Frequency-Domain Analysis of Control Systems

chapter 4

Reviews

- Root loci change after adding zeros or poles
- Basics about frequency domain analysis
  - Polar plot of the frequency response of a system
  - Nyquist plot
  - Nyquist criterion(introduction)

Outlines

- Nyquist criterion(continued)
- Definition of Bode plot
- Bode plots of typical transfer functions
- How to compose the Bode plot of a system
- Minimum-phase system
- Relative stability: gain margin and phase margin
- Gain and phase margin on Bode plot

Nyquist criterion(continued)
Example 4.2

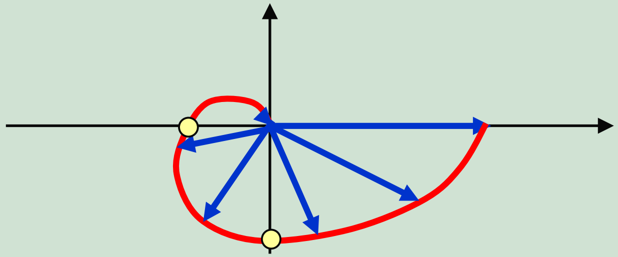
Q: Sketch the Nyquist plot of the system with the following open-loop transfer function and analyze the stability of the closed-loop system.

$$G_0(j\omega) = \frac{100}{(j\omega+1)(0.5j\omega+1)(0.2j\omega+1)}$$

A: substitute $\omega = 0$ in $G_0(j\omega)$ $\quad G_0(j0) = 100\angle 0^\circ$

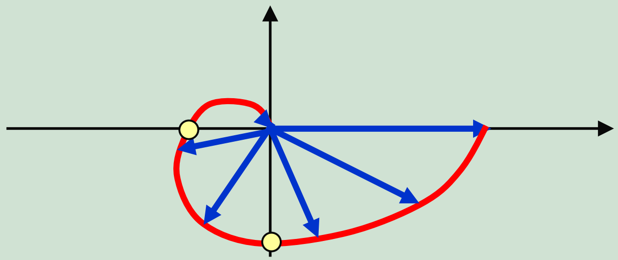
substitute $\omega = \infty$ in $G_0(j\omega)$ $\quad G_0(j\infty) = 0\angle -270^\circ$

when ω varies from 0 to ∞ $\quad \left|G_0(j\omega)\right|$ monotonically decreases from 100 to 0



Nyquist criterion(continued)
Example 4.2

In order to find the intersect(s) of the Nyquist plot with the real axis, multiply $G_0(j\omega)$ with the complex conjugate of the denominator.



$$G_0(j\omega) = \frac{100}{(j\omega+1)(0.5j\omega+1)(0.2j\omega+1)}$$

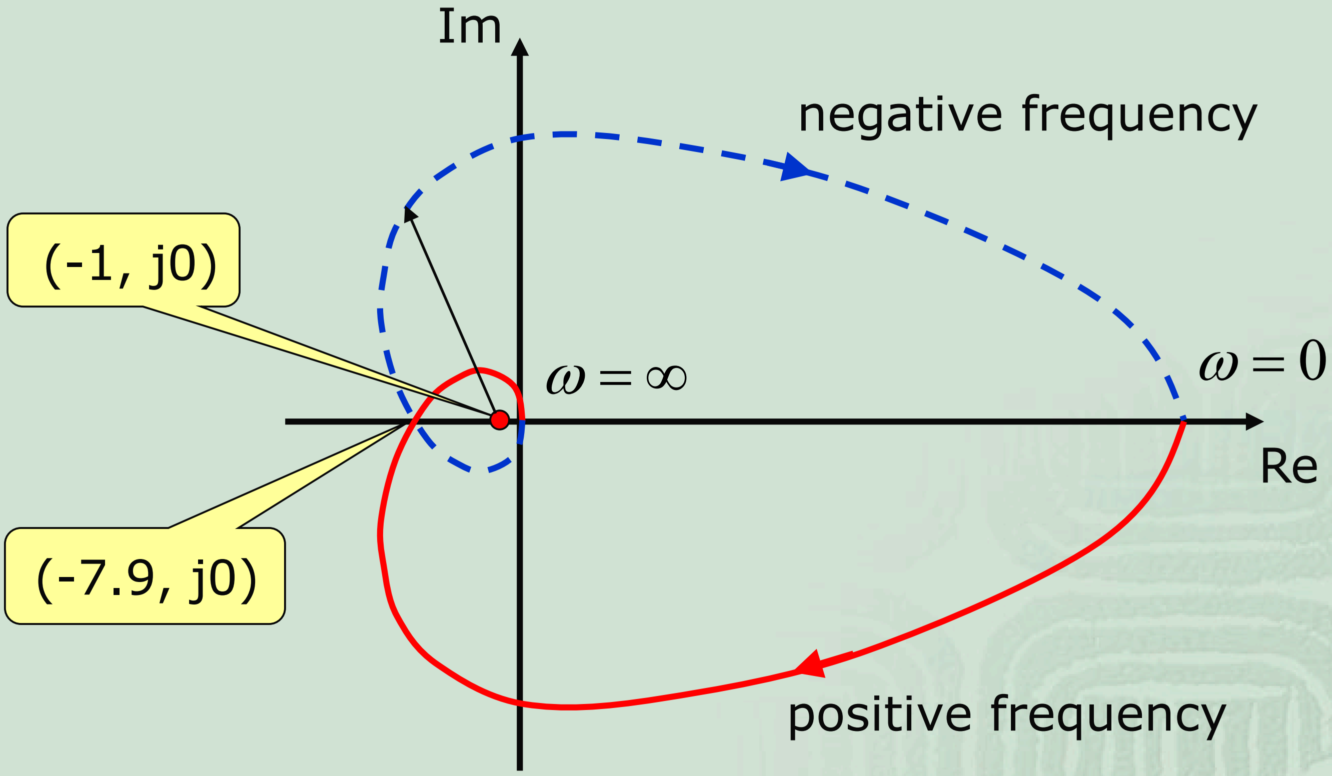
$$= \frac{100\left[(1-0.8\omega^2)+j(0.1\omega^3-1.7\omega)\right]}{(1+\omega^2)(1+0.25\omega^2)(1+0.04\omega^2)}$$

set the imaginary part to be zero $0.1\omega^3 - 1.7\omega = 0$

$$\omega = \sqrt{17}$$

intersect with the real axis: $G_0(j\sqrt{17}) = -7.9\angle -180^\circ$

Nyquist criterion(continued)
Example 4.2

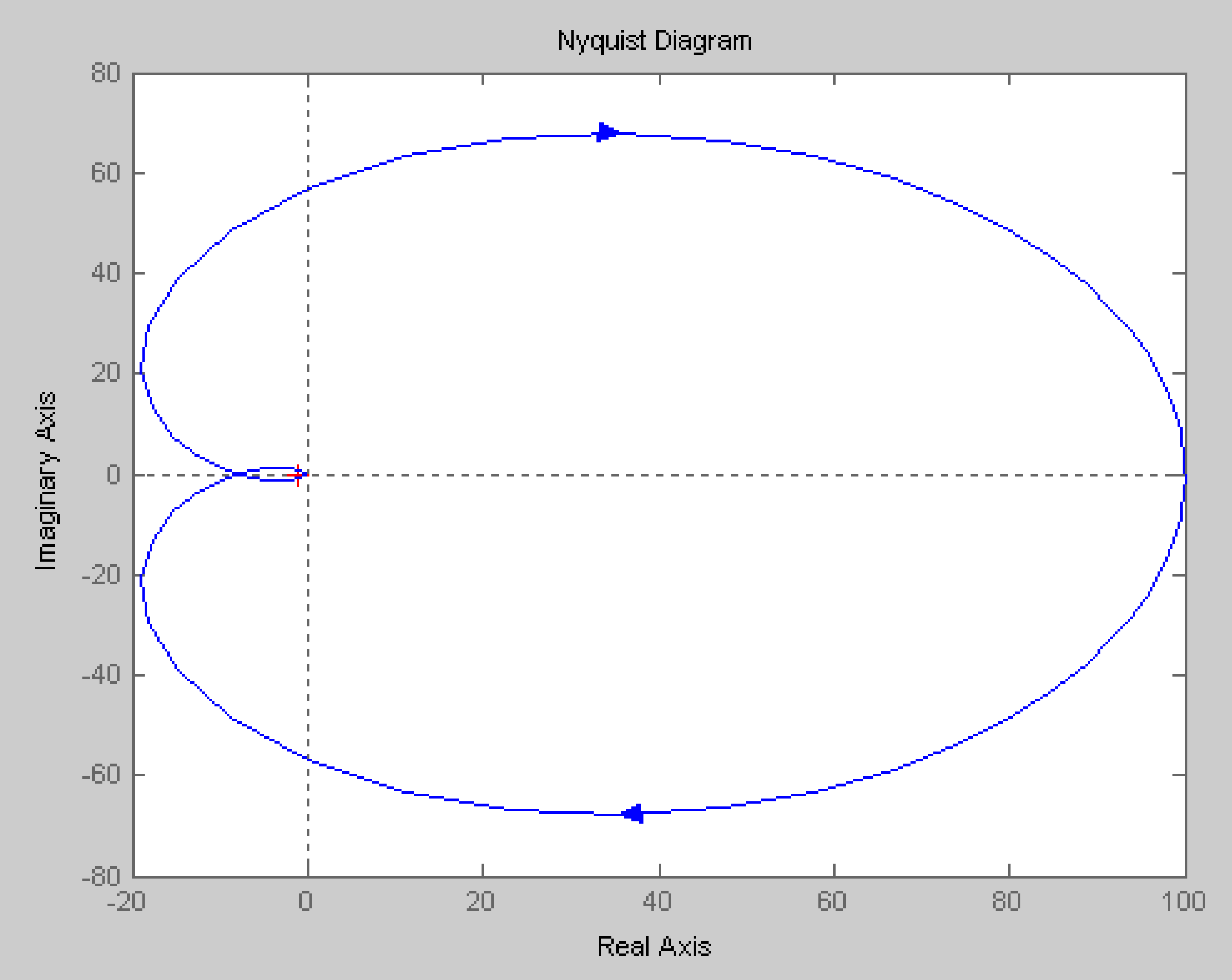


N=2, P=0, Z=2, that means there are two zeros of $1+G_0(s)$ in the right-half s-plane. So, there are two characteristic roots of the closed-loop system that are in the right-half s-plane. The system is unstable.

Question: what happens if decrease the open-loop gain?

Nyquist criterion(continued)
Example 4.2

Nyquist plot sketched by Matlab



Nyquist criterion(continued)
Example 4.4

Q: Sketch the Nyquist plot of the system with the following open-loop transfer function and analyze the stability of the closed-loop system.

$$G_0(s) = \frac{K}{s(s+1)}$$

A: substitute $s = j\omega$ in $G_0(s)$

$$G_0(j\omega) = \frac{K}{j\omega(j\omega+1)}$$

substitute $\omega = 0$ in $G_0(j\omega)$ $\quad G_0(j0) = \infty\angle -90^\circ$

substitute $\omega = \infty$ in $G_0(j\omega)$ $\quad G_0(j\infty) = 0\angle -180^\circ$

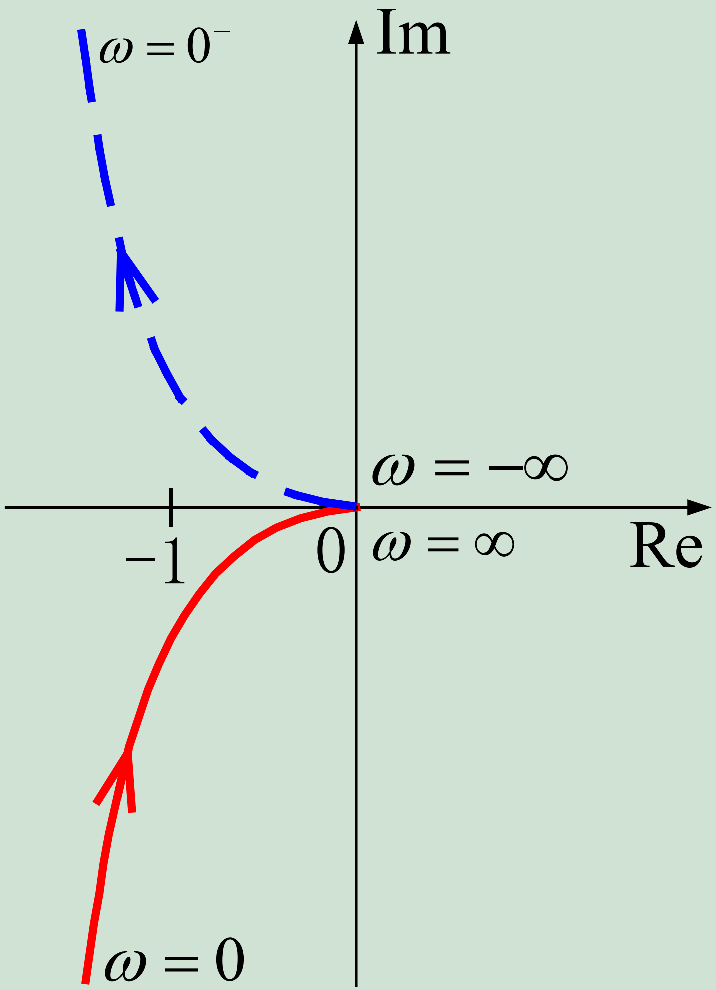
tendency analysis: when ω varies from 0 to ∞

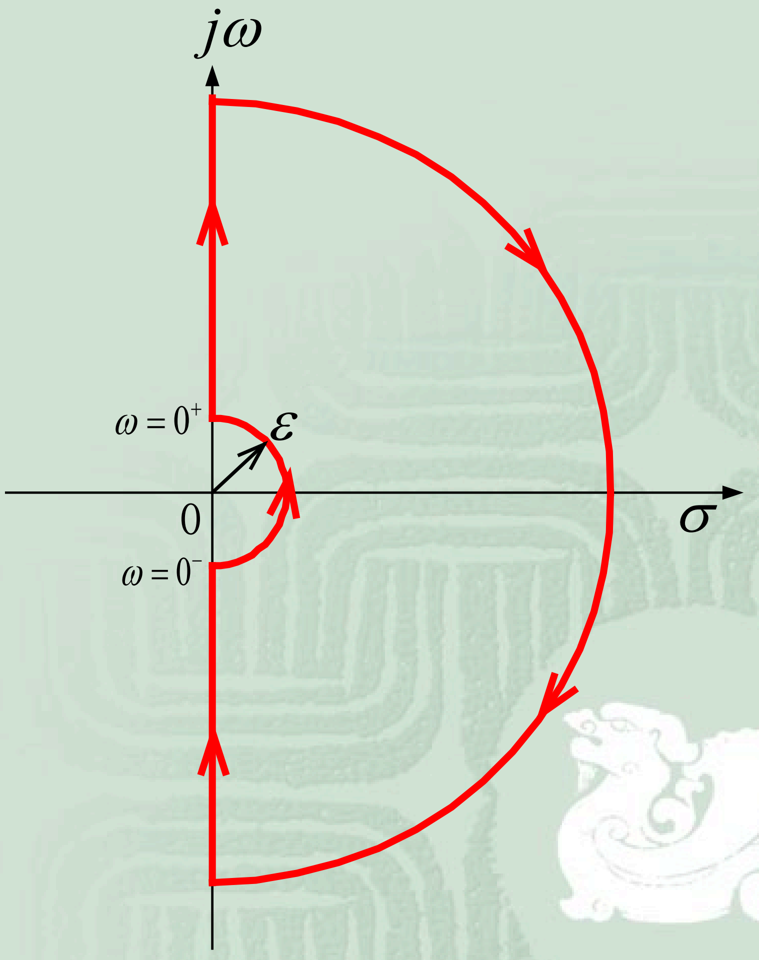
$\left|G_0(j\omega)\right|$ monotonically decrease from infinity to zero

$\angle G_0(j\omega) = -90 - \tan^{-1}\omega$ monotonically decrease from -90 to -180

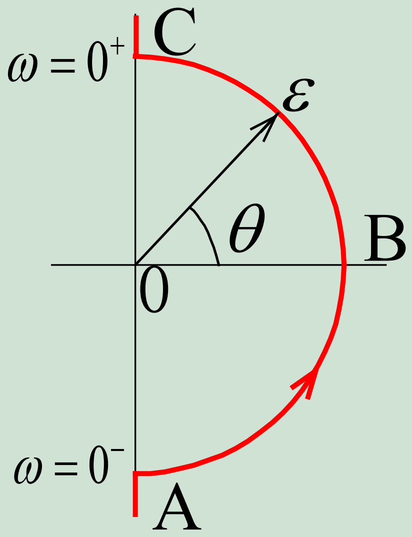
Nyquist criterion(continued)
Example 4.4

Question: how dose the plot get closed?





