Chapter 3

Time Domain Analysis of Control Systems

Review of Last Chapter

- How to model a LTI system
  - ODE, TF, SE
- Pictorial representation of a control system
  - Block diagrams
  - Signal flow graphs
  - State diagrams
- How to solve a LTI system
  - Time domain method
  - s domain method
- Stability
  - Zero-State & Zero-Input Response
  - Liapunov Stability
  - Asymptotic Stability
  - BIBO Stability

Outlines

- **Stability**
  - Stability condition
  - Routh-Hurwitz criterion
- **Steady-state performance**
  - Definition of time response;
  - Typical test signals;
  - Performance specifications of steady-state response – steady-state error;
  - Impact of disturbance to steady-state error;
  - Impact of parameter variation to steady-state error.
- **Transient Response**
  - Performance Criteria
  - Transient Response of 1^{st}-Order Systems
  - Transient Response of 2^{nd}-Order Systems

Routh-Hurwitz Criterion

- Routh-Hurwitz criterion is an algebraic method that provides information on the absolute stability of a linear time-invariant system.

- The system has a characteristic equation with constant coefficients.

- The criterion tests whether any of the roots of the characteristic equation lie in the right-half s-plane.

- The number of roots lie on the imaginary axis and in the right-half plane is also indicated.

Routh-Hurwitz Criterion

Consider that the characteristic equation of a LTI SISO system is of the form

$$F(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

where all the coefficients are real. In order that the above equation does not have roots with positive real parts, it is necessary that the following two conditions hold:

1. All the coefficient of the equation have the same sign.

2. None of the coefficients vanishes.

Routh-Hurwitz Criterion

The conditions above are based on the laws of algebra, which related to the coefficients of the above equation as follows:

$$\frac{a_{n-1}}{a_n} = -\sum all\ roots$$

$$\frac{a_{n-2}}{a_n} = \sum products\ of\ the\ roots\ taken\ two\ at\ a\ time$$

$$\frac{a_{n-3}}{a_n} = -\sum products\ of\ the\ roots\ taken\ three\ at\ a\ time$$

$$\vdots$$

$$\frac{a_0}{a_n} = (-1)^n\ products\ of\ all\ the\ roots$$

Remarks

All these ratios must be positive and nonzero unless at least one of the roots has a positive real part.

The two necessary conditions can easily be checked by inspection of the equation. However, these conditions are not sufficient, for it is quite possible that an equation with all its coefficients nonzero and of the same sign still may not have all the roots in the left half of the s-plane

Hurwitz Criterion

The Routh-Hurwitz criterion is based on the Hurwitz criterion, which is stated as follows:

The necessary and sufficient condition that all roots of the characteristic equation lie in the left half of the s-plane is that the equation's Hurwitz determinant $D_k, \quad k = 1, 2, \cdots, n$, must all be positive.

$$D_1 = a_{n-1} \qquad D_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} \qquad D_3 = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix}$$

$$D_n = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\ a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\ 0 & a_{n-1} & a_{n-3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{vmatrix}$$

Routh-Hurwitz Criterion

Routh-Hurwitz criterion is a simplification of the Hurwitz criterion.

The simplification is made by rearrange the coefficients of the characteristic equation into two rows, and form the following array of numbers by the indicated operations, as illustrated in the next slide.

Routh's Tabulation

$$a_6 s^6 + a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

s^6	a_6	a_4	a_2	a_0
s^5	a_5	a_3	a_1	0
s^4	$\dfrac{a_5 a_4 - a_6 a_3}{a_5} = A$	$\dfrac{a_5 a_2 - a_6 a_1}{a_5} = B$	$\dfrac{a_5 a_0 - a_6 \times 0}{a_5} = a_0$	0
s^3	$\dfrac{A a_3 - a_5 B}{A} = C$	$\dfrac{A a_1 - a_5 a_0}{A} = D$	$\dfrac{A \times 0 - a_5 \times 0}{A} = 0$	0
s^2	$\dfrac{B \times C - A \times D}{C} = E$	$\dfrac{C a_0 - A \times 0}{C} = a_0$	$\dfrac{C \times 0 - A \times 0}{C} = 0$	0
s^1	$\dfrac{ED - C a_0}{E} = F$	0	0	0
s^0	$\dfrac{F a_0 - E \times 0}{F} = a_0$	0	0	0

Routh-Hurwitz Criterion

Sufficient and necessary condition:

1. The roots of the equation are all in the left-half of the s-plane if all the elements of the first column of the Routh’s tabulation are of the same sign.

2. The number of changes of signs in the elements of the first column equals the number of roots with positive real parts or in the right-half s-plane.

Routh-Hurwitz Criterion

- The reason for the foregoing conclusion is simple, based on the requirements of the Hurwitz determinants. The ralations between the elements in the first column of the Routh’s tabulation and the Hurwitz determinants are:

$$
\begin{array}{cc}
s^6 & a_6 \\
s^5 & a^5 = D_1 \\
s^4 & \dfrac{D_2}{D_1} \\
s^3 & \dfrac{D_3}{D_2} \\
s^2 & \dfrac{D_4}{D_3} \\
s^1 & \dfrac{D_5}{D_4} \\
s^0 & \dfrac{D_6}{D_5}
\end{array}
$$

Example – 3.4

Q: Tell the stability of the following system with given characteristic equation.

$$2s^4 + 2s^3 + 8s^2 + 3s + 2 = 0$$

s^4	2	8	2
s^3	2	3	0
s^2	$\frac{2\times 8 - 2\times 3}{2} = 5$	$\frac{2\times 2 - 2\times 0}{2} = 2$	0
s^1	$\frac{5\times 3 - 2\times 2}{5} = \frac{11}{5}$	0	0
s^0	2	0	0

A: the elements in the first column of the Routh's tabulation are of the same sign, the system is stable.

Example – 3.5

Q: Tell the stability of the following system with given characteristic equation.

$$s^4 + 5s^3 + 8s^2 + 16s + 20 = 0$$

s^4	1	8	20
s^3	5	16	0
s^2	$\dfrac{5\times 8 - 1\times 16}{5} = \dfrac{24}{5} = 4.8$	20	0
s^1	$\dfrac{4.8\times 16 - 5\times 20}{4.8} = -4.83$	0	0
s^0	20	0	0

A: signs of elements in the first column changes twice, there are two roots in the right-half s-plane. The system is unstable.

Special Cases When Routh’s Tabulation Terminates Prematurely

- **The first element in any one row of Routh’s tabulation is zero, but the others are not.**

$$s^5 + 2s^4 + 2s^3 + 4s^2 + s + 1 = 0$$

s^5	1	2	1
s^4	2	4	1
s^3	$0 \approx \delta^+$	$\frac{1}{2}$	0
s^2	$4 - \frac{1}{\delta^+} \approx -\frac{1}{\delta^+}$	1	0
s^1	$\frac{1}{2}$	0	0
s^0	1	0	0

Special Cases When Routh's Tabulation Terminates Prematurely

- It should be noted that the δ method may not give correct results if the equation has pure imaginary roots.
- Further reading is recommended.

K.J. KHATWANI, "On Routh-Hurwitz Criterion," IEEE Trans. Automatic Control, Vol. AC-26, p.583, Apr. 1981

S.K. PILLAI, "The ε Method of the Routh-Hurwitz Criterion," IEEE Trans. Automatic Control, Vol. AC-26, p.584, Apr. 1981

Special Cases When Routh's Tabulation Terminates Prematurely

- **The elements in one row of Routh's tabulation are all zeros.**

$$s^3 + 2s^2 + 4s + k = 0$$

s^3	1	4
s^2	2	k
s^1	$\frac{8-k}{2}$	0
s^0	k	

When $k = 8$, the elements in s^1 row are all zero.

Auxiliary equation $A(s) = 0$, which is formed from the coefficients of the row just above the row of zeros, is needed to continue the Routh's tabulation.

Special Cases When Routh’s Tabulation Terminates Prematurely

- **Concrete steps**

$$s^3 + 2s^2 + 4s + k = 0$$

s^3	1	4
s^2	2	8
s^1	4	0
s^0	8	

1. Form the auxiliary equation $2s^2 + 8 = 0$
2. Take the derivative of the auxiliary equation respect to s $dA(s)/ds = 0$.

   $$4s = 0$$

3. Replace the row of zeros with the coefficients of $A(s) = 0$.
4. Continue the Routh’s tabulation in the usual manner with the newly formed row of coefficients replacing the row of zeros.
5. Interpret the changes of signs, if any, of the coefficients in the first column.

Remarks about the Auxiliary Equation

- That the elements in a row are all zeros indicates that one or more of the following conditions may exist:
  1. one pair of real roots with equal magnitude but opposite signs;
  2. one or more pairs of imaginary roots;
  3. pairs of complex-conjugate roots forming symmetry about the origin of the s-plane.

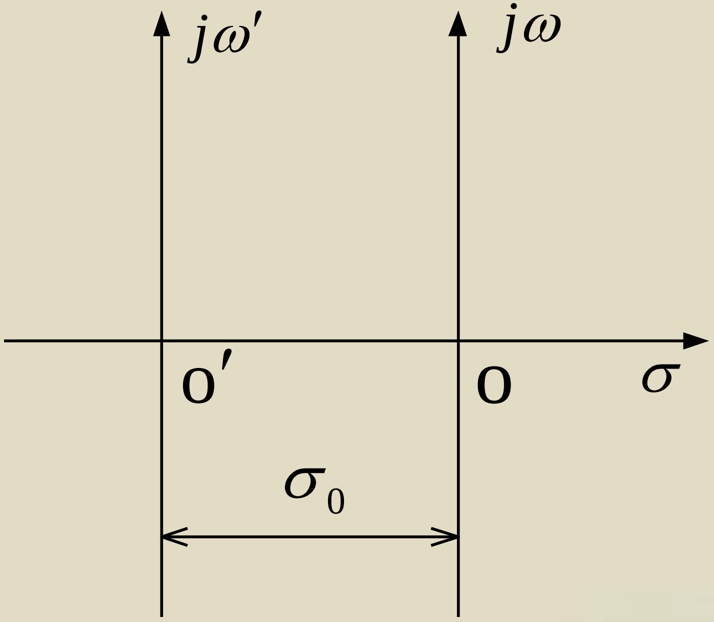
- The roots of the auxiliary equation also satisfy the original equation.
  1. Solving the auxiliary equation can get part of the roots of the original equation;
  2. Perform partial fraction expansion using the left side of the auxiliary equation and form a new equation with the left polynomial.
  3. Solving the new equation can get the left roots of the original equation.

Application of Routh-Hurwitz Criterion

- Estimate the stability margin

$$s^3 + 7s^2 + 17s + 11 = 0$$

s^3	1	17
s^2	7	11
s^1	$\frac{108}{7}$	0
s^0	11	0



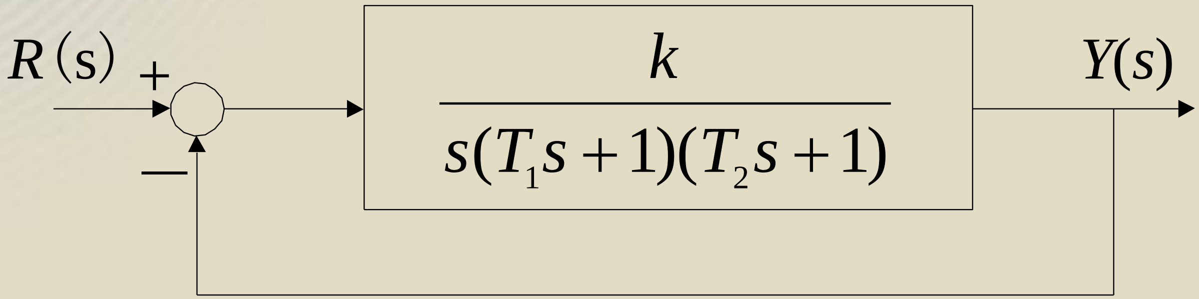
Set $s = s' - \sigma_0$, if $\sigma_0 = 1$, substitute $s = s' - 1$ into the characteristic equation。

$$(s'-1)^3 + 7(s'-1)^2 + 17(s'-1) + 11 = 0$$

$$s'^3 + 4s'^2 + 6s' = 0$$

Application of Routh-Hurwitz Criterion

- Determine the stability region of parameters



$$G(s) = \frac{k}{s(T_1 s + 1)(T_2 s + 1) + k}$$

$$T_1 T_2 s^3 + (T_1 + T_2)s^2 + s + k = 0$$

Application of Routh-Hurwitz Criterion

$$T_1T_2s^3+(T_1+T_2)s^2+s+k=0$$

s^3	T_1T_2	1
s^2	T_1+T_2	k
s^1	$\dfrac{(T_1+T_2)-kT_1T_2}{T_1+T_2}$	0
s^0	k	0

$T_1>0$、$T_2>0$

$T_1+T_2>kT_1T_2$

$k>0$

$$\Longrightarrow \quad \frac{1}{T_1}+\frac{1}{T_2}>k>0$$

Next Milestone

- **Steady-state performance**
  - Definition of time response;
  - Typical test signals;
  - Performance specifications of steady-state response – steady-state error;
  - Impact of disturbance to steady-state error;
  - Impact of parameter variation to steady-state error.

