Control Systems Design

Chapter 5

Review

- Three steps involved in control system design
- Commonly used specifications
- Controller configuration
- Typical Controllers
- Control system design using time-domain method

Outlines

- How to design a controller using frequency-domain method;

Controller Design with Frequency-Domain Methods

- Performance specifications in frequency-domain
  - phase margin γ or gain margin k_g
  - gain or phase crossover frequency ω_c

- General relations between frequency-domain specifications and time-domain specifications
  - the smaller the γ is, the larger the overshoot $\sigma\%$ is
  - the bigger the M_r is, the larger the overshoot $\sigma\%$ is
  - the faster a system responds to a unit-step input, the higher the bandwidth ω_B is
  - the shorter the settling time t_s is, the higher the bandwidth ω_B is
  - the bigger the gain crossover frequency ω_c is, the bigger the bandwidth ω_B is

General Guidelines

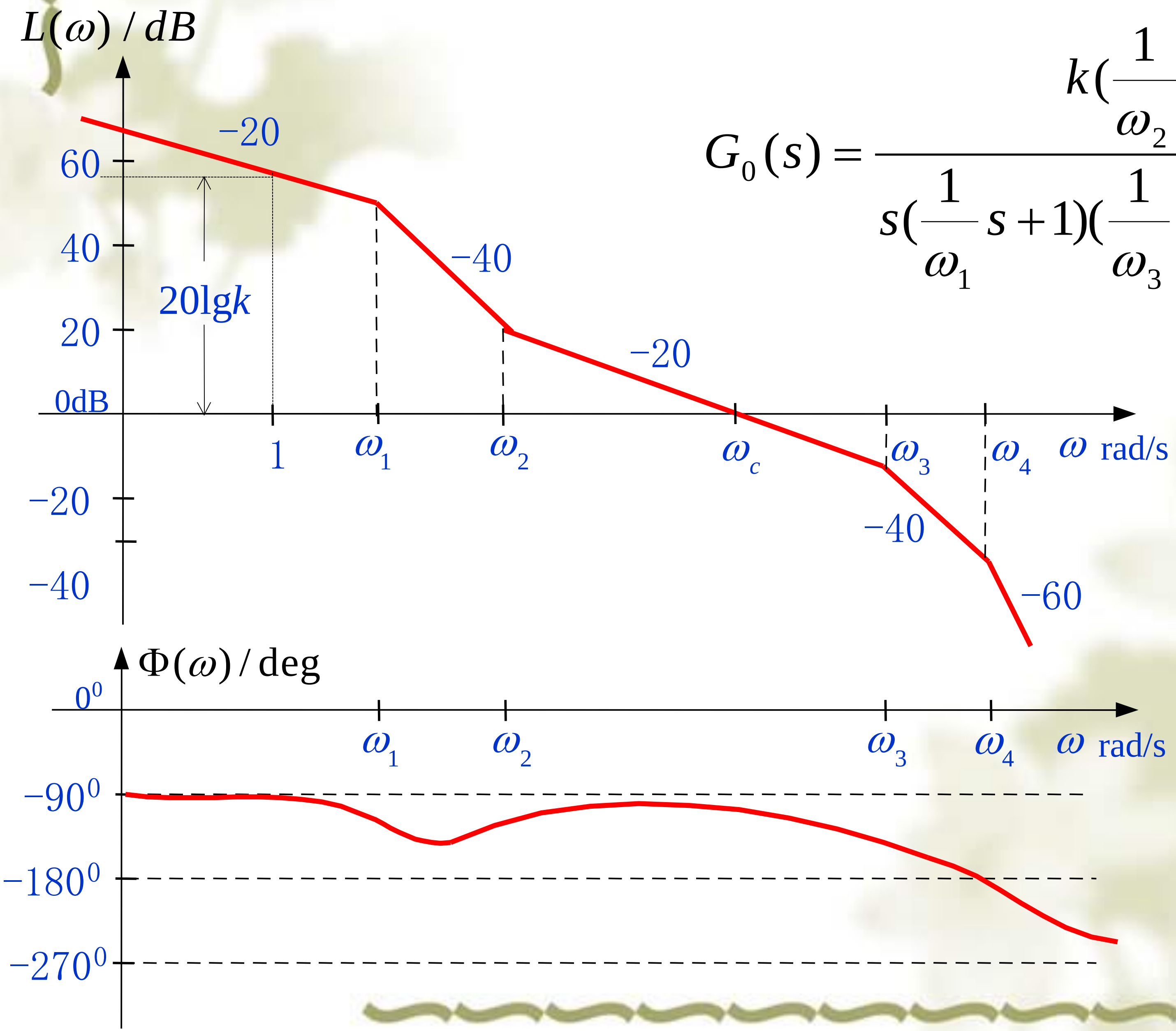
- The magnitude plot should pass through the real axis in the intermediate frequency band and with a slop of -20dB per decade. The pass-through section should have a width of 1 decade;

- The low-frequency band should have enough height;

- In the high-frequency band, the magnitude plot should decay with an appropriate cutoff rate.

General Guidelines

$$G_0(s) = \frac{k(\frac{1}{\omega_2}s+1)}{s(\frac{1}{\omega_1}s+1)(\frac{1}{\omega_3}s+1)(\frac{1}{\omega_4}s+1)}$$



Example - 5.3

Q: the open-loop transfer function of a unit feedback system is shown as follows. Please design a series compensator so that the following performance specifications are satisfied: the ramp-error constant $K_v = 20$, the phase margin $\gamma \geq 50^\circ$

$$G_0(s) = \frac{K}{s(0.5s+1)}$$

A: 1st step, choose K to satisfy the steady-state error requirement

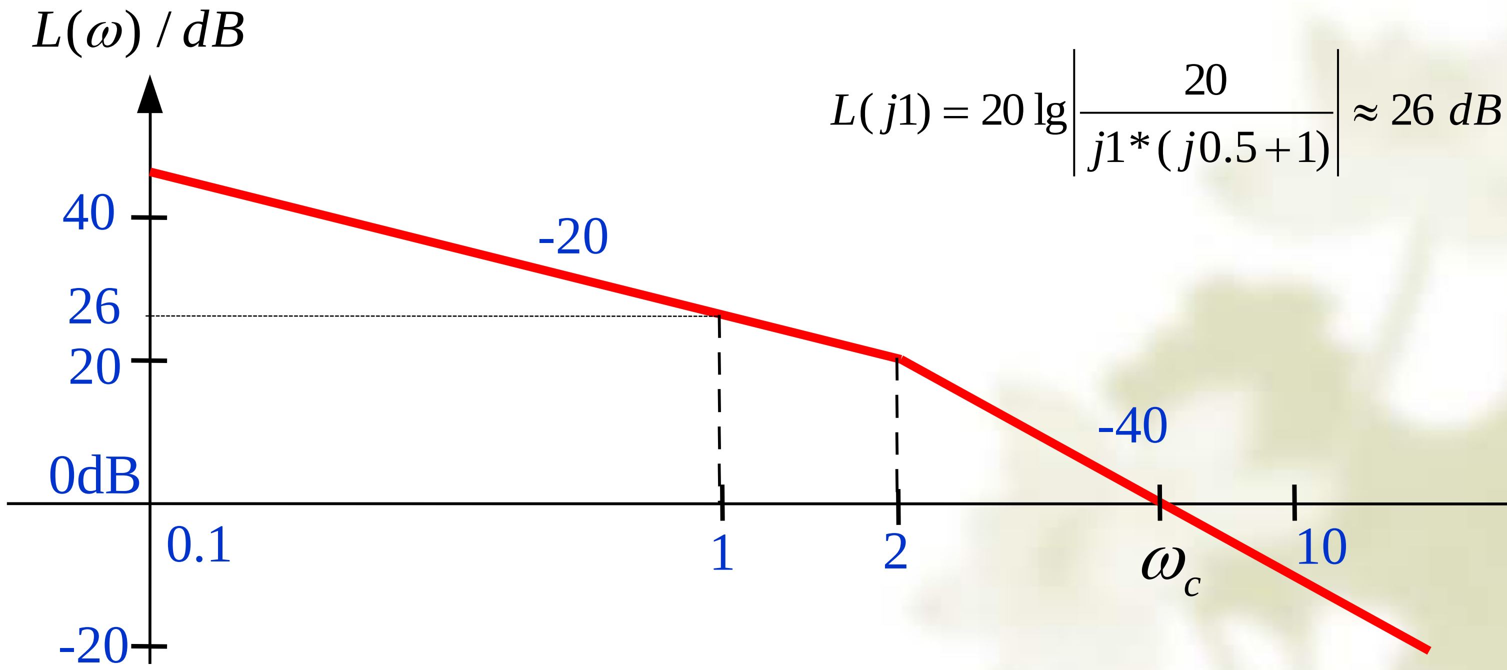
$$\because \quad K_v = \lim_{s \to 0} sG_0(s)$$

$$\therefore \quad K_v = K = 20$$

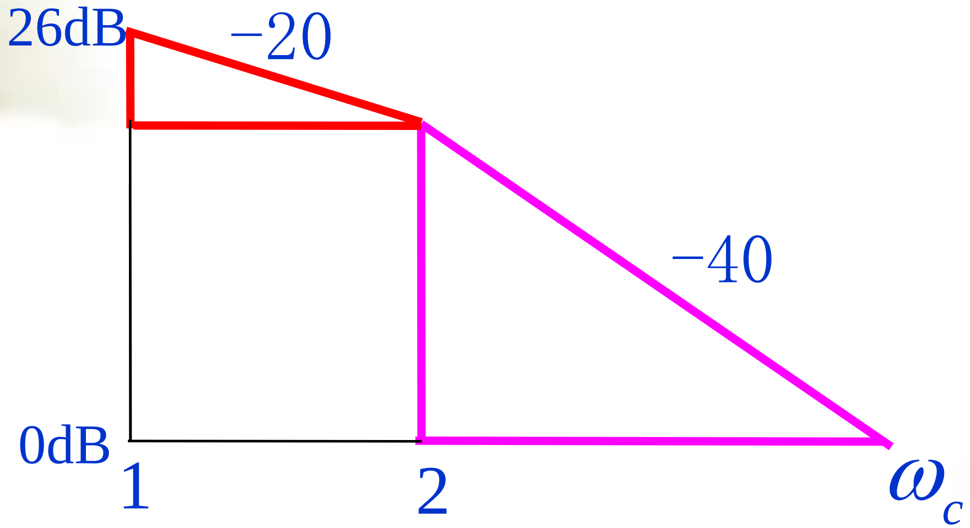
2nd step, sketch the bode plot and check the phase margin:

$$G_0(j\omega)=\frac{20}{j\omega(0.5j\omega+1)}$$

$$\omega=1 \qquad L(j1)=20\lg\left|\frac{20}{j1}\right|\approx 26dB$$



the gain crossover frequency is:



$$20\lg 20 = 20\lg 2 + 40\lg\frac{\omega_c}{2} = 20\lg 2 + 20\lg\left[\frac{\omega_c}{2}\right]^2 = 20\lg\frac{\omega_c^2}{2}$$

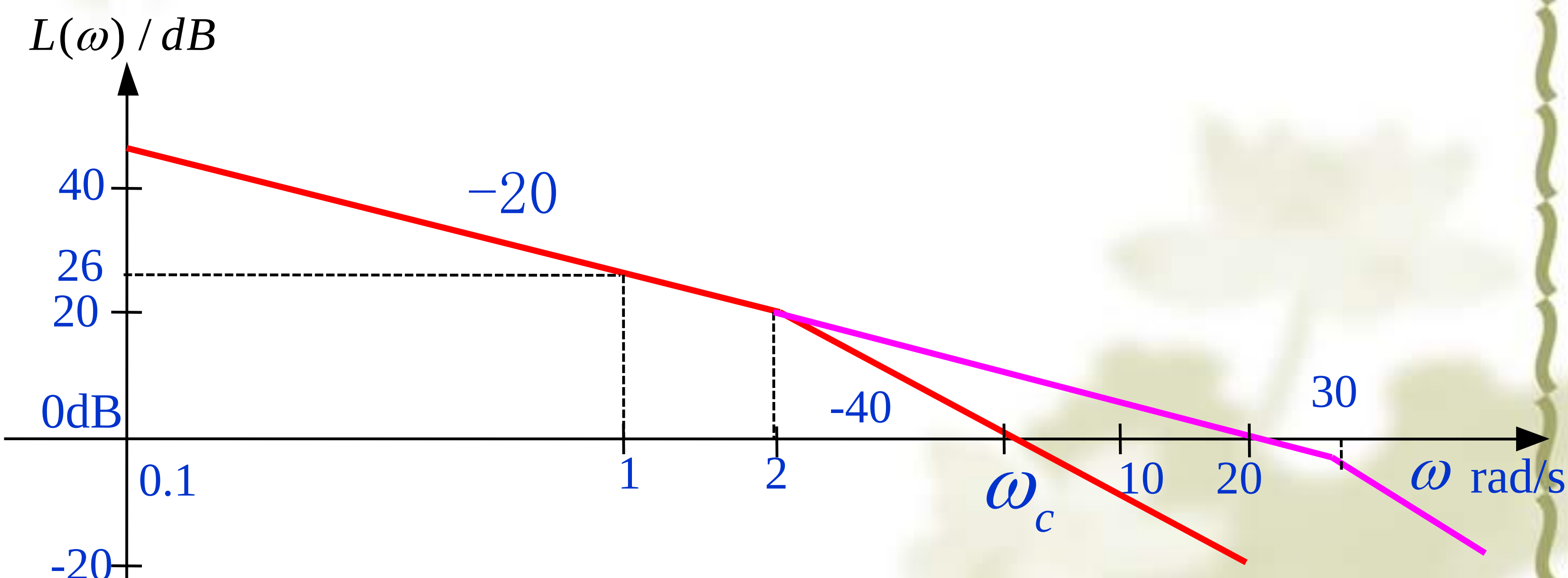
$$20 = \frac{\omega_c^2}{2} \qquad \omega_c = \sqrt{40} = 6.3$$

$$\gamma = 180^0 + \Phi(\omega_c) = 180^0 - 90^0 - tg^{-1}(0.5\times 6.3) = 18^0$$

Cancellation method can be used.

