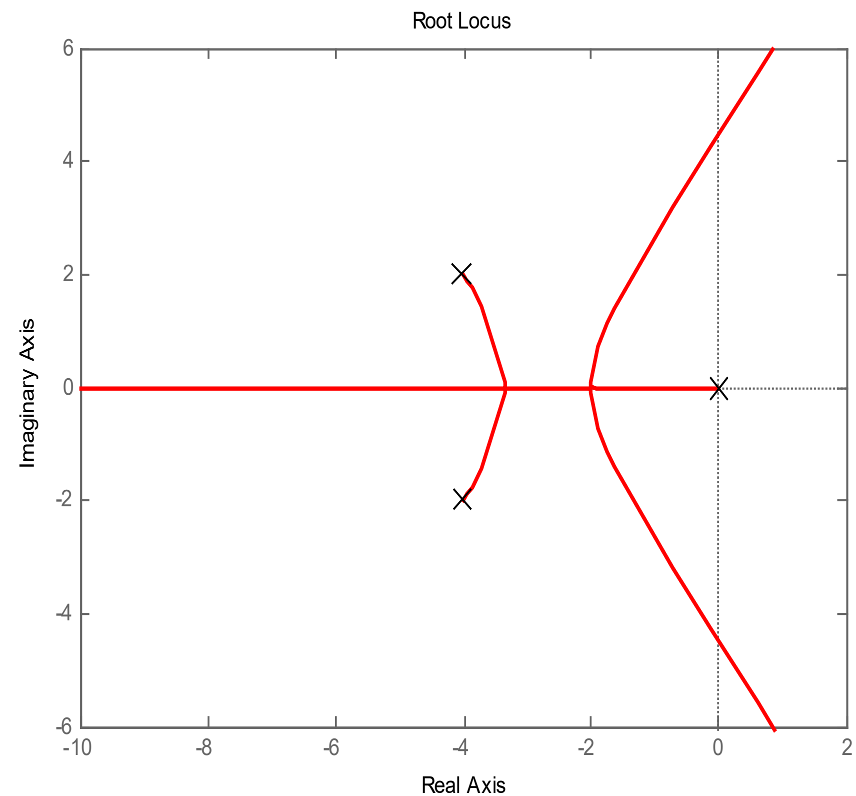

```
s=tf('s');
g=10*(s+2)/(s^2+2*s+3);
rlocus(g);
```

