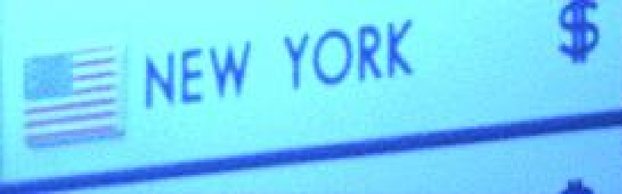
State Feedback Control Systems

Chapter 6



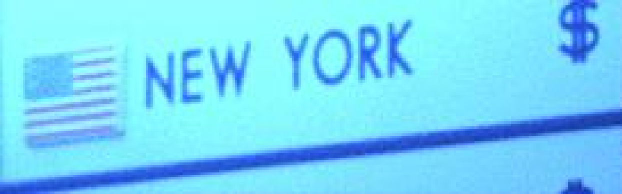
Review

- Controllability and Observability versus Zero-Pole Cancellation

- Controllability and Observability Decomposition

- State-feedback control

- Output-feedback control



Outlines

- How to design a state feedback controller
- How to design a state feedback with integral controller
- What’s a state observer
- How to design a state observer
- How to design a state feedback controller with feedback states from an observer

Example – 6.16

Q: The state equation of a system is shown as follows. Please find appropriate feedback gains to make the unit-step response of the system satisfy the given specifications: $\sigma\% \le 5\%$, $t_s = 0.5s$

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 0 & -30 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 256 & 0 \end{bmatrix} X$$

A: check the controllability of the system

$$S = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -30 \end{bmatrix} \qquad rank(S) = 2$$

System is controllable

Introducing state feedback

$$\dot{X} = (A - BK^T)X + Br = \begin{bmatrix} 0 & 1 \\ -k_1 & -30 - k_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

Example – 6.16

$$G(s) = C(sI - A + BK^{\mathrm{T}})^{-1}B = \begin{bmatrix}256 & 0\end{bmatrix}\begin{bmatrix} s & -1 \\ k_1 & s+30+k_2 \end{bmatrix}^{-1}\begin{bmatrix}0 \\ 1\end{bmatrix}$$

$$G(s) = \frac{256}{s^2 + (k_2+30)s + k_1}$$

$$\boxed{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

$$\sigma\% \le 5\%$$

$$\boxed{\sigma = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}$$

$$t_s = \frac{4}{\zeta\omega_{\mathrm{n}}} = 0.5s$$

$$\boxed{t_s = \frac{4}{\zeta\omega_{\mathrm{n}}}}$$

Example – 6.16

Method 1:

$$G(s) = \frac{256}{s^2 + (k_2 + 30)s + k_1}$$

$$\boxed{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

$$\sigma\% = 5\% \longrightarrow \zeta = 0.46$$

$$\boxed{\sigma = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}$$

$$t_s = 0.5s \longrightarrow \omega_n = 17.39$$

$$\boxed{t_s = \frac{4}{\zeta\omega_\mathrm{n}}}$$

Example – 6.16

$$G(s) = \frac{256}{s^2 + (k_2 + 30)s + k_1}$$

$$\boxed{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

$$\zeta = 0.46 \qquad \omega_n = 17.39$$

$$k_1 = \omega_n^2 = 302.41$$

$$2\zeta\omega_n = k_2 + 30 = 16 \quad \Rightarrow \quad G(s) = \frac{256}{s^2 + 16s + 302.41}$$

$$k_2 = -14$$

Example – 6.16

$$G(s)=\frac{256}{s^2+16s+302.41}$$

For unit-step input, the final value is:

$$y_{ss}=\lim_{s\to 0} sY(s)=\lim_{s\to 0} sG(s)\cdot\frac{1}{s}=\lim_{s\to 0} G(s)=G(0)=0.85$$

Not errorless!

Example – 6.16

Method 2:

$$G(s) = \frac{256}{s^2 + (k_2 + 30)s + k_1}$$

$$\boxed{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

In order to make the steady-state error equal to zero, according to the final-value theorem of Laplace transformation:

$$y_{ss} = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sG(s) \cdot \frac{1}{s} = \lim_{s \to 0} G(s) = G(0) = 1$$

$$k_1 = 256$$

Example – 6.16

$$D(s) = s^2 + (k_2 + 30)s + 256 \qquad \omega_n = \sqrt{256} = 16$$

Set $$t_s = \frac{4}{\zeta\omega_n} = 0.5s$$

$$\boxed{\sigma = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}$$

$$\zeta = 0.5$$

$$2\zeta\omega_n = k_2 + 30 \qquad k_2 = -14$$

$$\sigma\% = 16.3\%$$

Does not satisfy the requirement!

Example – 6.16

$$D(s) = s^2 + (k_2 + 30)s + 256 \qquad \omega_n = \sqrt{256} = 16$$

Comparing with the results gotten from method 1, ω_n is smaller, in order to satisfy the settling time requirement, the damping ratio should be increased. Therefore, the overshot will be decreased, which does not conflict with other requirements.

$$\boxed{\sigma = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}$$

Set $\sigma\% = 4\%$ $\quad \zeta = 0.707$

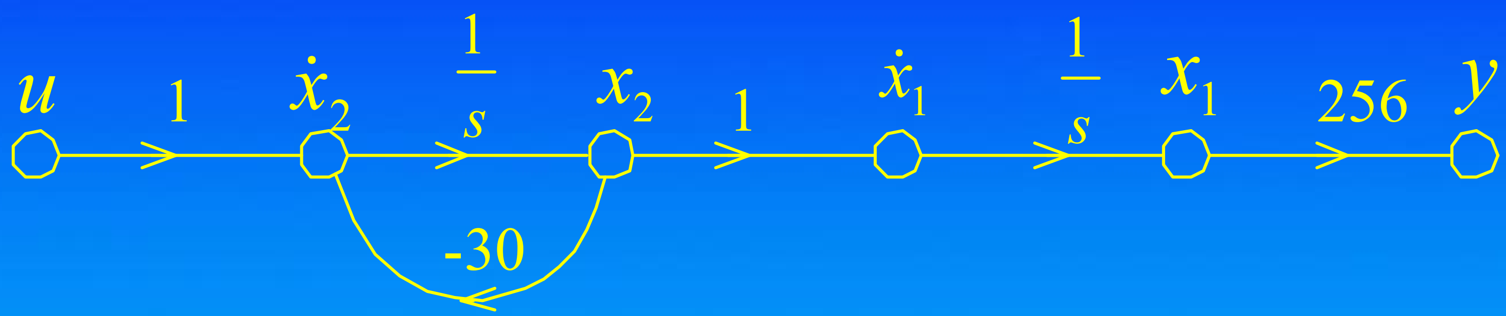
$$2\zeta\omega_n = k_2 + 30 \qquad k_2 = -7.37$$

$$t_s = \frac{4}{\zeta\omega_n} = 0.36s$$

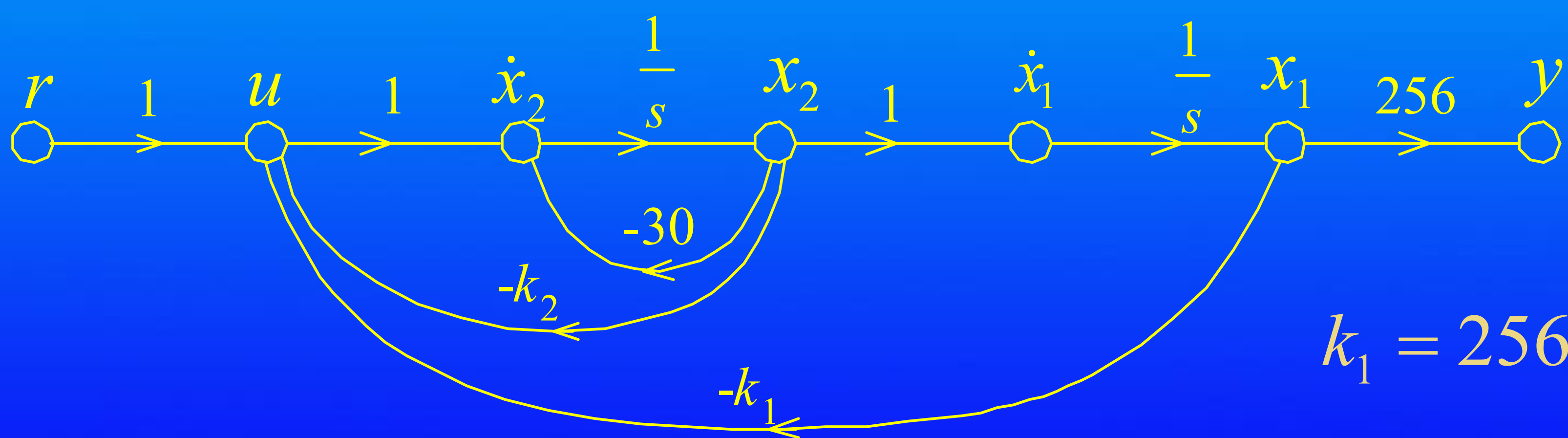
satisfy the requirement

Example – 6.16

The state diagram of the original system:



