Digital Control Systems

Chapter 7

Review of Last Chapter

- Controllability and observability
- What is state feedback control
- How to design a state feedback controller
- State feedback with integral control
- How to design a state observer
- How to design a state feedback controller with feedback states coming from an observer

Outlines

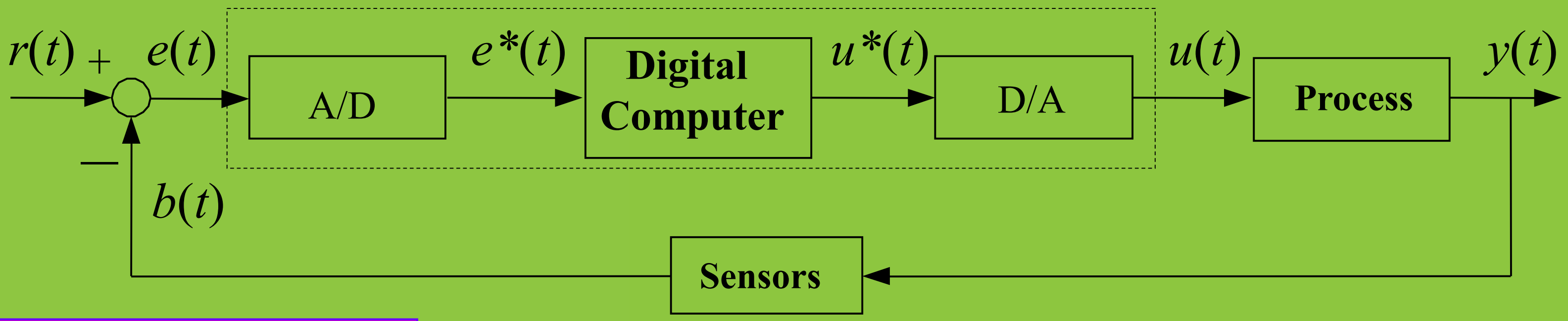
- Basics about digital control systems (discrete-data control systems)
- Interfaces between continuous and discrete-data systems ----- the Sample-and-hold device
- Tools for analyzing and synthesizing discrete-data systems ----- the z-transform and inverse z-transform
- Discrete-data models of control systems
- Pulse-transfer functions
- Discrete state equation
- Discretization of continuous system
  - from transfer function to z-transfer function
  - from state equation to discrete state equation

Basics about Digital Control Systems

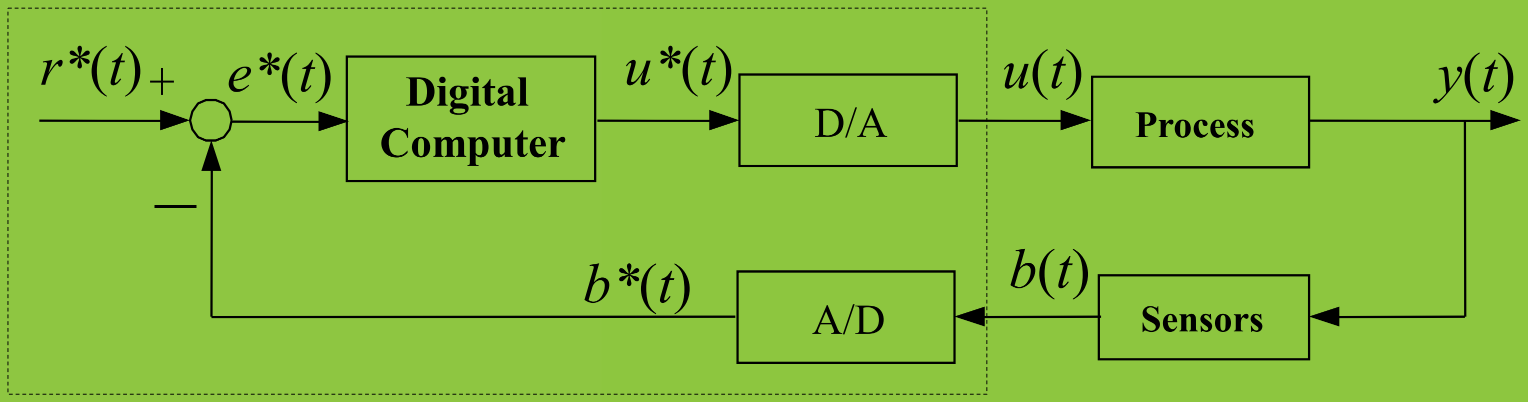
A system with digital controller is called as a digital control system.

The most distinct characteristic of a digital control system is that there exist discrete blocks in the control system.

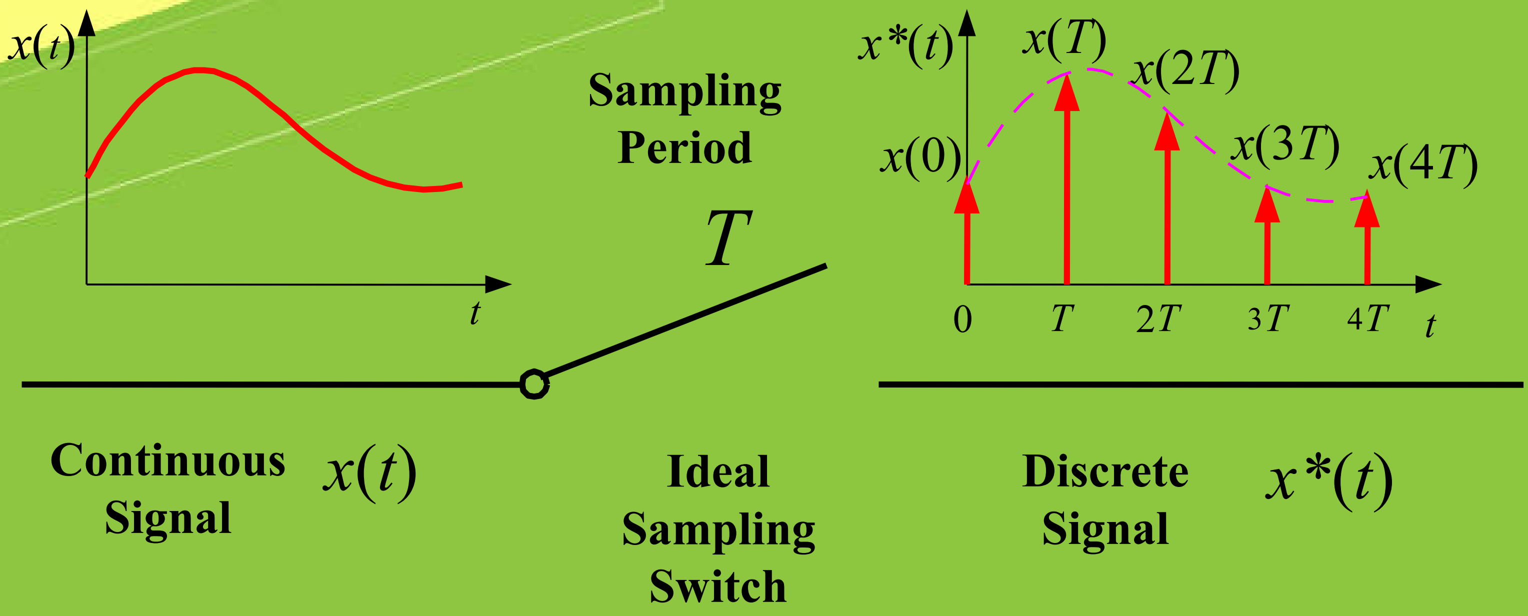
Inputs are analog signals



Inputs are digital signals



A/D Converter

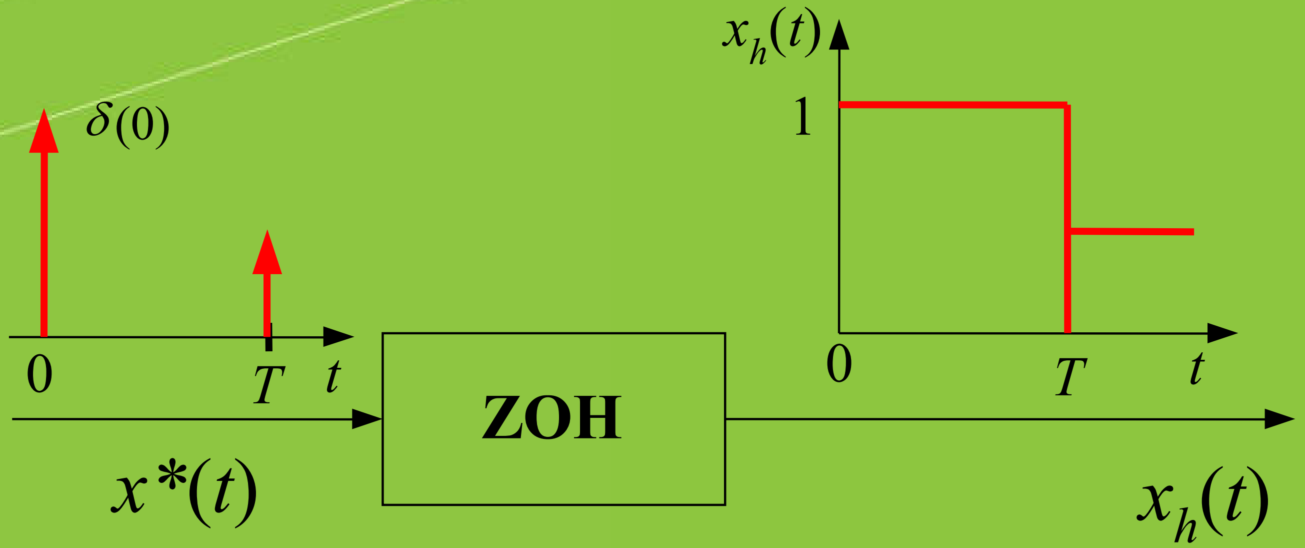


The function of an A/D converter can be represented by an ideal sampling switch. The output of the ideal sampler is a train of impulses with the magnitude of x(t) at each sampling time carried by the strength of the impulse.

