# Digital Control Systems

## Chapter 7

# Review

- Basics about discrete-data control systems (digital control systems)
- Interfaces between continuous and discrete-data systems ----- the Sample-and-hold device
- Tools for analyzing and synthesizing discrete-data systems ----- the z-transform and inverse z-transform
- Discrete-data models of control systems
- Pulse-transfer functions
- Discrete state equation
- Discretization of continuous system
  - from transfer function to z-transfer function
  - from state equation to discrete state equation

# Outlines

- Stability assessment of discrete-data systems
- Steady-state error of discrete-data systems
- Transient response of a discrete-data system
- Mapping between s-plane and z-plane trajectory
- Design a discrete-data controller through discretizing a continuous-data controller
- Design a discrete-data controller in the z-plane

# How to assess the stability of a digital system

- For a continuous system, we can use Routh-Hurwitz criterion to assess the stability of the system.
- Can we use Routh-Hurwitz criterion to assess the stability of a digital system?

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设置

Can we use Routh-Hurwitz criterion to evaluate the stability of a discrete-data system?

- A Yes, we can!
- B No, we can not

提交

Is there any difference between the following two characteristic equations?

Continuous system $s^2 + 1.2s + 0.96 = 0$

Digital system $z^2 + 1.2z + 0.96 = 0$

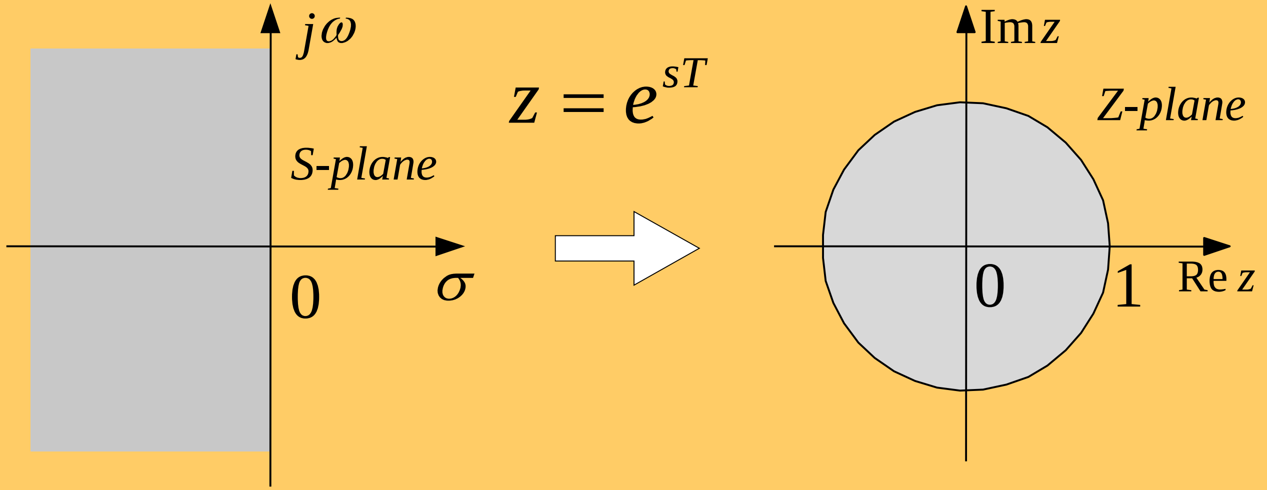
A Yes

B No

提交

# Analysis of Discrete-Date Control Systems

Stability of discrete-data control systems



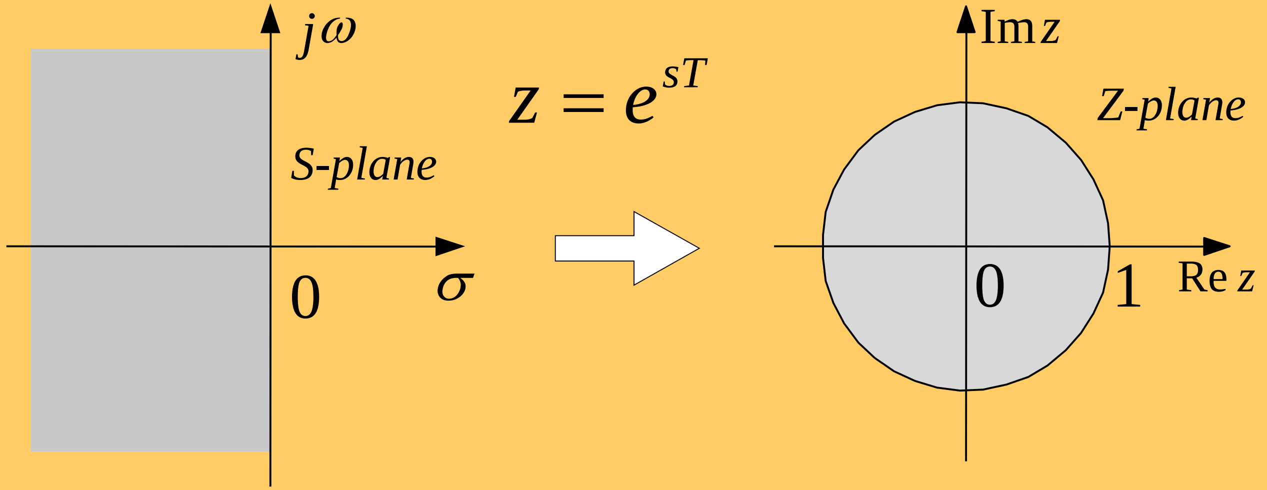