How to Use Matlab to Construct the Root Loci

$$G_0(s) = \frac{k}{s(s^2 + 2s + 5)}$$

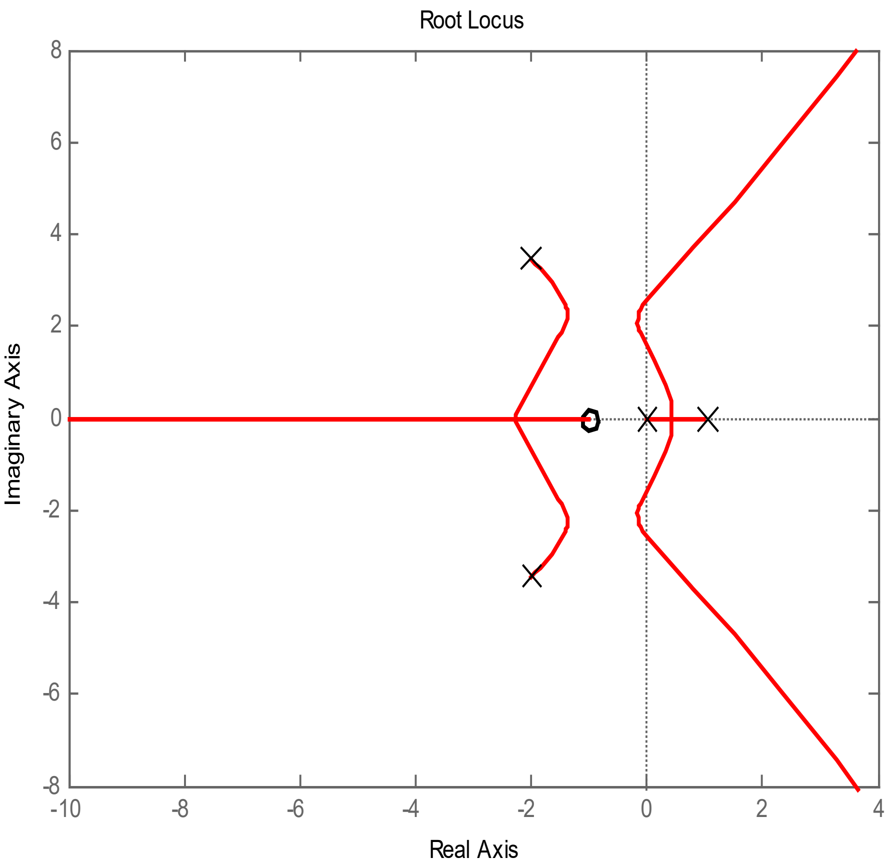
$$G_0(s) = \frac{k}{s(s^2 + 8s + 20)}$$



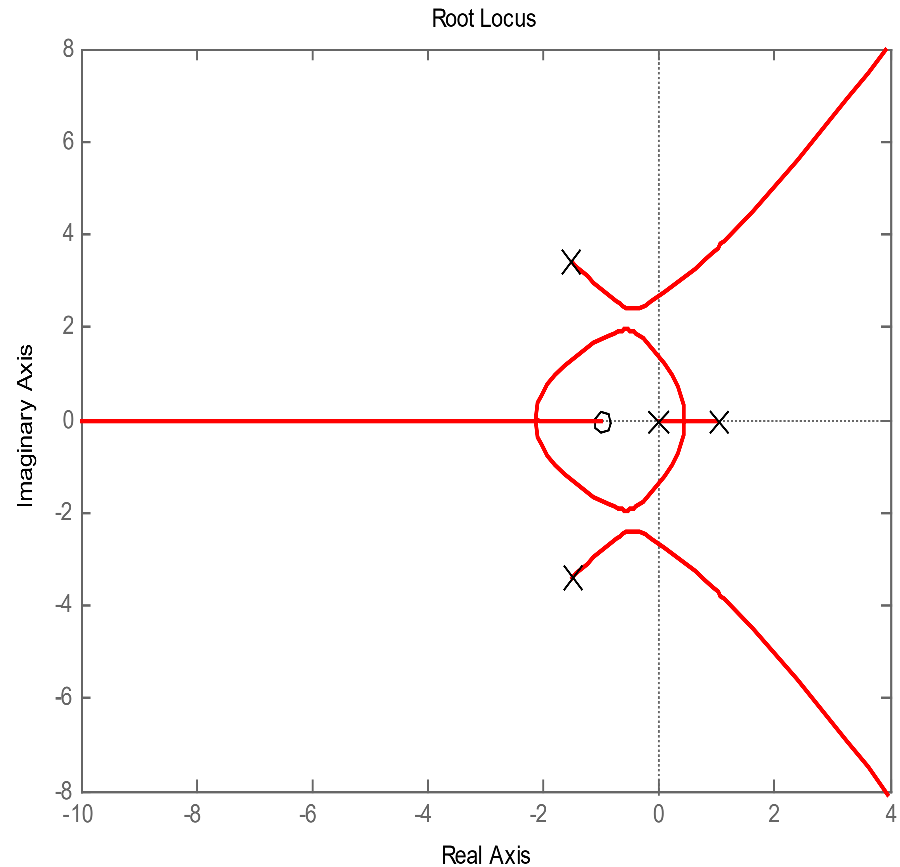


How to Use Matlab to Construct the Root Locus

$$G_0(s)=\frac{k(s+1)}{s(s-1)(s^2+4s+16)}$$



$$G_0(s)=\frac{k(s+1)}{s(s-1)(s^2+3s+14)}$$



Wrap-up

- **Transient Response**
  - Approximation of High-Order Systems
- **Root loci**
  - Basic concept about root loci;
  - Rules for constructing root loci;

Assignment

- Page 73
  - Q16
  - Q17

Review questions

Q1: The maximum overshoot of a unit-step response of the second order prototype system will never exceed 100 percent when the damping ratio ζ and the natural frequency ω_n are all positive. (T) (F)

$$\sigma\% = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

Review questions

Q2: For the second-order prototype system, when the undamped natural frequency ω_n increases, the maximum overshoot of the output stays the same.

(T) (F)

$$\sigma\% = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

Review questions

Q3: The maximum overshoot of the following system will never exceed 100% when ζ , ω_n and T are all positive.