Mathematically, the sampled discrete signal sequence can be expressed by the convolution of the continuous signal and an impulse train with each impulse separated by sampling period T.

$$x^*(t) = x(0)\delta(t) + x(T)\delta(t-T) + x(2T)\delta(t-2T) + \cdots$$

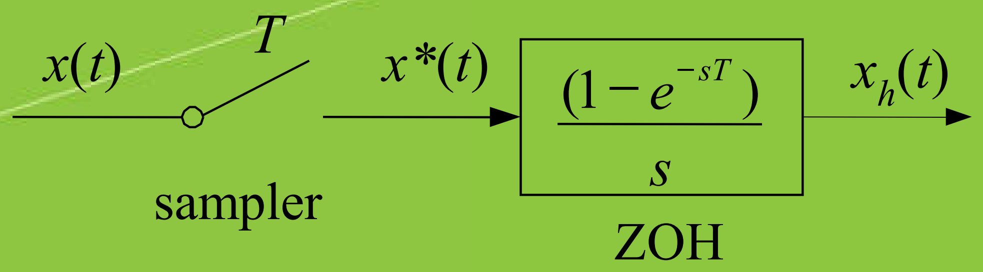
D/A Converter

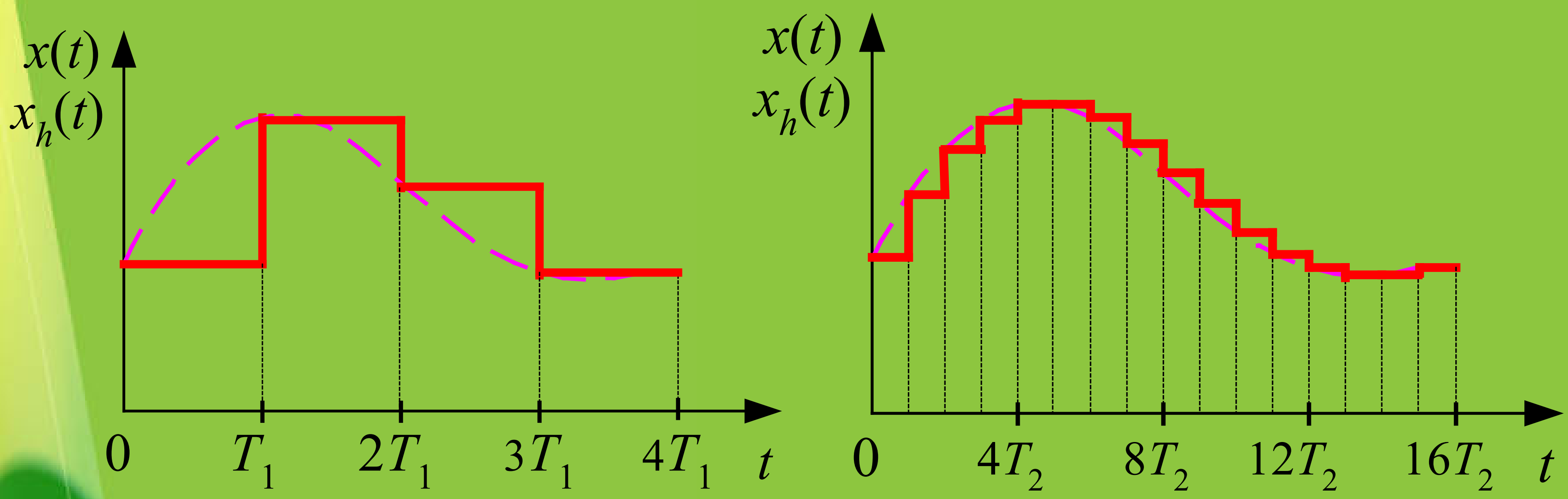


The D/A converter is usually a zero-order hold (ZOH). The ZOH simply holds the magnitudes of the signal carried by the incoming impulse at a given time instant for the entire sampling period until the next impulse arrives. The output of the ZOH is a staircase approximation of the input impulse sequence.

$$x_h(t) = 1(t) - 1(t-T) \qquad X_h(s) = \frac{1}{s} - \frac{1}{s}e^{-Ts} = \frac{1-e^{-Ts}}{s}$$

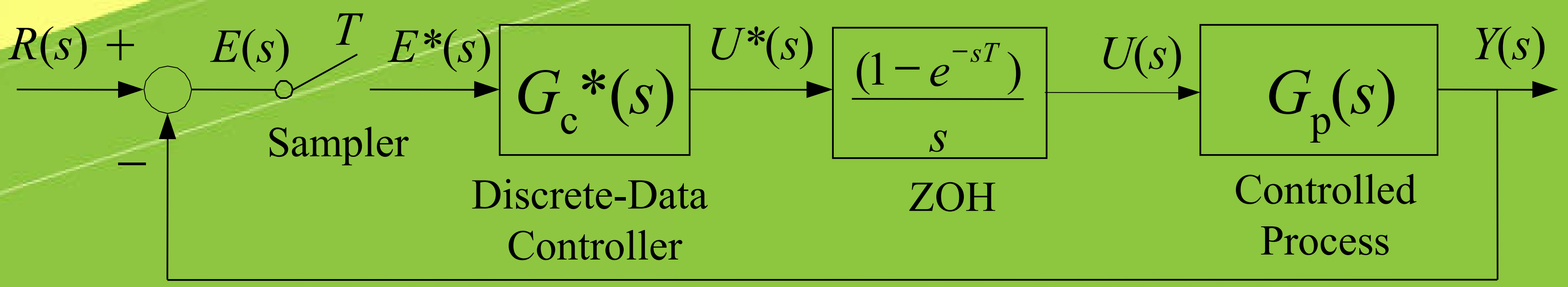
Sample-and-hold Device





The output of the ZOH is a staircase approximation of the input to the ideal sampler. As the sampling period approaches zero, the output of the ZOH approaches the input.

Block Diagrams of Digital Control Systems



Discrete-data sequence and its Laplace transformation:

$$x^*(t) = x(0)\delta(t) + x(T)\delta(t-T) + x(2T)\delta(t-2T) + \cdots$$

$$X^*(s) = L[x^*(t)] = L[x(0)\delta(t) + x(T)\delta(t-T) + x(2T)\delta(t-2T) + \cdots]$$

$$= x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \cdots$$

$$X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

Z - Transform

Consider the sequence $x(k),\ k = 0, 1, 2, \cdots$, the Z-transform of $x(k)$ is defined as:

$$X(z) = z - transform\ \ of\ \ x(k) = Z[x(k)]$$

$$= \sum_{k=0}^{\infty} x(k) z^{-k}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

where z is a complex variable with real and imaginary parts.

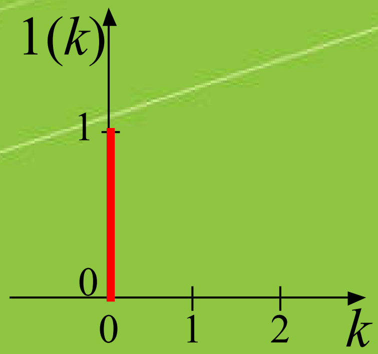
Comparing with the Laplace transform of a discrete-data sequence

$$X^*(s) = \sum_{k=0}^{\infty} x(kT) e^{-kTs}$$

$$z^{-1} = e^{-Ts}$$

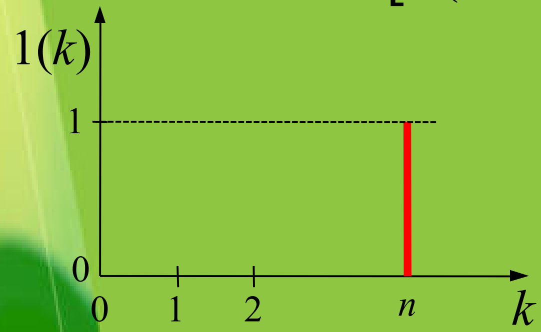
Z-transform may be regarded as a special case of the Laplace transform of a discrete-date sequence with sampling time T equal to 1.

Z - Transform



$$Z[\delta(k)] = \delta(0)z^{0} + \delta(1)z^{-1} + \delta(2)z^{-2} + \cdots = 1$$

$$Z[\delta(k-n)] = \delta(-n)z^{0} + \delta(1-n)z^{-1} + \cdots + \delta(0)z^{-n} + \cdots = z^{-n}$$



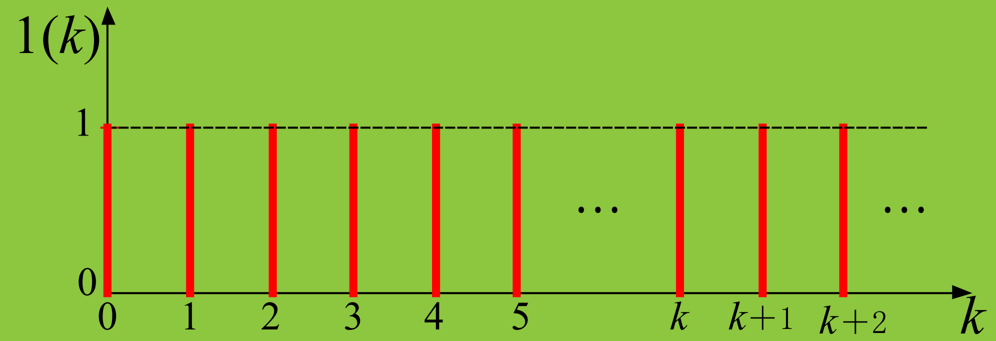
Example – 7.1

Q: please find the z-transform of the unit-step function $x(t) = 1(t)$

A: the unit-step function is a continuous function. It needs to be discretized before applying z-transform.

Assume the sampling period is T, after discretized, the sequence becomes:

$$x(kT) = 1 \qquad k = 0, 1, 2\cdots$$



$$X(z) = Z\left[x(kT)\right] = \sum_{k=0}^{\infty} 1 \cdot z^{-k} = \sum_{k=0}^{\infty} z^{-k} = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

$$X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z-1} \qquad if \quad |z| > 1$$

geometric progression

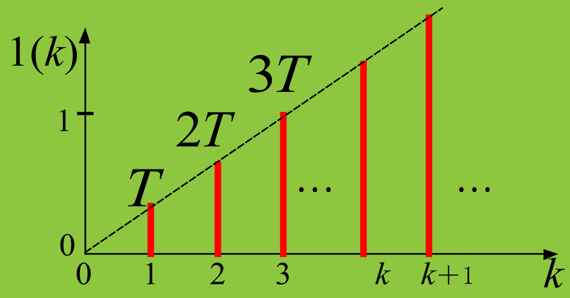
Example – 7.2

Q: please find the z-transform of the unit-ramp function $x(t) = t \quad t \geq 0$

A: the unit-ramp function is a continuous function. It needs to be discretized before applying z-transform.

Assume the sampling period is T, after discretized, the sequence becomes:

$$x(kT) = kT \qquad k = 0,1,2,\cdots$$



$$X(z) = \sum_{k=0}^{\infty} x(kT) \cdot z^{-k} = \sum_{k=0}^{\infty} kTz^{-k} = T(z^{-1} + 2z^{-2} + 3z^{-3} + \cdots)$$

$$X(z) = \frac{Tz^{-1}}{(1-z^{-1})^2} = \frac{Tz}{(z-1)^2} \qquad if \quad |z| > 1$$

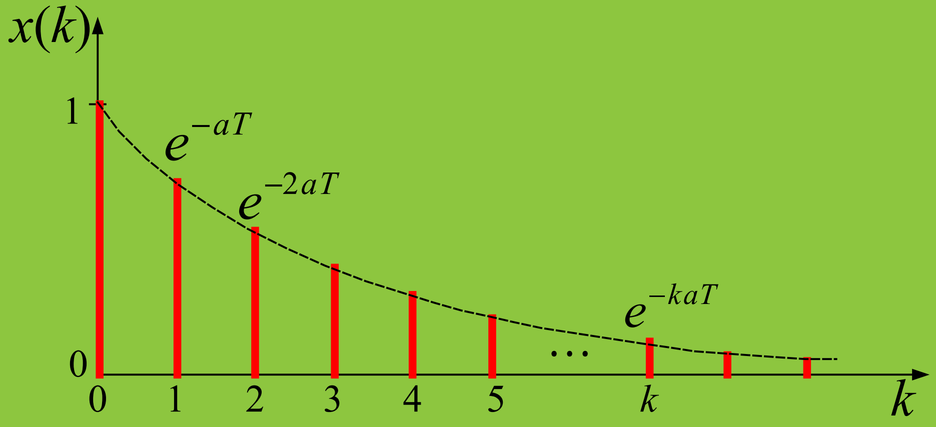
Example – 7.3

Q: please find the z-transform of the exponential function $x(t) = e^{-at} \quad t \geq 0$

A: the exponential function is a continuous function. It needs to be discretized before applying z-transform.