For a discrete-data control system to be stable, it is sufficient and necessary that all the characteristic roots of the system are located inside unit-circle on the z-plane

$$G(z) = \frac{N(z)}{D(z)} \qquad D(z) = 0$$

$$X(k+1) = GX(k) + HU(k) \qquad |zI - G| = 0$$

# Analysis of Discrete-Date Control Systems

Stability of discrete-data control systems




If we apply Routh-Hurwitz criterion to a z-transfer function, what do we get?

# Stability Tests of Discrete-Date Systems

We can still apply the Routh-Hurwitz criterion to discrete-data systems if we could find a transformation that transforms the unit circle in the z-plane onto the imaginary axis of another complex plane. We cannot use z - transform relation $z = \exp(Ts)$ or $s = (\ln z)/T$, since it would transform an algebraic equation in z into a nonalgebraic equation in s, and the Routh test still cannot be applied. However, there are many bilinear transformations of the form

$$z = \frac{aw+b}{cw+d}$$

where a, b, c, d are real constants, and w is a complex variable, that will transform circles in the z-plane onto straight lines in the w-plane. One such transformation that transforms the interior of the unit circle of the z-plane onto the left half of the w-plane

$$z = \frac{1+w}{1-w}$$

# Stability Tests of Discrete-Date Systems

Once the characteristic equation in z domain is transformed into the w domain, the Routh-Hurwitz criterion can again be applied to the equation in w.

Example – 7.17 Evaluate the stability of the following discrete-data systems.

(1) $$z^2 + 1.2z + 0.96 = 0$$

A: calculate the characteristic roots directly

$$z_1 = -0.6 + j0.77, z_2 = -0.6 - j0.77$$

they lie inside the unit circle, so the system is stable.