Time Response

- Time Response: the state and output responses with respect to time.
- The transient response and the steady-state response

$$y(t) = y_t(t) + y_{ss}(t)$$

- Transient response is defined as the part of the time response that goes to zero as time goes to infinity

$$\lim_{t \to \infty} y_t(t) = 0$$

- Steady-state response is the part of the total response that remains after the transient has died out

Remarks

- All real stable control systems exhibit transient phenomena to some extend before the steady state is reached. Since inertia, mass, and inductance are unavoidable in physical systems, the responses of a typical control system can not follow sudden changes in the input instantaneously, and transients are usually observed.
- The control of the transient response is necessarily important, as it is a significant behavior of the system; and the deviation between the output response and the input or the desired response, before the steady state is reached, must be closely controlled.

Remarks

- The steady-state response of a control system is also very important, since it indicates where the system output ends up at when time becomes large.
- For a position control system, the steady-state position gives an indication of the final accuracy of the system. In general, if the steady-state response of the output does not agree with the desired reference exactly, the system is said to have a steady-state error.

Remarks

- The study of a control system in the time domain essentially involves the evaluation of the transient and steady-state responses of the system.
- In the design problem, specifications are usually given in terms of the transient and steady-state performances, and controllers are designed so that the specifications are all met by the designed system.
- In the following lessons, we are going to discuss the performances of a system in its steady-state and transient response.

Typical Test Signals

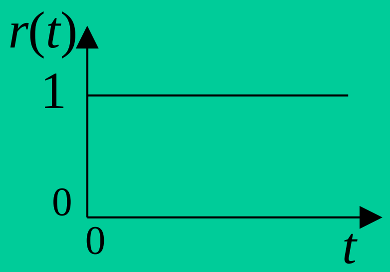
- Response to what?
  - Inputs

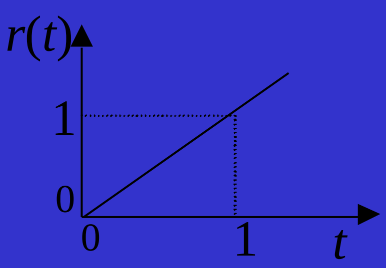
- What's the problem?
  - The inputs to many practical control systems are not exactly known ahead of time. It is difficult to design a control system so that it will perform satisfactorily for all possible forms of input signals.

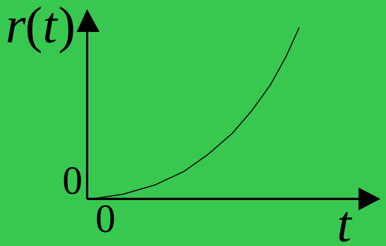
- How to solve the problem?
  - To assume some basic types of test inputs so that the performance of a system can be evaluated.

Typical Test Signals

Test signal	$r(t)$ $t<0,\ r(t)=0$	$R(s)$
Step-function	$r(t)=1(t)$	$R(s)=\frac{1}{s}$
Ramp-function	$r(t)=t$	$R(s)=\frac{1}{s^2}$
Parabolic-function	$r(t)=\frac{1}{2}t^2$	$R(s)=\frac{1}{s^3}$



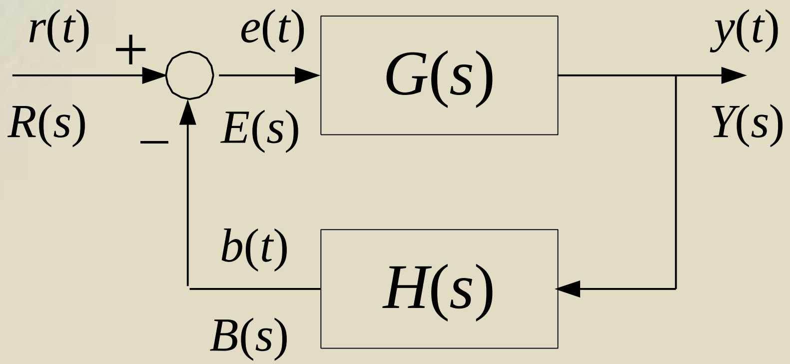




Remarks

- The step function is very useful as a test signal since its initial instantaneous jump in amplitude reveals a great deal about a system’s quickness in responding to inputs with abrupt changes.
- The ramp function is a signal that changes constantly with time. The ramp function has the ability to test how the system would respond to a signal that changes linearly with time.
- The parabolic function represents a signal that is one order faster than the ramp function.
- From the step function to the parabolic function, the signals become progressively faster with respect to time.
- Theoretically, we can define signals with still higher rates. However, in reality, we seldom find it necessary or feasible to use a test signal faster than a parabolic function.

An Important Performance Specification of Steady-state Response - Steady-state Error



Defined by output: $e(t) = y_r(t) - y(t)$

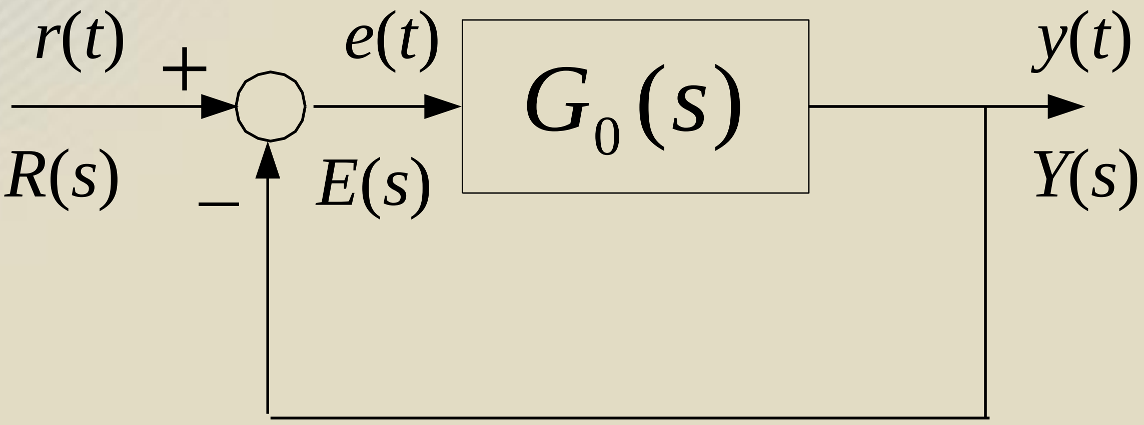
Defined by input: $e(t) = r(t) - b(t)$

$$e_{ss} = \lim_{t \to \infty} e(t)$$

In reality, when time lasts more than 4~5 times of the time constant of the system, we can think that the steady-state is reached.

Steady-state Error

For unit feedback system:



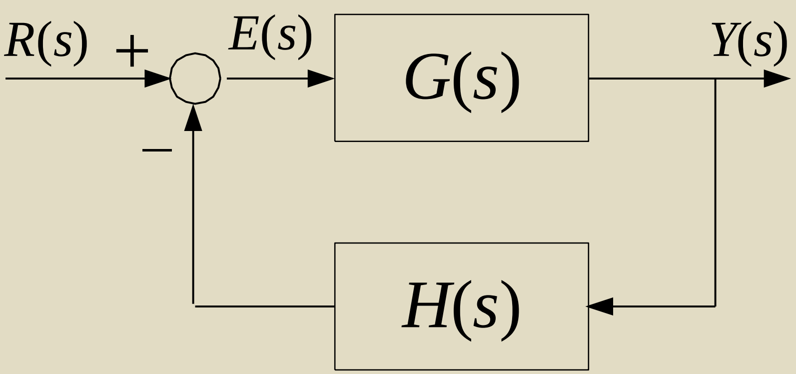
$$e(t) = r(t) - y(t)$$

Calculation of the Steady-state Error

Laplace Final-value Theorem:

If the Laplace transform of $f(t)$ is $F(s)$, and if $sF(s)$ does not have any pole whose real part is zero or positive, then

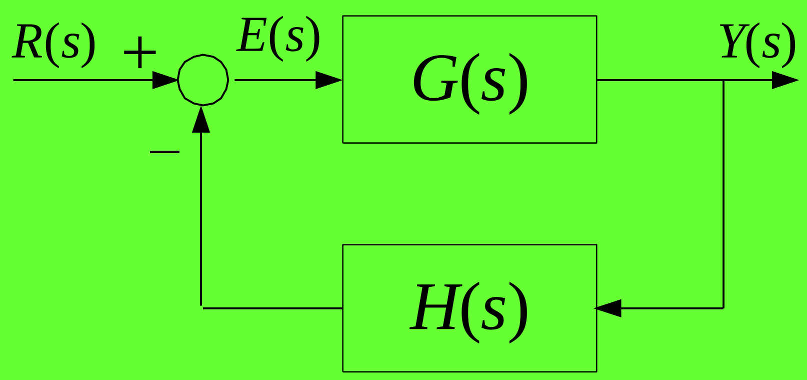
$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$



$$E(s) = \frac{1}{1+G(s)H(s)}R(s) = \frac{R(s)}{1+G_0(s)}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G_0(s)}$$

Example - find the final value of the following systems



(1) $G(s) = \dfrac{1}{s+1}$

$H(s) = \dfrac{1}{s+2}$

$R(s) = \dfrac{1}{s}$

(2) $G(s) = \dfrac{\omega}{s^2+\omega^2}$

$H(s) = 1$

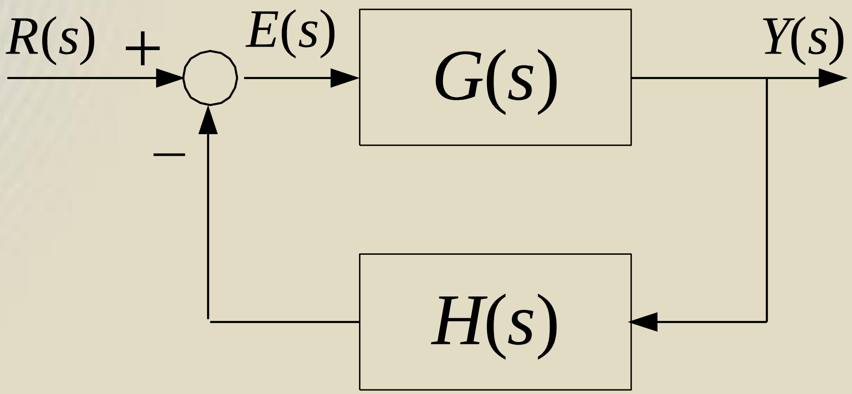
$R(s) = \dfrac{1}{s}$

(1) $$E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{(s+1)(s+2)}{s(s^2+3s+3)} \qquad e_{ss} = \lim_{s\to 0} sE(s) = \frac{2}{3}$$

(2) $$E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{s^2+\omega^2}{s(s^2+\omega^2+\omega)}$$

Laplace final-value theorem is not applicable.

Type of Control Systems



$$E(s) = \frac{1}{1+G(s)H(s)}R(s) = \frac{R(s)}{1+G_0(s)}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G_0(s)}$$

Clearly, e_{ss} depends on the characteristics of $G_0(s)$. More specifically, e_{ss} depends on the number of poles that $G_0(s)$ has at s=0. The number is known as the type of the control system.