Assume the sampling period is T, after discretized, the sequence becomes:

$$x(kT) = e^{-akT} \qquad k = 0,1,2,\cdots$$



$$X(z) = \sum_{k=0}^{\infty} x(kT) \cdot z^{-k} = \sum_{k=0}^{\infty} e^{-akT} z^{-k} = 1 + e^{-aT} z^{-1} + e^{-2aT} z^{-2} + \cdots$$

$$= \frac{1}{1 - e^{-aT} z^{-1}} = \frac{z}{z - e^{-aT}} \qquad if \quad |z| > e^{-aT}$$

Some Important Theorems of the z-Transform

Real Translation (Time delay) $Z\left[x(k-m)\right]=z^{-m}X\left(z\right)$

Real Translation (Time advance)

$$Z\left[x(k+m)\right]= z^{m}X(z)-z^{m}x(0)-z^{m-1}x(1)-\cdots-zx(m-1)$$

Complex translation $Z\left[e^{-akT}x(k)\right]=X\left(ze^{aT}\right)$

Initial-value theorem $\lim_{k\to 0}x\left(kT\right)=\lim_{z\to\infty}X\left(z\right)$

Final-value theorem $\lim_{k\to\infty}x\left(kT\right)=\lim_{z\to 1}\left(1-z^{-1}\right)X\left(z\right)$

Real Convolution

$$Z\left[x_1(k)*x_2\left(k\right)\right]=Z\left[\sum_{k=0}^{m}x_1\left(k\right)x_2\left(k-m\right)\right]=X_1\left(z\right)X_2\left(z\right)$$

	Wrong	Correct
Table 7.1 Row 3 Column 4	z^{-1}	z^{-k}
Table 7.1 Row 6 Column 4	$\frac{z}{z-e^{aT}}$	$\frac{z}{z-e^{-aT}}$
Table 7.1 Last Row Column 3	a^{kT}	a^{k}
Table 7.2 Row 7 Column 3	$X(ze^{-aT})$	$X(ze^{aT})$

Some More Examples

Q: please find the z-transform of the following functions

$1(k-2)$

$$Z[1(k-2)] = z^{-2}Z[1(k)] = z^{-2}\frac{1}{1-z^{-1}} = \frac{z^{-2}}{1-z^{-1}}$$

$e^{-ak}\sin\omega k$

$$Z[\sin\omega k] = \frac{z\sin\omega}{z^2-2z\cos\omega+1}$$

using $e^{a}z$ to replace z

$$Z\left[e^{-ak}\sin\omega k\right] = \frac{e^{a}z\sin\omega}{e^{2a}z^2-2e^{a}z\cos\omega+1}$$

Inverse z-Transform

Just as in the Laplace transform, one of the major objectives of the z-transform is that algebraic manipulation can be made first in the z-domain, and then the final time response is determined by the inverse z-transform.

In general, the inverse z-transform of a function $X(z)$ yields information on $x(kT)$ only, not on $x(t)$. In other words, the z-transform carries information only at the sampling instants.

The inverse z-transform can be carried out by one of the following three methods:

- Power-series method
- Partial-fraction expansion
- Residue method

Example – 7.7

Q: please find the inverse z-transform of $X(z)=\dfrac{1}{(z-1)(z-0.5)}$

dividend

divisor

A: use power-series method $(z-1)(z-0.5)=z^2-1.5z+0.5$

$$
\begin{array}{rl}
 & z^{-2}+1.5z^{-3}+1.75z^{-4}+\cdots \\
z^2-1.5z+0.5 \big) & 1 \\
 & \underline{1-1.5z^{-1}+0.5z^{-2}} \\
 & 1.5z^{-1}-0.5z^{-2} \\
 & \underline{1.5z^{-1}-2.25z^{-2}+0.75z^{-3}} \\
 & 1.75z^{-2}-0.75z^{-3} \\
 & \underline{1.75z^{-2}-2.625z^{-3}+0.875z^{-4}} \\
 & \vdots \qquad\qquad \vdots \qquad\qquad \vdots
\end{array}
$$

quotient

$$X(z)=z^{-2}+1.5z^{-3}+1.75z^{-4}+\cdots$$

$$x(0)=x(1)=0,\quad x(2)=1,\quad x(3)=1.5,\quad x(4)=1.75,\quad \cdots$$

Example – 7.7

Q: please find the inverse z-transform of $X(z)=\dfrac{1}{(z-1)(z-0.5)}$

A: use partial-fraction expansion

$$X(z)=\frac{2}{z-1}+\frac{-2}{z-0.5}$$ not in the look-up table

$$\frac{X(z)}{z}=\frac{1}{z(z-1)(z-0.5)}=\frac{2}{z}+\frac{2}{z-1}-\frac{4}{z-0.5}$$

$$\frac{z}{z-1}$$

$$X(z)=2+\frac{2z}{z-1}-\frac{4z}{z-0.5}$$

$$\frac{z}{z-a}$$

$$x(k)=2\delta(k)+2(k)-4\times(0.5)^{k}$$

Example – 7.7

Q: please find the inverse z-transform of $X(z) = \dfrac{1}{(z-1)(z-0.5)}$

A: use residue method

$$x(k) = \sum_m RES[x(z)z^{k-1}]_{z=z_m} = \sum_m RES\left[\frac{z^{k-1}}{(z-1)(z-0.5)}\right]_{z=z_m}$$

where z_m is the poles of the expression $x(z)z^{k-1}$

when $k \geq 1$, $x(z)z^{k-1}$ has two poles:

$$x(k) = \left[(z-1)\frac{z^{k-1}}{(z-1)(z-0.5)}\right]_{z=1} + \left[(z-0.5)\frac{z^{k-1}}{(z-1)(z-0.5)}\right]_{z=0.5}$$

$$= 2 - 2\times(0.5)^{k-1} = 2 - 4\times(0.5)^k$$

Example – 7.7

when $k=0$, $x(z)z^{k-1}$ has three poles:

$$x(0)=\left[z\frac{1}{z(z-1)(z-0.5)}\right]_{z=0}+\left[(z-1)\frac{1}{z(z-1)(z-0.5)}\right]_{z=1}+\left[\frac{z-0.5}{z(z-1)(z-0.5)}\right]_{z=0.5}$$

$$=2+2-4=0$$

$$x(k)=\begin{cases}0 & k=0\\ 2-4\times(0.5)^k & k\geq 1\end{cases}$$

Example – 7.8

Q: please find the inverse z-transform of $X(z) = \dfrac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$

A: use partial-fraction expansion

$$\frac{X(z)}{z} = \frac{(1-e^{-aT})}{(z-1)(z-e^{-aT})} = \frac{1}{z-1} - \frac{1}{z-e^{-aT}}$$

$$X(z) = \frac{z}{z-1} - \frac{z}{z-e^{-aT}}$$

$$x(k) = 1 - e^{-akT}$$

Example – 7.8

Q: please find the inverse z-transform of $X(z)=\dfrac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$

A: use residue method

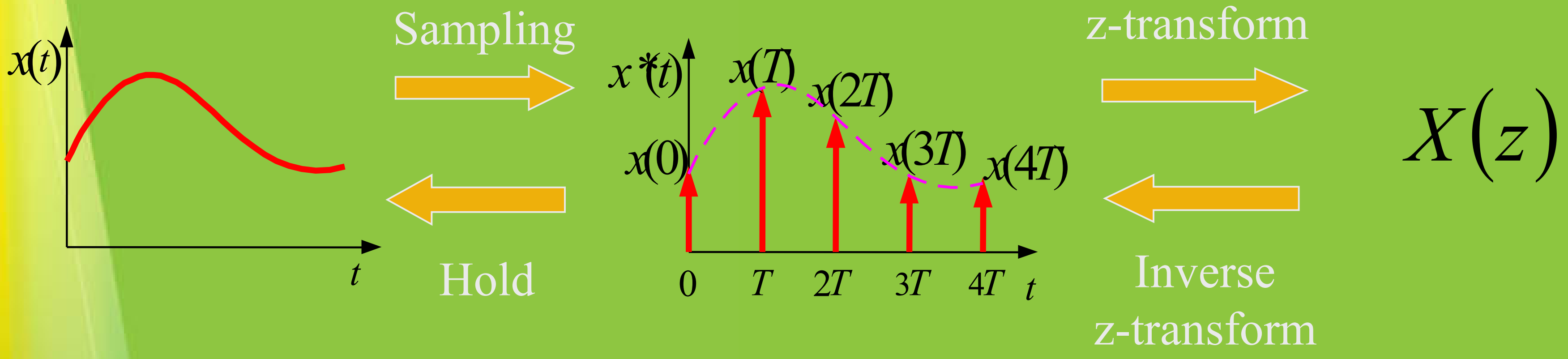
$$x(k)=\sum_{m=1}^{2}\mathrm{Res}\left[\frac{(1-e^{-aT})z\cdot z^{k-1}}{(z-1)(z-e^{-aT})}\right]_{z=z_m}$$

$$x(k)=\left[(z-1)\frac{(1-e^{-aT})z^{k}}{(z-1)(z-e^{-aT})}\right]_{z=1}+\left[(z-e^{-aT})\frac{(1-e^{-aT})z^{k}}{(z-1)(z-e^{-aT})}\right]_{z=e^{-aT}}$$

$$=1-e^{-akT}$$

Remarks

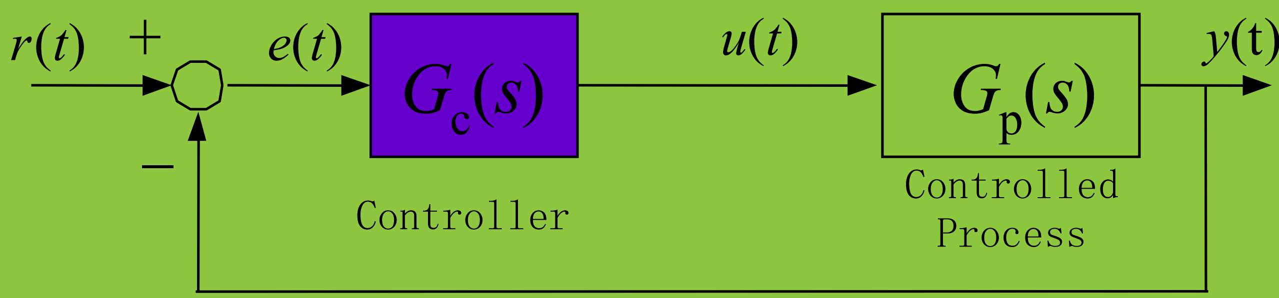
z-transform and inverse z-transform are transformations between discrete-data sequences and z-expression. The z-expression has nothing to do with the continuous signals

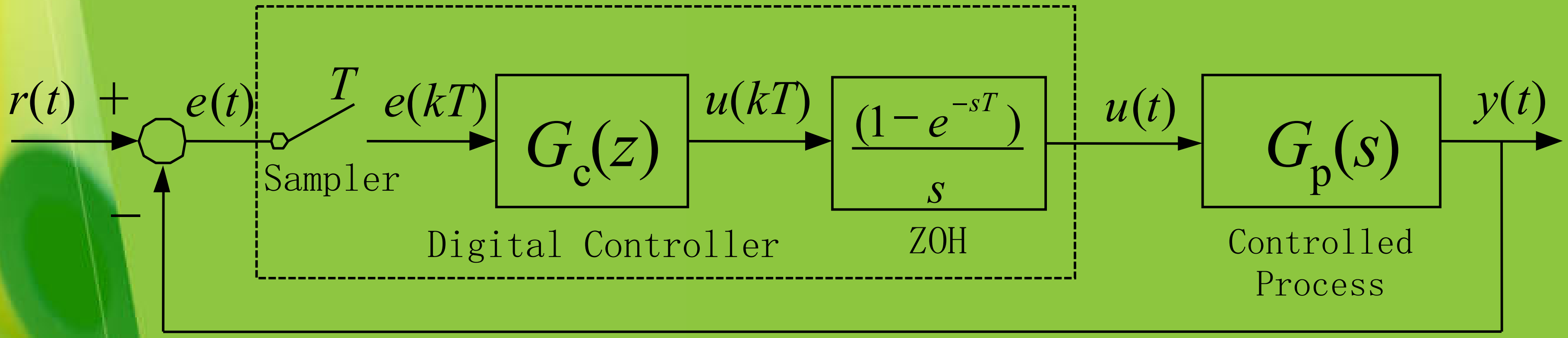


Discrete-Data Models of Control Systems

Why discrete-data models?