Nyquist criterion(continued)
Example 4.4



set $s = \varepsilon e^{j\theta}$ $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} G_0(\varepsilon e^{j\theta}) = \lim_{\varepsilon \to 0} \frac{K}{\varepsilon e^{j\theta}} = \lim_{\varepsilon \to 0} \frac{K}{\varepsilon} e^{-j\theta}$$

$$\left| G_0(\varepsilon e^{j\theta}) \right| \to \infty$$

when θ varies from -90 degree to 90 degree

$\angle G_0(\varepsilon e^{j\theta})$ varies from 90 to 0 then to -90

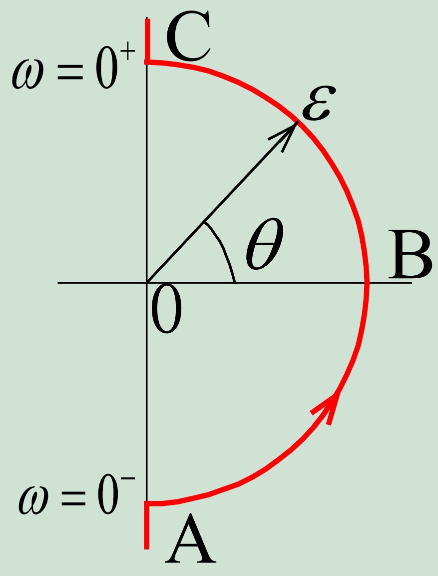
Nyquist criterion(continued)

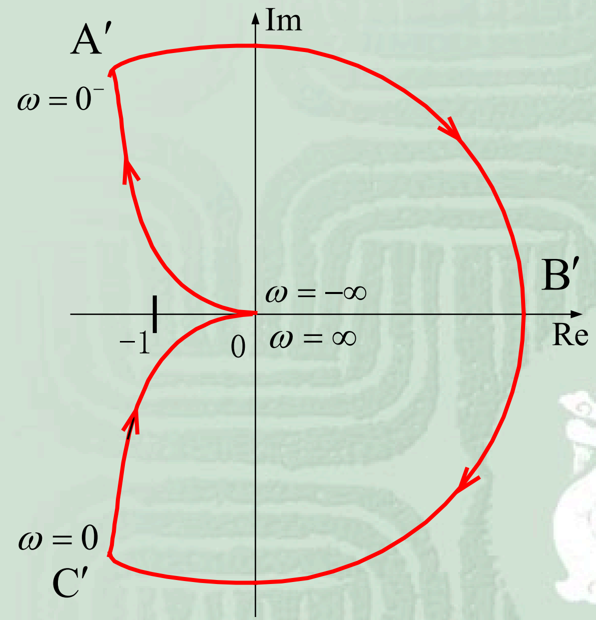
Example 4.4

① A $\omega = 0^- \quad \theta = -90^0$ $\qquad \lim\limits_{\varepsilon \to 0} \dfrac{k}{\varepsilon} e^{-j(-90^0)} = \infty\angle 90^0 \Rightarrow A'$

② B $s = \varepsilon \quad \theta = 0^0$ $\qquad \lim\limits_{\varepsilon \to 0} \dfrac{k}{\varepsilon} e^{-j0^0} = \infty\angle 0^0 \Rightarrow B'$

③ C $\omega = 0^+ \quad \theta = 90^0$ $\qquad \lim\limits_{\varepsilon \to 0} \dfrac{k}{\varepsilon} e^{-j(90^0)} = \infty\angle -90^0 \Rightarrow C'$





N=0, system is stable

Nyquist criterion(continued)
Example 4.5

Q: Sketch the Nyquist plot of the system with the following open-loop transfer function and analyze the stability of the closed-loop system.

$$G_0(s) = \frac{K}{s^2(Ts+1)}$$

A: substitute $s = j\omega$ in $G_0(s)$

$$G_0(j\omega) = \frac{K}{-\omega^2(Tj\omega+1)}$$

substitute $\omega = 0$ in $G_0(j\omega)$ $\quad G_0(j0) = \infty\angle -180^\circ$

substitute $\omega = \infty$ in $G_0(j\omega)$ $\quad G_0(j\infty) = 0\angle -270^\circ$

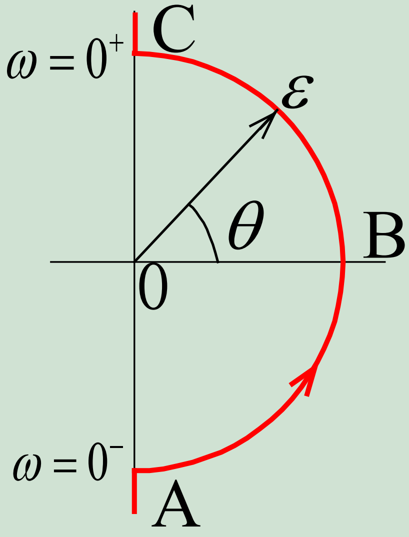
tendency analysis: when ω varies from 0 to ∞

$\left|G_0(j\omega)\right|$ monotonically decrease from infinity to zero

$\angle G_0(j\omega)$ monotonically decrease from -180 to -270

Nyquist criterion(continued)
Example 4.5

Two poles on the origin. set $s = \varepsilon e^{j\theta}$ $\varepsilon \to 0$



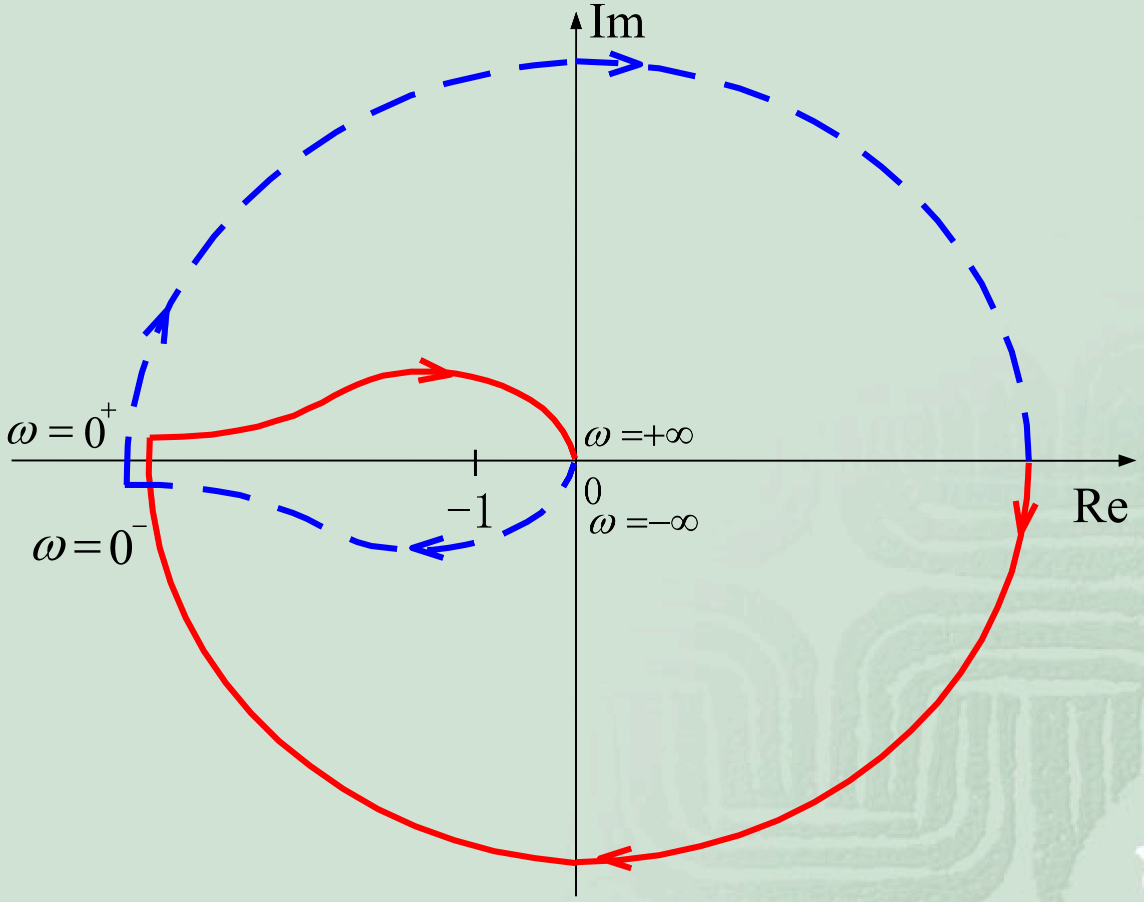
$$\lim_{\varepsilon \to 0} G_0(\varepsilon e^{j\theta}) = \lim_{\varepsilon \to 0} \frac{k}{(\varepsilon e^{j\theta})^2} = \lim_{\varepsilon \to 0} \frac{k}{\varepsilon^2} e^{-j2\theta}$$

$$\left| G_0(\varepsilon e^{j\theta}) \right| \to \infty$$

when θ varies from -90 degree to 90 degree

$\angle G_0(\varepsilon e^{j\theta})$ varies from 180 to 0 then to -180

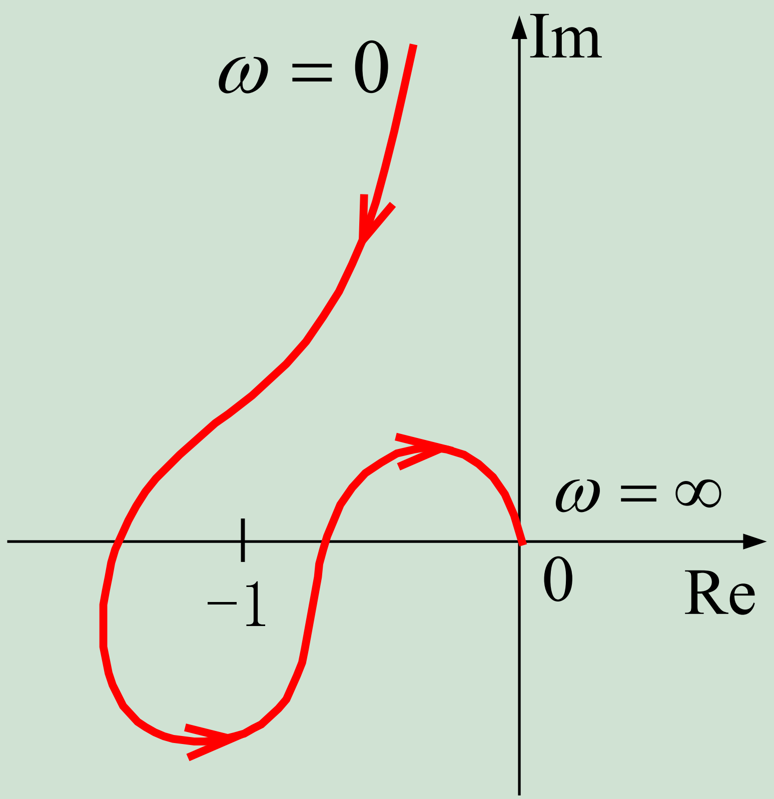
Nyquist criterion(continued)
Example 4.5



N=2, P=0, Z=2, system is unstable.

Nyquist criterion(continued) Example 4.7

Q: analyze the stability of the given system. The Nyquist plot of $G_0(s)$ is shown as follows. The open-loop TF does not have poles in the right-half s-plane, and the system is a type-3 system.



A: analyze how the plot get closed at $\omega = 0$

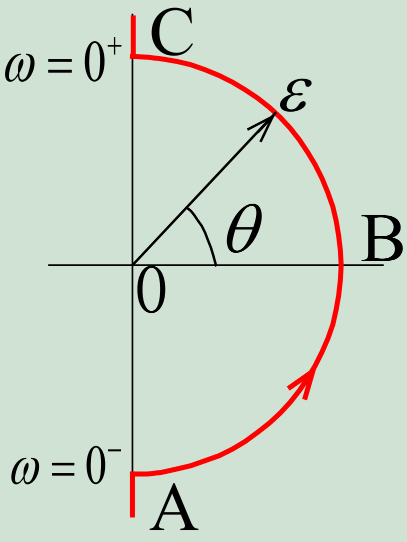
For a type-3 system, when $s \to 0$

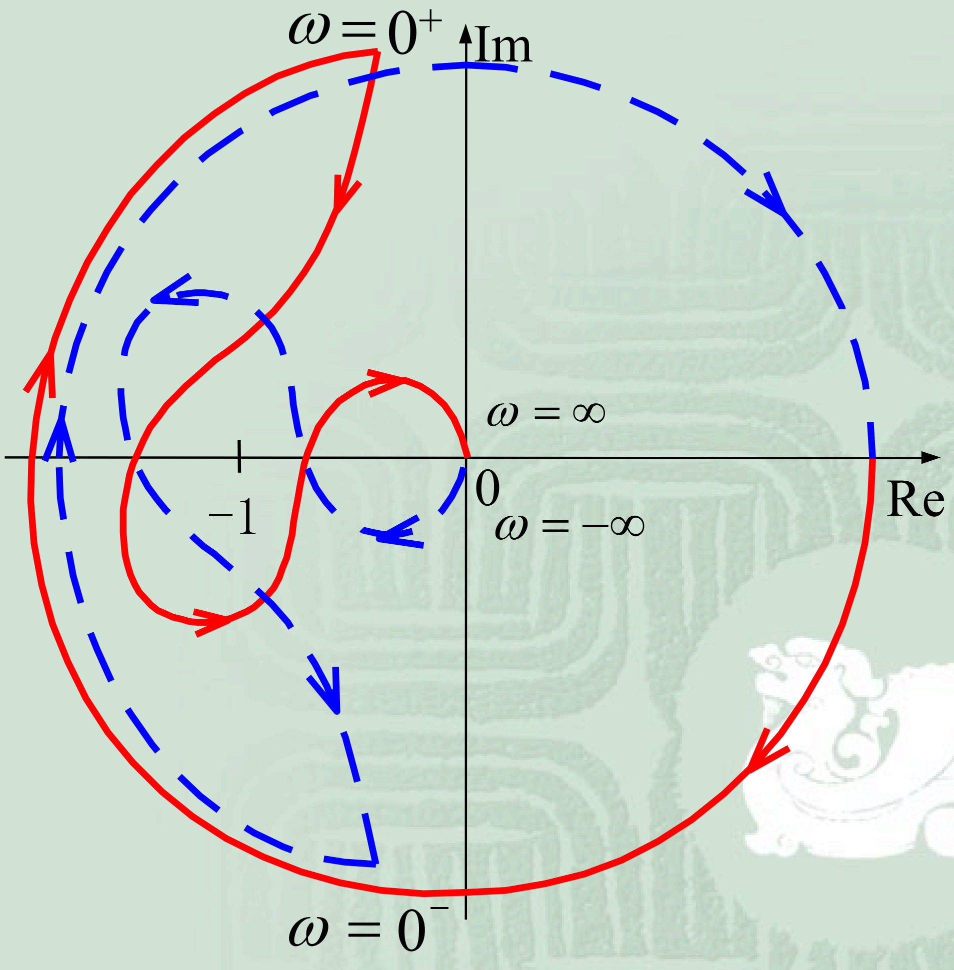
set $s = \varepsilon e^{j\theta}$ $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} G_0(\varepsilon e^{j\theta}) = \lim_{\varepsilon \to 0} \frac{K}{(\varepsilon e^{j\theta})^3} = \lim_{\varepsilon \to 0} \frac{K}{\varepsilon^3} e^{-j3\theta}$$

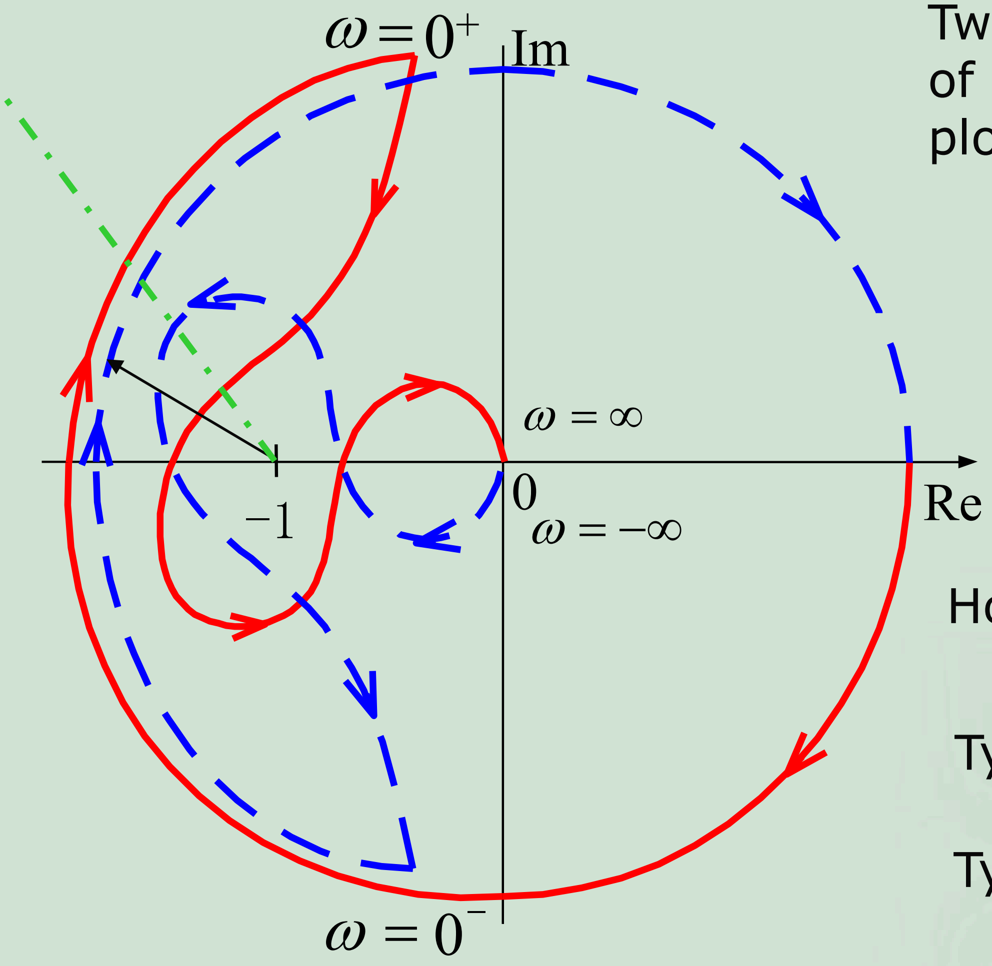
Nyquist criterion(continued)
Example 4.7

θ	-90	-60	-30	0	30	60	90
$\angle G_0(\varepsilon e^{j\theta})$	270	180	90	0	-90	-180	-270





Nyquist criterion(continued)
Example 4.7



Two ways to count the encirclement of (-1,j0) point made by the Nyqusit plot.

N=0, P=0, Z=0

System is stable.

