State Feedback Control Systems

Chapter 6

Review

- Definition of controllability
- Controllability condition and its demonstration
- Controllability canonical form and its controllability

Outlines to this Lesson

- Definition of observability
- Observability condition and its demonstration
- Observability canonical form and its observability
- Controllability and Observability versus Zero-Pole Cancellation
- Controllability and Observability Decomposition
- State-feedback control
- Output-feedback control

Observability

Definition:

Given a linear time-invariant system that is described by the following dynamic equations, the state $X(t_0)$ is said to be observable if given any input $U(t)$, there exists a finite time $t_f \geq t_0$ such that the knowledge of $U(t)$ for $t_0 \leq t < t_f$; matrices A, B, C, and D; and the output $Y(t)$ for $t_0 \leq t < t_f$ are sufficient to determine $X(t_0)$. If every state of the system is observable for a finite t_f, we say that the system is completely observable, or simply observable.

$$\dot{X}(t) = AX(t) + BU(t)$$
$$Y(t) = CX(t) + DU(t)$$

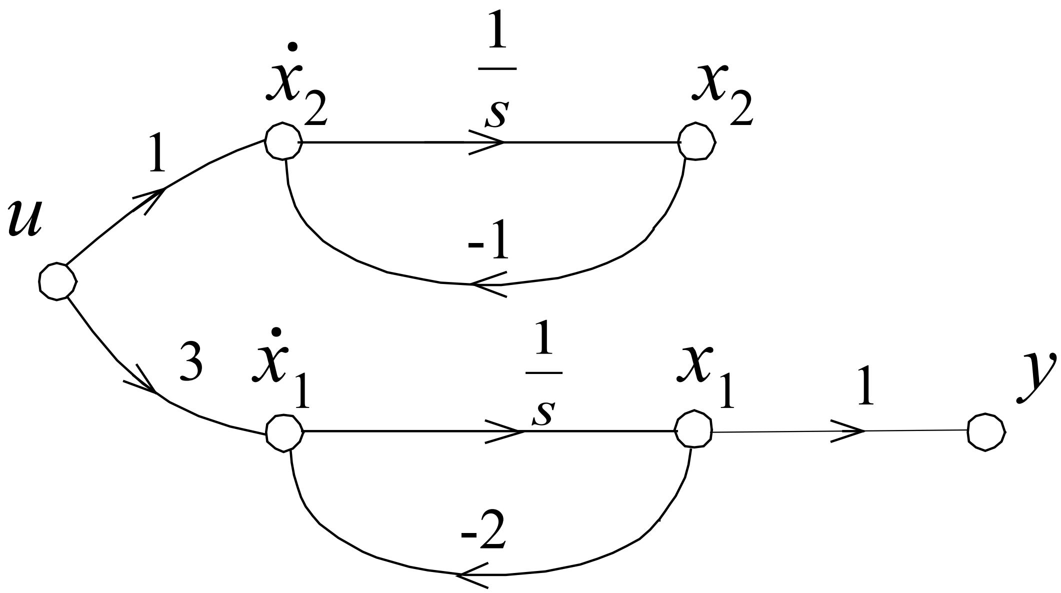
$X(t)$: $n \times 1$

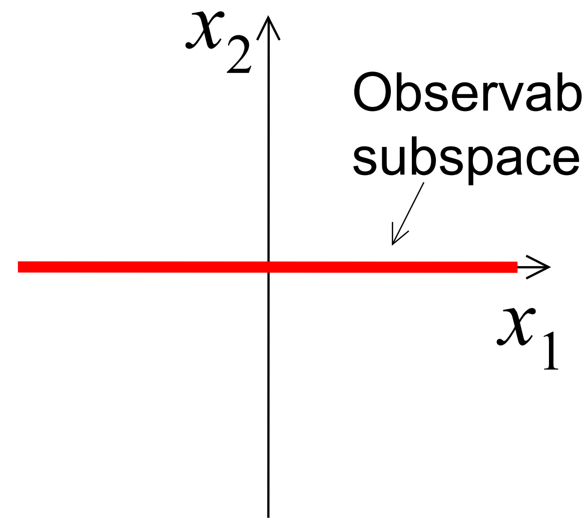
$U(t)$: $r \times 1$

$Y(t)$: $p \times 1$

Examples of Unobservable

Example 6.7 Find the given system is observable or unobservable.

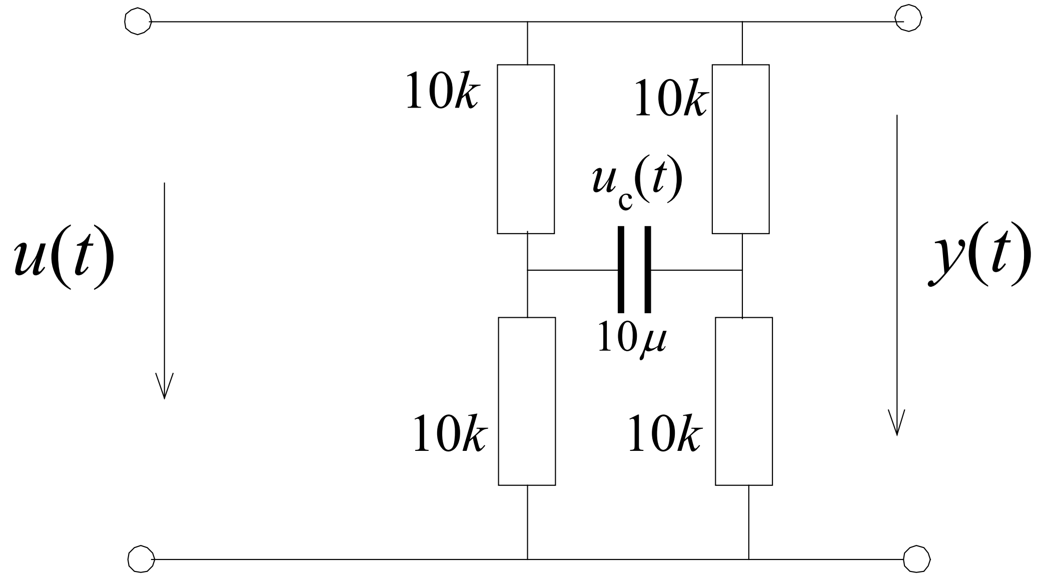




y is independent of x_2 , system is unobservable.

Examples of Unobservable

Example 6.8 Find the given system is observable or unobservable.



$y(t)$ does not have any information of $u_c(t)$, system is unobservable.

Observability Condition

For the system described by the given state equation

$$\dot{X}(t) = AX(t) + BU(t)$$
$$Y(t) = CX(t) + DU(t)$$

$X(t):\quad n \times 1$

$U(t):\quad r \times 1$

$Y(t):\quad p \times 1$

to be completely observable, it is necessary and sufficient that the following $np \times n$ observability matrix V has a rank of n:

$$V = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{np \times n}$$

Observability Example

Example 6.9

Find the given system is observable or unobservable.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t) \qquad y = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

A: $C = \begin{bmatrix} 1 & 2 \end{bmatrix}$ $\qquad CA = \begin{bmatrix} -2 & -2 \end{bmatrix}$

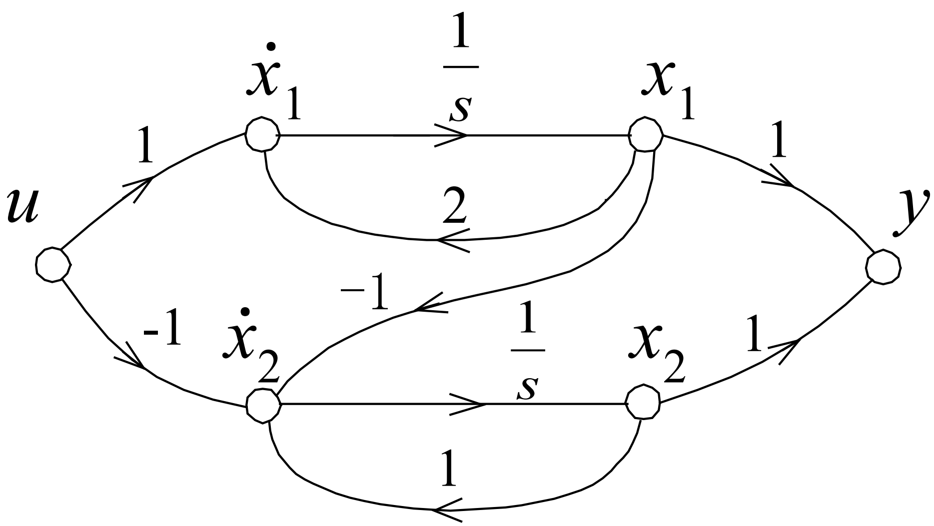
$$V = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & -2 \end{bmatrix}$$

$rank(V) = 2$ $\qquad$ system is observable

Observability Example

Example 6.10

Find the given system is observable or unobservable.



A:

$$\dot{X} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} X + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} X$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$V = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$rank(V) = 1 < 2$$

system is unobservable

Proof of the Observability Condition

To prove the observability of a system, we need to demonstrate that by the knowledge of $U(t), A, B, C, D$, and $Y(t)$, we can determine $X(t_0)$.

For a LTI system:

$$\dot{X}(t) = AX(t) + BU(t)$$
$$Y(t) = CX(t) + DU(t)$$

$X(t):\quad n\times 1$

$U(t):\quad r\times 1$

$Y(t):\quad p\times 1$

Set $U(t) = 0$

$$Y = CX$$
$$\dot{Y} = C\dot{X} = CAX$$
$$\ddot{Y} = CA\dot{X} = CA^2X$$
$$\vdots$$
$$Y^{(n-1)} = CA^{n-1}Y$$

$$\begin{bmatrix} Y \\ \dot{Y} \\ \vdots \\ Y^{(n-1)} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} X = VX$$

For an single output system:

$$\begin{bmatrix} Y(t_0) \\ \dot{Y}(t_0) \\ \vdots \\ Y^{(n-1)}(t_0) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{(n-1)} \end{bmatrix}_{n\times n} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_3(t_0) \end{bmatrix}_{n\times 1}$$

There are n unknown variables and n equations.

If $rank(V) = n$, the inverse of V exists, the equation has unique solution, and the system is observable.

If $rank(V) < n$, the inverse of V does not exist,we can not find a unique solution through the above equation.

For a multiple-output system, assume there are q outputs:

$$\begin{bmatrix} Y(t_0) \\ \dot{Y}(t_0) \\ \vdots \\ Y^{(n-1)}(t_0) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{nq\times n} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{bmatrix}_{n\times 1}$$

There are n unknown variables and nq equations.

If $rank(V) = n$, the equations have solutions, and the system is observable.

If $rank(V) < n$, we can not find a unique solution through the above equation.

Observability does not change
after non-singular linear transformation

Observability Condition 2

For a linear time-invariant system with distinct eigenvalues to be observable, it is necessary and sufficient that the C matrix in its diagonal canonical form does not have a column with all elements to be zero.

For example:

A single output system

$$\dot{x} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} X$$

if C does not have any zero element, the system is observable.

$$\lambda_i \neq \lambda_j \quad i, j = 1, 2, \cdots, n \qquad c_i \neq 0, \quad i = 1, 2, \cdots, n$$

$$C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

$$CA = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1 c_1 & \lambda_2 c_2 & \cdots & \lambda_n c_n \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}^2 = \begin{bmatrix} \lambda_1^2 c_1 & \lambda_2^2 c_2 & \cdots & \lambda_n^2 c_n \end{bmatrix}$$

$$V = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ \lambda_1 c_1 & \lambda_2 c_2 & \cdots & \lambda_n c_n \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} c_1 & \lambda_2^{n-1} c_2 & \cdots & \lambda_n^{n-1} c_n \end{bmatrix}$$

if $c_i \neq 0$, $\quad i = 1,2,\cdots,n$

$\lambda_i \neq \lambda_j \quad i \neq j$

$i, j = 1,2,\cdots,n$

then $rank(V) = n$

Example 6.11 Use Matlab tools to find out the observability of the give system.

$$\dot{X}=\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}X+\begin{bmatrix}1\\ -1\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 1\end{bmatrix}X$$

A:

```
a=[-1 0;0  -2]
c=[1   1]
v=[c ; c*a]
rank (v)
```

```
v=

     1      1
    -1     -2

rank (v)=2
```

Observability Canonical Form

$$\dot{X} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_1 \\ 1 & 0 & \cdots & 0 & -a_2 \\ 0 & 1 & \cdots & 0 & -a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_n \end{bmatrix} X + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ 0 \\ 0 \end{bmatrix} U$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} X$$

Systems in Observability canonical form must be observable.

$$V = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & & 1 & -a_n \\ \vdots & \iddots & \iddots & \vdots \\ 1 & -a_n & \cdots & \cdots \end{bmatrix} \qquad rank(V) = n$$

Controllability and Observability versus Zero-Pole Cancellation

Why can a transfer function always be transformed into a state space model in CCF or OCF?