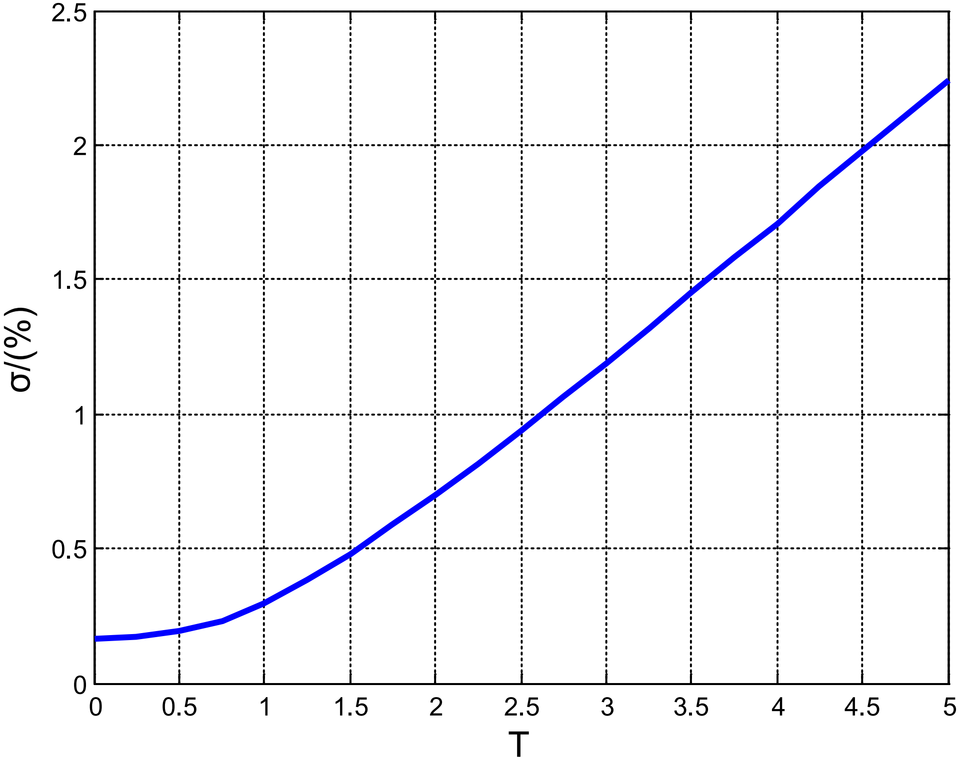
(T) (F)

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2(1+Ts)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Reference answers:

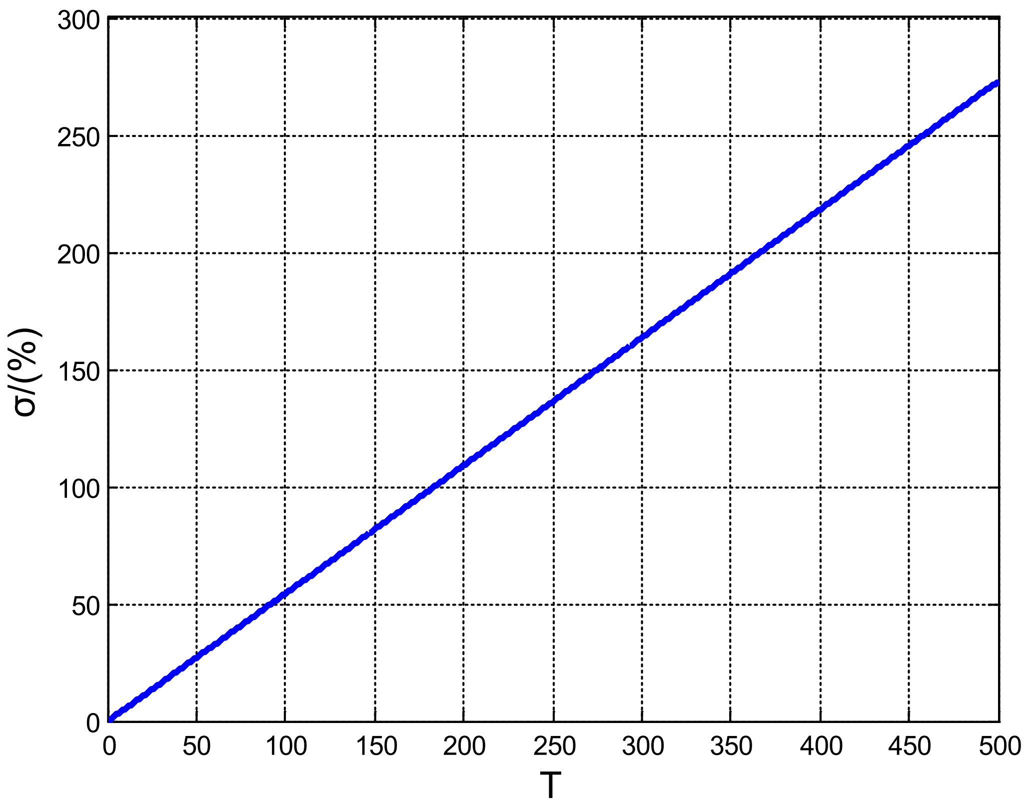
Q3: False. $$\frac{Y(s)}{R(s)} = \frac{\omega_n^2(1+Ts)}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Let $\varpi_n = 1, \zeta = 0.5$, and T varies from 0 to 5, and then the relationship between T and the overshoot can be shown in the following graph:



Reference answers:

Let $\varpi_n = 1, \zeta = 0.5$, and T varies from 0 to 500, and then the relationship between T and the overshoot can be shown in the following graph:



Review questions

Q4: Increasing the undamped natural frequency will generally reduce the rise time of the step response

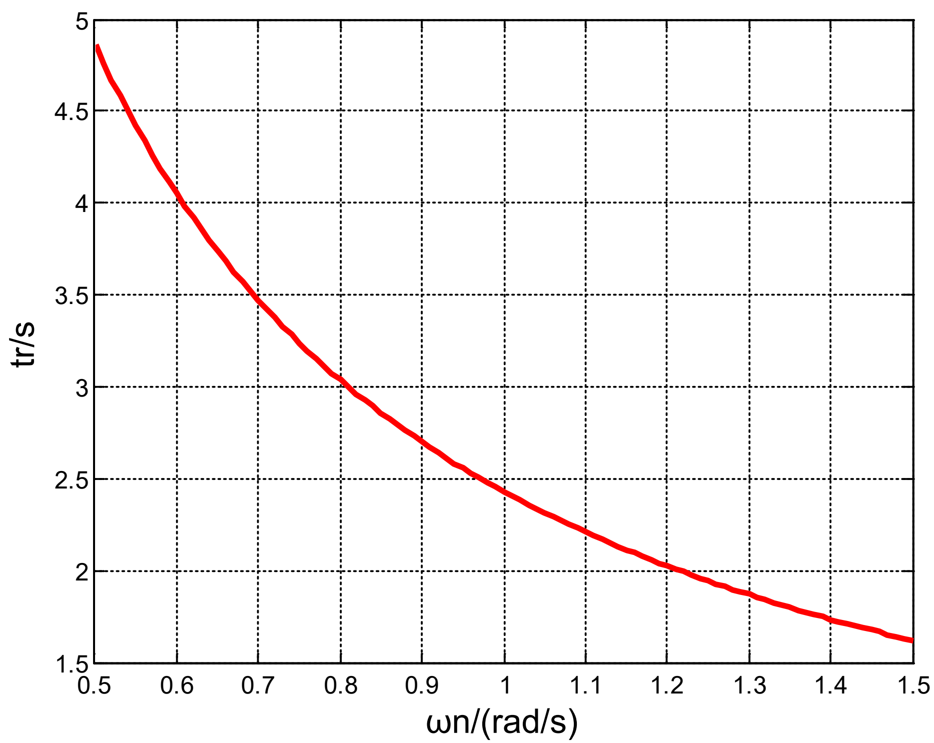
(T) (F)

Reference answers:

Q4: True.