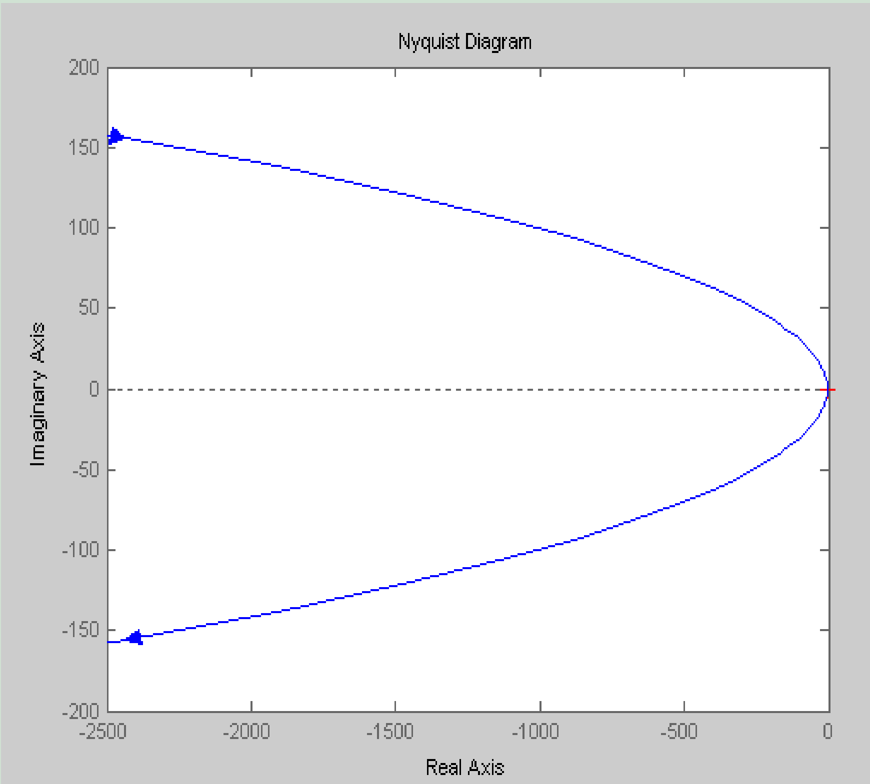
How the locus get closed at infinity:

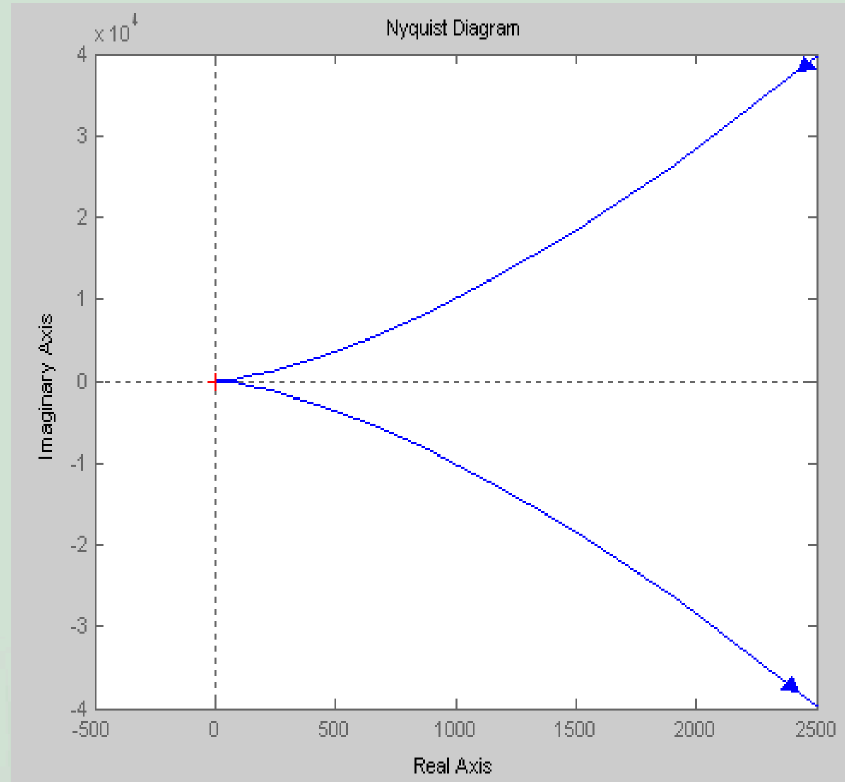
Type 1 system: pass through 180^0

Type 2 system: pass through 360^0

Type 3 system: pass through 540^0

Drawing Nyquist Plot Using Matlab



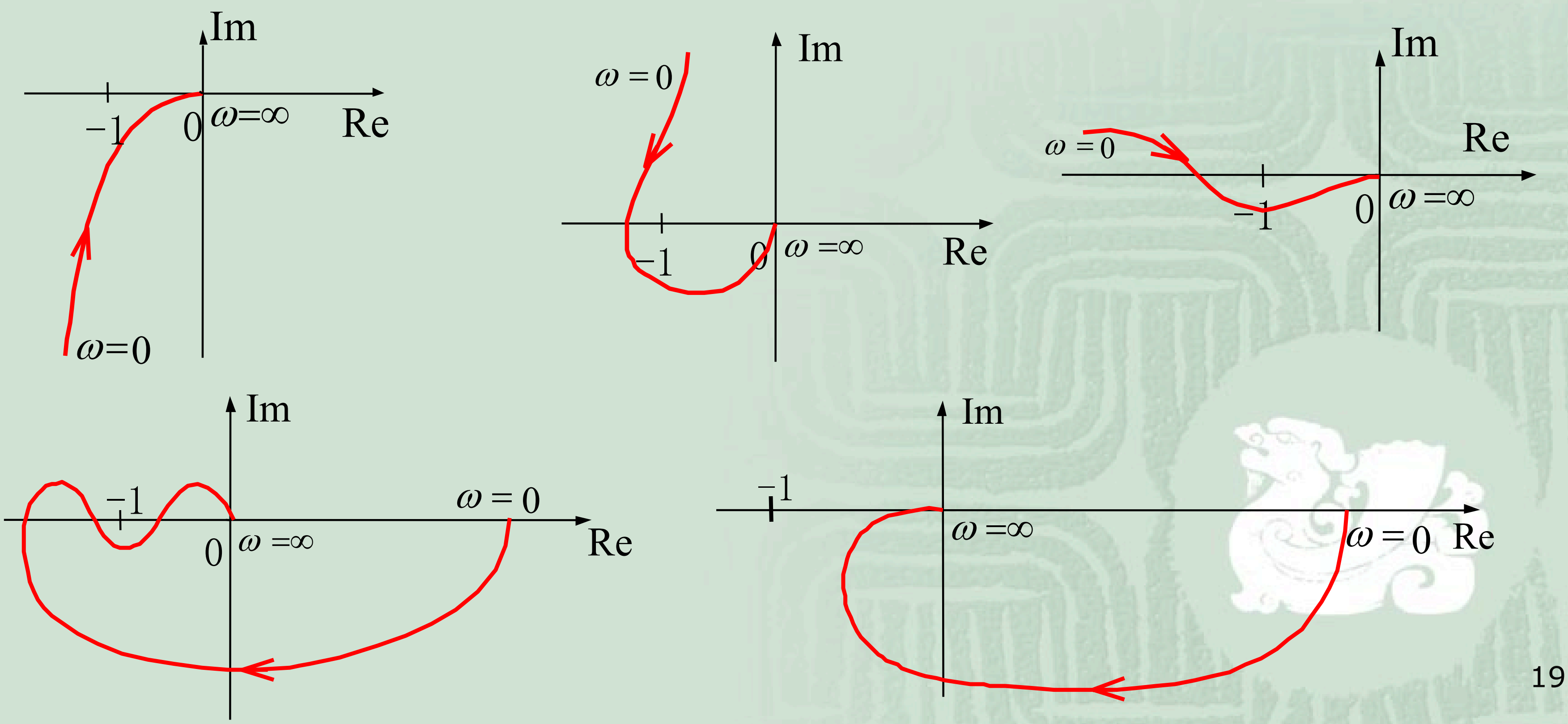



```
s=tf('s');
g=10/(s^2*(s+1));
nyquist(g)
```

```
s=tf('s');
g=10/(s^3*(s+1));
nyquist(g)
```

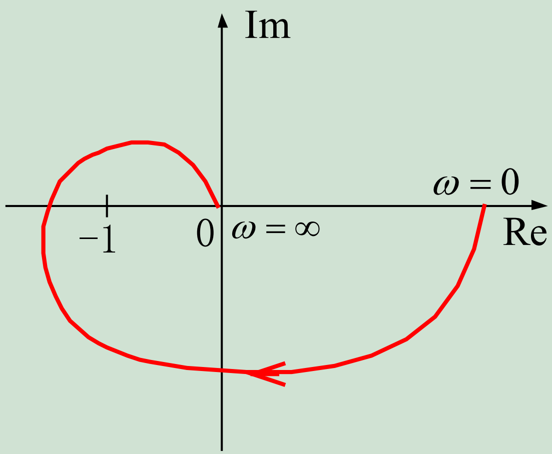
Mapping region and system stability

- Stable region excludes the (-1,0) point or the (-1,0) point stays on the left side of Nyquist plot.

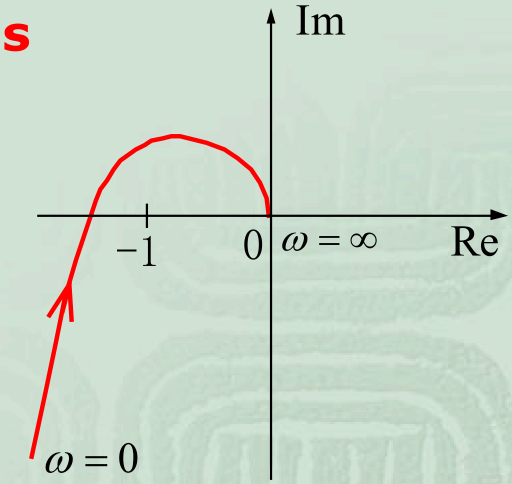


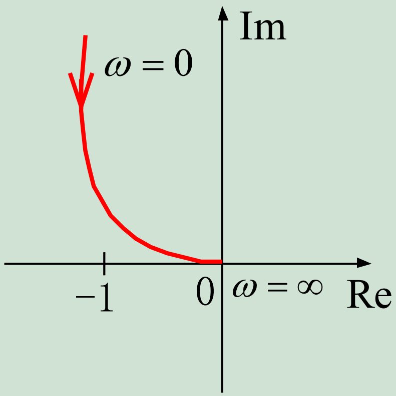
Mapping region and system stability

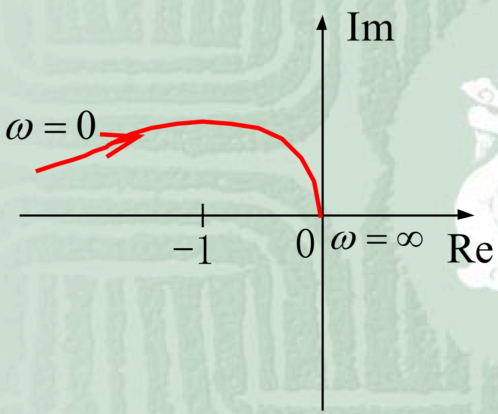
- Unstable region encloses the (-1,0) point or the (-1,0) point stays on the right side of Nyquist plot.



Not rigorous







Bode Plot

- Bode plot is another graphical expression of the frequency response of a system

- The Bode plot of the function $G(j\omega)$ is composed of two plots, one with the magnitude of the $G(j\omega)$ in decibels (dB) versus $\lg\omega$ or ω, and the other with the phase of $G(j\omega)$ in degrees as a function of $\lg\omega$ or ω.

What advantages does the bode plot have?

Definition of the Magnitude and Angle Plot

Consider a general function

$$G(s)=\frac{K(1+T_1 s)(1+T_2 s)}{s(1+T_a s)(1+2\zeta s/\omega_n+s^2/\omega_n^2)}e^{-T_d s}$$

The magnitude of G in dB is obtained by multiplying the logarithm to the base 10 of $|G(j\omega)|$ by 20

$$\begin{aligned}|G(j\omega)|_{dB} &= 20\lg|G(j\omega)| = 20\lg|K| + 20\lg|1+j\omega T_1| + 20\lg|1+j\omega T_2| \\ &\quad - 20\lg|j\omega| - 20\lg|1+j\omega T_a| - 20\lg|1+j2\zeta\omega/\omega_n - \omega^2/\omega_n^2|\end{aligned}$$

Definition of the Magnitude and Phase Plot

The phase of G is

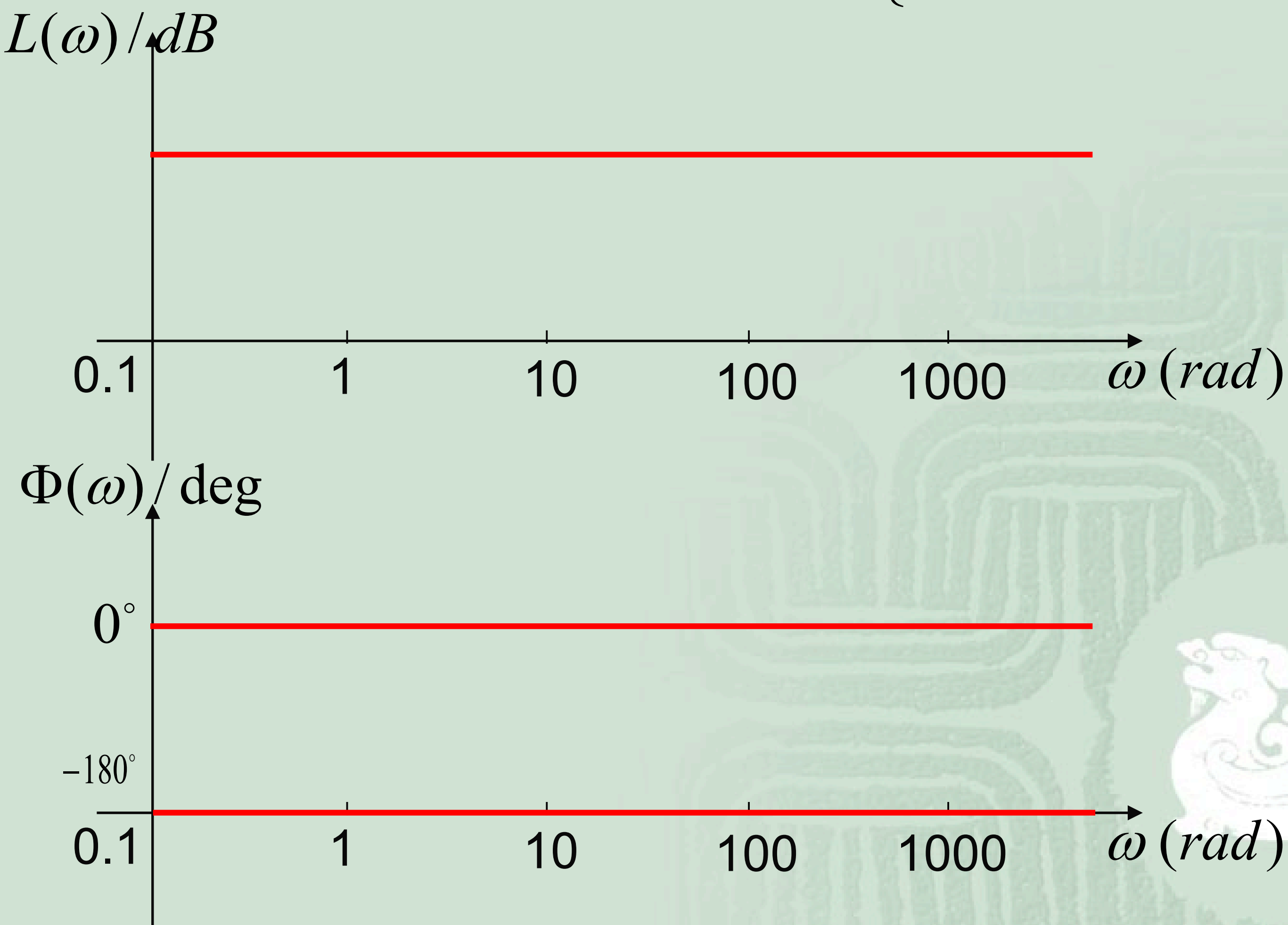
$$\angle G(j\omega) = \angle K + \angle(1 + j\omega T_1) + \angle(1 + j\omega T_2)$$
$$- \angle j\omega - \angle(1 + j\omega T_a) - \angle(1 + j2\zeta\omega / \omega_n - \omega^2 / \omega_n^2) - \omega T_d$$

In general, $G(j\omega)$ can contain just five simple kinds of factors:

1. Constant factor: K
2. Poles or zeros at the origin of order p: $(j\omega)^{\pm p}$
3. Poles or zeros at $s = -1/T$ of order q: $(1 + j\omega T)^{\pm q}$
4. Complex poles and zeros of order r: $(1 + j2\zeta\omega / \omega_n - \omega^2 / \omega_n^2)^{\pm r}$
5. Pure time delay: $e^{-j\omega T_d}$

Bode Plot of K

$$K_{dB} = 20\lg|K| = const. \qquad \angle K = \begin{cases} 0^\circ & K > 0 \\ -180^\circ & K < 0 \end{cases}$$



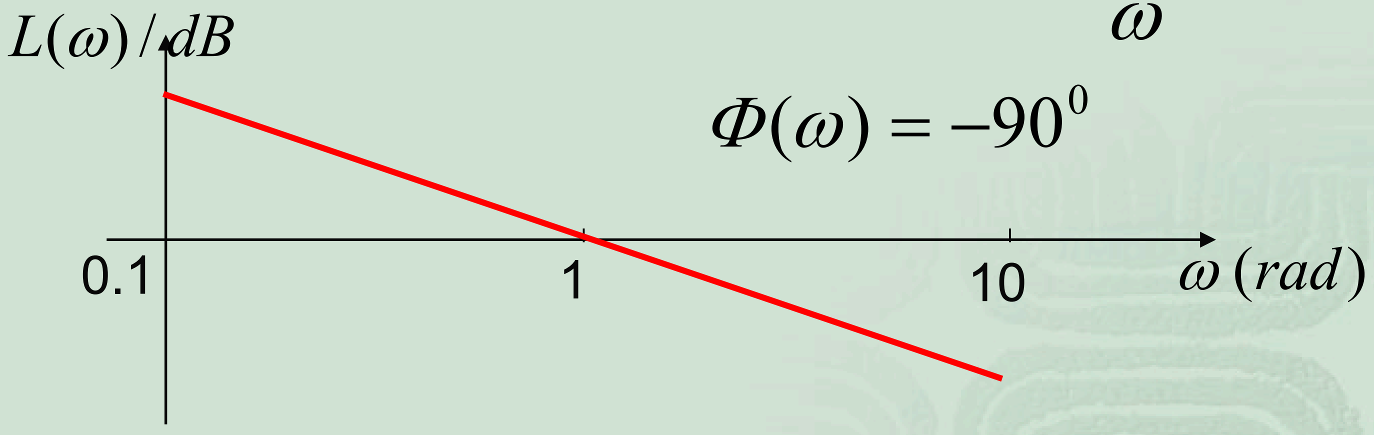
Bode Plot of Integration Block

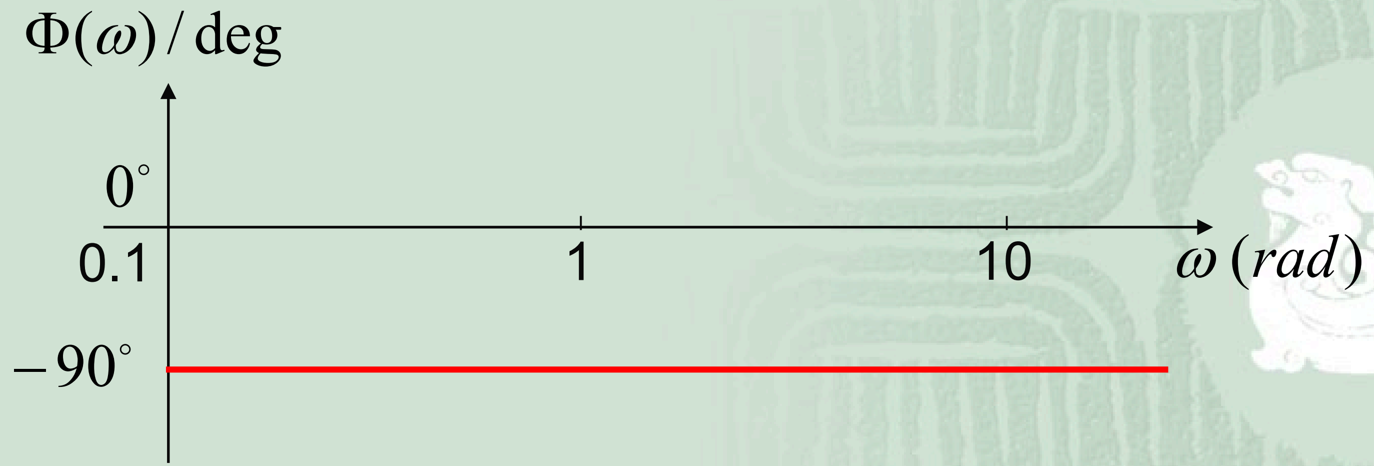
$$G(s) = \frac{1}{s}$$

$$G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega} e^{-j90^0}$$

$$L(\omega) = 20\lg\frac{1}{\omega} = -20\lg\omega$$

$$\Phi(\omega) = -90^0$$





Bode Plot of Inertia Block

$$G(s)=\frac{1}{Ts+1} \qquad G(j\omega)=\frac{1}{j\omega T+1}$$

$$L(\omega)=20\lg\frac{1}{\sqrt{(\omega T)^2+1}}=-20\lg\left[(\omega T)^2+1\right]^{\frac{1}{2}}$$