$$G_c(s)=\frac{(0.5s+1)}{(0.033\,s+1)}$$

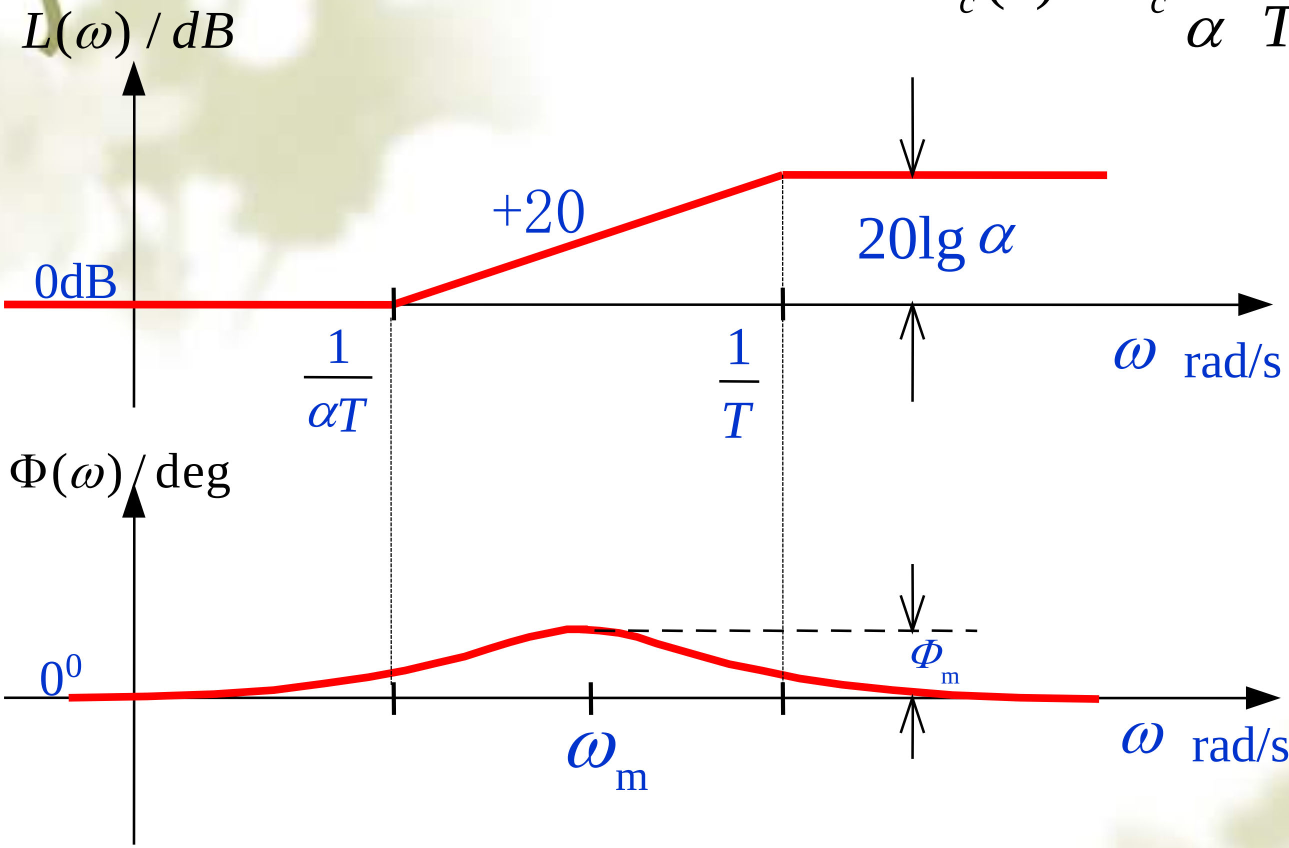
The numerator cancels the corner frequency $\omega_c = 2$, new corner frequency is determined through trial-and-error.



$$\gamma = 180^0 + \Phi(\omega_c) = 180^0 - 90^0 - tg^{-1}(0.033\times 20) = 56.3^0$$

Choose phase-lead controller

$$G_c(s) = k_c \frac{1}{\alpha}\frac{\alpha Ts+1}{Ts+1} \qquad \alpha > 1$$



set $\quad k_c \frac{1}{\alpha} = 1$

then $\quad k_c = \alpha$

If the phase margin

$\gamma \geq 50^\circ$

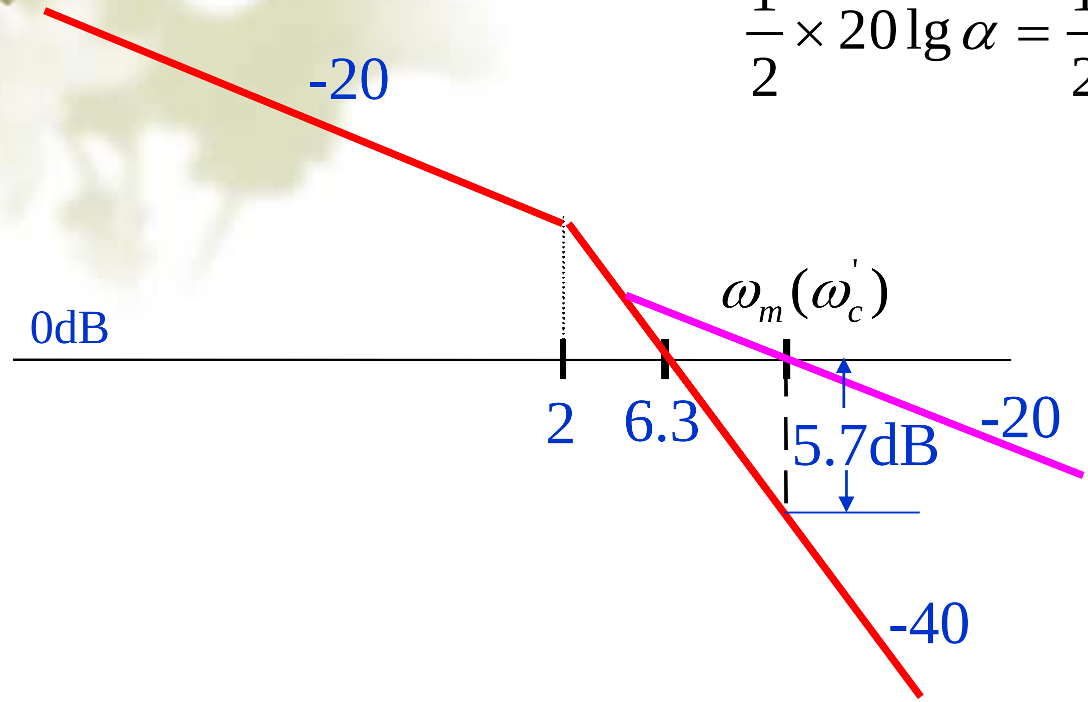
Then, the controller should provide:

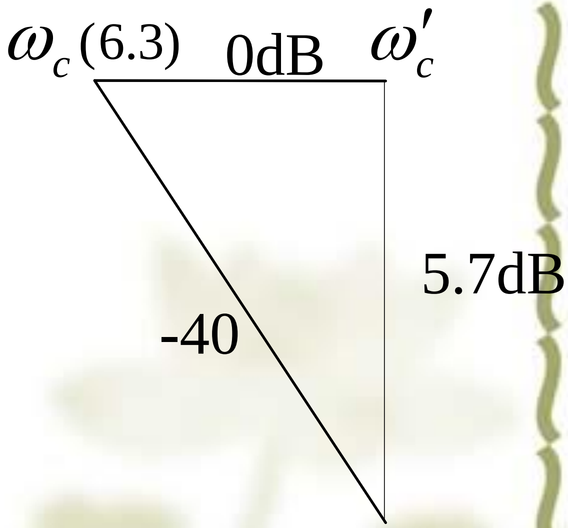
$$\Phi_m \approx 50^\circ - 18^\circ + 3^\circ$$
$$= 35^\circ$$

$$\sin\Phi_m = \frac{\alpha-1}{\alpha+1} = \sin 35^0 = 0.574 \qquad \alpha = 3.7$$

On the magnitude plot, the biggest lift occurs at where the ω_m is.

$$\frac{1}{2}\times 20\lg\alpha = \frac{1}{2}\times 20\lg 3.7 = 5.7\text{dB}$$





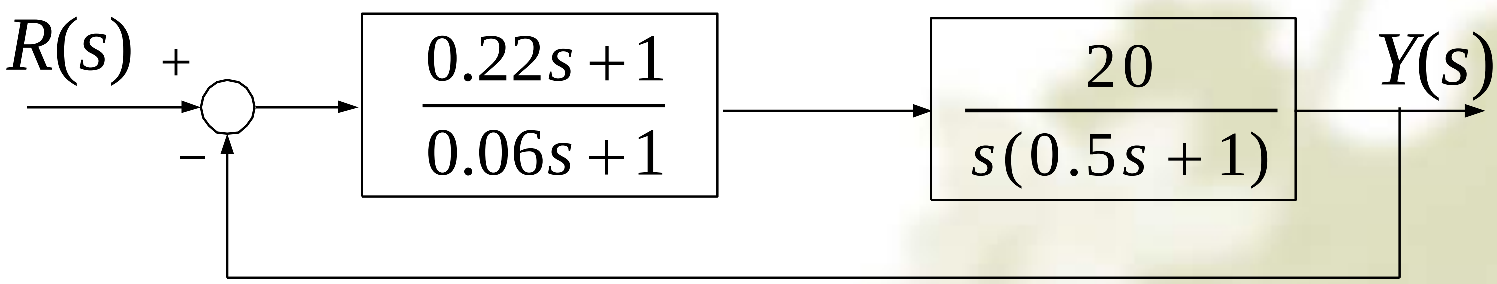
$$5.7 = 40(\lg\omega_c' - \lg\omega_c) = 40\lg\left[\frac{\omega_c'}{\omega_c}\right]$$

$$\because \quad \omega_c = 6.3 \qquad \therefore \quad \omega_c' = 8.8$$

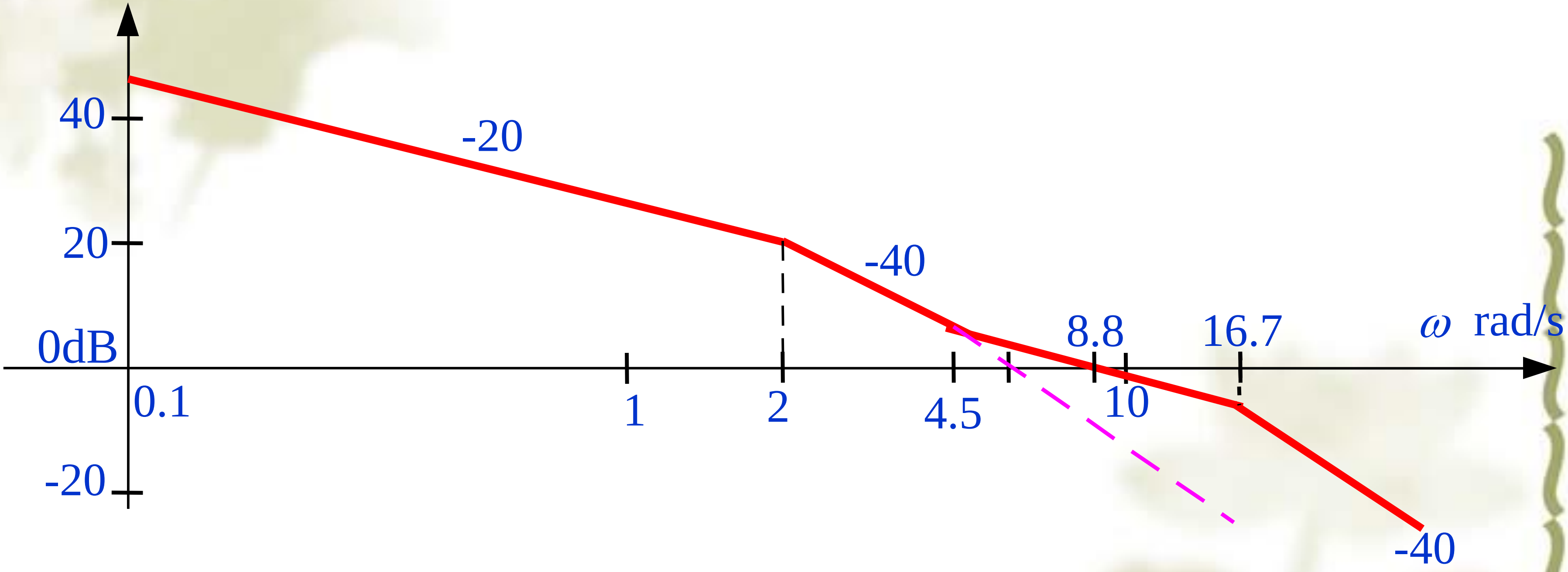
$$\omega_m^2 = \frac{1}{\alpha T} \cdot \frac{1}{T} \qquad \omega_m = \frac{1}{T\sqrt{\alpha}} \qquad T = \frac{1}{\omega_m \sqrt{\alpha}} = \frac{1}{8.8 \times \sqrt{3.7}} = 0.06$$

$$\alpha T = 0.22$$

$$G_c(s) = \frac{\alpha T s + 1}{Ts + 1} = \frac{0.22s + 1}{0.06s + 1} = 3.7 \frac{s + 4.5}{s + 16.7}$$



Bode plot of the compensated system:



Phase margin of the compensated system:

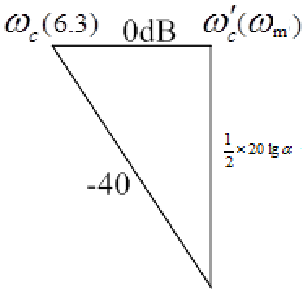
$$\gamma = 180^0 - 90^0 - tg^{-1}(0.5\times 8.8) - tg^{-1}(0.06\times 8.8) + tg^{-1}(0.22\times 8.8) = 47.4^0$$

$\gamma = 47.4^0 < 50^0$ Adjust the design (do not need in the exam of this course)

Go back to Φ_m which the controller should provide, and increase it

$$\Phi_m \approx 50^\circ - 18^\circ + 8^\circ = 40^\circ$$

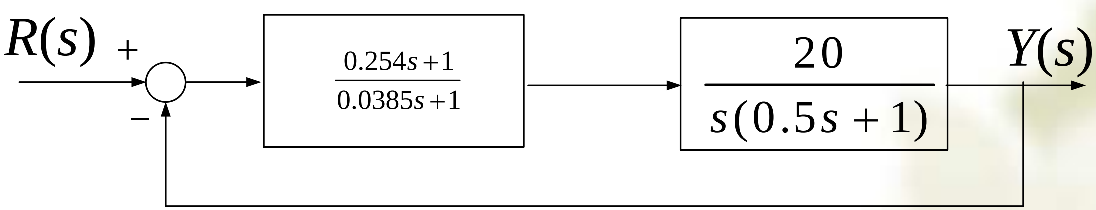
$$\sin \Phi_m = \frac{\alpha - 1}{\alpha + 1} = \sin 40^0 \qquad \alpha = 6.599$$



$$\frac{1}{2} \times 20\lg\alpha = 40(\lg\omega'_c - \lg\omega_c) \qquad \omega'_c = 10.10 = \omega_m$$

$$T = \frac{1}{\omega_m\sqrt{\alpha}} = 0.0385 \qquad \alpha T = 0.254$$

$$G_c(s) = \frac{\alpha Ts + 1}{Ts + 1} = \frac{0.254s + 1}{0.0385s + 1} = 6.599\frac{s + 4.08}{s + 25.97}$$



$$\gamma = 180^0 - 90^0 - tg^{-1}(0.5\times 10.10) - tg^{-1}(0.0385\times 10.10) + tg^{-1}(0.254\times 10.10)$$

$= 58.66^0 > 50^0$ Meet the requirements.

