Time Domain Analysis
of
Control Systems

Chapter 3

Reviews

- **Transient Response**
  - Approximation of High-Order Systems
- **Basic concept about root loci**
- **Rules for constructing root loci**
  - Start and end point
  - Number of branches
  - Asymptotes
  - Symmetry
  - Root loci on real axis
  - Breakaway points
  - Angles of departure and arrival
  - Intersection with the imaginary axis

Outlines

- Root loci change after adding zeros or poles
- Basics about frequency domain analysis

Parameter Design Using Root Locus Method

Root loci are the trajectories of roots of system characteristic equation when k' varies from 0 to infinity. The characteristic equation is:

$$1 + G_0(s) = 0 \qquad \text{if} \quad G_0(s) = \frac{k' N_0(s)}{D_0(s)}$$

then, the characteristic equation is rewritten as:

$$1 + \frac{k' N_0(s)}{D_0(s)} = 0$$

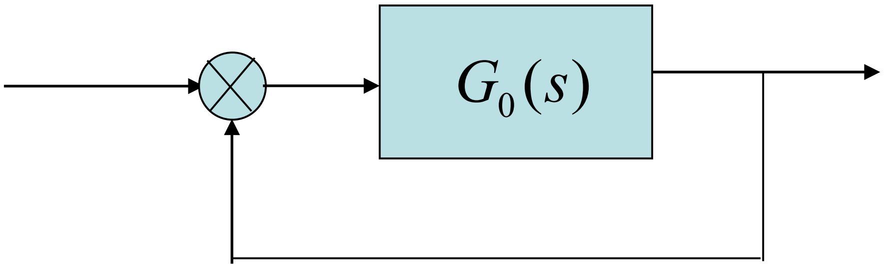
If we have a characteristic equation and want to study the impact of the variation of a coefficient to the characteristic equation, we can use the root locus method.

$$s^n + a_n s^{n-1} + \cdots + a_3 s^2 + a_2 s + a_1 = 0$$

If want to study a_2

$$1 + \frac{a_2 s}{s^n + a_n s^{n-1} + \cdots + a_3 s^2 + a_1} = 0$$

Adding Poles and Zeros to $G_0(s)$



The characteristic equation:

$$1+G_0(s)=0$$

The characteristic equation after adding poles or zeros:

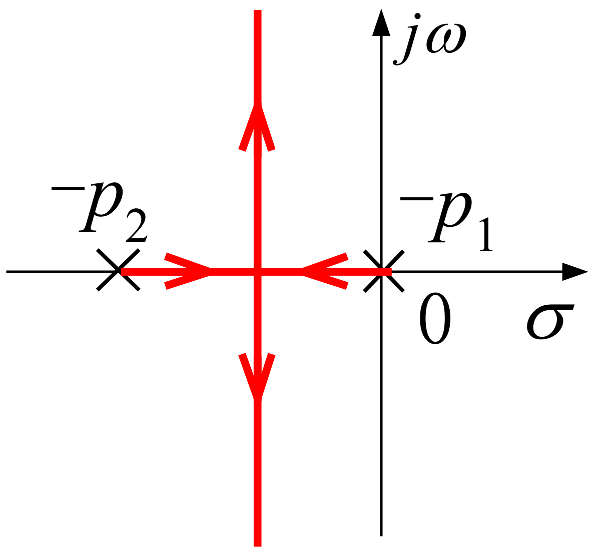
$$1+G_0(s)T(s)=0$$

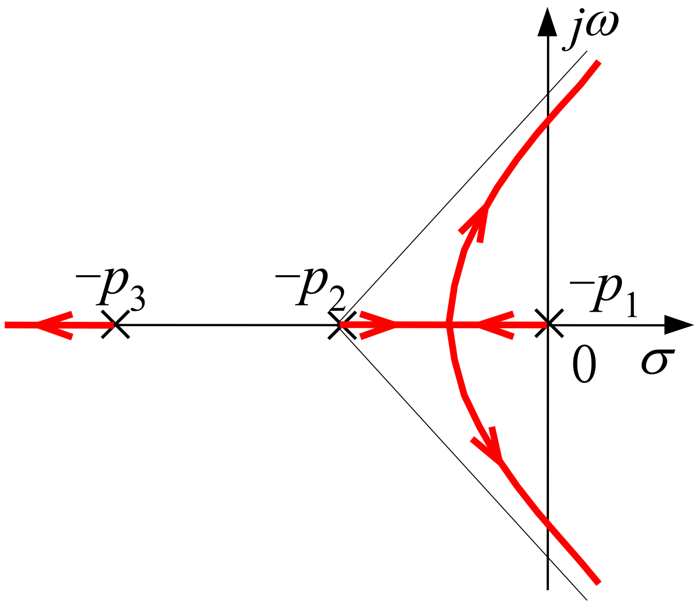
Three cases:

1. Adding poles;
2. Adding zeros;
3. Adding zero-pole pairs;

Effects of Adding Poles

Adding a pole to the left

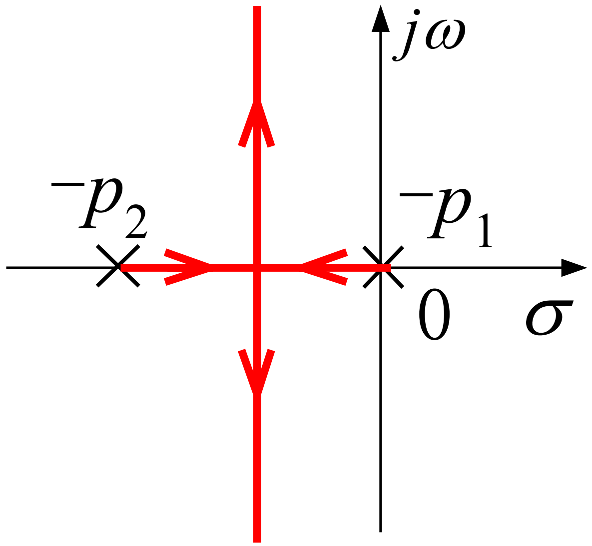


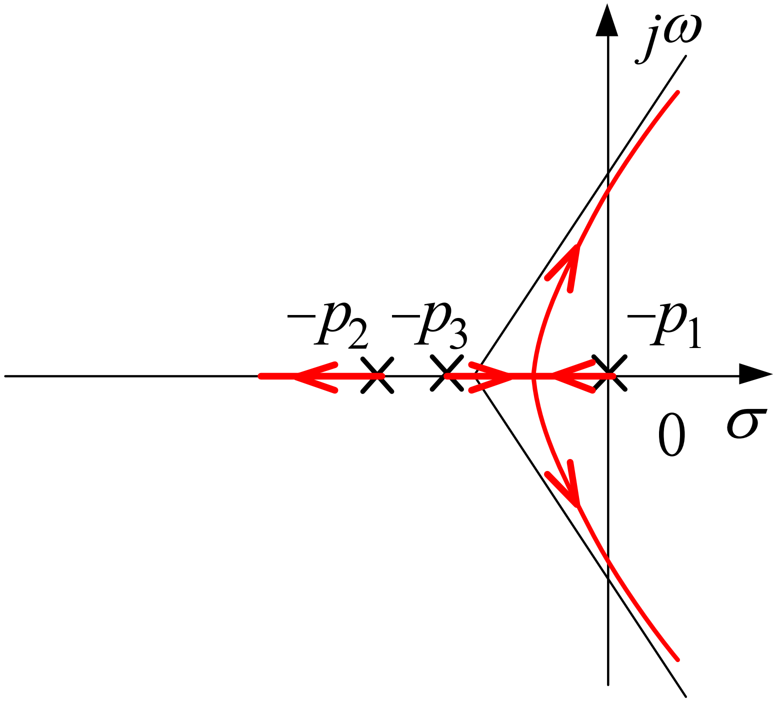


Adding a pole to $G_0(s)$ has the effect of pushing the root loci toward the right-half plane.

Effects of Adding Poles

Adding a pole in the middle

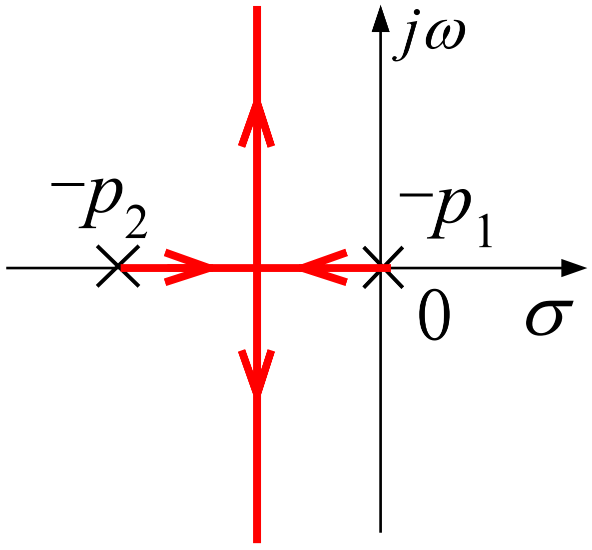


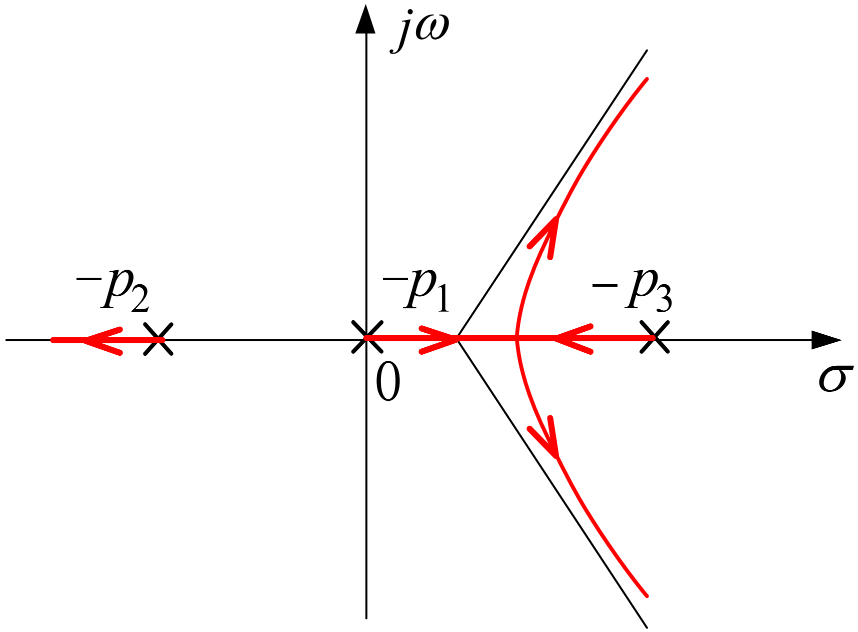


Adding a pole to $G_0(s)$ has the effect of pushing the root loci toward the right-half plane.

Effects of Adding Poles

Adding a pole to the right

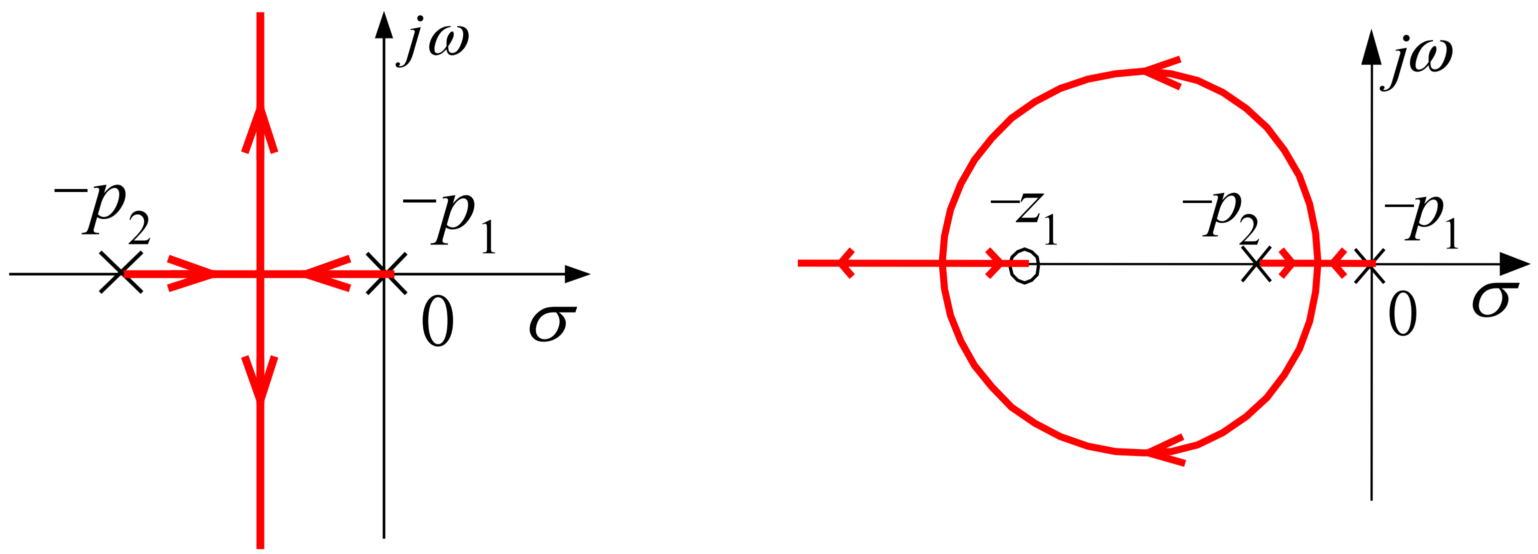




Adding a pole to $G_0(s)$ has the effect of pushing the root loci toward the right-half plane.