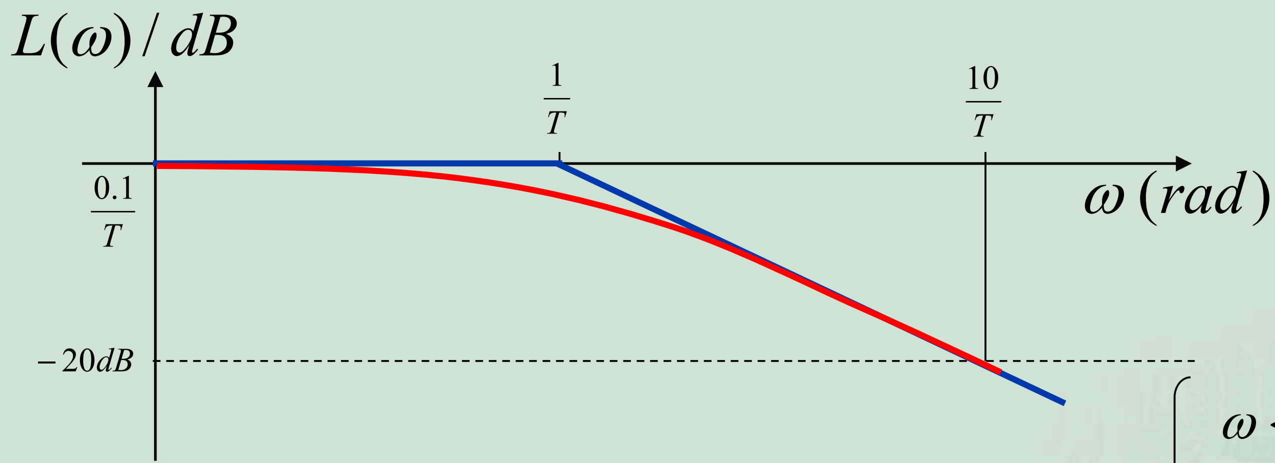
$$\varPhi(\omega)=-tg^{-1}(\omega T)$$

$\omega << 1/T \quad (\omega T << 1)$ $\qquad L(\omega)\approx 0dB$ $\qquad \Phi(\omega)=0^\circ$

$\omega >> 1/T \quad (\omega T >> 1)$ $\qquad L(\omega)\approx -20\lg\omega T$ $\qquad \Phi(\omega)=-90^\circ$

$\omega = 1/T \quad (\omega T = 1)$ $\qquad L(\omega)=-3dB$ $\qquad \Phi(\omega)=-45^\circ$

Bode Plot of Inertia Block

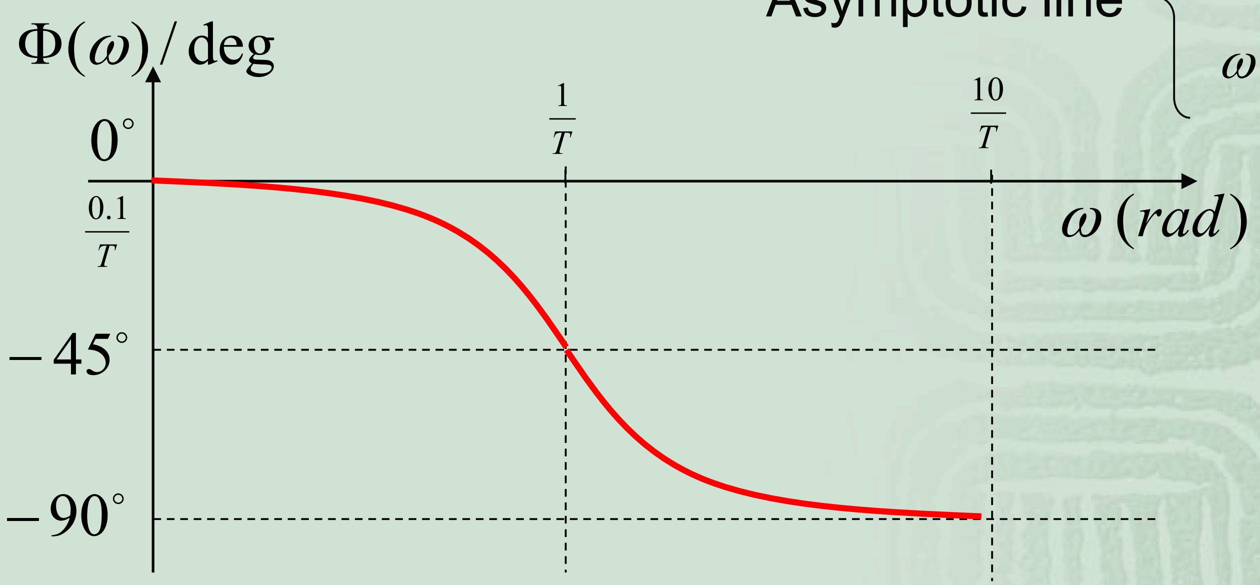


$$\omega T = 0.1$$

$$L(\omega) = -0.04dB$$

$$\Phi(\omega) = 5.7^0$$

Asymptotic line
$$\begin{cases} \omega < \dfrac{1}{T} & L(\omega) = 0dB \\ \omega > \dfrac{1}{T} & L(\omega) = -20\lg \omega T \end{cases}$$



$$\omega T = 10$$

$$L(\omega) = -20.04dB$$

$$\Phi(\omega) = -84.3^0$$

Bode Plot of 2nd-Order Block

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{1}{T^2s^2+2\zeta Ts+1}$$

$$G(j\omega)=\frac{1}{(1-\omega^2T^2)+j2\zeta\omega T}$$

$$L(\omega)=-20\lg\left[(1-\omega^2T^2)^2+(2\zeta\omega T)^2\right]^{\frac{1}{2}}$$

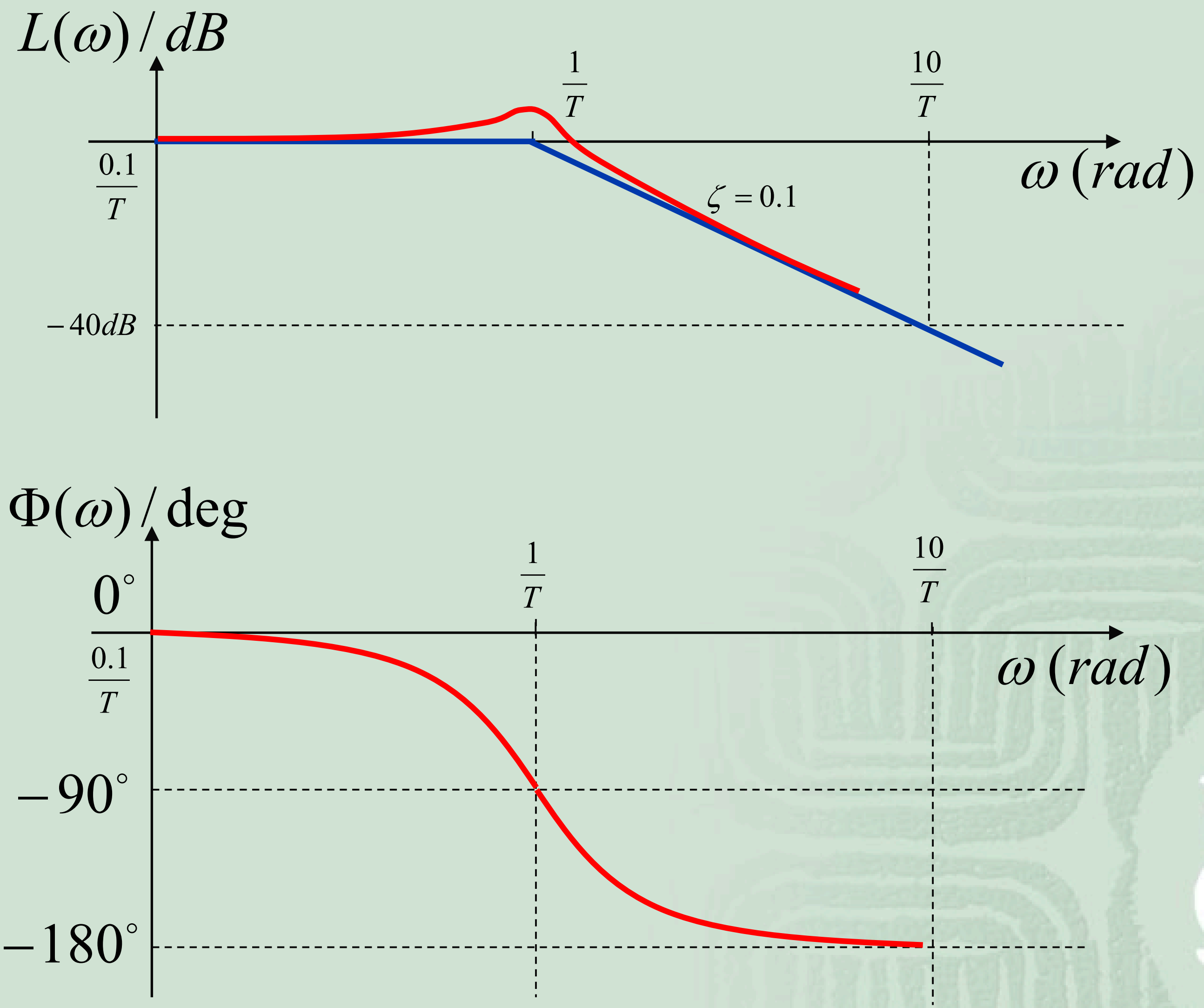
$$\Phi(\omega)=-tg^{-1}\frac{2\zeta\omega T}{1-\omega^2T^2}\approx-2tg^{-1}(\omega T)$$

$\omega << 1/T \quad (\omega T << 1)$ $\qquad L(\omega)\approx 0dB$ $\qquad \Phi(\omega)=0^\circ$

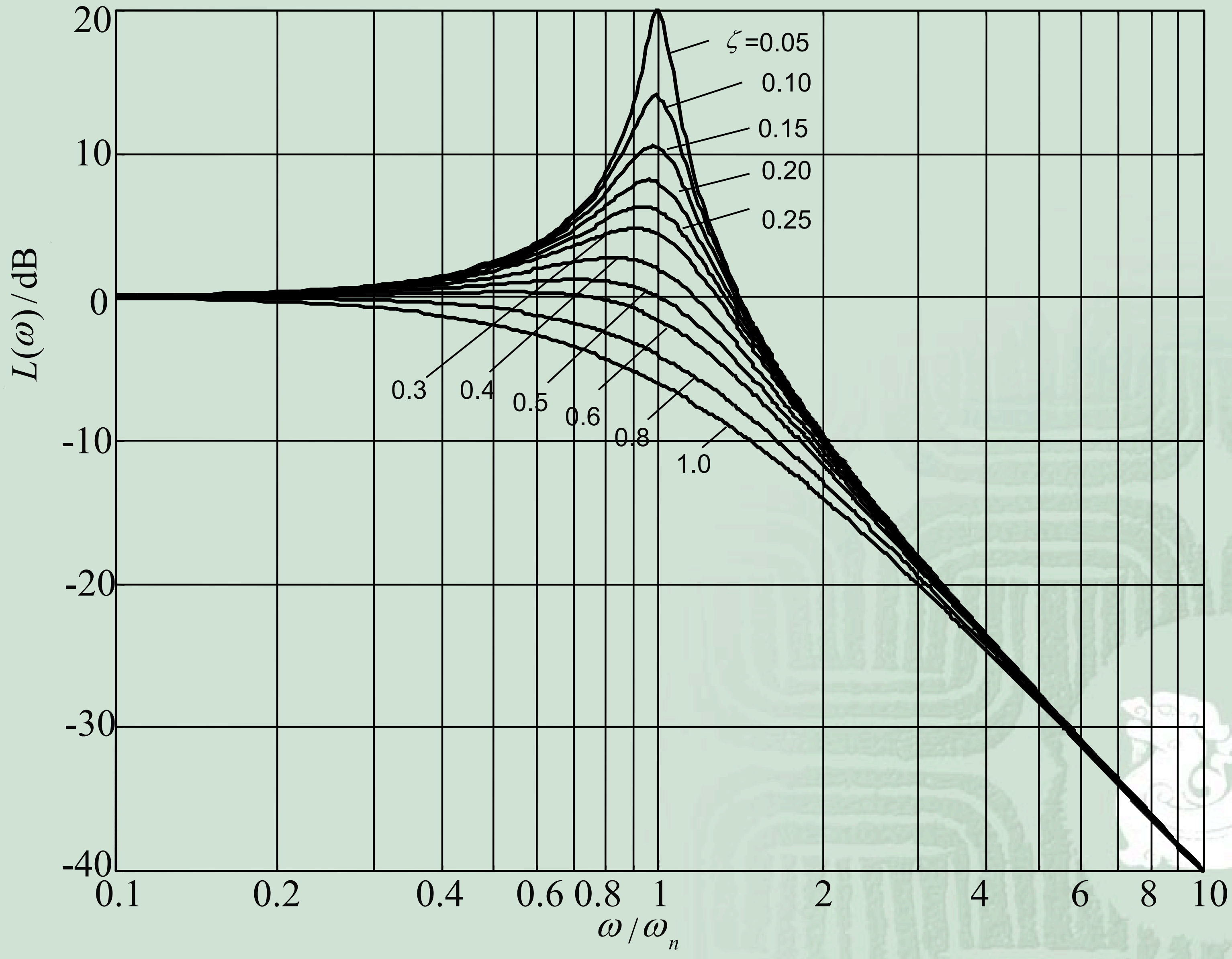
$\omega >> 1/T \quad (\omega T >> 1)$ $\qquad L(\omega)\approx -40\lg\omega T$ $\qquad \Phi(\omega)=-180^\circ$

$\omega = 1/T \quad (\omega T = 1)$ $\qquad L(\omega) = -20\lg 2\zeta$ $\qquad \Phi(\omega)=-90^\circ$

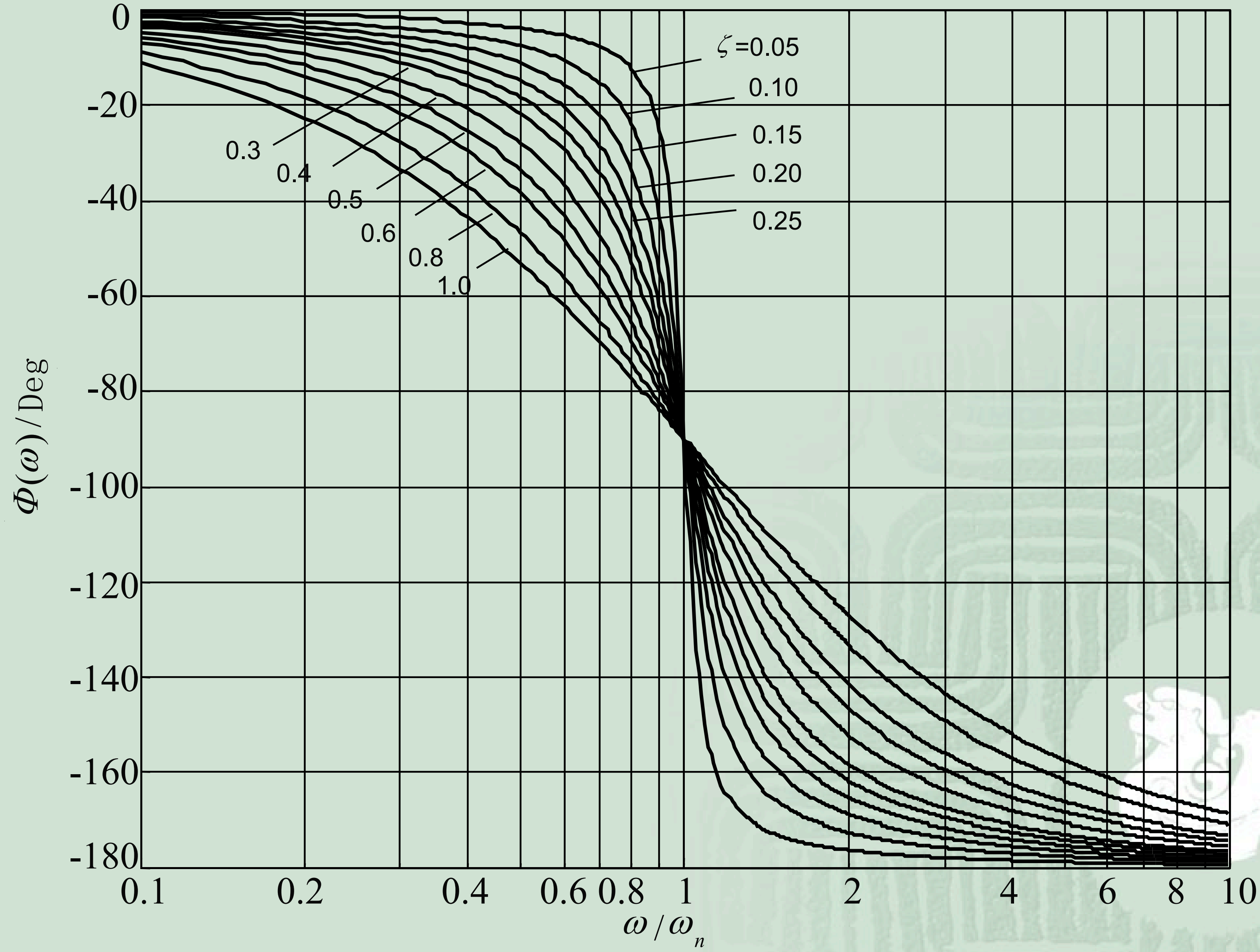
Bode Plot of 2nd-Order Block



Bode Plot of 2nd-Order Block



Bode Plot of 2nd-Order Block



Bode Plot of 2nd-Order Block

Peak value of the magnitude plot:

$$\left|G(j\omega)\right| = \frac{\omega_n^2}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