(2) $$z^3 + 6.3z^2 + 8.2z + 0.4 = 0$$

A: use Routh criterion. Do the w-transformation first

$$2.5w^3 - 10.3w^2 - 0.1w + 15.9 = 0$$

$$2.5w^3 - 10.3w^2 - 0.1w + 15.9 = 0$$

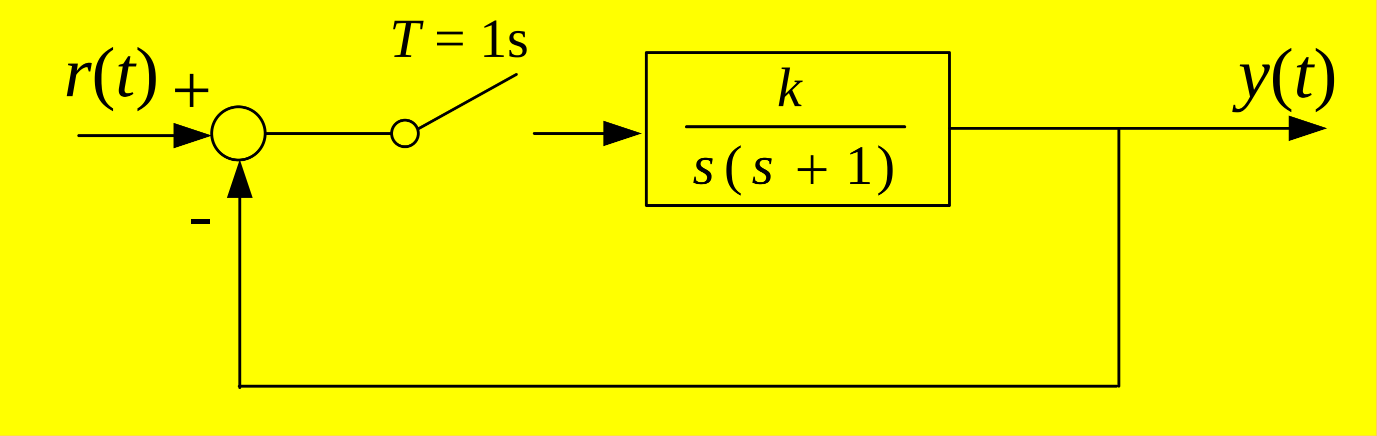
Create the Routh table:

| | | |
|---|---|---|
| $w^3$ | 2.5 | -0.1 |
| $w^2$ | -10.3 | 15.9 |
| $w^1$ | -3.76 | 0 |
| $w^0$ | 15.9 | 0 |

The sign of the first column changes twice, there are two characteristic roots lie in the right half of the w-plane, so the system is unstable.

# Example - 7.18

Q: please find the feasible range of k to make the following system stable.




A: for continuous system $G_0(s) = \dfrac{k}{s(s+1)} = k(\dfrac{1}{s} - \dfrac{1}{s+1})$

for discrete-data system

$$G_0(z) = k(\frac{z}{z-1} - \frac{z}{z-e^{-T}}) = \frac{kz(1-e^{-T})}{(z-1)(z-e^{-T})}$$

The characteristic equation of the closed-loop system

$$D(z) = (z-1)(z-e^{-T}) + kz(1-e^{-T}) = 0$$

Considering T=1s,

$$D(z) = z^2 + (0.63k - 1.37)z + 0.37 = 0$$

Apply w-transformation

$$z = \frac{1+w}{1-w}$$

yields:

$$D_s(w) = (2.74 - 0.63k)w^2 + 1.26w + 0.63k = 0$$

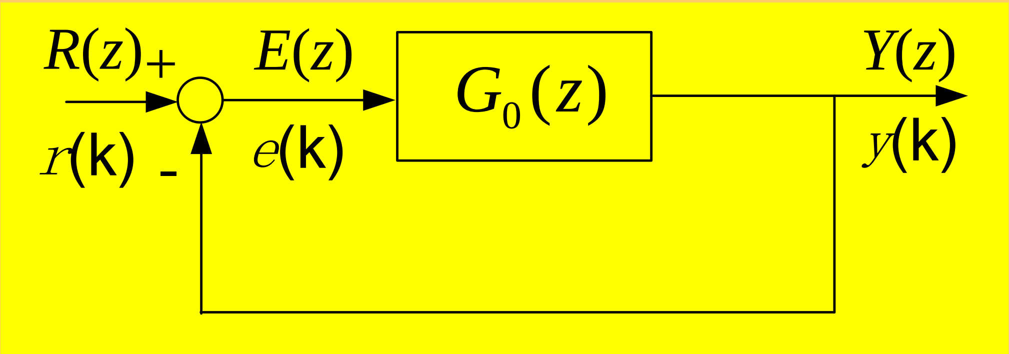
stability condition: $\begin{cases} 2.74 - 0.63k > 0 \\ 0.63k > 0 \end{cases} \Longrightarrow 0 < k < 4.35$

when T=0.1s, the stability region of k is $0 < k < 40$

Conclusion: the smaller the sampling period is, the larger the stability region is.