Effects of Adding Zeros

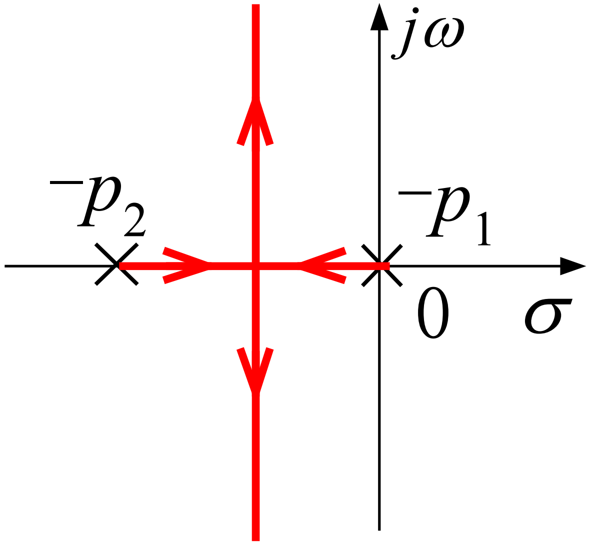
Adding a zero to the left

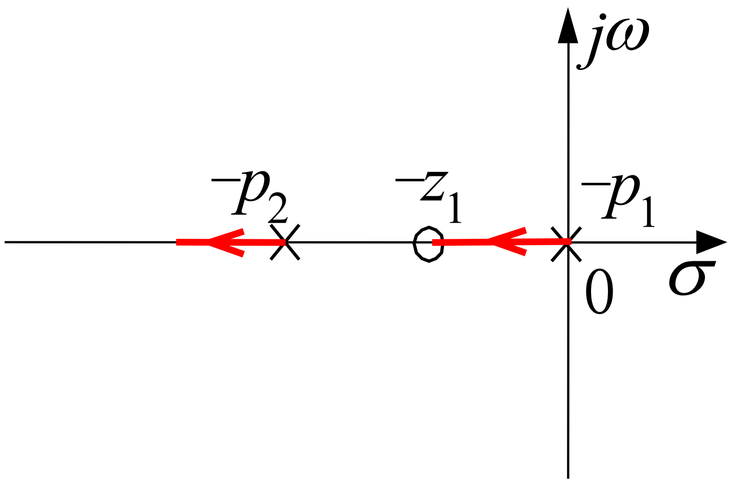


Adding a zero to $G_0(s)$ has the effect of pushing the root loci toward the left-half plane.

Effects of Adding Zeros

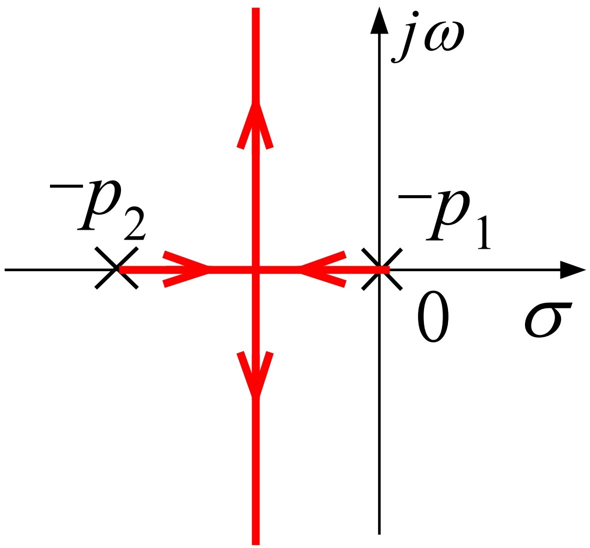
Adding a zero in the middle

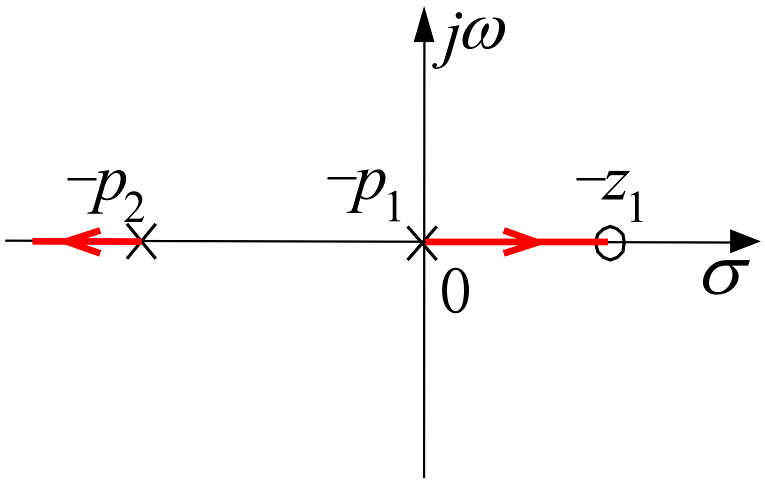




Effects of Adding Zeros

Adding a zero to the right





Effects of Adding Zero-Pole Pairs

$$G_c(s) = k_c \frac{s + z_c}{s + p_c}$$

If both z_c, p_c are located in the left half s plane

$|z_c| < |p_c|$ The zero is more close to the imaginary axis. --- Leading Pair

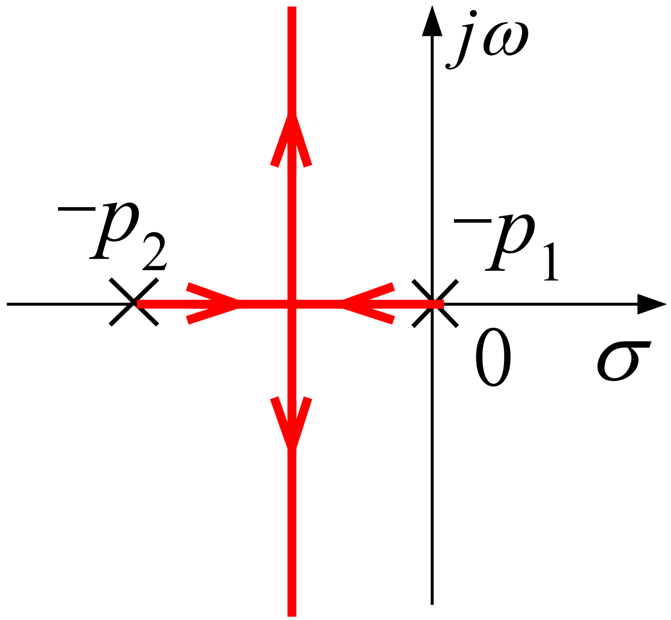
$|z_c| > |p_c|$ The pole is more close to the imaginary axis. --- Lagging Pair

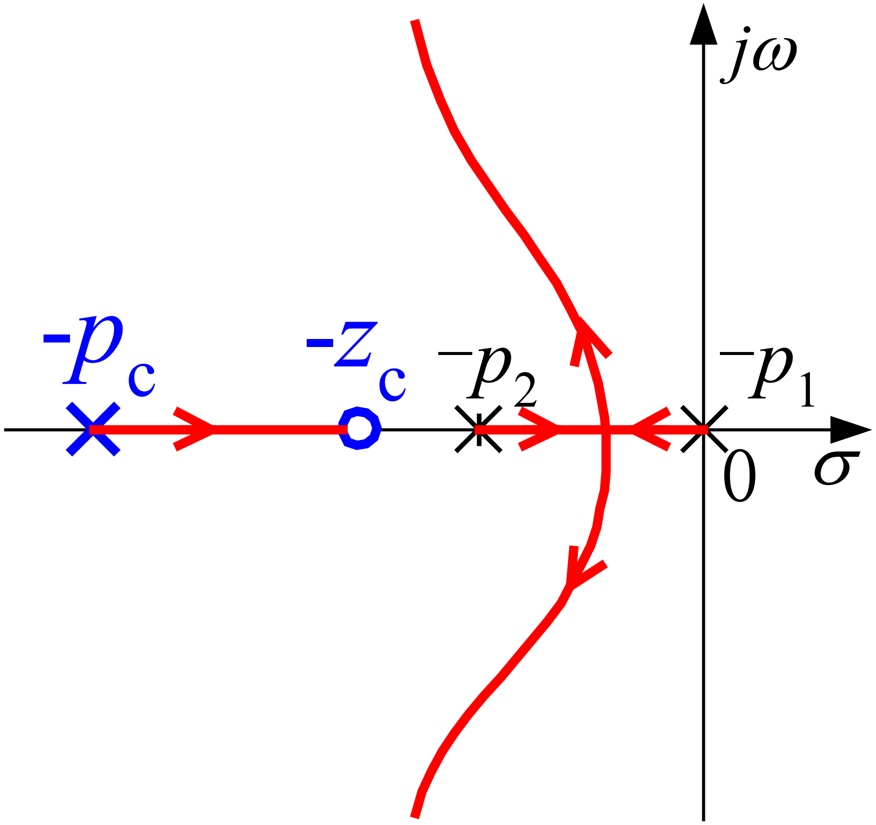
The effects of adding a leading pair are similar to those of adding a zero.

The effects of adding a lagging pair are similar to those of adding a pole.

Adding a Leading Zero-Pole Pair

（1） $-p_c$ $-z_c$ $-p_2$ $-p_1$

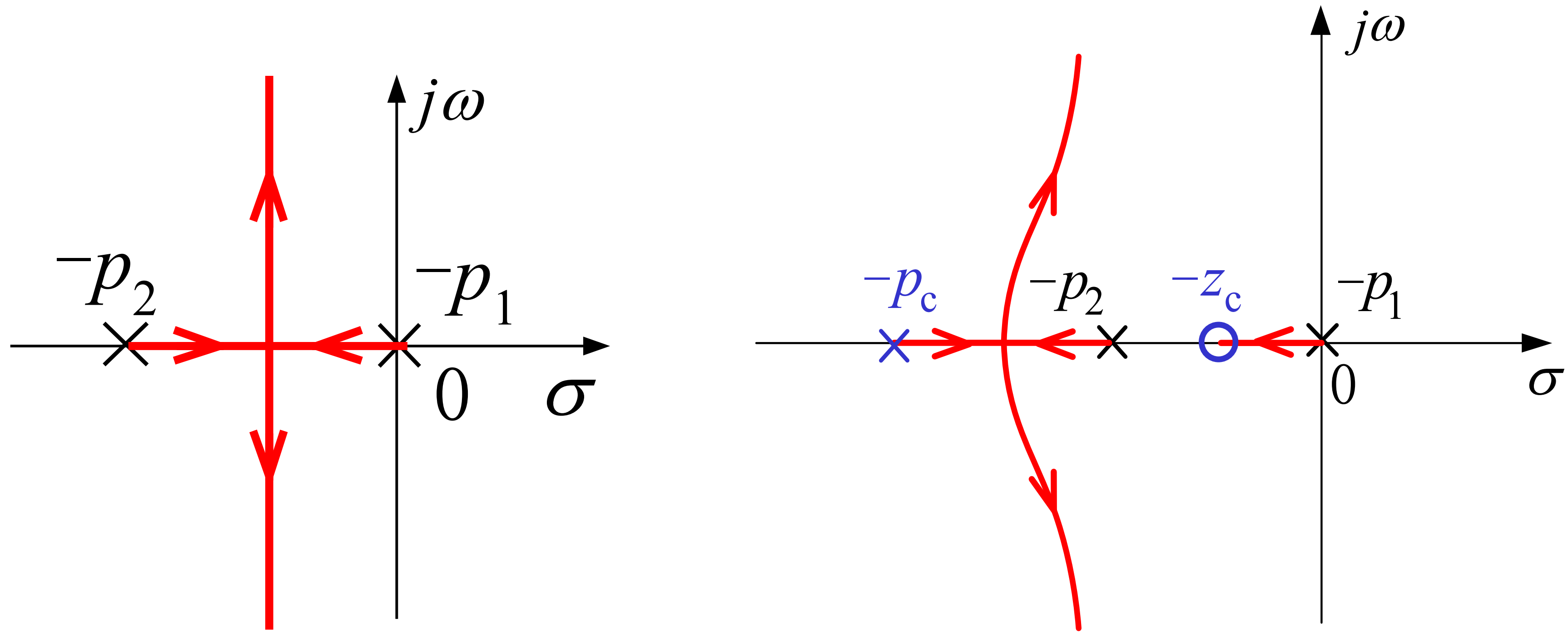




How about adding the pair to other positions?