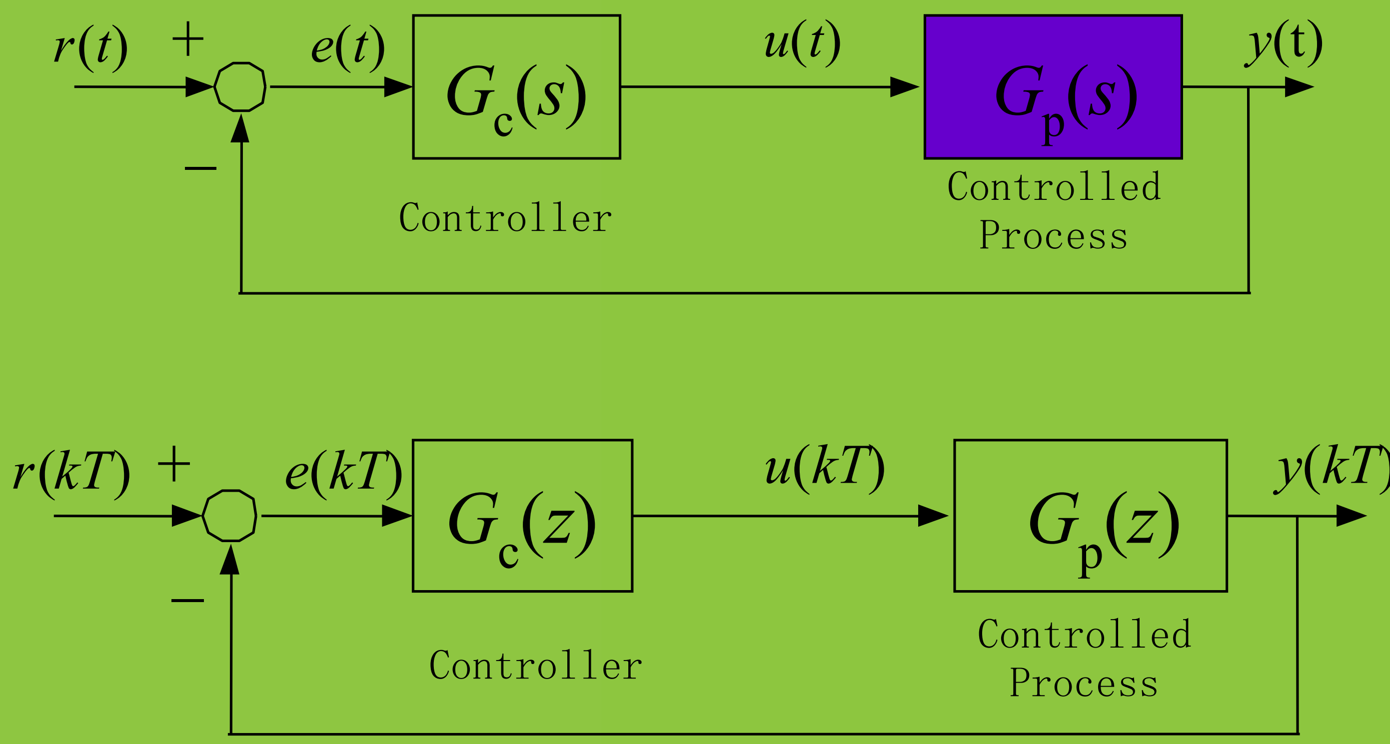
To implement a controller using digital computer





Discrete-Data Models of Control Systems

To design a digital controller using discrete-data methods



How to Get Discrete-Data Models of Control Systems

Forward Difference Method

$$\frac{dy(t)}{dt} + ay(t) = u(t)$$

$$\frac{y(k+1) - y(k)}{T} + ay(k) = u(k)$$

$$y(k+1) + (aT - 1)y(k) = Tu(k)$$

Backward Difference Method

$$\frac{y(k) - y(k-1)}{T} + ay(k) = u(k)$$

How to Get Discrete-Data Models of Control Systems

For a single-input-single-output linear time-invariant system, its discrete-data model can be expressed as:

Backward difference form:

$$y(k)+a_1 y(k-1)+\cdots+a_n y(k-n) \qquad n \geq m$$

$$=b_0 u(k)+b_1 u(k-1)+\cdots+b_m u(k-m)$$

Foreword difference form:

$$y(k+n)+a_1 y(k+n-1)+\cdots+a_n y(k) \qquad n \geq m$$

$$=b_0 u(k+m)+b_1 u(k+m-1)+\cdots+b_m u(k)$$

Example – 7.9

Q: Use z-transform and inverse z-transform to solve the given difference equation

$$y(k+2) - 5y(k+1) + 6y(k) = u(k)$$

$$u(k) = \delta_0(k) \qquad y(0) = 1, \quad y(1) = 0$$

A: $$Z[y(k+1)] = zY(z) - zy(0)$$

$$Z[y(k+2)] = z^2Y(z) - z^2y(0) - zy(1)$$

$$z^2Y(z) - z^2y(0) - zy(1) - 5zY(z) + 5zy(0) + 6Y(z) = U(z)$$

$$[z^2 - 5z + 6]Y(z) = z^2y(0) + zy(1) - 5zy(0) + U(z)$$

$$\because \quad u(k) = \delta_0(k) \qquad \therefore \quad U(z) = 1$$

Example – 7.9

$$Y(z)=\frac{z^2y(0)+z[y(1)-5y(0)]}{z^2-5z+6}+\frac{1}{z^2-5z+6}=\frac{z^2-5z+1}{z^2-5z+6}$$

$$\frac{Y(z)}{z}=\frac{z^2-5z+1}{z(z-2)(z-3)}=\frac{\frac{1}{6}}{z}+\frac{\frac{5}{2}}{z-2}+\frac{-\frac{5}{3}}{z-3}$$

$$Y(z)=\frac{1}{6}+\frac{\frac{5}{2}z}{z-2}-\frac{\frac{5}{3}z}{z-3}$$

$$y(k)=\frac{1}{6}\delta_0(k)+\frac{5}{2}\times(2)^k-\frac{5}{3}\times(3)^k$$

Pulse-Transfer Function

Difference equation of a SISO LTI discrete system:

$$y(k+n)+a_1 y(k+n-1)+\cdots+a_n y(k)$$
$$=b_0 u(k+m)+b_1 u(k+m-1)+\cdots+b_m u(k) \qquad n \geq m$$

Apply z-transform to both sides:

$$(z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n) Y(z) = (b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m) U(z)$$

Define the z-transfer function of the discrete system:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n} \qquad n \geq m$$

Pulse-Transfer Function

By definition:

$$Y(z) = G(z)U(z)$$

If $u(k) = \delta(k)$ then $U(z) = 1$

It yields: $$Y(z) = G(z)$$

So, the z-transfer function of the discrete system is also called as the pulse transfer function.

Discrete State Equations

Example 7.10: please find the discrete state equation of the system with the given difference equation.

$$y(k+2)+3y(k+1)+2y(k)=u(k)$$

$$\dot{X}=AX+BU$$

$$X(k+1)=AX(k)+BU(k)$$

Discrete State Equations

Example 7.10: please find the discrete state equation of the system with the given difference equation.

$$y(k+2)+3y(k+1)+2y(k)=u(k)$$

A: choose $x_1(k)=y(k) \qquad x_2(k)=y(k+1)$

$$x_1(k+1)=x_2(k)$$

$$x_2(k+1)=-2x_1(k)-3x_2(k)+u(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\cdot\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Discrete State Equations

$$X(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \qquad G = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$X(k+1) = GX(k) + Hu(k)$$

$$y(k) = CX(k)$$

Discrete State Equations

Example 7.11: please find the discrete state equation of the system with the given difference equation.

$$y(k+3)+3y(k+2)+8y(k+1)+12y(k)$$
$$=u(k+2)+7u(k+1)+13u(k)$$

A: Applying z-transform to the above difference equation

$$(z^3+3z^2+8z+12)Y(z)=(z^2+7z+13)U(z)$$

$$G(z)=\frac{Y(z)}{U(z)}=\frac{z^2+7z+13}{z^3+3z^2+8z+12}$$

$$\frac{Y(z)}{U(z)}=\frac{Y(z)}{W(z)}\cdot\frac{W(z)}{U(z)}$$

$$\frac{W(z)}{U(z)}=\frac{1}{z^3+3z^2+8z+12}$$

$$\frac{Y(z)}{W(z)}=z^2+7z+13$$

Discrete State Equations

$$\frac{W(z)}{U(z)} = \frac{1}{z^3 + 3z^2 + 8z + 12} \qquad \frac{Y(z)}{W(z)} = z^2 + 7z + 13$$

