1.

Sol: (a):

$$A = \left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & b_1 \\ 1 & 1 & 2 & 1 & b_2 \\ 1 & 2 & 2 & 0 & b_3 \\ 2 & 0 & 3 & 2 & b_4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & b_2 \\ 0 & 1 & 1 & 1 & b_1 \\ 1 & 2 & 2 & 0 & b_3 \\ 2 & 0 & 3 & 2 & b_4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & b_2 \\ 0 & 1 & 1 & 1 & b_1 \\ 0 & 1 & 0 & -1 & b_3-b_2 \\ 0 & -2 & -1 & 0 & b_4-2b_2 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & b_2 \\ 0 & 1 & 1 & 1 & b_1 \\ 0 & 0 & -1 & -2 & b_3-b_2-b_1 \\ 0 & 0 & 1 & 2 & b_4-2b_2+2b_1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & b_2 \\ 0 & 1 & 1 & 1 & b_1 \\ 0 & 0 & -1 & -2 & b_3-b_2-b_1 \\ 0 & 0 & 0 & 0 & b_1-3b_2+b_3+b_4 \end{array}\right]$$

When $b_1-3b_2+b_3+b_4=0$, the system has at least one solution.

(b) For $(b_1,b_2,b_3,b_4)=(1,1,1,1)$.

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & -3 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1-2x_4=-1 \\ x_2-x_4=0 \\ x_3+2x_4=1. \end{cases}$$

Let $x_4=0$

$$\begin{cases} x_1=-1 \\ x_2=0 \\ x_3=1 \end{cases} \quad \vec{x_p}=\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$N(A)$.

$$\begin{cases} x_1-2x_4=0 \\ x_2-x_4=0 \\ x_3+2x_4=0 \end{cases} \Rightarrow \begin{cases} x_1=2x_4 \\ x_2=x_4 \\ x_3=-2x_4 \end{cases} \quad \therefore \vec{x_n}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 2x_4 \\ x_4 \\ -2x_4 \\ x_4 \end{bmatrix}=x_4\begin{bmatrix} 2 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore \vec{x}=\vec{x_p}+\vec{x_n}=\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}+x_4\begin{bmatrix} 2 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

2. (a).

Sol:
$$\det A = \begin{vmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{vmatrix} = \begin{vmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & b-r & c-r & t-r \\ 0 & b-r & c-r & d-r. \end{vmatrix} = \begin{vmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-r \\ 0 & 0 & c-s & d-r \end{vmatrix}$$

$$= \begin{vmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-r \\ 0 & 0 & 0 & d-t \end{vmatrix} = a(b-r)(c-s)(d-t).$$

If A is invertible., that means $a(b-r)(c-s)(d-t) \neq 0$,

which means: $\begin{cases} a \neq 0 \\ b \neq r \\ c \neq s \\ d \neq t \end{cases}$

(b)

$$\det C = (-1)^{1+1} a \begin{vmatrix} b-r & s-r & s-r \\ 0 & c-s & t-r \\ 0 & 0 & d-t \end{vmatrix}$$

$$= a(b-r)(c-s)(d-t).$$

3. (a).

$$A = \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & b_1 \\ 2 & 3 & 4 & 5 & 6 & b_2 \\ 3 & 4 & 5 & 6 & 7 & b_3 \\ 4 & 5 & 6 & 7 & 8 & b_4 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & b_1 \\ 0 & -1 & -2 & -3 & -4 & b_2-2b_1 \\ 0 & -2 & -4 & -6 & -8 & b_3-3b_1 \\ 0 & -3 & -6 & -9 & -12 & b_4-4b_1 \end{array}\right] \quad 2b_1-b_2$$

$$\longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & b_1 \\ 0 & 1 & 2 & 3 & 4 & 2b_1-b_2 \\ 0 & -2 & -4 & -6 & -8 & b_3-3b_1 \\ 0 & -3 & -6 & -9 & -12 & b_4-4b_1 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & b_1 \\ 0 & 1 & 2 & 3 & 4 & 2b_1-b_2 \\ 0 & 0 & 0 & 0 & 0 & b_1-2b_2+b_3 \\ 0 & 0 & 0 & 0 & 0 & 2b_1-3b_2+b_4 \end{array}\right]$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} - R.$$

$$R = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b).

$$N(A): \begin{cases} x_1 - x_3 - 2x_4 - 3x_5 = 0 \\ x_2 + 2x_3 + 3x_4 + 4x_5 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 + 2x_4 + 3x_5 \\ x_2 = -2x_3 - 3x_4 - 4x_5 \end{cases}$$

$$\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 + 3x_5 \\ -2x_3 - 3x_4 - 4x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Bases for $N(A)$

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \right\}$$

Bases for $C(A^T)$.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \right\}$$

Bases for $C(A)$.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \right\}$$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

Bases for $N(A^T)$

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

4 Sol.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 3 & -1 & 6 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & 2 & -2 & 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

Basis of $\mathbb{R}^3$: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \right\}$.

$$\vec{v}_3 = -\vec{v}_2 + 2\vec{v}_1 \qquad \vec{v}_5 = -3\vec{v}_2 + \vec{v}_1 + 3\vec{v}_4$$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \qquad \begin{bmatrix} 1 \\ 4 \\ 10 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

5.(a) Sol: Since if we write these two set of vectors in determinant and transpose one of them, we will get the same determent as another one, and the result of transposing a determinent will not change, so. these two sets of vectors have equal volumes.