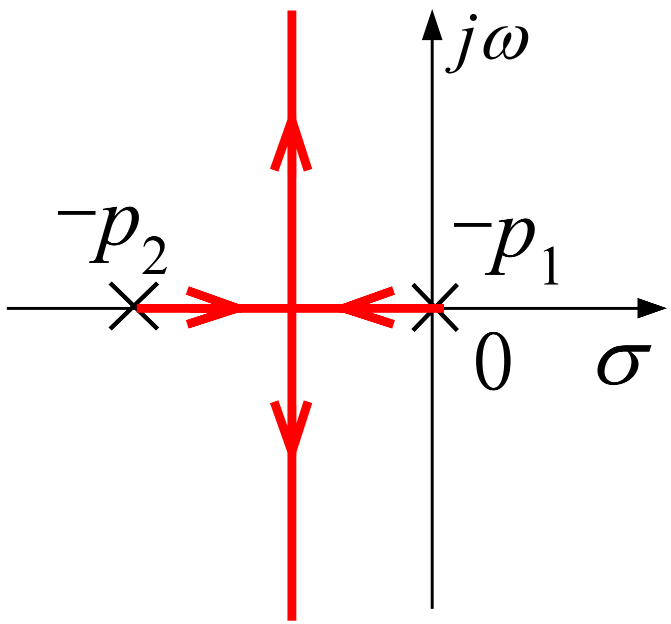
Adding a Leading Zero-Pole Pair

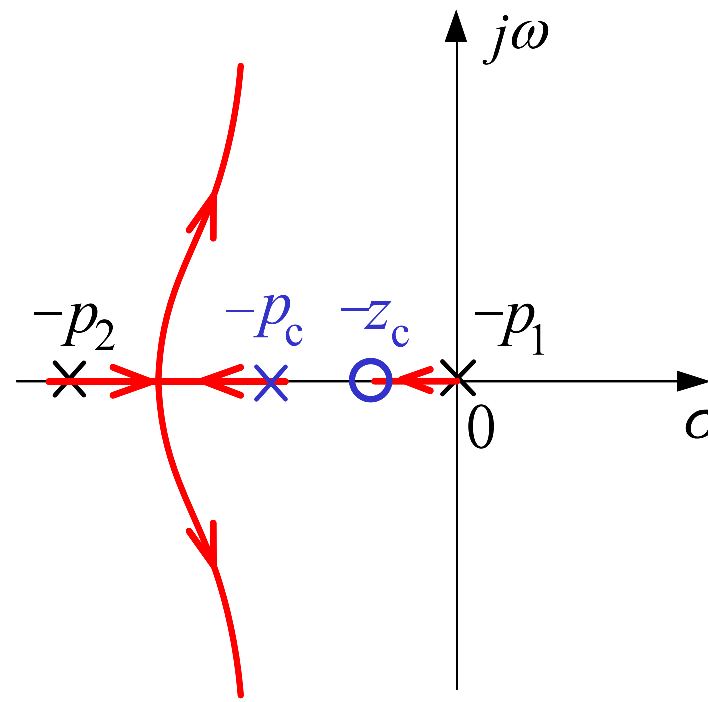
（2） $-p_c \quad -p_2 \quad -z_c \quad -p_1$



Adding a Leading Zero-Pole Pair

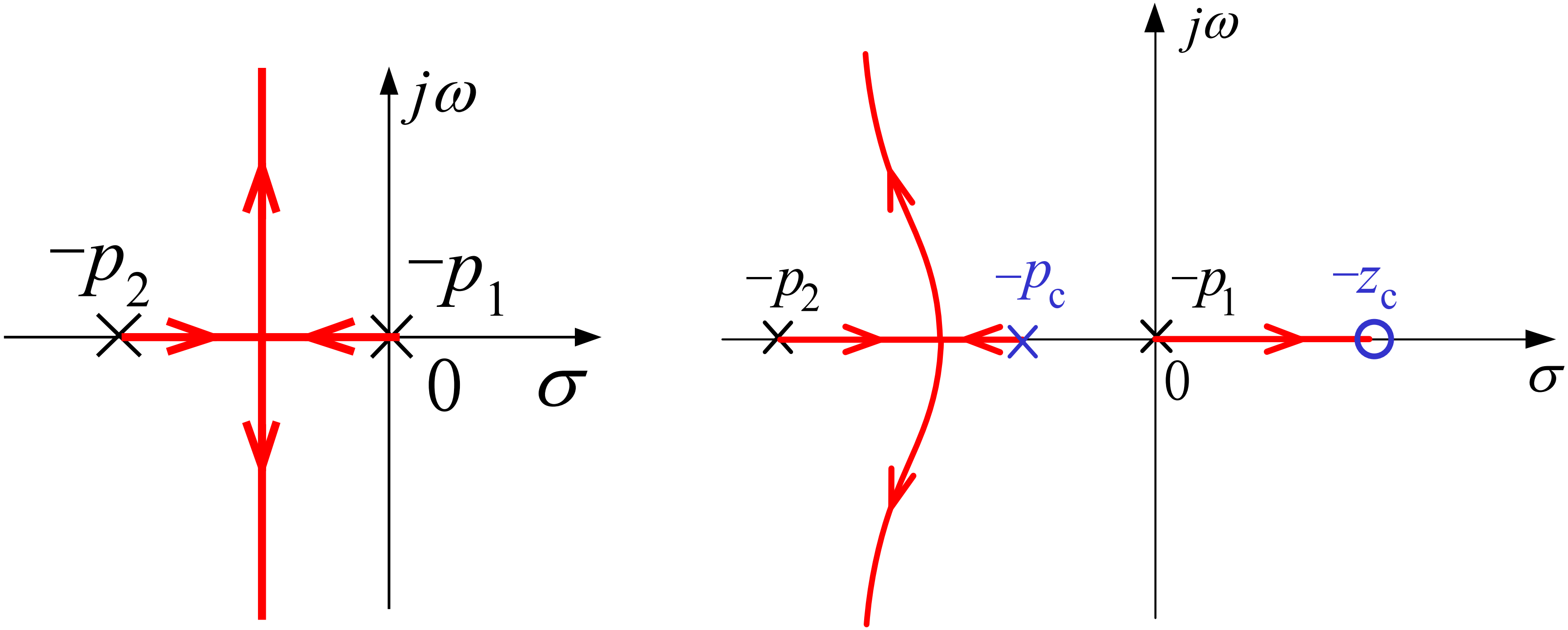
（3） $-p_2 \quad -p_c \quad -z_c \quad -p_1$





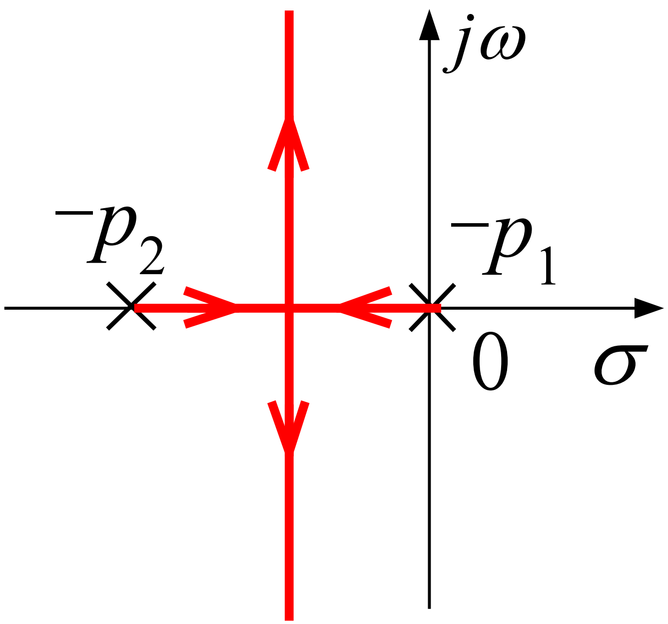
Adding a Leading Zero-Pole Pair

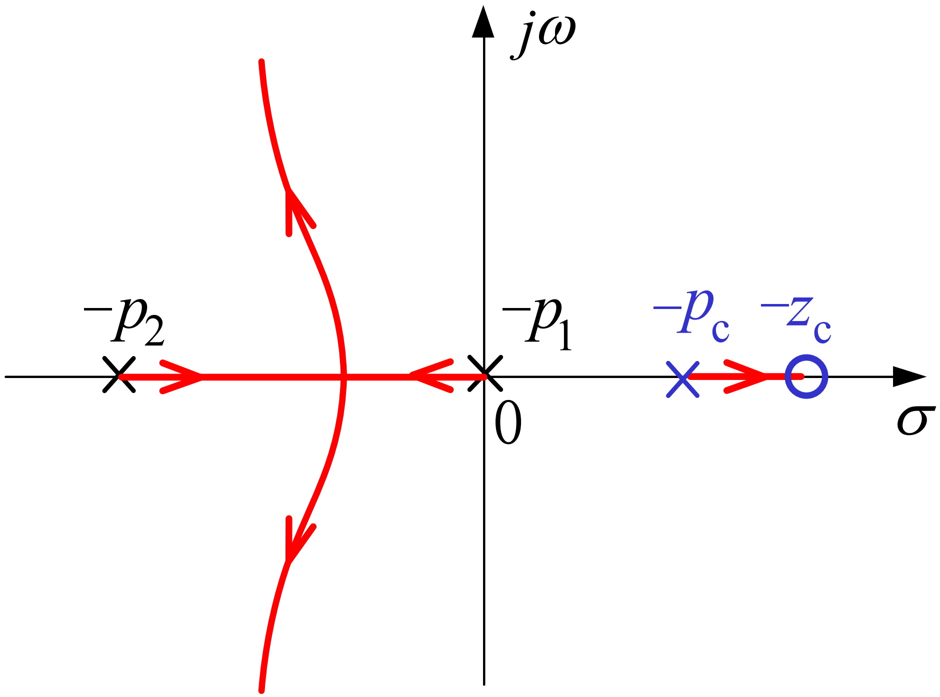
（4） $-p_2 \quad -p_c \quad -p_1 \quad -z_c$



Adding a Leading Zero-Pole Pair

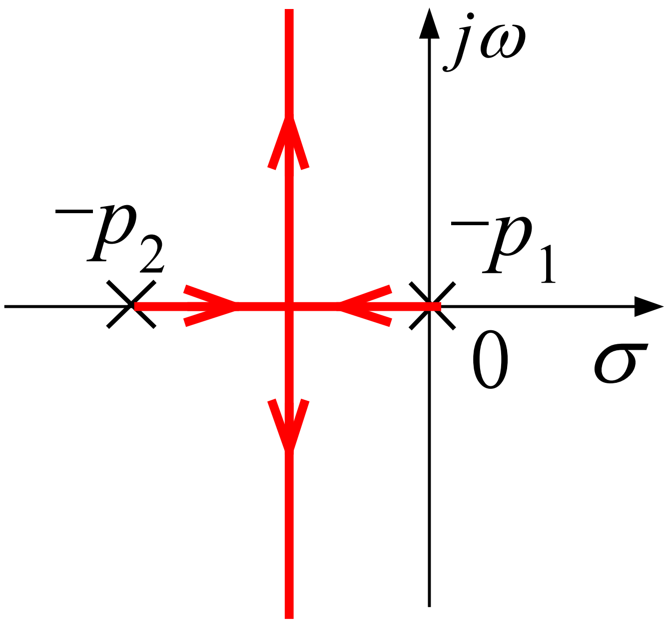
（5） $-p_2 \quad -p_1 \quad -p_c \quad -z_c$