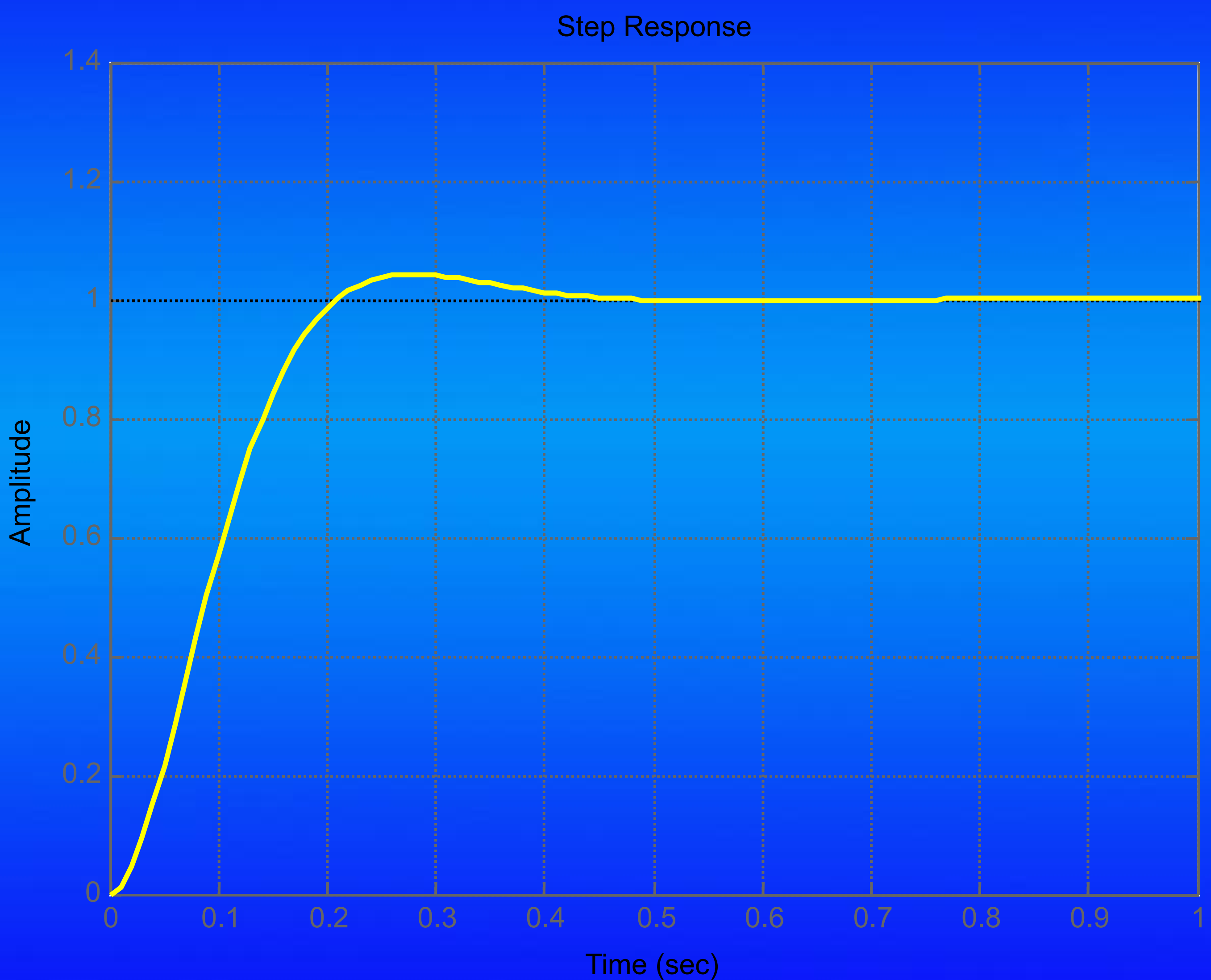
The state diagram of the compensated system:



$$k_1 = 256$$

$$k_2 = -7.37$$

Example – 6.16




```
a=[0  1; 0  -30];
b=[0;1];
c=[256  0];
d=0;
k=[256  -7.37];
A=(a-b*k);
g=ss(A,b,c,d);
step(g,1)
grid on
```

SJTU 2000

Q: The state equation of the system is shown as follows. Please :

（1）Identify the controllability and observability of the system

（2）If the system is uncontrollable or unobservable, do controllability or observability decomposition

（3）Design a state feedback control to move the poles of the system to $-1 \pm j1$

$$\dot{X} = AX + Bu$$

$$Y = CX + Du$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

SJTU 2000

A: （1）construct the controllability matrix

$$AB=\begin{bmatrix}0&1&0\\0&0&0\\0&0&-3\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}\qquad A^2B=\begin{bmatrix}0&1&0\\0&0&0\\0&0&-3\end{bmatrix}^2\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$S=\begin{bmatrix}B & AB & A^2B\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}$$

$$rank(S)=2<3$$

The system is uncontrollable

SJTU 2000

construct the observability matrix

$$CA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$V = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & 9 \end{bmatrix}^T$$

$rank(V) = 3$ The system is observable

（2）Do controllablility decomposition. Take two independent columns from S and select another independent one to construct a non-singular matrix

$$T=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} \quad \text{and} \quad T^{-1}=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\hat{A}=T^{-1}AT=\left[\begin{array}{cc:c}0 & 0 & 0\\ 1 & 0 & 0\\ \hdashline 0 & 0 & -3\end{array}\right] \qquad \hat{B}=T^{-1}B=\left[\begin{array}{c}1\\ 0\\ \hdashline 0\end{array}\right] \qquad \hat{C}=CT=\left[\begin{array}{cc:c}1 & 0 & 0\\ 0 & 1 & 1\end{array}\right]$$

A second order sub-system is controllable

$$\dot{\hat{X}}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\hat{X}+\begin{bmatrix}1\\ 0\end{bmatrix}u \qquad \hat{Y}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\hat{X}+\begin{bmatrix}1\\ 2\end{bmatrix}u$$

（3）Design a state feedback controller for the controllable part

The characteristic equation of the controllable part is:

$$f(s) = |sI - A_1| = \begin{vmatrix} s & 0 \\ -1 & s \end{vmatrix} = s^2$$

The desired characteristic equation is:

$$f_d(s) = (s+1-j)(s+1+j) = s^2 + 2s + 2$$

The characteristic equation of the controllable part with state feedback is:

$$f_c(s) = |sI - (A - BK^T)| = s^2 + k_1 s + k_2$$

$$k_1 = 2, \quad k_2 = 2$$

Q: The state equation of the system is shown as follows. Please :

（1）Identify the controllability and observability of the system

（2）Design a state feedback control to move the poles of the system to (-3,-2)

$$\dot{X} = AX + Bu$$

$$Y = CX$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

BIT 2000

A: （1）construct the controllability matrix

$$AB=\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}-1\\ 0\end{bmatrix}$$

$$S=\begin{bmatrix}B & AB\end{bmatrix}=\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$$

$$rank(S)=1<2$$

The system is uncontrollable

（2）Design a state feedback controller for the controllable part

The system is in diagonal cononical form:

$$\dot{X} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

The controllable state corresponds to the undesired mode.

The mode corresponding to the uncontrollable state has already been at the desired location. We only need to move the pole at -1 to -3.

The desired characteristic equation is:

$$f_d(s) = s + 3$$

The characteristic equation of the controllable part with state feedback is:

$$f_c(s) = s - (-1 - k) = s + k + 1 \qquad k = 2$$

Question:

Will state feedback change the type of a control system?

投票 最多可选1项

设置

Will state feedback change the type of a control system?