Type of Control Systems

In general, $G_0(s)$ can be expressed for convenience as:

$$G_0(s) = G(s)H(s) = \frac{k\prod_{i=1}^{m}(T_i s+1)}{s^r\prod_{j=1}^{v}(\tau_j s+1)} = \frac{k'\prod_{i=1}^{m}(s+z_i)}{s^r\prod_{j=1}^{v}(s+p_j)}$$

k is the open loop gain $\qquad k = \lim_{s\to 0} s^r G_0(s)$

The system order $\qquad r+v$

Type 0 $\qquad r=0$

Type 1 $\qquad r=1$

Type 2 $\qquad r=2$

Steady-State Error of a System with a Step Input

Laplace transform of the step function: $R(s) = \dfrac{1}{s}$

Steady-state error of a closed loop control system:

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G_0(s)} = \lim_{s \to 0} \frac{1}{1 + G_0(s)} = \frac{1}{1 + \lim\limits_{s \to 0} G_0(s)}$$

$$e_{ss} = \frac{1}{1 + \lim\limits_{s \to 0} \dfrac{k \prod\limits_{i=1}^{m} (T_i s + 1)}{s^r \prod\limits_{j=1}^{v} (\tau_j s + 1)}}$$

Steady-State Error of a System with a Step Input

$$e_{ss} = \frac{1}{1+\lim_{s\to 0}\dfrac{k\prod_{i=1}^{m}(T_i s+1)}{s^r\prod_{j=1}^{v}(\tau_j s+1)}}$$

If set: $k_p = \lim_{s\to 0} G_0(s)$ $\quad e_{ss} = \dfrac{1}{1+k_p}$ $\quad k_p$ is named as step-error constant

$r=0 \quad k_{\mathrm{p}} = k \quad e_{ss} = \dfrac{1}{1+k}$ Type 0 system

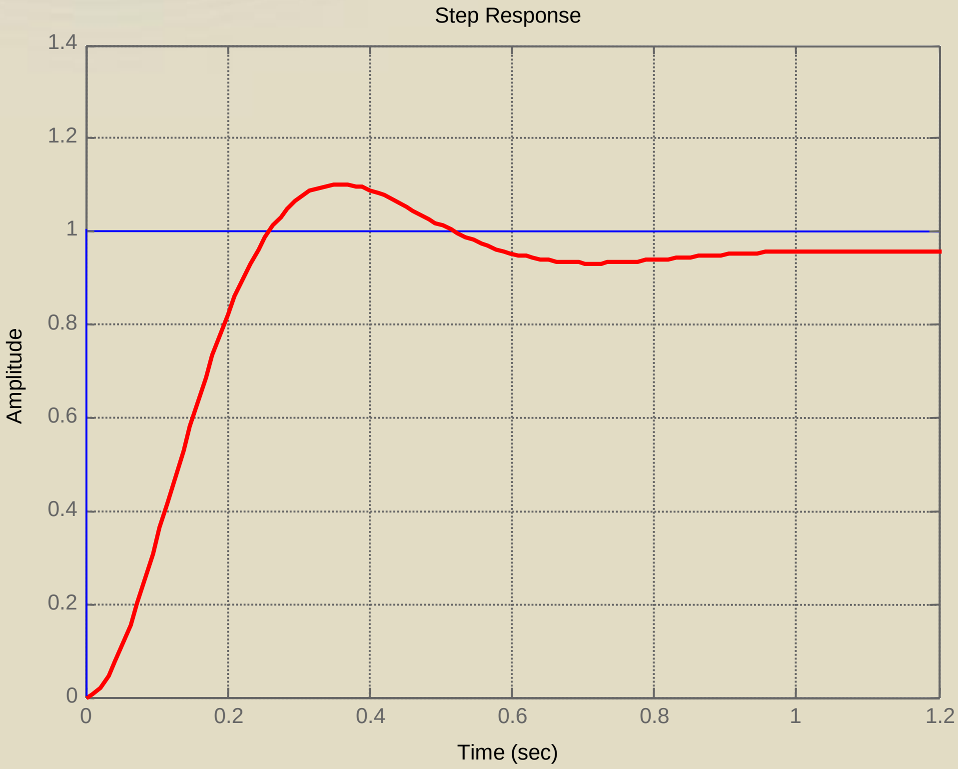
$r \geq 1 \quad k_p \to \infty \quad e_{ss} = 0$ Type 1 or higher system

Steady-State Error of System with a Step-Input

$$G_0(s) = \frac{20}{(2s+1)(0.1s+1)}$$

$$e_{ss} = \frac{1}{1+k_p} = \frac{1}{21} = 0.0476$$

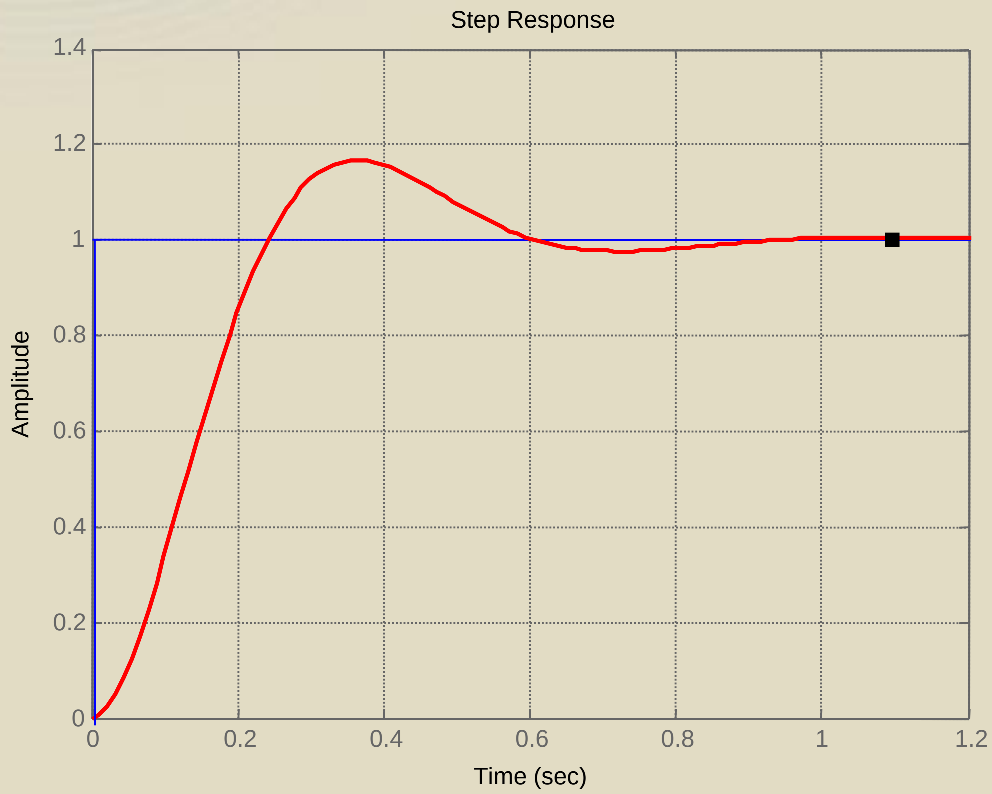
$$y(t) = 1 - 0.0476 = 0.9524$$




```
s=tf('s');
g0=20/((2*s+1)*(0.1*s+1));
sys=feedback(g0,1);
t=0:0.01:1.2;
step(sys,'r',t);
Td=1*(t>=0);
hold on;
plot(t,Td,'b');
grid on;
```

Steady-State Error of System with a Step-Input

$$G_0(s) = \frac{100}{s(s+10)} \qquad e_{ss} = 0$$



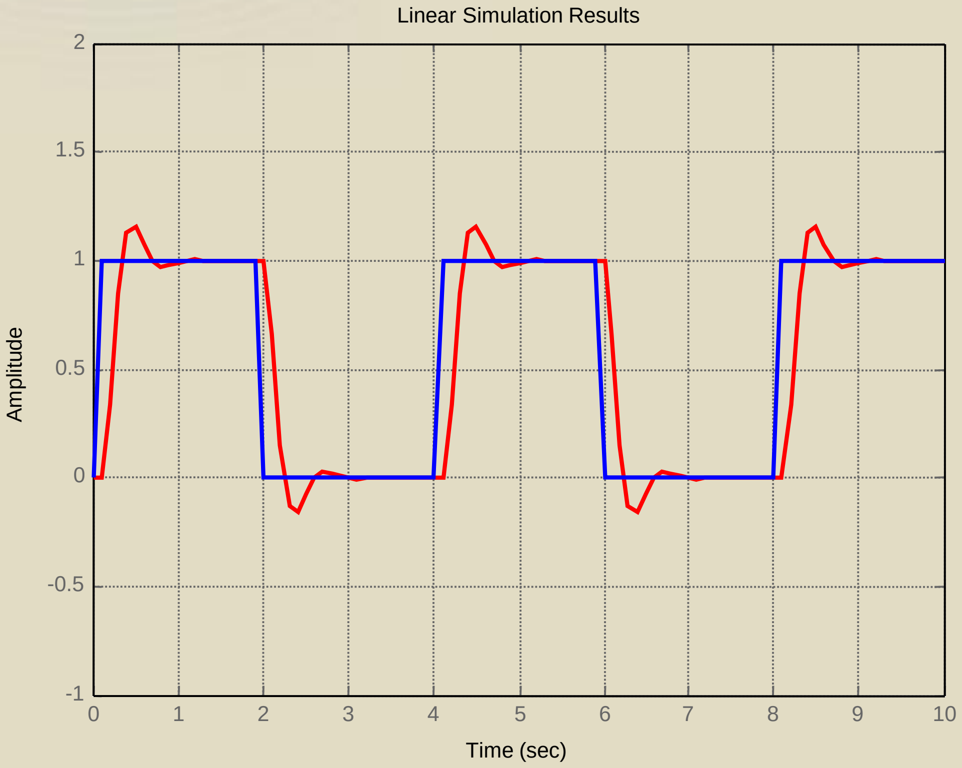

```
s=tf('s');
g0=100/(s*(s+10));
sys=feedback(g0,1);
t=0:0.01:1;
step(sys,'r',t);
Td=1*(t>=0);
hold on;
plot(t,Td,'b');
grid on;
```

Example

A system with zero steady-state error is not necessarily to be a system without errors in transient.

$$G_0(s) = \frac{100}{s(s+10)}$$

Input is a square waveform




```
s=tf('s');
g0=100/(s*(s+10));
sys=feedback(g0,1);
t=0:0.1:10;
Td=1*((t>0 & t<2) |( t>4 & t<6) |
t>8) ;
lsim (sys,'r', Td, t);
axis([0 10 -1 2]);
grid on;
hold on;
plot(t,Td,'b');
```

Steady-State Error of a System with Ramp Input

Laplace transform of the ramp function: $R(s) = \dfrac{1}{s^2}$

Steady-state error of a closed loop control system:

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G_0(s)} = \lim_{s \to 0} \frac{1}{s[1 + G_0(s)]} = \lim_{s \to 0} \frac{1}{sG_0(s)}$$

$$e_{ss} = \frac{1}{\lim\limits_{s \to 0} \dfrac{k \prod\limits_{i=1}^{m} (T_i s + 1)}{s^{r-1} \prod\limits_{j=1}^{v} (\tau_j s + 1)}}$$

Steady-State Error of a System with Ramp Input

$$e_{ss} = \frac{1}{\lim\limits_{s\to 0} \dfrac{k\prod\limits_{i=1}^{m}(T_i s+1)}{s^{r-1}\prod\limits_{j=1}^{v}(\tau_j s+1)}}$$

If set: $k_v = \lim\limits_{s\to 0} sG_0(s)$ $\quad e_{ss} = \dfrac{1}{k_v}$ $\quad k_v$ is named as ramp-error constant

Type 0 system $\quad r=0 \quad k_v = 0 \quad e_{ss} = \infty$

Type 1 system $\quad r=1 \quad k_v \to Const. \quad e_{ss} = 1/k_v$

Type 2 or higher system $r \geq 2 \quad k_v = \infty \quad e_{ss} = 0$

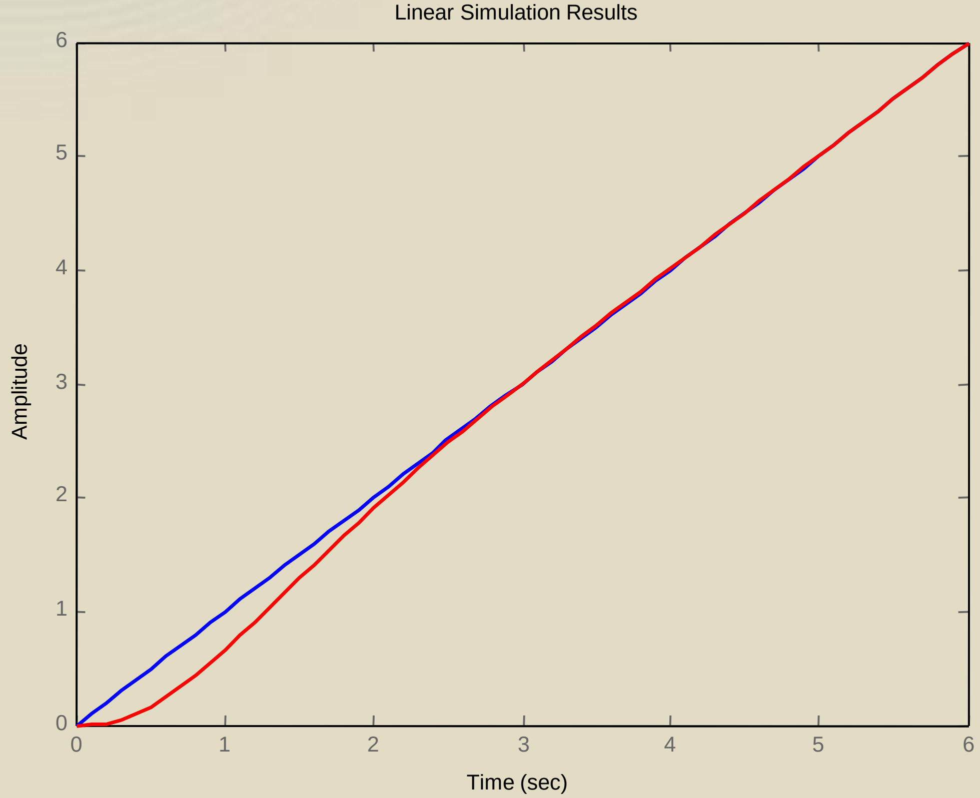
Steady-State Error of System with a Ramp Input

$$G_0(s) = \frac{20}{s(s+10)} \qquad e_{ss} = \frac{1}{k_v} = \frac{1}{2} = 0.5$$


```
s=tf('s');
g0=20/(s*(s+10));
sys=feedback(g0,1);
t=0:0.1:6;
u1=t;
lsim(sys,'r',u1,t);
hold on;
plot(t,u1,'b');
grid on;
```