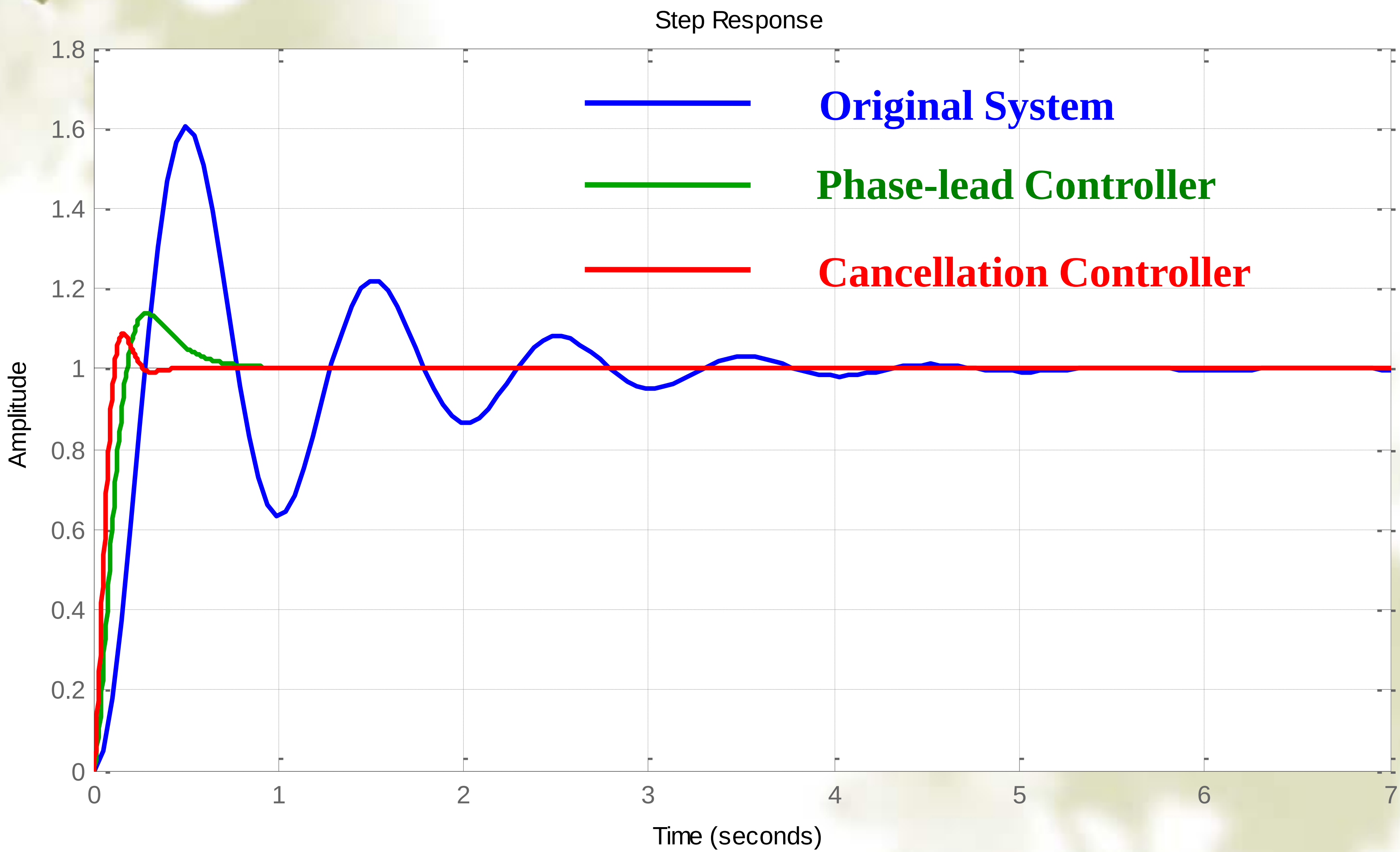
Remarks:

Principle of compensation: make the magnitude plot pass through the real axis with a slop of -20dB/decade, and keep enough distance away from sections with other slops. Trial-and-error is usually used at this time.

If ω_1 and ω_2 are two corner frequencies of the compensator, then $\omega_m = \sqrt{\omega_1 \times \omega_2}$ is where the maximum phase lift is.

Matlab Verification

```
s=tf('s');
k=20;
m=k/(s*(0.5*s+1));
sys=feedback(m,1);
step(sys,'b');
grid on;
hold on;
m1=((0.254*s+1)/(0.0385*s+1))*k/(s*(0.5*s+1));
sys1=feedback(m1,1);
step(sys1,'g');
m2=((0.5*s+1)/(0.033*s+1))*k/(s*(0.5*s+1));
sys2=feedback(m2,1);
step(sys2,'r');
```



Step Response
Original System
Phase-lead Controller
Cancellation Controller
Amplitude
Time (seconds)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
0
1
2
3
4
5
6
7

Example - 5.4

Q: the open-loop transfer function of a unit feedback system is shown as follows. Please design a series compensator so that the ramp-error constant $K_v = 10$

$$G_0(s) = \frac{1}{s(0.5s+1)(0.1s+1)}$$

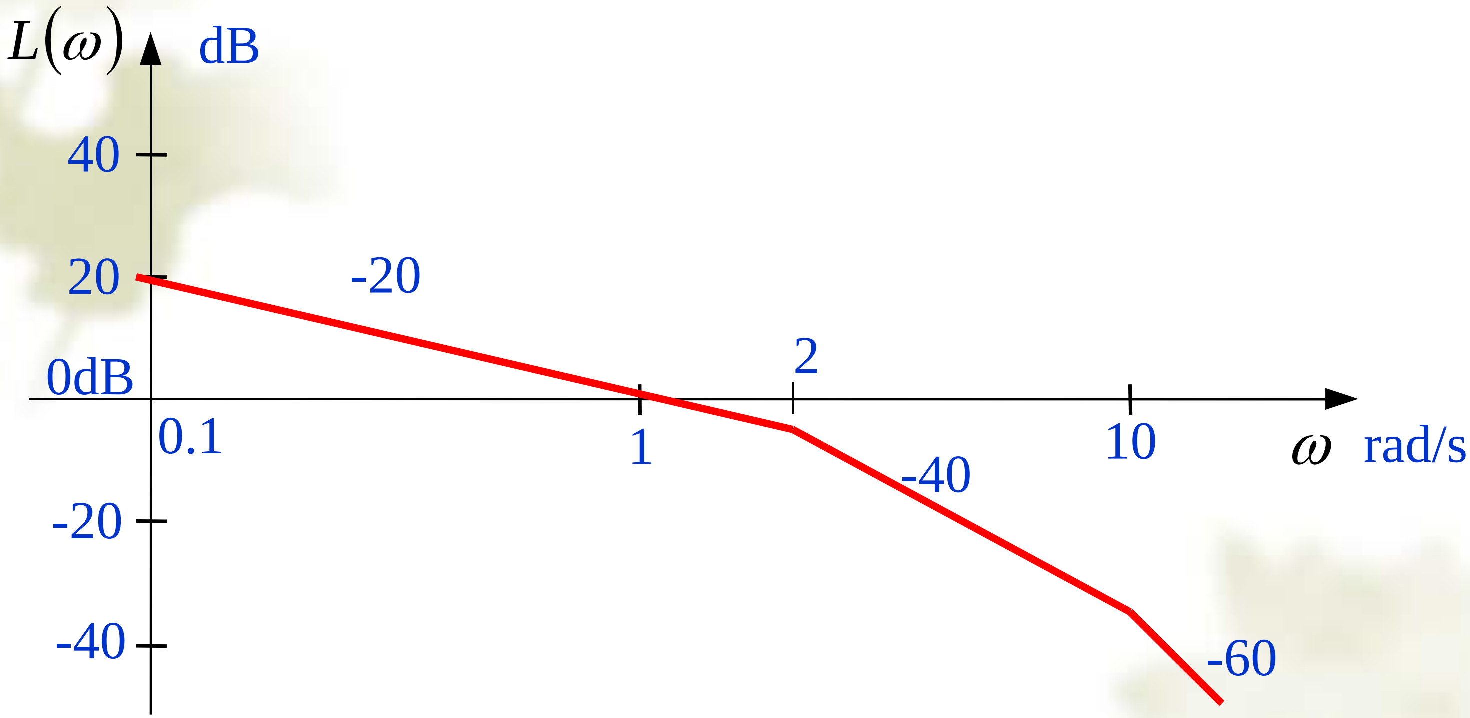
A: 1st step, choose performance specifications of the original system

Low-frequency band: $\omega = 1 \qquad 20\lg 1 = 0$

Corner frequency: $\omega_1 = 2 \qquad slope = -20\,dB$

$\omega_2 = 10 \qquad slope = -20\,dB$

$$\gamma = 180^0 + \Phi(\omega_c) = 180^0 - 90^0 - tg^{-1}(0.5) - tg^{-1}(0.1) = 57.7^0$$



$$\gamma = 180^0 + \Phi(\omega_c) = 180^0 - 90^0 - tg^{-1}(0.5) - tg^{-1}(0.1) = 57.7^0$$

$$K_v = \lim_{s \to 0} sG_0(s) = 1$$ Not satisfy the requirement

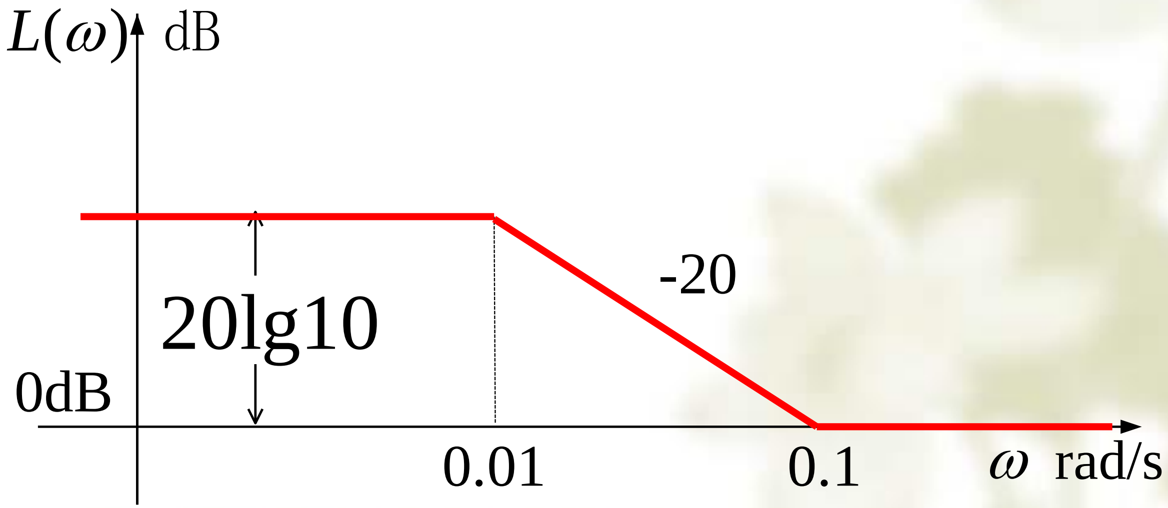
Low-frequency gain should be increased.

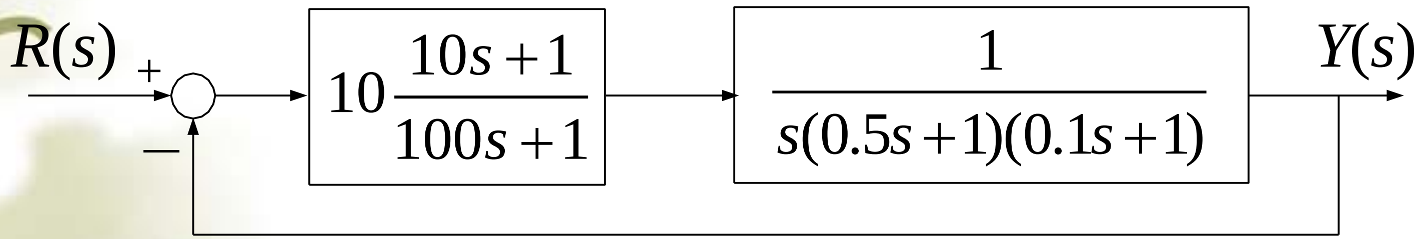
A phase-lag controller is chosen

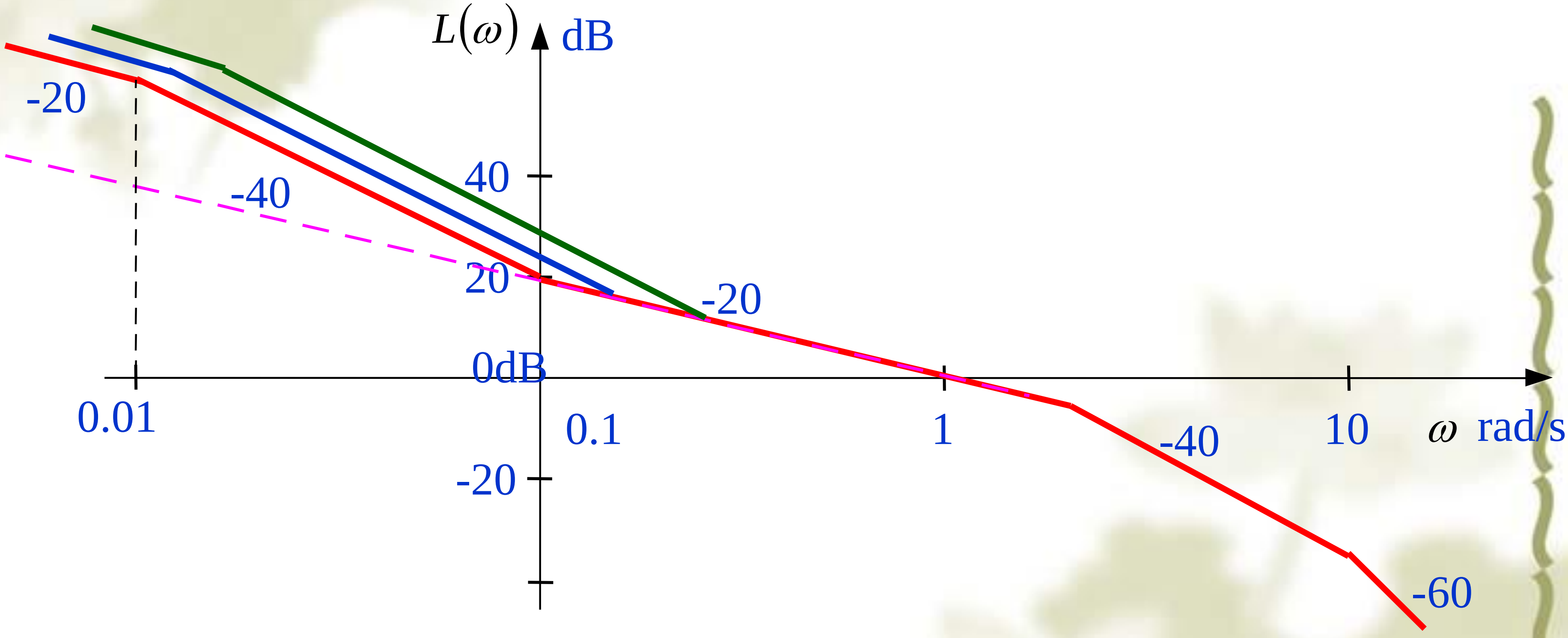
$$G_c(s) = k_c \frac{\alpha T s + 1}{T s + 1} \qquad \alpha < 1 \qquad k_c = \frac{1}{\alpha}$$