A Yes

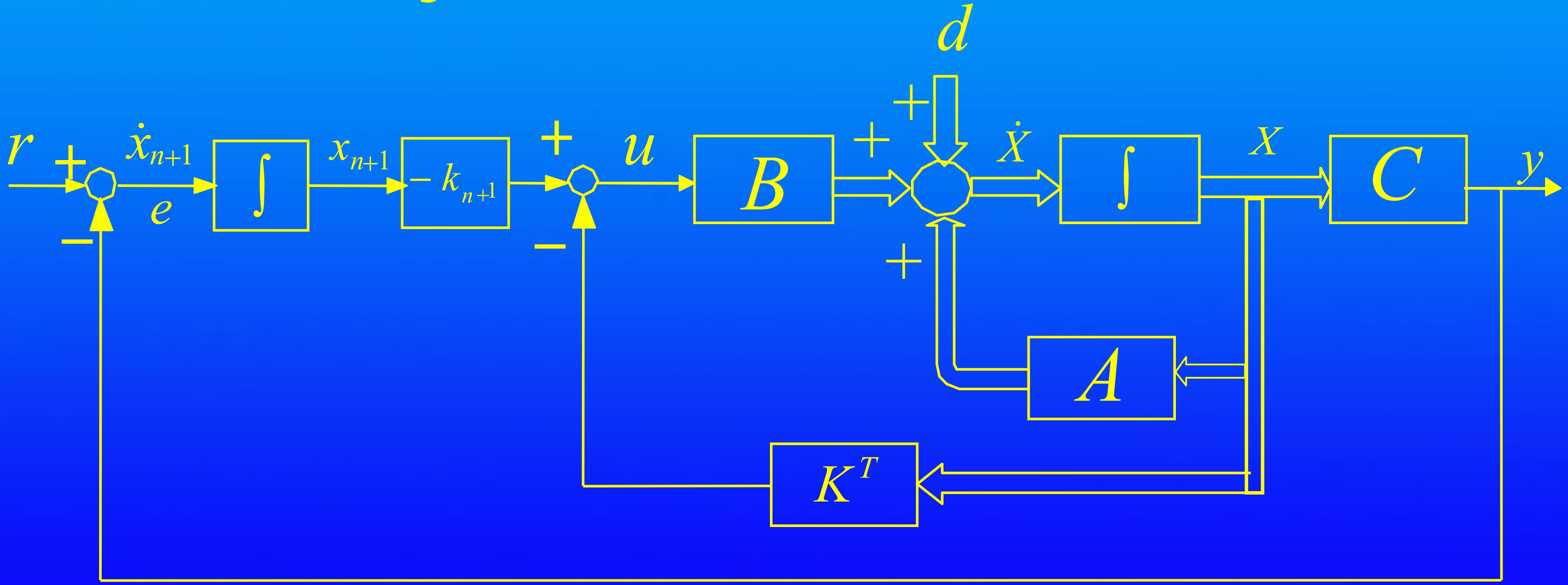
B No

提交

State Feedback with Integral Control

The state-feedback control introduced before has one deficiency in that it is not easy to improve the type of the system. As a result, the state-feedback control with constant gain feedback is generally used in the situation where the system does not track inputs.

In general, most control systems must track inputs. One solution to this problem is to introduce integral control, just as with PI controller, together with the constant-gain state feedback.



State Feedback with Integral Control

$$\dot{X} = AX + Bu + d$$

$$x_{n+1} = \int e\,dt$$

$$\dot{x}_{n+1} = e = r - y = r - CX$$

$$\dot{\bar{X}} = \bar{A}\bar{X} + \bar{B}u + \begin{bmatrix} d \\ r \end{bmatrix} \qquad y = \bar{C}\bar{X}$$

$$\bar{X} = \begin{bmatrix} X \\ x_{n+1} \end{bmatrix} \qquad \bar{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \qquad \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}$$

$$\bar{K} = \begin{bmatrix} K & k_{n+1} \end{bmatrix}^T = \begin{bmatrix} k_1 & k_1 & \cdots & k_n & k_{n+1} \end{bmatrix}^T$$

$$u = -K^T X - k_{n+1}x_{n+1} = -\bar{K}^T\bar{X} \qquad \dot{\bar{X}} = (\bar{A} - \bar{B}\bar{K}^T)\bar{X} + \begin{bmatrix} d \\ r \end{bmatrix}$$

Example - 6.17

Q: The state equation of the system is shown as follows. Please find appropriate feedback gains so that the steady-state error of the system to the unit-step input is zero, and the poles of the compensated system are located at $-20 \pm j20, \quad -100$

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 0 & -30 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 256 & 0 \end{bmatrix} X$$

A: check the controllability of the system

$$S = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -30 \end{bmatrix} \qquad rank(S) = 2$$

System is controllable

Refer to example 6.16, the poles can't be placed at the given location just by introducing state feedback.

Example - 6.17

Introducing integrator

$$\dot{\overline{X}} = \overline{A}\,\overline{X} + \overline{B}u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \qquad y = \overline{C}\,\overline{X} \qquad u = -K^T X - k_{n+1} x_{n+1} = -\overline{K}^T \overline{X}$$

$$\overline{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -30 & 0 \\ -256 & 0 & 0 \end{bmatrix} \qquad \overline{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\overline{C} = \begin{bmatrix} C & 0 \end{bmatrix} = \begin{bmatrix} 256 & 0 & 0 \end{bmatrix}$$

Example - 6.17

Check the controllability of the expanded system:

$$S' = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \bar{A}^2\bar{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -30 \\ 1 & -30 & 900 \\ 0 & 0 & -256 \end{bmatrix}$$

$rank\left(S'\right) = 3$ System is controllable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -30 & 0 \\ -256 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -k_1 & -k_2 & -k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -k_1 & -k_2 - 30 & -k_3 \\ -256 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

Example - 6.17

$$\left|sI-\bar{A}+\bar{B}\bar{K}\right| = \begin{vmatrix} s & -1 & 0 \\ k_1 & s+30+k_2 & k_3 \\ 256 & 0 & s \end{vmatrix} = s^3+(30+k_2)s^2+k_1 s-256k_3$$

$$G(s)=\frac{Y(s)}{R(s)}=\frac{-256k_3}{s^3+(30+k_2)s^2+k_1 s-256k_3}$$

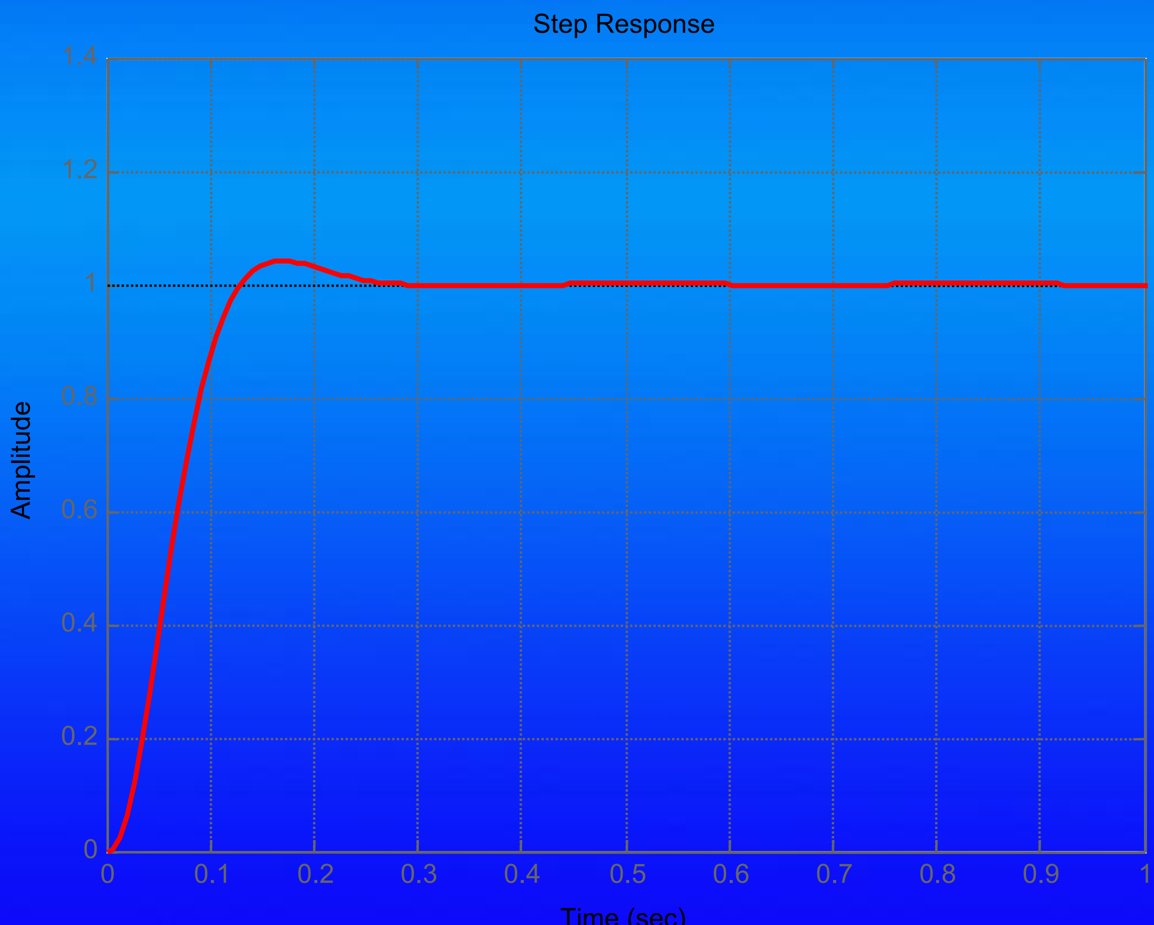
Characteristic equation of the desired system:

$$(s+20+\mathrm{j}20)(s+20-\mathrm{j}20)(s+100)=s^3+140s^2+4800s+80000=0$$

$$k_1=4800, \qquad k_2=110 \qquad k_3=-312.5$$

Example - 6.17

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -k_1 & -30-k_2 & -k_3 \\ -256 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r = \begin{bmatrix} 0 & 1 & 0 \\ -4800 & -140 & 312.5 \\ -256 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$