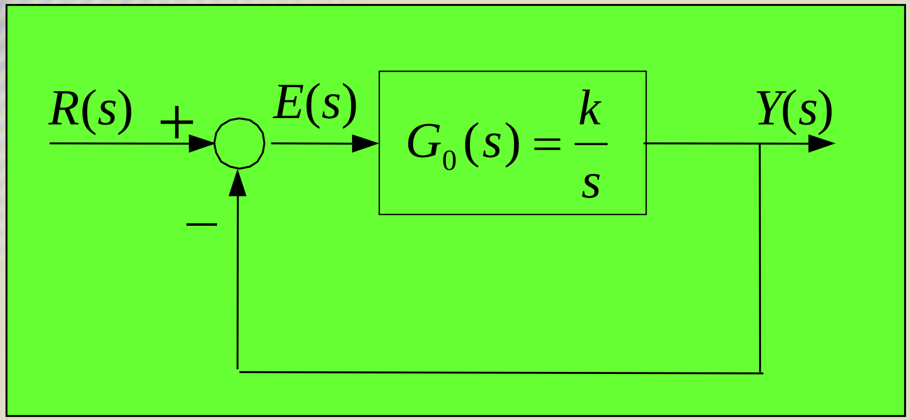
Steady-State Error of System with a Ramp Input

$$G_0(s) = \frac{2(s+1)}{s^2(0.1s+1)} \qquad e_{ss} = \frac{1}{k_v} = 0$$




```
s=tf('s');
g0=2*(s+1)/(s^2*(0.1*s+1));
sys=feedback(g0,1);
t=0:0.1:6;
u1=t;
lsim(sys,'r',u1,t);
hold on;
plot(t,u1,'b');
grid on;
```

Example - find the final value of the following system with the given inputs



(1) $r(t) = \frac{1}{2}t^2$

(2) $r(t) = \sin \omega t$

A: first, check the stability of the system. The TF of the system is:

$$G(s) = \frac{G_0(s)}{1+G_0(s)} = \frac{k/s}{1+k/s} = \frac{k}{s+k}$$

The system is stable when k>0.

$$E(s) = R(s) - Y(s) = R(s) - G(s)R(s) = [1-G(s)]R(s)$$

$$= [1 - \frac{G_0(s)}{1+G_0(s)}]R(s) = \frac{R(s)}{1+G_0(s)}$$

(1) When the input is $r(t)=\frac{1}{2}t^2 \qquad R(s)=\frac{1}{s^3}$

$$sE(s)=\frac{sR(s)}{1+G_0(s)}=\frac{s}{s^3(1+k/s)}=\frac{1}{s(s+k)}$$

Laplace final value theorem is not applicable

$$e_{ss}=\lim_{s\to 0}\frac{sR(s)}{1+G_0(s)}=\lim_{s\to 0}\frac{1}{s^2\left[1+G_0(s)\right]}=\lim_{s\to 0}\frac{1}{s^2\left[1+k/s\right]}\to\infty$$

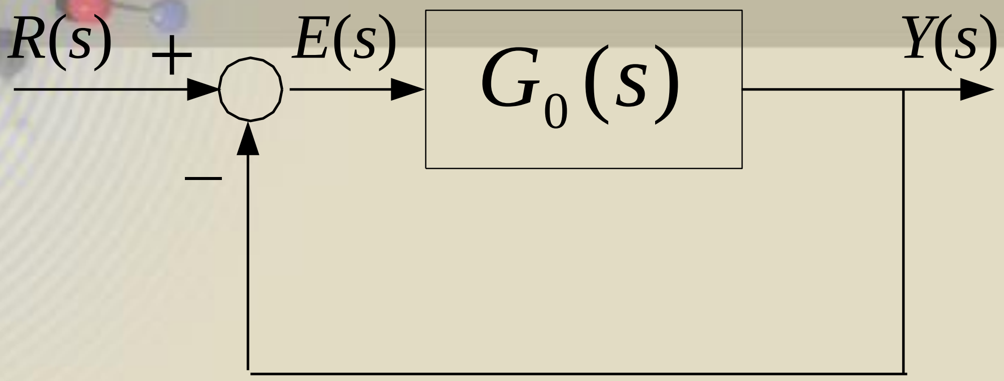
coincidence

(2) When input is $r(t)=\sin\omega t \qquad R(s)=\frac{\omega}{s^2+\omega^2}$

$$E(s)=\frac{R(s)}{1+G_0(s)}=\frac{\frac{\omega}{s^2+\omega^2}}{1+k/s}=\frac{s\omega}{(s+k)(s^2+\omega^2)}$$

Laplace final-value theorem is not applicable.
Other method should be used.

$$\sin\omega t \longrightarrow \boxed{G(j\omega)} \longrightarrow \left|G(j\omega)\right|\sin[\omega t+\angle G(j\omega)]$$



Close-loop TF from R(s) to E(s) $G(s) = \dfrac{E(s)}{R(s)} = \dfrac{1}{1+G_0(s)} = \dfrac{s}{s+k}$

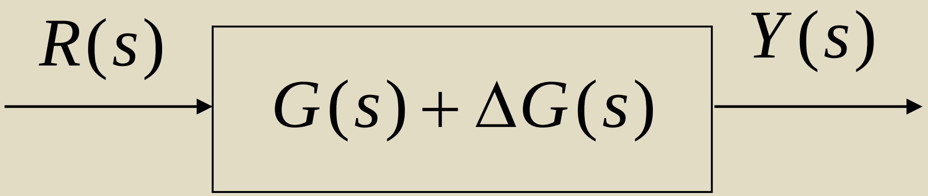
$$G(j\omega) = \frac{j\omega}{j\omega + k} = \frac{\omega\angle 90^0}{\sqrt{\omega^2 + k^2}\angle tg^{-1}\frac{\omega}{k}} = \frac{\omega}{\sqrt{\omega^2 + k^2}}\angle(90^0 - tg^{-1}\frac{\omega}{k})$$

$$e_{ss} = ?$$

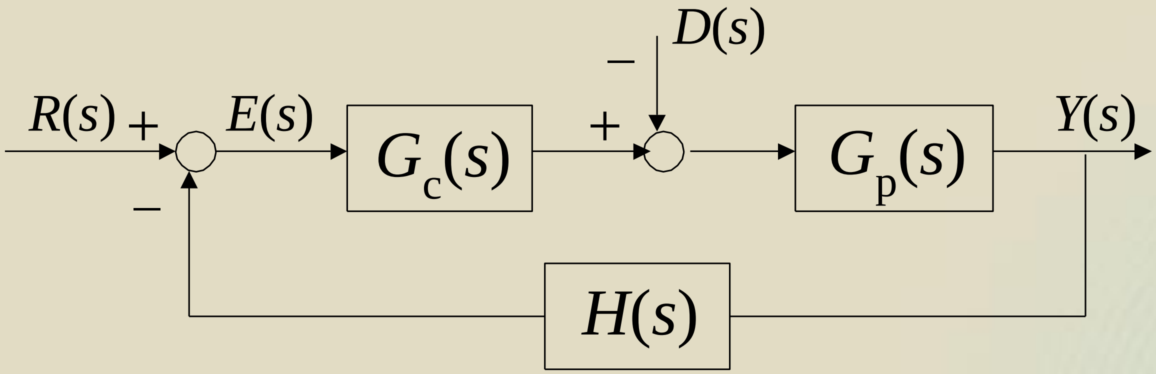
How do disturbances impact steady-state error?

- Disturbances are from two origins

1. Inside
   - Nonlinearity
   - Parameter variation

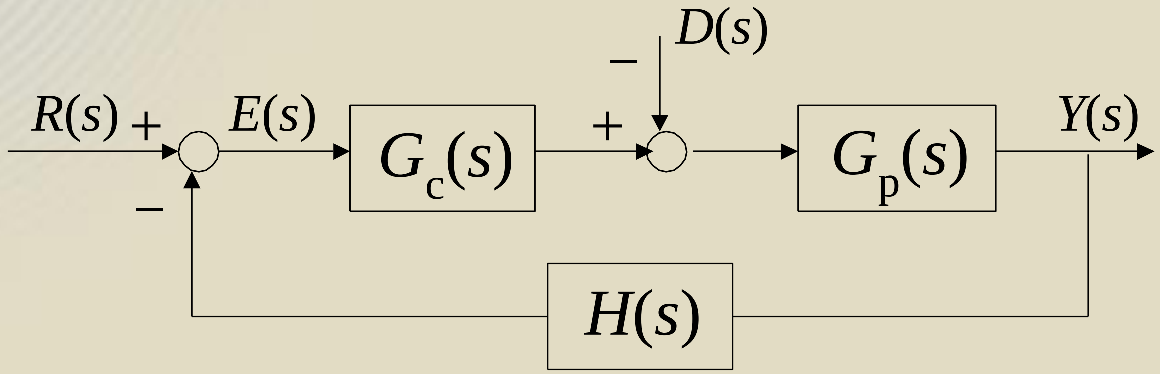


2. Outside
   - Environment



Disturbances

- Disturbances are from outside environment



TF from D(s) to E(s):

$$E(s) = \frac{G_p(s)H(s)}{1 + G_c(s)G_p(s)H(s)} = \frac{G_p(s)H(s)}{1 + G_0(s)}$$

How do the different blocks act on the disturbances?

Disturbances

Set the disturbance as: $d = d \cdot 1(t)$

TF of the disturbances: $D(s) = d / s$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \cdot \frac{G_{\mathrm{p}}(s)H(s)}{1+G_0(s)} \cdot D(s)$$

$$= \frac{G_{\mathrm{p}}(0)H(0)}{1+G_0(0)} \cdot d = \frac{G_0(0)}{G_{\mathrm{c}}(0)\left[1+G_0(0)\right]} \cdot d$$

$$G_0(s) = \frac{k\prod_{i=1}^{m}(T_i s+1)}{s^r \prod_{j=1}^{v}(\tau_j s+1)}$$

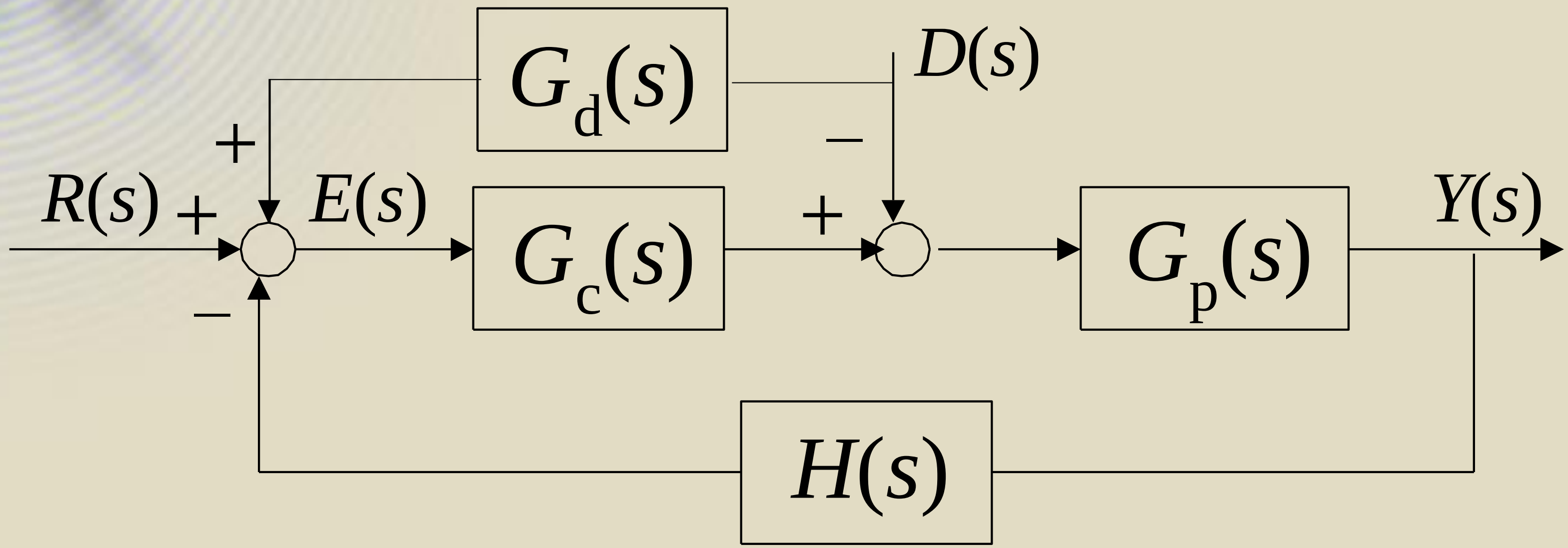
$$G_0(0) >> 1$$

$$e_{ss} \approx \frac{d}{G_{\mathrm{c}}(0)}$$

Conclusion on how to attenuate the impact of disturbances

1. Increase the gain of the controller before the disturbances;
2. Add integration block in the controller before the disturbances;

Composite Control to Disturbances

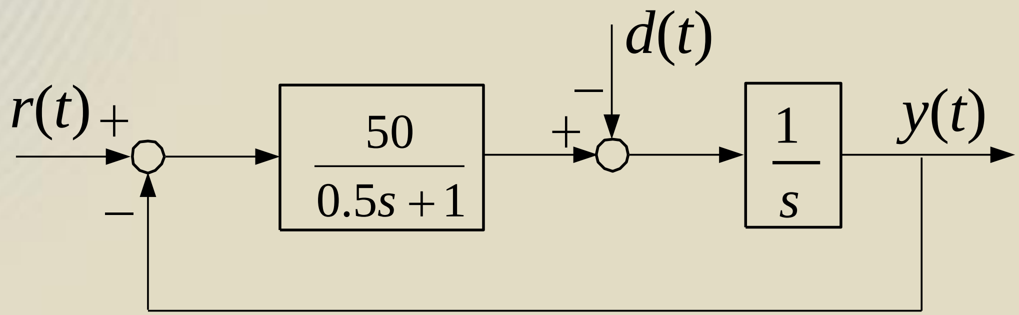


$$\frac{Y(s)}{D(s)} = \frac{G_d(s)G_c(s)G_p(s) - G_p(s)}{1 + G_c(s)G_p(s)H(s)}$$

Ideally: $G_d(s) = 1 / G_c(s)$ ⇨ $\dfrac{Y(s)}{D(s)} = 0$

Example - 3.11

Q: when the input r(t)=t, the disturbance d(t)=0.5*1(t), please find the steady-state error of the given system, where the error is defined as e(t)=r(t)-y(t).