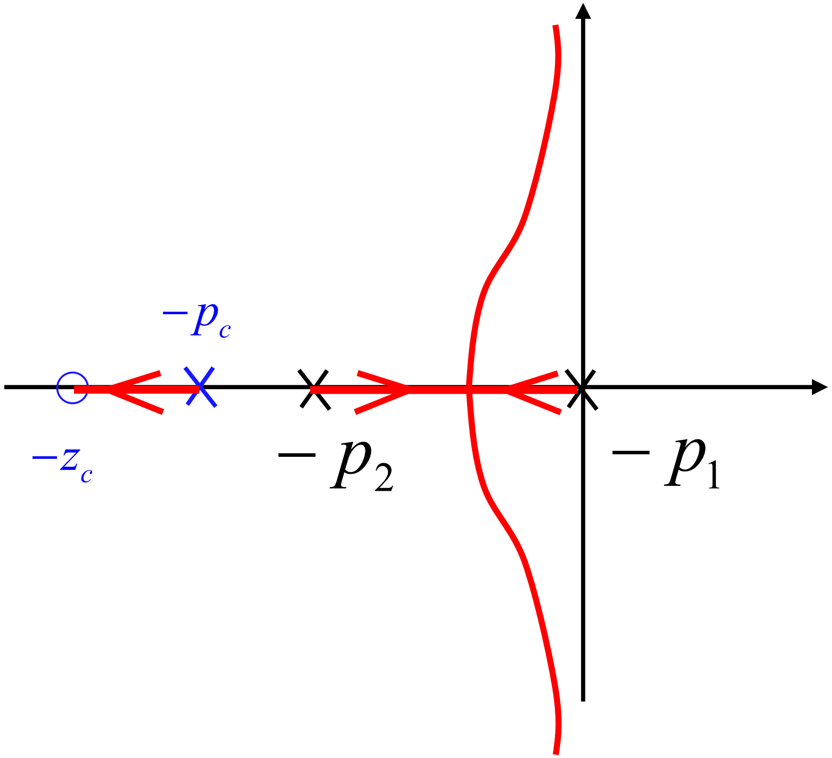




Adding a Lagging Zero-Pole Pair

(1) $-z_c$ $-p_c$ $-p_2$ $-p_1$

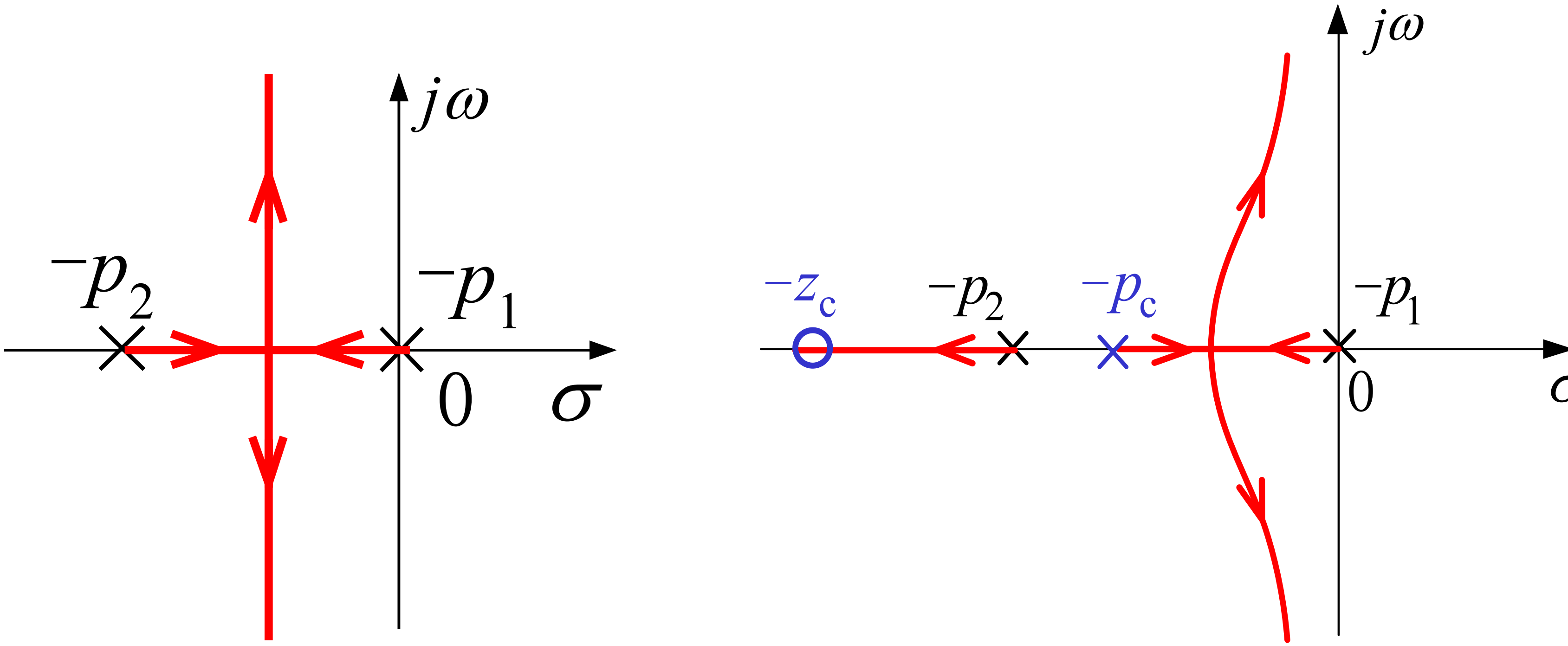




How about adding the pair to other positions?

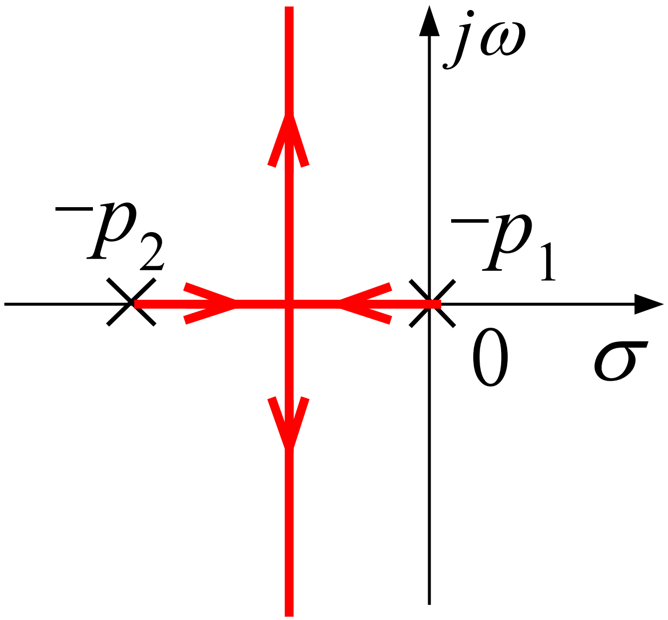
Adding a Lagging Zero-Pole Pair

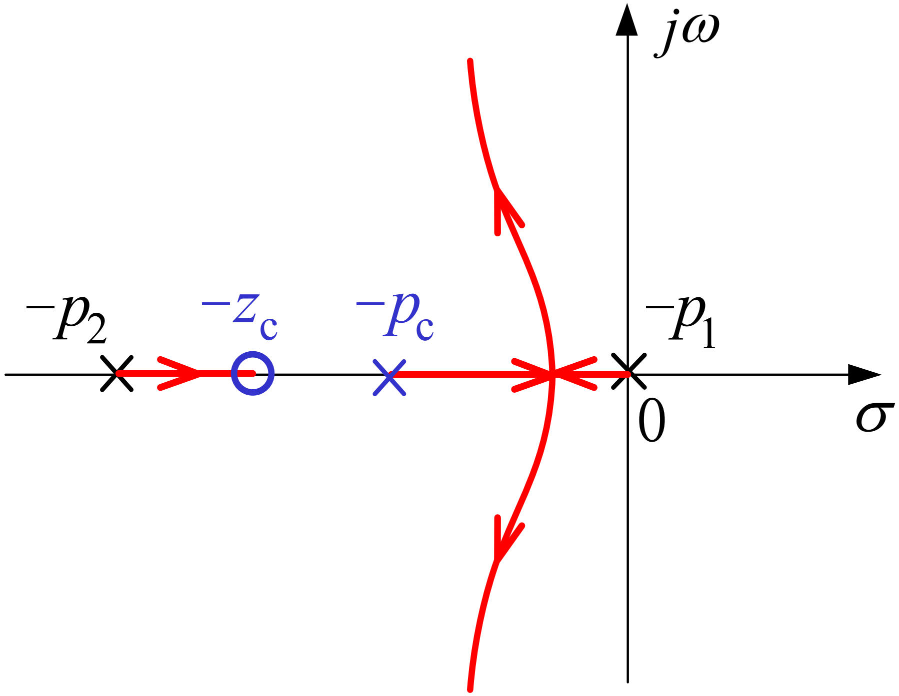
（2） $-z_c \quad -p_2 \quad -p_c \quad -p_1$



Adding a Lagging Zero-Pole Pair

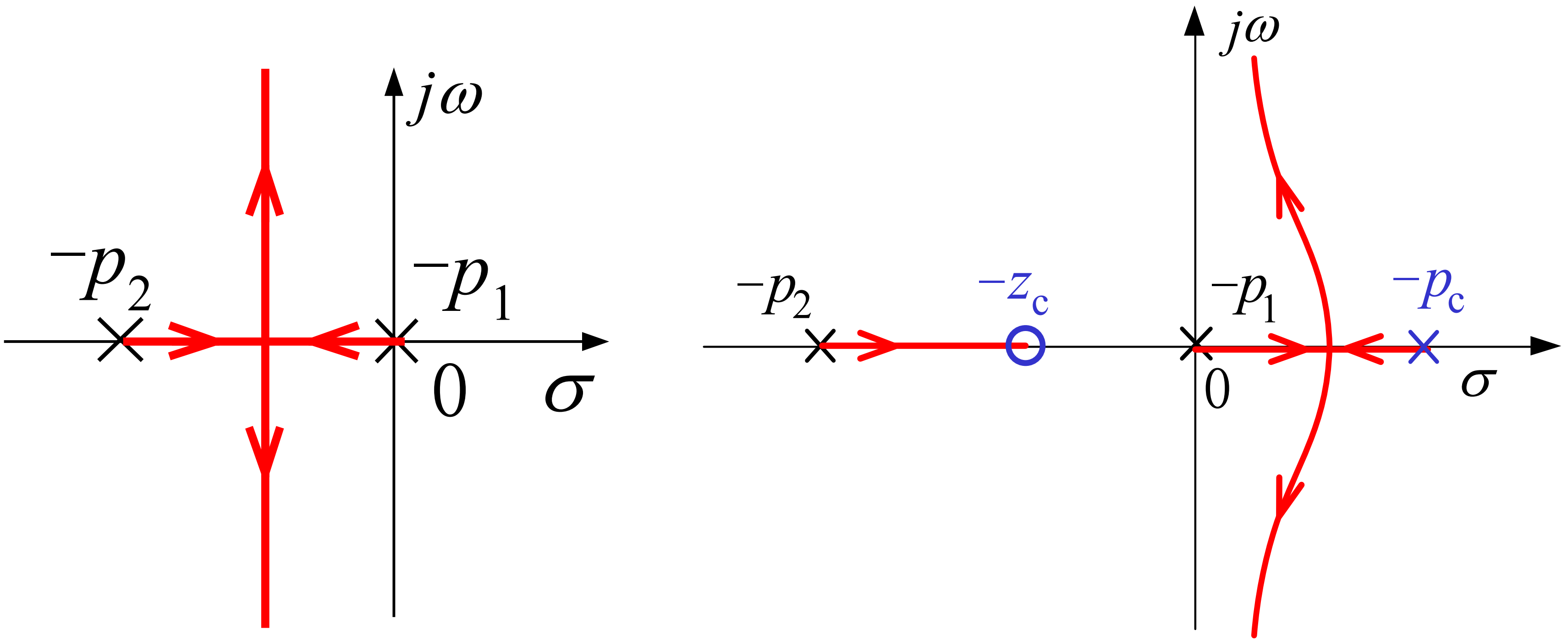
（3）　$-p_2$　$-z_c$　$-p_c$　$-p_1$





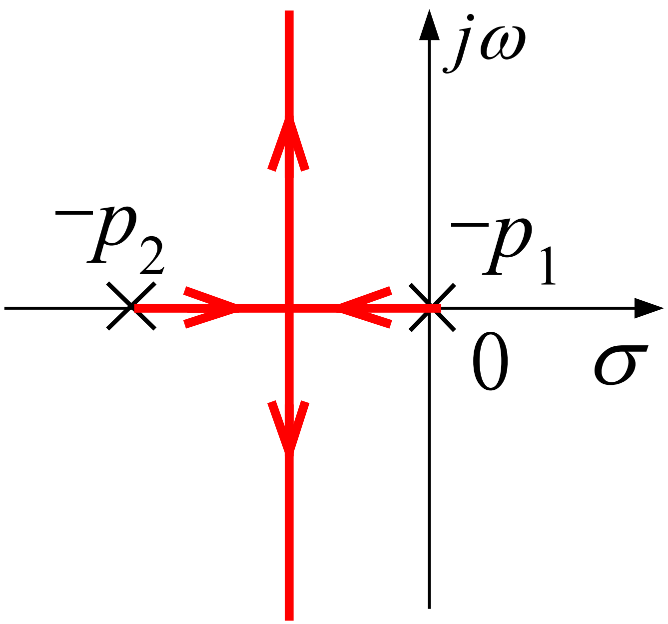
Adding a Lagging Zero-Pole Pair

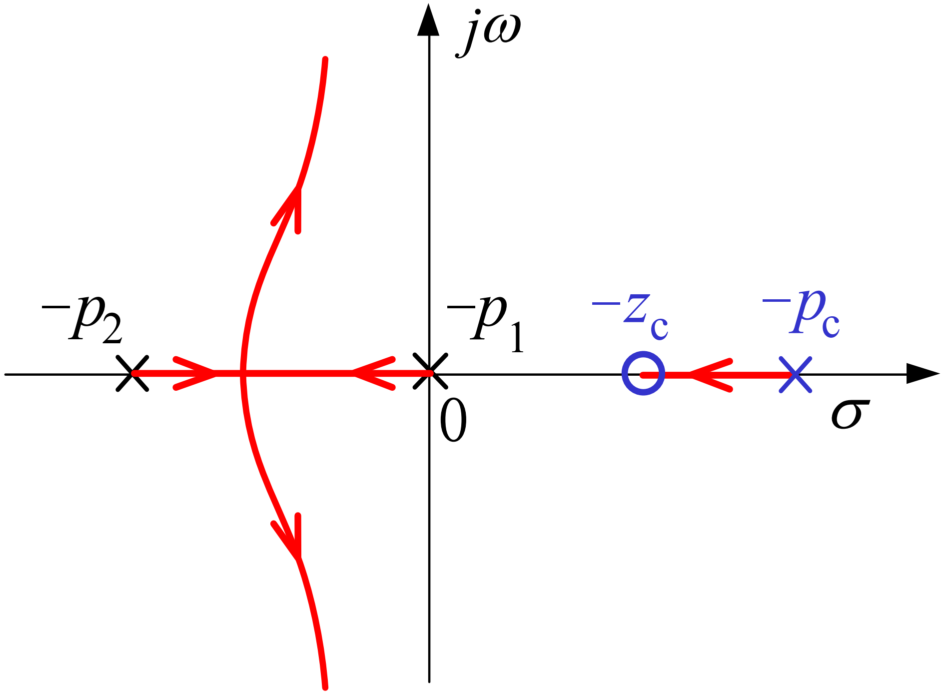
（4） $-p_2 \quad -z_c \quad -p_1 \quad -p_c$