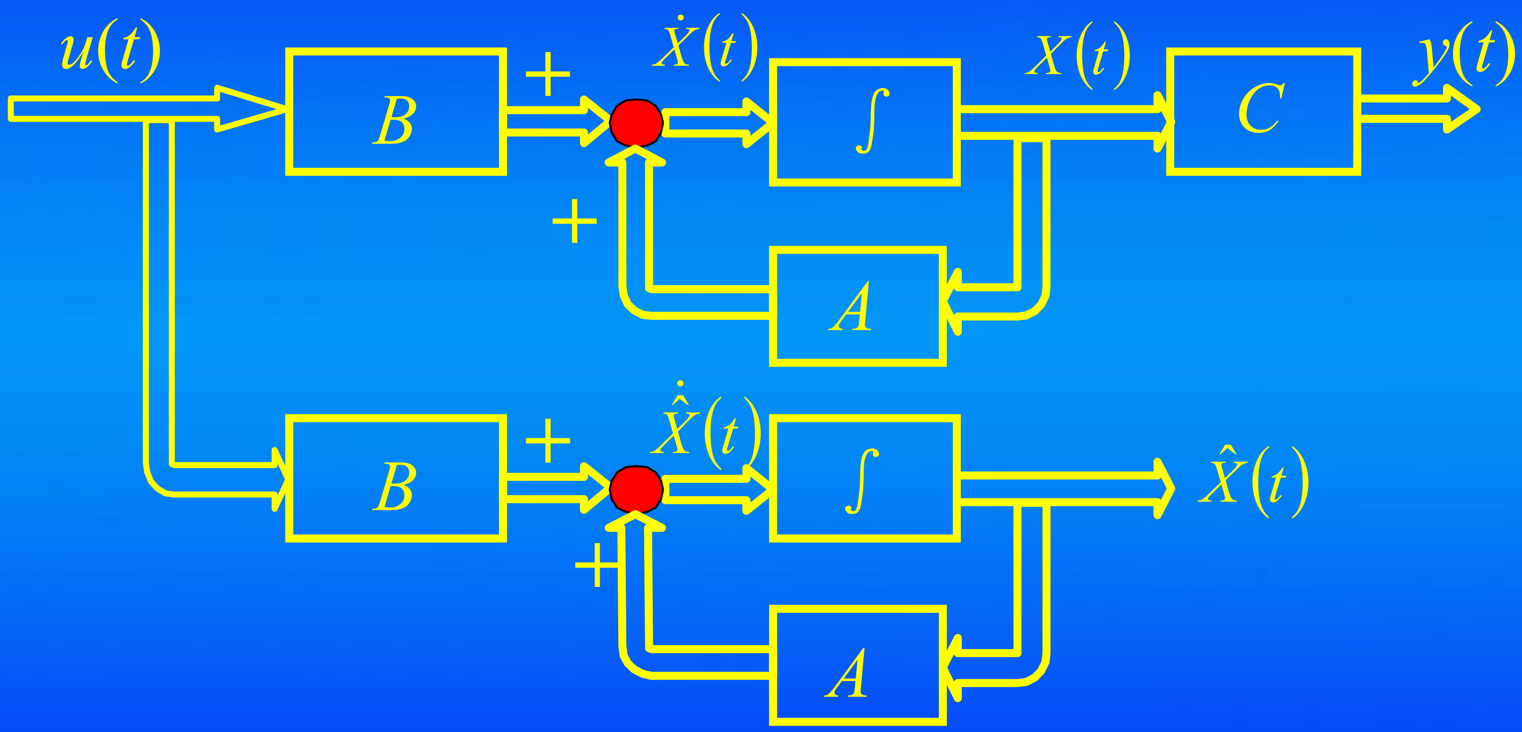

```
a=[0  1  0;
-4800  -140  312.5;
-256  0  0];
b=[0; 0; 1];
c=[256  0  0];
d=0;
g=ss(a,b,c,d);
step(g, 1);
grid on;
```

State Observer

- Why state observer?
  - Not all the states are easy to obtain
- How to get all the states of a system?
  - reconstruction of all the states using measurable inputs and outputs
  - limited to observable systems
  - also known as state estimator

How to construct state observer ?

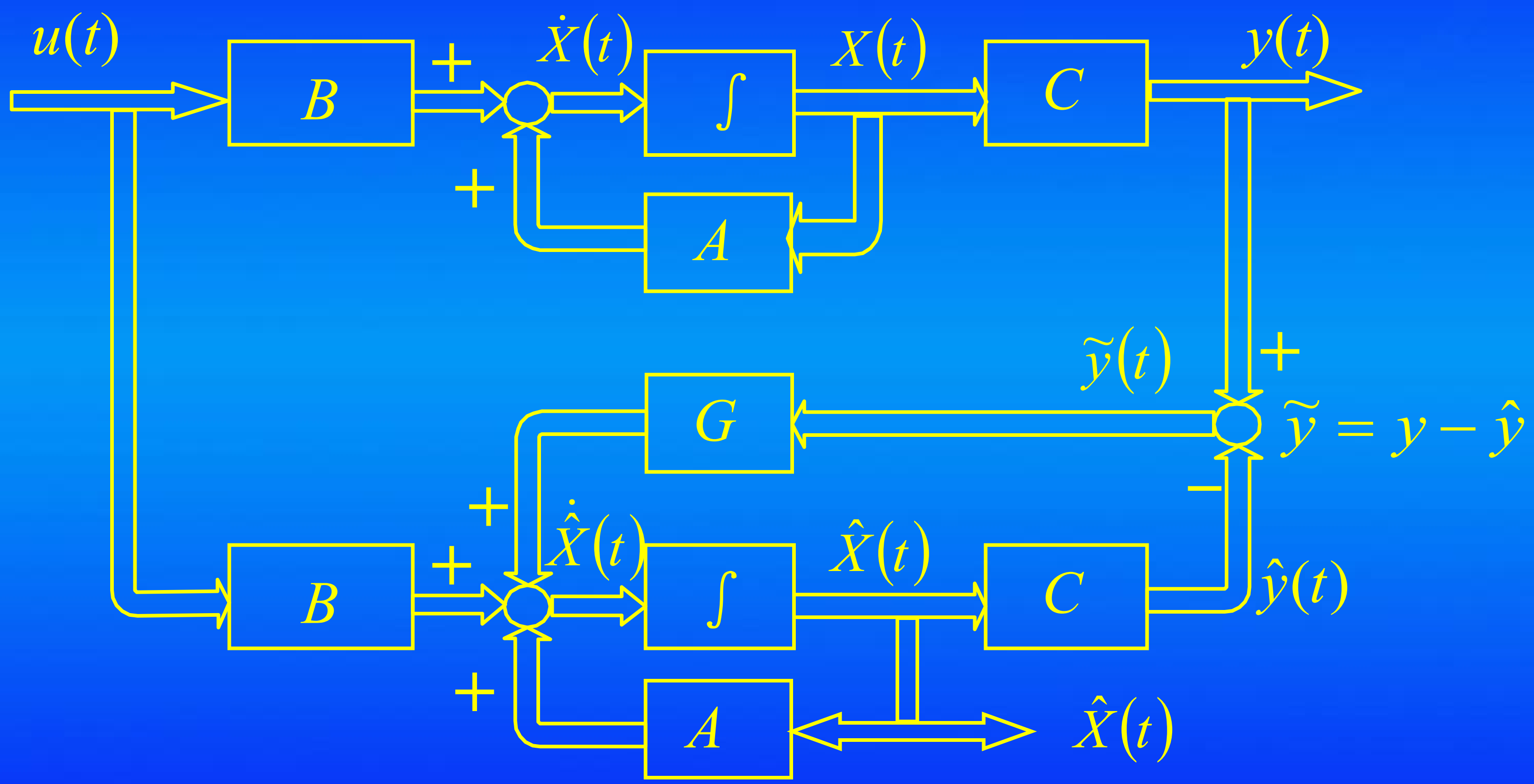
Open-loop estimation:



The error of estimation is: $\tilde{X} = X - \hat{X}$

How to construct state observer ?

Closed-loop estimation:



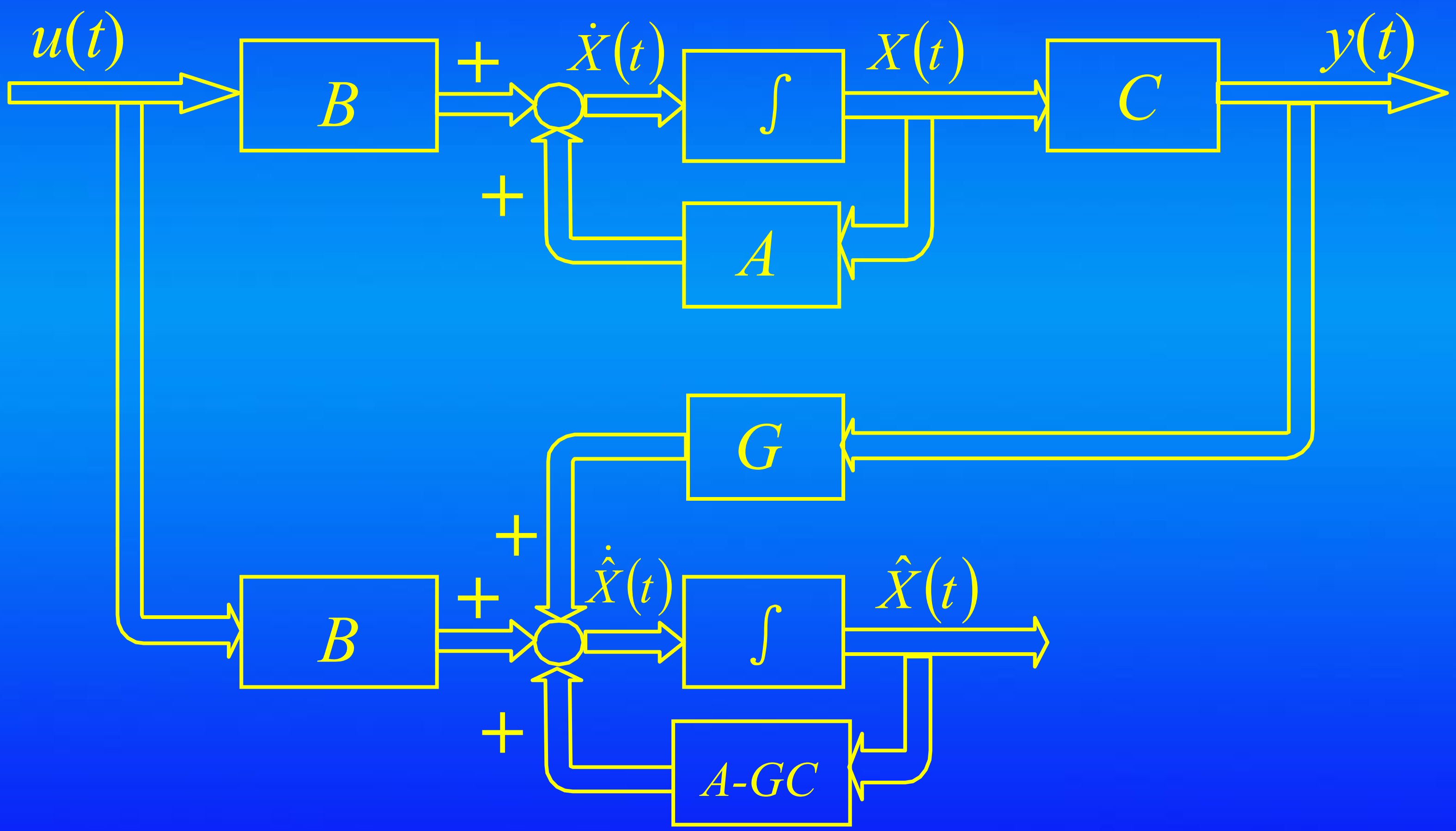
$$\dot{\hat{X}} = A\hat{X} + Bu + G\tilde{y} = A\hat{X} + Bu + G(y - C\hat{X})$$

$$\dot{\hat{X}} = (A - GC)\hat{X} + Bu + Gy$$

G is called the linear feedback matrix

How to construct state observer ?

$$\dot{\hat{X}} = (A - GC)\hat{X} + Bu + Gy$$

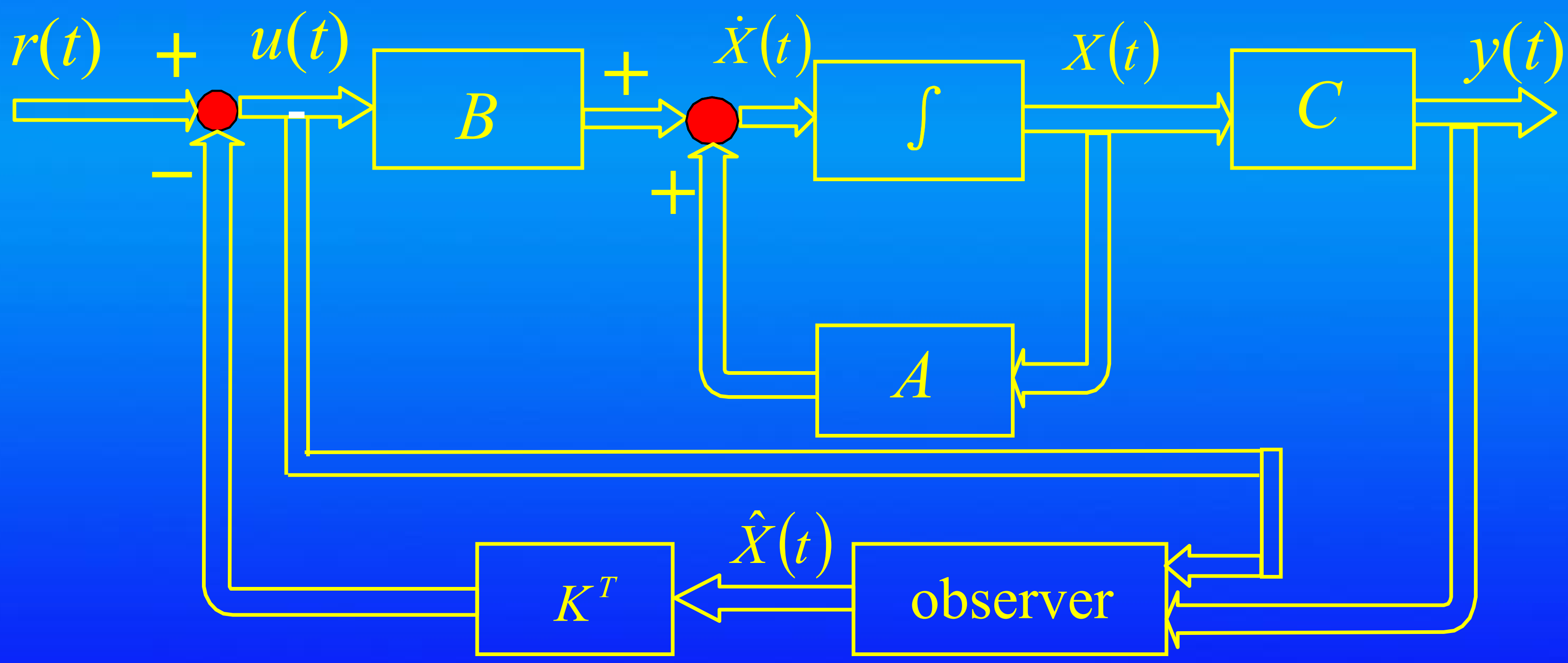


G is called the linear feedback matrix or gain matrix of observer.

State Observer

$$\dot{X} = AX + Bu$$

$$y = CX$$



$\hat{X}(t)$ are estimated system states

How to construct state observer ?

The error of estimation is: $\tilde{X} = X - \hat{X}$

$$\dot{\tilde{X}} = (AX + Bu) - \left[(A - GC)\hat{X} + Bu + GCX\right]$$

$$\dot{\tilde{X}} = (A - GC)\tilde{X} \qquad \tilde{X}(t) = e^{(A-GC)t}\,\tilde{X}(0)$$

So long as the real part of the characteristic roots of (A-GC) are all negative, the estimation error will approach zero when time lasts long enough.

If the poles of (A-GC) can be arbitrarily placed, the estimation time can be adjusted freely.

Theorem about the Existence of Estimators

For a system (A, C) to have a gain matrix G, and the poles of (A-GC) can be arbitrarily placed, it is sufficient and necessary that (A, C) is observable.

Proof: sufficiency:

According to duality principle, (A, C) is observable means its dual system (A^T, C^T) is controllable. So, there must exist a state feedback gain K, which can arbitrarily place the pole of (A^T, C^T).

$$\left|sI-(A^T-C^TK^T)\right|=\left|sI-(A-KC)\right|$$

Set G=K, the sufficiency is proved.

Necessity:

Omitted.

Example - 6.18

Q: The state equation of the system is shown as follows. Please design an observer and place its poles at $s_1 = s_2 = -3$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

A: check the observability of the system

$$V = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad rank(V) = 2$$

System is observable

Example - 6.18

$$\dot{\hat{X}} = (A - GC)\hat{X} + Bu + Gy$$

$$A - GC = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \end{bmatrix} = \begin{bmatrix} -2g_1 & 1 \\ -2-2g_2 & -3 \end{bmatrix}$$

$$\left| sI - (A - GC) \right| = \begin{vmatrix} s + 2g_1 & -1 \\ 2 + 2g_2 & s + 3 \end{vmatrix} = s^2 + (3 + 2g_1)s + (6g_1 + 2g_2 + 2)$$