Considering that the corner frequencies should be 1/5~1/10 of gain crossover frequency and enough low-frequency gain, the following parameter is chosen:

$$G_c(s) = 10 \cdot \frac{10s + 1}{100s + 1} = \frac{s + 0.1}{s + 0.01}$$





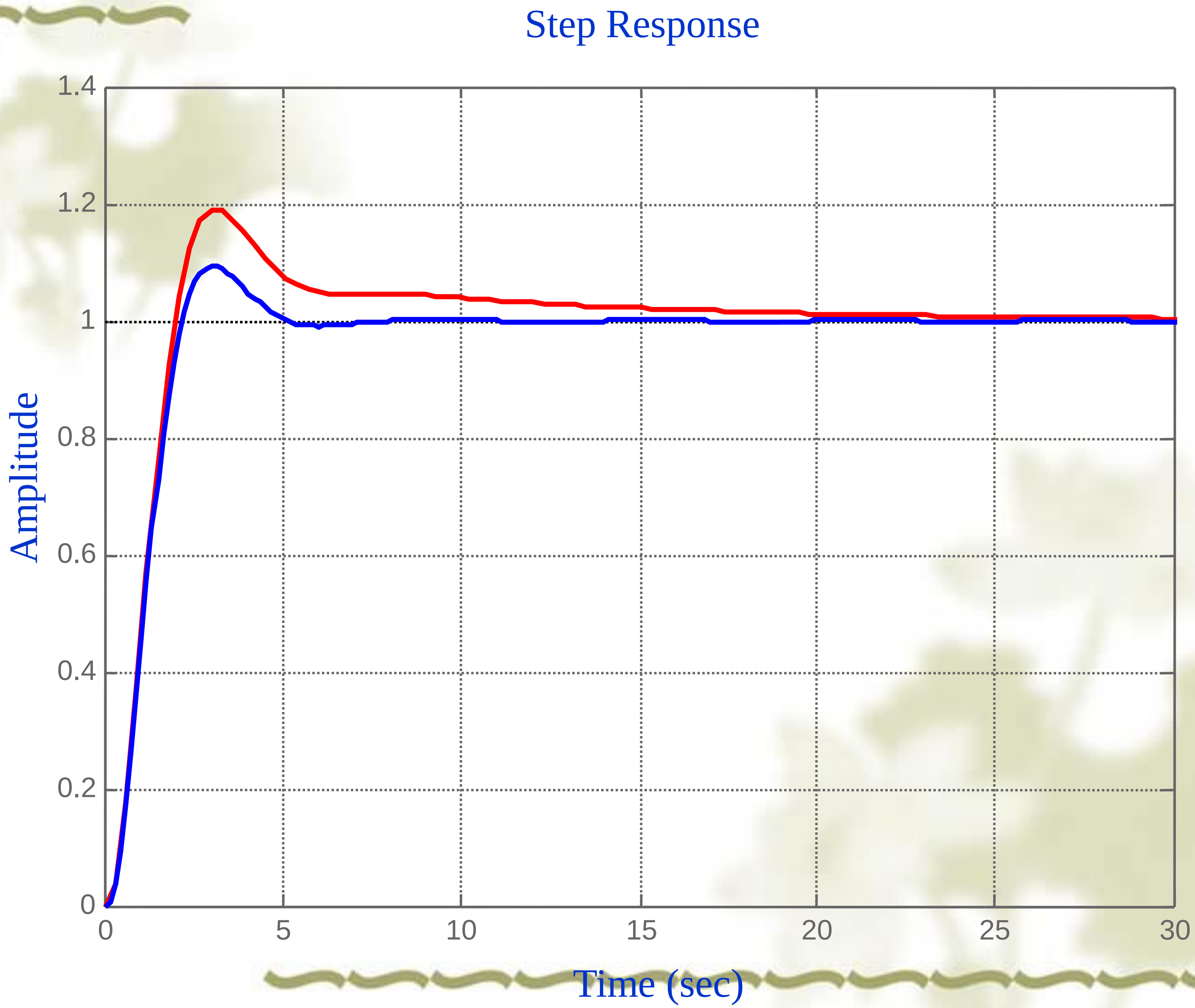


$$\gamma = 180^0 + \Phi(\omega_c) = 180^0 - 90^0 - tg^{-1}(0.5) - tg^{-1}(0.1) - tg^{-1}(100) + tg^{-1}(10) = 52.6^0$$

Solution is not unique!

Time response to unit-step signal

```
s=tf('s');
k=1;
m=k/(s*(0.5*s+1)*(0.1*s+1));
sys=feedback(m,1);
step(sys,’b’);
grid on ;
hold on;
m1=( (s+0.1)/(s+0.01))*k/(s*(0.55*s+1)*(0.1*s+1));
sys1=feedback(m1,1);
step(sys1,’r’);
```

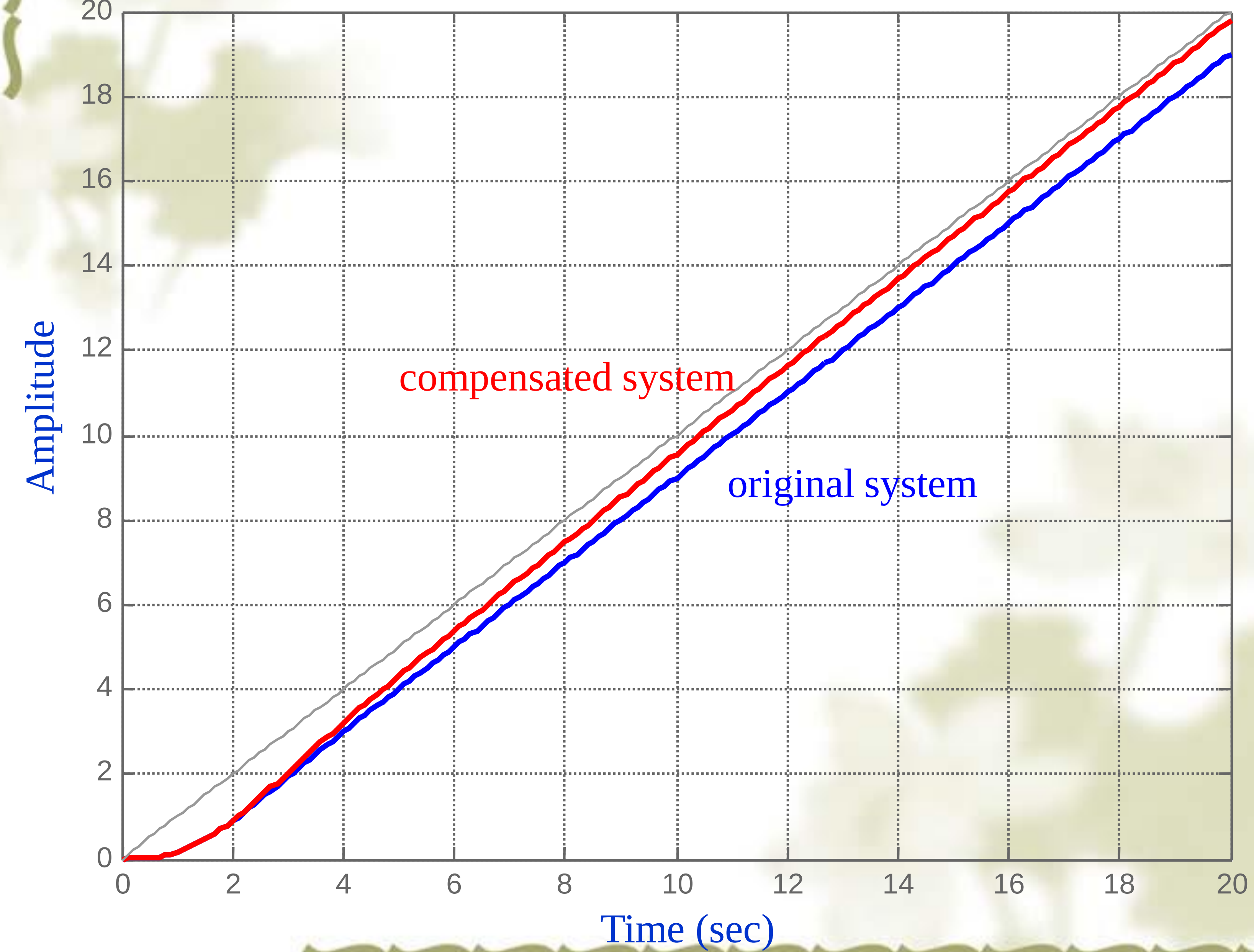


Step Response
Amplitude
Time (sec)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25
30

Time response to unit-ramp signal

```
s=tf('s');
m=1/(s*(0.5*s+1)*(0.1*s+1));
sys=feedback(m,1);
t=0:0.1:20;
lsim(sys,'b',t,t);
grid on;
hold on;
m1=( (s+0.1)/(s+0.01))*1/(s*(0.55*s+1)*(0.1*s+1));
sys1=feedback(m1,1);
lsim(sys1,'r',t,t);
```

Linear Simulation Results



Example - 5.5

Q: the open-loop transfer function of a unit feedback system is shown as follows. Please design a serious compensator so that the following specifications are satisfied: $K_v = 180, \quad \gamma = 45^\circ, \quad \omega_c = 3.5$

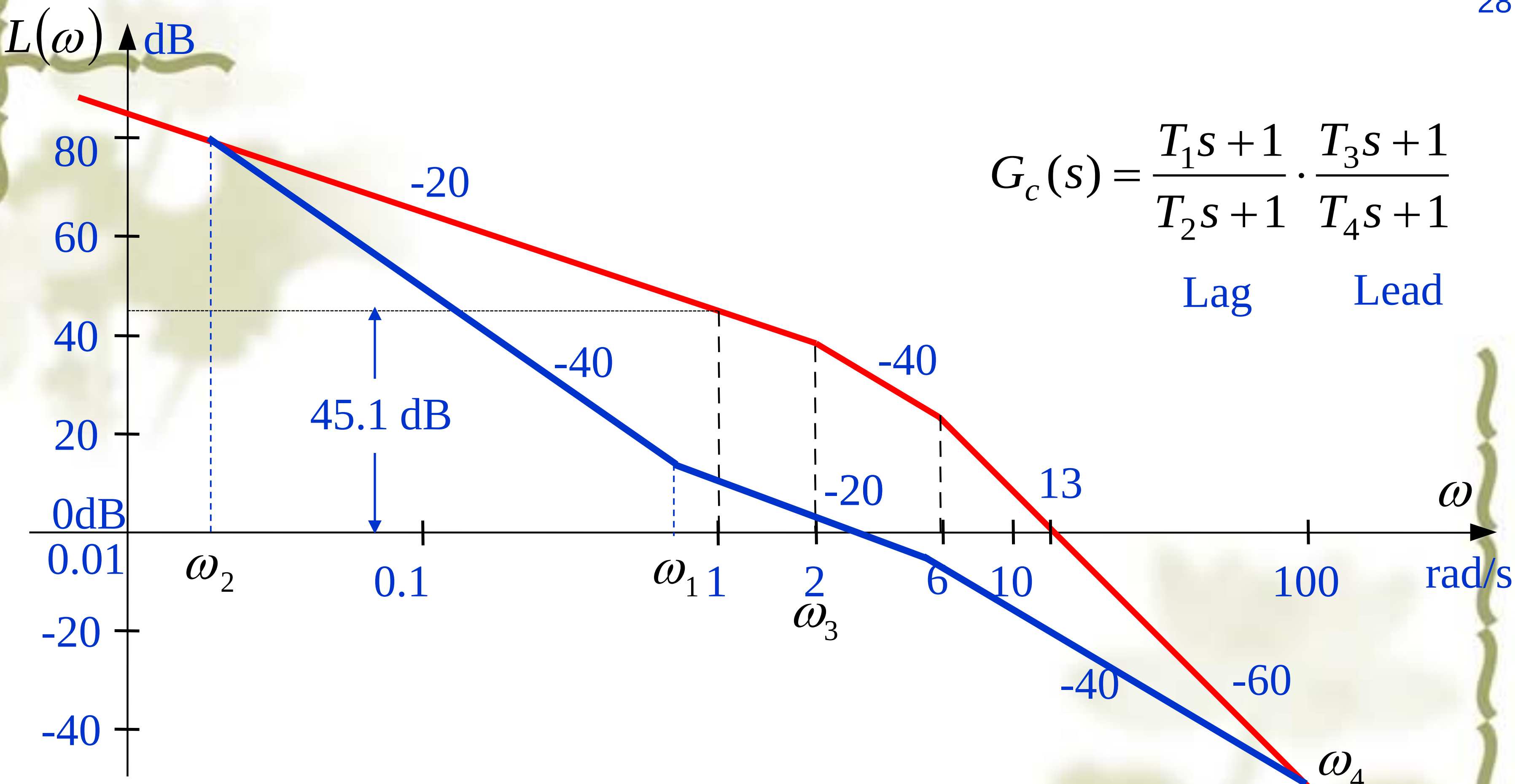
$$G_0(s) = \frac{K_v}{s(0.5s+1)(\frac{1}{6}s+1)}$$

A: 1st step, check the performance specifications of the original system

Low-frequency band: $\omega = 1 \qquad 20\lg 180 = 45.1$

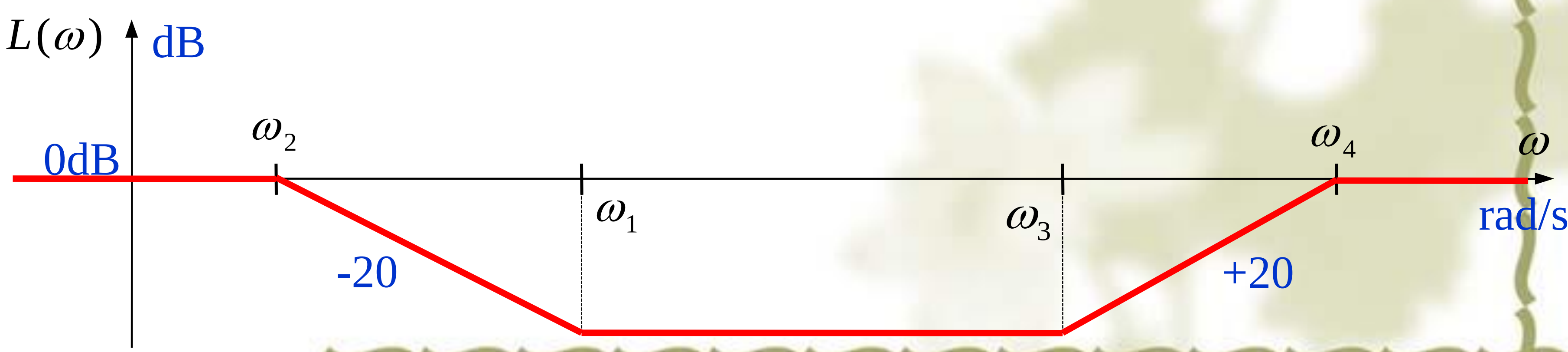
Corner frequency: $\omega_{c1} = 2 \qquad slope = -20dB$

$\omega_{c2} = 6 \qquad slope = -20dB$



$$G_c(s)=\frac{T_1s+1}{T_2s+1}\cdot\frac{T_3s+1}{T_4s+1}$$

Lag Lead



A lead-lag controller is chosen. Starting from the given gain crossover frequency determine the parameter values of the controller.