Controllability and Observability versus Zero-Pole Cancellation

Example 6.12 Please convert the transfer function of the given system into diagonal canonical form, then discuss its controllability and observability.

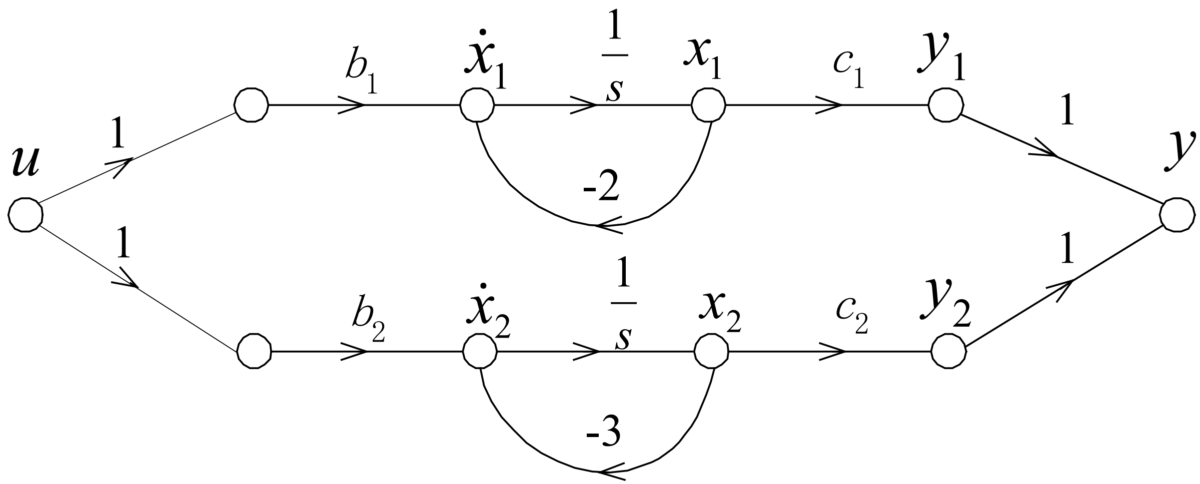
$$G(s)=\frac{(s+a)}{(s+2)(s+3)}$$

A: $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}b_1\\ b_2\end{bmatrix}u \qquad y=\begin{bmatrix}c_1 & c_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$Y(s)=G(s)U(s)=\frac{a-2}{s+2}U(s)+\frac{-a+3}{s+3}U(s)=Y_1(s)+Y_2(s)$$

Controllability and Observability versus Zero-Pole Cancellation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u \qquad y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

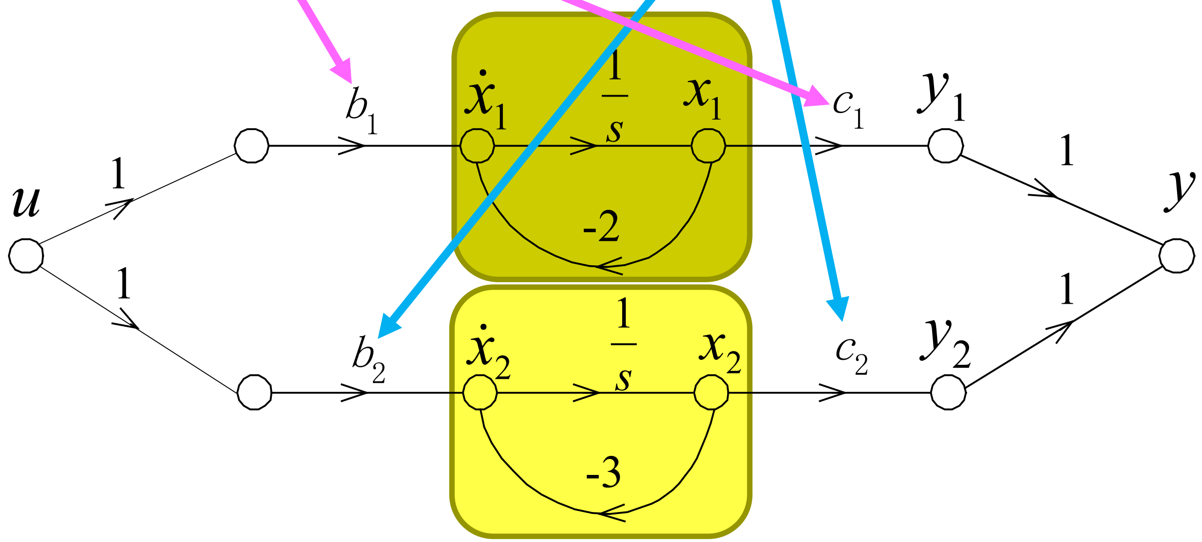


$$Y(s) = G(s)U(s) = \frac{a-2}{s+2}U(s) + \frac{-a+3}{s+3}U(s) = Y_1(s) + Y_2(s)$$

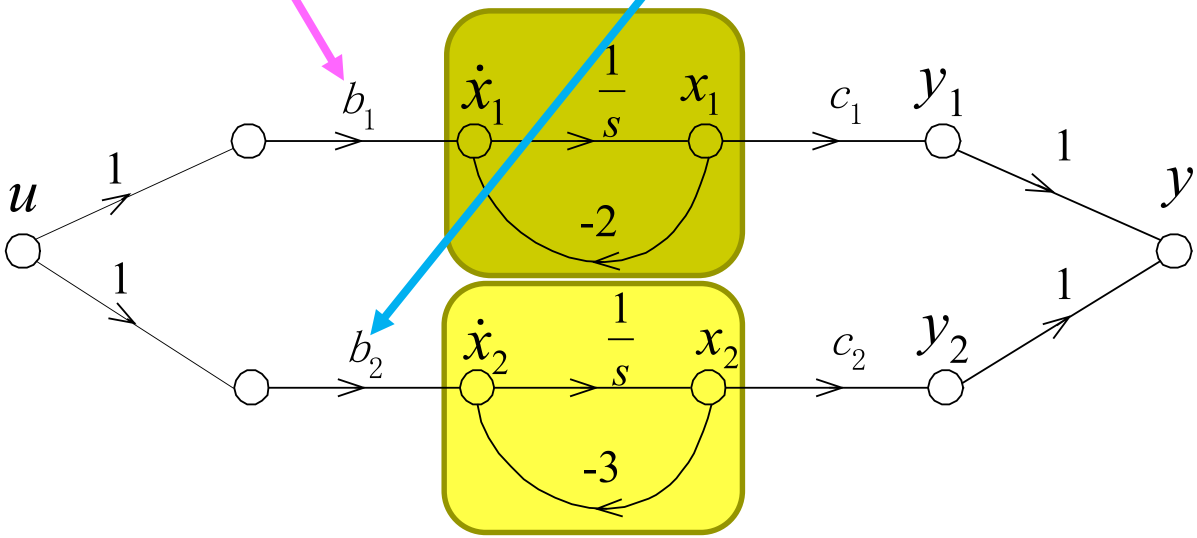
$$Y(s) = \frac{1}{s+2} \cdot (a-2) \cdot U(s) + \frac{1}{s+3} \cdot (-a+3) \cdot U(s) = Y_1(s) + Y_2(s)$$

Controllability and Observability versus Zero-Pole Cancellation

$$Y(s) = \frac{1}{s+2}\cdot(a-2)\cdot U(s) + \frac{1}{s+3}\cdot(-a+3)\cdot U(s) = Y_1(s) + Y_2(s)$$



$$Y(s)=\boxed{\frac{1}{s+2}}\cdot\boxed{(a-2)}\cdot U(s)+\boxed{\frac{1}{s+3}}\cdot\boxed{(-a+3)}\cdot U(s)=Y_1(s)+Y_2(s)$$



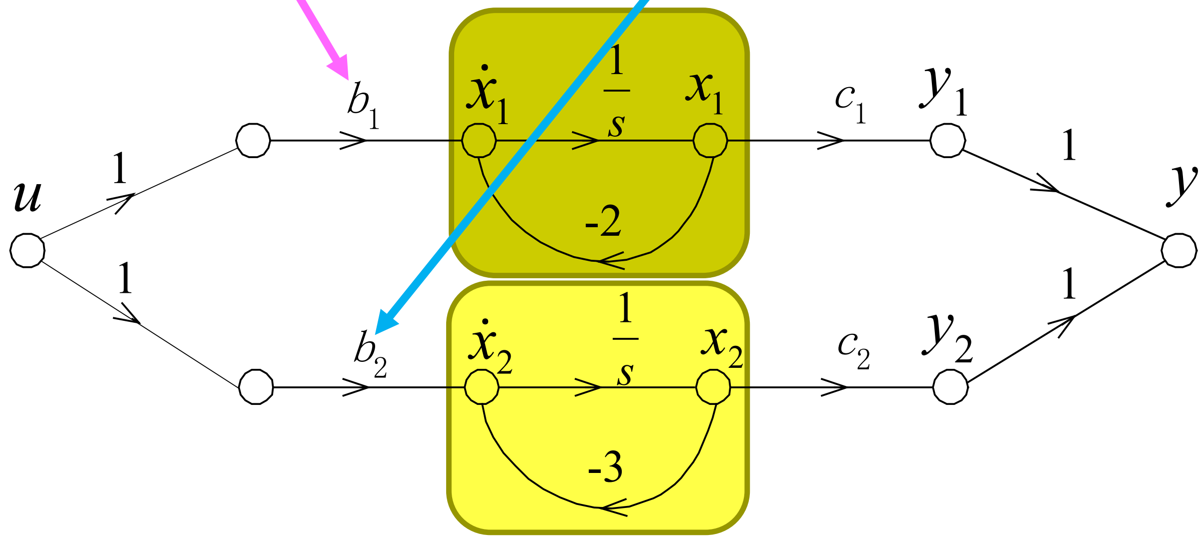
If $\quad b_1=a-2,\quad b_2=-a+3 \qquad c_1=1,\quad c_2=1$

when $\quad a\neq 2 \quad a\neq 3$

system is both controllable and observable

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-2 & 0\\ 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}a-2\\ -a+3\end{bmatrix}u$$

$$Y(s)=\boxed{\frac{1}{s+2}}\cdot\boxed{(a-2)}\cdot U(s)+\boxed{\frac{1}{s+3}}\cdot\boxed{(-a+3)}\cdot U(s)=Y_1(s)+Y_2(s)$$