The characteristic polynomial of the desired observer is:

$$(s+3)^2 = s^2 + 6s + 9$$

Example - 6.18

$$3 + 2g_1 = 6 \qquad\qquad g_1 = 1.5$$

$$6g_1 + 2g_2 + 2 = 9 \qquad\qquad g_2 = -1$$

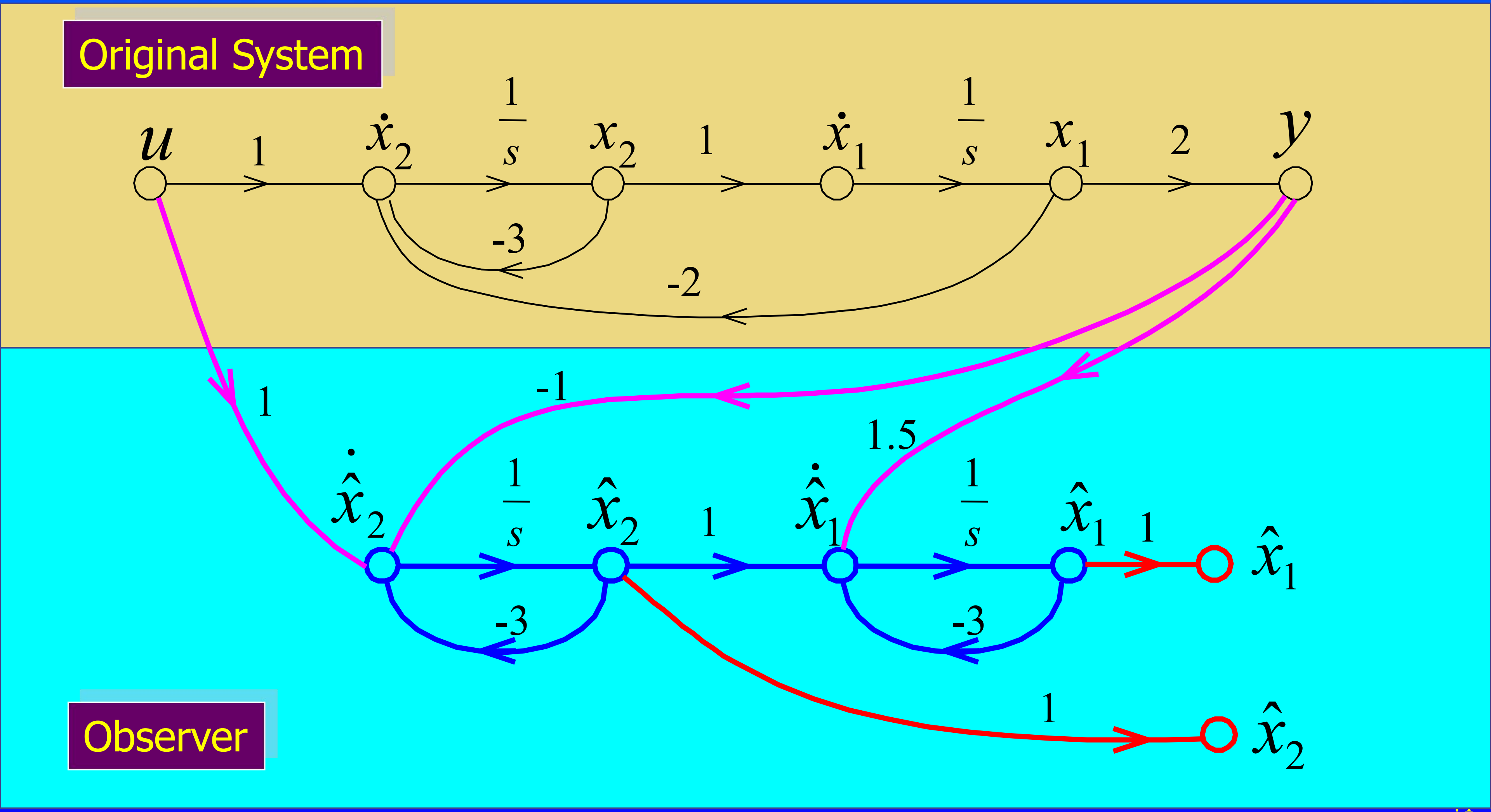
$$A - GC = \begin{bmatrix} -2g_1 & 1 \\ -2-2g_2 & -3 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}$$

State equation of the desired observer is:

$$\dot{\hat{X}} = (A - GC)\hat{X} + Bu + Gy$$

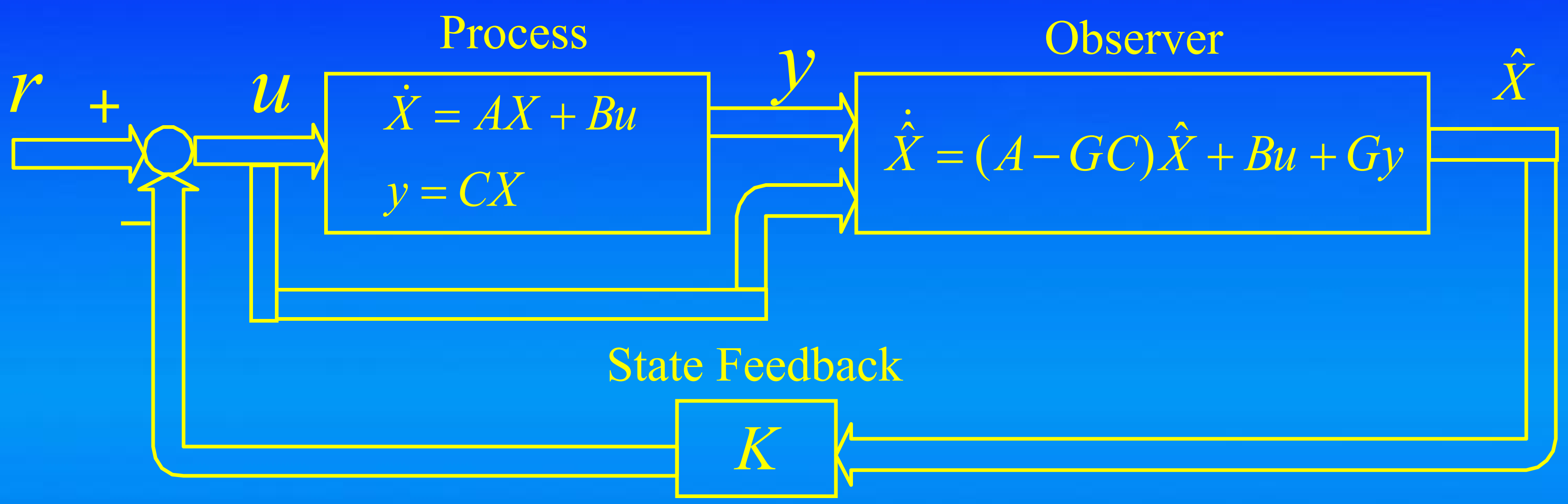
$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1.5 \\ -1 \end{bmatrix} y$$

Example - 6.18



Original System
u
1
$\dot{x}_2$
$\frac{1}{s}$
x_2
1
$\dot{x}_1$
$\frac{1}{s}$
x_1
2
y
-3
-2
1
-1
1.5
$\dot{\hat{x}}_2$
$\frac{1}{s}$
$\hat{x}_2$
1
$\dot{\hat{x}}_1$
$\frac{1}{s}$
$\hat{x}_1$
1
$\hat{x}_1$
-3
-3
1
$\hat{x}_2$
Observer

State Feedback Control with Observer



Theorem: If a LTI system is both controllable and observable, it is feasible to design a state feedback control with an observer for the system. At this time, the pole-placement of observer and controller can be done independently.

State Feedback Control with Observer

Process needs to be controlled:

$$\dot{X} = AX + Bu$$

$$y = CX$$

Observer:

$$\dot{\hat{X}} = (A - GC)\hat{X} + Bu + Gy$$

Control:

$$u = r - K^T \hat{X}$$

State estimation error:

$$\tilde{X} = X - \hat{X}$$

→ new states:

$$\dot{\tilde{X}} = (AX + Bu) - \left[(A - GC)\hat{X} + Bu + GCX\right] = (A - GC)\tilde{X}$$

$$\dot{X} = AX + B(r - K^T \hat{X}) = AX + Br - BK^T (X - \tilde{X})$$

$$= (A - BK^T)X + BK^T \tilde{X} + Br$$

State Feedback Control with Observer

$$\dot{X} = (A - BK^T)X + BK^T\tilde{X} + Br$$

$$\dot{\tilde{X}} = (A - GC)\tilde{X}$$

$$\begin{bmatrix} \dot{X} \\ \dot{\tilde{X}} \end{bmatrix} = \begin{bmatrix} A - BK^T & BK^T \\ 0 & A - GC \end{bmatrix} \begin{bmatrix} X \\ \tilde{X} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

The characteristic equation is:

$$| sI - (A - BK^T) | . | sI - (A - GC) | = 0$$

State Feedback Control with Observer

$$\begin{bmatrix} \dot{X} \\ \dot{\tilde{X}} \end{bmatrix} = \begin{bmatrix} A - BK^T & BK^T \\ 0 & A - GC \end{bmatrix} \begin{bmatrix} X \\ \tilde{X} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} X \\ \tilde{X} \end{bmatrix}$$

The controller and the observer can be designed independently.

1. the new system with the observer is not controllable.
2. the observer is not controllable.
3. the transfer function is not affected by the observer.