Adding a Lagging Zero-Pole Pair

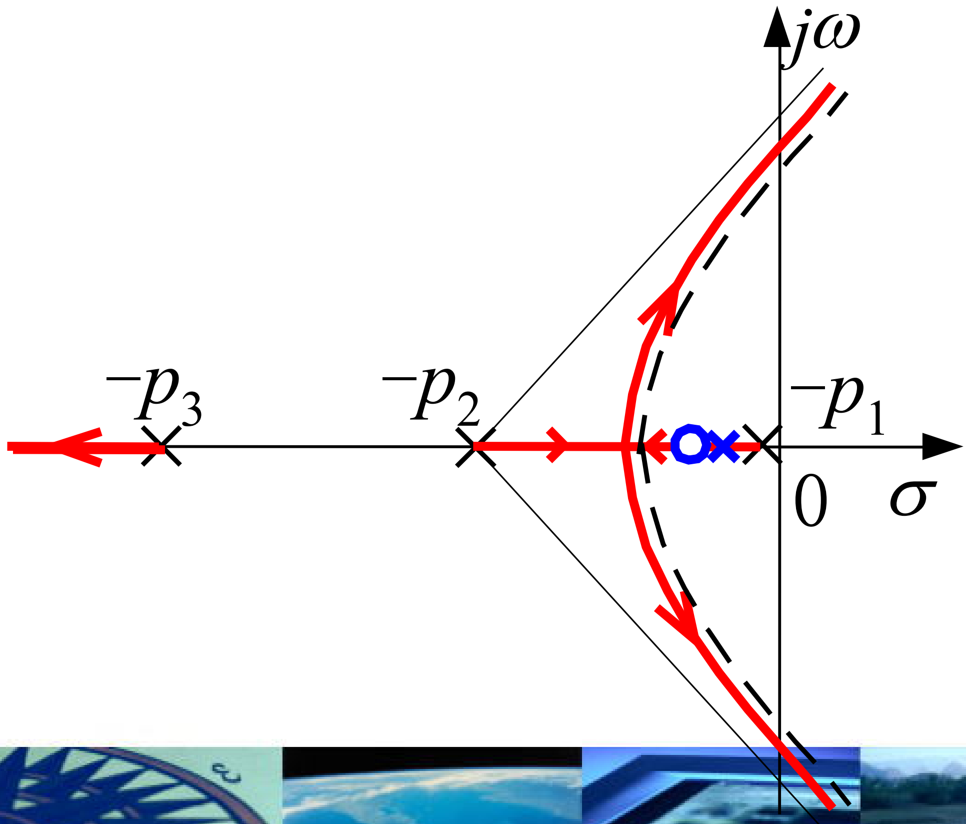
（5） $-p_2$ $-p_1$ $-z_c$ $-p_c$

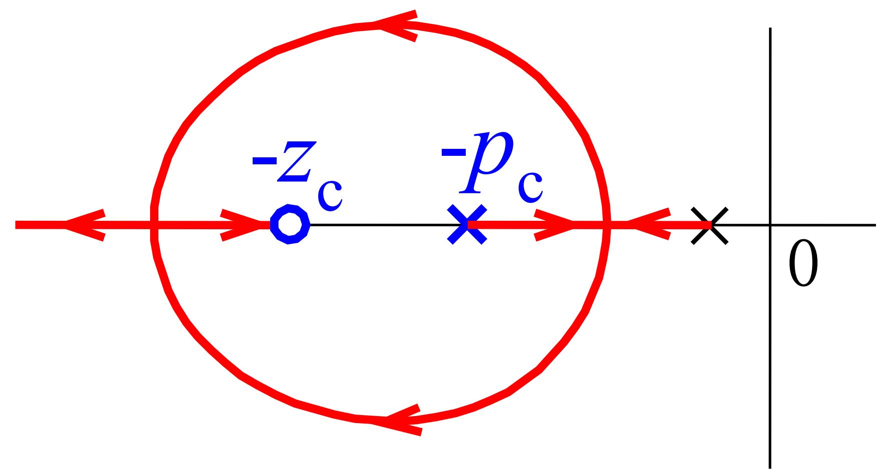




Adding a Lagging Zero-Pole Pair Close to the Origin

$$G_c(s) = k_c \frac{s + z_c}{s + p_c}$$





Adding a Lagging Zero-Pole Pair Close to the Origin

Step-error constant

$$\lim_{s\to 0} G_0(s) = k' \frac{\prod_{i=1}^{m} z_i}{\prod_{j=1}^{v} p_j} \qquad k_f = k' \frac{\prod_{i=1}^{m} z_i}{\prod_{j=1}^{n} p_j} \cdot \frac{z_c}{p_c}$$

$$|z_c| > |p_c| \quad \Longrightarrow \quad k_f > k_b$$

Adding a lagging zero-pole pair close to the origin can decrease the steady-state error of a type-zero system.

Wrap-up for root loci

- Purposes of introducing root loci
- Conditions and rules for constructing root loci
- Expanded usages of root loci method

Frequency-Domain Analysis of Control Systems

chapter 4

Review of Time Domain Analysis

- Basics about time response
- Stability definition and criterion
- Steady-state and transient performance
- Root locus

Outlines

- Basics about frequency domain analysis
- Polar plot of the frequency response of a system
- Nyquist plot
- Nyquist criterion(introduction)

Introduction

- What is frequency response?
  - the frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal

- What is frequency-domain analysis
  - Study the response characteristics of a system to sinusoidal waveform inputs of different frequencies.

Introduction (Continue)

Consider a system response $Y(s)=G(s)U(s)$, the input is sinusoidal with amplitude U and frequency ω_0

$$u(t)=U\sin\omega_0 t$$

The Laplace transform of the input is

$$U(s)=\frac{U\omega_0}{s^2+\omega_0^2}$$

Assume $$G(s)=\frac{m(s)}{\prod_{i=1}^{n}(s+p_i)}$$

where p_i are assumed to be distinct poles. Then in partial fraction form we have

$$Y(s)=\frac{k_1}{s+p_1}+\cdots+\frac{k_n}{s+p_n}+\frac{\alpha s+\beta}{s^2+\omega_0^2}$$

Introduction (Continue)

Taking the inverse Laplace transform yields

$$y(t) = k_1 e^{-p_1 t} + \cdots + k_n e^{-p_n t} + L^{-1}\left\{\frac{\alpha s + \beta}{s^2 + \omega_0^2}\right\}$$

where α and β are constants which are problem dependent. If the system is stable, then all p_i have negative nonzero real parts, and

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} L^{-1}\left\{\frac{\alpha s + \beta}{s^2 + \omega_0^2}\right\}$$

for $t \to \infty$

$$y(t) = L^{-1}\left\{\frac{\alpha s + \beta}{s^2 + \omega_0^2}\right\} = U\left|G(j\omega_0)\right|\sin(\omega_0 t + \phi)$$

$$\phi = \angle G(j\omega_0)$$

Introduction (Continue)

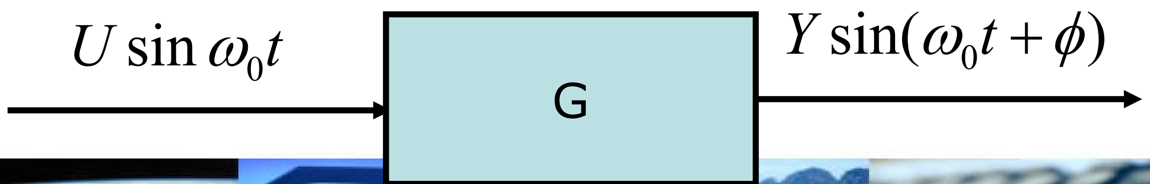
If the input to a linear time-invariant system is sinusoidal with amplitude U and frequency ω_0

$$u(t) = U \sin \omega_0 t$$

the steady-state output of the system, $y(t)$, will be a sinusoid with the same frequency ω_0, but possibly with different amplitude and phase; that is

$$y(t) = Y \sin(\omega_0 t + \phi)$$

When the frequency of the input sinusoid changes, the amplitude and phase shift of the output change with it.



Introduction (Continue)

Obviously, the changes of the output are determined by the internal characteristics of the system. The characteristics can be obtained by the transfer function of the system.

$$Y(s) = G(s)U(s)$$

For sinusoidal steady-state analysis, we replace s by $j\omega$,

$$Y(j\omega) = G(j\omega)U(j\omega)$$

By writing the function $Y(j\omega)$ as

$$Y(j\omega) = \left|Y(j\omega)\right| \angle Y(j\omega)$$

with similar definitions for $G(j\omega)$ and $U(j\omega)$, we can get the magnitude and phase relation between the input and the output

$$\left|Y(j\omega)\right| = \left|G(j\omega)\right|\left|U(j\omega)\right| \qquad \angle Y(j\omega) = \angle G(j\omega) + \angle U(j\omega)$$

Introduction (Continue)

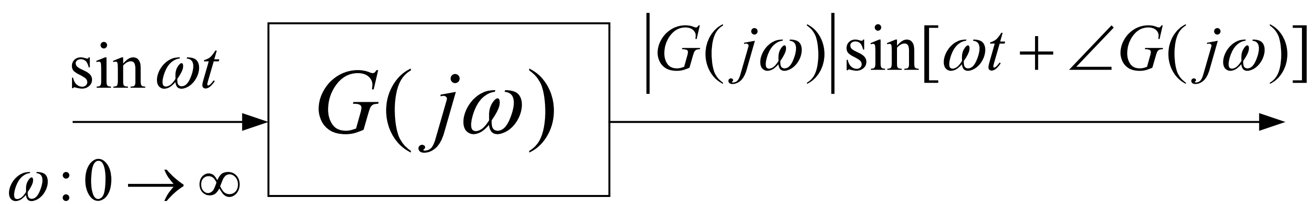
The magnitude-frequency relation of the system is

$$\left|G(j\omega)\right| = \frac{\left|Y(j\omega)\right|}{\left|U(j\omega)\right|}$$

The phase-frequency relation of the system is

$$\angle G(j\omega) = \angle Y(j\omega) - \angle U(j\omega)$$

The above two relations can be gotten by measurement



Introduction

- Why frequency-domain analysis?
  - Advantages
    - Very clear physical fundamentals;
    - Ready availability of sinusoid test signals;
    - A wealthy of graphical methods not limited to low-order systems;
    - More convenient for measurements of system sensitivity to noise and parameter variation;
    - Presents an alternative point of view to control system problem.
  - Disadvantages
    - Only applicable to LTI system;
    - Hard to get accurate time-domain performance criteria

Frequency Response Plots

The transfer function of a system $G(s)$ can be described in the frequency domain by the relation

$$G(j\omega) = G(s)\Big|_{s=j\omega} = R(\omega) + jX(\omega)$$

where $R(\omega) = \mathrm{Re}[G(j\omega)]$ and $X(\omega) = \mathrm{Im}[G(j\omega)]$

Alternatively the transfer function can be represented by a magnitude and a phase as

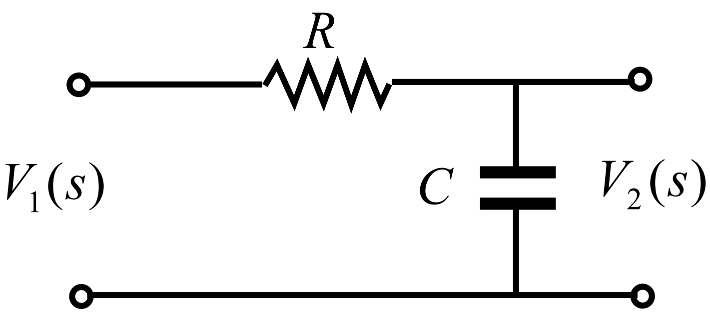
$$G(j\omega) = \left|G(j\omega)\right| e^{j\phi(j\omega)} = \left|G(\omega)\right| \angle \phi(j\omega)$$

where $\phi(j\omega) = \tan^{-1}\dfrac{X(\omega)}{R(\omega)}$ and $\left|G(\omega)\right|^2 = [R(\omega)]^2 + [X(\omega)]^2$

The polar plot are usually used to graphically represent the frequency response of the system. The coordinates of the polar plot can be obtained from the above equations.