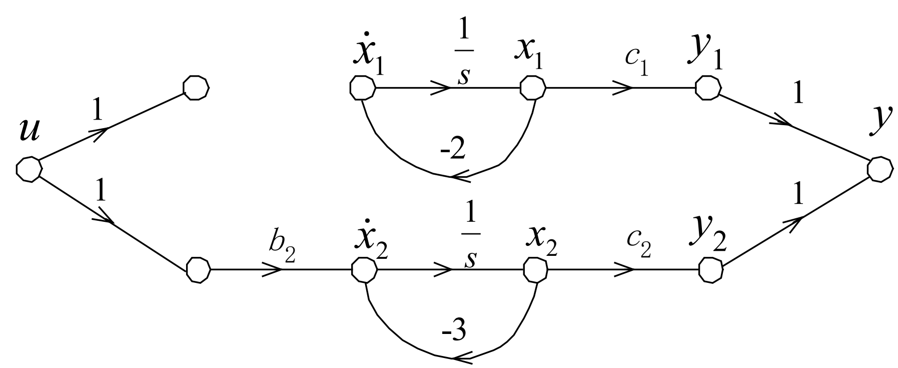


If $\quad b_1=a-2,\quad b_2=-a+3 \qquad c_1=1,\quad c_2=1$

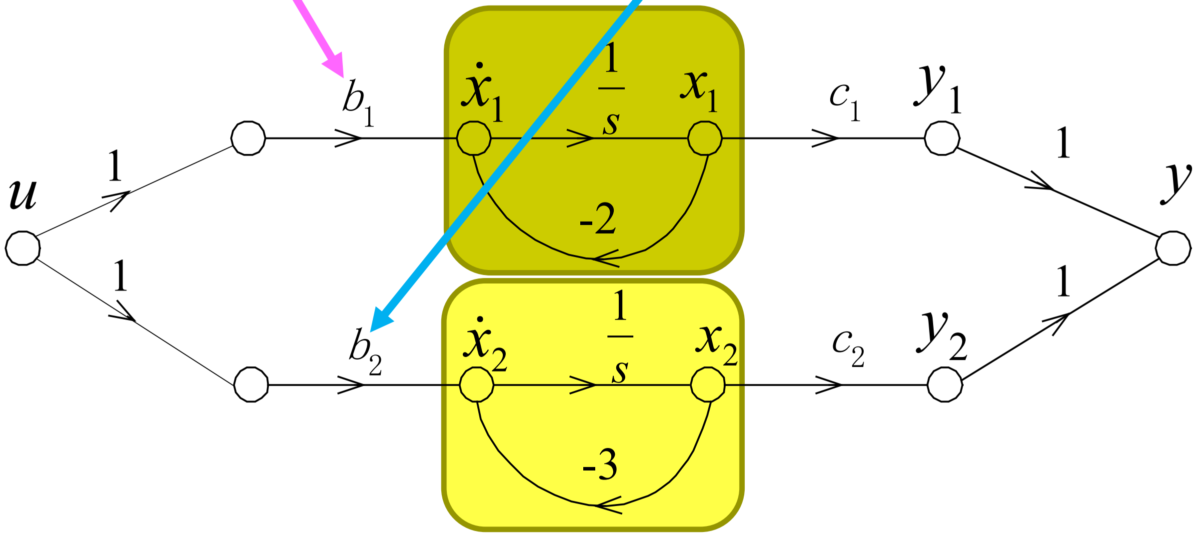
$a=2$

$$G(s)=\frac{1}{s+3}$$

uncontrollable



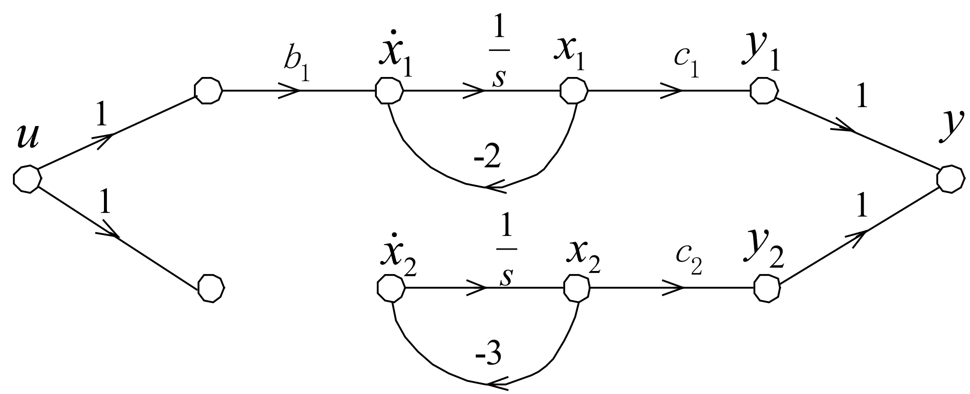
$$Y(s)=\boxed{\frac{1}{s+2}}\cdot\boxed{(a-2)}\cdot U(s)+\boxed{\frac{1}{s+3}}\cdot\boxed{(-a+3)}\cdot U(s)=Y_1(s)+Y_2(s)$$



If $\quad b_1=a-2,\quad b_2=-a+3 \qquad c_1=1,\quad c_2=1$

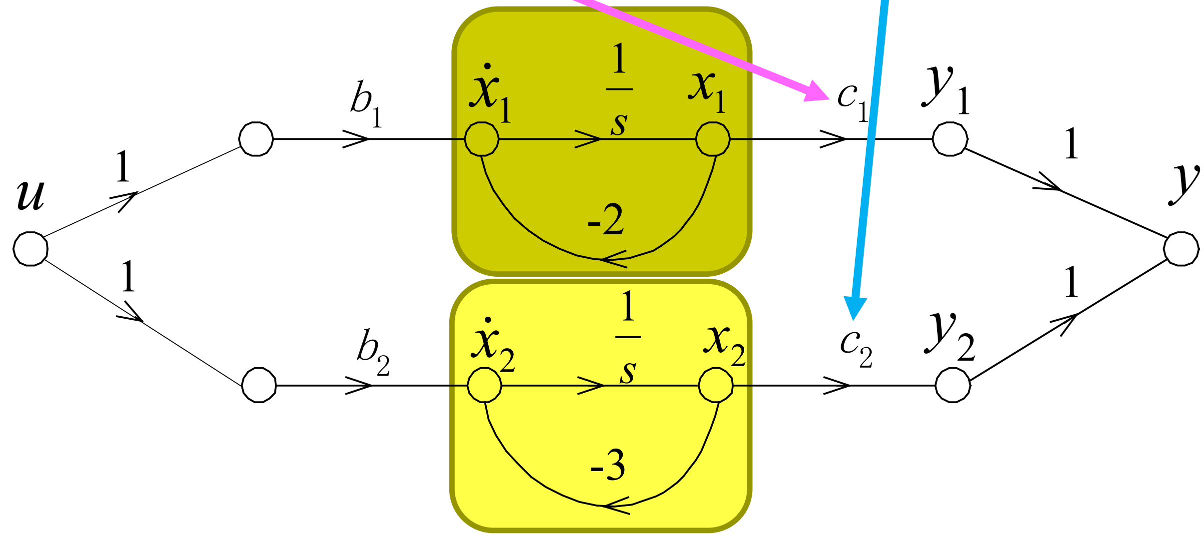
$$a=3$$

$$G(s)=\frac{1}{s+2}$$



uncontrollable

$$Y(s)=\boxed{\frac{1}{s+2}}\cdot\boxed{(a-2)}\cdot U(s)+\boxed{\frac{1}{s+3}}\cdot\boxed{(-a+3)}\cdot U(s)=Y_1(s)+Y_2(s)$$



If $b_1=1,\quad b_2=1\qquad c_1=a-2,\quad c_2=-a+3$

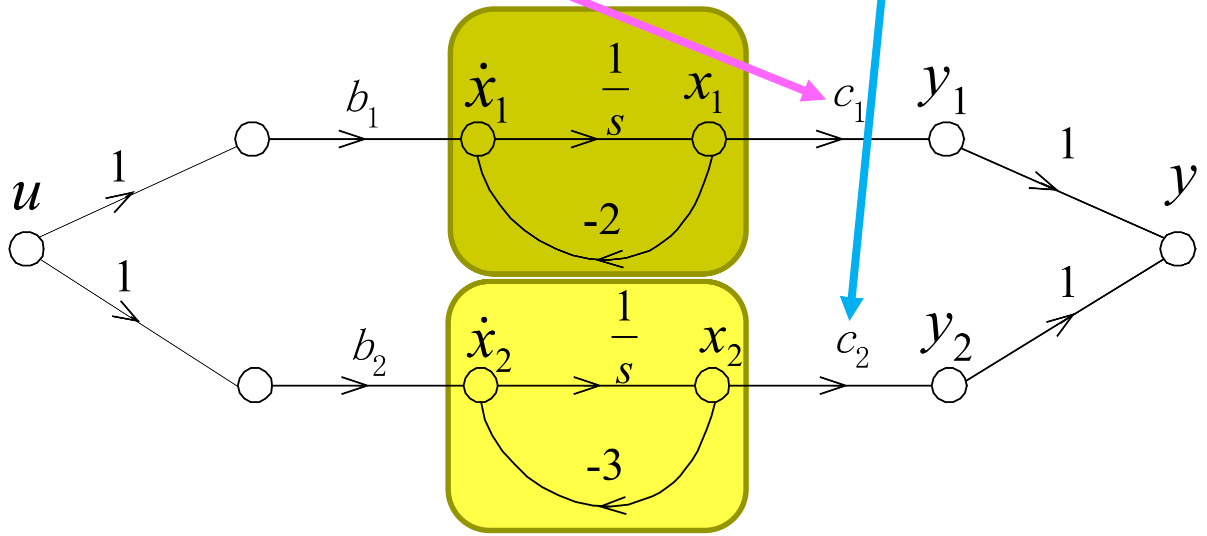
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-2 & 0\\ 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}a-2 & -a+3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

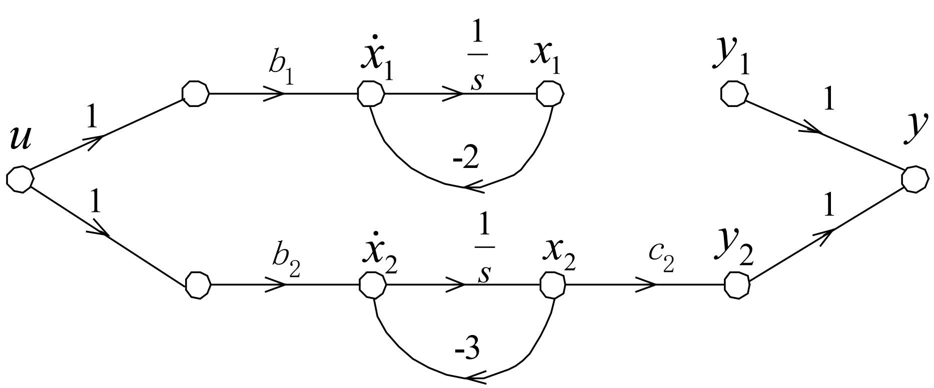
when $a\neq 2\quad a\neq 3$

system is both controllable and observable

$$Y(s) = \boxed{\frac{1}{s+2}} \cdot \boxed{(a-2)} \cdot U(s) + \boxed{\frac{1}{s+3}} \cdot \boxed{(-a+3)} \cdot U(s) = Y_1(s) + Y_2(s)$$