# Steady-State Error of Discrete-Date Systems

Definition of the steady-state error: $e_{ss} = \lim_{k \to \infty} e(k)$




$$E(z) = R(z) - Y(z) = \frac{R(z)}{1 + G_0(z)}$$

The final-value theorem of z-transform is applied here to calculate the steady-state error of a discrete-data system. When the discrete data system is stable

$$e_{ss} = \lim_{k \to \infty} e(kT) = \lim_{z \to 1} (1 - z^{-1}) E(z)$$

$$e_{ss} = \lim_{k \to \infty} e(kT) = \lim_{z \to 1} (1 - z^{-1}) \frac{R(z)}{1 + G_0(z)}$$

The number of the z=1 poles of the open-loop z-transfer function determines the type of the system.

# Steady-State Error of Discrete-Date Systems

The steady-state error to a unit-step input:

$$r(k) = 1(kT) \qquad R(z) = \frac{z}{z-1}$$

$$e_{ss} = \lim_{z\to 1}\left[(1-z^{-1})\cdot\frac{1}{1+G_0(z)}\cdot\frac{z}{z-1}\right] = \frac{1}{1+\lim\limits_{z\to 1} G_0(z)} = \frac{1}{1+k_p}$$

$k_p$ is named as step-error constant. $k_p = \lim\limits_{z\to 1} G_0(z)$

Type 0 system: $k_p = G_0(1) \qquad e_{ss} = \dfrac{1}{1+k_p}$

Type**I** and **II** system $k_p = \infty \qquad e_{ss} = 0$

# Steady-State Error of Discrete-Date Systems

The steady-state error to a unit-ramp input:

$$r(k) = kT \qquad R(z) = \frac{Tz}{(z-1)^2}$$

$$e_{ss} = \lim_{z\to 1}\left[(1-z^{-1})\cdot\frac{1}{1+G_0(z)}\cdot\frac{Tz}{(z-1)^2}\right] = T\lim_{z\to 1}\frac{1}{(z-1)G_0(z)}$$

$$k_v = \frac{1}{T}\lim_{z\to 1}[(z-1)G_0(z)]$$ $k_v$ is named as ramp-error constant.

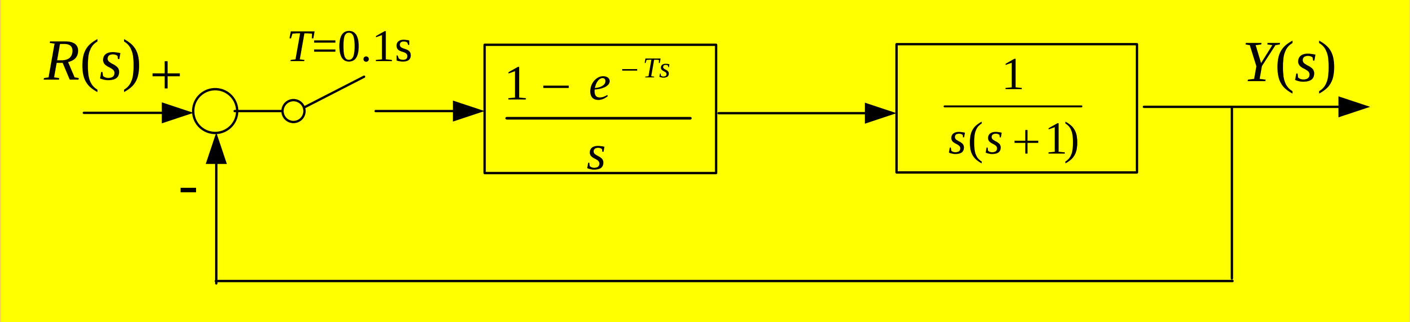
Type 0 system: $k_v = 0$ $\quad e_{ss} = \dfrac{1}{k_v} = \infty$