$$w(k+3) + 3w(k+2) + 8w(k+1) + 12w(k) = u(k)$$

$$y(k) = w(k+2) + 7w(k+1) + 13w(k)$$

$$x_1(k) = w(k) \qquad x_1(k+1) = x_2(k)$$

$$x_2(k) = w(k+1) \qquad x_2(k+1) = x_3(k)$$

$$x_3(k) = w(k+2) \qquad x_3(k+1) = -12x_1(k) - 8x_2(k) - 3x_3(k) + u(k)$$

$$y(k) = 13x_1(k) + 7x_2(k) + x_3(k)$$

$$x_1(k+1) = x_2(k)$$

$$x_2(k+1) = x_3(k)$$

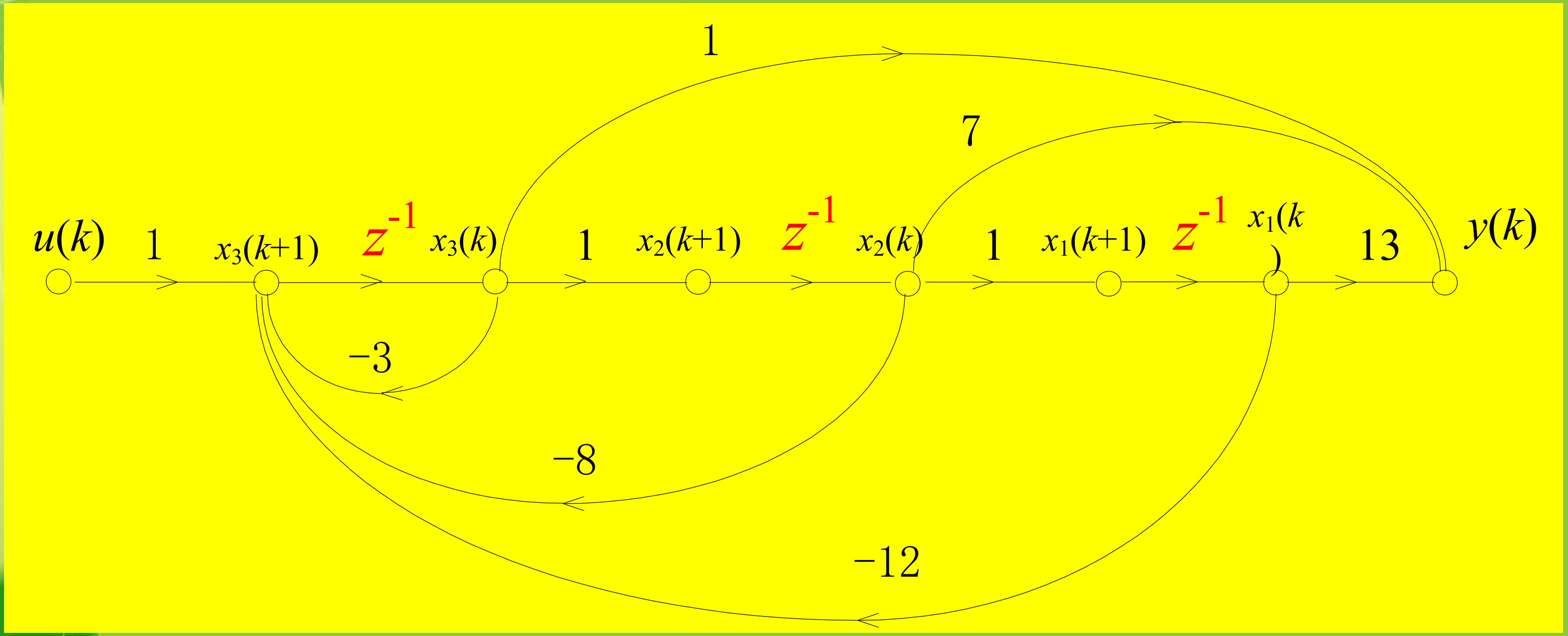
$$x_3(k+1) = -12x_1(k) - 8x_2(k) - 3x_3(k) + u(k)$$

$$y(k) = 13x_1(k) + 7x_2(k) + x_3(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -8 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 13 & 7 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -8 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k) \qquad y(k) = \begin{bmatrix} 13 & 7 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$



$$G(z) = \frac{Y(z)}{U(z)} = \frac{z^{-1} + 7z^{-2} + 13z^{-3}}{1 + 3z^{-1} + 8z^{-2} + 12z^{-3}}$$

Discrete State Equations

Example 7.12: please find the discrete state equation of the system with the given z-transfer function

$$\frac{Y(z)}{U(z)} = G(z) = \frac{2z^2 + 7z + 6}{z^2 + 3z + 2}$$

A: The denominator and numerator have the same order. It is not a proper fraction. Need to convert it into mixed fraction first.

$$G(z) = \frac{2z^2 + 7z + 6}{z^2 + 3z + 2} = 2 + \frac{z + 2}{z^2 + 3z + 2}$$

$$\begin{bmatrix} x_1(k+1) \\ x_1(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + 2u(k)$$

Get z-Transfer Function From Discrete State Equation

$$X(k+1) = GX(k) + HU(k)$$
$$Y(k) = CX(k) + DU(k)$$

$$zX(z) = GX(z) + HU(z)$$
$$Y(z) = CX(z) + DU(z)$$

$$G(z) = C(zI - G)^{-1} H + D$$

Example – 7.12

Q: please find the z-transfer function of the system with the given discrete state equation.

$$\begin{bmatrix} x_1(k+1) \\ x_1(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + 2u(k)$$

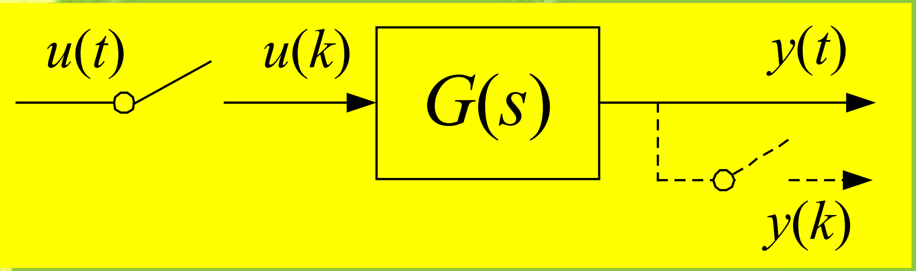
A:

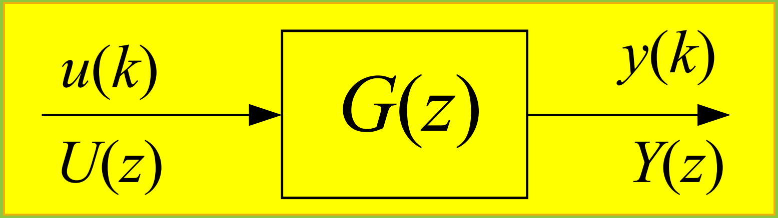
$$G(z) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} z & -1 \\ 2 & z+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2 = \frac{\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} z+3 & 1 \\ -2 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{z^2+3z+2} + 2$$

$$= \frac{z+2}{z^2+3z+2} + 2$$

Discretization of a continuous model

From transfer function to z-transfer function





$$Z[G(s)] = G(z)$$

Method 1: look up tables

Method 2: step 1: $g(t) = L^{-1}\left[G(s)\right]$

step 2: sample $g(t)$ and obtain the corresponding pulse sequence $g(k)$

step 3: apply z-transform to $g(k)$ and obtained z-transfer function $G(z)$

Example – 7.13

Q: please find the corresponding z-transfer function of the given transfer function.

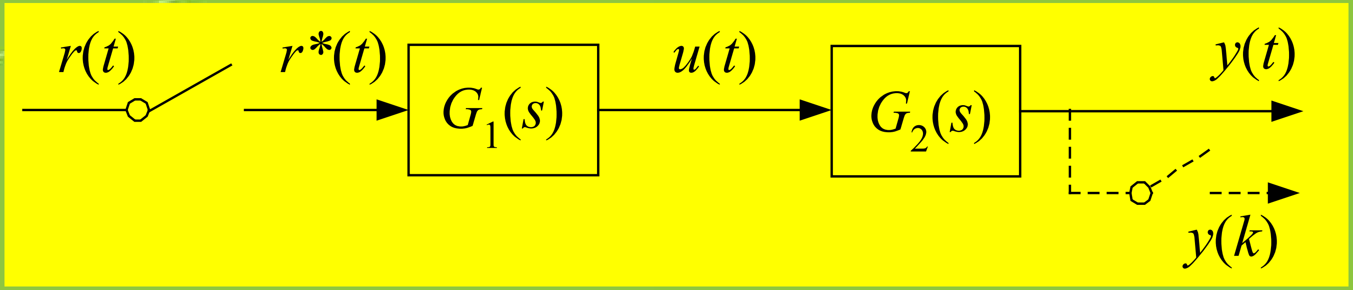
$$G(s) = \frac{a}{s(s+a)}$$

A:

$$G(z) = Z\left[\frac{a}{s(s+a)}\right] = Z\left[\frac{1}{s} - \frac{1}{s+a}\right]$$

$$= \frac{z}{z-1} - \frac{z}{z-e^{-aT}} = \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$$

Example for method 2



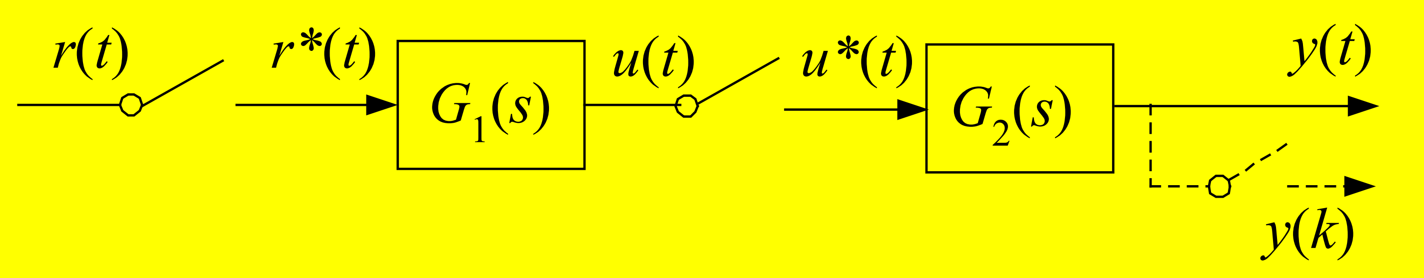
$$Y(s) = G(s)R(s) = G_1(s)G_2(s)R(s)$$

Discretization: $$Y^*(t) = G^*(t) \otimes R^*(t) = \left[G_1(t) \otimes G_2(t)\right]^* \otimes R^*(t)$$

$$Y(z) = G(z) * R(z) = \left[G_1G_2(z)\right]R(z)$$

$$G(z) = Z\left[G(s)\right] = Z\left[G_1(s)G_2(s)\right] = G_1G_2(z)$$

Example for method 2



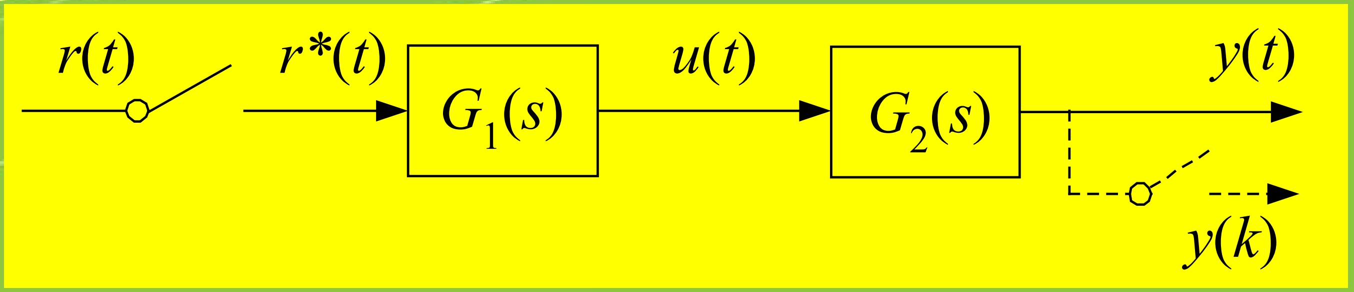
$$Y(s) = G(s)R(s) = G_1(s)G_2(s)R(s)$$

$$Y^*(t) = G^*(t) \otimes R^*(t) = G_1^*(t) \otimes G_2^*(t) \otimes R^*(t)$$

$$Y(z) = G(z) * R(z) = G_1(z)G_2(z)R(z)$$

$$G(z) = Z\left[G_1(s)\right] \cdot Z\left[G_2(s)\right] = G_1(z) \cdot G_2(z)$$

Differences between $G_1G_2(z)$ and $G_1(z)G_2(z)$



$$G_1(s)=\frac{1}{s+a}、G_2(s)=\frac{1}{s+b}$$

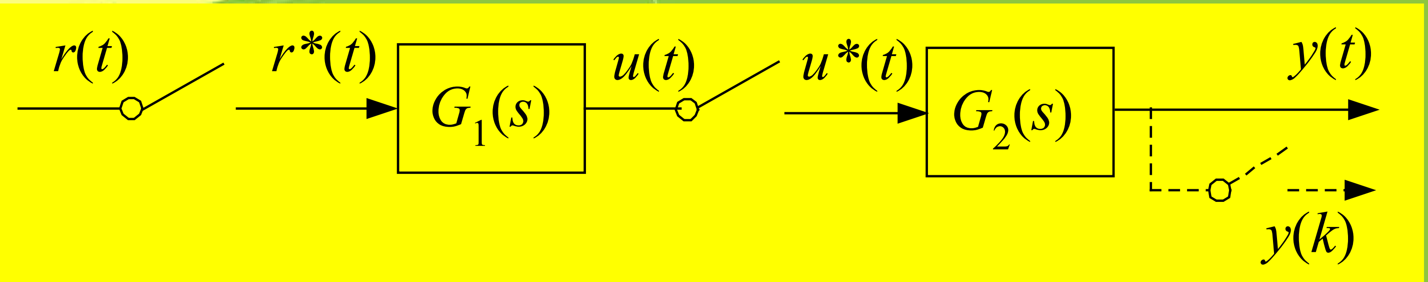
Please find the z-transfer function of the open loop transfer function