A: the output is determined by both the input and the disturbance. Superposition principle should be used here to calculate the output.

the TF from input to output

$$G_1(s) = \frac{Y(s)}{R(s)} = \frac{\dfrac{50}{s(0.5s+1)}}{1+\dfrac{50}{s(0.5s+1)}} = \frac{50}{s(0.5s+1)+50}$$

The system is stable.

the TF from disturbance to output

$$G_2(s)=\frac{Y(s)}{D(s)}=\frac{-1/s}{1+\dfrac{50}{s(0.5s+1)}}=\frac{(0.5s+1)}{s(0.5s+1)+50}$$

the system error:

$$\begin{aligned}E(s)=R(s)-Y(s)&=R(s)-G_1(s)R(s)-G_2(s)D(s)\\&=\left[1-G_1(s)\right]R(s)-G_2(s)D(s)\\&=\frac{0.5s^2+s}{0.5s^2+s+50}\cdot\frac{1}{s^2}+\frac{0.5s+1}{0.5s^2+s+50}\cdot\frac{0.5}{s}\end{aligned}$$

Check poles of sE(s) and calculate the system's steady-state error:

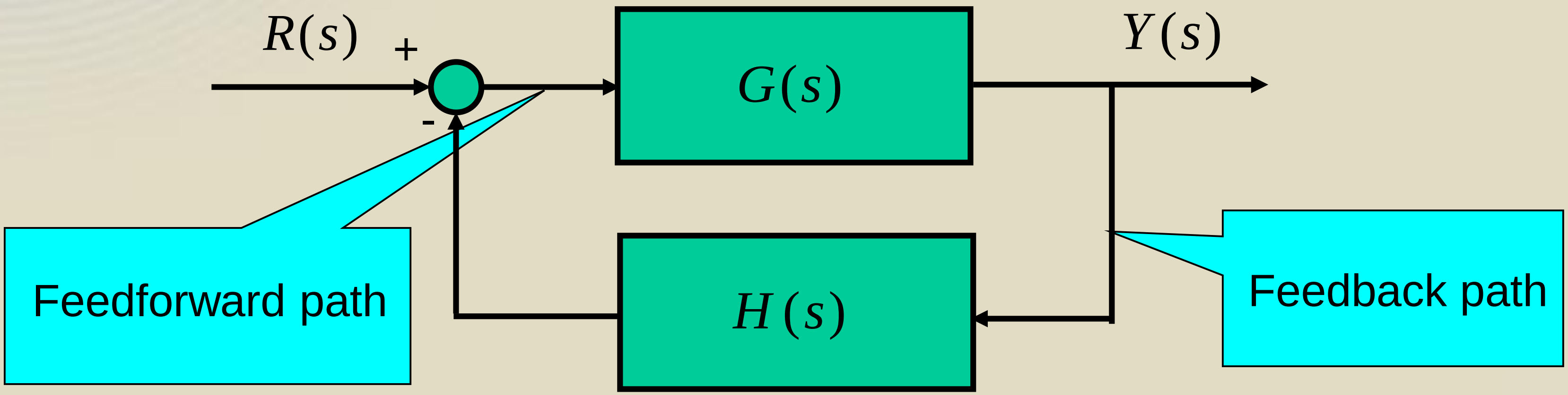
$$e_{ss}=\lim_{s\to 0} s\cdot E(s)=\lim_{s\to 0}\left(\frac{0.5s+1}{0.5s^2+s+50}+\frac{0.5s+1}{0.5s^2+s+50}\times 0.5\right)$$

$$=\frac{1}{50}+\frac{1}{50}\times 0.5=0.03$$

How to evaluate the impact of parameter variation to system steady-state error?

Sensitivity Analysis

- What does sensitivity analysis do?
  - Analyze the impact of parameter variation to the control systems



TF of the system: $$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

Sensitivity of System TF to Parameter Variation in Feedforward Path

Definition: $s_G = \dfrac{\Delta T(s)/T(s)}{\Delta G(s)/G(s)}$

Calculation:

$$s_G = \frac{\partial T / T}{\partial G / G} = \frac{\partial T}{\partial G} \times \frac{G}{T} = \frac{(1+GH) - GH}{(1+GH)^2} \times \frac{G}{G/(1+GH)}$$

$$s_G = \frac{1}{1+G(s)H(s)}$$

$$\boxed{T = \frac{G}{1+GH}}$$

Sensitivity of System TF to Parameter Variation in Feedback Path

Definition: $$s_H = \frac{\Delta T(s)/T(s)}{\Delta H(s)/H(s)}$$

Calculation:

$$s_H = \frac{\partial T}{\partial H} \times \frac{H}{T} = \frac{-G^2}{(1+GH)^2} \times \frac{H}{\frac{G}{1+GH}} = \frac{-GH}{1+GH}$$

$$s_H = \frac{-G(s)H(s)}{1+G(s)H(s)}$$

Sensitivity of System TF to Parameter Variation

For a negative feedback system

$$s_G = \frac{1}{1+G(s)H(s)} \qquad s_H = \frac{-G(s)H(s)}{1+G(s)H(s)}$$

When there exists integration block in the G(s), at steady-state

$$G(0) \to \infty$$

$$\boxed{\lim_{s\to 0} \frac{k\prod_{i=1}^{m}(T_i s+1)}{s^r \prod_{j=1}^{v}(\tau_j s+1)}}$$

$$s_G \approx 0 \qquad s_H \approx -1$$

Conclusion:
negative feedback systems can effectively attenuate the impact of the parameter variation in the feedforward path, but can do nothing to that in the feedback path.

Next Milestone

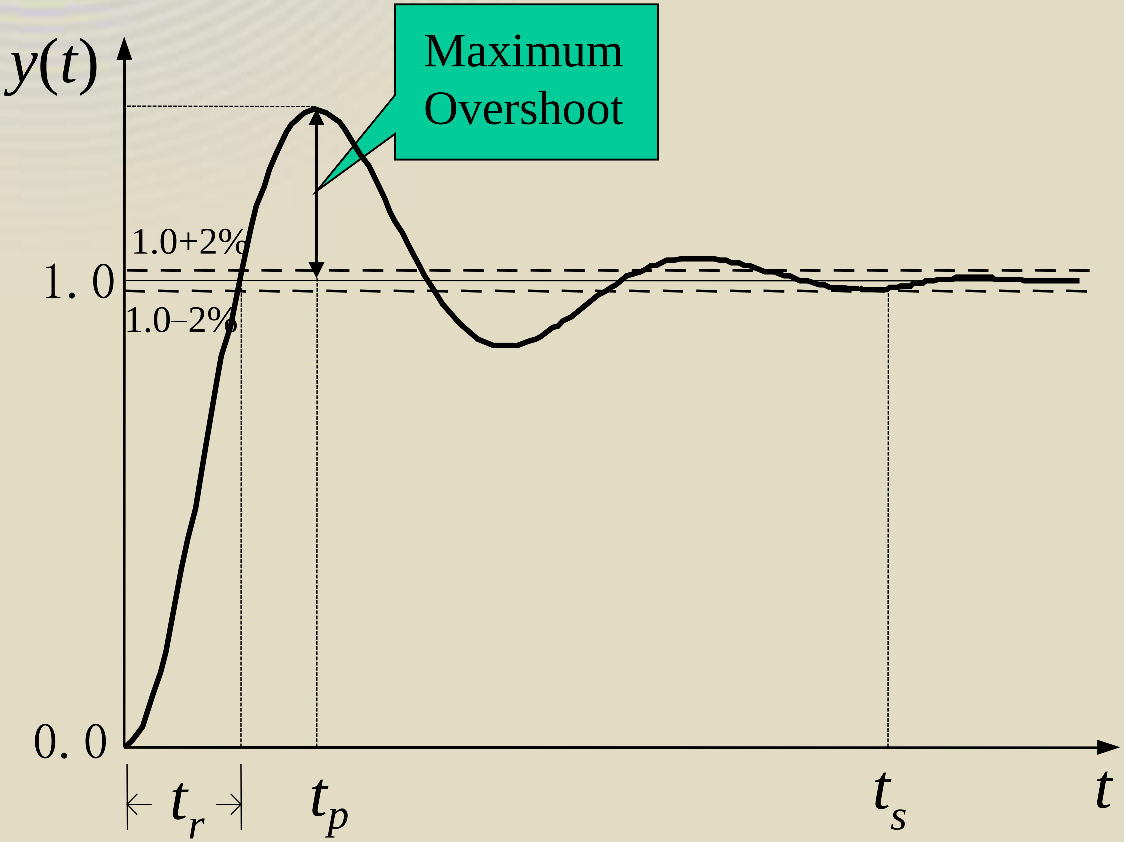
- **Transient Response**
  - Performance Criteria
  - Transient Response of 1st-Order Systems
  - Transient Response of 2nd-Order Systems

Transient Response

- Transient response is the response of a system to a certain input which dies out as time becomes large;
- For linear control systems, the characterization of the transient response is often done by use of the unit-step function as input;
- The response of a control system when the input is a unit-step function is called the unit-step response.

Performance Criteria

Commonly used performance criteria characterizing transient response



$\sigma\%$	Maximum Overshoot
t_r	Rise Time
t_p	Peak Time
t_s	Settling Time

Performance Criteria

Maximum Overshoot $\sigma\%$

The difference between maximum value and the steady-state value of output.

Rise Time t_r

Rise time is defined as the time required for the step response to rise from zero to the final value of output. (In some other situation, rise time is defined as the time required for the step response to rise from 10 to 90 percent of the final value.)

Peak Time t_p

Peak time is defined as the time required for the step response to rise from zero to the maximum value.

Settling Time t_s

Settling time is defined as the time required for the step response to decrease and stay within a specified percentage of its final value. A frequently used figure is 2%

Performance Criteria

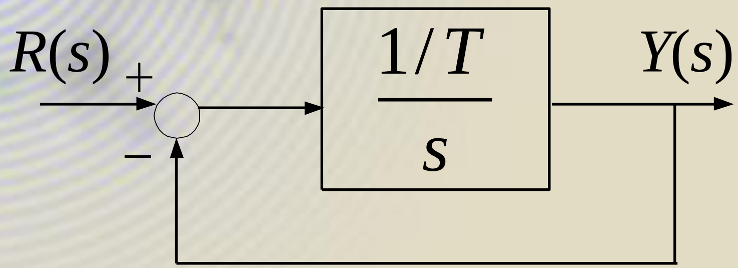
Maximum Overshoot $\sigma\%$ is used to measure the relative stability of a control system. A system with a large overshoot is usually undesirable.

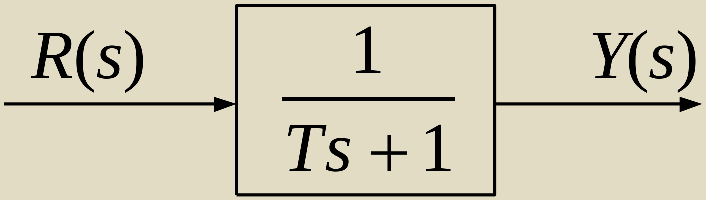
Rise time t_r , peak time t_p , settling Time t_s are used to measure the quickness that a system responds to the unit-step input.