If the new gain crossover frequency is $\omega_c = 3.5$

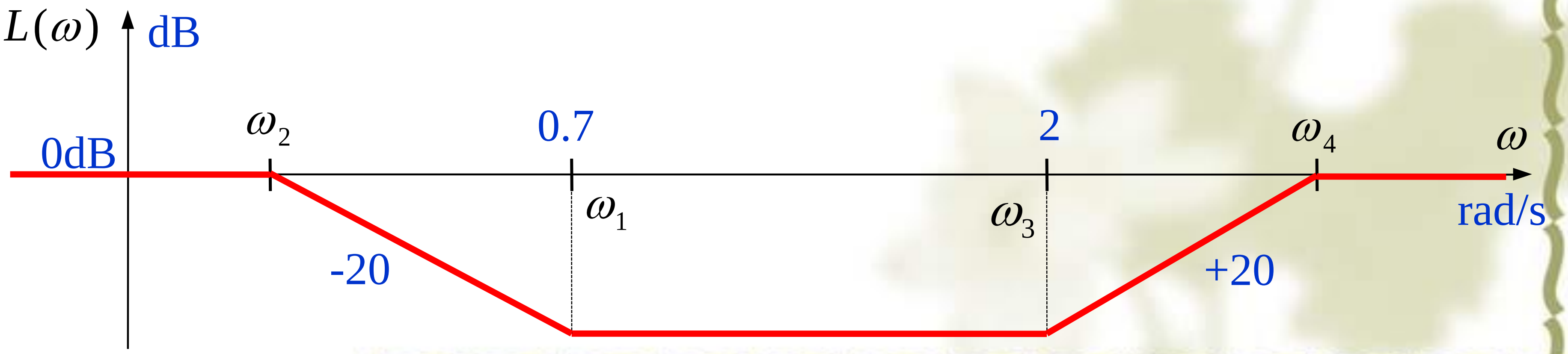
choose the corner frequency of the lag block to be: $\omega_1 = 3.5/5 = 0.7$

The other corner frequency of the lag block satisfies:

$$20(\lg \omega_4 - \lg \omega_3) = 20(\lg 0.7 - \lg \omega_2)$$

set $\omega_3 = 2, \quad \omega_4 = 100$

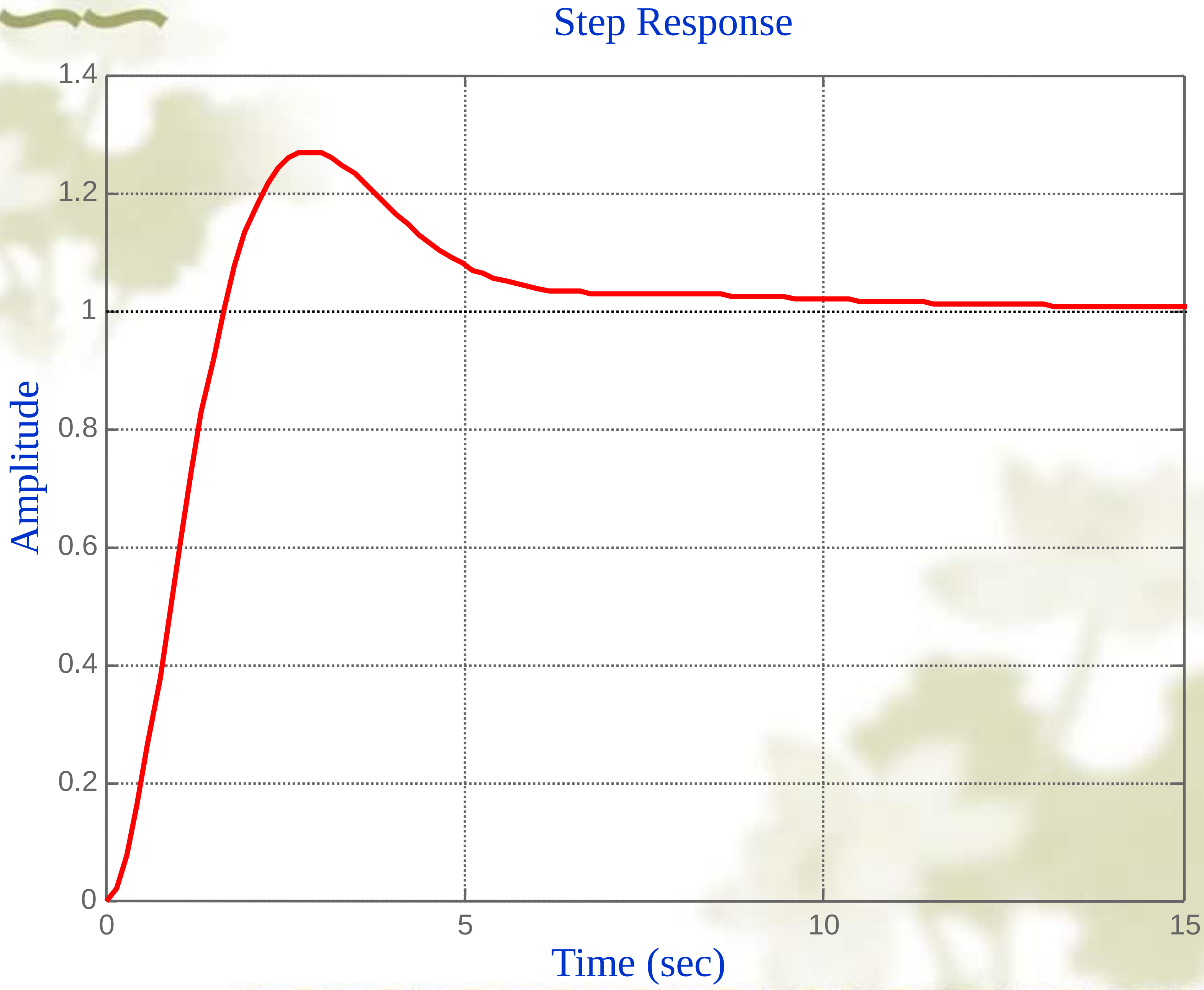
$$\omega_2 = 0.014$$



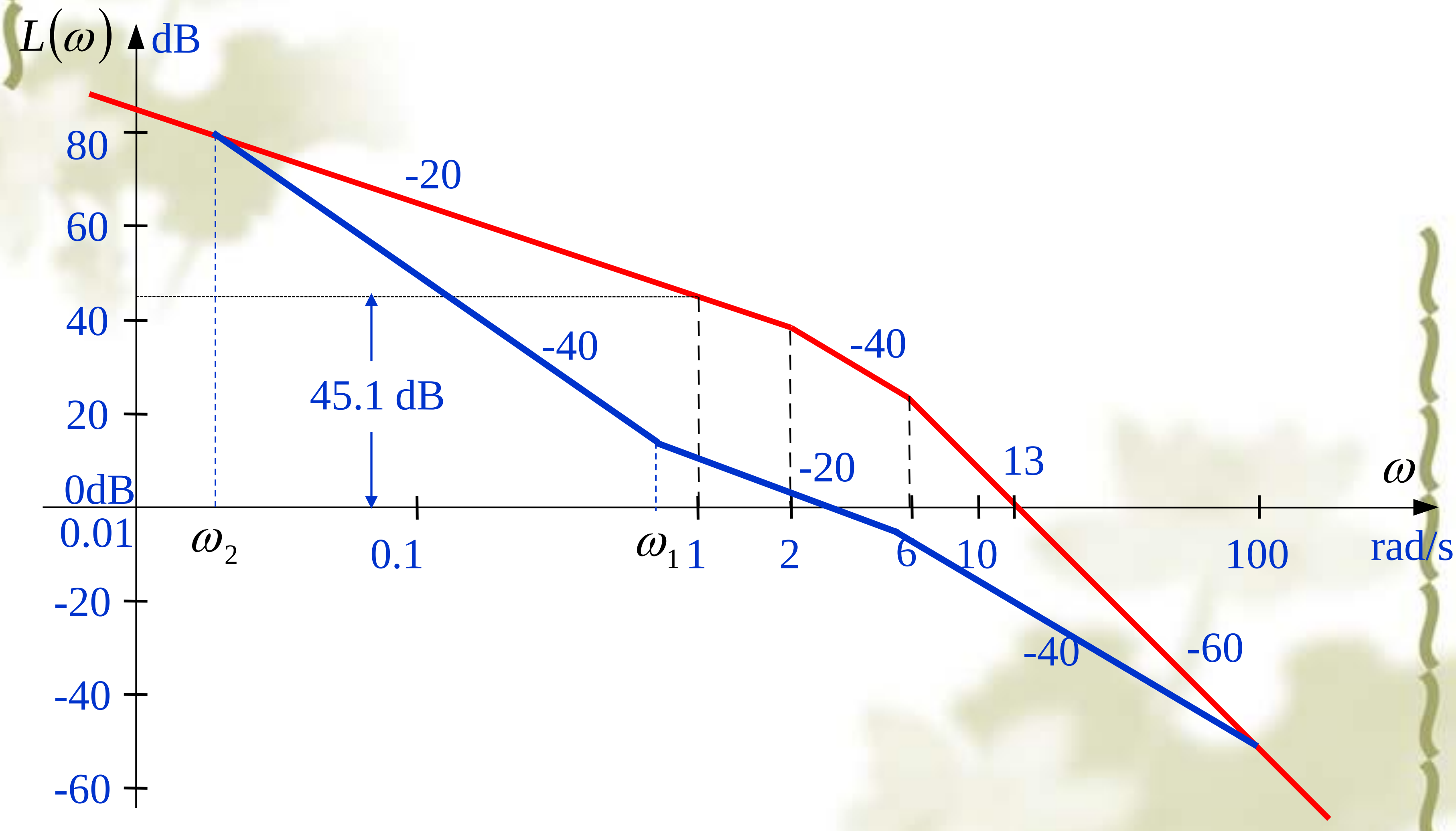
The final controller is:

$$G_c(s) = \frac{\frac{1}{2}s+1}{\frac{1}{100}s+1} \cdot \frac{\frac{1}{0.7}s+1}{\frac{1}{0.014}s+1} = \frac{0.5s+1}{0.01s+1} \cdot \frac{1.43s+1}{71.4s+1}$$

```
s=tf('s');
k=180;
m=((0.5*s+0.1)/(0.01*s+1))*((1.43*s+1)/(71.4*s+1))
*k/(s*(0.5*s+1)*(1.6667*s+1));
sys=feedback(m,1);
step(sys,15);
grid on;
```



Step Response
Amplitude
Time (sec)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15



$$\gamma = 180^0 - 90^0 + tg^{-1}(3.5\times1.43) - tg^{-1}(3.5\times0.01) - tg^{-1}(3.5\times71.4) - tg^{-1}(3.5\times\frac{1}{6}) = 46.7^0$$

Example - 5.6

The open-loop transfer function of a unit feedback system is shown as follows.

$$G_0(s) = \frac{Ke^{-0.03s}}{s(s+1)(0.2s+1)}$$

Requirements: $K = 30, \quad \omega_c \geq 2.5, \quad \gamma = 40^\circ \pm 5^\circ$

Q: 1. Determine the type of the compensator (phase-lead, phase-lag, or lead-lag controller).

2. If the Lead-Lag controller is applied and shown as follows.

$$G_c(s) = \frac{(2s+1)(s+1)}{(20s+1)(0.01s+1)}$$

Calculate ω_c and γ ; check the requirements.

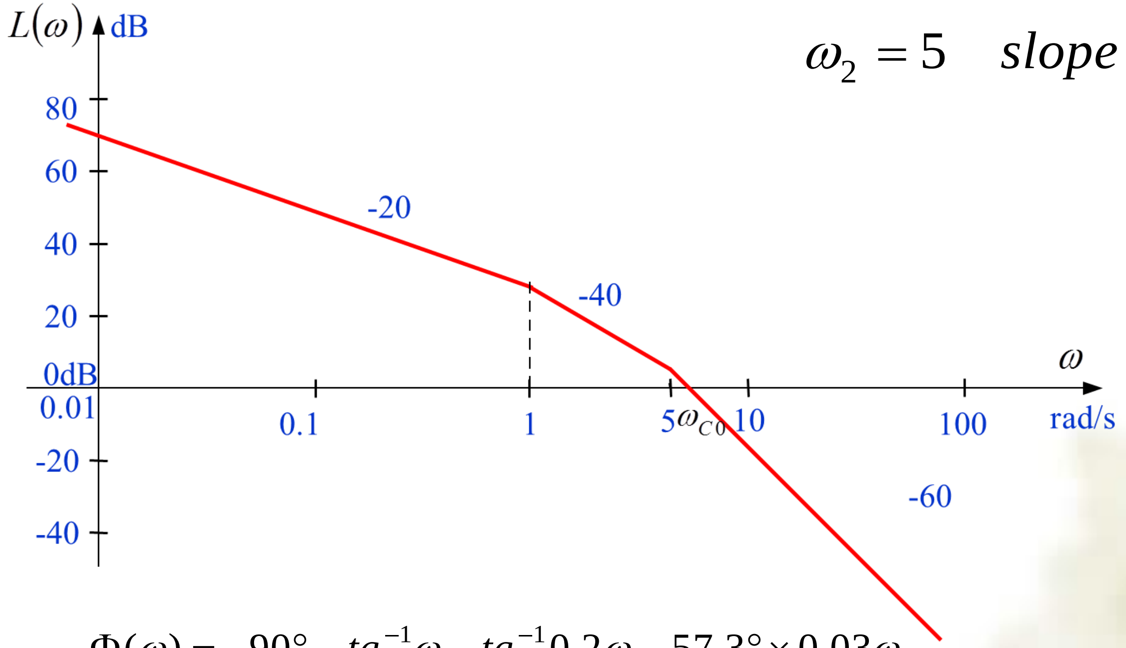
A: 1. *K*=30, the open loop transfer function is $G_0(s)=\dfrac{30e^{-0.03s}}{s(s+1)(0.2s+1)}$

Its magnitude Bode plot is the same as $G_0(s)=\dfrac{30}{s(s+1)(0.2s+1)}$

Low-frequency band: $\omega=0.01 \quad 20\lg 3000=69.5$

Corner frequency: $\omega_1=1 \quad slope=-40dB$

$\omega_2=5 \quad slope=-60dB$



$$\omega_{C0}=\sqrt[3]{150}=5.3$$

$$\Phi(\omega)=-90^\circ-tg^{-1}\omega-tg^{-1}0.2\omega-57.3^\circ\times0.03\omega$$

$$\gamma_0=180^\circ+\Phi(\omega_{C0})=90^\circ-tg^{-1}\omega_{C0}-tg^{-1}0.2\omega_{C0}-57.3^0\times0.03\omega_{C0}=-45^\circ$$

The system is not stable.
If a phase-lead controller is chosen, Φ_m should be more than 85° !
If a phase-lag controller is chosen，

not considering the phase lag it brings, let $\omega_c = 2.5$

$$\gamma = 180° + \Phi(\omega_C) = 90° - tg^{-1}\omega_C - tg^{-1}0.2\omega_C - 57.3^0 \times 0.03\omega_C = -9.1°$$

could not meet $\omega_c \geq 2.5$ and $\gamma = 40^\circ \pm 5^\circ$ at the same time.

So a lead-lag controller is chosen.