$$G(s)=G_1(s)G_2(s)=\left(\frac{1}{s+a}\right)\left(\frac{1}{s+b}\right)=\frac{1}{b-a}\left(\frac{1}{s+a}-\frac{1}{s+b}\right)$$

$$G(z)=Z\left[G(s)\right]=Z\left[\frac{1}{b-a}\left(\frac{1}{s+a}-\frac{1}{s+b}\right)\right]$$

$$=\frac{1}{b-a}\left(\frac{z\left(e^{-aT}-e^{-bT}\right)}{\left(z-e^{-aT}\right)\left(z-e^{-bT}\right)}\right)$$

Differences between $G_1G_2(z)$ and $G_1(z)G_2(z)$



$$G_1(s)=\frac{1}{s+a}、G_2(s)=\frac{1}{s+b}$$

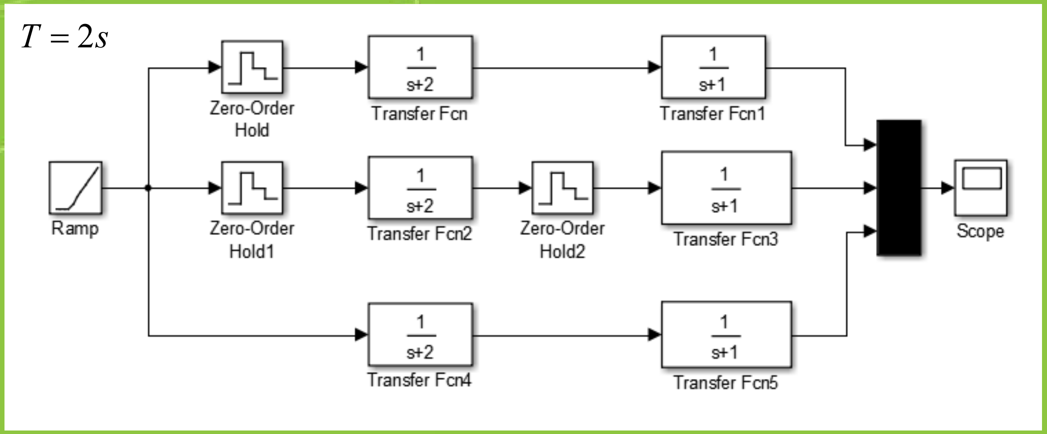
Please find the z-transfer function of the open loop transfer function

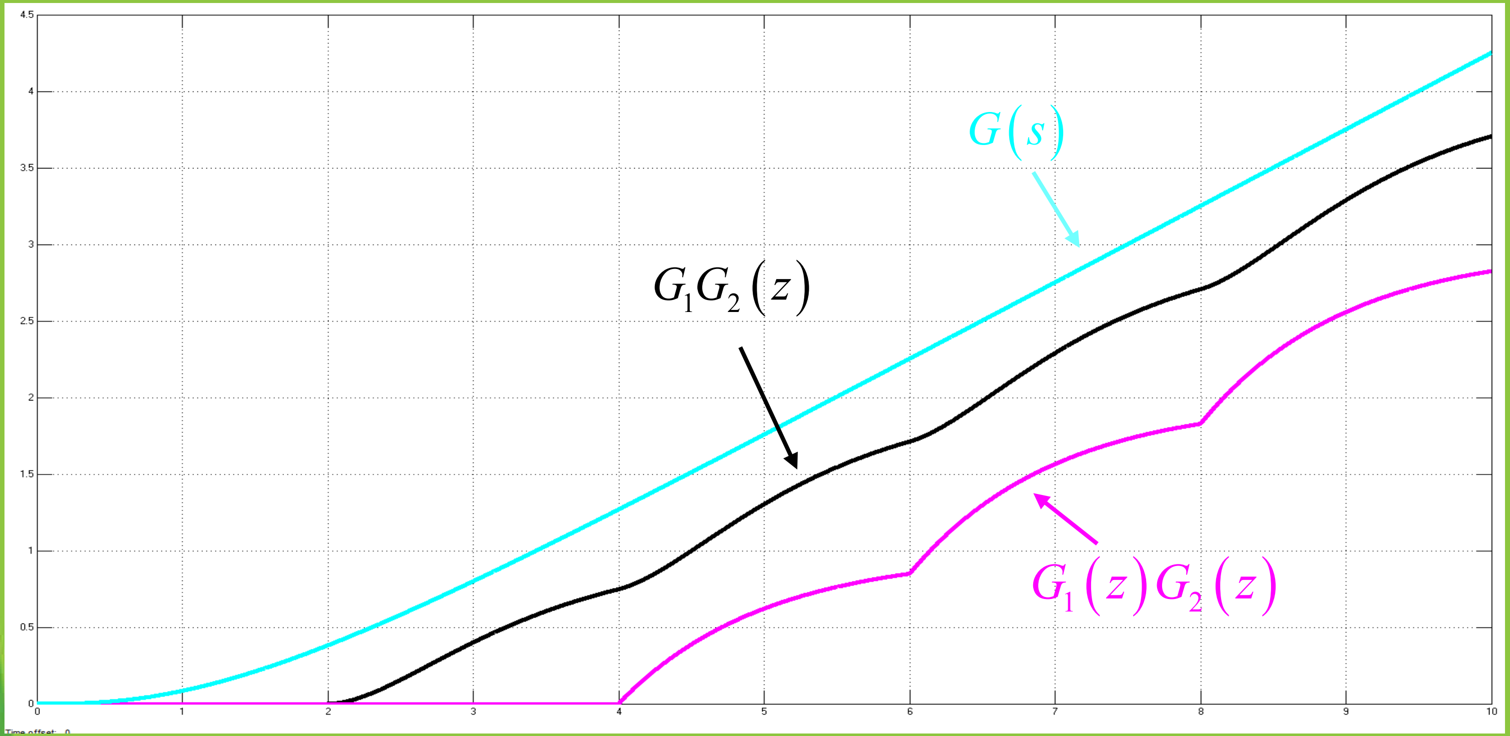
$$G_1(z)=Z\left[G_1(s)\right]=\frac{z}{z-e^{-aT}} \qquad G_2(z)=Z\left[G_2(s)\right]=\frac{z}{z-e^{-bT}}$$

$$G(z)=G_1(z)G_2(z)=\frac{z^2}{\left(z-e^{-aT}\right)\left(z-e^{-bT}\right)}$$

Obviously, $G_1G_2(z)\neq G_1(z)G_2(z)$

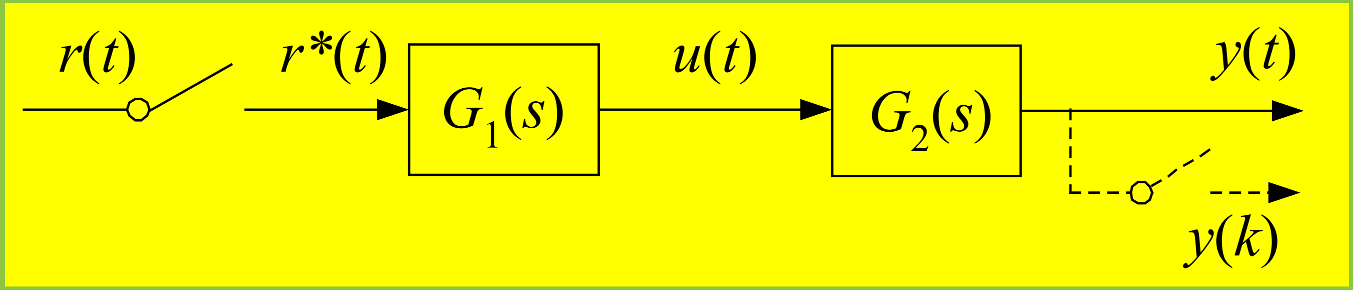
Differences between $G_1G_2(z)$ and $G_1(z)G_2(z)$



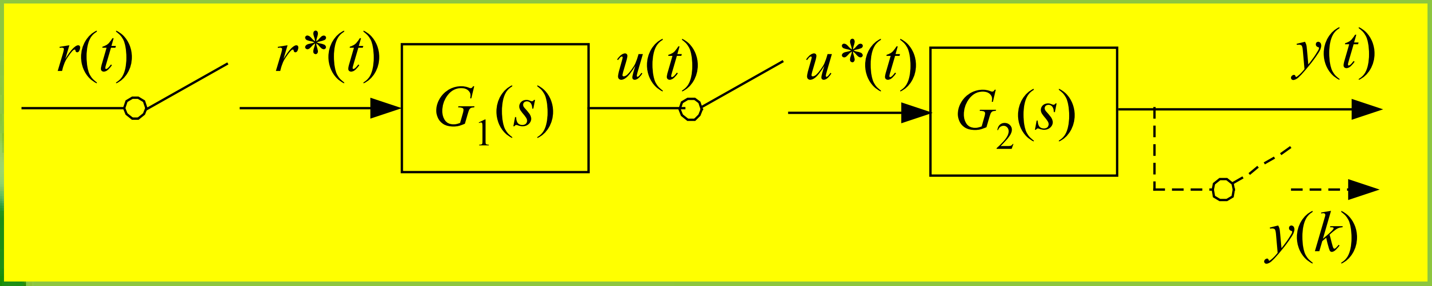


Remarks

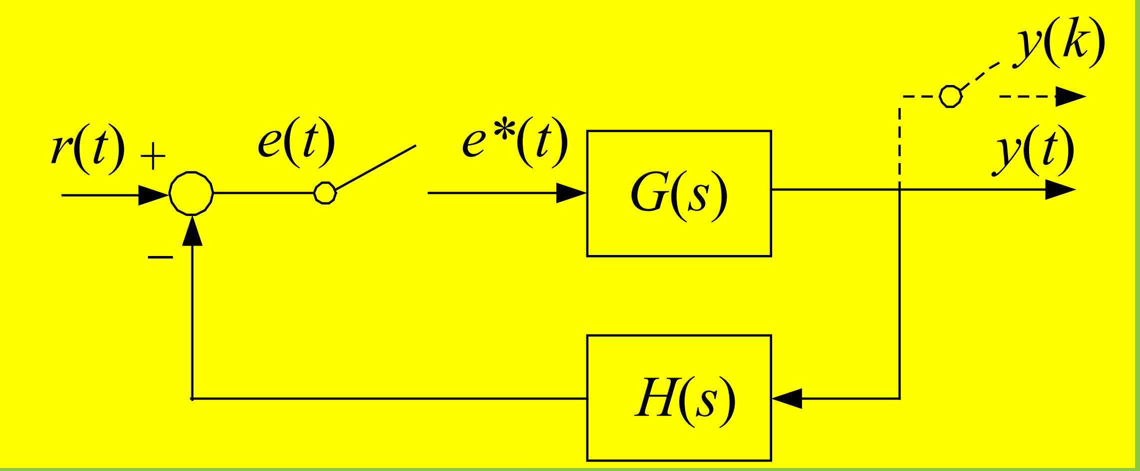
When deducing the corresponding z-transfer function according to a given transfer function, it is important to pay attention to the location of ideal samplers.