Set $\frac{d\left|G(j\omega)\right|}{\omega}=0$

$$\omega_m = \omega_n\sqrt{1-2\zeta^2}$$

$$\left|G(j\omega_m)\right| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

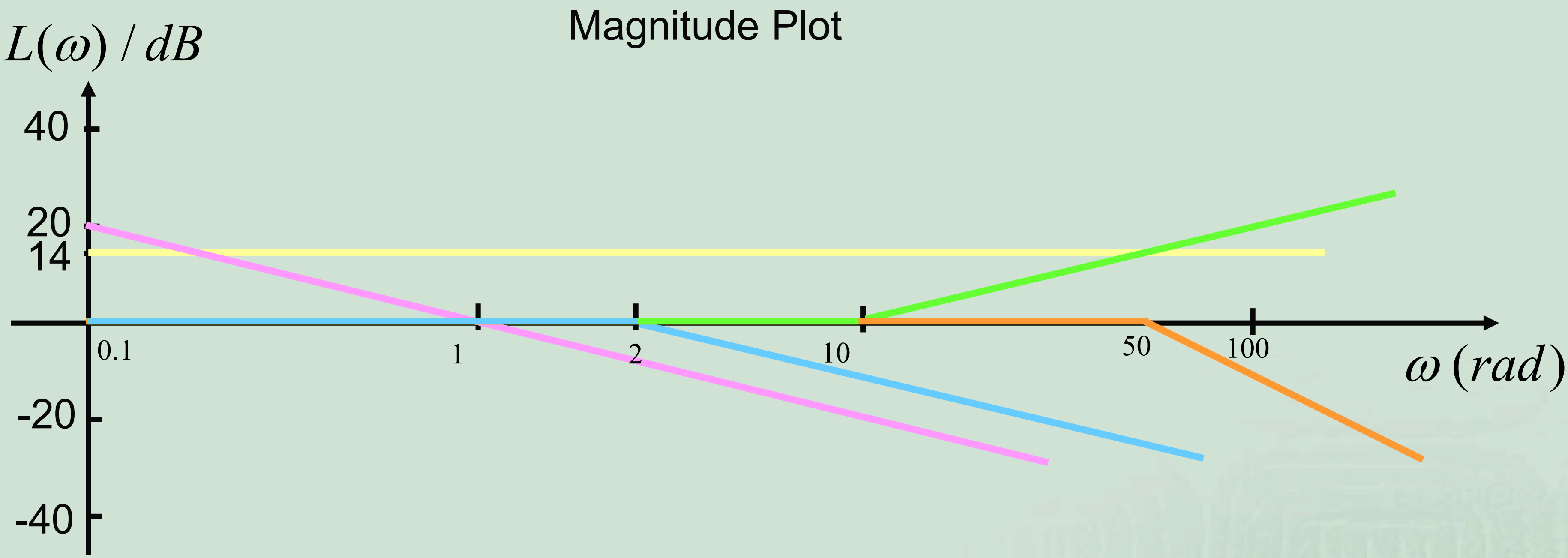
Composing the Bode Plot of a System

Example 4.8: Compose the bode plot of the given system

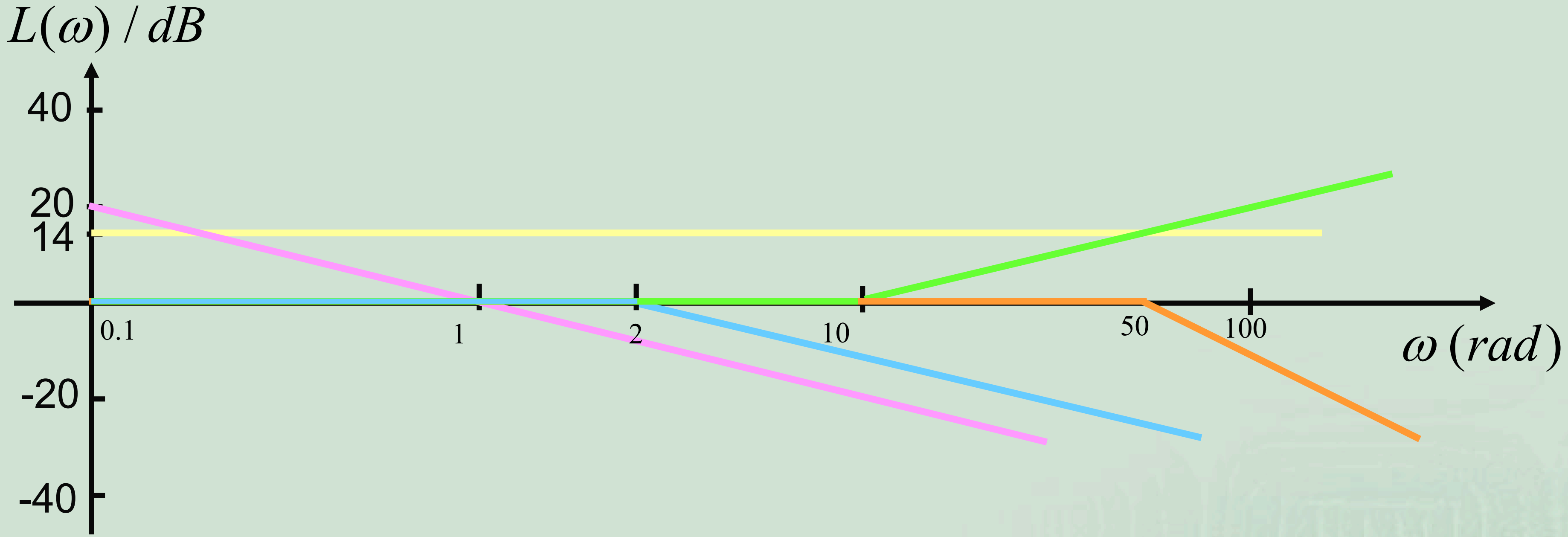
$$G(s)=\frac{5(0.1s+1)}{s(0.5s+1)(\frac{1}{2500}s^2+\frac{6}{50}s+1)}$$

A: analyze the factors of the given system

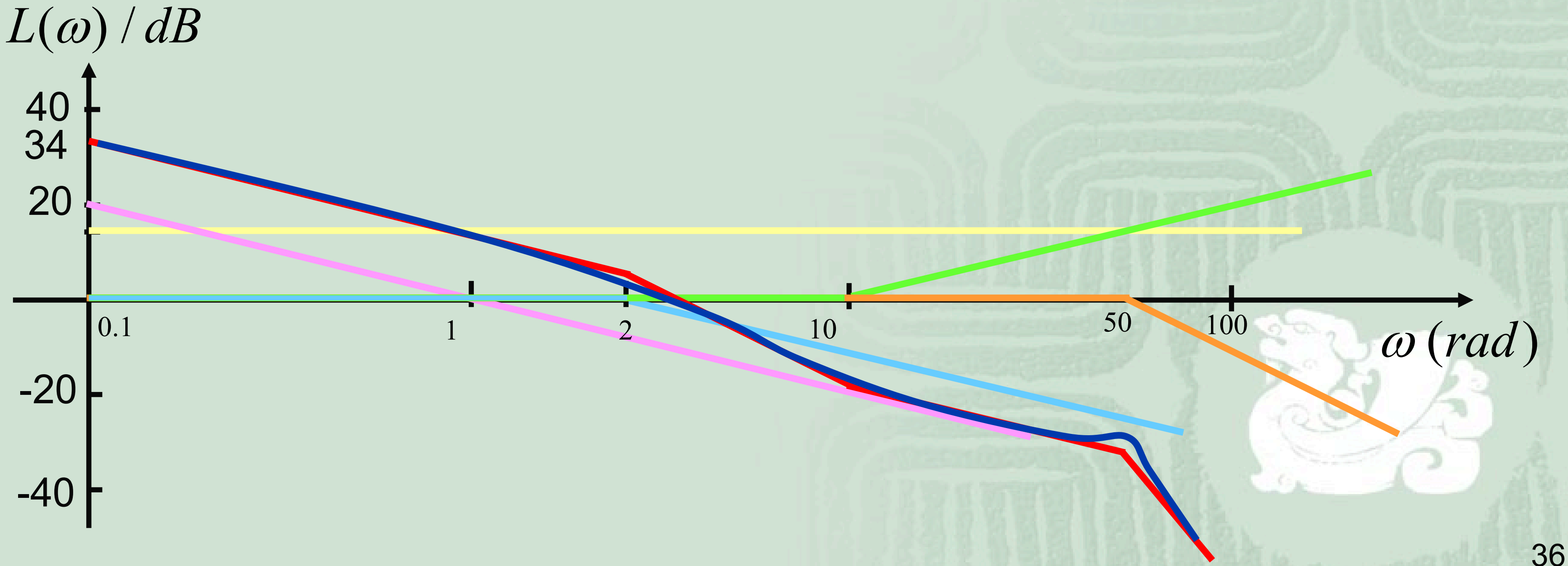
(1) proportional block	$K=5$	$20\lg K=20\lg 5=14dB$		$\Phi_1=0^\circ$
(2) integration block	$1/s$	$-20\lg\omega$	$\omega_c=1$	$\Phi_1=-90^\circ$
(3) inertia block	$1/(0.5s+1)$	$-20dB/Dec$	$\omega_1=1/T_1=2$	
(4) proportional and differential block	$(0.1s+1)$	$+20dB/Dec$	$\omega_2=1/T_2=10$	
(5) oscillation block	$({s^2}/{2500}+{6s}/{50}+1)$	$-40dB/Dec$ $\omega_3=1/T_3=50$		



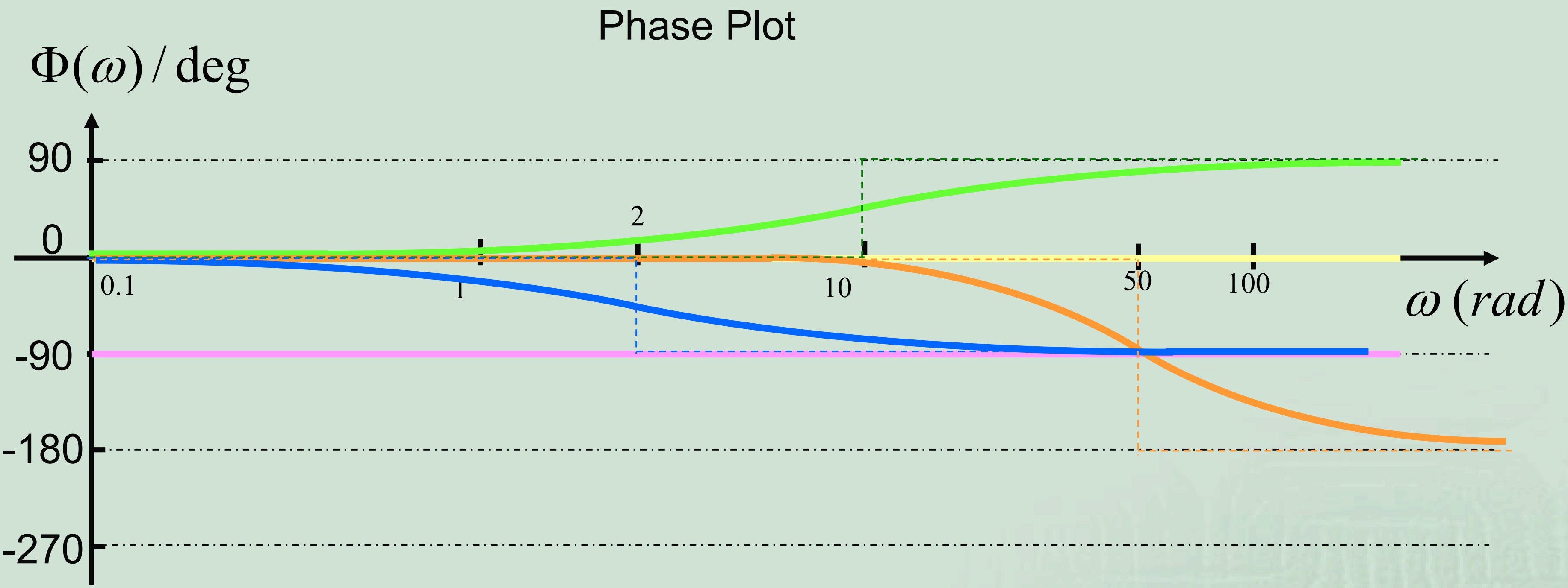
(1) proportional block	$K=5$	$20\lg K = 20\lg 5 = 14dB$		$\Phi_1 = 0^\circ$
(2) integration block	$1/s$	$-20\lg\omega$	$\omega_c = 1$	$\Phi_1 = -90^\circ$
(3) inertia block	$1/(0.5s+1)$	$-20dB/Dec$	$\omega_1 = 1/T_1 = 2$	
(4) proportional and differential block	$(0.1s+1)$	$+20dB/Dec$	$\omega_2 = 1/T_2 = 10$	
(5) oscillation block	$({s^2}/{2500} + {6s}/{50} + 1)$	$-40dB/Dec$	$\omega_3 = 1/T_3 = 50$	



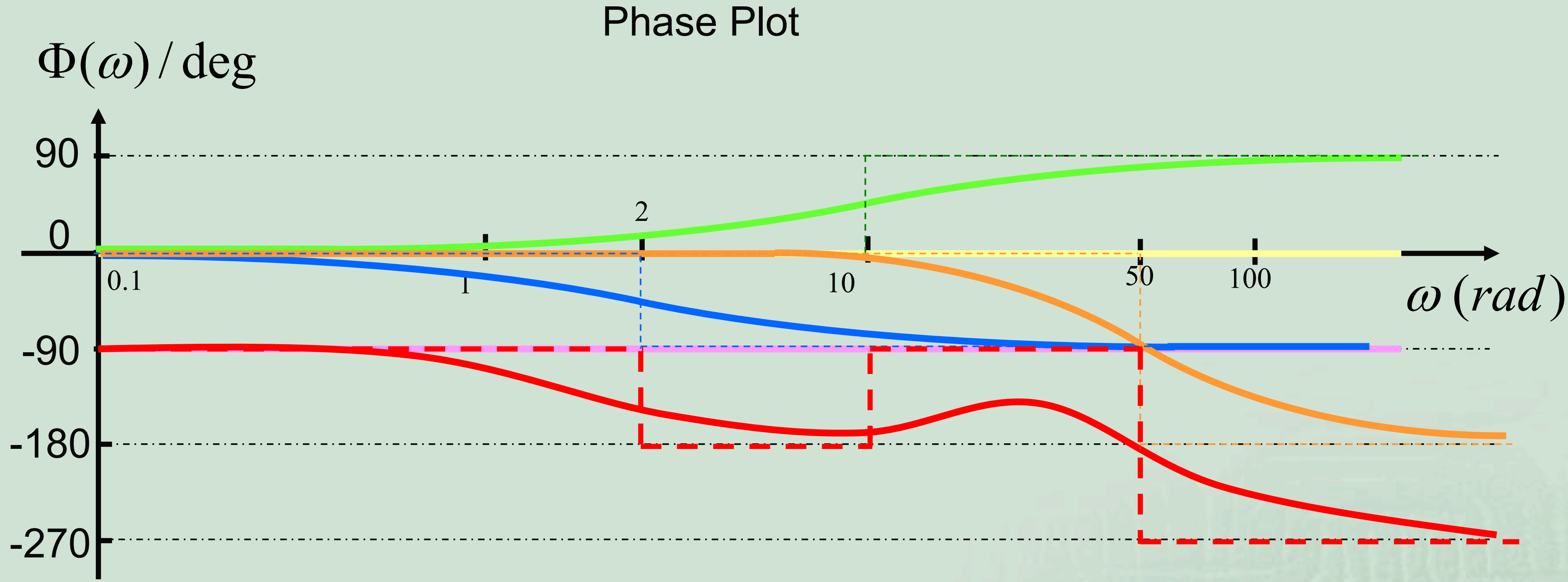
L(ω) / dB
40
20
14
-20
-40
0.1
1
2
10
50
100
ω (rad)



L(ω) / dB
40
34
20
-20
-40
0.1
1
2
10
50
100
ω (rad)



(1) proportional block	$K=5$	$20\lg K = 20\lg 5 = 14dB$		$\Phi_1 = 0^\circ$
(2) integration block	$1/s$	$-20\lg\omega$	$\omega_c = 1$	$\Phi_1 = -90^\circ$
(3) inertia block	$1/(0.5s+1)$	$-20dB/Dec$	$\omega_1 = 1/T_1 = 2$	
(4) proportional and differential block	$(0.1s+1)$	$+20dB/Dec$	$\omega_2 = 1/T_2 = 10$	
(5) oscillation block	$(s^2/2500 + 6s/50 + 1)$	$-40dB/Dec$	$\omega_3 = 1/T_3 = 50$	



(1) proportional block	$K=5$	$20\lg K = 20\lg 5 = 14dB$		$\Phi_1 = 0^\circ$
(2) integration block	$1/s$	$-20\lg\omega$	$\omega_c = 1$	$\Phi_1 = -90^\circ$
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(4) proportional and differential block	$(0.1s+1)$	$+20dB/Dec$	$\omega_2 = 1/T_2 = 10$	
(5) oscillation block	$(s^2/2500 + 6s/50 + 1)$	$-40dB/Dec$ $\omega_3 = 1/T_3 = 50$		

Remarks

- How to sketch the Bode plot of a system?
  - Decompose the factors in the given system transfer function
  - Sketch the asymptotic line of the magnitude and phase plot of each block
  - Compose the asymptotic line of the magnitude and phase plot of the system by adding those of individual blocks
  - Sketch the magnitude and phase plot by revising the errors on the asymptote near the corner frequency of each block.

Advantages of Bode Plot

- Since the magnitude of $G(j\omega)$ in the Bode plot is expressed in dB, product and division factors in $G(j\omega)$ become additions and subtractions, respectively. The phase relations are also added and subtracted from each other algebraically.

- The magnitude plot of the Bode plot of $G(j\omega)$ can be approximated by straight-line segments, which allow the simple sketching of the plot without detailed computation.

Disadvantages of Bode Plot ?

Minimum-Phase and Nonminimum-Phase Functions

A majority of the process transfer functions encountered in linear control systems do not have poles or zeros in the right-half s-plane. This class of transfer functions is called the minimum-phase transfer functions. When a transfer function has either a pole or a zero in the right-half s-plane, it is called a nonminimum-phase transfer function.

Example 4.11 Sketch the Bode plot of the following three transfer functions.