2. If the lead-lag controller is applied $G_c(s) = \dfrac{(2s+1)(s+1)}{(20s+1)(0.01s+1)}$

The open loop transfer function is

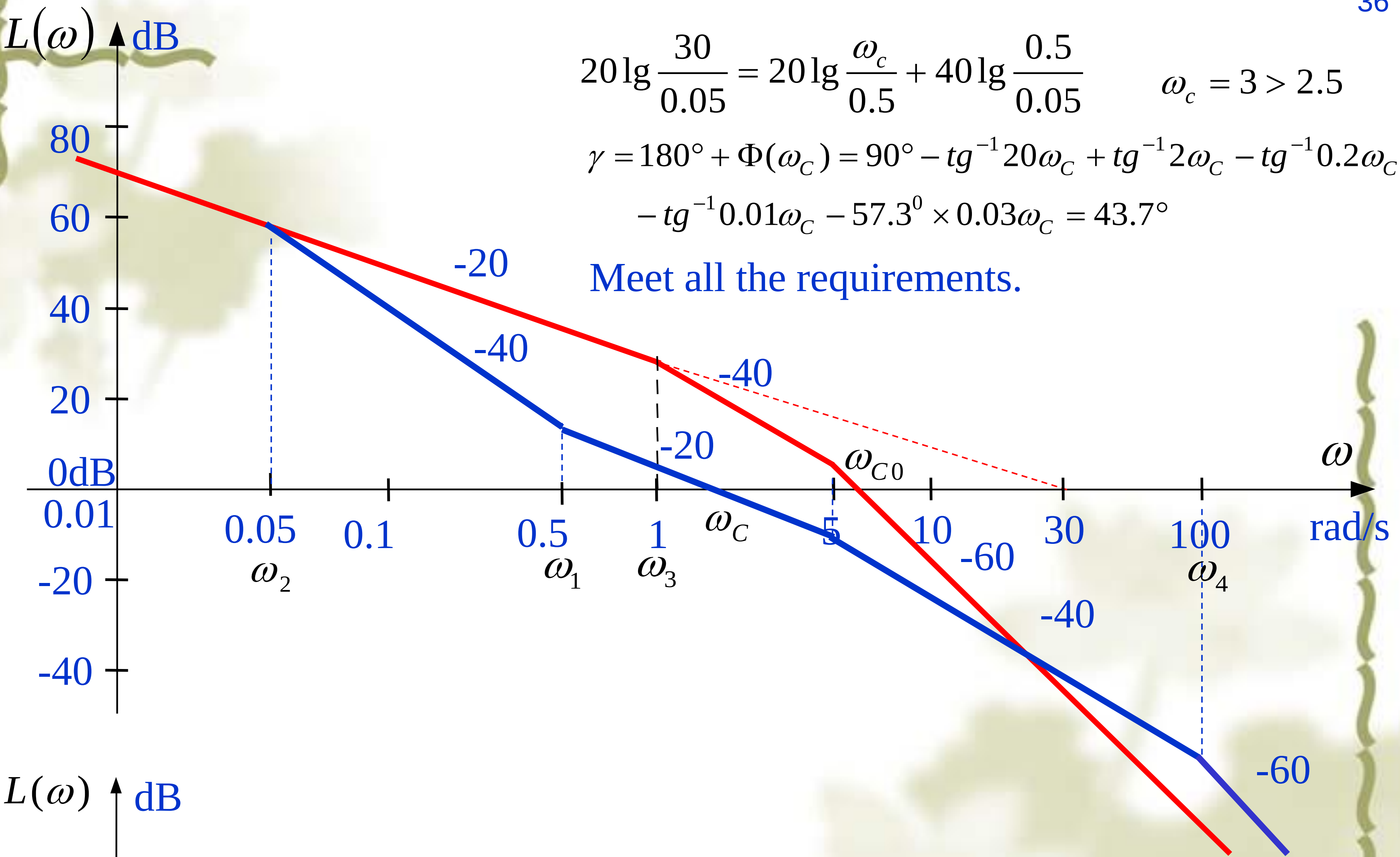
$$G_c(s)G_0(s) = \frac{30(2s+1)e^{-0.03s}}{s(20s+1)(0.2s+1)(0.01s+1)}$$

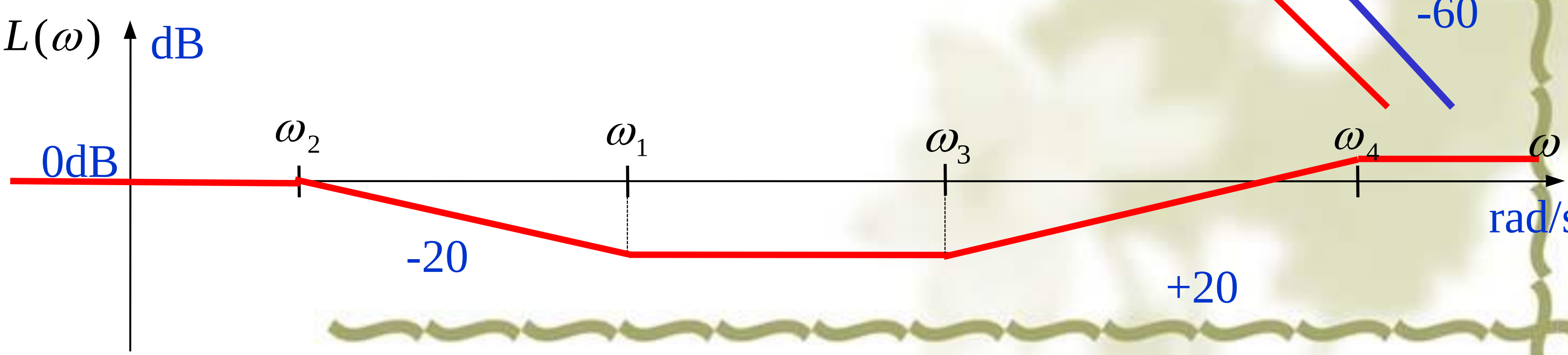
Draw its magnitude plot and calculate.

$$20\lg\frac{30}{0.05} = 20\lg\frac{\omega_c}{0.5} + 40\lg\frac{0.5}{0.05} \qquad \omega_c = 3 > 2.5$$

$$\gamma = 180^\circ + \Phi(\omega_C) = 90^\circ - tg^{-1}20\omega_C + tg^{-1}2\omega_C - tg^{-1}0.2\omega_C$$
$$- tg^{-1}0.01\omega_C - 57.3^0 \times 0.03\omega_C = 43.7^\circ$$

Meet all the requirements.





Example - 5.7

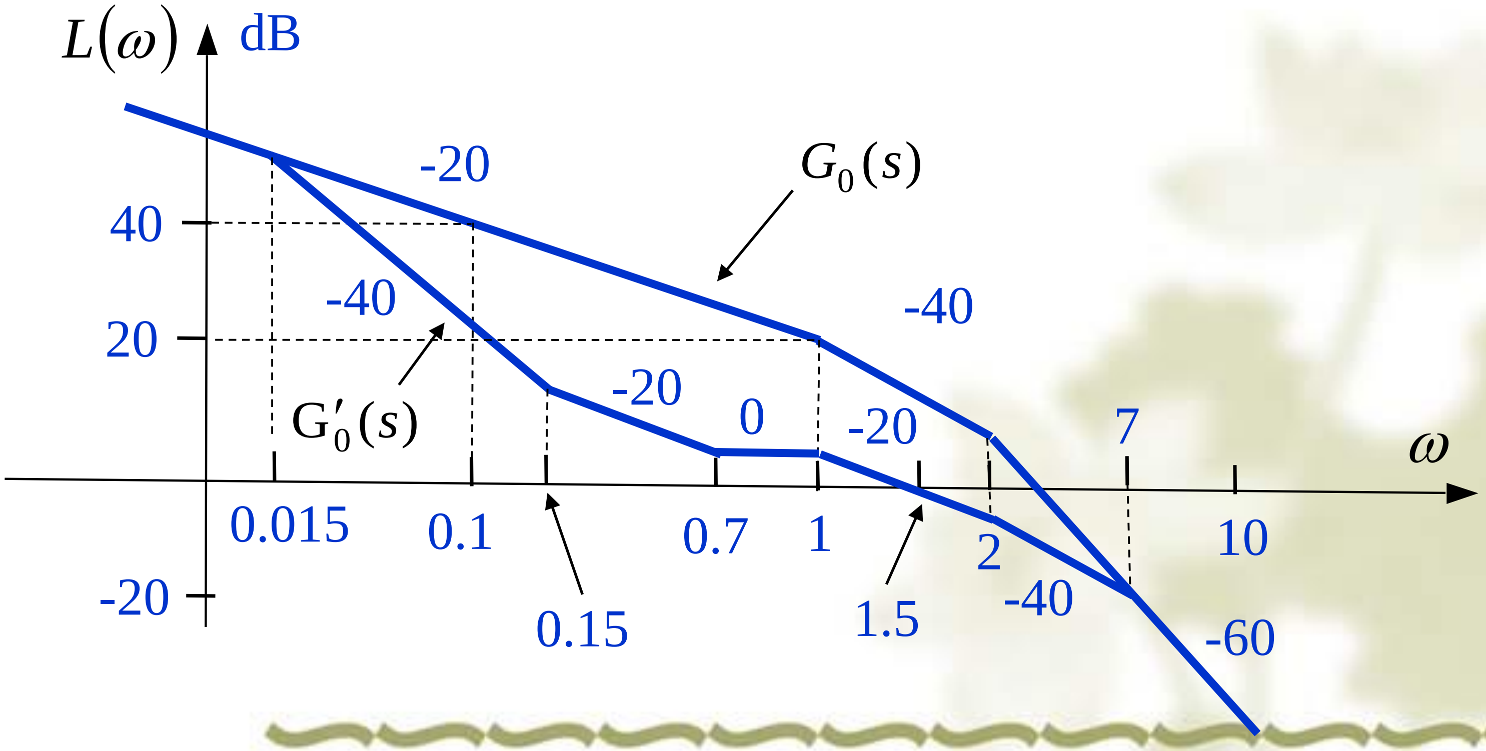
The original open-loop transfer function of a minimum-phase system is shown as follows：$G_0(s)$

The compensated open-loop transfer function is shown as follows：$G_0'(s)$

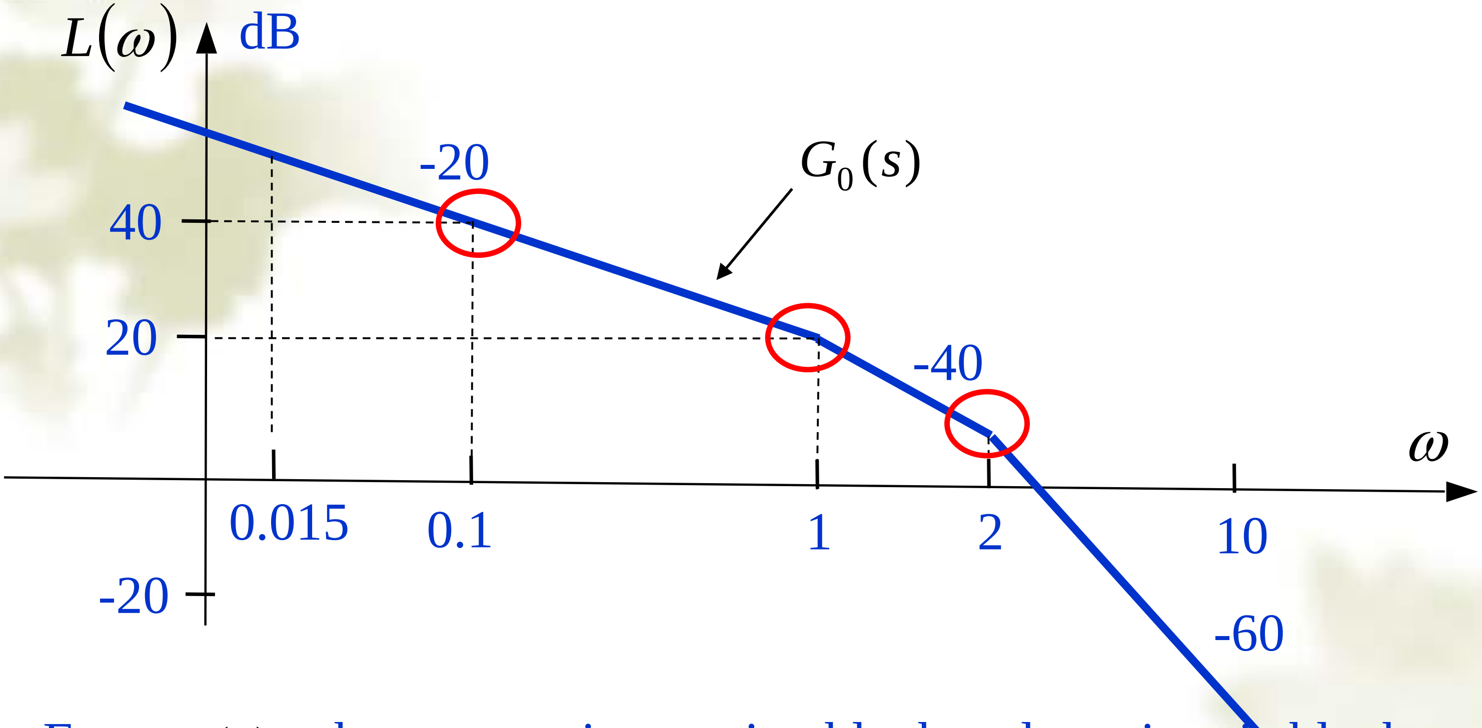
Q: 1. Find the expression of $G_0(s)$ and $G_0'(s)$