Example - 6.19

Q: The state equation of the given system is shown as follows. Please design a controller with poles at $s_{1,2} = -1 \pm j$, and an observer with poles at $s_1 = s_2 = -3$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

A: check the controllability and observability of the system

$$S = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \qquad rank(S) = 2$$

System is controllable

$$V = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad rank(V) = 2$$

System is observable

Example - 6.19

Design the controller:

$$A - BK^T = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 - 2 & -k_2 - 3 \end{bmatrix}$$

$$\left| sI - (A - BK^T) \right| = \begin{vmatrix} s & -1 \\ k_1 + 2 & s + k_2 + 3 \end{vmatrix} = s^2 + (k_2 + 3)s + (k_1 + 2)$$

The desired characteristic polynomial of the controller is:

$$(s + 1 + j)(s + 1 - j) = s^2 + 2s + 2$$

$$k_1 = 0, \quad k_2 = -1 \qquad K^T = [0 \quad -1]$$

Example - 6.19

Design the observer:

$$A-GC=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}-\begin{bmatrix}g_1\\ g_2\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}=\begin{bmatrix}-2g_1 & 1\\ -2-2g_2 & -3\end{bmatrix}$$

$$|sI-(A-GC)|=\begin{vmatrix}s+g_1 & -1\\ 2+g_2 & s+3\end{vmatrix}=s^2+(2g_1+3)s+(6g_1+2g_2+2)$$

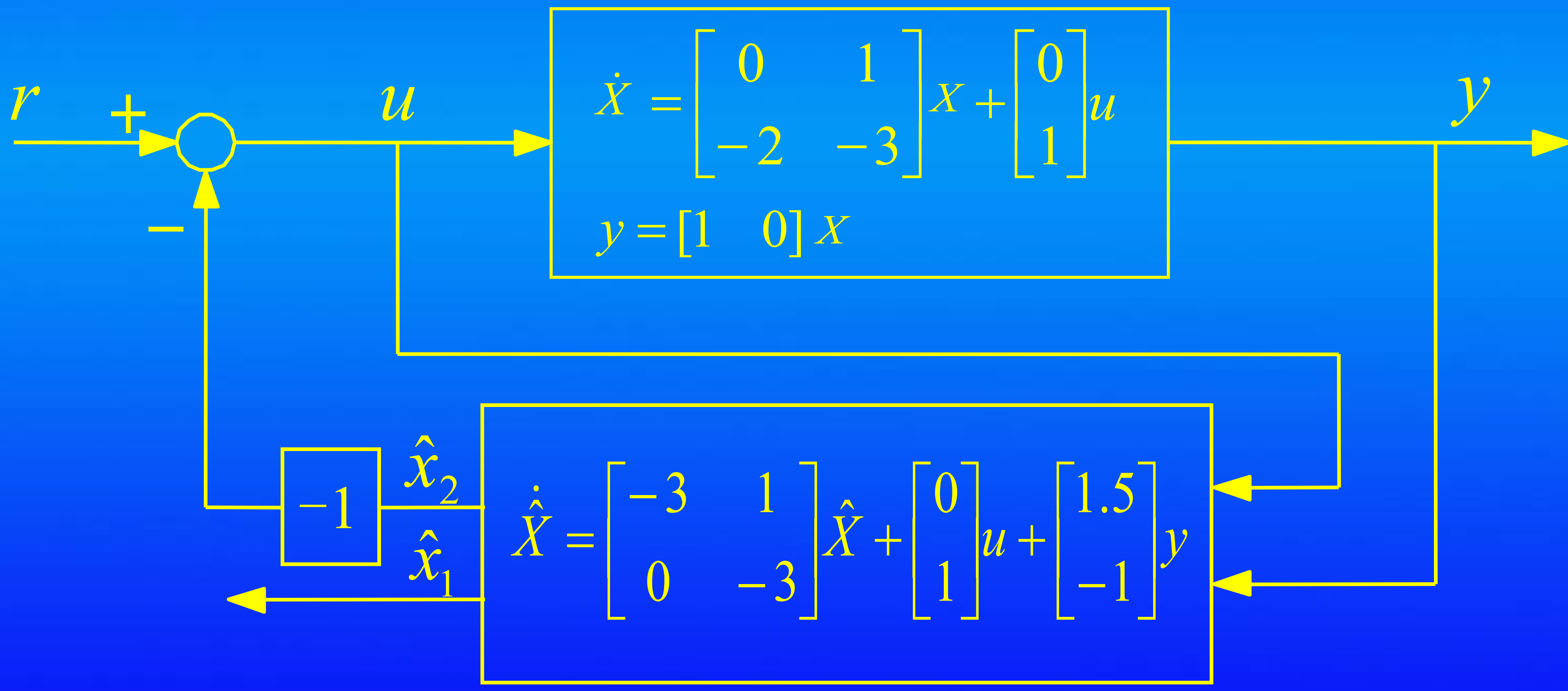
The desired characteristic polynomial of the observer is:

$$(s+3)^2=s^2+6s+9$$

$$g_1=1.5 \qquad g_2=-1 \qquad G^T=[1.5 \quad -1]$$

Example - 6.19

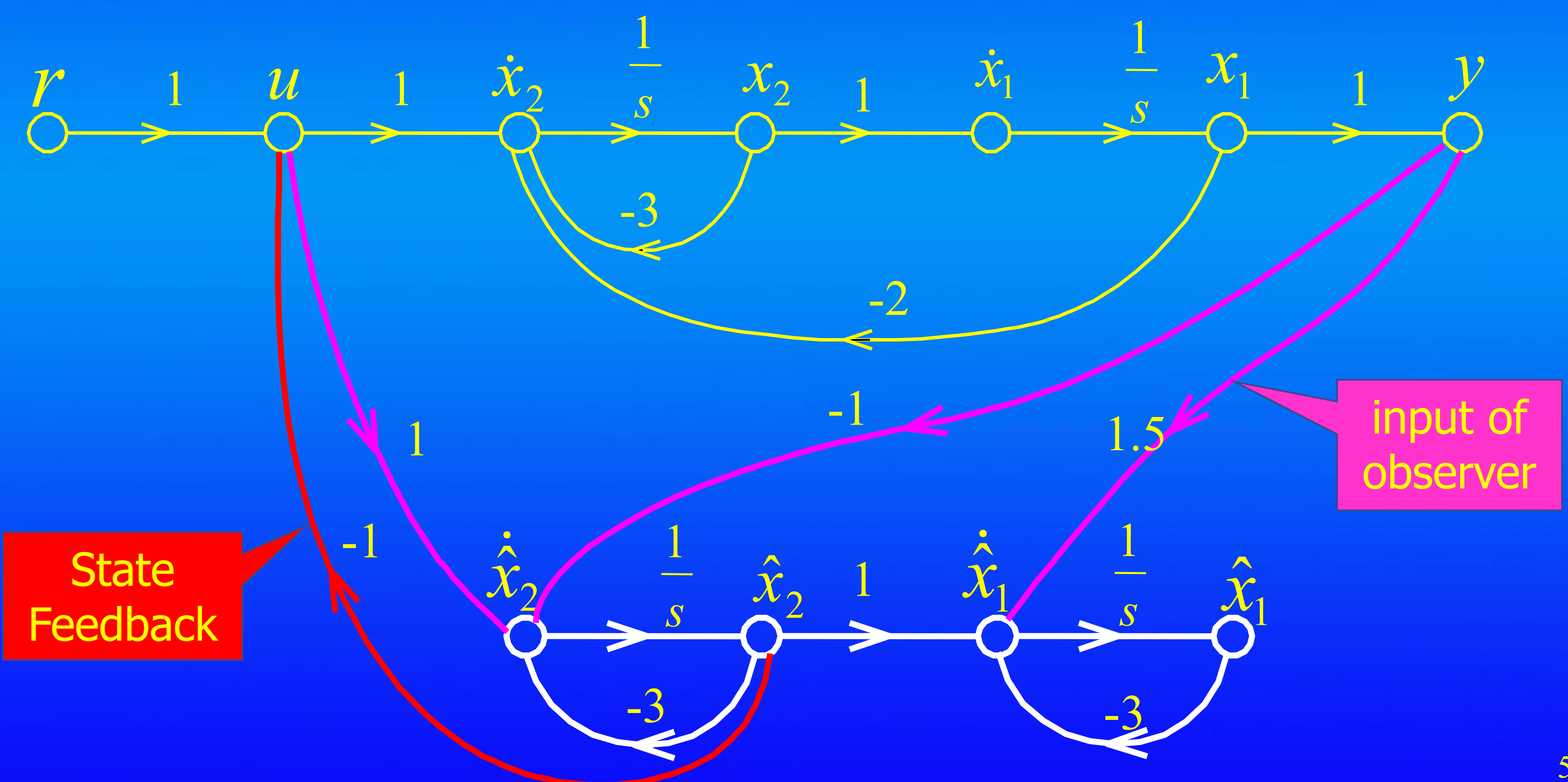
Observer equation: $$\dot{\hat{X}} = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}\hat{X} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u + \begin{bmatrix} 1.5 \\ -1 \end{bmatrix}y$$



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u,$$

$$y=\begin{bmatrix}2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\dot{\hat{X}}=\begin{bmatrix}-3 & 1\\ 0 & -3\end{bmatrix}\hat{X}+\begin{bmatrix}0\\ 1\end{bmatrix}u+\begin{bmatrix}1.5\\ -1\end{bmatrix}y$$



Matlab Simulation

```
A=[0 1;-2 -3];
B=[0 1]';
C=[2 0];
K=[0 -1]';
G=[1.5 -1]';
sys=ss([A-B*K' B*K'; zeros(2,2) A-G*C], eye(4),eye(4),eye(4));
t=0:0.01:4;
z=initial(sys,[1;0;0.5;0.5],t);
x1=[1 0 0 0]*z';
x2=[0 1 0 0]*z';
e1=[0 0 1 0]*z';
e2=[0 0 0 1]*z';
```