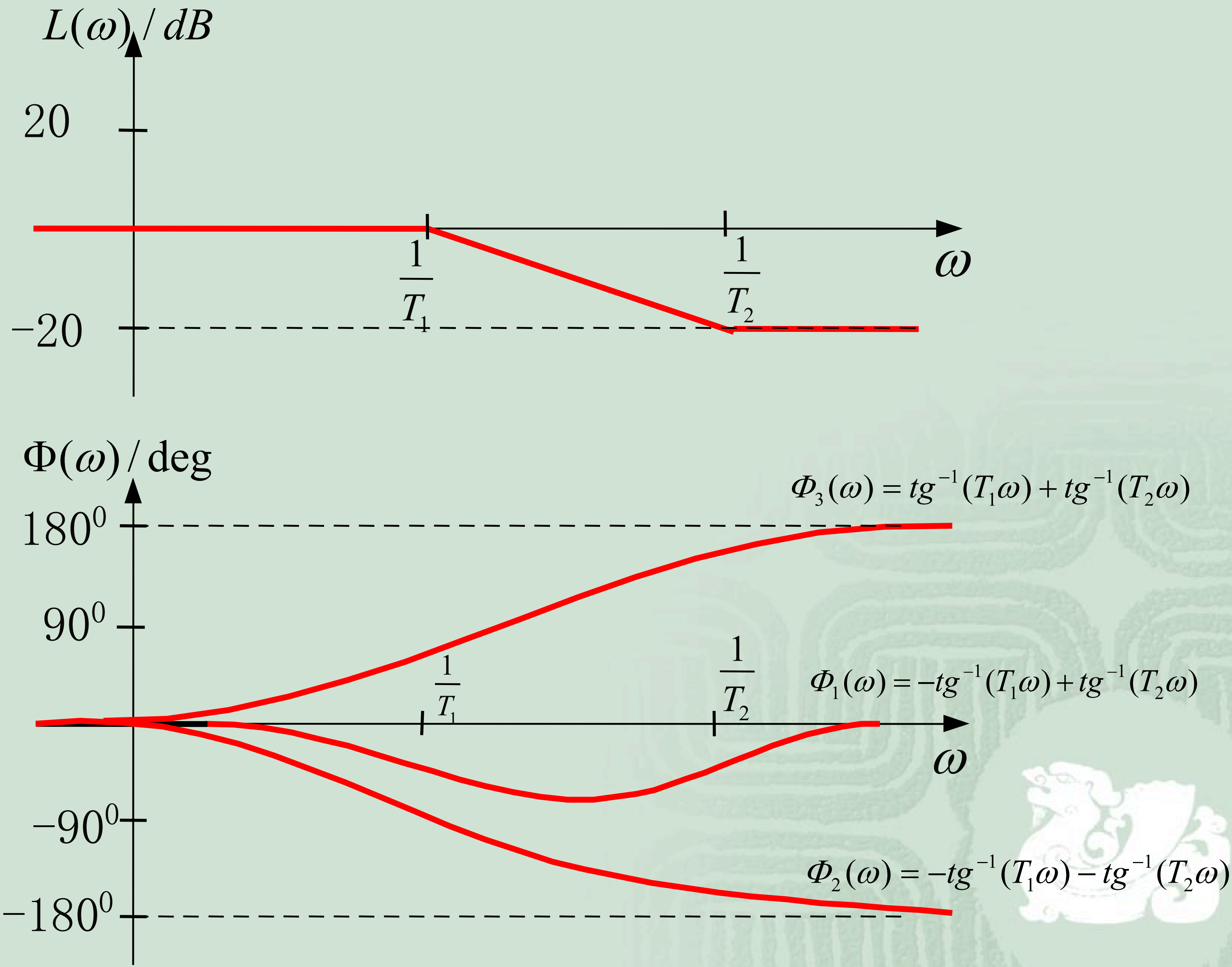
$$G_1(s) = \frac{T_2 s + 1}{T_1 s + 1} \qquad G_2(s) = \frac{1 - T_2 s}{T_1 s + 1} \qquad G_3(s) = \frac{T_2 s + 1}{1 - T_1 s}$$

A: $$L_1(\omega) = L_2(\omega) = L_3(\omega)$$

$$\Phi_1(\omega) = -tg^{-1}(T_1\omega) + tg^{-1}(T_2\omega)$$

$$\Phi_2(\omega) = -tg^{-1}(T_1\omega) - tg^{-1}(T_2\omega)$$

$$\Phi_3(\omega) = tg^{-1}(T_1\omega) + tg^{-1}(T_2\omega)$$



$L(\omega)/dB$
20
-20
$\frac{1}{T_1}$
$\frac{1}{T_2}$
ω
$\Phi(\omega)/\deg$
$\Phi_3(\omega)=tg^{-1}(T_1\omega)+tg^{-1}(T_2\omega)$
180^0
90^0
$\frac{1}{T_1}$
$\frac{1}{T_2}$
$\Phi_1(\omega)=-tg^{-1}(T_1\omega)+tg^{-1}(T_2\omega)$
ω
-90^0
$\Phi_2(\omega)=-tg^{-1}(T_1\omega)-tg^{-1}(T_2\omega)$
-180^0

Properties of Minimum-Phase Functions

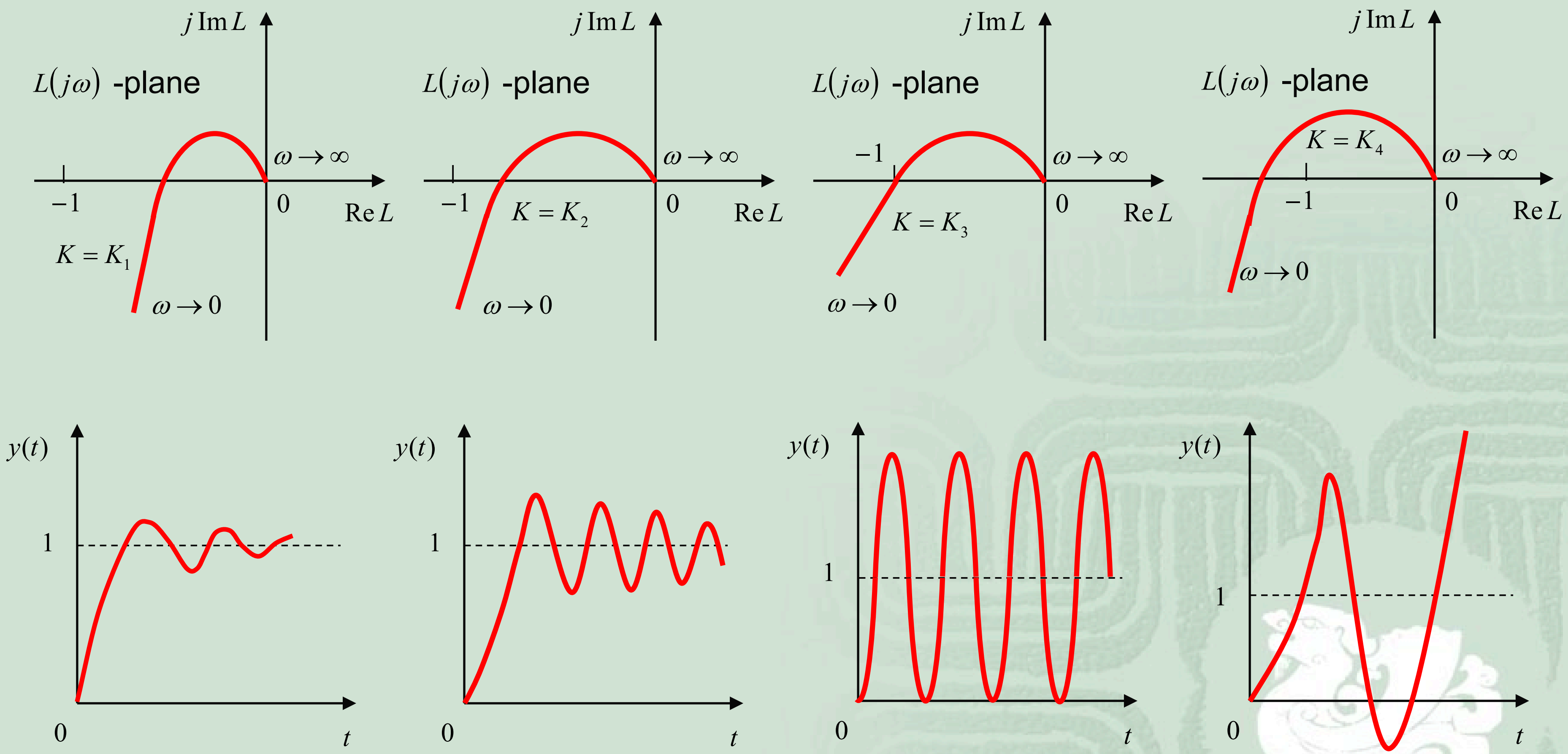
Minimum-phase transfer functions have some important properties :

(1) The magnitude and phase characteristics are uniquely related.

(2) For a minimum-phase transfer function $G(s)$ with m zeros and n poles, excluding the poles at $s = 0$, if any, when $s = j\omega$, and as ω varies from 0 to ∞ , the total phase variation of $G(j\omega)$ is $(n-m)\pi/2$

Relative Stability

Nyquist plot and time-domain response of a third-order system



Stable and well damped system

Stable but oscillatory system

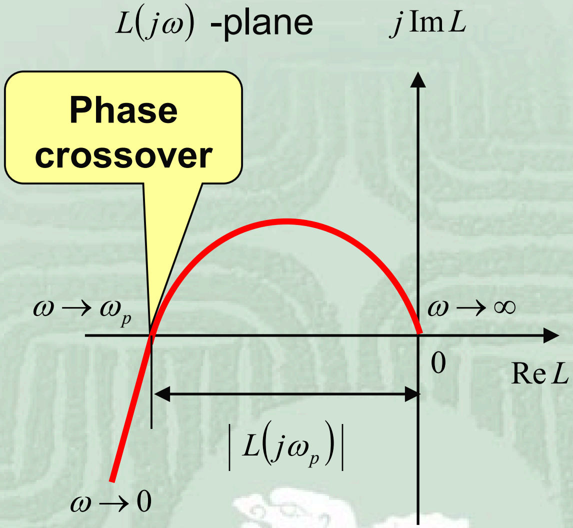
Marginally unstable system

unstable system

Gain Margin

Gain margin (GM) is one of the most frequently used criteria for measuring the relative stability of control systems. In the frequency domain, gain margin is used to indicate the closeness of the intersection of the negative real axis made by the Nyquist plot of $L(j\omega)$ to the $(-1, j0)$ point.

Phase Crossover is a point on the $L(j\omega)$ plot at which the plot intersect the negative real axis.



Phase Crossover Frequency is the frequency at the phase crossover, or where $\angle L(j\omega) = 180^\circ$

$$GM = 20\lg \frac{1}{|L(j\omega_p)|} = -20\lg |L(j\omega_p)| \quad dB$$

Gain Margin

On the basis of the definition, we can draw the following conclusions about the gain margin of the system depending on the properties of the Nyquist plot

1. The $L(j\omega)$ plot does not intersect the negative real axis (no finite nonzero phase crossover).

$$\left| L(j\omega_p) \right| = 0 \qquad GM = \infty \ dB$$

2. The $L(j\omega)$ plot intersects the negative real axis between 0 and the -1 point.

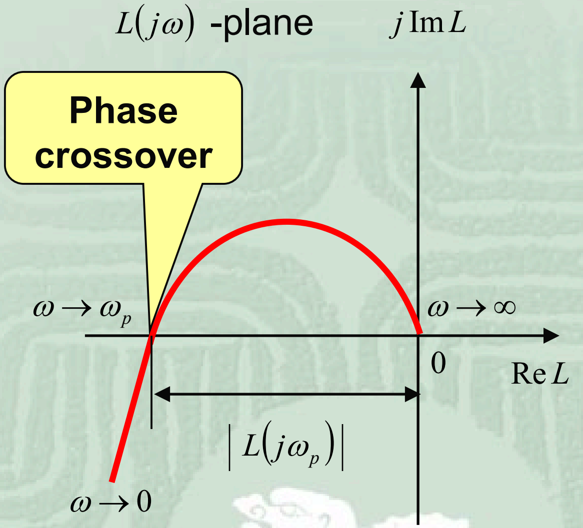
$$0 < \left| L(j\omega_p) \right| < 1 \qquad GM > 0 \ dB$$

3. The $L(j\omega)$ plot passes through the (-1,j0) point.

$$\left| L(j\omega_p) \right| = 1 \qquad GM = 0 \ dB$$

4. The $L(j\omega)$ plot encloses the (-1,j0) point.

$$\left| L(j\omega_p) \right| > 1 \qquad GM < 0 \ dB$$



Gain Margin

Based on the forgoing discussions, the physical significance of gain margin can be summarized as:

Gain margin is the amount of gain in decibels (dB) that can be added to the loop before the closed-loop system becomes unstable.

▲ When the Nyquist plot does not intersect the negative real axis at any finite nonzero frequency, the gain margin is infinite in dB; this means that, theoretically, the value of the open-loop gain can be increased to infinity before instability occurs.

▲ When the Nyquist plot of $L(j\omega)$ passes through the (-1, j0) point, the gain margin is 0 dB, which implies that the open-loop gain can not be increased, as the system is at the margin of instability.

▲ When the phase crossover is to the left of the (-1, j0) point, the phase margin is negative in dB, and the loop gain must be reduced by the gain margin to achieve stability.

Gain Margin of Nonminimum-Phase Systems

Care must be taken when attempting to extend gain margin as a measure of relative stability to systems with nonminimum-phase open-loop transfer functions.

For a system with a nonminimum-phase open-loop transfer function, the system may be unstable even when the phase-crossover point is to the right of (-1, j0), and thus a positive margin may still correspond to an unstable system.

Nevertheless, the closeness of the phase crossover to the (-1, j0) point still gives an indication of relative stability.

Phase Margin

The gain margin is only one-dimensional representation of the relative stability of a closed-loop system. As the name implies, gain margin indicates system stability with respect to the variation in open-loop gain only.

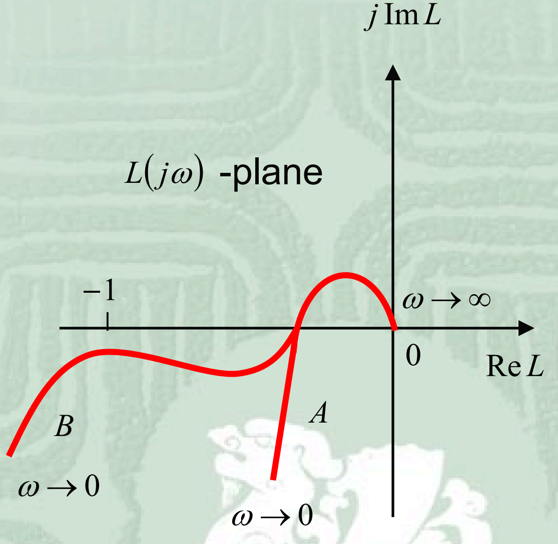
Gain margin is inadequate to indicate relative stability when system parameters other than the open-loop gain are subject to variation.

To include the effect of phase shift on stability, phase margin is introduced.

Gain Crossover is a point on the $L(j\omega)$ at which the magnitude of $L(j\omega)$ is equal to 1.

Gain Crossover Frequency is the frequency ω_g at the gain crossover, or where

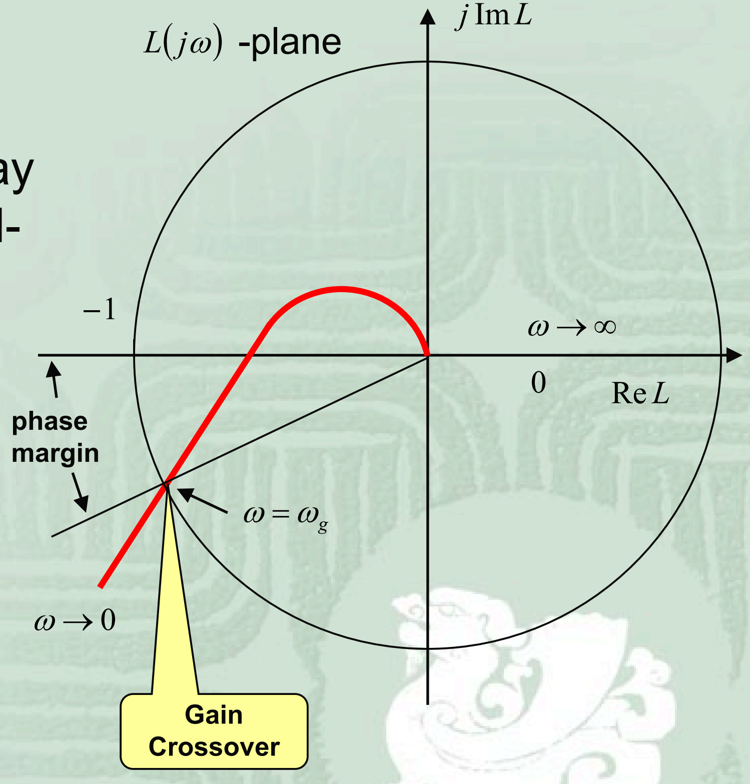
$$\left| L(j\omega_g) \right| = 1$$



Phase Margin

Phase margin (PM) is defined as the angle in degrees through which $L(j\omega)$ plot must be rotated about the origin so that the gain crossover passes through the (-1, j0) point.

Phase margin is the amount of pure phase delay that can be added to the loop before the closed-loop system becomes unstable.



When the system is of minimum-phase type, the analytical expression of the phase margin can be expressed as

$$PM = \angle L(j\omega_g) - 180^\circ$$

Phase Margin of Nonminimum-Phase Systems

Care should be taken when interpreting the phase margin from the Nyquist plot of a nonminimum-phase transfer function. When the open-loop transfer function is of the nonminimum-phase type, the gain crossover can occur in any quadrant of the $L(j\omega)$ -plane, and the definition of phase margin given in the above equation is no longer valid.

Example 1

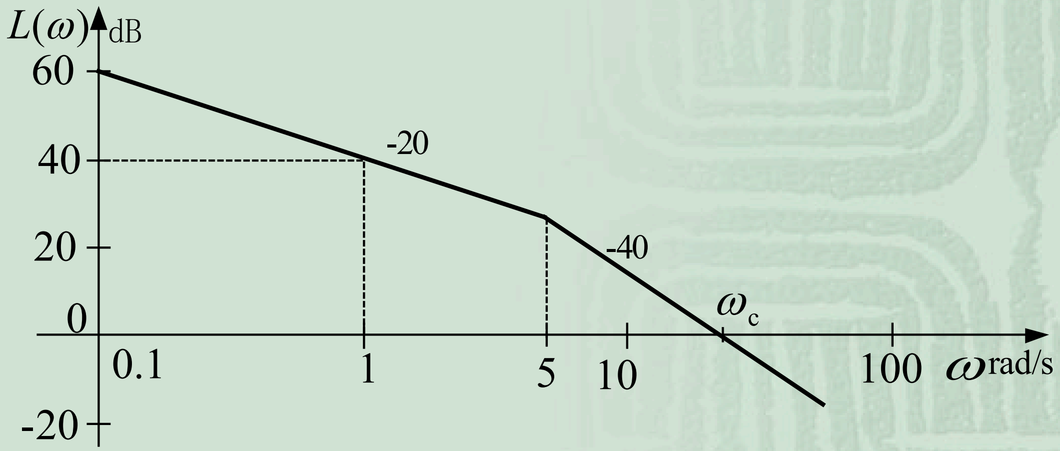
Q: Compose the magnitude bode plot of the following minimum-phase system

$$G_0(s) = \frac{100}{s(0.2s+1)}$$

A: (1) There is an integration block. So, the low-frequency section has a -20dB per decade slope.

(2) The corner frequency is $\omega_1 = 5$. The slope changes to be -40dB/dec.

(3) In the low-frequency section where $\omega=1$, the $L(\omega) = 20\lg 100 = 40\text{dB}$



Example 2

Q: Compose the bode plot of the following minimum-phase system

$$G_0(s) = \frac{10}{s(0.1s+1)(0.25s+1)}$$

A: (1) There is an integration block. So, the low-frequency section has a -20dB per decade slope.

(2) In the low-frequency section where $\omega=1$, the $L(\omega) = 20\lg 10 = 20\text{dB}$

(3) The first corner frequency is $\omega_1=4$. The slope changes to be -40dB/dec.

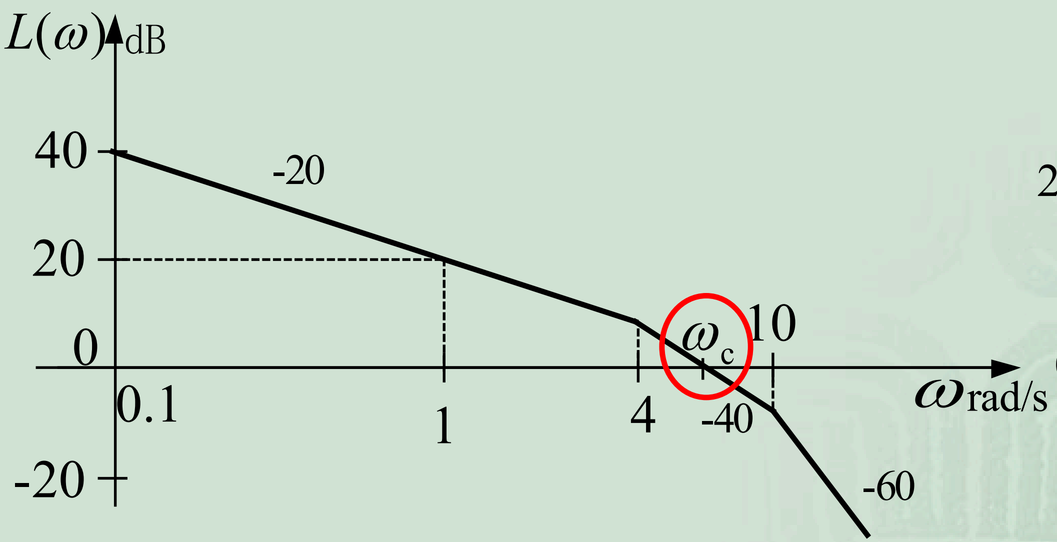
(4) The second corner frequency is $\omega_2=10$. The slope changes to be -60dB/dec.