2. Find the transfer function of the compensator

3. Calculate the phase margin and gain margin of $G_0'(s)$



1. Find the expression of $G_0(s)$ and $G'_0(s)$



For $G_0(s)$, there are an integration block and two inertia blocks

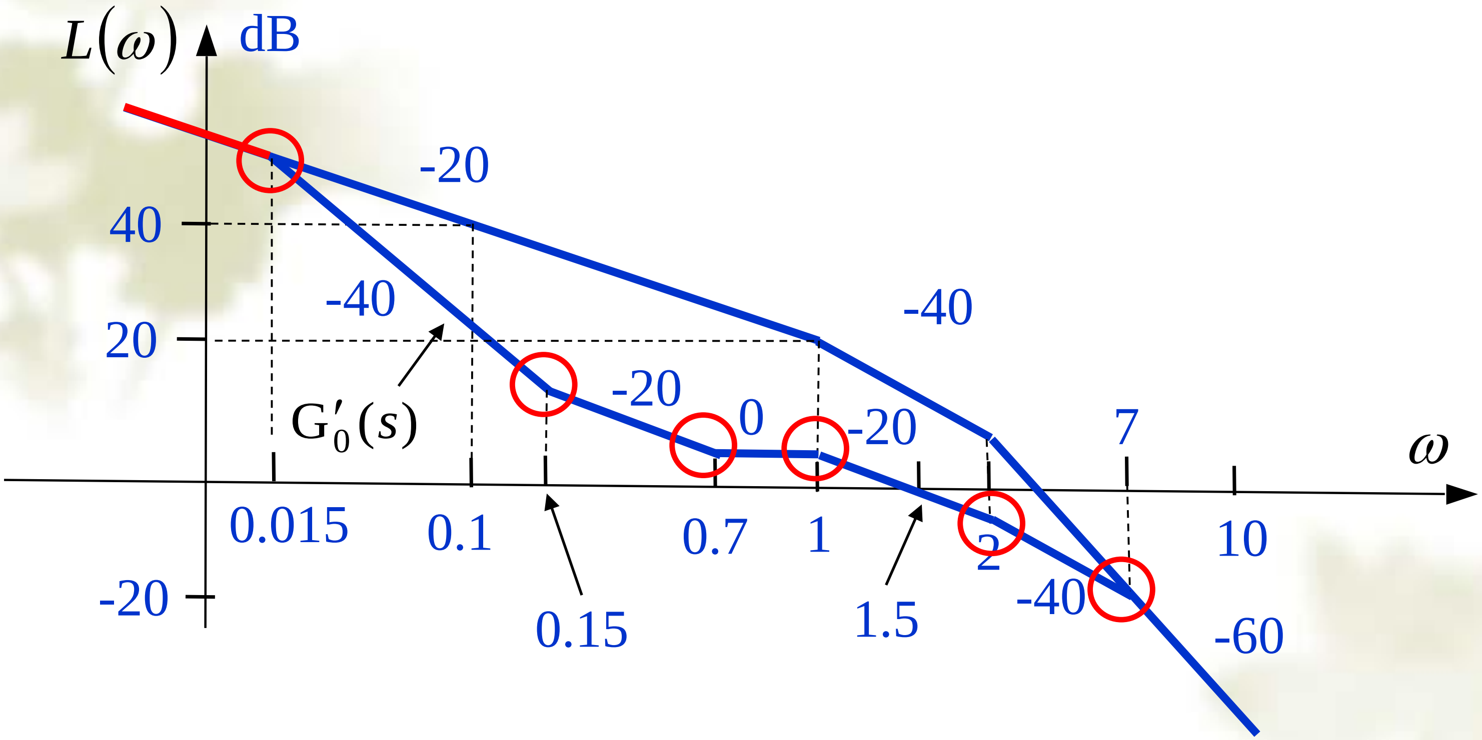
（1） $40 = 20\log\left(\dfrac{K}{0.1}\right) \quad \Rightarrow \quad K=10$

（2） $\dfrac{1}{T_1} = \omega_1 = 1 \quad \Rightarrow \quad T_1 = 1$

$\dfrac{1}{T_2} = \omega_2 = 2 \quad \Rightarrow \quad T_2 = 0.5$

$$G_0(s) = \frac{10}{s(s+1)(0.5s+1)}$$

1. Find the expression of $G_0(s)$ and $G_0'(s)$

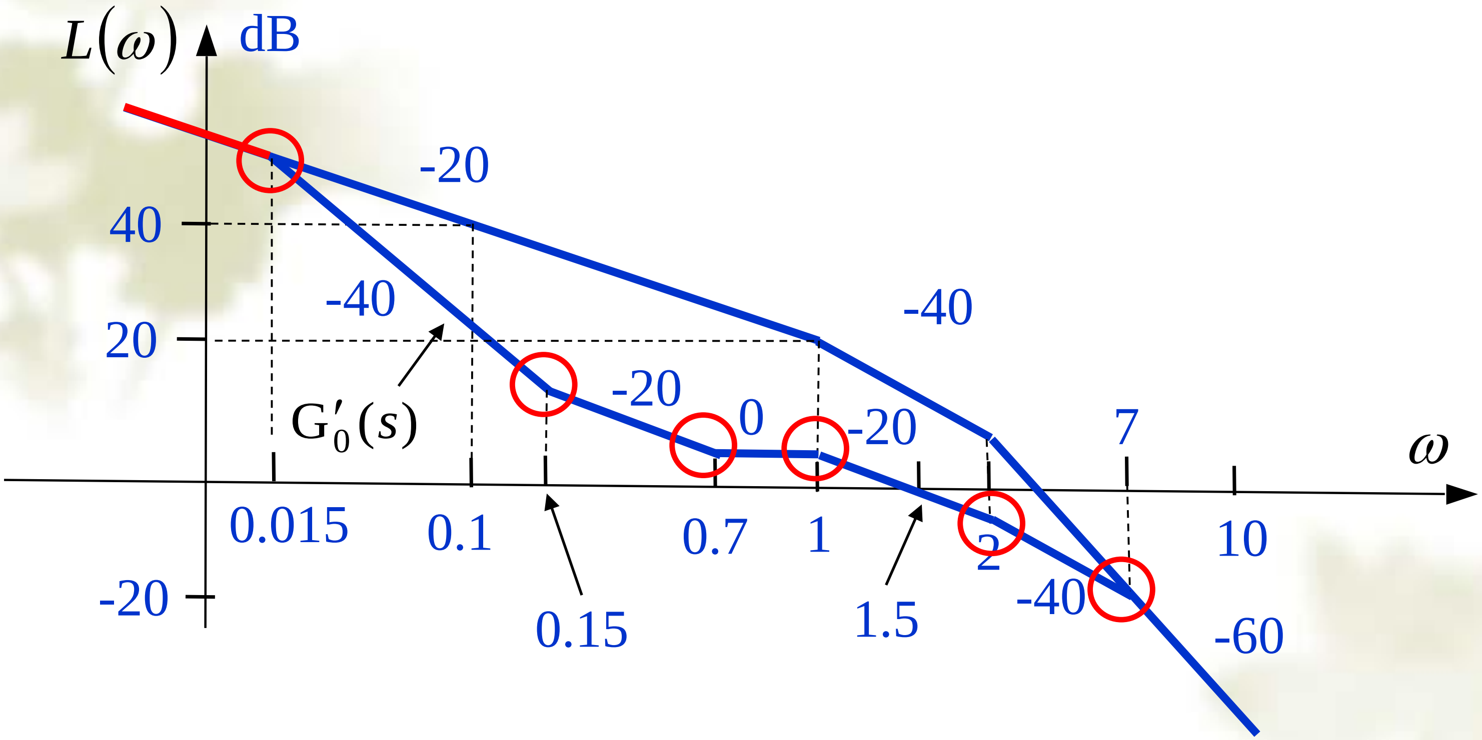


(1) For $G_0'(s)$, the low-frequency section is the same as $G_0(s)$
So, $G_0'(s)$ has an integration block and same $K'=10$

(2)

$$\begin{matrix} \omega_1=0.015 & \omega_4=1 & & T_1=66.7 & T_4=1 \\ \omega_2=0.15 & \omega_5=2 & \Rightarrow & T_2=6.67 & T_5=0.5 \\ \omega_3=0.7 & \omega_6=7 & & T_3=1.43 & T_6=0.143 \end{matrix}$$

1. Find the expression of $G_0(s)$ and $G'_0(s)$



$$G'_0(s)=\frac{10}{s}\frac{(6.67s+1)(1.43s+1)}{(66.7s+1)(s+1)(0.5s+1)(0.143s+1)}$$

2. Find the transfer function of the compensator

In order to find the transfer function of the compensator

We use $G_0'(s) \div G_0(s)$

Thus, transfer function of the compensator is

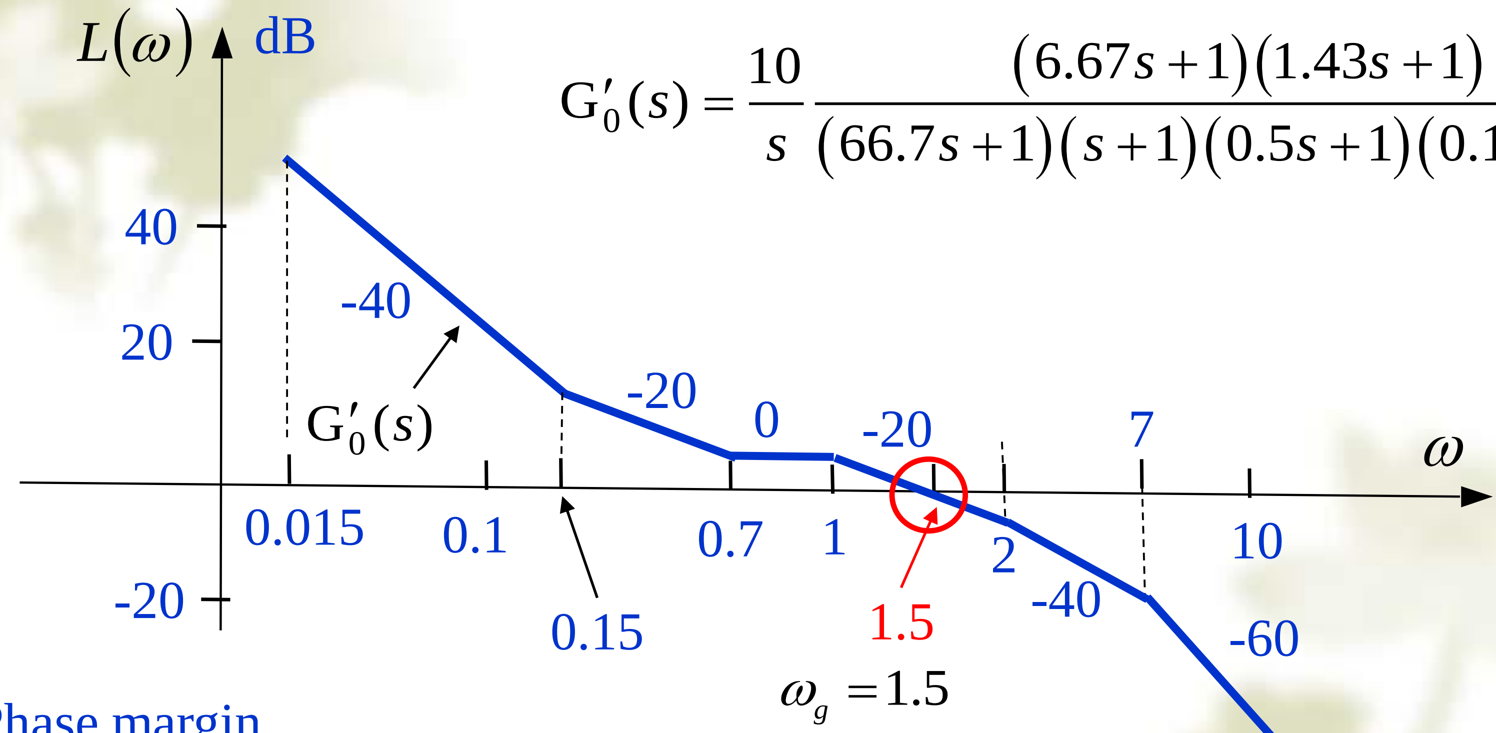
$$F(s) = \frac{10}{s}\frac{(6.67s+1)(1.43s+1)}{(66.7s+1)(s+1)(0.5s+1)(0.143s+1)} \div \frac{10}{s(s+1)(0.5s+1)}$$

$$= \frac{(6.67s+1)(1.43s+1)}{(66.7s+1)(0.143s+1)}$$

This is a Lead-Lag compensator

3. Calculate the phase margin and gain margin of $G_0'(s)$

$$G_0'(s)=\frac{10}{s}\frac{(6.67s+1)(1.43s+1)}{(66.7s+1)(s+1)(0.5s+1)(0.143s+1)}$$



$\omega_g = 1.5$

Phase margin

$$\gamma = 180^\circ + \angle G_0'(\omega_g) = 180^\circ - 90^\circ + \arctan(6.67\omega_g) + \arctan(1.43\omega_g)$$
$$- \arctan(66.7\omega_g) - \arctan(\omega_g)$$
$$- \arctan(0.5\omega_g) - \arctan(0.143\omega_g)$$

$\gamma = 44.5^\circ$

3. Calculate the phase margin and gain margin of $G_0'(s)$

$$\angle G_0'(\omega_g) = -90^\circ + \arctan(6.67\omega_g) + \arctan(1.43\omega_g) - \arctan(66.7\omega_g) - \arctan(\omega_g) - \arctan(0.5\omega_g) - \arctan(0.143\omega_g)$$

Let $\angle G_0'(\omega_g) = -180^\circ$

We get $\omega_g = 3.91$ (by try and error)

Gain margin

$$k_g = -20\lg\left|G_0'(\omega_k)\right| = -20\lg 0.143 = 16.89\text{dB}$$

Wrap-up

- How to design a controller using time-domain method;

- How to design a controller using frequency-domain method;

投票 最多可选1项

设置

Review questions

If a PD controller is so designed that the characteristic-equation roots have better damping than the original system, the maximum overshoot of the system is always reduced

A T

B F

提交

Review questions

- If a PD controller is so designed that the characteristic-equation roots have better damping than the original system, the maximum overshoot of the system is always reduced

(T) (F)

Review questions

- A system compensated with a PD controller is usually more robust than a system compensated with a PI controller.

A T

B F

提交

Review questions

- A system compensated with a PD controller is usually more robust than a system compensated with a PI controller.

(T) (F)

Review questions

- The maximum phase that is available from single-stage phase-lead control is 90 degree.

A T

B F

提交

Review questions

- The maximum phase that is available from single-stage phase-lead control is 90 degree.

(T) (F)

投票 最多可选1项

设置

Review questions

- The design objective of the phase-lead controller is to place the maximum phase lead at the new gain-crossover frequency.

A T

B F

提交

Review questions

- The design objective of the phase-lead controller is to place the maximum phase lead at the new gain-crossover frequency.

(T) (F)

State Feedback Control Systems

Chapter 6

Review of the Last Chapter

- Three steps involved in control system design
- Control system configuration
- Typical controller types
- Parameter value determination
- Design controller using time-domain method
- Design controller using frequency-domain method

Introduction to this Chapter

- Controllability concept and condition
- Observability concept and condition
- Controllability & observability versus zero-pole cancellation
- Controllability & observability decomposing
- State-feedback and pole-placement design
- Observer and state-feedback control system with an observer

Outlines to this Class

- **Definition of controllability**
- **Controllability condition**

Advantages of State Equation

- Be able to depict complex systems
- Be able to provide internal information of a control system
- When certain conditions are satisfied, the poles of a system can be arbitrarily placed so that desirable performance can be obtained

Controllability

Definition:

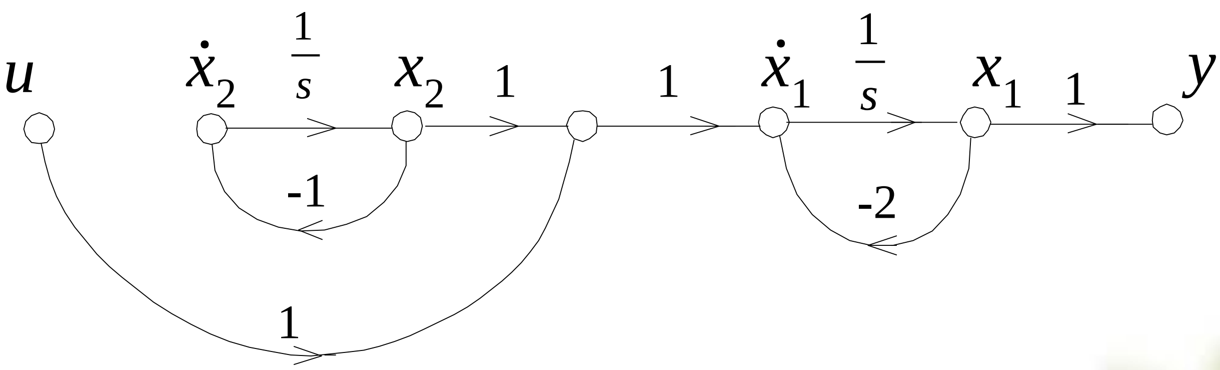
A process is said to be completely controllable if every state variable of the process can be controlled to reach a certain objective in finite time by some unconstrained control.