$$G(z) = Z\left[G(s)\right] = Z\left[G_1(s)G_2(s)\right] = G_1G_2(z)$$



$$G(z) = Z\left[G_1(s)\right] \cdot Z\left[G_2(s)\right] = G_1(z) \cdot G_2(z)$$



$$e(t) = r(t) - h(t) \otimes y(t)$$

$$y(t) = g(t) \otimes e(t)$$

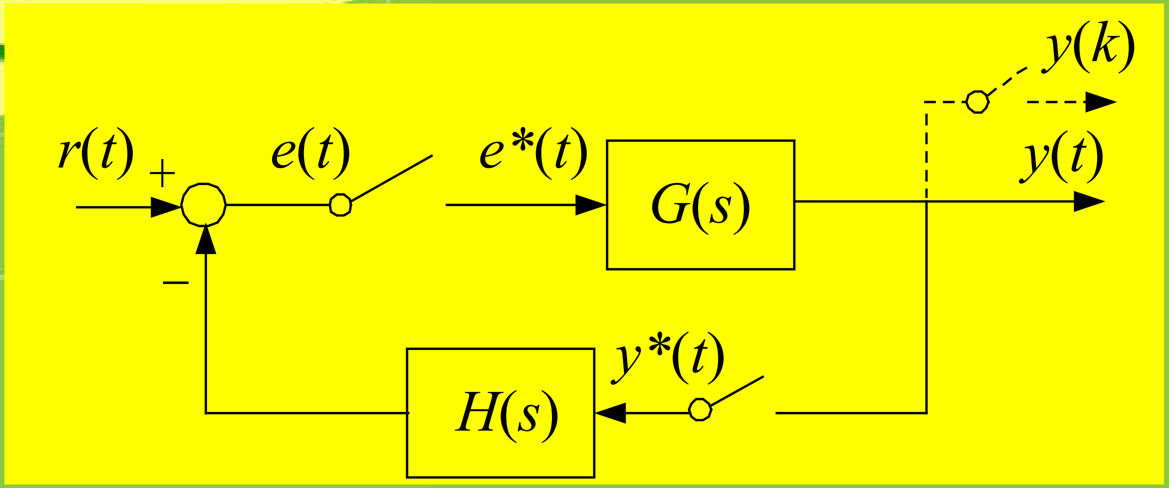
$$(1 + g(t) \otimes h(t)) \otimes y(t) = g(t) \otimes r(t)$$

Discretization:

$$(1 + \left[g(t) \otimes h(t)\right]^{*}) \otimes y^{*}(t) = g^{*}(t) \otimes r^{*}(t)$$

$$(1 + \left[GH(z)\right])Y(z) = G(z)R(z)$$

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$



$$e(t) = r(t) - h(t) \otimes y(t)$$

$$y(t) = g(t) \otimes e(t)$$

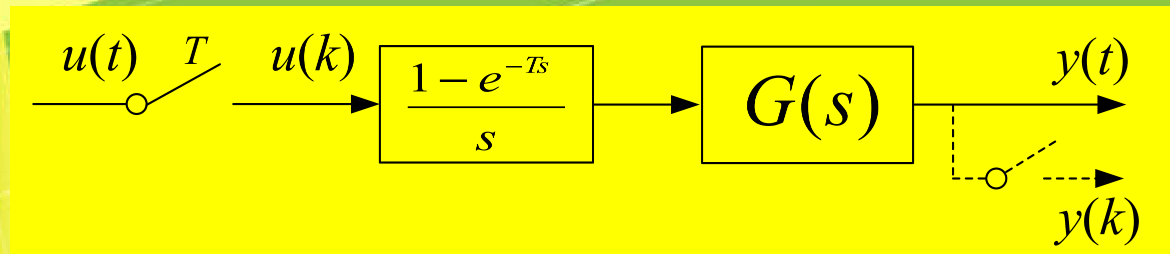
$$(1 + g(t) \otimes h(t)) \otimes y(t) = g(t) \otimes r(t)$$

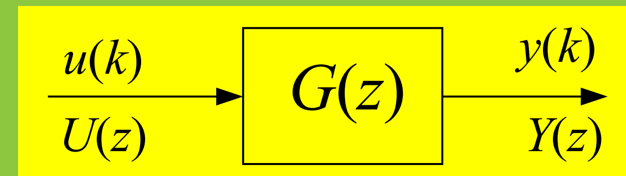
Discretization: $$(1 + g^*(t) \otimes h^*(t)) \otimes y^*(t) = g^*(t) \otimes r^*(t)$$

$$(1 + [G(z)H(z)])Y(z) = G(z)R(z)$$

$$\frac{G(z)}{1 + G(z)H(z)}$$

From transfer function to z-transfer function





$$Z[\frac{1-e^{-Ts}}{s}\cdot G(s)]=G(z)$$

Method 1: look up tables

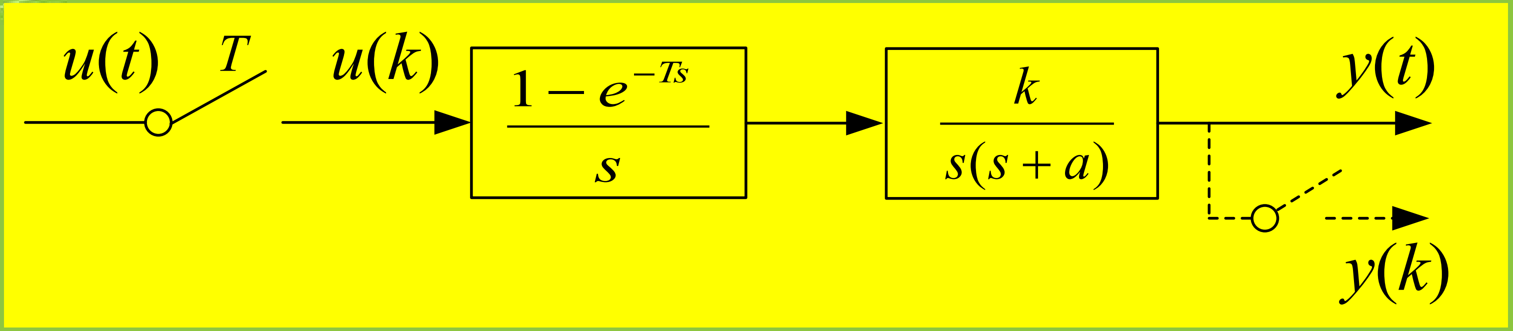
Method 2: step 1: $g(t)=L^{-1}\left[\frac{1-e^{-Ts}}{s}G(s)\right]$

step 2: sample $g(t)$ and obtain the corresponding pulse sequence $g(k)$

step 3: apply z-transform to $g(k)$ and obtained z-transfer function $G(z)$

Example – 7.14

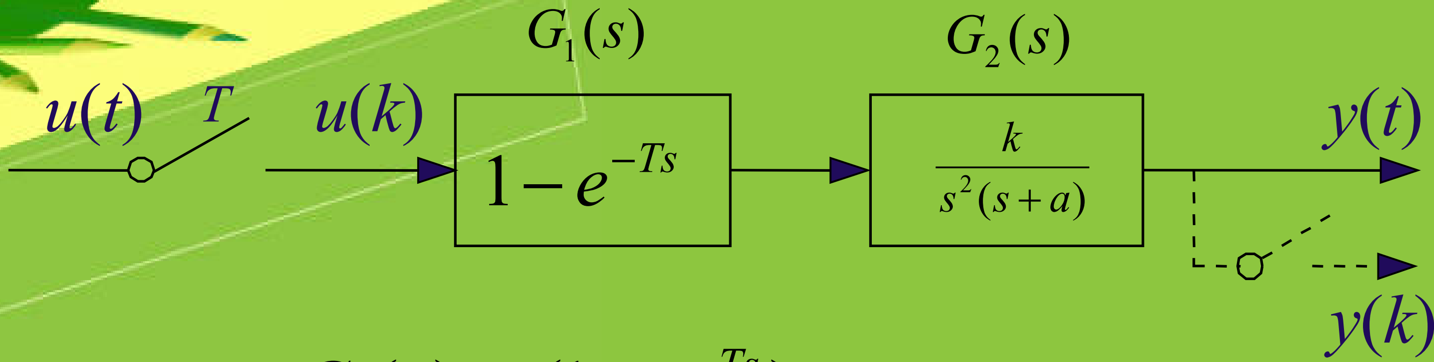
Q: please find the corresponding z-transfer function of the system with the following block diagram.



A: $$G_0(s) = G_h(s)G_p(s) = \frac{1-e^{-Ts}}{s}\cdot\frac{k}{s(s+a)} = (1-e^{-Ts})\frac{k}{s^2(s+a)}$$

$$G_1(s) = (1-e^{-Ts}) \qquad G_2(s) = \frac{k}{s^2(s+a)}$$

$$G_0(s) = G_1(s)G_2(s)$$

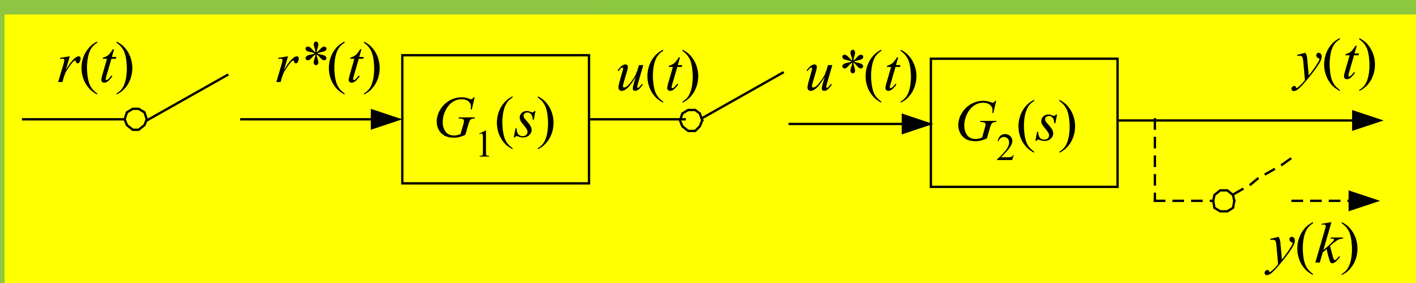


Considering $G_1(s) = (1 - e^{-Ts})$

$$g_1(t) = [\delta(t) - \delta(t - T)]$$

We can consider $g_1(t)$ as a discrete signal. Therefore, we can assume there is a ideal sampler before $G_2(s)$