If $\quad b_1 = 1, \quad b_2 = 1 \qquad c_1 = a-2, \quad c_2 = -a+3$

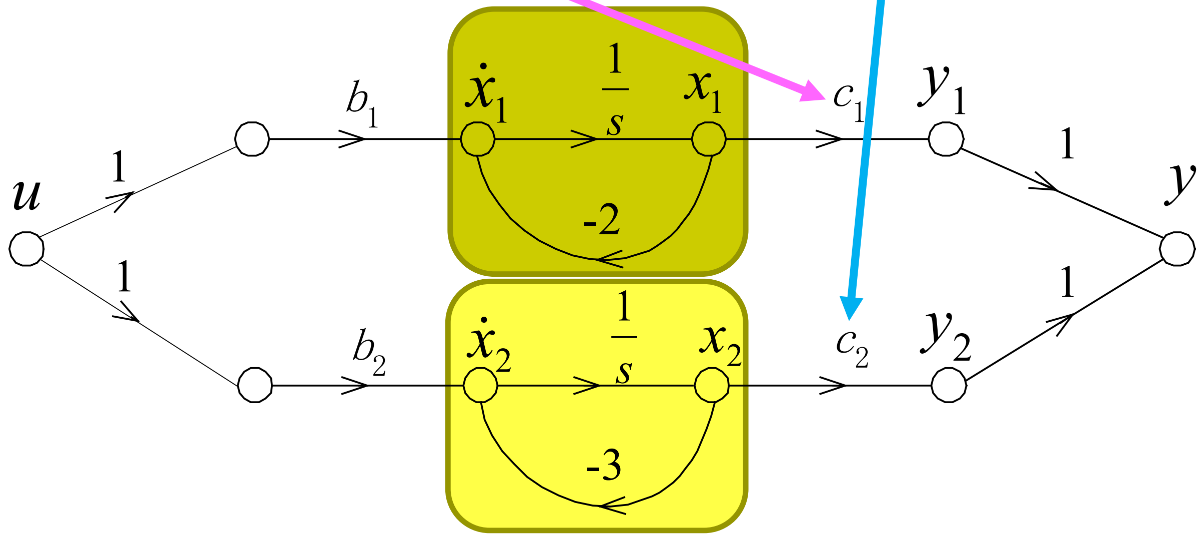


$$a = 2$$

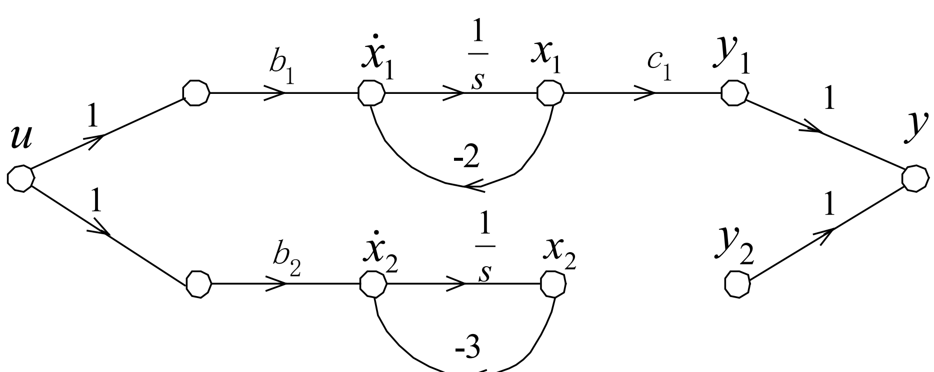
$$G(s) = \frac{1}{s+3}$$

unobservable

$$Y(s)=\frac{1}{s+2}\cdot(a-2)\cdot U(s)+\frac{1}{s+3}\cdot(-a+3)\cdot U(s)=Y_1(s)+Y_2(s)$$



If $\quad b_1=1,\quad b_2=1 \qquad c_1=a-2,\quad c_2=-a+3$

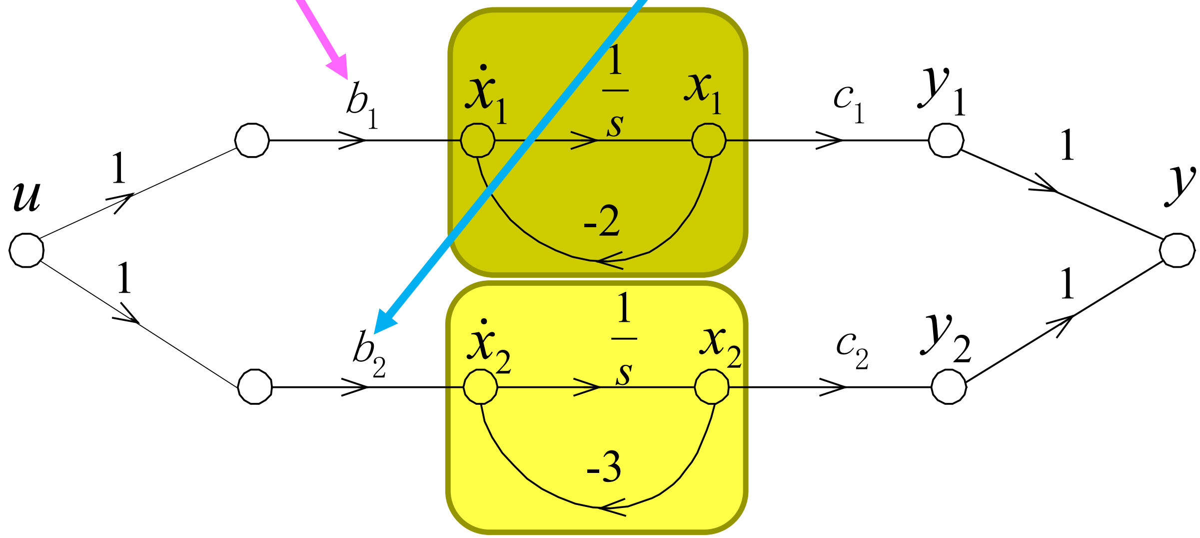


$$a=3$$

$$G(s)=\frac{1}{s+2}$$

unobservable

$$Y(s)=\frac{1}{s+2}\cdot(a-2)\cdot U(s)+\frac{1}{s+3}\cdot(-a+3)\cdot U(s)=Y_1(s)+Y_2(s)$$



If choose $b_1=(a-2)$ and $a=2,$ then $b_1=0$

$$Y(s)=\frac{1}{s+3}\cdot(-a+3)\cdot U(s)=Y_2(s)$$

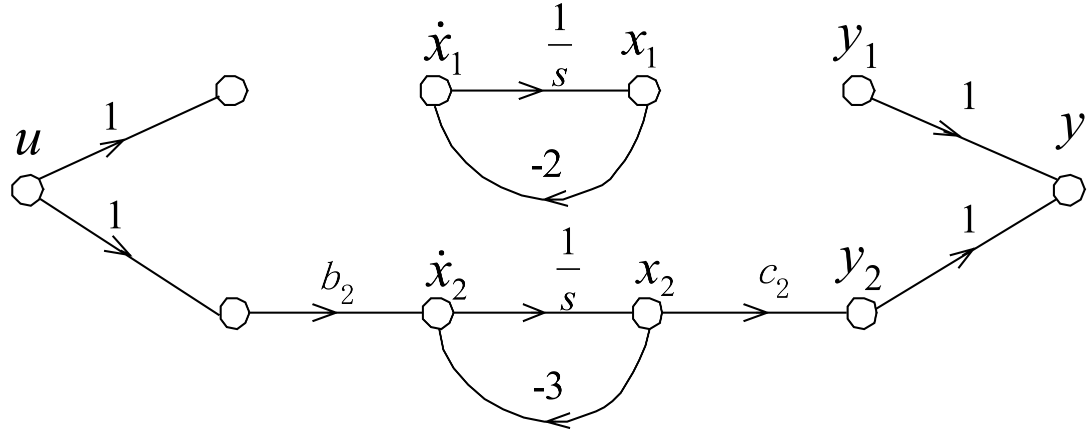
and further $c_1=0,\quad c_2=1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 0 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+3 & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{(s+2)(s+3)}$$

$$= \frac{s+2}{(s+2)(s+3)} = \frac{1}{s+3}$$

neither controllable
nor observable



Remarks

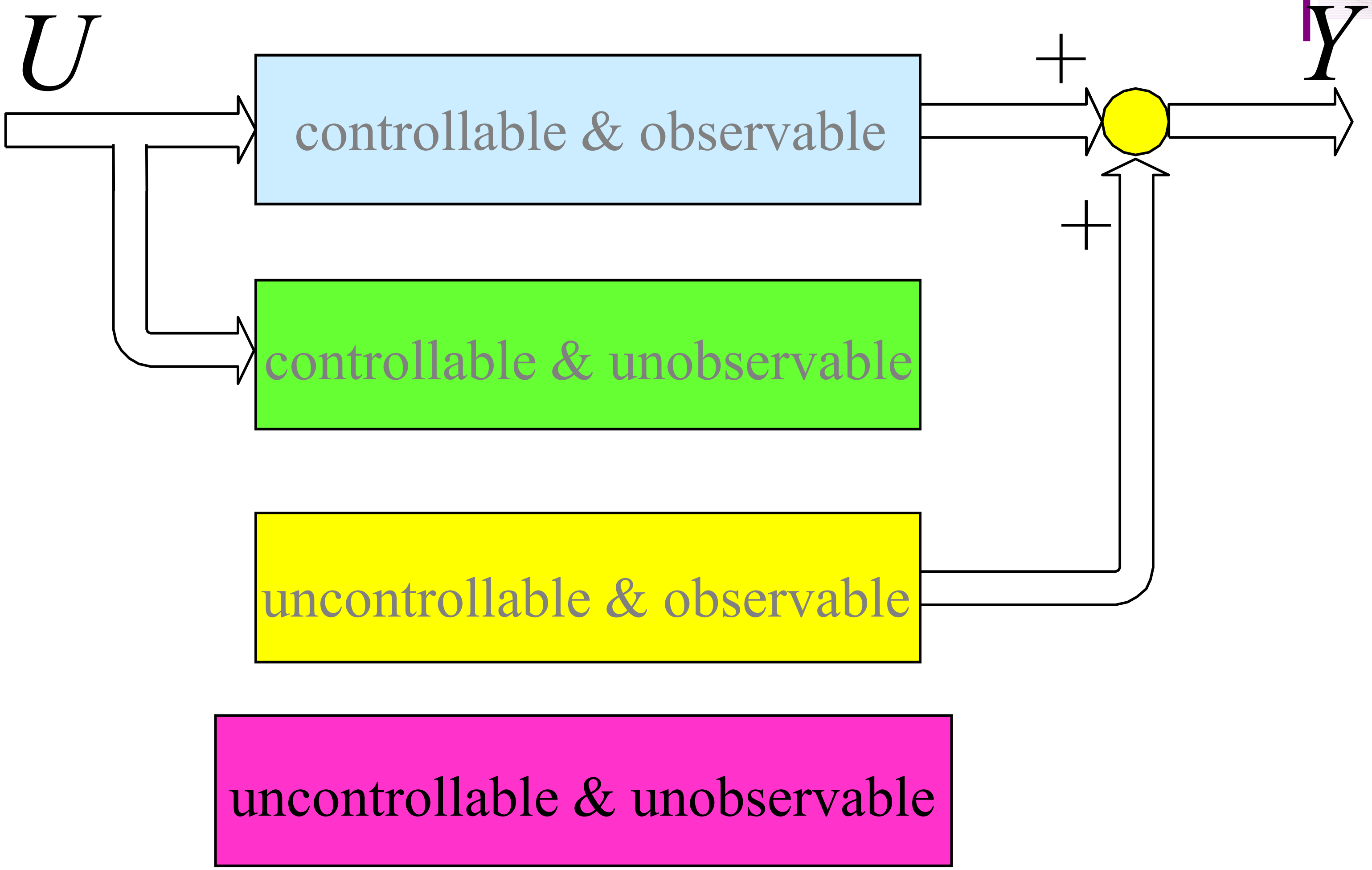
- For a SISO system (A, B, C) to be controllable and observable, it is sufficient and necessary that its transfer function does not have zero-pole cancellation.

$$G(s) = C(sI - A)^{-1}B = \frac{C \cdot \operatorname{adj}(sI - A) \cdot B}{|sI - A|}$$

Remarks

- In the transfer function of a SISO system, a cancelled pole could relate to an uncontrollable, or an unobservable, or an uncontrollable and unobservable mode. The transfer function only depicts the controllable and observable modes.

Controllability and Observability Decomposition



Controllability and Observability Decomposition

For a LTI system (A, B, C), rank(S) = m < n , there must exist a linear non-singular transformation P, X=PX', which can transform the system into a new one (A',B',C') that explicitly comprise a controllable part and an uncontrollable part.

$$\begin{bmatrix} \dot{X}_1' \\ \dot{X}_2' \end{bmatrix} = \begin{bmatrix} A_{11}' & A_{12}' \\ 0 & A_{22}' \end{bmatrix} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} + \begin{bmatrix} B_1' \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} C_1' & C_2' \end{bmatrix} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix}$$

$$\dot{X}_1' = A_{11}'X_1' + A_{12}'X_2' + B_1'U \quad \Rightarrow \quad \text{controllable states}$$

$$\dot{X}_2' = A_{22}'X_2' \quad \Rightarrow \quad \text{uncontrollable states}$$

Controllability and Observability Decomposition

How to construct the linear non-singular transformation P ?

$$P = [p_1, \cdots, p_m, p_{m+1}, \cdots, p_n]$$

where:

$[p_1, \cdots, p_m]$ are m linearly independent columns in S

$[p_{m+1}, \cdots, p_n]$ are n-m linearly independent columns in R^n, which are also independent of $p_1, \ldots, p_m$.