Example 1

Q: Construct the polar plot of the frequency response of the RC circuit



A: the transfer function is

$$G(s) = \frac{1}{RCs + 1}$$

The sinusoidal steady-state transfer function is

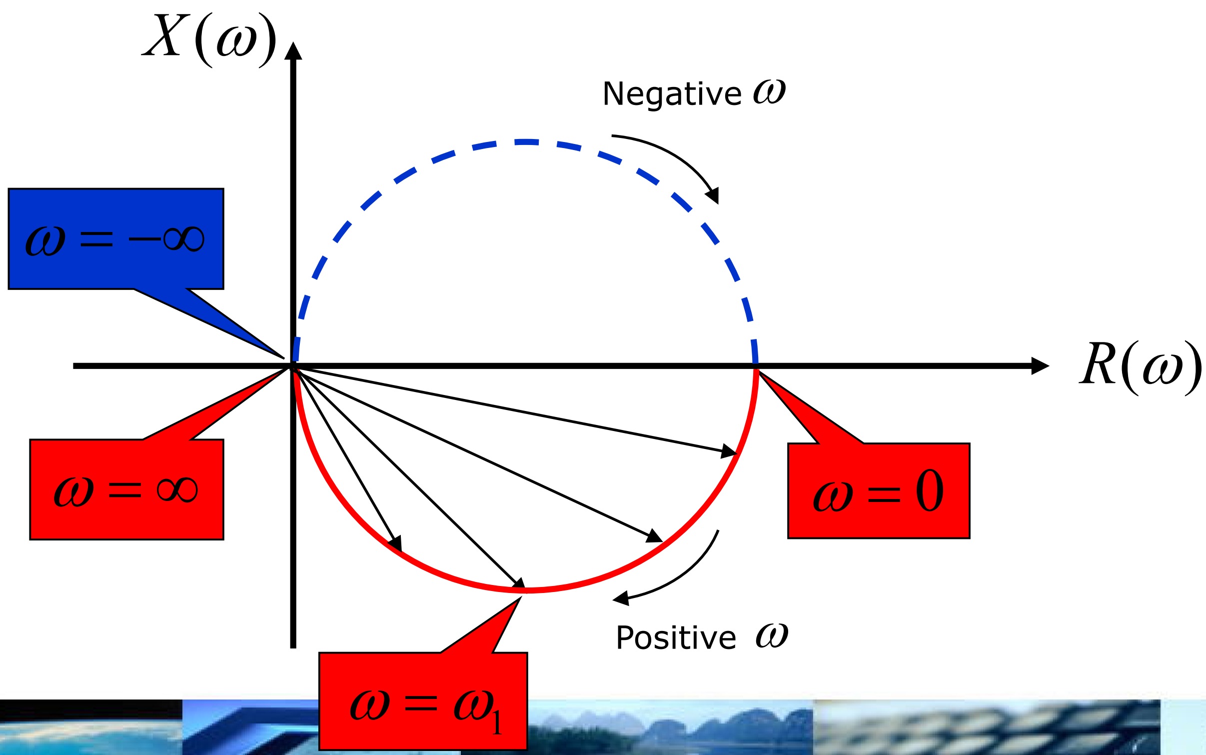
$$G(j\omega) = \frac{1}{j\omega(RC) + 1} = \frac{1}{j(\omega/\omega_1) + 1} \qquad \text{where} \qquad \omega_1 = \frac{1}{RC}$$

In magnitude-angle form $G(j\omega) = |G(\omega)| \angle \phi(\omega)$

$$|G(\omega)| = \frac{1}{[1 + (\omega/\omega_1)^2]^{1/2}} \qquad \phi(\omega) = -\tan^{-1}(\omega/\omega_1)$$

Example 1

$$|G(\omega)| = \frac{1}{[1+(\omega/\omega_1)^2]^{1/2}} \qquad \phi(\omega) = -\tan^{-1}(\omega/\omega_1)$$



Example 2

Q: Construct the polar plot of the frequency response of the following transfer function

$$G(s) = \frac{T_1 s + 1}{T_2 s + 1}$$

A: The sinusoidal steady-state transfer function is

$$G(\omega) = \frac{j\omega T_1 + 1}{j\omega T_2 + 1} \qquad \phi(\omega) = -\tan^{-1}(\omega / \omega_1)$$

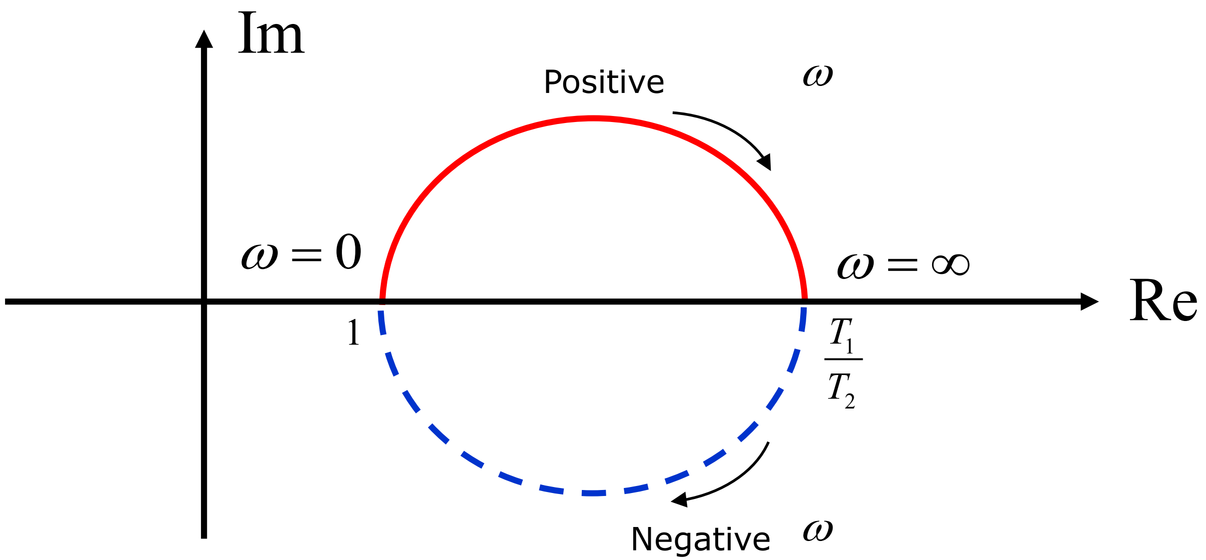
In magnitude-angle form $G(j\omega) = |G(\omega)| \angle \phi(\omega)$

$$|G(s)| = \sqrt{\frac{\omega^2 T_1^2 + 1}{\omega^2 T_2^2 + 1}} \qquad \phi(\omega) = \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

Example 2

$$\left|G(s)\right| = \sqrt{\frac{\omega^2 T_1^2 + 1}{\omega^2 T_2^{\,2} + 1}} \qquad \phi(\omega) = \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

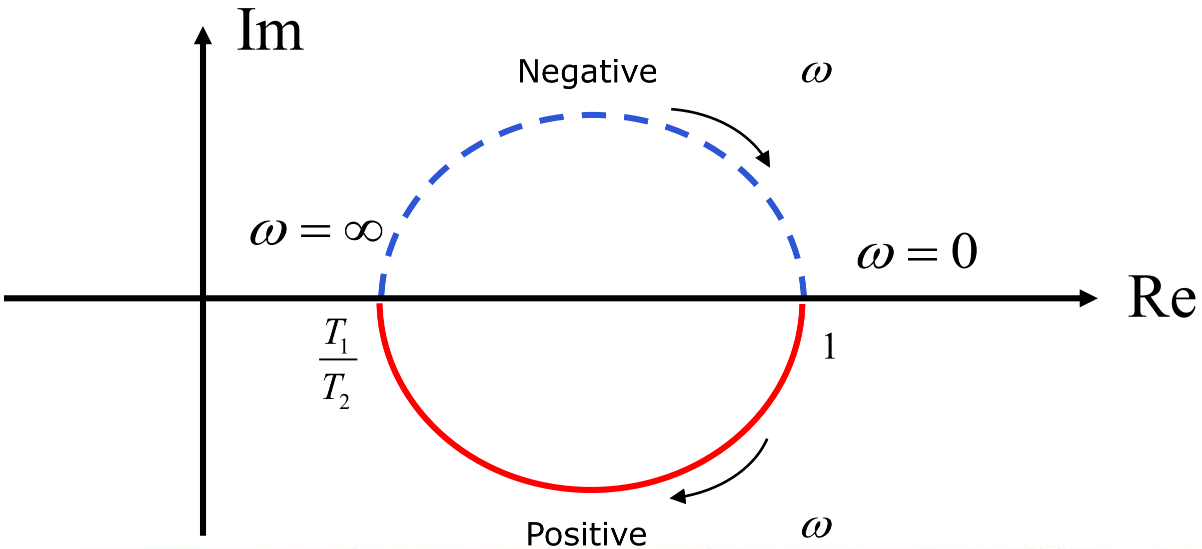
Case 1: $T_1 > T_2$



Example 2

$$|G(s)| = \sqrt{\frac{\omega^2 T_1^2 + 1}{\omega^2 T_2^2 + 1}} \qquad \phi(\omega) = \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

Case 2: $T_1 < T_2$



Stability Analysis

Question: (1) How to analyze the stability of a system in frequency domain?

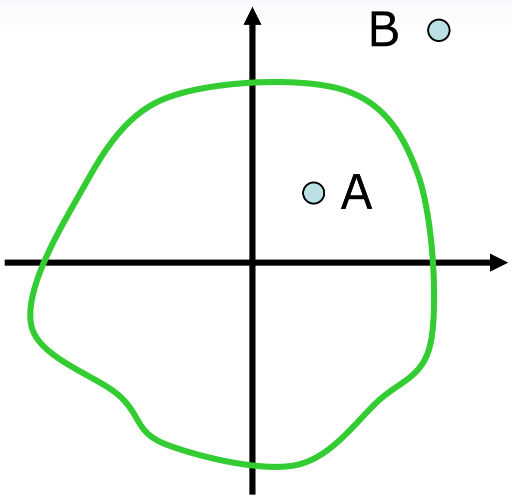
(2) Do we have any convenient tools to do that?

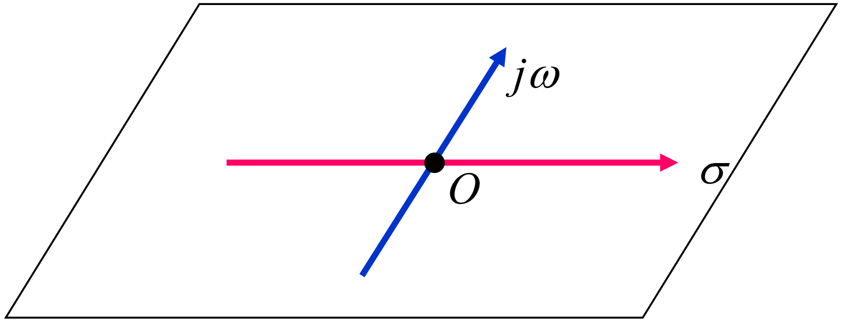
Nyquist Criterion

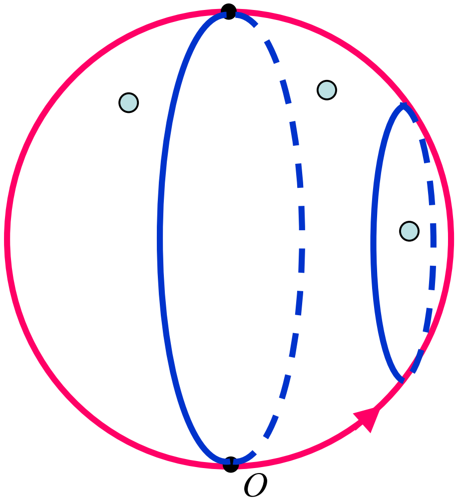
Nyquist Stability Criterion: Fundamentals

Encircled

A point or region in a complex function plane is said to be encircled by a closed path if it is found **inside** the path



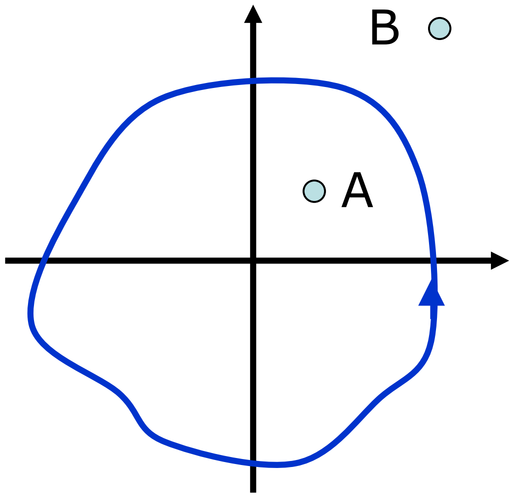


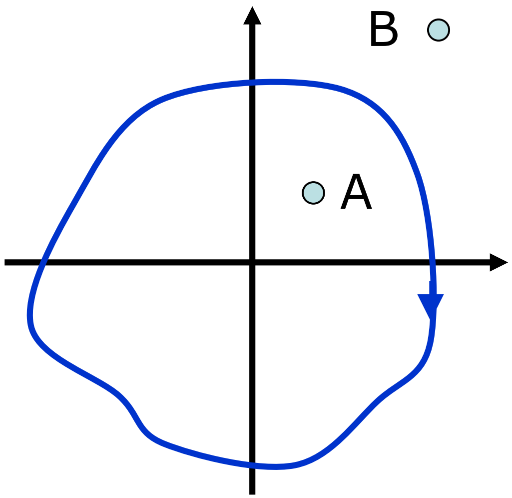


Nyquist Stability Criterion: Fundamentals

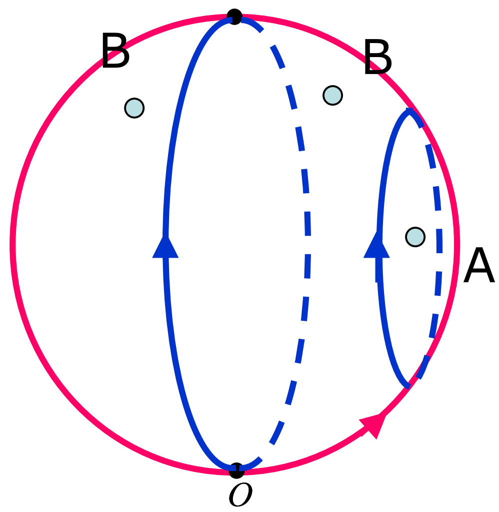
Enclosed

A point or region is said to be enclosed by a closed path if it is encircled in counter clockwise (CCW) direction, or the point or region lies to the left of the path when the path is traversed in the prescribed direction.