Now, we can use $G(z) = Z\left[G_1(s)\right] \cdot Z\left[G_2(s)\right] = G_1(z) \cdot G_2(z)$



$$G_0(z)=Z\left[(1-e^{-Ts})\frac{k}{s^2(s+a)}\right]=Z(1-e^{-Ts})\cdot Z\left[\frac{k}{s^2(s+a)}\right]$$

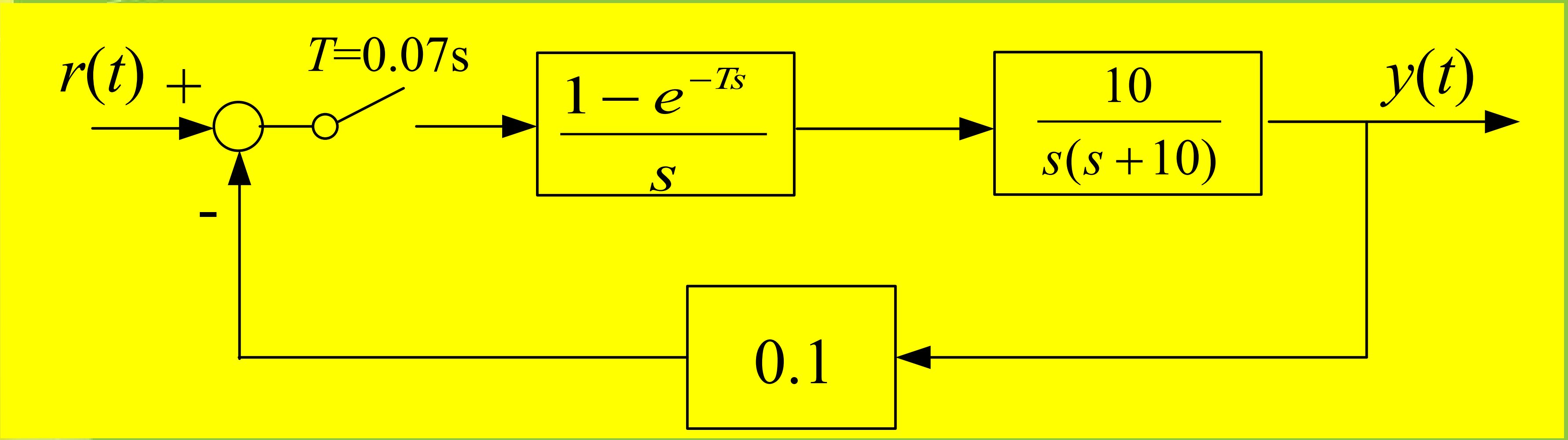
$$=Z(1-e^{-Ts})\cdot Z\left[\frac{k}{a^2}(\frac{a}{s^2}-\frac{1}{s}+\frac{1}{s+a})\right]$$

$$G_0(z)=(1-z^{-1})\left[\frac{k}{a^2}(\frac{aTz}{(z-1)^2}-\frac{z}{z-1}+\frac{z}{z-e^{-aT}})\right]$$

$$=(1\ -z^{-1})\left[\frac{k}{a}\frac{Tz}{(z-1)^2}-\frac{k}{a^2}\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}\right]$$

Example – 7.15

Q: please find the corresponding z-transfer function of the system with the following block diagram, then deduce the unit-step response of the system.



A: according to the block diagram, $T = 0.07, \quad a = 10, \quad k = 10, \quad e^{-aT} \approx 0.5$

$$G_0(z) = (1 - z^{-1})\left[\frac{k}{a}\frac{Tz}{(z-1)^2} - \frac{k}{a^2}\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}\right]$$

$$G_0(z)=(1-z^{-1})\left[\frac{0.07z}{(z-1)^2}-\frac{z(1-0.5)}{10(z-1)(z-0.5)}\right]=\frac{0.2z+0.15}{10z^2-15z+5}$$

$$G(z)=\frac{G_0(z)}{1+0.1G_0(z)}=\frac{0.2z+0.15}{10z^2-15z+5+0.02z+0.015}=\frac{0.2z+0.15}{10z^2-14.98z+5.015}$$

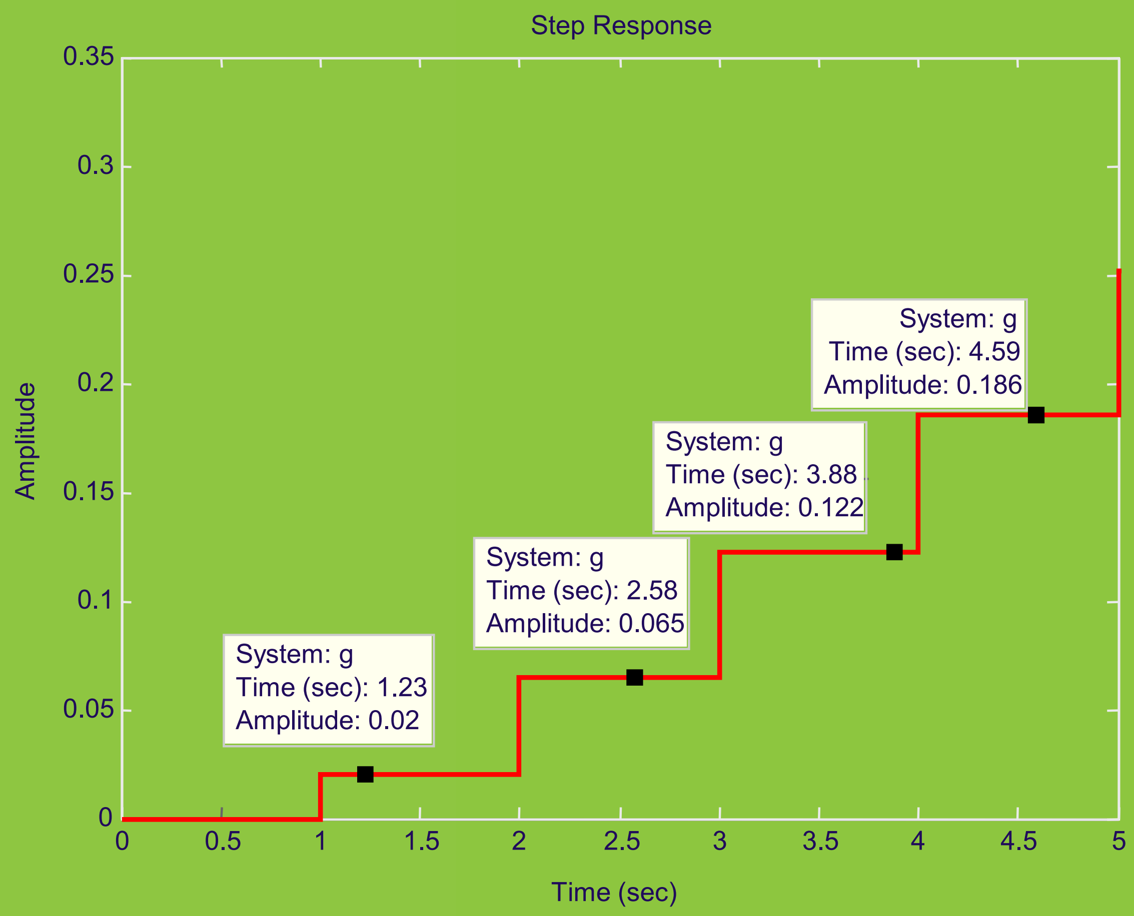
$$r(t)=1(t) \qquad R(z)=\frac{z}{z-1}$$

$$Y(z)=G(z)R(z)=\frac{0.2z^2+0.15z}{10z^3-24.98z^2+19.995z-5.015}$$

$$Y(z)\approx 0.02z^{-1}+0.07z^{-2}+0.12z^{-3}+0.19z^{-4}+\cdots$$

$$y(0)=0, y(T)=0.02, y(2T)=0.07,\cdots$$

```
z=tf('z');
g = (0.2*z+0.15)/(10*z^2-14.98*z+5.015);
step(g)
```



From State Equation to Discrete State Equation

$$\dot{X}(t) = AX(t) + BU(t)$$

$$Y(t) = CX(t) + DU(t)$$

Solve the above equation:

$$X(t) = e^{A(t-t_0)}X(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}BU(\tau)\,d\tau$$

Set $t_0 = kT, \quad t = (k+1)T$ and assume $U(t) = U(k)$ when $kT \le t \le (k+1)T$

$$X(k+1) = e^{AT}X(k) + \int_{kT}^{(k+1)T} e^{A(kT+T-\tau)}BU(k)\,d\tau$$

Select a new variable of integration: $t = kT + T - \tau$

$$X(k+1) = e^{AT}X(k) + \int_{T}^{0} e^{At}(-dt)BU(k)$$

$$= e^{AT}X(k) + \int_{0}^{T} e^{At}BU(k)dt$$

From State Equation to Discrete State Equation

$$X(k+1) = e^{AT}X(k) + \int_0^T e^{At}dtBU(k)$$

Set $G = e^{AT}$ $\quad H = \int_0^T e^{At}dtB$

$$X(k+1) = GX(k) + HU(k)$$

$$Y(k) = CX(k) + DU(k)$$

Example – 7.16

Q: please find the discrete state equation of the given state equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\vdots$$

A:

$$G = e^{AT} = I + AT + \frac{1}{2} A^2 T^2 + \cdots = I + AT = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

$$H = \int_0^T e^{At} dt \cdot B = \int_0^T \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} dt \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u(k) \qquad y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Another way to calculate e^{AT}

$$e^{At} = L^{-1}[(sI - A)^{-1}]$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} = \frac{1}{s^2} \cdot \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$e^{AT} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

From State Equation to Discrete State Equation

$$\dot{X}(t) = AX(t) + BU(t)$$

$$Y(t) = CX(t) + DU(t)$$

Question:

Can we just use difference to approximate derivative, then get the discrete state equation?

单选题 1分

设置

Can we just use difference to approximate derivative, then get the discrete state equation?

A Yes

B No

提交

From State Equation to Discrete State Equation

$$\dot{X}(t) = AX(t) + BU(t)$$

$$Y(t) = CX(t) + DU(t)$$

$$\frac{X(kT+T) - X(kT)}{T} = AX(kT) + BU(kT)$$

$$X(kT+T) - X(kT) = ATX(kT) + BTU(kT)$$

$$X(kT+T) = ATX(kT) + X(kT) + BTU(kT)$$

$$X(kT+T) = (I + AT)X(kT) + BTU(kT)$$

$$X(kT+T)=(I+AT)X(kT)+BTU(kT)$$

Set $T=1$

$$X(k+1)=(I+A)X(k)+BU(k)$$

Wrap-Up

- Basics about discrete-data control systems (digital control systems)
- Interfaces between continuous and discrete-data systems ----- the Sample-and-hold device
- Tools for analyzing and synthesizing discrete-data systems ----- the z-transform and inverse z-transform
- Discrete-data models of control systems
- Pulse-transfer functions
- Discrete state equation
- Discretization of continuous system
  - from transfer function to z-transfer function
  - from state equation to discrete state equation

Assignment

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- 1，(1) (4)
- 2，(3) (4)

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- 3，(1) (2)
- 4