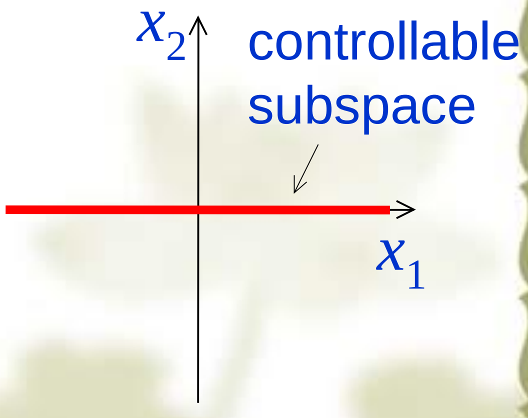
Intuitively:

If any one of the state variables is independent of the control, there would be no way of driving this particular state variable to a desired state in finite time by means of a control effort. Therefore, the particular state is said to be uncontrollable.

Examples of Uncontrollable

Example 6.1 Find the given system is controllable or uncontrollable.

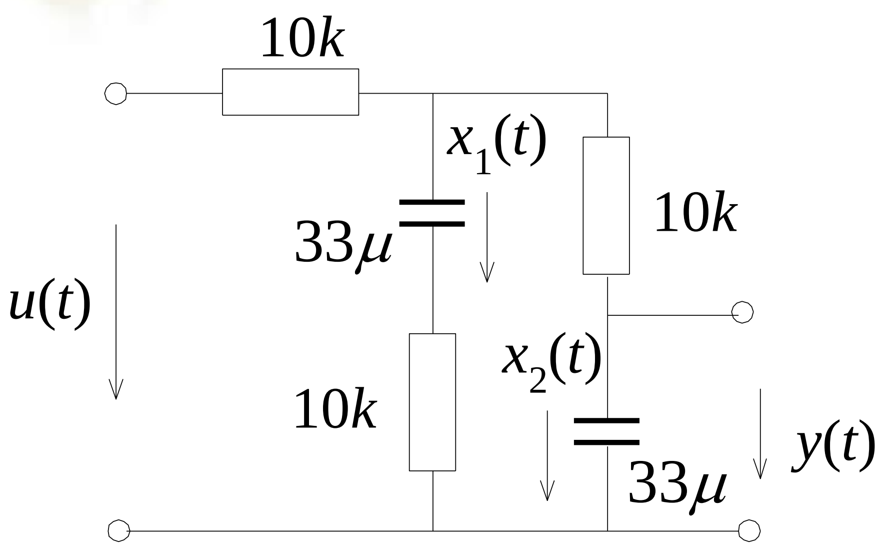


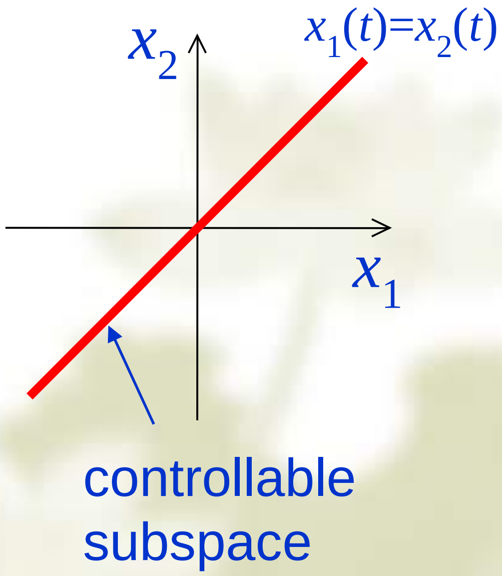


x_2 is independent of the control u

Examples of Uncontrollable

Example 6.2 Find the given system is controllable or uncontrollable.





Controllability Condition

For the system described by the given state equation

$$\dot{X}(t) = AX(t) + BU(t)$$
$$Y(t) = CX(t) + DU(t)$$

$X(t): \quad n\times 1$

$U(t): \quad r\times 1$

$Y(t): \quad p\times 1$

to be completely controllable, it is necessary and sufficient that the following $n\times nr$ controllability matrix S has a rank of n:

$$S = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

Controllability Example

Example 6.3 Find the given system is controllable or uncontrollable.

$$\dot{X} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

A:

$$B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad AB = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \qquad S = [B, AB] = \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix}$$

$$rank(S) = 2$$ system is controllable

Controllability Example

Example 6.4

Find the given system is controllable or uncontrollable.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

A:

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} \qquad AB = \begin{bmatrix} -3 & 3 \\ 0 & 0 \\ -2 & 0 \end{bmatrix} \qquad A^2B = \begin{bmatrix} 9 & -9 \\ 0 & 0 \\ 2 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & -1 & -3 & 3 & 9 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 & 2 & 0 \end{bmatrix}$$

$$rank(S) = 2 < 3$$

system is uncontrollable

Proof of the Controllability Condition

To prove the controllability of a system, we need to demonstrate that by some ways we can make the system reach any arbitrary point. For a linear time-invariant system, we can always set the coordinate origin to be the select destination point. Therefore, we just need to prove that the system can be driven to the origin by certain control in finite time.

For a LTI system: $\dot{X}(t) = AX(t) + BU(t)$

its state response is: $X(t) = e^{At}X(0) + \int_0^t e^{A(t-\tau)}BU(\tau)d\tau$

Assume: $t = T \quad X(T) = 0$ ⇨ Prove $U(t)$ does exist!

$$e^{AT}X(0) = -\int_0^T e^{A(T-\tau)}Bu(\tau)d\tau$$

$$X(0) = -\int_0^T e^{-A\tau}BU(\tau)d\tau$$

Cayley-Hamilton Theorem

For a n$\times$n square matrix A, it must satisfy its own characteristic equation.

If: $$|\lambda I - A| = \lambda^n + a_n \lambda^{n-1} + \cdots + a_2 \lambda + a_1 = 0$$

Then: $$A^n + a_n A^{n-1} + \cdots + a_2 A + a_1 I = 0$$

Corollary 1: For a n$\times$n square matrix A, it's nth or higher order power can be expressed by the linear combination of its n-1 lower order powers.

$$A^k = \sum_{j=0}^{n-1} a_{kj} A^j \qquad k \geq n$$

For example:

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \qquad n = 2$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 3 & \lambda + 4 \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0$$

$$A^2 + 4A + 3I = 0$$

$$A^2 = -4A - 3I$$

$$A^3 = A \cdot A^2 = A \cdot (-4A - 3I) = -4A^2 - 3A$$

$$= -4(-4A - 3I) - 3A = 13A + 12I$$

$$A^4 = -40A - 39I$$

Corollary 2: matrix exponential function e^{At} can be expressed by the linear combination of the n powers, from 0 to n-1th order, of square matrix A.

$$e^{At} = 1 + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!}A^kt^k$$

$$e^{At} = \sum_{k=0}^{\infty}\frac{t^k}{k!}\cdot A^k = \sum_{k=0}^{\infty}\frac{t^k}{k!}\cdot\sum_{j=0}^{n-1}\alpha_{kj}A^j = \sum_{j=0}^{n-1}\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}\alpha_{kj}\right)A^j$$

$$e^{At} = \sum_{j=0}^{n-1}\beta_j(t)A^j$$

where $\beta_j(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\alpha_{kj}$ $\qquad j = 0 \sim n-1$

Back to the proof:

$$X(0) = -\int_0^{\mathrm{T}} e^{-A\tau} BU(\tau)d\tau$$

Substitute $e^{At} = \sum_{j=0}^{n-1} \beta_j(t)A^j$ into the above equation

$$X(0) = -\int_0^{\mathrm{T}} \sum_{j=0}^{n-1} \beta_j(-\tau)A^j \cdot BU(\tau)d\tau$$

$$X(0) = -\sum_{j=0}^{n-1} A^j B \int_0^{\mathrm{T}} \beta_j(-\tau)U(\tau)d\tau = -\sum_{j=0}^{n-1} A^j B \cdot f_j(\mathrm{T})$$

where $f_j(\mathrm{T}) = \int_0^{\mathrm{T}} \beta_j(-\tau)U(\tau)d\tau \Rightarrow p \times 1$

$$X(0) = -\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}_{n\times np} \begin{bmatrix} f_0(\mathrm{T}) \\ f_1(\mathrm{T}) \\ \vdots \\ f_{n-1}(\mathrm{T}) \end{bmatrix}_{np\times 1}$$

For a single input system:

$$X(0) = -\begin{bmatrix} b & Ab & A^2b & \cdots & A^{n-1}b \end{bmatrix}_{n\times n} F_{n\times 1}$$

There are n unknown variables and n equations.

If $rank(S) = n$, the inverse of S exists, the equation has unique solution, and the system is controllable.

If $rank(S) < n$, the inverse of S does not exist,we can not find a unique solution through the above equation.

For a multiple-input system, assume there are p inputs:

$$X(0) = -\begin{bmatrix} \underbrace{\underset{p}{B} \quad \underset{p}{AB} \quad \underset{p}{A^2B} \quad \cdots \quad \underset{p}{A^{n-1}B}}_{n} \end{bmatrix}_{n\times np} \begin{bmatrix} f_0(\mathrm{T}) \\ f_1(\mathrm{T}) \\ \vdots \\ f_{n-1}(\mathrm{T}) \end{bmatrix}_{np\times 1}$$

There are np unknown variables and n equations.

If $rank(S) = n$, the equation has infinite solutions, and the system is controllable.

If $rank(S) < n$,we can not find a unique solution through the above equation.

Choosing different state variable set will lead to different state equations.

One form of state equation of a LTI system can be transformed into another one by non-singular linear-transformation.

Question:

Will non-singular linear-transformation change the controllability of a LTI system?

Non-Singular Linear Transformation

$$\dot{X} = AX + BU$$

$$Y = CX$$

The non-singular linear transformation is:

$$X = PX^{'} \qquad X^{'} = P^{-1}X$$

$$\dot{X}^{'} = P^{-1}\dot{X} = P^{-1}(AX + BU) = P^{-1}AX + P^{-1}BU$$

$$= P^{-1}APX^{'} + P^{-1}BU = A^{'}X^{'} + B^{'}U$$

$$Y = CX = CPX^{'} = C^{'}X^{'}$$

$$A^{'} = P^{-1}AP, \quad B^{'} = P^{-1}B \qquad C^{'} = CP$$

System Controllability after Linear Transformation

For the original system (A,B): $S = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$

For the system after transformation (A', B'):

$$S' = \begin{bmatrix} B' & A'B' & A'^2B' & \cdots & A'^{n-1}B' \end{bmatrix}$$

$$A' = P^{-1}AP, \quad B' = P^{-1}B$$

$$A'B' = P^{-1}AP \cdot P^{-1}B = P^{-1}AB$$

$$A'^2B' = P^{-1}AP \cdot P^{-1}AP \cdot P^{-1}B = P^{-1}A^2B$$

$$A'^{n-1}B' = P^{-1}A^{n-1}B$$

$$S' = P^{-1}S \qquad S = PS'$$

$$rank(S') = rank(S)$$

Controllability does not change
after non-singular linear transformation

Controllability Condition 2

For a linear time-invariant system with distinct eigenvalues to be controllable, it is necessary and sufficient that the B matrix in its diagonal canonical form does not have a row with all elements to be zero.

For example:

A single input system

$$\dot{X} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} X + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u$$

if B does not have any zero element, the system is controllable.

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad Ab = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} \lambda_1 b_1 \\ \lambda_2 b_2 \\ \vdots \\ \lambda_n b_n \end{bmatrix}$$

$$A^2 b = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n^2 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} \lambda_1^2 b_1 \\ \lambda_2^2 b_2 \\ \vdots \\ \lambda_n^2 b_n \end{bmatrix}$$

$$S = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 & \cdots & \lambda_1^{n-1} b_1 \\ b_2 & \lambda_2 b_2 & \lambda_2^2 b_2 & \cdots & \lambda_2^{n-1} b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_n & \lambda_n b_n & \lambda_n^2 b_n & \cdots & \lambda_n^{n-1} b_n \end{bmatrix}$$

if $b_i \neq 0, \quad i = 1,2,\cdots,n$

$\lambda_i \neq \lambda_j \quad i, j = 1,2,\cdots,n$

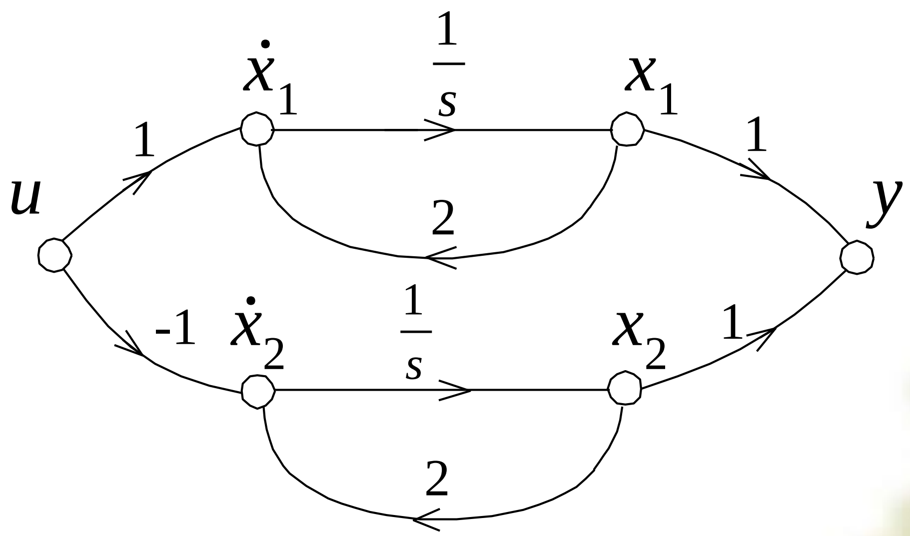
then $rank(S) = n$

Example 6.5:

$$\dot{X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} X + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} u$$

controllable

Example 6.6:



$$\dot{X} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} X + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} X$$

$$S = \begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$rank(S) = 1 < 2$ uncontrollable

Controllability Canonical Form

SISO:
$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

Systems in controllability canonical form must be controllable.

$$b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad Ab = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -a_n \end{bmatrix} \quad A^2 b = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ -a_n \\ -a_{n-1} + a_n^2 \end{bmatrix} \quad S = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & & 1 & -a_n \\ \vdots & \therefore & \therefore & \vdots \\ 1 & -a_n & \cdots & \cdots \end{bmatrix}$$

Wrap-up

- Definition of controllability
- Controllability condition and its demonstration
- Controllability canonical form and its controllability

Assignment Review

page_72_ex10_ex11

Assignment

Page 127

1， 2

Page 128

7，8

Requirement

1. Sketch Bode plots for the original and modified system;
2. Calculate the system performance indices after modification;
3. Simulate the system before and after modification using Matlab.

Page 149-150

1，(2)、(4)

2，(1)