Type**I** system: $k_v = G_0(1)/T$ $\quad e_{ss} = T/G_0(1)$

Type **II** system: $k_v = \infty$ $\quad e_{ss} = 0$

# Example - 7.19

Q: please find the steady-state error of the following system to a unit-step input.




A: $$G_0(z) = Z\left[(1-e^{-Ts})\frac{1}{s^2(s+1)}\right] = Z(1-e^{-Ts})\cdot Z\left[(\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1})\right]$$

$$=(1-z^{-1})\left[\frac{Tz}{(z-1)^2}-\frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}\right]$$

Substitute T=0.1 into the above equation:

$$G_0(z)=\frac{0.005(z+0.9)}{(z-1)(z-0.905)}$$

z-transform of the unit-step function: $R(z)=\frac{z}{z-1}=\frac{1}{1-z^{-1}}$

$$e_{ss}=\lim_{z\to 1}\left[(1-z^{-1})\cdot\frac{1}{1+G_0(z)}\cdot\frac{1}{1-z^{-1}}\right]=\frac{1}{1+\lim\limits_{z\to 1}G_0(z)}=\frac{1}{1+k_p}$$

$$k_p=\lim_{z\to 1}G_0(z)=\lim_{z\to 1}\frac{0.005(z+0.9)}{(z-1)(z-0.905)}=\infty$$

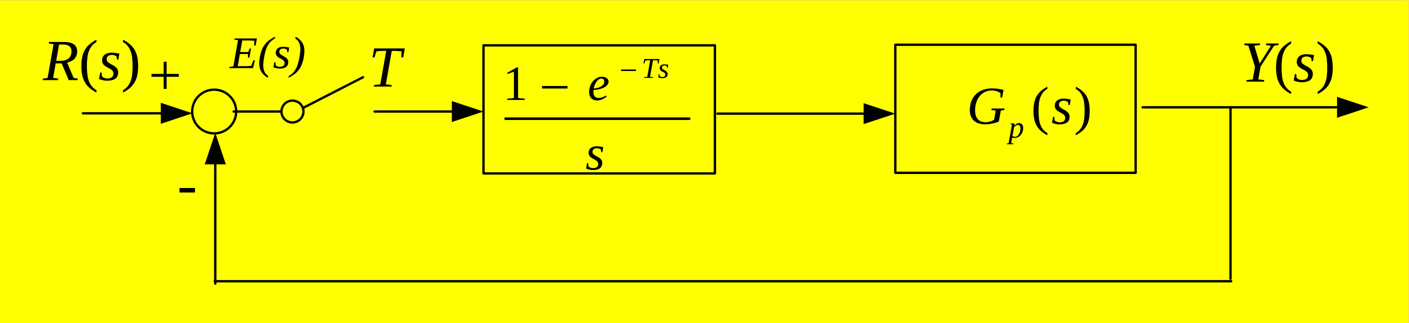
$$e_{ss}=0$$

# Example

Q: A discrete data system is shown as follows. Given the transfer function

$$G_p(s) = \frac{K(1+T_a s)(1+T_b s)\cdots(1+T_m s)}{(1+T_1 s)(1+T_2 s)\cdots(1+T_n s)s^j}$$

Prove the type 0 system has steady-state error $e_{ss} = \frac{R}{1+K}$ when the input is a step function $r(t) = R(t)$