The methods commonly used to study the performance of a system in transient:

1. Computer simulation – numerical method;
2. Referring to transient performance of the first-order and second order system, estimate those of higher order systems

Transient Response of the 1st-Order Systems

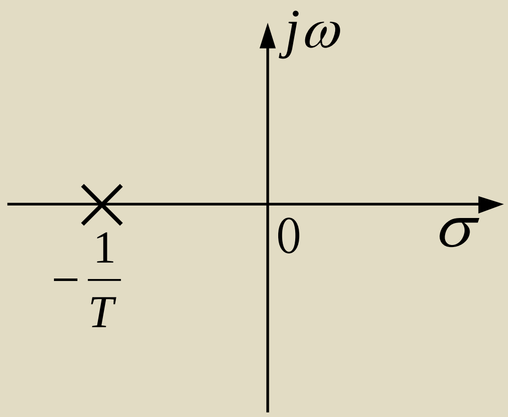


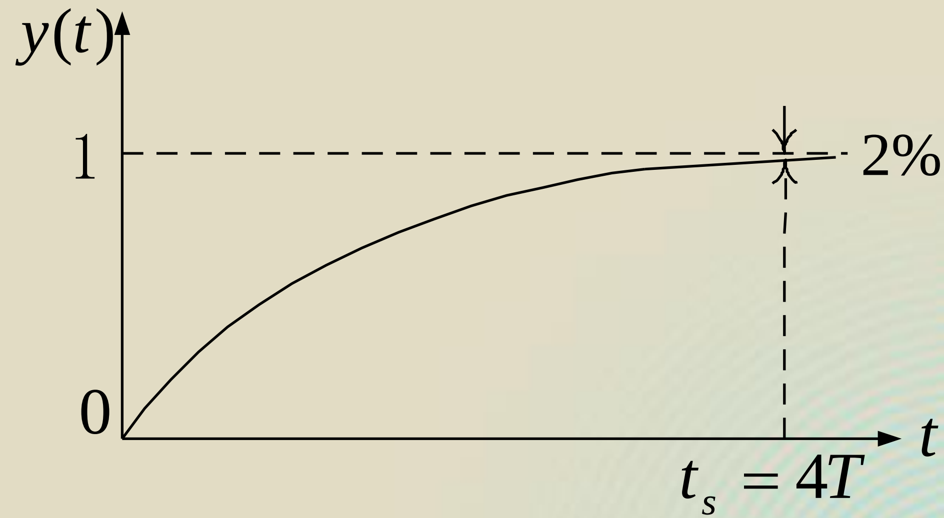


$$G(s) = \frac{Y(s)}{R(s)} = \frac{\frac{1}{T}}{s + \frac{1}{T}} = \frac{1}{Ts+1}$$

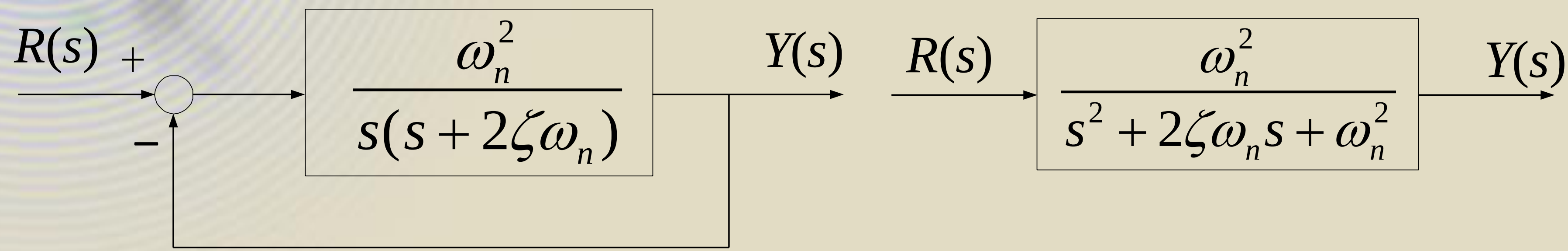
$$y(t) = 1 - e^{-\frac{1}{T}t}$$

$$y(t) = 1 - e^{-4t} \approx 0.9817$$





Transient Response of the 2nd-Order Systems



$$G(s)=\frac{Y(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{1}{T^2s^2+2\zeta Ts+1}$$

ω_n Natural undamped frequency

$T = 1/\omega_n$ Time constant, reciprocal of the natural undamped frequency

ζ Damping ratio

Transient Response of the 2nd-Order Systems

$$G(s)=\frac{Y(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{1}{T^2s^2+2\zeta Ts+1}$$

Roots of the characteristic equation:

$$s_1,s_2=-\zeta\omega_n\pm j\omega_n\sqrt{1-\zeta^2}$$

Set $\sigma=\zeta\omega_n$ Real part of the roots

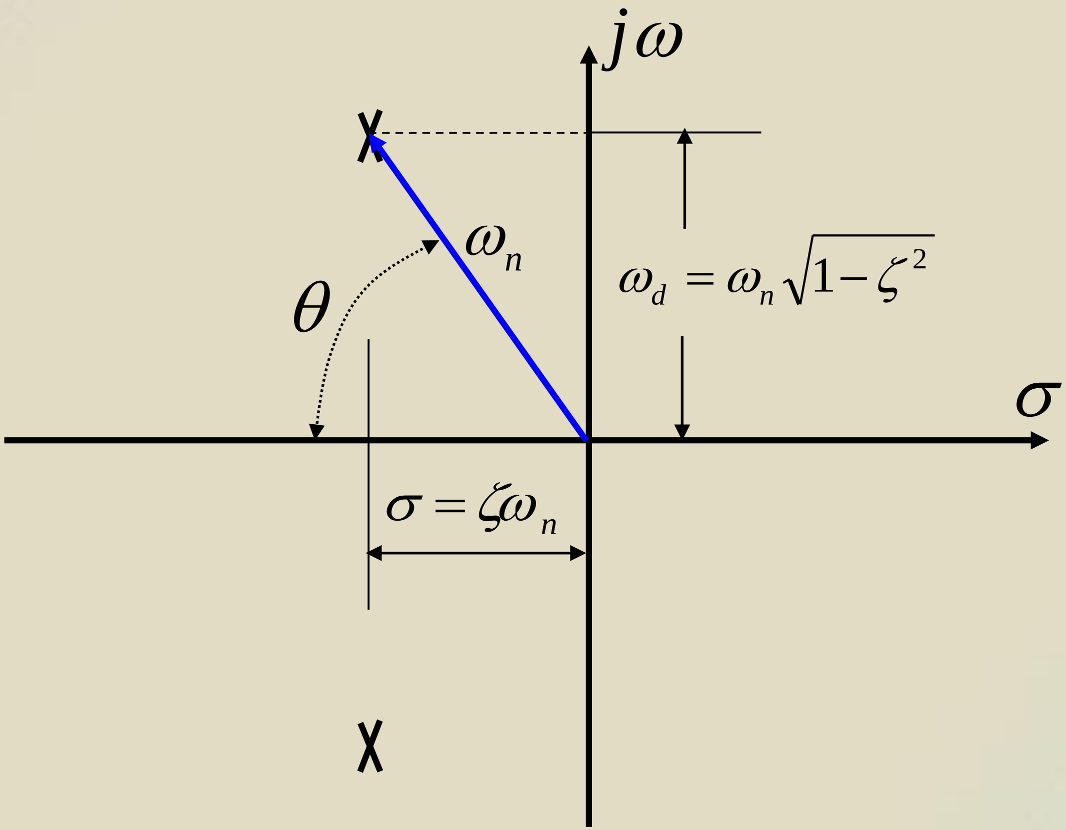
$\omega_d=\omega_n\sqrt{1-\zeta^2}$ Imaginary part of the roots

$$s_1,s_2=-\sigma\pm j\omega_d$$

Transient Response of the 2nd-Order Systems

Relationship between the characteristic-equation roots and σ, ζ, ω_n and ω_d

$$\zeta = \cos\theta$$

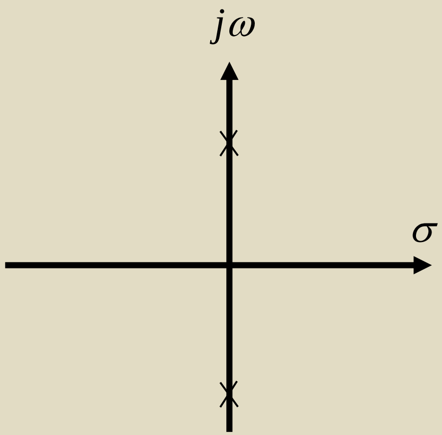


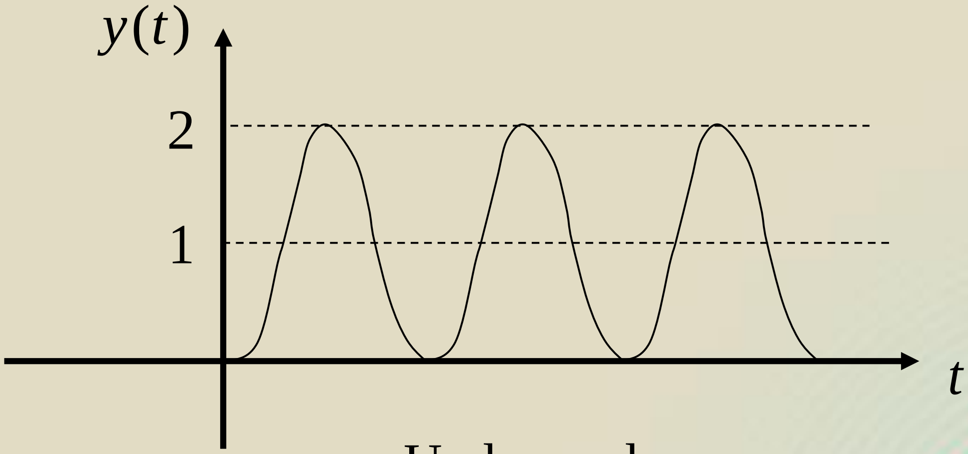
Transient Response of the 2nd-Order Systems

When $\zeta = 0,$ the unit-step response is:

$$s_{1,2} = \pm j\omega_n = \pm j\frac{1}{T}$$

$$y(t) = 1 - \cos\omega_n t$$





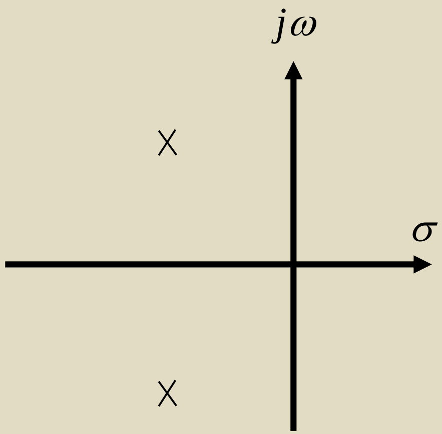
Undamped

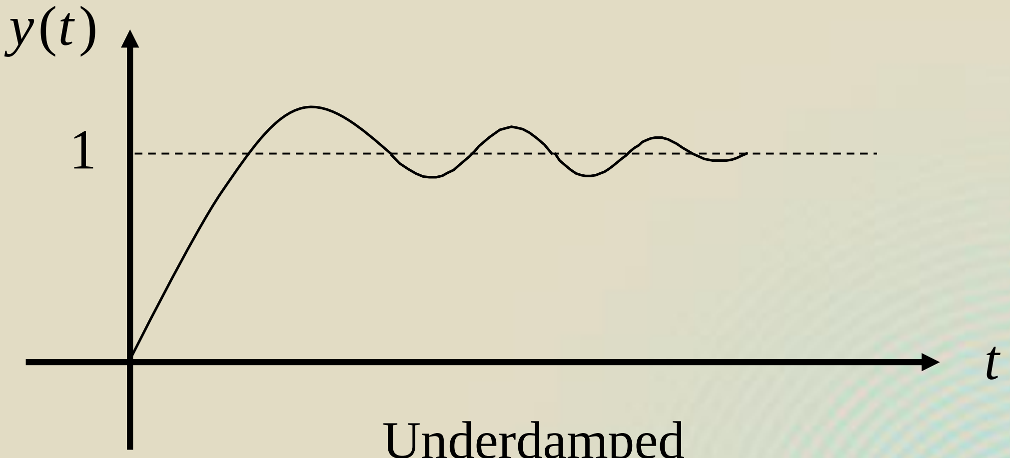
Transient Response of the 2nd-Order Systems