Example – 6.13

$$\dot{X} = \begin{bmatrix} -2 & 1 & -1 \\ 0 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} X$$

$$S = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$\mathrm{rank}(S) = 2 < 3$,

System is uncontrollable

$$P = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\dot{X}' = P^{-1}APX' + P^{-1}Bu$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 & -1 \\ 0 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} X' + \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1' \\ \dot{x}_2' \\ \dot{x}_3' \end{bmatrix} = \left[\begin{array}{cc|c} 0 & -1 & -1 \\ 1 & -2 & -1 \\ \hline 0 & 0 & -2 \end{array}\right] \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} + \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right] u$$

$$y = CPX' = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} X' = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

Controllability and Observability Decomposition

For a LTI system (A, B, C), rank(V) = m < n , there must exist a linear non-singular transformation Q, X=QX', which can transform the system into a new one (A',B',C') that explicitly comprise an observable part and an unobservable part.

$$\begin{bmatrix} \dot{X}_1' \\ \dot{X}_2' \end{bmatrix} = \begin{bmatrix} A_{11}' & 0 \\ A_{21}' & A_{22}' \end{bmatrix} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} + \begin{bmatrix} B_1' \\ B_2' \end{bmatrix} U \qquad Y = \begin{bmatrix} C_1' & 0 \end{bmatrix} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix}$$

$$\dot{X}_1' = A_{11}' X_1' + B_1' U$$ observable states

$$\dot{X}_2' = A_{21}' X_1' + A_{22}' X_2' + B_2' U$$ unobservable states

$$Y = C_1' X_1'$$

Controllability and Observability Decomposition

How to construct the linear non-singular transformation Q ?

$$Q^{-1} = \left[q_1^T, \cdots, q_m^T, q_{m+1}^T, \cdots, q_n^T \right]^T$$

where:

$\left[q_1^T, \cdots, q_m^T \right]^T$ are m linearly independent rows in V

$\left[q_{m+1}^T, \cdots, q_n^T \right]^T$ are n-m linearly independent rows in R^n, which are also independent of $q_1,\ldots,q_m$.

Example

$$\dot{X} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} X$$

$$V = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 2 \\ 4 & -7 & 4 \end{bmatrix}$$

$\mathrm{rank}(V) = 2 < 3$,

System is unobservable

$$Q^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 3 & -1 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{X}' = Q^{-1}AQX' + Q^{-1}Bu$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} X' + \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1' \\ \dot{x}_2' \\ \dot{x}_3' \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 0 \\ \hline -5 & 3 & 2 \end{array}\right] \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} u$$

$$y = CQX' = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} X' = \left[\begin{array}{cc|c} 1 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

Controllability and Observability Decomposition

If a LTI system (A, B, C) is neither controllable nor observable, there must exist a linear non-singular transformation P, X=PX’, which can transform the system into a new one (A’,B’,C’) that has the following form:

$$\begin{bmatrix} \dot{x}_1' \\ \dot{x}_2' \\ \dot{x}_3' \\ \dot{x}_4' \end{bmatrix} = \begin{bmatrix} A_{11}' & 0 & A_{13}' & 0 \\ A_{21}' & A_{22}' & A_{23}' & A_{24}' \\ 0 & 0 & A_{33}' & 0 \\ 0 & 0 & A_{43}' & A_{44}' \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} + \begin{bmatrix} B_1' \\ B_2' \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = [C_1' \quad 0 \quad C_3' \quad 0] \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix}$$

Controllable & observable

Controllable & unobservable

Uncontrollable & observable

Uncontrollable & unobservable

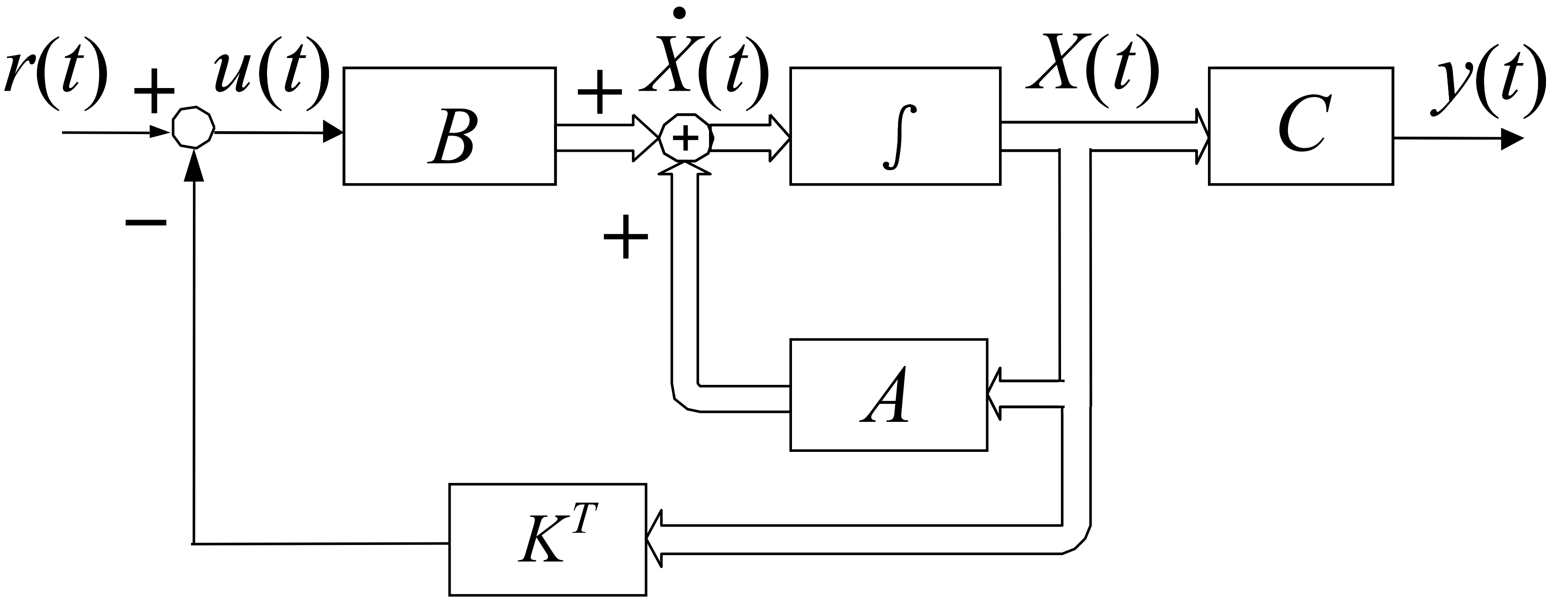
State-Feedback Control

Instead of using controllers with fixed configurations in the forward or feedback path, control is achieved by feeding back the state variables through real constant gains.

SISO: $\dot{X} = AX + Bu$ $\quad y = CX$

state feedback gain

If choose: $u = r - K^T X$ where $K^T = [k_1, k_2, \cdots, k_n]$



Pole-Placement Design through State Feedback

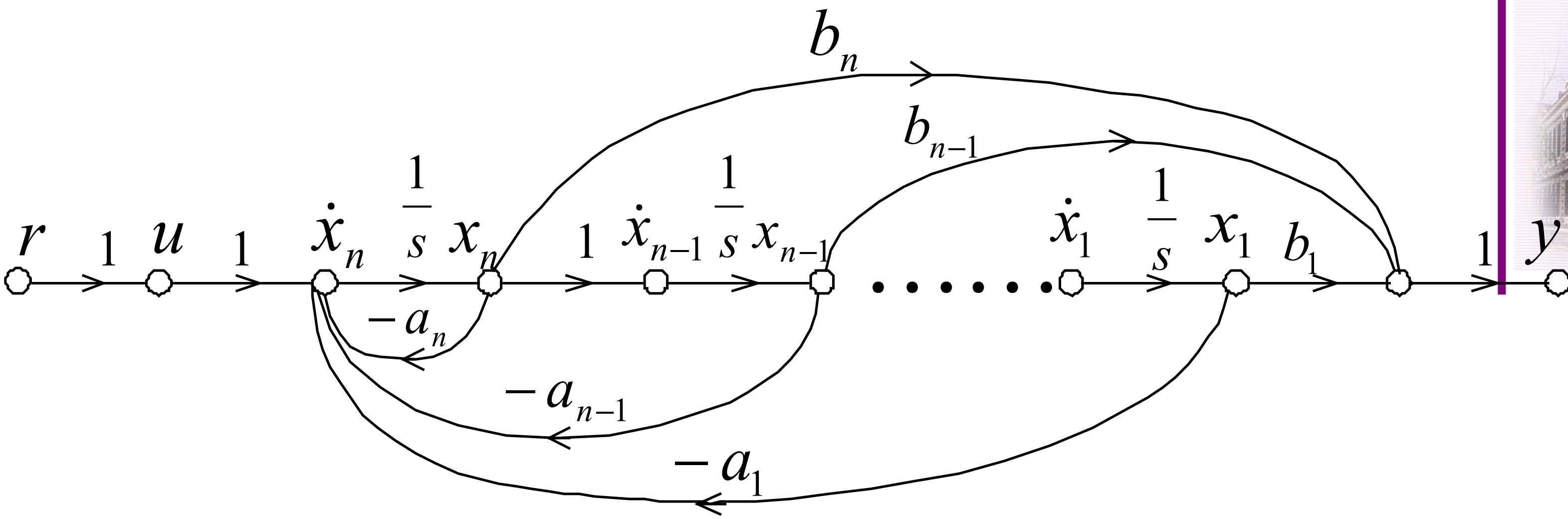
Knowing the relation between the closed-loop poles and the system performance, we can effectively carry out the design by specifying the location of these poles.

Theorem: For a SISO system (A,B,C), if its poles can be arbitrarily placed, it is a sufficient and necessary condition that the system is controllable.