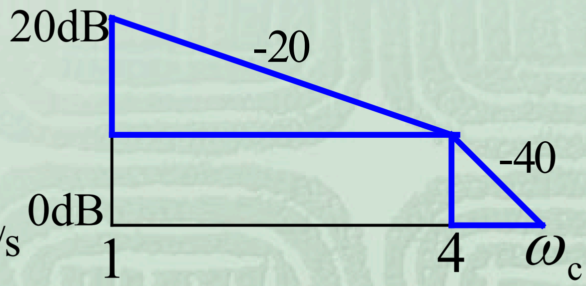
Example 2

Q: Compose the bode plot of the following minimum-phase system

$$G_0(s)=\frac{10}{s(0.1s+1)(0.25s+1)}$$

A:





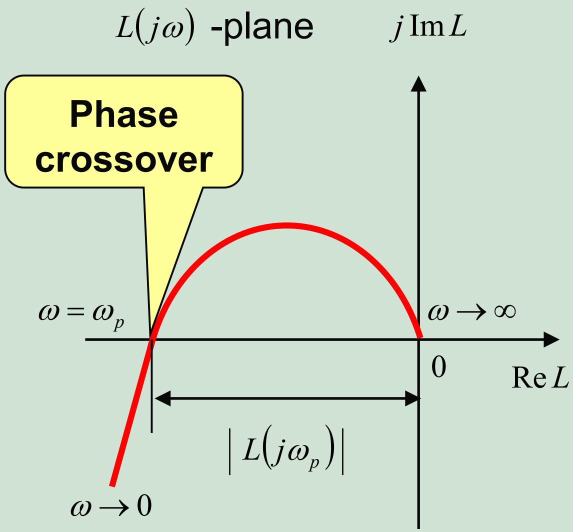
(5) To find the crossover frequency.

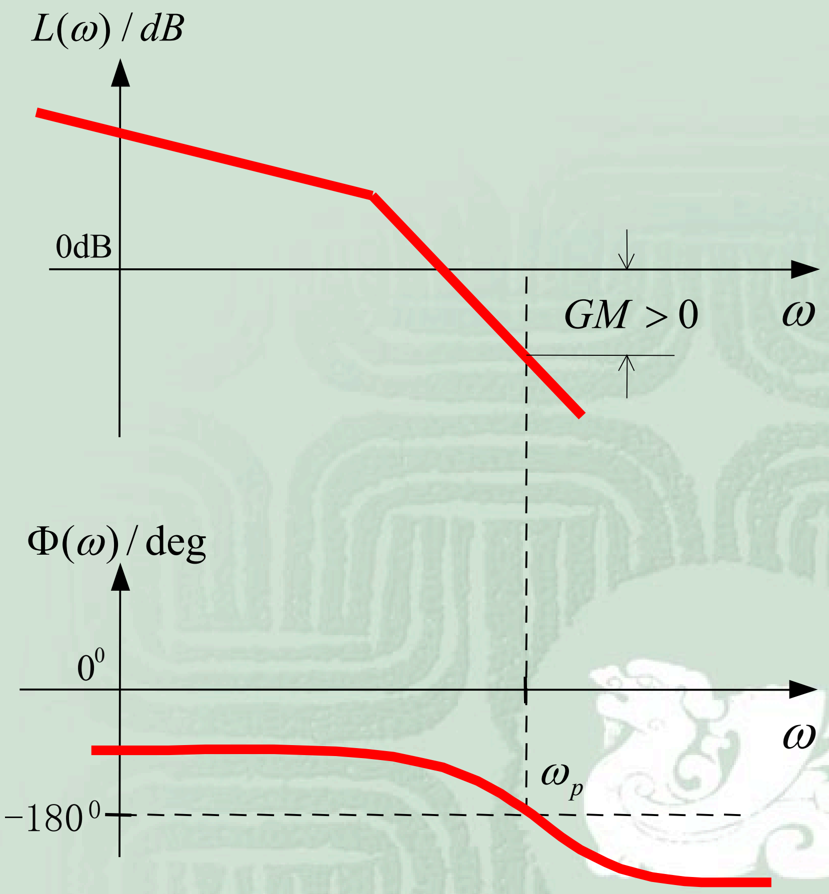
$$40(\lg\omega_c-\lg 4)+20(\lg 4-\lg 1)=20\text{dB}$$

$$\omega_c=6.32$$

Gain and Phase Margin on Bode Plot

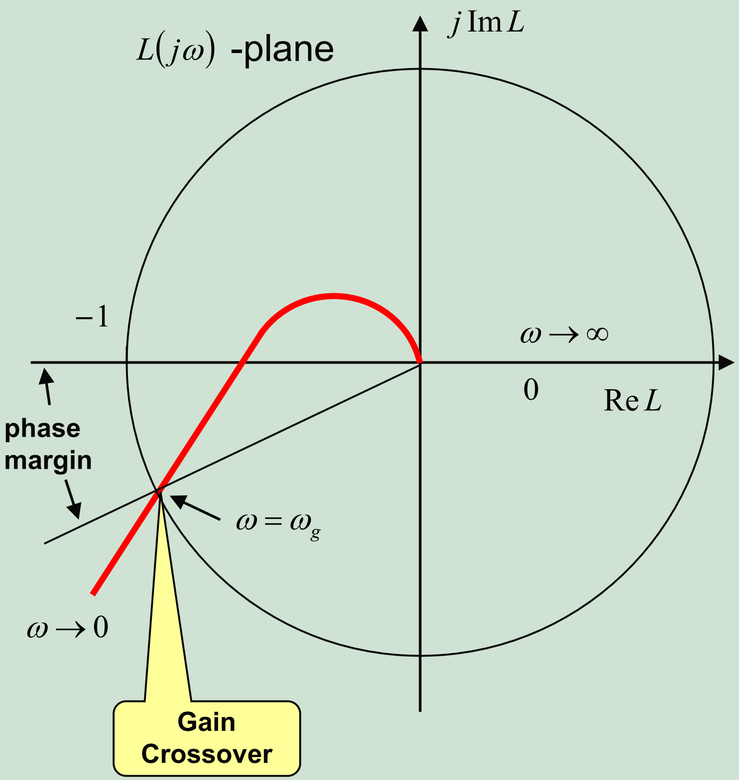
The intersection with the negative real axis of the Nyquist plot is corresponding to the 180° line on the phase plot.

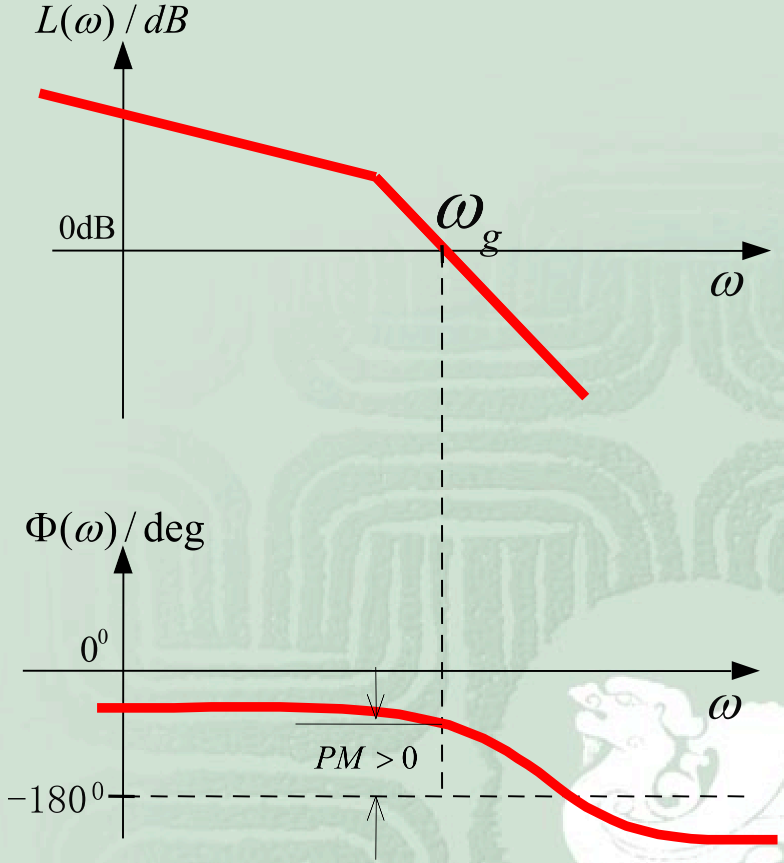




Gain and Phase Margin on Bode Plot

The intersection of the Nyquist plot with the unit circle is corresponding to the zero dB line on the magnitude plot.

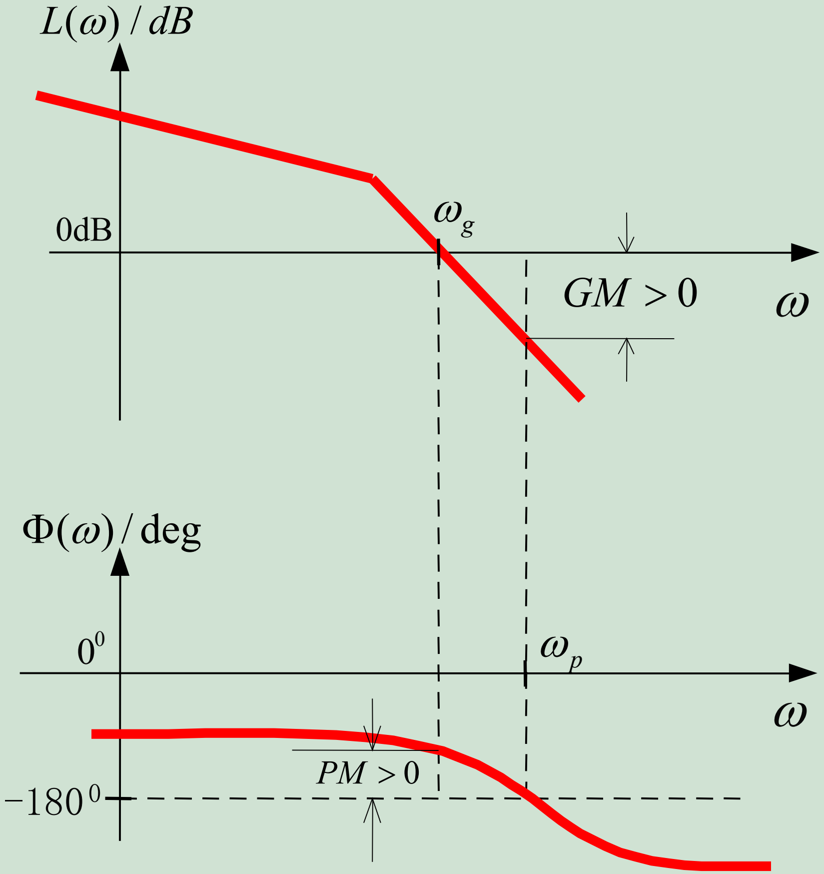




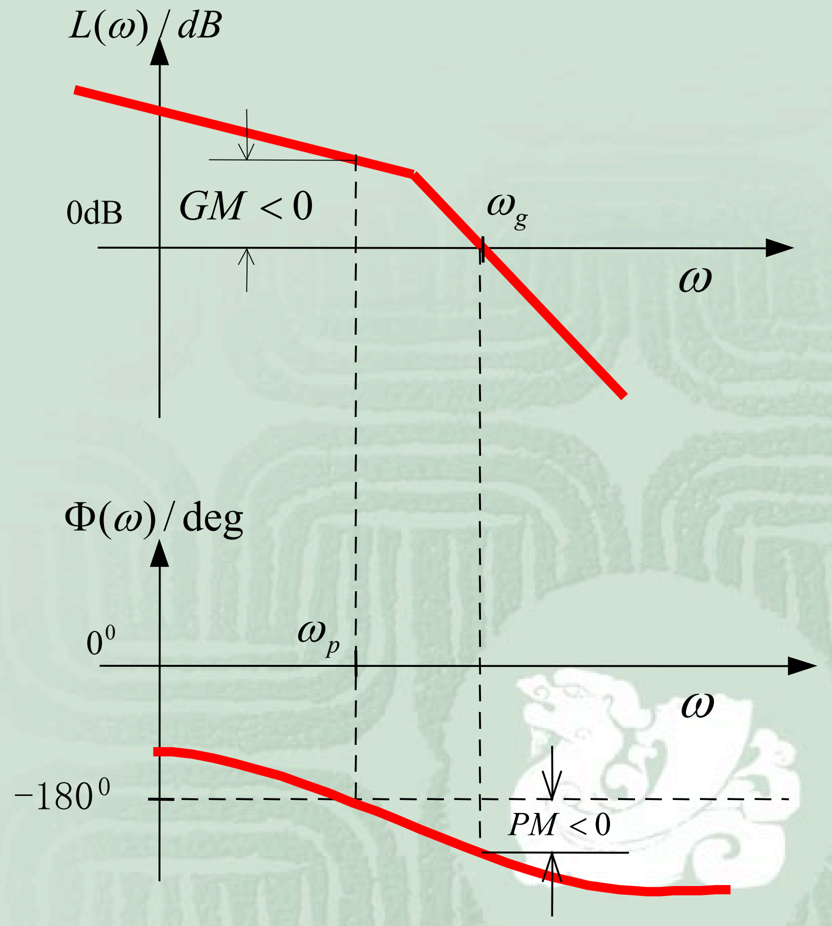
Stability Criterion

If the phase angle of the open-loop transfer function of a minimum-phase system is greater than -180 degrees at the gain crossover frequency, the system is stable.

If the magnitude of the open-loop transfer function of a minimum-phase system is less than zero at the phase crossover frequency, the system is stable.

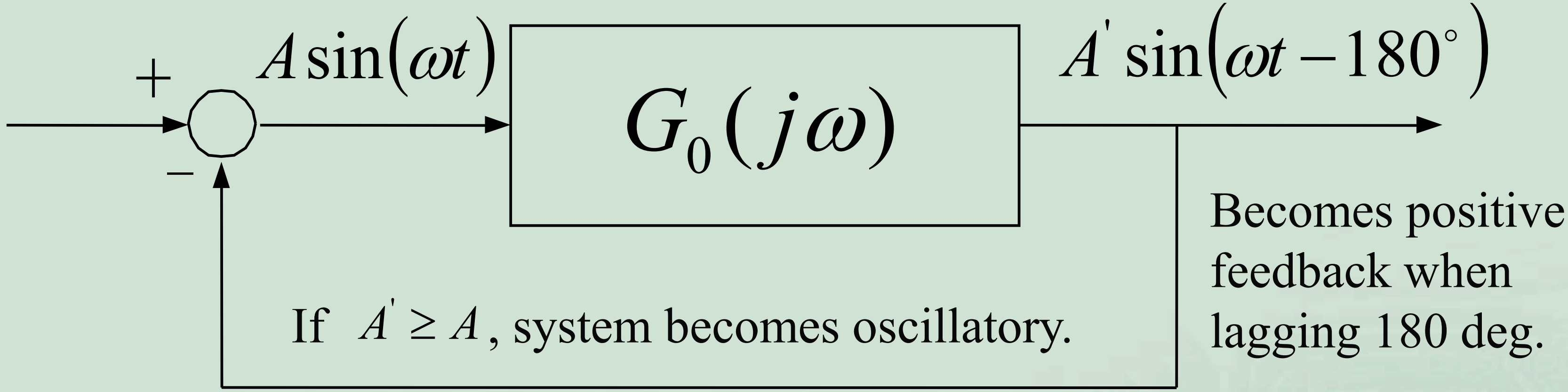


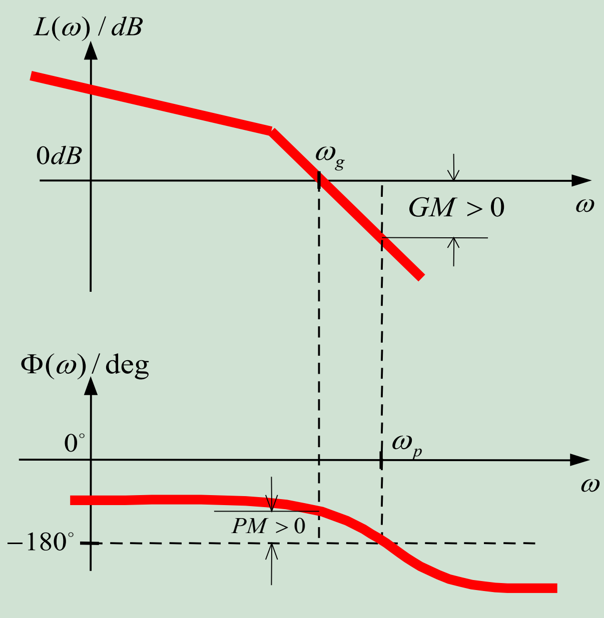
Stable System



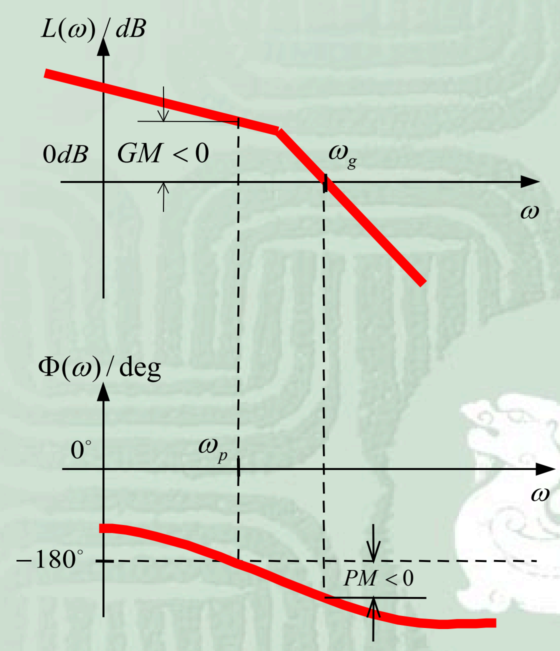
Unstable System

Physical Significance of the Stability Criteria





Stable System



Unstable System

Stability Analysis with the Bode Plot

Advantages of the Bode Plot

1. In the absence of a computer, a Bode diagram can be sketched by approximating the magnitude and phase with straight-line segments.

2. Gain crossover, phase crossover, gain margin, and phase margin are more easily determined on the Bode plot than from the Nyquist plot.

3. For design purposes, the effects of adding controllers and their parameters are more easily visualized on the Bode plot than on the Nyquist plot

Disadvantages of the Bode Plot

Absolute and relative stability of only minimum-phase systems can be determined from the Bode plot.