When $0 < \zeta < 1,$ the unit-step response is:

$$Y(s) = G(s)R(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \theta)$$





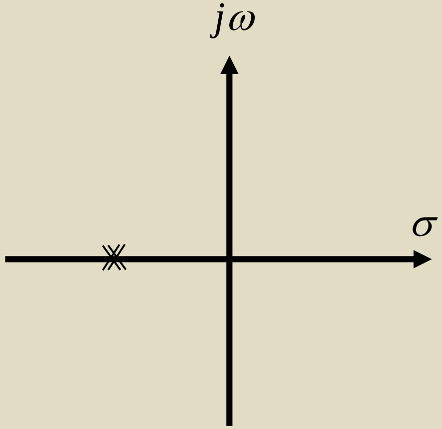
Underdamped

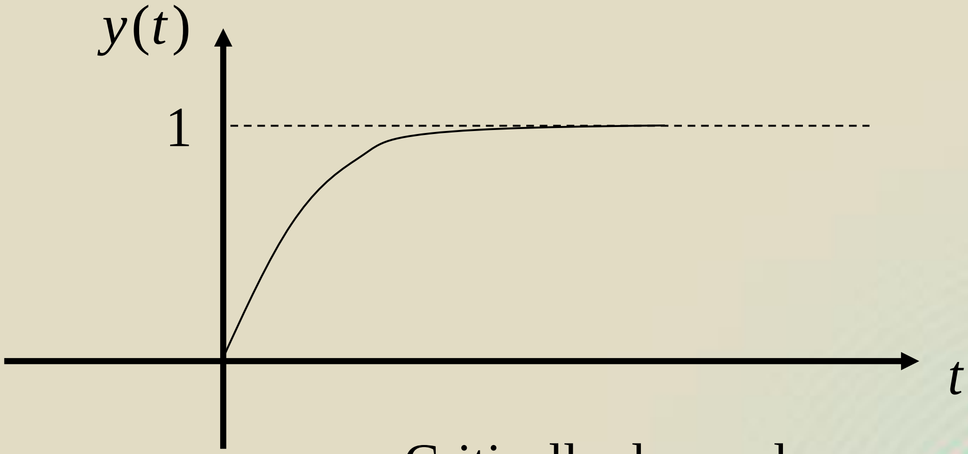
Transient Response of the 2nd-Order Systems

When $\zeta = 1,$ the unit-step response is:

$$s_{1,2} = -\omega_n = -\frac{1}{T}$$

$$y(t) = 1 - (1 + \omega_n t)e^{-\omega_n t}$$





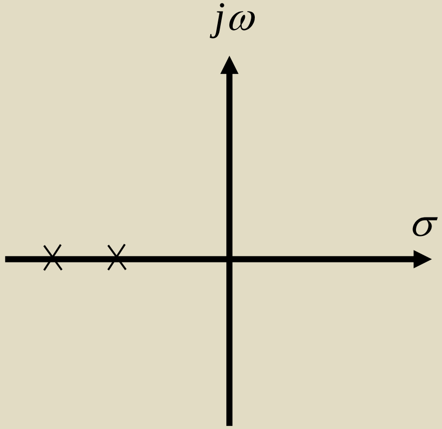
Critically damped

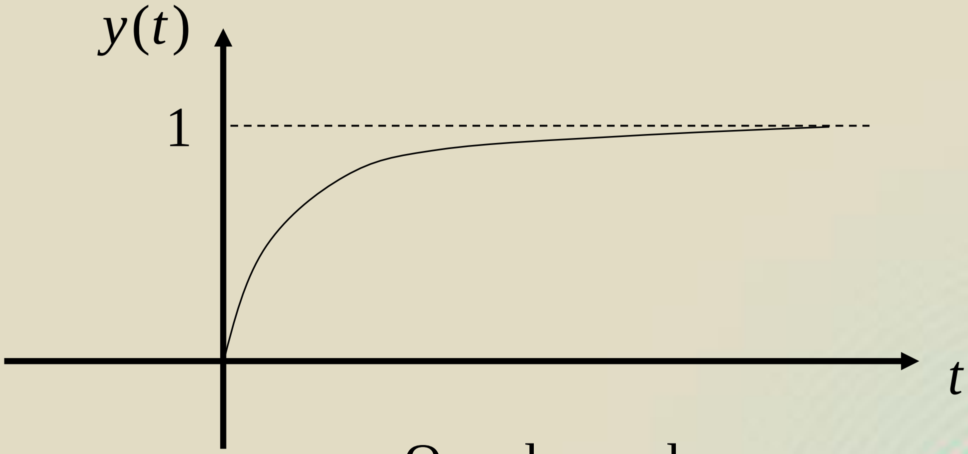
Transient Response of the 2nd-Order Systems

When $\zeta > 1,$ the unit-step response is:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

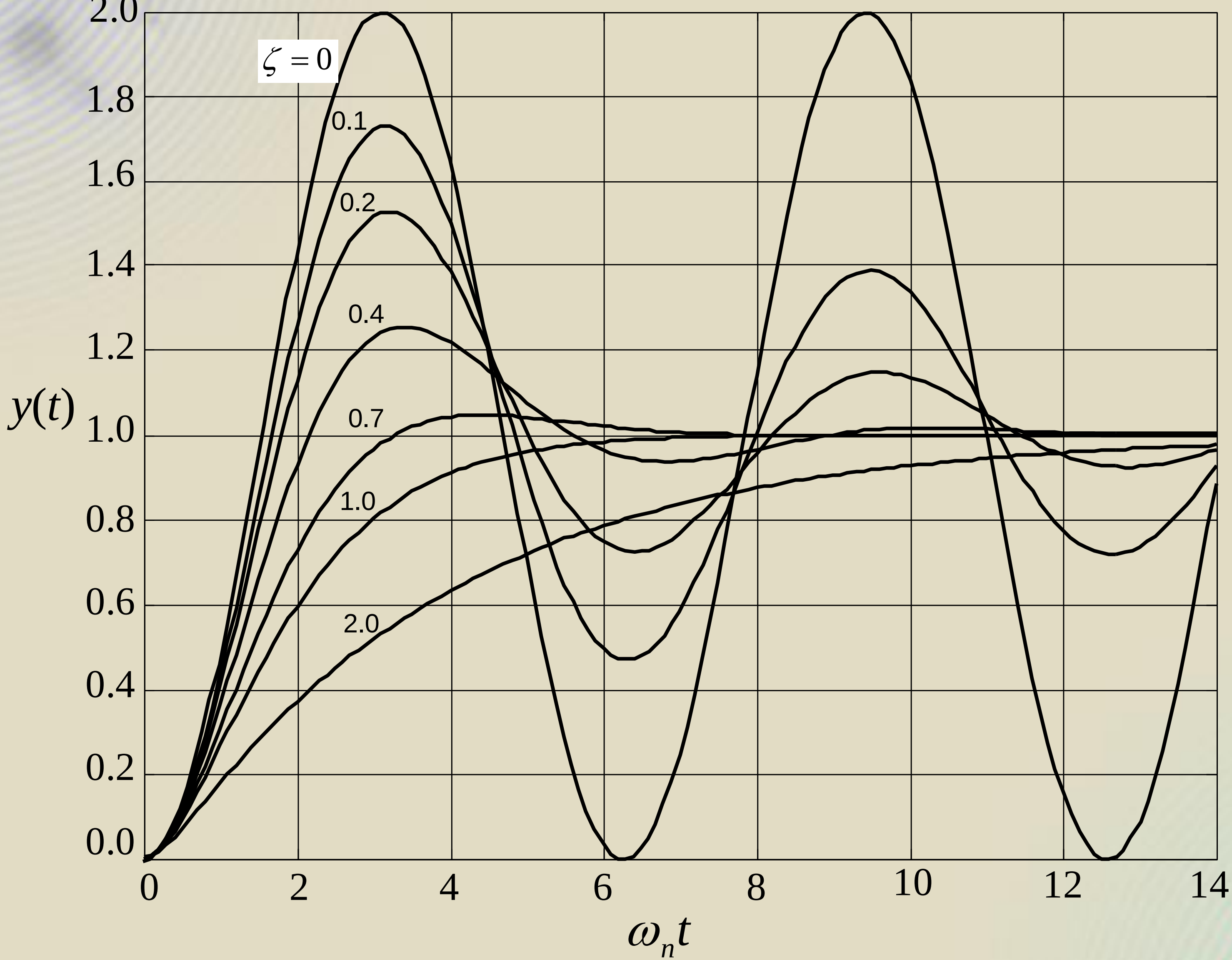
$$y(t) = 1 + k_1 e^{s_1 t} + k_2 e^{s_2 t}$$





Overdamped

Step Response of Systems with Different Damping Ratios



Transient Response of the 2nd-Order Systems

Two useful experimental formulas to estimate the maximum overshoot and settling time of the step response of 2nd-order under damped systems

$$\sigma\% = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

$$t_s = \frac{4}{\zeta\omega_n}$$

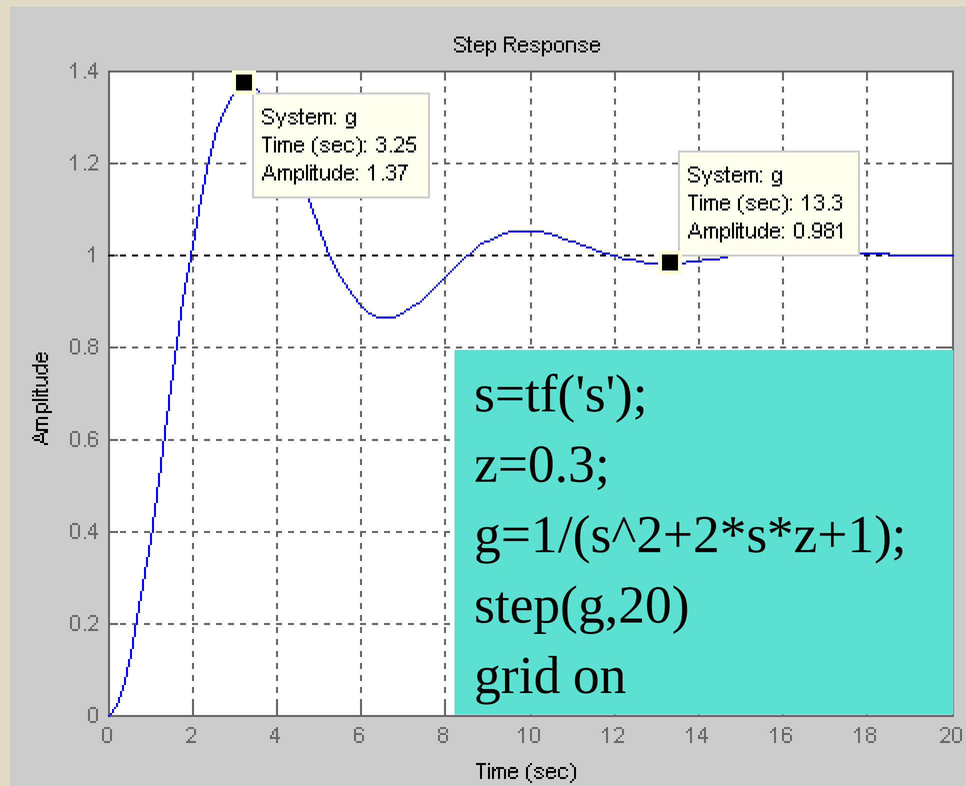
Example

Q: please find the maximum overshoot and settling time of the following system.

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad \text{where} \qquad \omega_n=1 \qquad \zeta=0.3$$

A: $\sigma\%=e^{-\frac{0.3\pi}{\sqrt{1-0.3^2}}}\times 100\%=37.2\%$

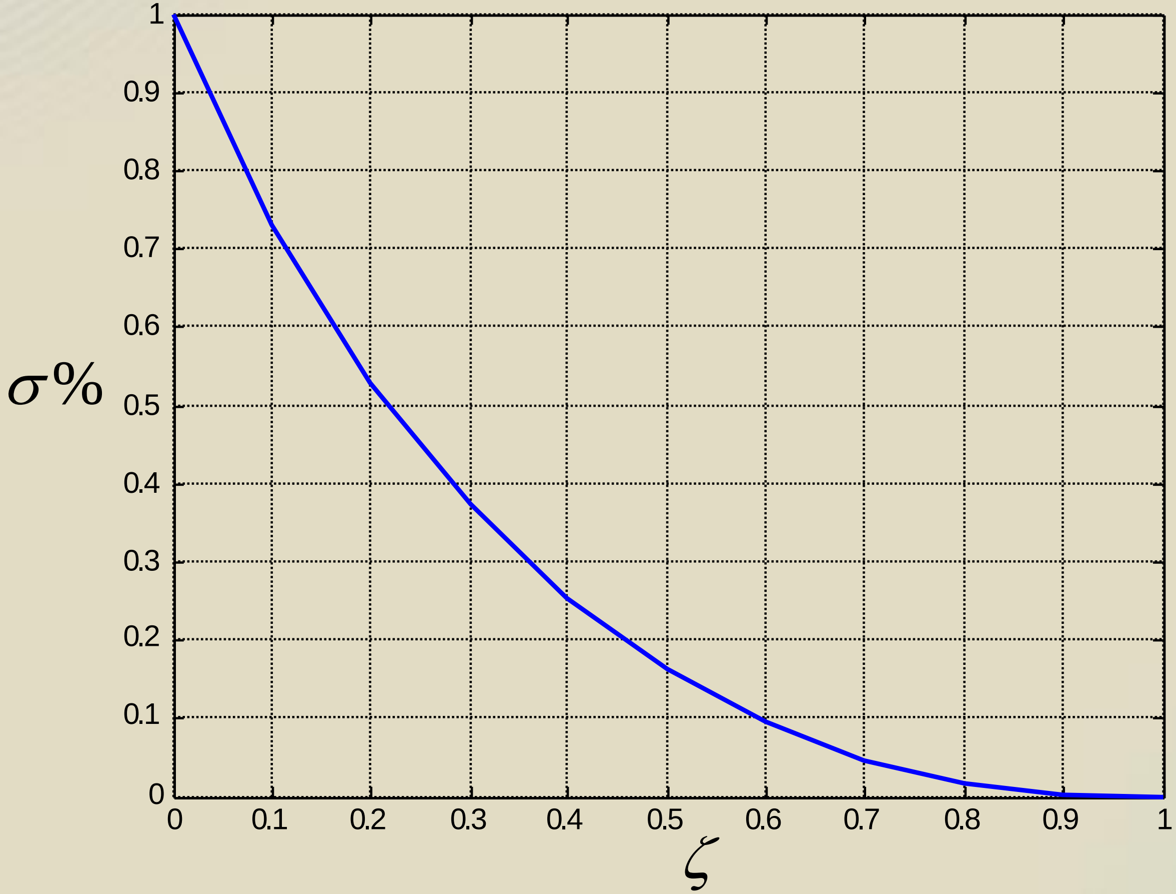
$$t_s=\frac{4}{\zeta\omega_n}=\frac{4}{0.3\times 1}=13.3\text{s}$$