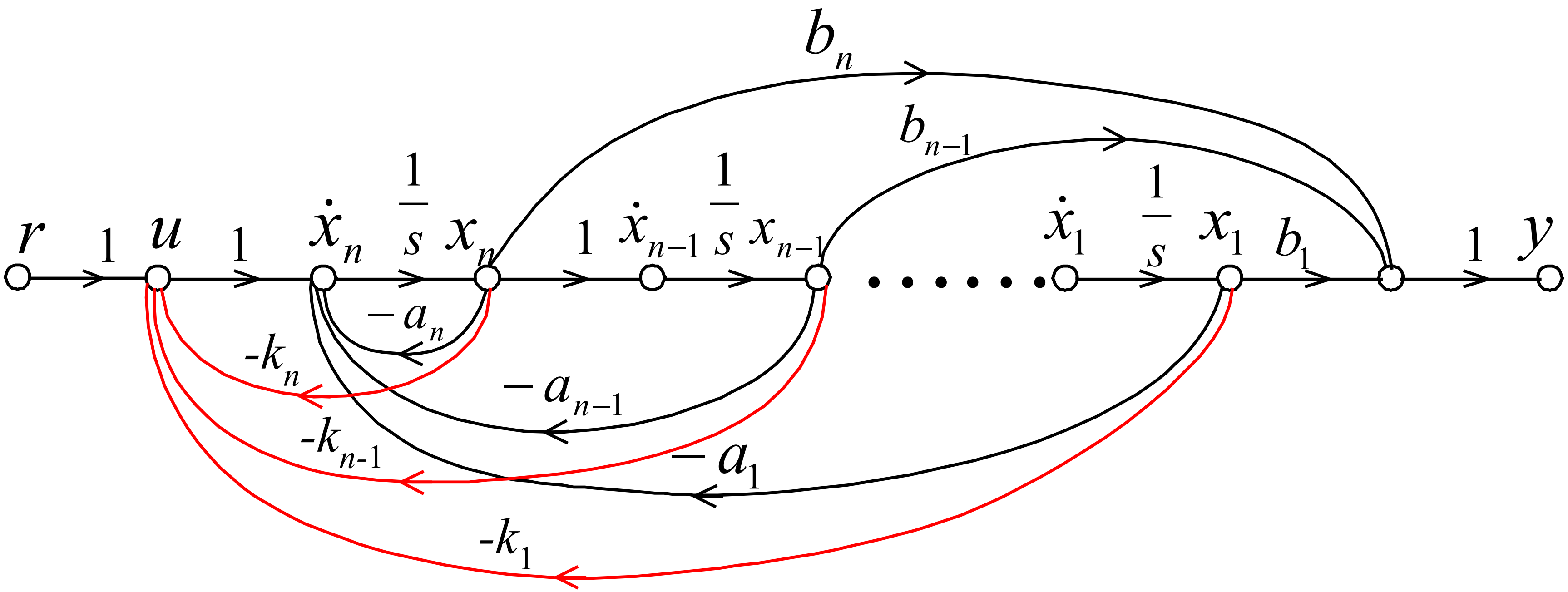
Proof: sufficiency

If a system is controllable, it can be transformed into CCF

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} X$$



Introducing state feedback: $u = r - K^T X$



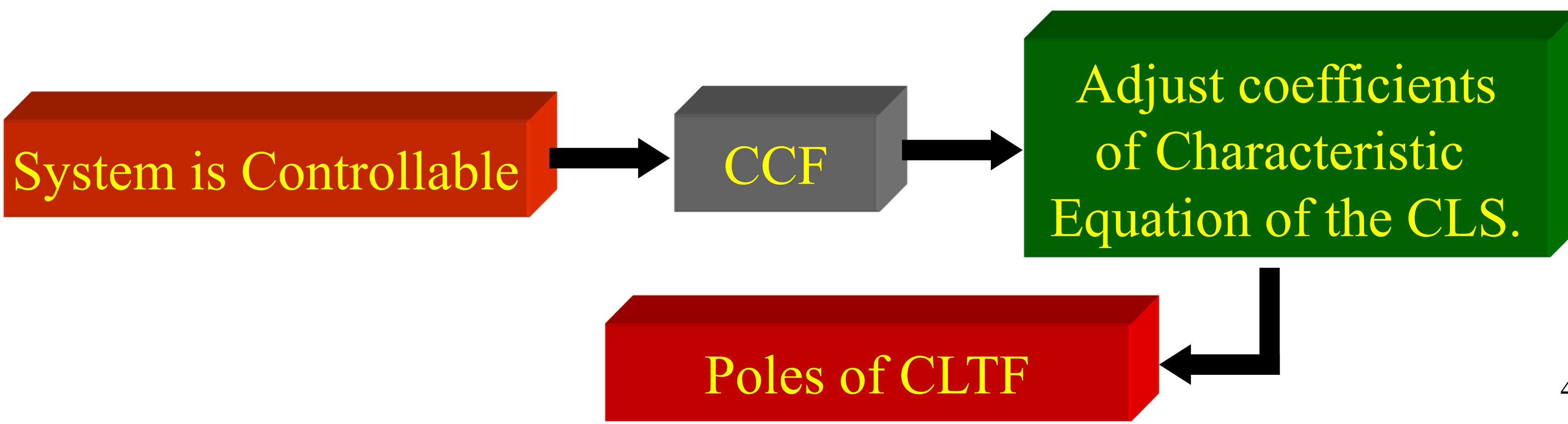
$$A - BK^T = A - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} = A + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ -k_1 & -k_2 & \cdots & -k_n \end{bmatrix}$$

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -k_1 - a_1 & -k_2 - a_2 & -k_3 - a_3 & \cdots & -k_n - a_n \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} r$$

$$G(s)=\frac{\dfrac{b_n}{s}+\dfrac{b_{n-1}}{s^2}+\cdots+\dfrac{b_1}{s^n}}{1+\dfrac{k_n}{s}+\dfrac{a_n}{s}+\dfrac{k_{n-1}}{s^2}+\dfrac{a_{n-1}}{s^2}\cdots\dfrac{k_1}{s^n}+\dfrac{a_1}{s^n}}$$

$$G(s)=\frac{Y(s)}{R(s)}=\frac{b_n s^{n-1}+b_{n-1}s^{n-2}+\cdots+b_1}{s^n+(a_n+k_n)s^{n-1}+\cdots+(a_1+k_1)}$$

n arbitrary poles can determine n coefficients.



Necessity:

If the system is uncontrollable, it must have uncontrollable states. The poles corresponding to these states can't be arbitrarily placed.

$$\begin{bmatrix} \dot{X}_1' \\ \dot{X}_2' \end{bmatrix} = \begin{bmatrix} A_{11}' & A_{12}' \\ 0 & A_{22}' \end{bmatrix} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} + \begin{bmatrix} B_1' \\ 0 \end{bmatrix} U$$

U can't affect X_2'

How to determine the Feedback Gain ?

Method of undetermined coefficient:

Step 1: Substitute $u = r - K^T X$ into the state equation

$$\dot{X} = AX + B(r - K^T X) = (A - BK^T)X + Br$$

Characteristic equation:

$$\Delta(s) = \left| sI - (A - BK^T) \right| = 0$$

Step 2: Calculate the characteristic equation of the desired system

$$\Delta'(s) = (s + p_1)(s + p_2)\cdots(s + p_n) = s^n + a_n s^{n-1} + \cdots + a_2 s + a_1 = 0$$

Step 3: Compare the two characteristic equations:

$$\left| sI - (A - BK^t) \right| = s^n + a_n s^{n-1} + \cdots + a_2 s + a_1$$

Example – 6.14

Q: Giving a system with the following state equation, please find the feedback gain that can make the poles of the modified system at $-2 \pm j2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

A: check the controllability of the original system

$$S = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

$$rank(S) = 2$$

system is controllable, so its poles can be arbitrarily placed.

The characteristic equation after introducing state feedback:

$$A - BK^T = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 - 2 & -k_2 - 3 \end{bmatrix}$$

$$\left| sI - (A - BK^T) \right| = \begin{vmatrix} s & -1 \\ k_1 + 2 & s + k_2 + 3 \end{vmatrix} = s^2 + (k_2 + 3)s + k_1 + 2$$

The characteristic equation of the desired system:

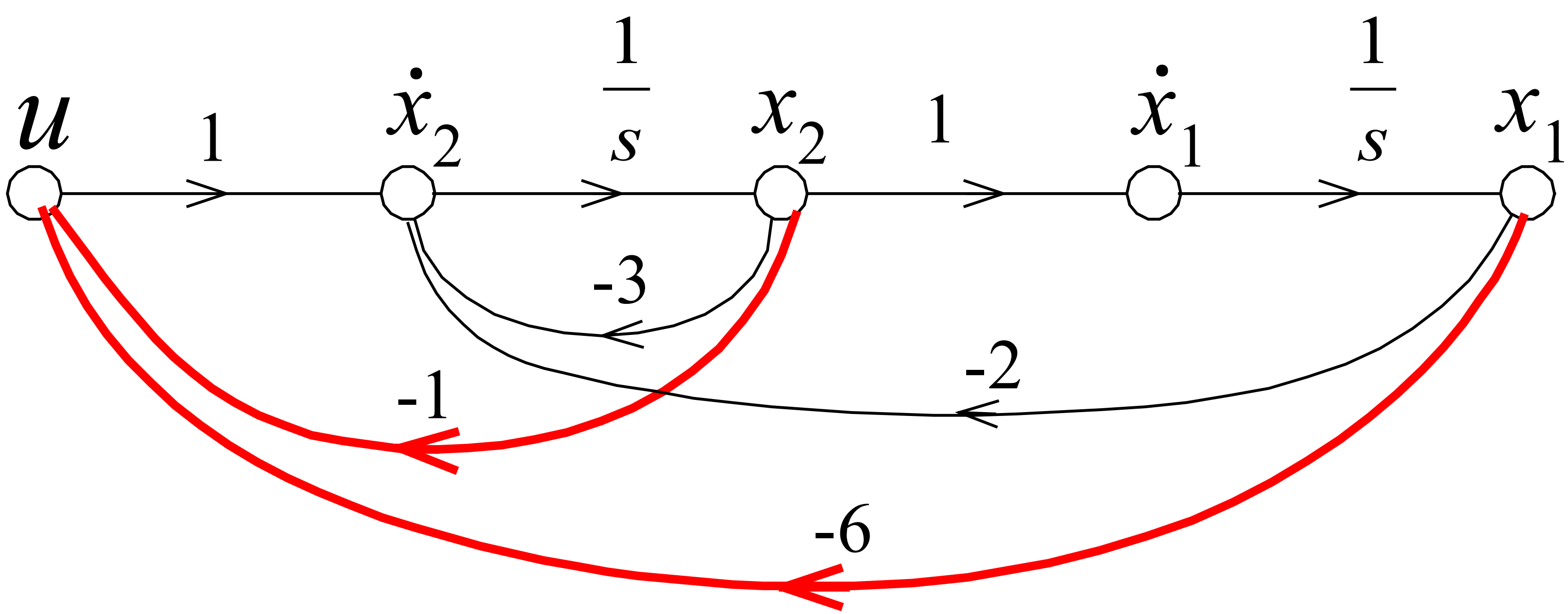
$$(s + 2 - j2)(s + 2 + j2) = s^2 + 4s + 8$$

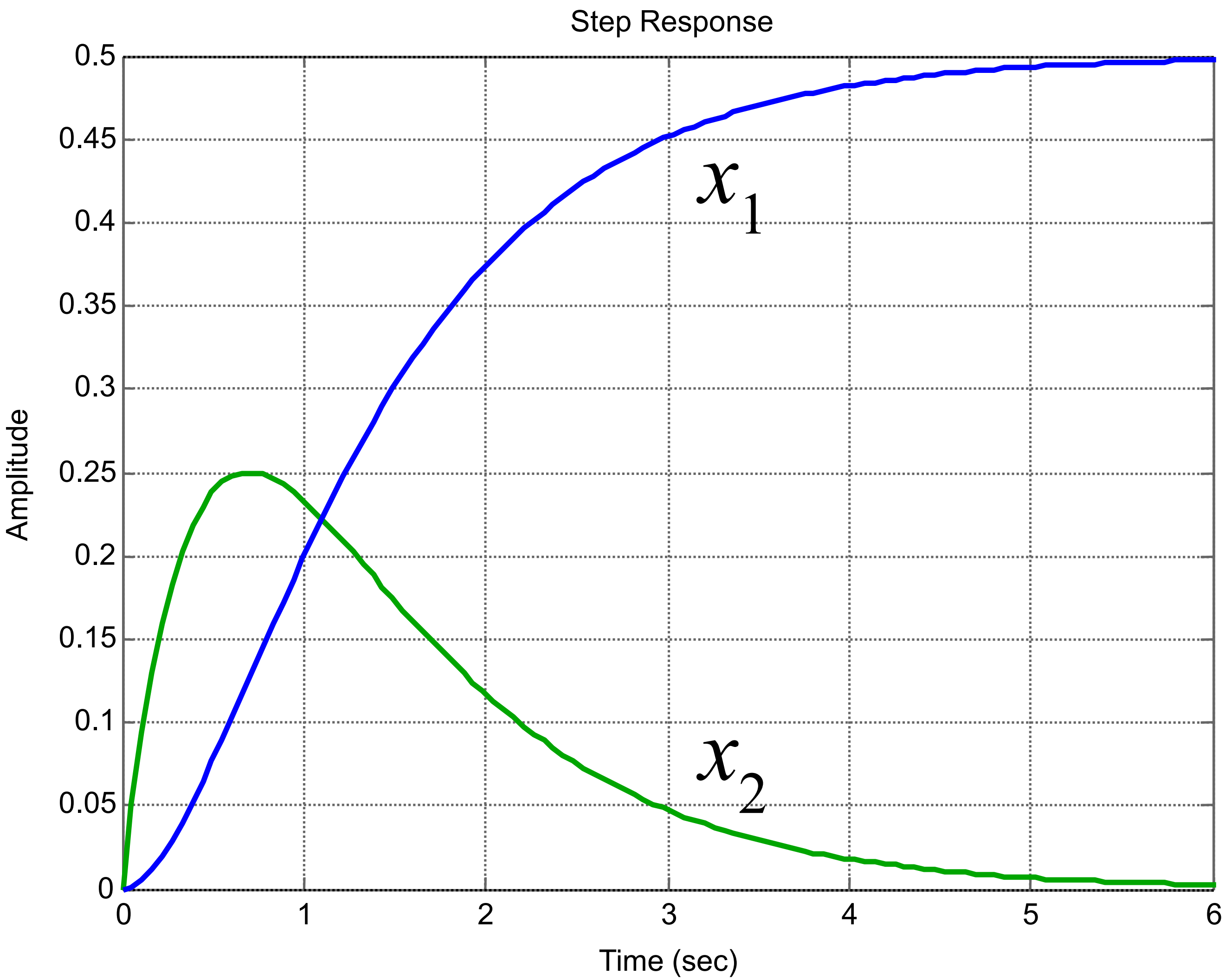
Compare the above two equations, we get:

$$k_1 = 6, \quad k_2 = 1$$

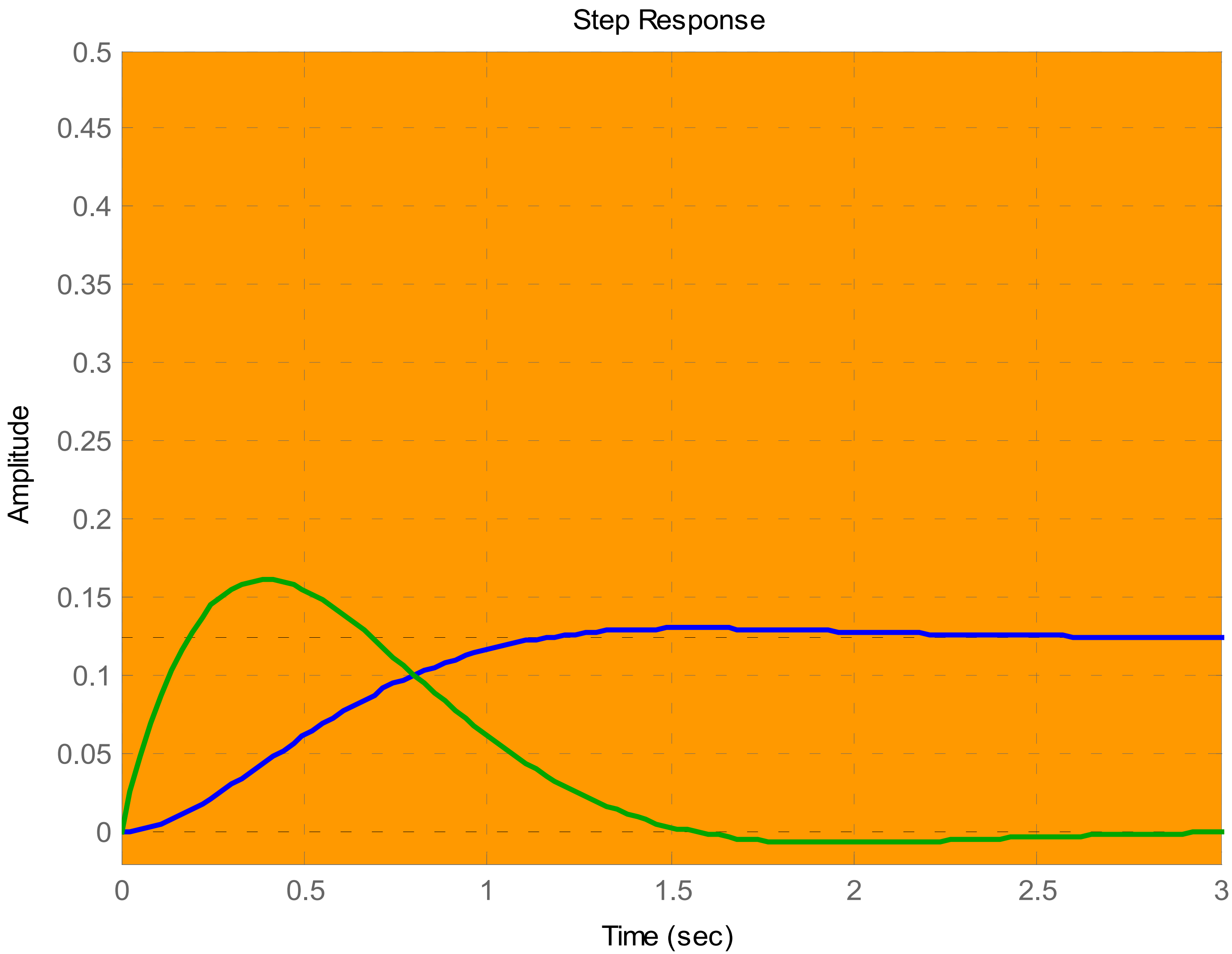
The original system: $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

The state feedback gain: $K^T = [6, 1]$





```
a=[0  1;-2  -3];
b=[0;1];
c1=[1  0];
c2=[0  1];
d=0;
g1=ss(a,b,c1,d);
g2=ss(a,b,c2,d);
step(g1,g2)
```


Step Response
Amplitude
Time (sec)
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.5
1
1.5
2
2.5
3

Example – 6.15

Q: Given a system with the following state equation, please find a feedback gain that can make the poles of the modified system at $-2,\ -1\pm j$

$$\dot{X}=\begin{bmatrix}0 & 0 & 0\\ 1 & -6 & 0\\ 0 & 1 & -12\end{bmatrix}X+\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}u$$

A: check the controllability of the original system

$$S=\left[B, AB, A^2B\right]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -6\\ 0 & 0 & 1\end{bmatrix}$$

$$rank(S)=3$$

system is controllable, so its poles can be arbitrarily placed.

The characteristic equation after introducing state feedback:

$$A-BK^T=\begin{bmatrix}0&0&0\\1&-6&0\\0&1&-12\end{bmatrix}-\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}k_1&k_2&k_3\end{bmatrix}=\begin{bmatrix}-k_1&-k_2&-k_3\\1&-6&0\\0&1&-12\end{bmatrix}$$

$$\left|sI-(A-BK^T)\right|=\begin{vmatrix}s+k_1&k_2&k_3\\-1&s+6&0\\0&-1&s+12\end{vmatrix}$$

$$=s^3+(k_1+18)s^2+(18k_1+k_2+72)s+72k_1+12k_2+k_3$$

The characteristic equation of the desired system:

$$(s+2)(s+1-j)(s+1+j)=s^3+4s^2+6s+4$$

Compare the above two equations, we get:

$$k_1+18=4$$

$$18k_1+k_2+72=6$$

$$72k_1+12k_2+k_3=4$$

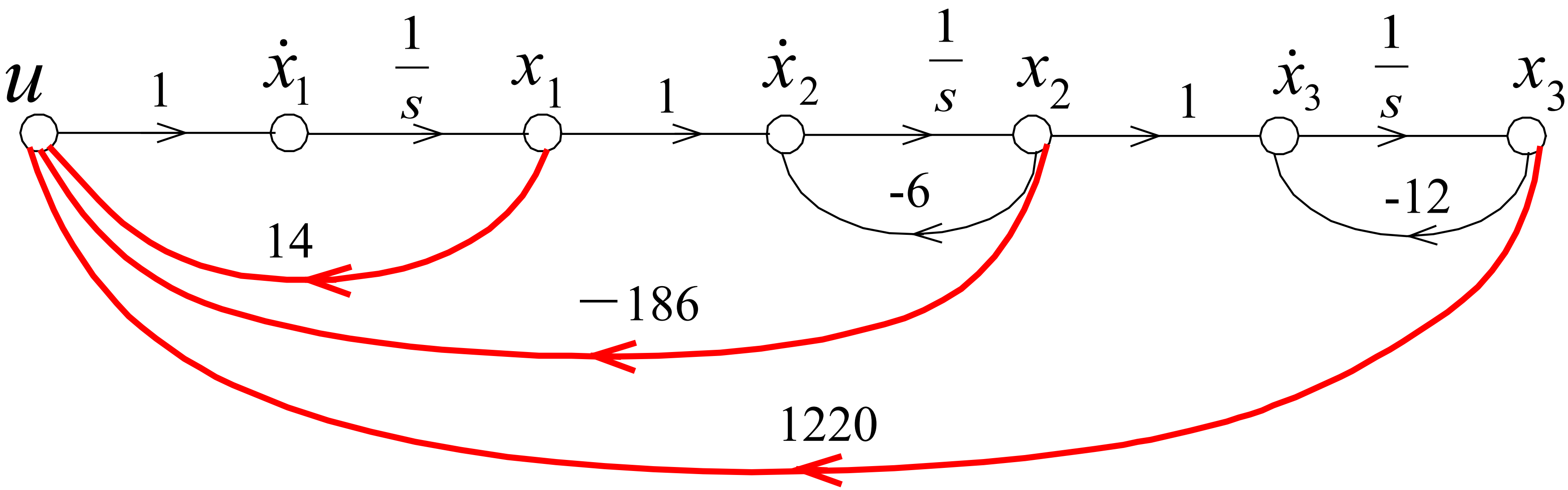
$$k_1=-14$$

$$k_2=186$$

$$k_3=-1220$$

The original system: $$\dot{X}=\begin{bmatrix}0 & 0 & 0\\ 1 & -6 & 0\\ 0 & 1 & -12\end{bmatrix}X+\begin{bmatrix}1\\0\\0\end{bmatrix}u$$

The state feedback gain: $$K^T=\begin{bmatrix}-14 & 186 & -1220\end{bmatrix}$$



投票 最多可选1项

设置

Can state feedback change the controllability of a system?

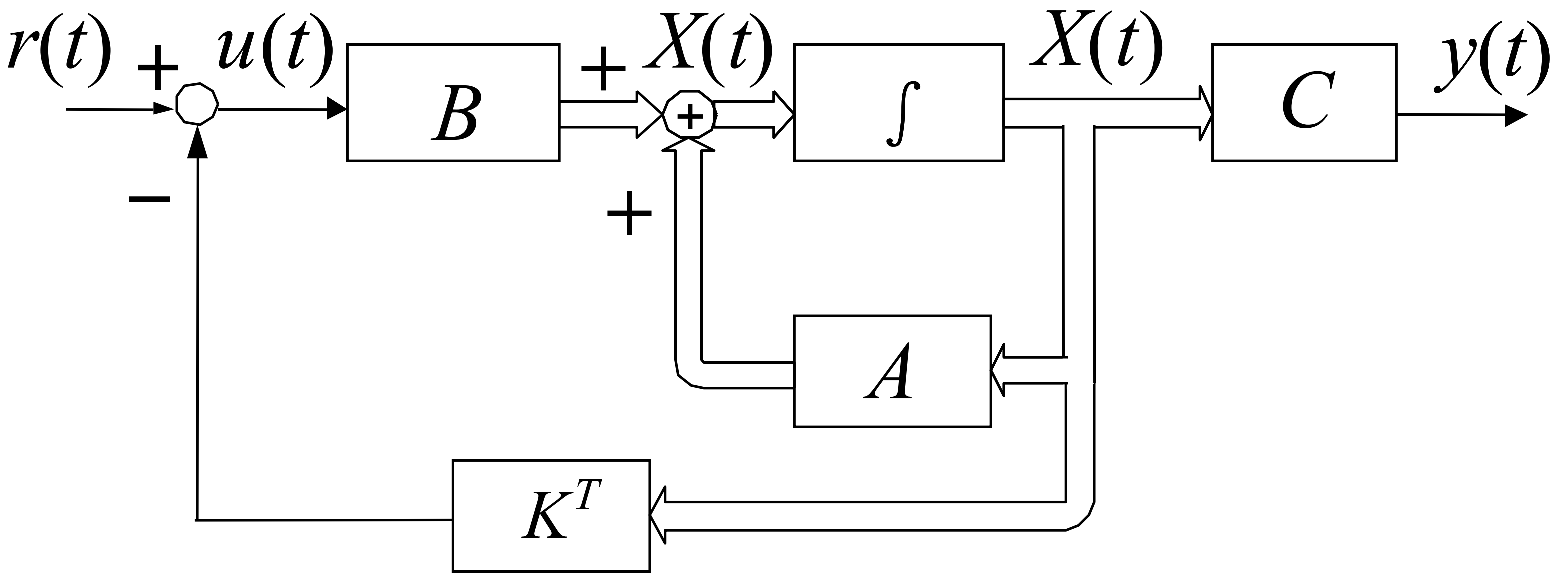
A Yes

B No

提交

State Feedback and Controllability

State feedback can not change the controllability of a system



$$\dot{X} = AX + Bu$$

$$y = CX$$

$$u = r - K^T X$$

$$\dot{X} = AX + B(r - K^T X)$$

$$= (A - BK^T)X + Br$$

The controllability matrix of the original system:

$$S = [B \quad AB \quad \cdots \quad A^{n-1}B]$$

The controllability matrix of the compensated system:

$$S' = [B \quad (A - BK^T)B \quad \cdots \quad (A - BK^T)^{n-1}B]$$

$$(A - BK^T)B = AB - B \cdot \underline{\underline{K^T B}}$$

$$(A - BK^T)^2 B = (A^2 - ABK^T - BK^T A + BK^T BK^T)B$$

$$= A^2 B - AB \cdot \underline{\underline{K^T B}} - B \cdot \underline{\underline{K^T AB}} + B \cdot \underline{\underline{K^T B}} \cdot \underline{\underline{K^T B}}$$

Notice that the double underscored is real numbers. So each column of S' can be linearly represented by the columns of S . Therefore:

$$rank(S) = rank(S')$$

投票 最多可选1项

设置

Can state feedback change the observability of a system?

A Yes

B No

提交

State Feedback and Observability

Can state feedback change the observability of a system?

Because the system poles can be arbitrarily placed through state feedback, when a replaced pole cancels a zero, the system observability will be changed.

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设置

Will state feedback change the order of a system?

A Yes

B No

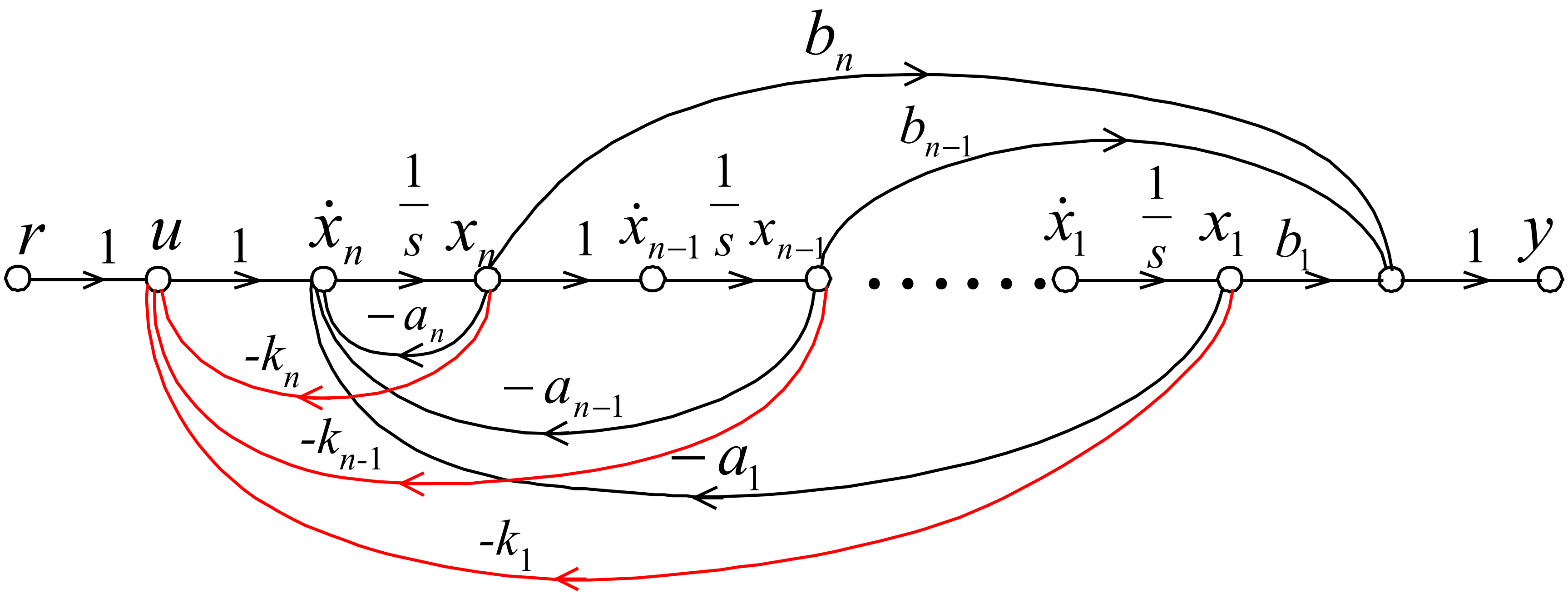
提交

State Feedback and Observability

Will state feedback change the order of a system?

State feedback will not change the order of a system.

Introducing state feedback: $u = r - K^T X$

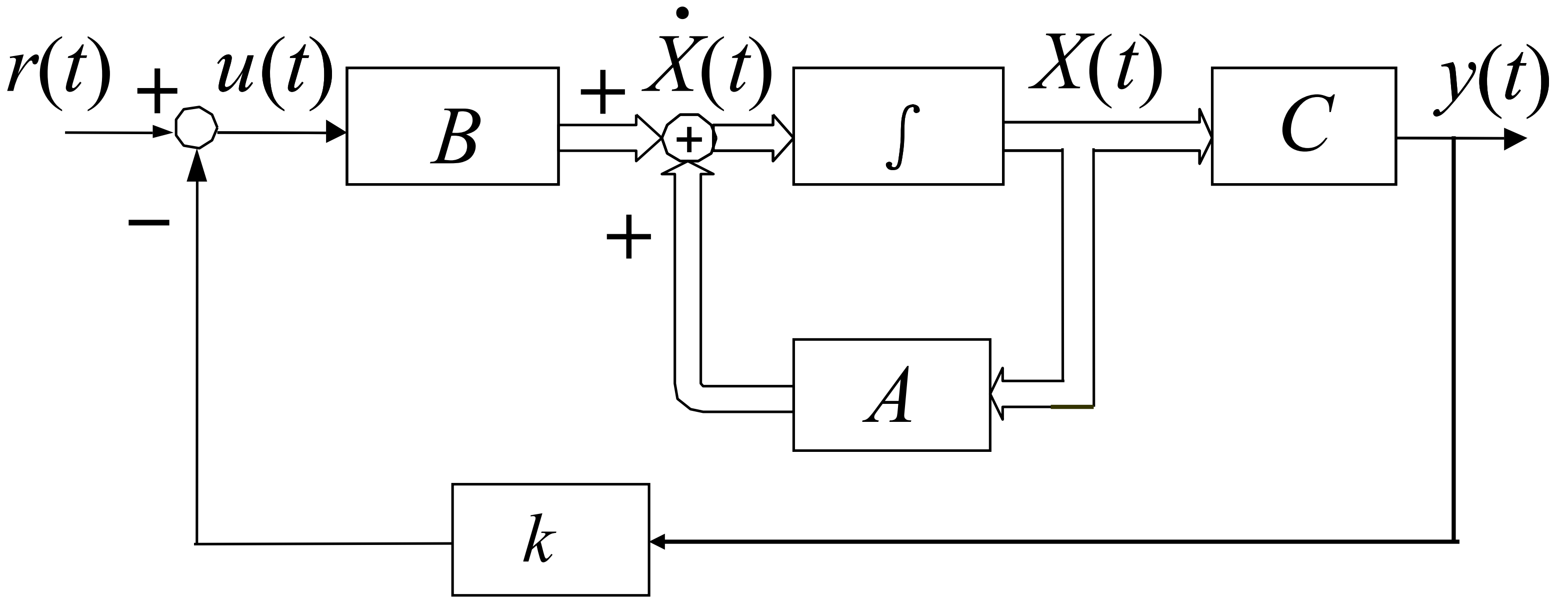


Output Feedback

Control is achieved by feeding back the outputs through real constant gains.

SISO: $\dot{X} = AX + Bu$ $\quad y = CX$

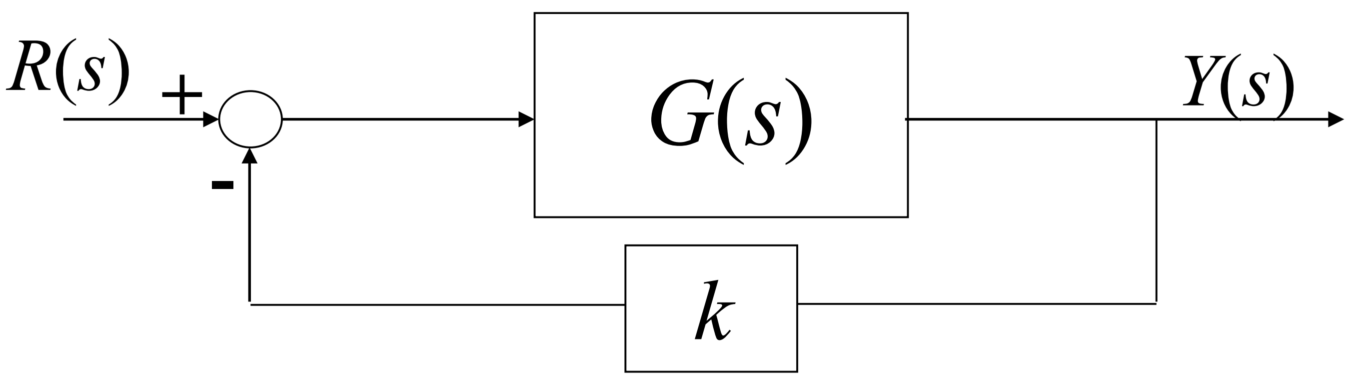
If choose: $u = r - ky$ $\quad$ where k output feedback gain



Output Feedback

$$\dot{X} = AX + B(r - ky) = AX + Br - BkCX$$

$$= (A - BkC)X + Br$$



General Conclusions about Output Feedback

If a SISO system is both controllable and observable, after introducing the output feedback, the new system is also controllable and observable.

When there is not output feedback: $$G(s) = \frac{N(s)}{D(s)}$$

Because the system is both controllable and observable, there is no common factor in numerator and denominator.

When there is output feedback:

$$M(s) = \frac{G(s)}{1+kG(s)} = \frac{\dfrac{N(s)}{D(s)}}{1+k\dfrac{N(s)}{D(s)}} = \frac{N(s)}{D(s)+kN(s)}$$

there is no common factor in numerator and denominator either. So, there is no zero-pole cancellation.

General Conclusions about Output Feedback

If a SISO system is uncontrollable (unobservable), after introducing the output feedback, the new system is still uncontrollable (unobservable).

There exists zero-pole cancellation in $\dfrac{C\,adj(sI-A)B}{|sI-A|}$

When there is output feedback:

$$\frac{\dfrac{C\,adj(sI-A)B}{|sI-A|}}{1+k\dfrac{C\,adj(sI-A)B}{|sI-A|}}=\frac{C\,adj(sI-A)B}{|sI-A|+kC\,adj(sI-A)B}$$

The common factors don't change, zero-pole cancellation does not disappear.

Wrap-up

- Definition of observability
- Observability condition and its demonstration
- Observability canonical form and its observability
- Controllability and Observability versus Zero-Pole Cancellation
- Controllability and Observability Decomposition
- State-feedback control
- Output-feedback control

Assignment

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9，(1)