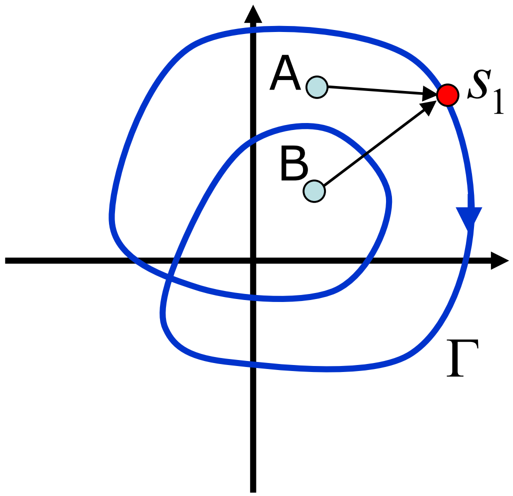
Nyquist Stability Criterion: Fundamentals

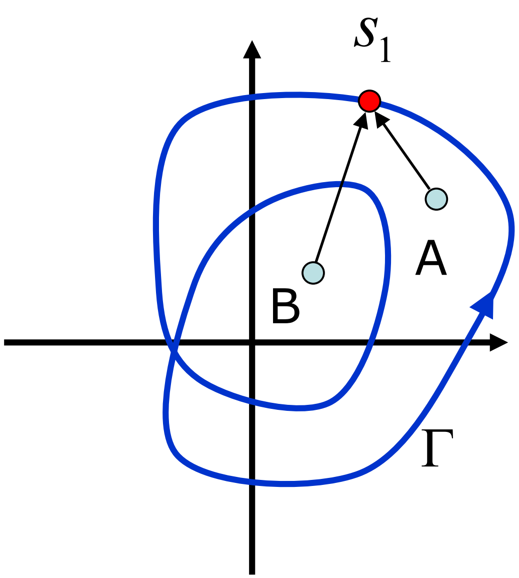


Nyquist Stability Criterion: Fundamentals

Number of Encirclements and Enclosures

When a point is encircled by a closed path Γ, a number N can be assigned to the number of times it is encircled. The magnitude of N can be determined by drawing an arrow from the point to any arbitrary point s_1 on the closed path Γ and then let s_1 follow the path in the prescribed direction until it returns to the starting point. The total net number of revolution traversed by the arrow is N, or the net angle is $2\pi \bullet N$ radian.

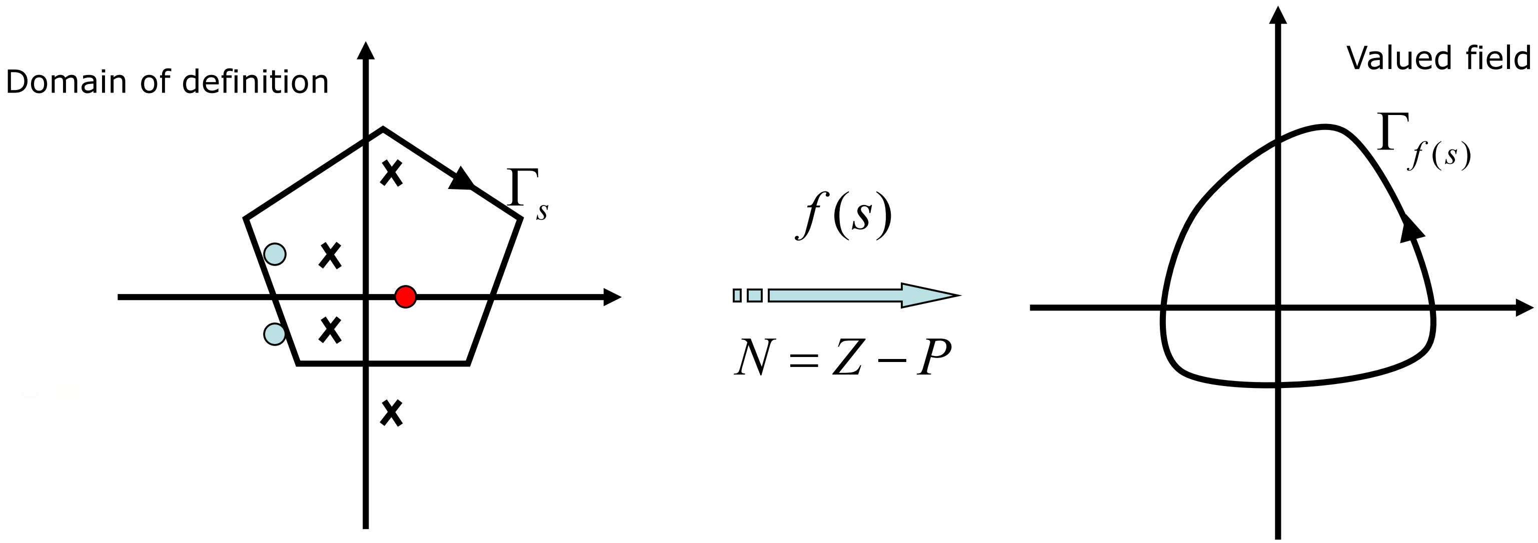




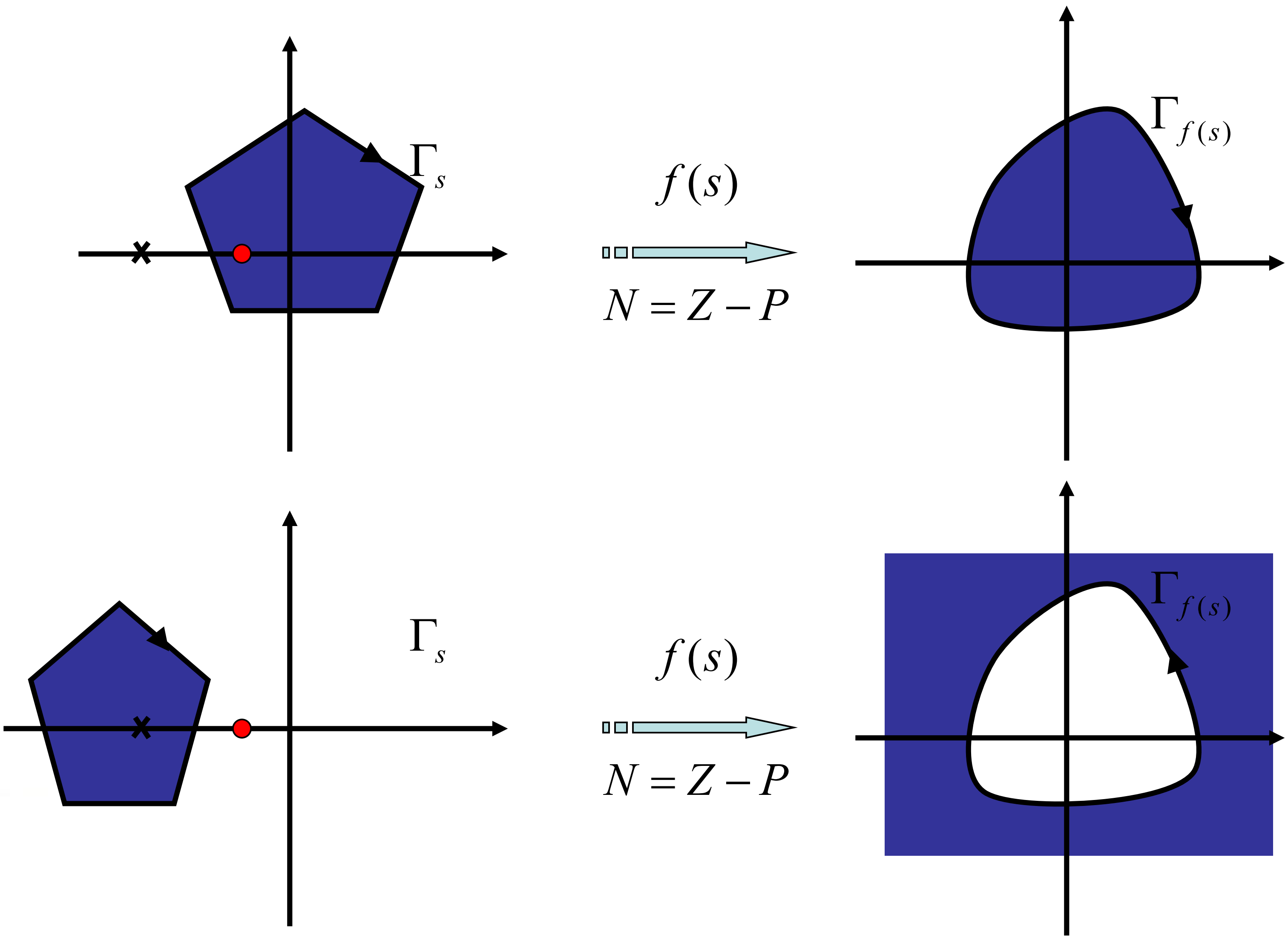
Nyquist Stability Criterion: Fundamentals

Principle of Argument

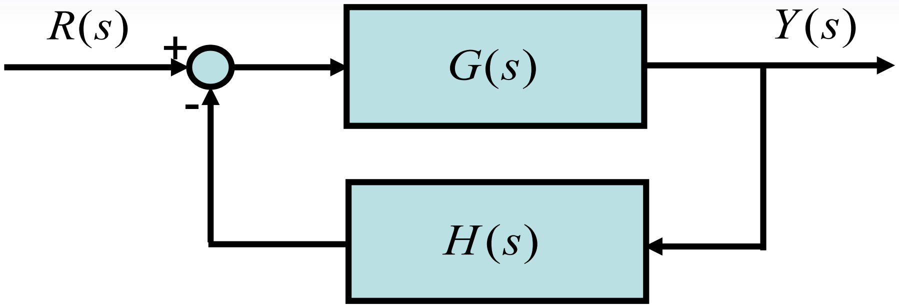
Let $f(s)$ be a single-valued function that has a finite number of poles in the s-plane. Suppose that an arbitrary closed path Γ_s is chosen in the s-plane so that the path dose not go through any one of the poles or zeros of $f(s)$; the corresponding locus $\Gamma_{f(s)}$ mapped in the $f(s)$ -plane will encircle the origin as many times as the difference between the number of zeros and poles of $f(s)$ that are encircled by the s-plane locus



Nyquist Stability Criterion: Fundamentals



Stability Related



$$T(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{G(s)}{1+G_0(s)}$$

$$1+G_0(s) = 1+\frac{N_0(s)}{D_0(s)} = \frac{D_0(s)+N_0(s)}{D_0(s)} = \frac{D_C(s)}{D_0(s)}$$

$$T(s) = \frac{G(s)\cdot D_0(s)}{D_c(s)}$$

closed-loop transfer function poles: roots of the characteristic equation

zeros of $1+G_0(s)$

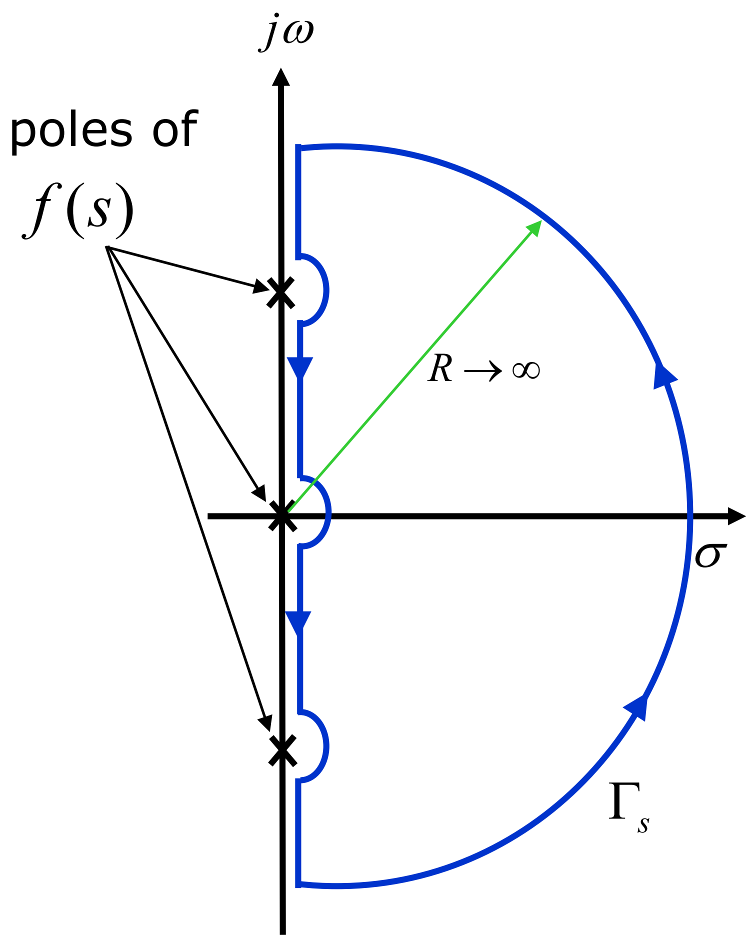
Stability condition: zeros of $1+G_0(s)$ are all in the left-half s-plane

Nyquist Path

Nyquist path is the s-plane locus that encircles the entire right-half s-plane.

The Nyquist path does not go through any poles or zeros of the concerned function of s.