Prove: The open loop transfer function of the above system is:

$$G(s) = G_h(s)G_p(s) = \frac{1-e^{-Ts}}{s}\cdot\frac{K(1+T_a s)(1+T_b s)\cdots(1+T_m s)}{(1+T_1 s)(1+T_2 s)\cdots(1+T_n s)s^j}$$

$$G(s)=G_h(s)G_p(s)=\frac{1-e^{-Ts}}{s}\cdot\frac{K(1+T_a s)(1+T_b s)\cdots(1+T_m s)}{(1+T_1 s)(1+T_2 s)\cdots(1+T_n s)s^j}$$

Discretize the above transfer function:

$$G(z)=Z[G_h(s)G_p(s)]=K(1-z^{-1})Z[\frac{(1+T_a s)(1+T_b s)\cdots(1+T_m s)}{(1+T_1 s)(1+T_2 s)\cdots(1+T_n s)s^{j+1}}]$$

For type 0 system, we have:

$$G(z)=Z[G_h(s)G_p(s)]=K(1-z^{-1})Z[\frac{(1+T_a s)(1+T_b s)\cdots(1+T_m s)}{(1+T_1 s)(1+T_2 s)\cdots(1+T_n s)s}]$$

$$=K(1-z^{-1})Z[\frac{1}{s}+\frac{C_1}{s+\frac{1}{T_1}}+\frac{C_2}{s+\frac{1}{T_2}}+\cdots+\frac{C_n}{s+\frac{1}{T_n}}]$$

$$=K(1-z^{-1})[\frac{z}{z-1}+\frac{C_1 z}{z-e^{-\frac{T}{T_1}}}+\frac{C_2 z}{z-e^{-\frac{T}{T_2}}}+\cdots+\frac{C_n z}{z-e^{-\frac{T}{T_n}}}]$$

$$=K[1+\frac{C_1(z-1)}{z-e^{-\frac{T}{T_1}}}+\frac{C_2(z-1)}{z-e^{-\frac{T}{T_2}}}+\cdots+\frac{C_n(z-1)}{z-e^{-\frac{T}{T_n}}}]$$

In above equation:

$$C_i = \lim_{s\to -\frac{1}{T_i}} (s+\frac{1}{T_i})\cdot \frac{K(1+T_a s)(1+T_b s)\cdots(1+T_m s)}{(1+T_1 s)(1+T_2 s)\cdots(1+T_n s)s} \qquad (i=1,2,\cdots,n)$$

From the picture:

$$E(z) = R(z) - Y(z) = \frac{R(z)}{1+G(z)}$$

Suppose the system is stable:

$$e_{ss} = \lim_{t\to\infty} e^*(t) = \lim_{z\to 1}(1-z^{-1})E(z) = \lim_{z\to 1}(1-z^{-1})\frac{R(z)}{1+G(z)}$$

Because $r(t) = R$ therefore $R(z) = \dfrac{Rz}{z-1}$

$$e_{ss} = \lim_{z\to 1}(1-z^{-1})\frac{R(z)}{1+G(z)} = \lim_{z\to 1}\frac{R}{1+G(z)}$$

$$= \lim_{z\to 1}\frac{R}{1+K[1+\dfrac{C_1(z-1)}{z-e^{-\frac{T}{T_1}}}+\dfrac{C_2(z-1)}{z-e^{-\frac{T}{T_2}}}+\cdots++\dfrac{C_n(z-1)}{z-e^{-\frac{T}{T_n}}}]}$$

$$= \frac{R}{1+K}$$

# Transient Response of Discrete Control Systems

The time domain performance specifications such as maximum overshoot, rise time and settling time used in continuous system can still be applied to discrete control systems.

The transient performance of a discrete-data control system is characterized by poles and zeros of the transfer function in the z-plane.

For analysis and design purposes, it is important to study the relation between the location of the characteristic-equation roots in the z-plane and the time response of the discrete-date system.

The correlation of a point on the S plane and its mapping on the Z plane?

# Mapping between s-Plane and z-Plane Trajectories

$$z = e^{sT}$$

Where T is the sampling period.