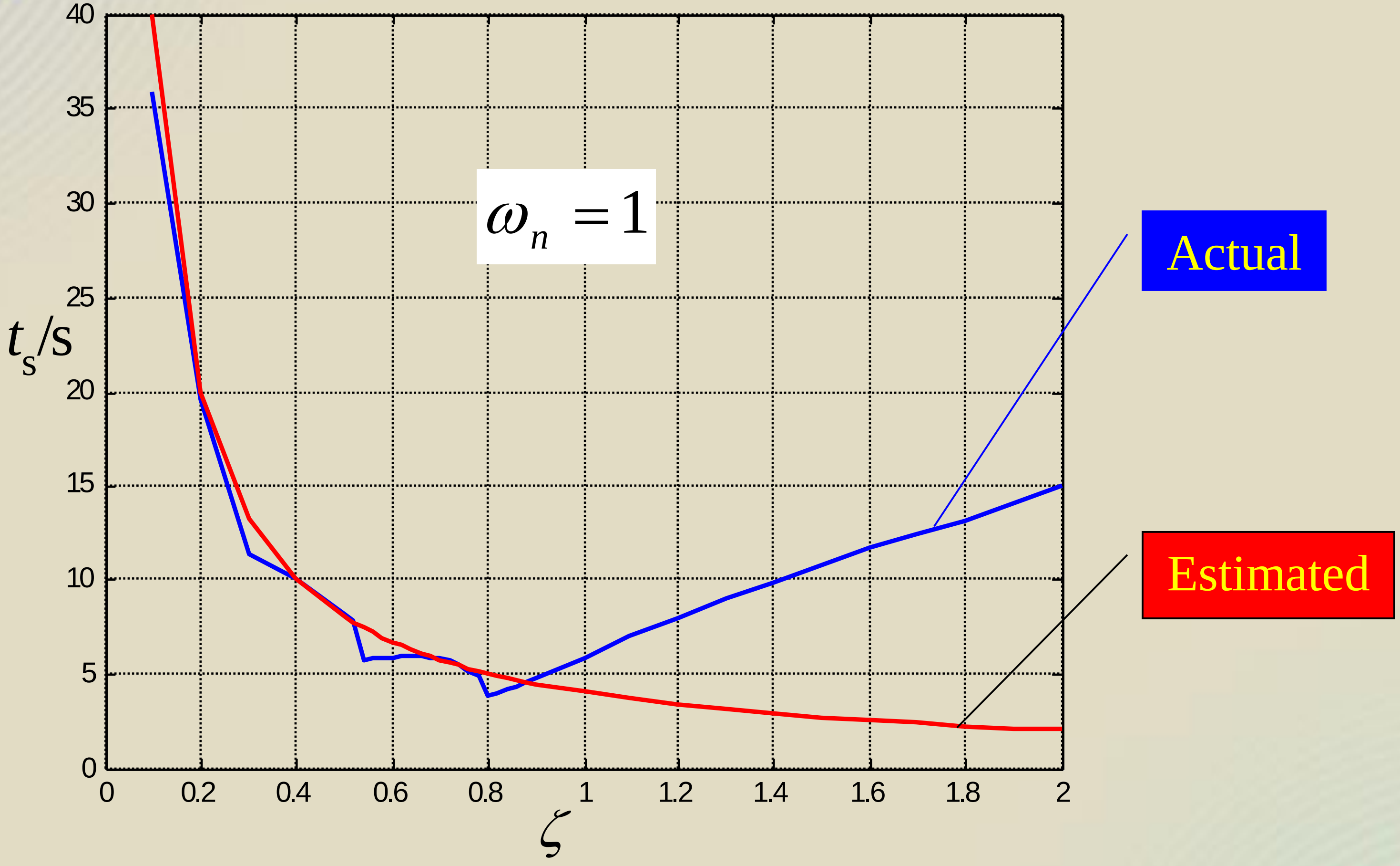

```
s=tf('s');
z=0.3;
g=1/(s^2+2*s*z+1);
step(g,20)
grid on
```

Relationship Between the Damping Ratio and the Maximum Overshoot

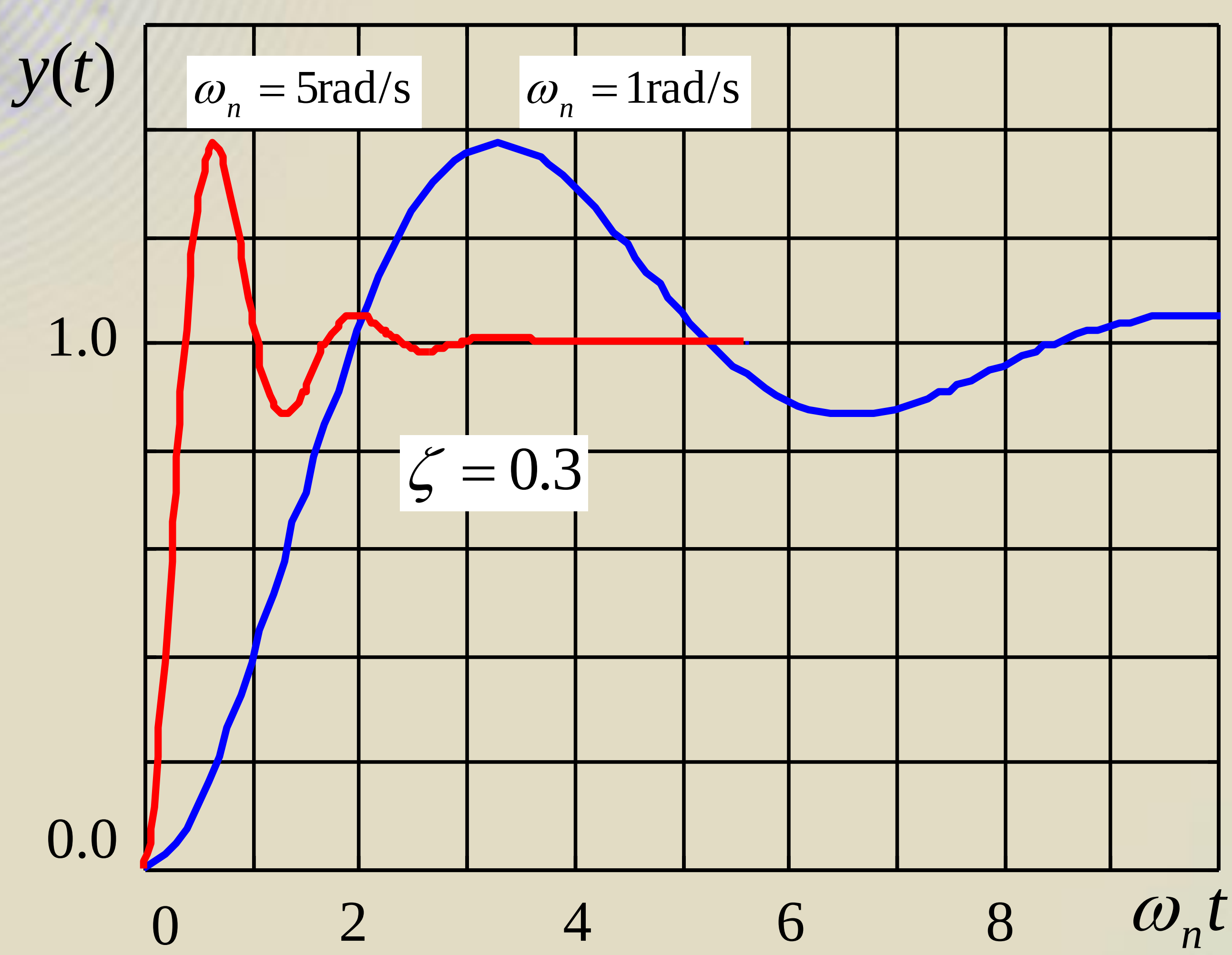
$$\sigma\% = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\% \qquad \zeta\uparrow \quad \sigma\%\downarrow$$



Relationship Between the Damping Ratio and the Settling Time



Relationship Between the Natural Undamped Frequency and Other Criteria



Example

Q: please design a second-order system with a damping ratio equal to 0.707 and a settling time equal to 0.5s. Find the poles of the system and calculate its natural undamped frequency.

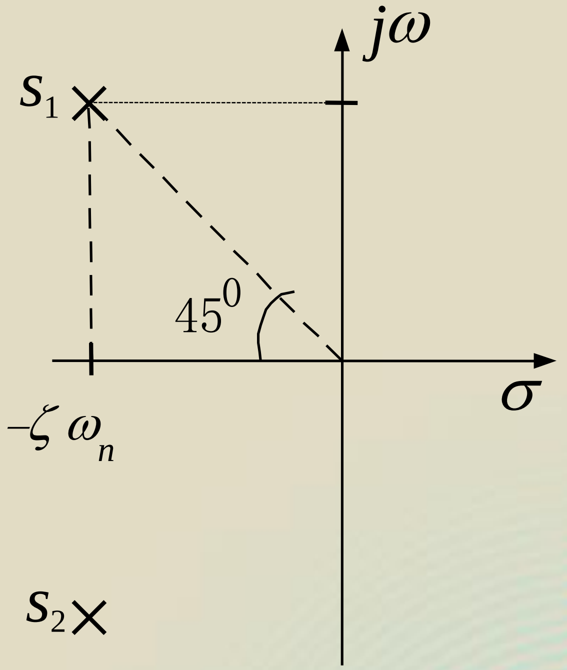
A: because $\zeta = 0.707 = \frac{\sqrt{2}}{2}$ and $\zeta = \cos\theta$

Then $\theta = 45^\circ$

According to $t_s = \frac{4}{\zeta\omega_n}$ we can get $\zeta\omega_n = 8$

Therefore $s_{1,2} = -8 \pm j8$

$$\omega_n = 8\sqrt{2}$$



Wrap-up

- **Stability**
  - S&N Conditions for System Stability
  - Routh-Hurwitz Criterion
  - Definition of time response;
  - Typical test signals;
  - Performance specifications of steady-state response – steady-state error;
  - Impact of disturbance to steady-state error;
  - Impact of parameter variation to steady-state error.
- **Transient Response**
  - Performance Criteria
  - Transient Response of 1st-Order Systems
  - Transient Response of 2nd-Order Systems

Assignment

Page 71

4， (3)， (4)

5， (2)

8, (1), (2), (3)

Page 72

9， (4)

10， 11， 12

单选题 1分

设置

Review questions

The Routh-Hurwitz criterion can be applied to the following system

$$s^4 + 7s^3 + 4s^2 + 17s + 11e^{-2s} = 0$$

A True

B False

提交

Review questions

When a row of Routh’ s tabulation contains all zeros before the tabulation ends, this means that the equation has roots on the imaginary axis of the s-plane.

A True

B False

提交

Review questions

If a unity-feedback control system is of type two, it is certain that the steady-state error of the system to a step input or a ramp input will be zero.

A True

B False

提交

Discussion

- If a system has a steady-state error equal to infinite, does that mean the system is unstable?
- For an unstable system, must it have an infinite steady-state error?
- If a system does not have a constant steady-state error, does that mean the system unstable?

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Discussion

If a system has a steady-state error equal to infinite, does that mean the system is unstable?

A Yes

B No

提交

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设置

Discussion

For an unstable system, must it have an infinite steady-state error?

- A Yes
- B No

提交

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Discussion

If a system does not have a constant steady-state error, does that mean the system unstable?

- A Yes
- B No

提交