(b). $Area = |\det(A)|$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1 \times 1 \times 1$$

$$= 1$$

6(a).

$$y = C + Dt.$$

$$\Rightarrow \begin{cases} C + 0 = 1 \\ C + D = 3 \\ C + 2D = 2 \\ C + 3D = 4 \\ C + 4D = 5. \end{cases}$$

$$A\hat{x} = \vec{b}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 5 \end{bmatrix}$$

$$A^T A \hat{x} = A^T \vec{b}.$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 5. & 10. \\ 10. & 30 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 15 \\ 39 \end{bmatrix}.$$

$$A^T A \hat{x} = A^T \vec{b}$$

$$\begin{bmatrix} 5 & 10 \\ 10 & 30 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 15 \\ 39 \end{bmatrix}$$

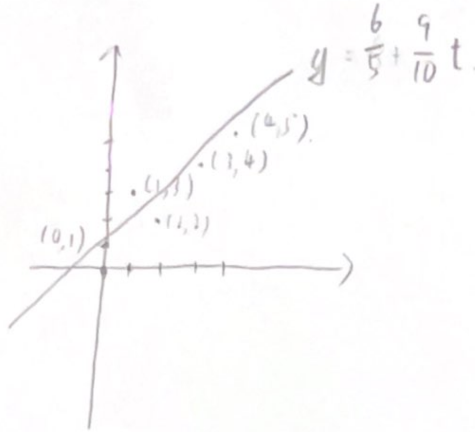
$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 10 & 30 \end{bmatrix}^{-1} \begin{bmatrix} 15 \\ 39 \end{bmatrix}$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{150 - 100} \begin{bmatrix} 30 & -10 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} 15 \\ 39 \end{bmatrix}$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 15 \\ 39 \end{bmatrix}$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ \frac{9}{10} \end{bmatrix} \qquad \therefore y = \frac{6}{5} + \frac{9}{10} t$$

(b).



(c). $\|\vec{e}\| = \|A\vec{x} - \vec{b}\|$

$$= \left\| \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \frac{6}{5} \\ \frac{9}{10} \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 5 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} \frac{6}{5} \\ \frac{21}{10} \\ 3 \\ \frac{39}{10} \\ \frac{48}{10} \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 5 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} \frac{6}{5} \\ -\frac{11}{10} \\ 1 \\ -\frac{1}{10} \\ -\frac{1}{5} \end{bmatrix} \right\|$$

$$= \sqrt{\left(\frac{6}{5}\right)^2 \left(-\frac{11}{10}\right)^2 + 1^2 + \left(\frac{1}{10}\right)^2 + \left(-\frac{1}{5}\right)^2}$$

$$= \frac{\sqrt{370}}{10}$$

7.

$$A-\lambda I=\begin{bmatrix} -\lambda & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\lambda \end{bmatrix}.$$

$$\det(A-\lambda I) = \left(-\lambda^3+\frac{1}{8}+\frac{1}{8}\right)-\left(-\frac{\lambda}{4}-\frac{\lambda}{4}-\frac{\lambda}{4}\right).$$

$$= -\lambda^3+\frac{1}{4}+\frac{3\lambda}{4}.$$

$$\lambda=1,\ \lambda=-\frac{1}{2}$$

① $\lambda=1$.

$A\vec{x}=\vec{x}$

$$\vec{x}(A-I)=\vec{0}=\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}.\rightarrow\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\rightarrow\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & -\frac{3}{2} \end{bmatrix}\rightarrow\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\rightarrow\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\Rightarrow\begin{cases} x_1=x_3 \\ x_2=x_3 \end{cases}\quad\therefore\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix}=x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

② $\lambda=-\frac{1}{2}$.

$A\vec{x}=-\frac{1}{2}\vec{x}$

$\vec{x}(A+\frac{1}{2}I)=\vec{0}$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\Rightarrow\left\{ x_1=-x_2-x_3 \right.$$

$$\therefore\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -x_2-x_3 \\ x_2 \\ x_3 \end{bmatrix}=x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}+x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{v_1} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \vec{v_2} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}.$$

$$\vec{e} = \begin{bmatrix}-1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix}\begin{bmatrix}-1 & 1 & 0\end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix}.$$

$$= \begin{bmatrix}-1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}-2\\0\\2\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

$$\therefore \vec{v_3} = \frac{1}{\|\vec{e}\|}\vec{e} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\-1\\2\end{bmatrix}.$$

An orthonormal basis of $\mathbb{R}^3$: $\left\{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\-1\\2\end{bmatrix}\right\}$