Matlab Simulation

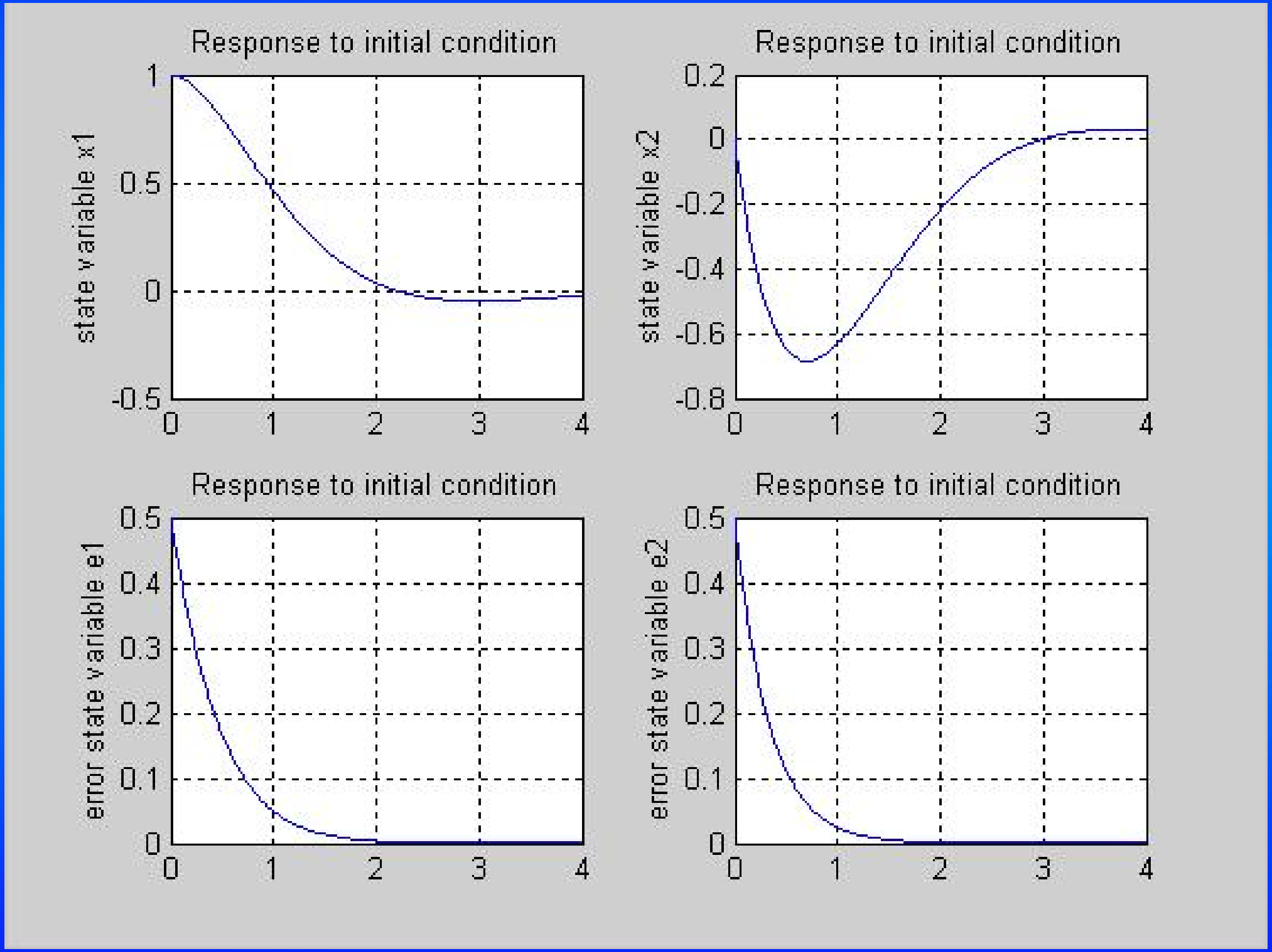
```
subplot(2,2,1); plot(t,x1);grid;
title('Response to initial condition');
ylabel('state variable x1');

subplot(2,2,2); plot(t,x2);grid;
title('Response to initial condition');
ylabel('state variable x2');

subplot(2,2,3); plot(t,e1);grid;
title('Response to initial condition');
ylabel('error state variable e1');

subplot(2,2,4); plot(t,e2);grid;
title('Response to initial condition');
ylabel('error state variable e2');
```

Matlab Simulation



Poles of controller

$$s_{1,2} = -1 \pm j$$

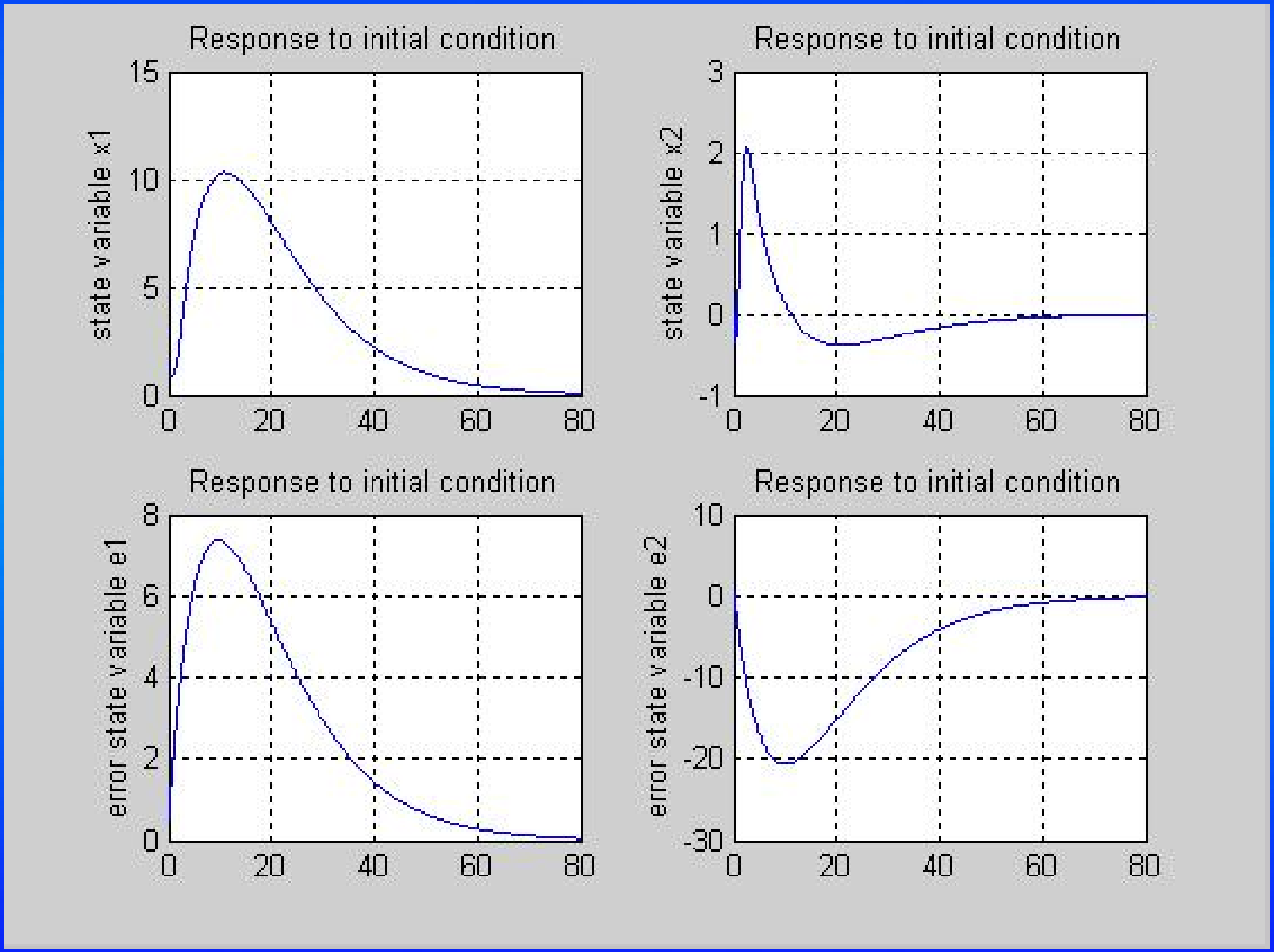
Poles of observer

$$s_1 = s_2 = -3$$

Initial condition

$$\begin{bmatrix} x_1 \\ x_2 \\ \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0.5 \\ 0.5 \end{bmatrix}$$

Matlab Simulation



Poles of controller

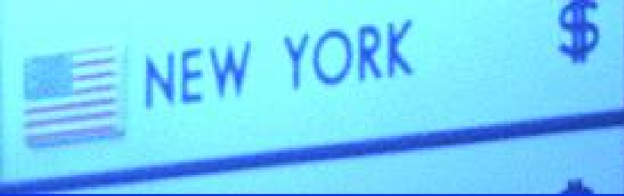
$$s_{1,2} = -1 \pm j$$

Poles of observer

$$s_1 = s_2 = -0.01$$

Initial condition

$$\begin{bmatrix} x_1 \\ x_2 \\ \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0.5 \\ 0.5 \end{bmatrix}$$



Wrap-up

- How to design a state feedback controller
- How to design a state feedback with integral controller
- What's a state observer
- How to design a state observer
- How to design a state feedback controller with feedback states from an observer

Assignment

Page 151

- 10, (1)
- 11, (2)
- 12