Calculation of Stability Margins

Example 4.12 Find the gain margin and phase margin of the system with the following open-loop transfer function, and estimate the system stability.

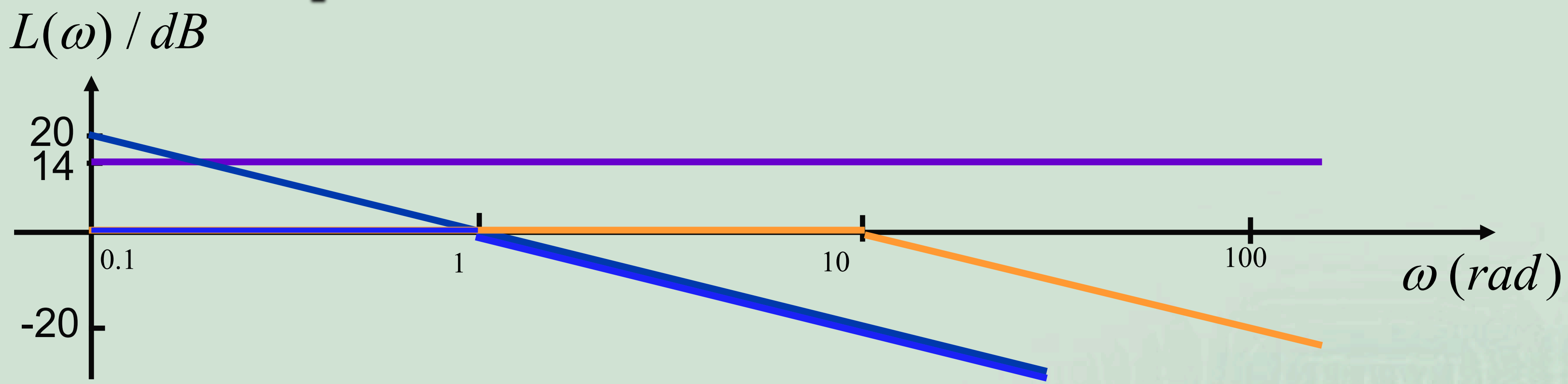
$$G_0(s) = \frac{K}{s(s+1)(0.1s+1)} \qquad K = 5 \quad \text{and} \quad K = 20$$

Q: when K=5

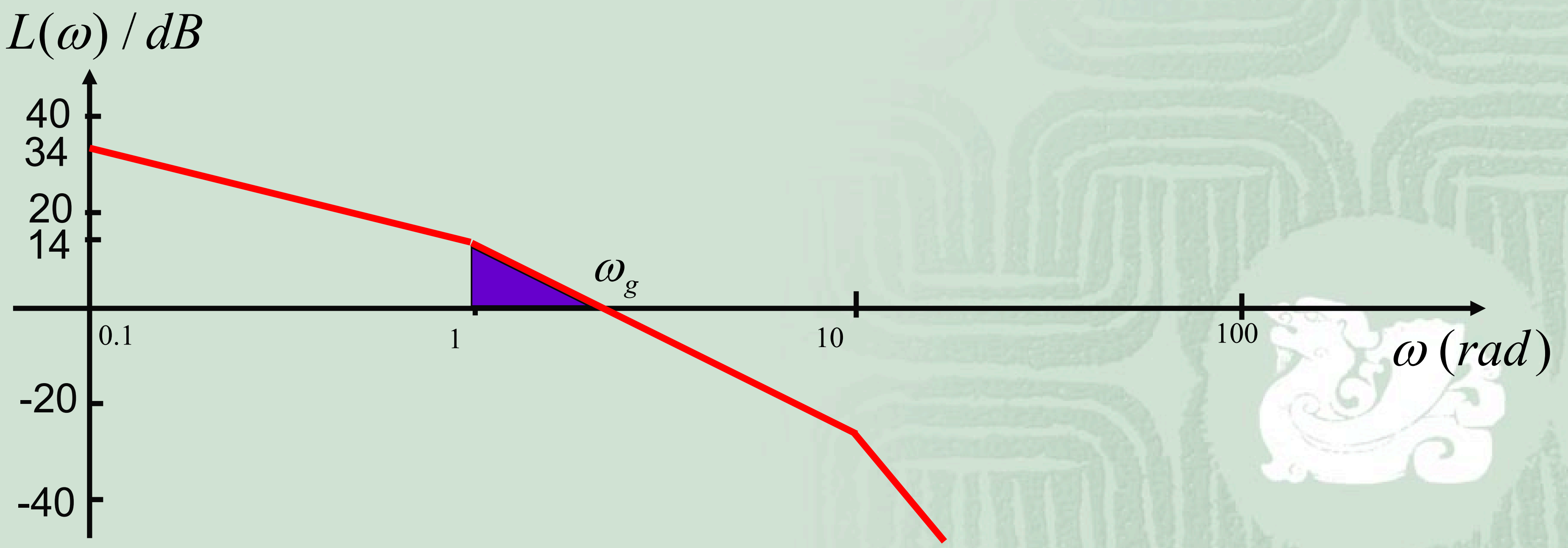
Four factor blocks

$K = 5$	$20\lg 5 = 14dB$	$\Phi_1 = 0^\circ$
$1/s$	$\omega_c = 1$	$slope = -20dB/Decade$
$1/(s+1)$	$\omega_c = 1$	$slope = -20dB/Decade$
$1/(0.1s+1)$	$\omega_c = 10$	$slope = -20dB/Decade$

Example 4.11

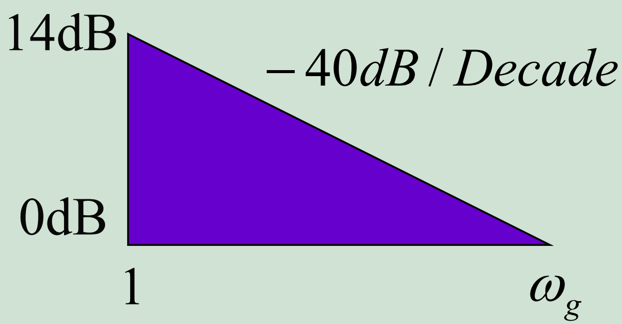


$L(\omega) / dB$
20
14
-20
0.1
1
10
100
$\omega\,(rad)$



$L(\omega) / dB$
40
34
20
14
-20
-40
ω_g
0.1
1
10
100
$\omega\,(rad)$

Example 4.11



calculate the gain crossover frequency

$$-40(\lg 1-\lg \omega_g)=14$$

$$40\lg \omega_g=14$$

$$\omega_g=2.24$$

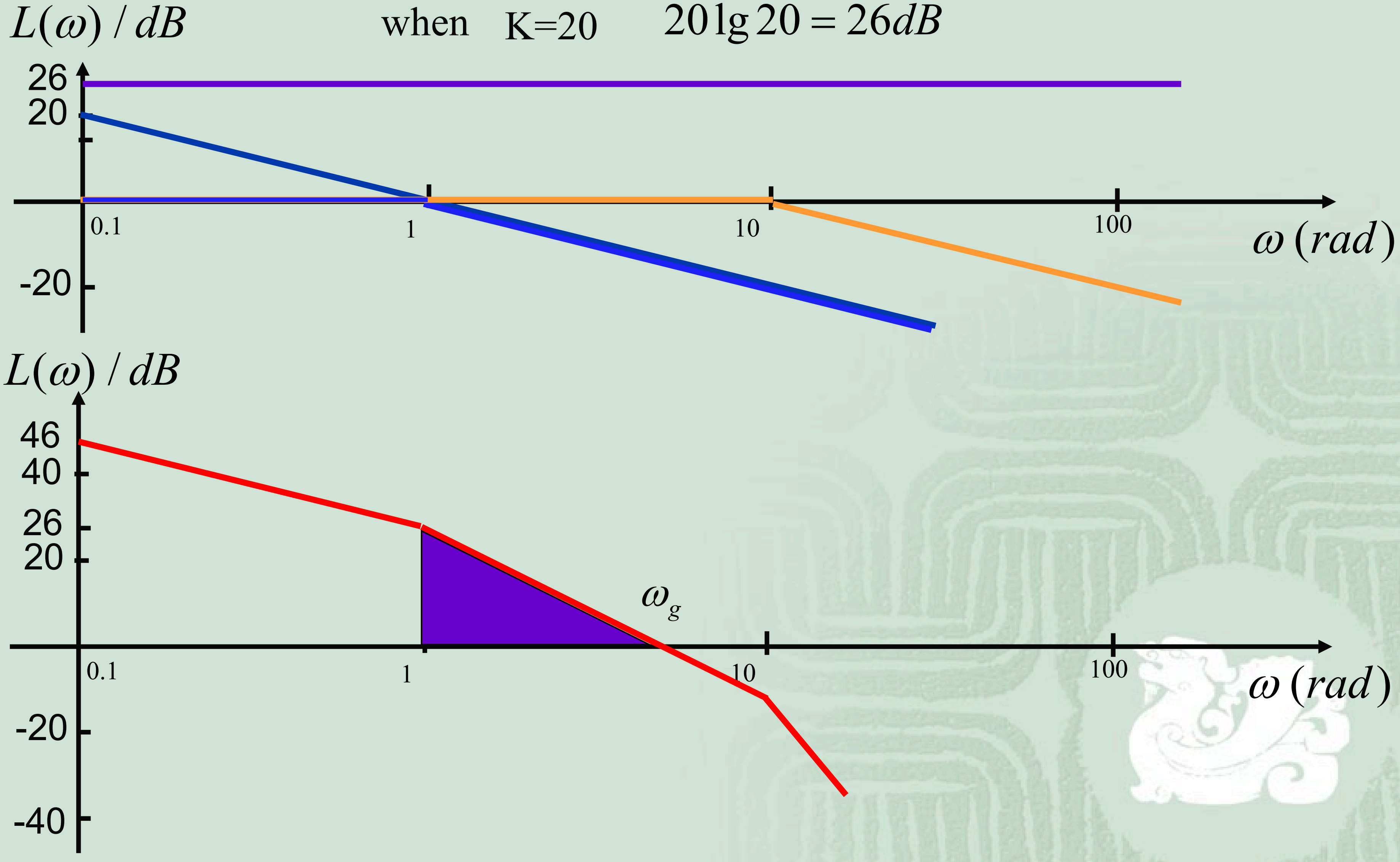
$$G_0(j\omega)=\frac{5}{j\omega(j\omega+1)(0.1j\omega+1)}$$

$$\Phi(\omega_g)=-90^0-tg^{-1}(\omega_g)-tg^{-1}(0.1\omega_g)=-168.6^0$$

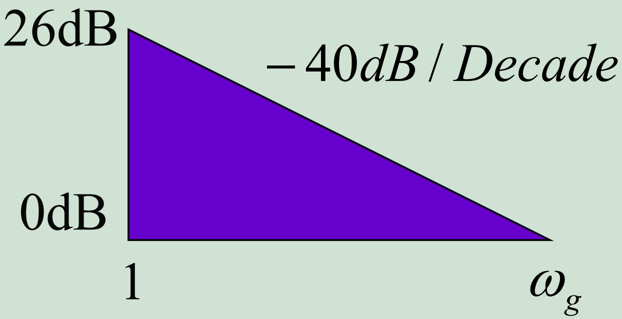
$$PM=180^0+\Phi(\omega_g)=11.4^0>0^0$$ ⇒ System is stable

Example 4.11

when K=20 $\quad 20\lg 20 = 26dB$



Example 4.11



calculate the gain crossover frequency

$$-40\left(\lg 1 - \lg \omega_g\right) = 26$$

$$40\lg \omega_g = 26$$

$$\omega_g = 4.47$$

$$G_0(j\omega) = \frac{20}{j\omega(j\omega+1)(0.1j\omega+1)}$$

$$\Phi(\omega_g) = -90^0 - tg^{-1}(\omega_g) - tg^{-1}(0.1\omega_g) = -191.5^0$$

$$PM = 180^0 + \Phi(\omega_g) = -11.5^0 < 0^0$$ ⇒ System is unstable

Example 4.11

calculate the phase crossover frequency

$$\Phi(\omega_p) = -90^0 - tg^{-1}(\omega_p) - tg^{-1}(0.1\omega_p) = -180^0$$

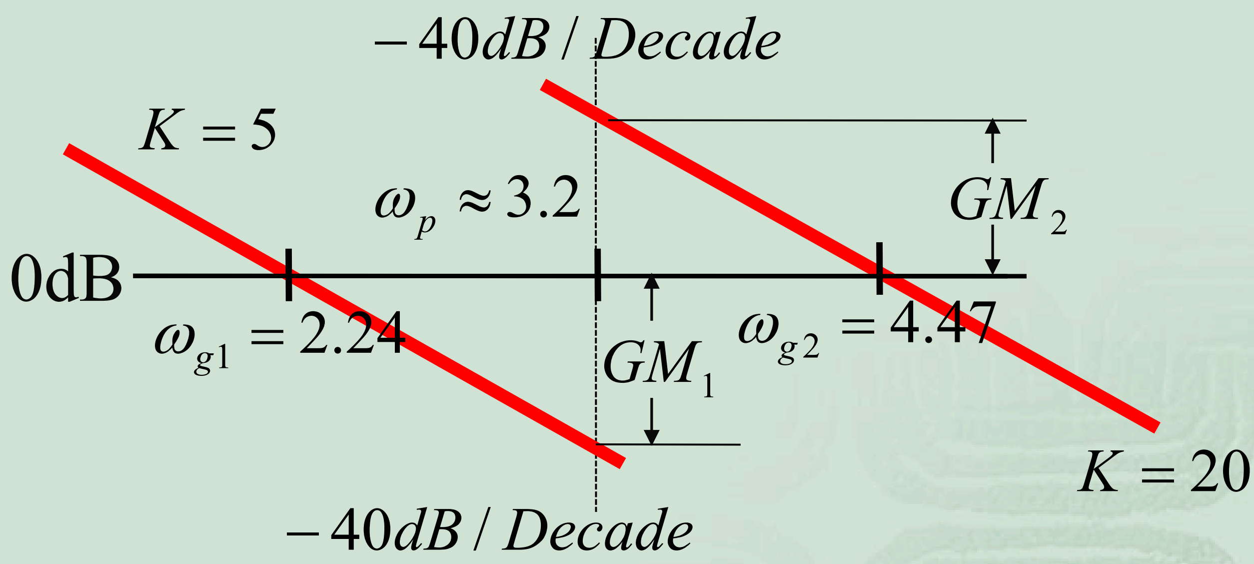
trial and error:

$$\Phi(3.3) = -90^0 - tg^{-1}(3.3) - tg^{-1}(0.1*3.3) = -178^0$$

$$\Phi(3.2) = -90^0 - tg^{-1}(3.2) - tg^{-1}(0.1*3.2) = -180.4^0$$

$$\omega_p \approx 3.2$$

Example 4.11



when $K=5$ $GM_1=-40(\lg 2.24-\lg 3.2)=6.2dB$

when $K=20$ $GM_2=-40(\lg 4.47-\lg 3.2)=-5.8dB$

Wrap-up

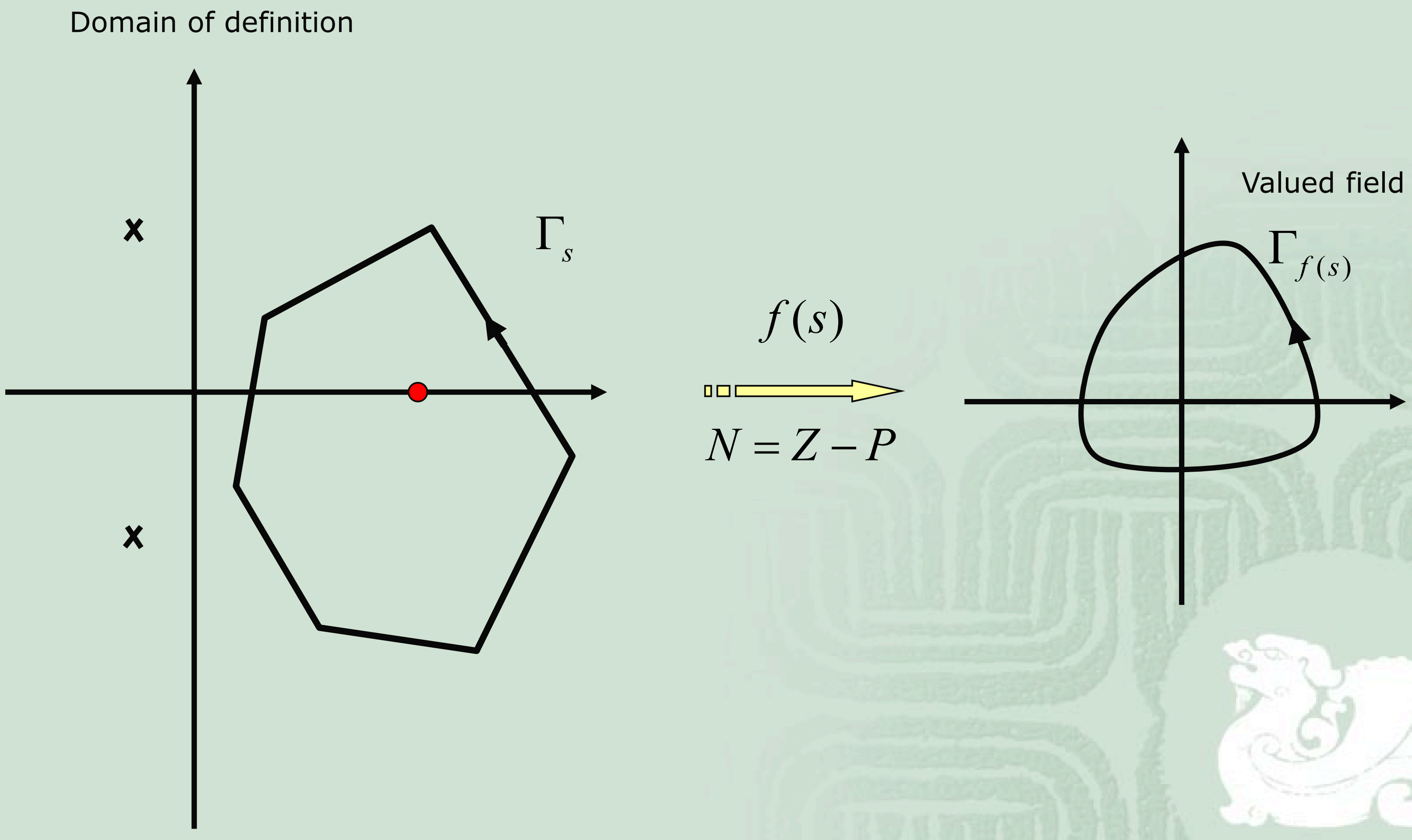
- Nyquist criterion(continued)
- Definition of Bode plot
- Bode plots of typical transfer functions
- How to compose the Bode plot of a system
- Minimum-phase system
- Relative stability: gain margin and phase margin
- Gain and phase margin on Bode plot

Assignment

Page 105~106

- 4,(1)
- 5
- 6
- 7，(5)、(6)

Discussion – Understand Principle of Argument



Discussion – Understand Principle of Argument

$$f(s)=\frac{K(s+z_1)}{(s+p_1)(s+p_2)}$$