(b). $A = Q\Lambda Q^T$

$$A = \begin{bmatrix}1 & -1 & -1\\1 & 1 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & -\frac{1}{2} & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}.$$

(c). $\lim_{n\to\infty} A^n = Q\Lambda^n Q^T$

$$\lim_{n\to\infty} A^n = \begin{bmatrix}1 & -1 & -1\\1 & 1 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}1^n & 0 & 0\\0 & -\frac{1}{2^n} & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$$

$$\lim_{n\to\infty} A^n = \begin{bmatrix}1 & -1 & -1\\1 & 1 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}1^n & 1^n & 1^n\\\frac{1}{2^n} & -\frac{1}{2^n} & 0\\0 & 0 & 0\end{bmatrix}$$

$$\lim_{n\to\infty} A^n = \begin{bmatrix}1^n - \frac{1}{2^n} & 1^n + \frac{1}{2^n} & 1^n\\1^n + \frac{1}{2^n} & 1^n - \frac{1}{2^n} & 1^n\\1^n & 1^n & 1^n\end{bmatrix}.$$

$$\lim_{n\to\infty} A^n = \begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix} = 3\begin{bmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix}.$$

8 (a). $A = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{7}{2} \end{bmatrix}$.

$A - \lambda I = \begin{bmatrix} \frac{1}{2}-\lambda & -\frac{3}{2} \\ \frac{3}{2} & \frac{7}{2}-\lambda \end{bmatrix}$.

$\det(A-\lambda I) = (\frac{1}{2}-\lambda)(\frac{7}{2}-\lambda) + \frac{9}{4}$

$= \frac{7}{4} - \frac{1}{2}\lambda - \frac{7}{2}\lambda + \lambda^2 + \frac{9}{4}$

$= \lambda^2 - 4\lambda + 4$

$= (\lambda - 2)^2$

$\therefore \lambda = 2$.

① $\lambda = 2$.

$A\vec{x} = 2\vec{x}$

$\vec{x}(A - 2I) = \vec{0}$

$\begin{bmatrix} \frac{1}{2}-2 & -\frac{3}{2} \\ \frac{3}{2} & \frac{7}{2}-2 \end{bmatrix} \to \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix} \to \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. $\Rightarrow \{ x_1 = -x_2$.

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. — one one vector, not enough to get bases for $\mathbb{R}^2$.

So, A is not diagonalizable.

(b). $A^T A = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{7}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{7}{2} \end{bmatrix}$

$= \begin{bmatrix} \frac{5}{2} & \frac{9}{2} \\ \frac{9}{2} & \frac{5}{2} \end{bmatrix}$

$\det(A^T A - \lambda I) = \begin{vmatrix} \frac{5}{2}-\lambda & \frac{9}{2} \\ \frac{9}{2} & \frac{5}{2}-\lambda \end{vmatrix}$

$= (\frac{5}{2}-\lambda)^2 - \frac{81}{4}$ $= \frac{25}{4} - 5\lambda + \lambda^2 - \frac{81}{4}$ $= (\lambda - 7)(\lambda + 2)$

$= \lambda^2 - 5\lambda + 14$ $\therefore \lambda = 7$ or $\lambda = 2$.

$\sigma_1 = \sqrt{7}$

$\sigma_2 = \sqrt{2}$.

① $\lambda = 7$.

$A\vec{x} = 7\vec{x}$

$\vec{x}(A-7I) = \vec{0}$

$$\begin{bmatrix} \frac{5}{2}-7 & \frac{9}{2} \\ \frac{9}{2} & \frac{5}{2}-7 \end{bmatrix} \to \begin{bmatrix} -\frac{9}{2} & \frac{9}{2} \\ \frac{9}{2} & -\frac{9}{2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow x_1 = x_2$$

$$\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{V_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

② $\lambda = -2$

$\vec{x}(A+2I) = \vec{0}$

$$\begin{bmatrix} \frac{5}{2}+2 & \frac{9}{2} \\ \frac{9}{2} & \frac{5}{2}+2 \end{bmatrix} \to \begin{bmatrix} \frac{9}{2} & \frac{9}{2} \\ \frac{9}{2} & \frac{9}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow x_1 = -x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{V_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{u_1} = \frac{1}{6_1} A\vec{V_1} = \frac{1}{\sqrt{7}} \begin{bmatrix} \frac{5}{2} & \frac{9}{2} \\ \frac{9}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{7}} \begin{bmatrix} \frac{7}{\sqrt{2}} \\ \frac{7}{\sqrt{2}} \end{bmatrix}$$

$$\vec{u_2} = \frac{1}{6_2} A\vec{V_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{5}{2} & \frac{9}{2} \\ \frac{9}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{2}{\sqrt{2}} \\ -\frac{2}{\sqrt{2}} \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$A = \begin{bmatrix} \frac{7}{\sqrt{14}} & 1 \\ \frac{7}{\sqrt{14}} & -1 \end{bmatrix} \begin{bmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

(c). $6_1 = \sqrt{7}$. A stretches the most.

$$\vec{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$