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2}$$

Let $\zeta=0.5$, and ω_n varies from 0.5 to 1.5, and then the relationship between ω_n and the rising time can be shown in the following graph:

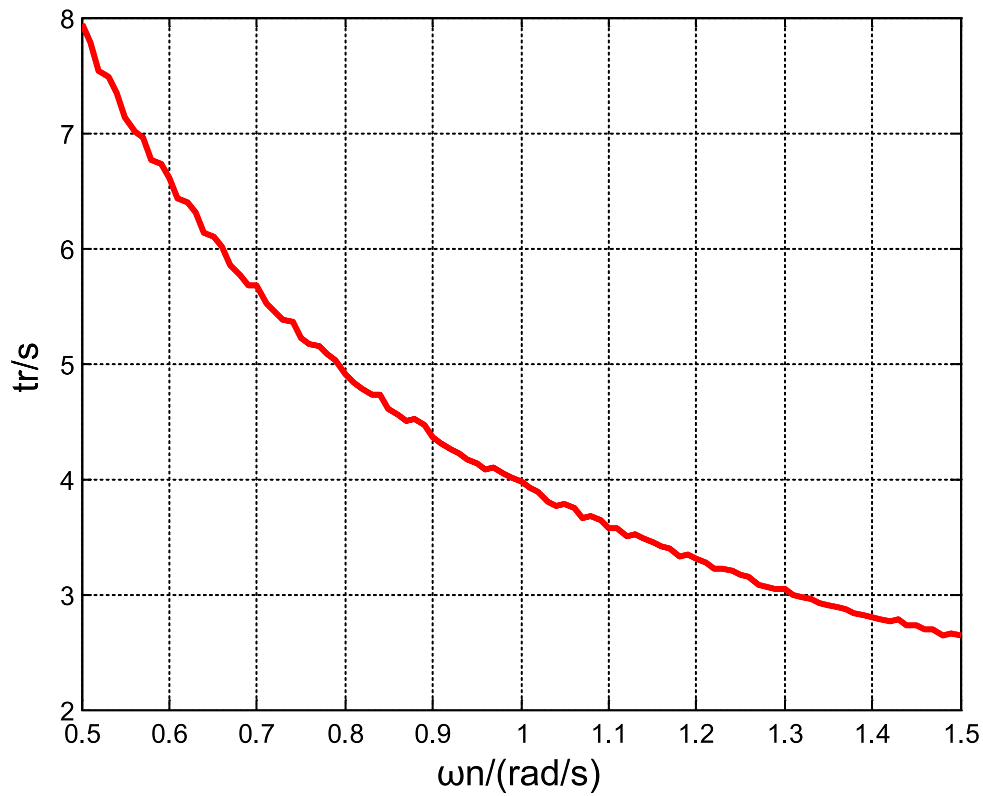


Reference answers:

Q4: True.

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2}$$

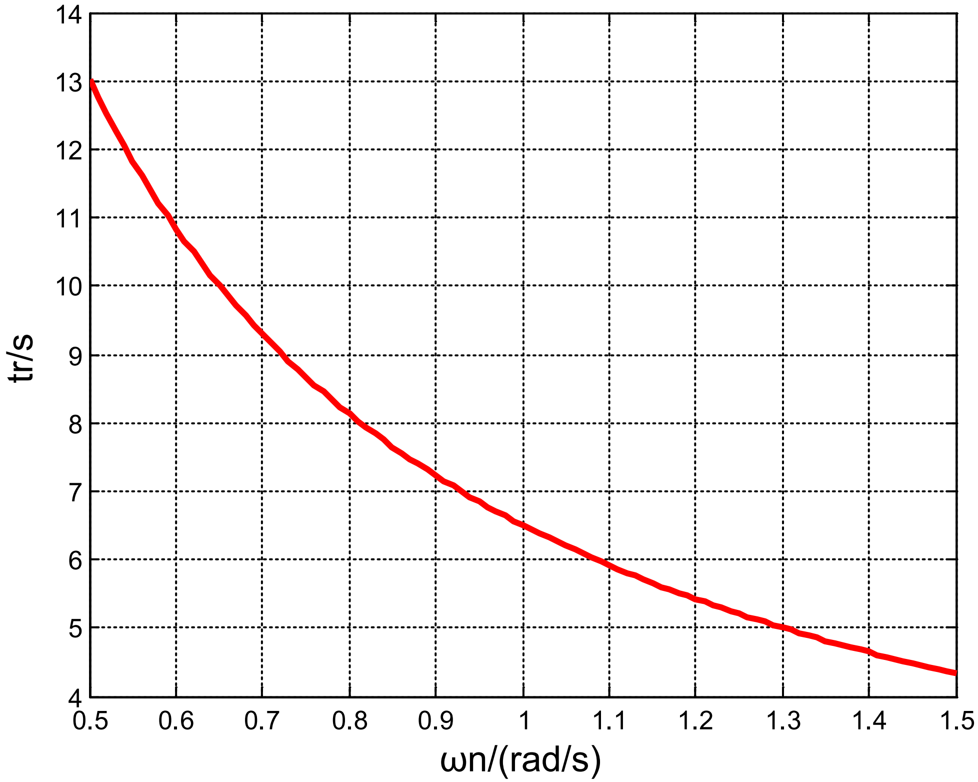
Let $\zeta=1$, and ω_n varies from 0.5 to 1.5, and then the relationship between ω_n and the rising time can be shown in the following graph:



Reference answers:

Q4: True. $$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2}$$

Let $\zeta=1.5$, and ω_n varies from 0.5 to 1.5, and then the relationship between ω_n and the rising time can be shown in the following graph:



Review questions

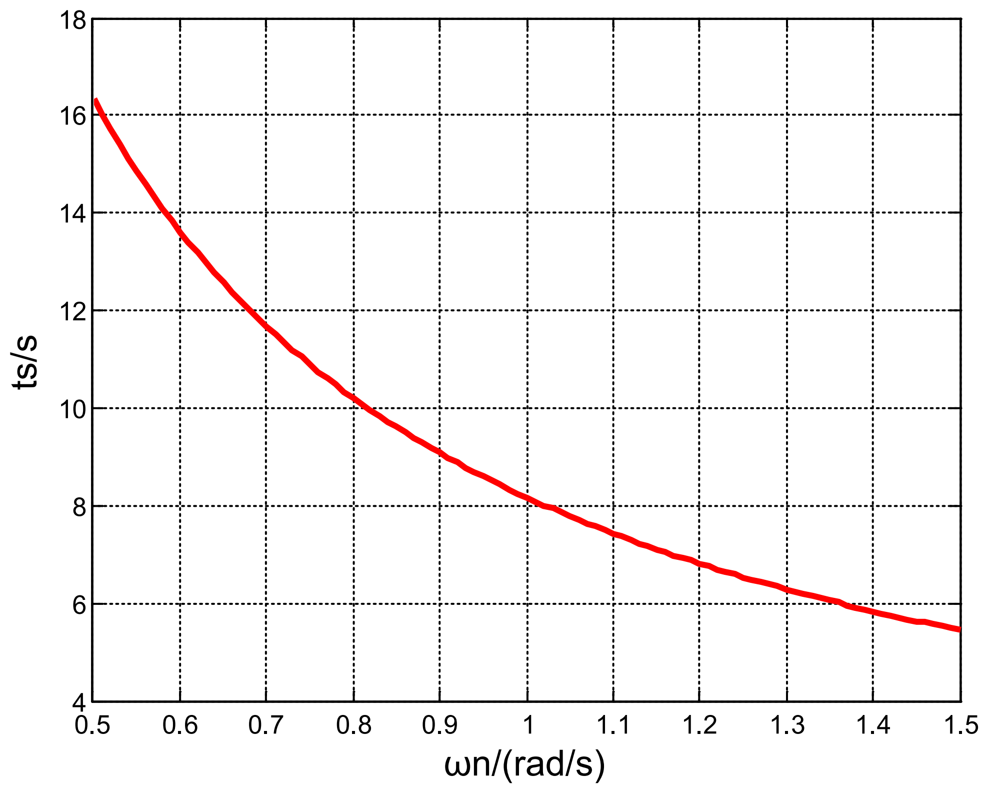
Q5: Increasing the undamped natural frequency will generally reduce the settling time of the step response

(T) (F)

Reference answers:

Q5: True. $$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2}$$

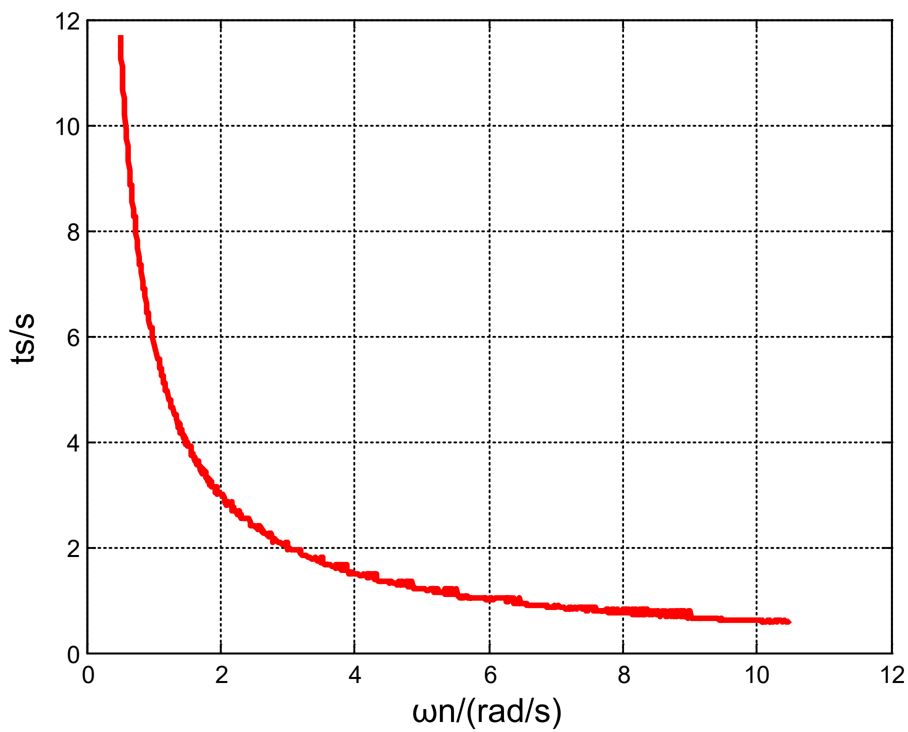
Let $\zeta=0.5$, and ω_n varies from 0.5 to 1.5, and then the relationship between ω_n and the settling time can be shown in the following graph:



Reference answers:

Q5: True. $$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

Let $\zeta = 1$, and ω_n varies from 0.5 to 11.5, and then the relationship between ω_n and the settling time can be shown in the following graph:

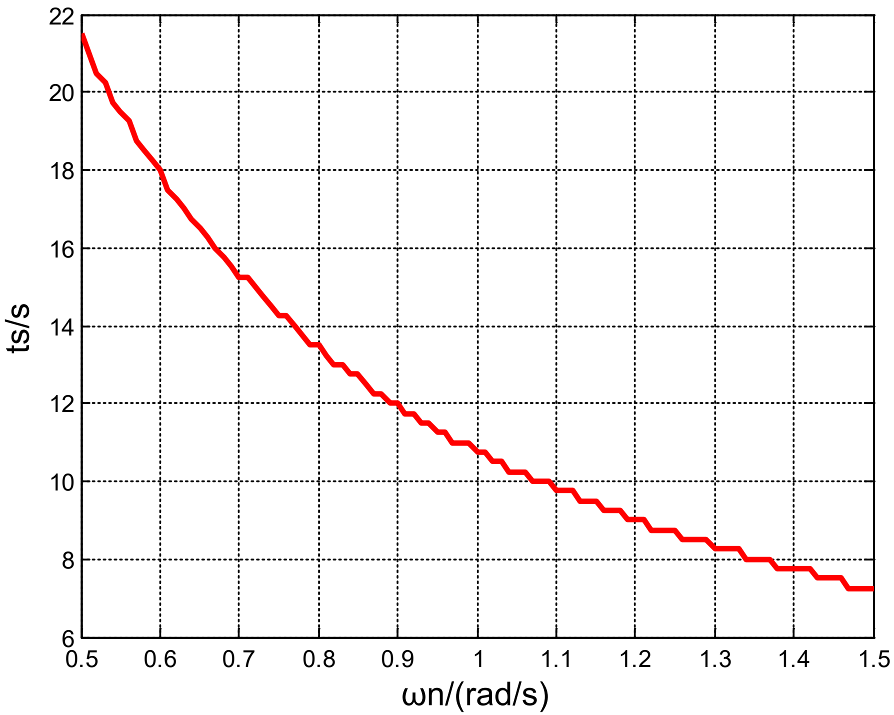


Reference answers:

Q5: True.

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2}$$

Let $\zeta=1.5$, and ω_n varies from 0.5 to 1.5, and then the relationship between ω_n and the settling time can be shown in the following graph:



Questions

- Why angle condition is a S&N condition?
- Why is the open loop gain is assigned to 10 instead of a variable?

```
s=tf('s');
g=10*(s+2)/(s^2+2*s+3);
rlocus(g);
```

- 增加正反馈系统根轨迹绘制规则与负反馈系统根轨迹绘制规则之间的对比表。
- 讲解第一次Quiz。