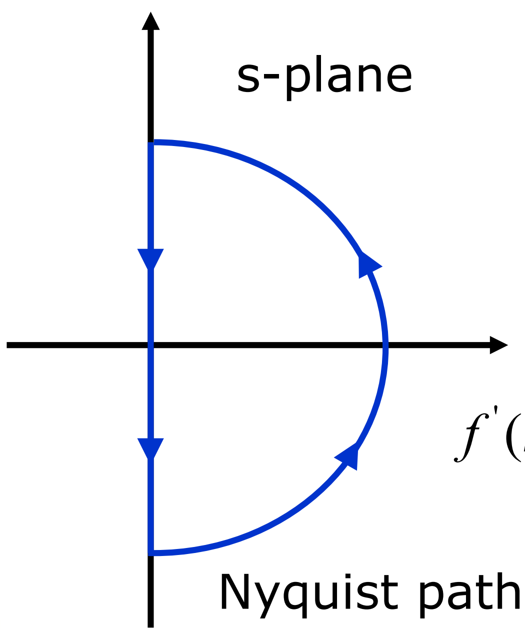
CCW is assigned as the positive direction



Nyquist Criterion

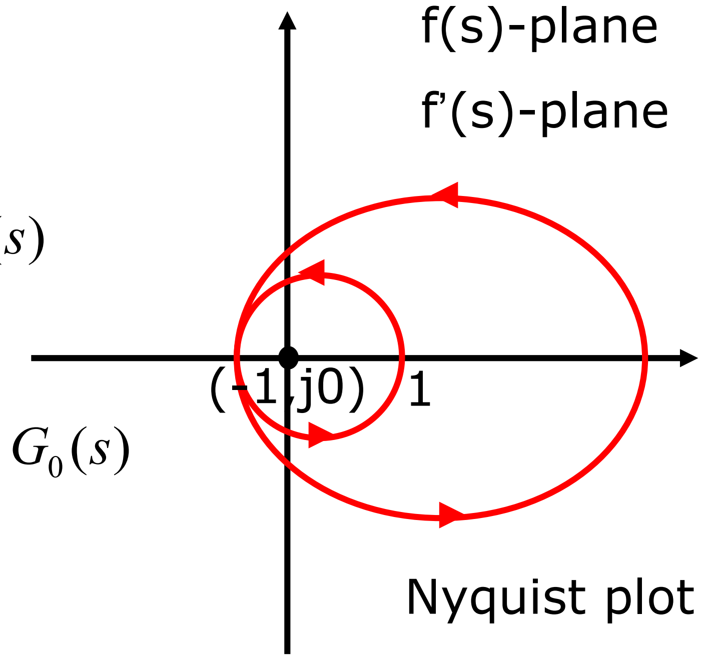
Set $f(s) = 1 + G_0(s) = \dfrac{D_c(s)}{D_0(s)}$ when s varies along the Nyquist path,

a corresponding locus is gotten in the f(s)-plane



$$f(s) = 1 + G_0(s)$$

$$f'(s) = f(s) - 1 = G_0(s)$$



(-1,j0) is the critical point

Nyquist Criterion

The application of the Nyquist criterion to the stability problem involves the following steps:

1. Define the Nyquist path in the s-plane;

2. Draw the $G_0(s)$ plot in the $G_0(s)$ plane;

3. The number of encirclements of the (-1,j0) point made by $G_0(s)$ is observed;

4. The Nyquist criterion follows the principle of argument

   N=Z-P

   where

   N = number of encirclements of the (-1,j0) point made by the $G_0(s)$ plot

   Z = number of zeros of $G_0(s)$ that are inside the Nyquist path

   P = number of poles of $G_0(s)$ that are inside the Nyquist path

Nyquist Criterion

N=Z-P

N = number of encirclements of the (-1,j0) point made by the $G_0(s)$ plot

Z = number of zeros of $G_0(s)$ that are inside the Nyquist path

P = number of poles of $G_0(s)$ that are inside the Nyquist path

$$1+G_0(s)=1+\frac{N_0(s)}{D_0(s)}=\frac{D_0(s)+N_0(s)}{D_0(s)}=\frac{D_C(s)}{D_0(s)}$$

$$T(s)=\frac{G(s)}{1+G_0(s)}=\frac{G(s)\cdot D_0(s)}{D_c(s)}$$

For closed-loop stability, Z must equal zero

Nyquist Criterion

Thus the condition of stability according to the Nyquist criterion is stated as

$$N = -P$$

For a closed-loop system to be stable, the $G_0(s)$ plot must encircle the (-1,j0) point as many times as the number of poles of $G_0(s)$ that are in the right-half s-plane, and the encirclement, if any, must be made in the clockwise direction (if Γ_s is defined in the CCW sense).

If the open-loop system is stable, the $G_0(s)$ plot should not encircle the (-1,j0) point.

$$N = 0$$

Example 4.1

Q: Sketch the Nyquist plot of the system with the following open-loop transfer function and analyze the stability of the closed-loop system.

$$G_0(s) = \frac{K}{(T_1 s + 1)(T_2 s + 1)}$$

where $K、T_1、T_2 > 0$

A: substitute $s = j\omega$ in $G_0(s)$

$$G_0(j\omega) = \frac{K}{(T_1 j\omega + 1)(T_2 j\omega + 1)}$$

$$G_0(j\omega) = \left|G_0(j\omega)\right| \angle G_0(j\omega)$$

substitute $\omega = 0$ in $G_0(j\omega)$ $\quad G_0(j0) = K\angle 0^\circ$

substitute $\omega = \infty$ in $G_0(j\omega)$ $\quad G_0(j\infty) = 0\angle -180^\circ$

Example 4.1

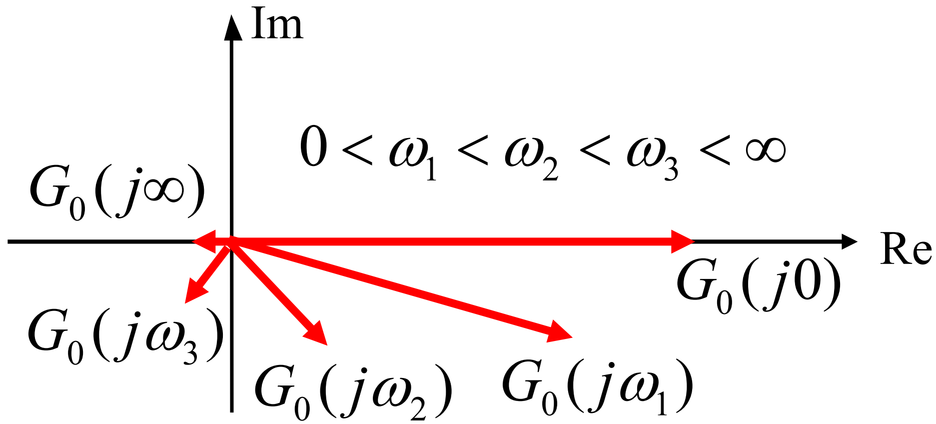
analyze the trend of $G_0(j\omega)$ when ω varies from 0 to ∞

$$\left|G_0(j\omega)\right| = \frac{K}{\left|T_1 j\omega + 1\right|\left|T_2 j\omega + 1\right|}$$

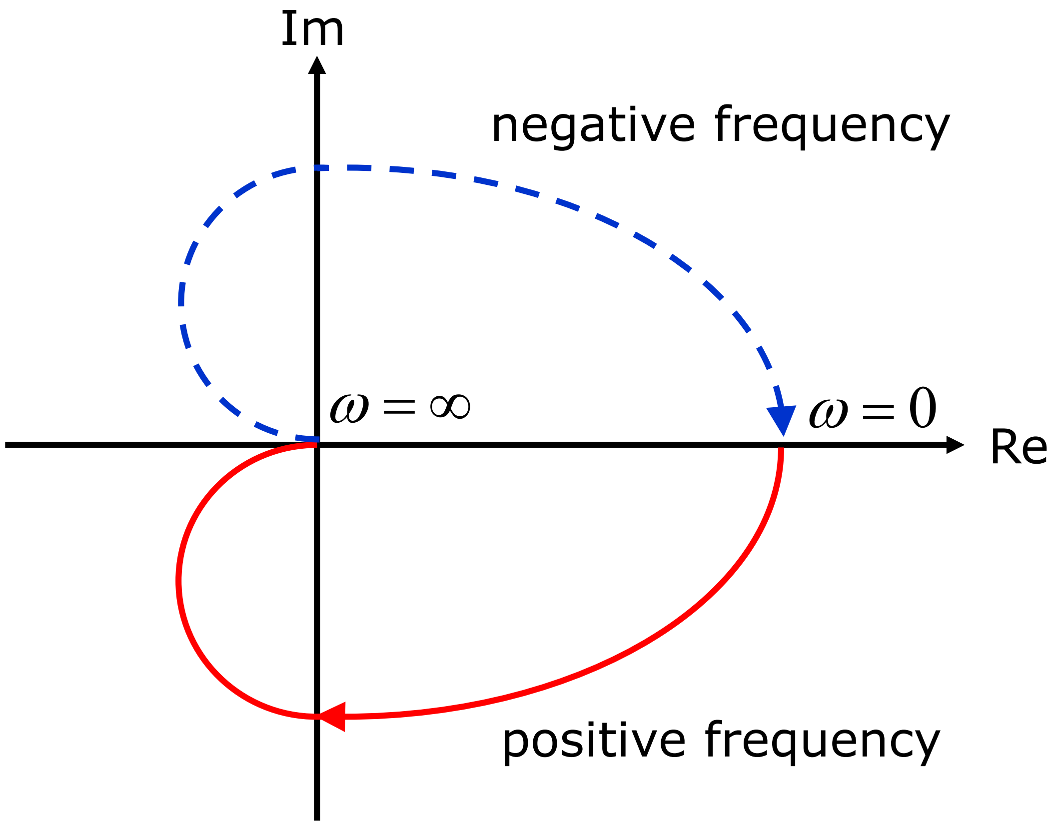
monotonically decreases from K to 0

$$\angle G_0(j\omega) = -tg^{-1}T_1\omega - tg^{-1}T_2\omega$$

varies from 0 to 180 degree



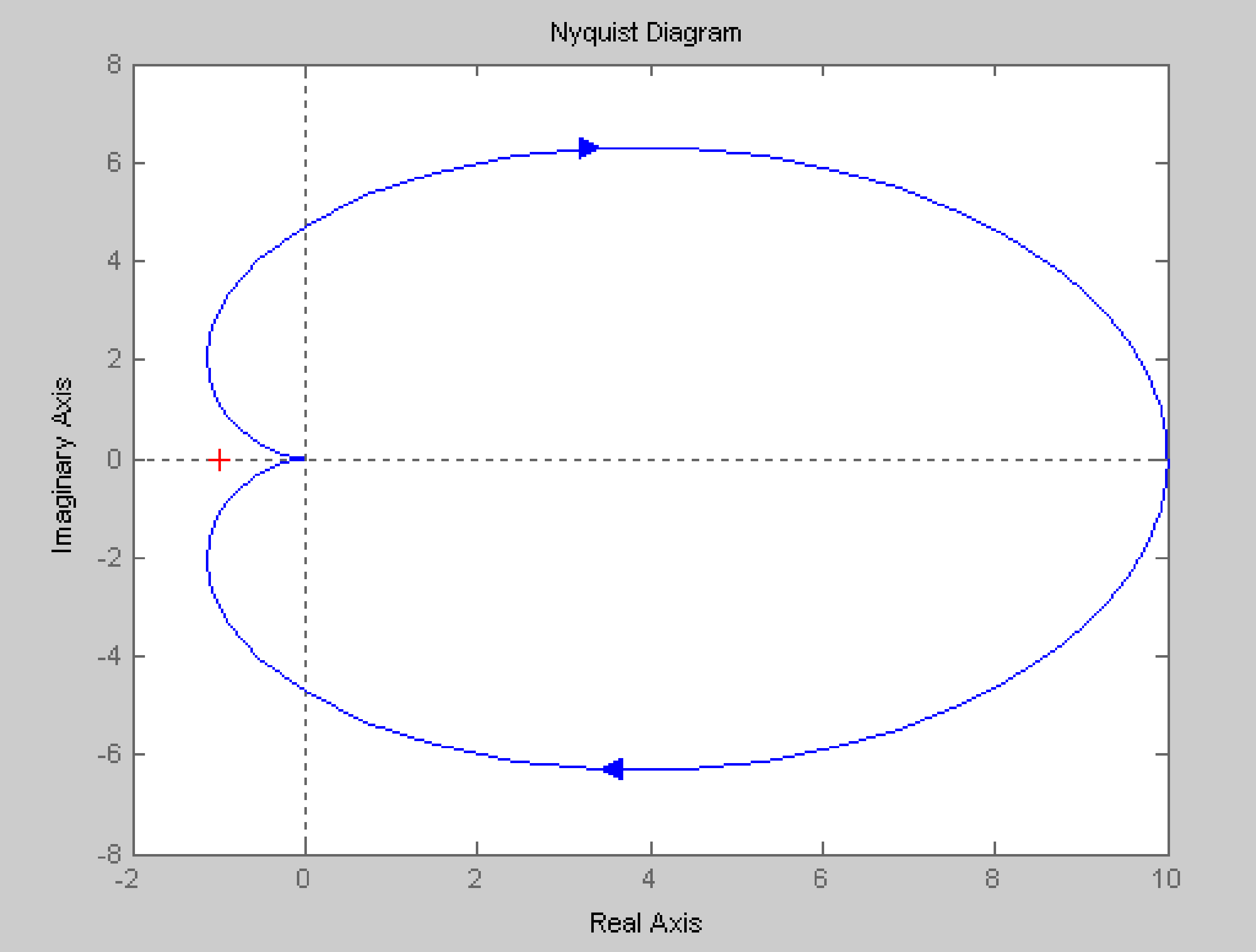
Example 4.1



N=0, P=0, so Z=0, system stable.

Example 4.1

Nyquist plot sketched by Matlab




```
s=tf('s');
g=10/((s+1)*(2*s+1));
nyquist(g);
```

Wrap-up

- Principle of argument
- Nyquist plot
- Nyquist criterion(introduction)

Assignment

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- 1,(5),(6)

Discussion

- When Nyquist path comes across pure imaginary poles or zeros of $1+G_0(s)$, can we just let the path go around those poles and zeros from the left side of the imaginary axis?