Set $s = \sigma + j\omega$

$$z = e^{(\sigma + j\omega)T} = e^{\sigma T} \cdot e^{j\omega T} = e^{\sigma T} \angle \omega T$$

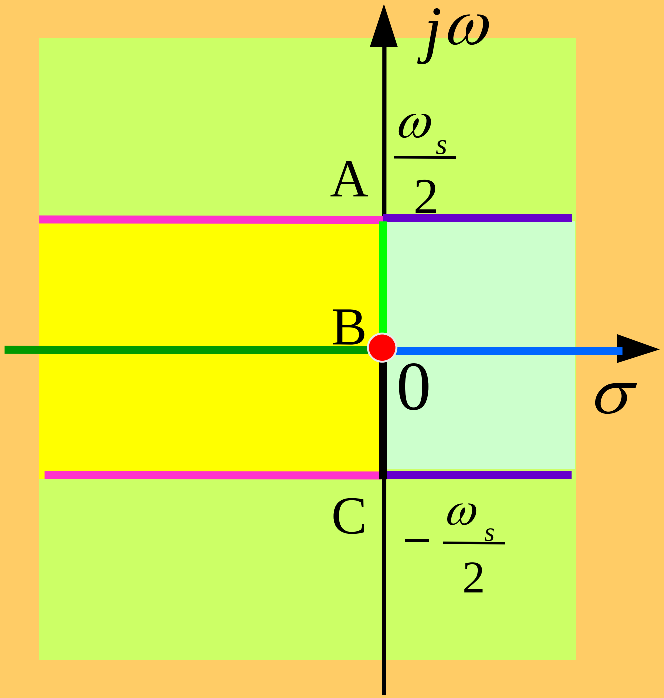
Set $\omega_s = \dfrac{2\pi}{T}$ $\qquad z = re^{j\theta}$

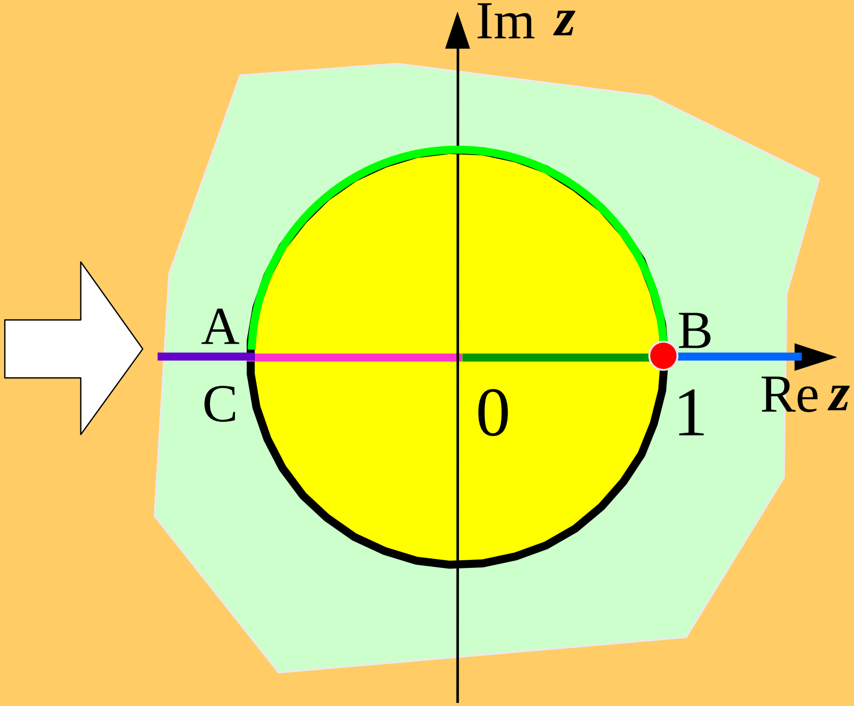
$$r = \left| z \right| = e^{\sigma T} = e^{\frac{2\pi}{\omega_s}\sigma}$$

$$\theta = \angle z = \omega T = 2\pi \frac{\omega}{\omega_s}$$

s-Plane

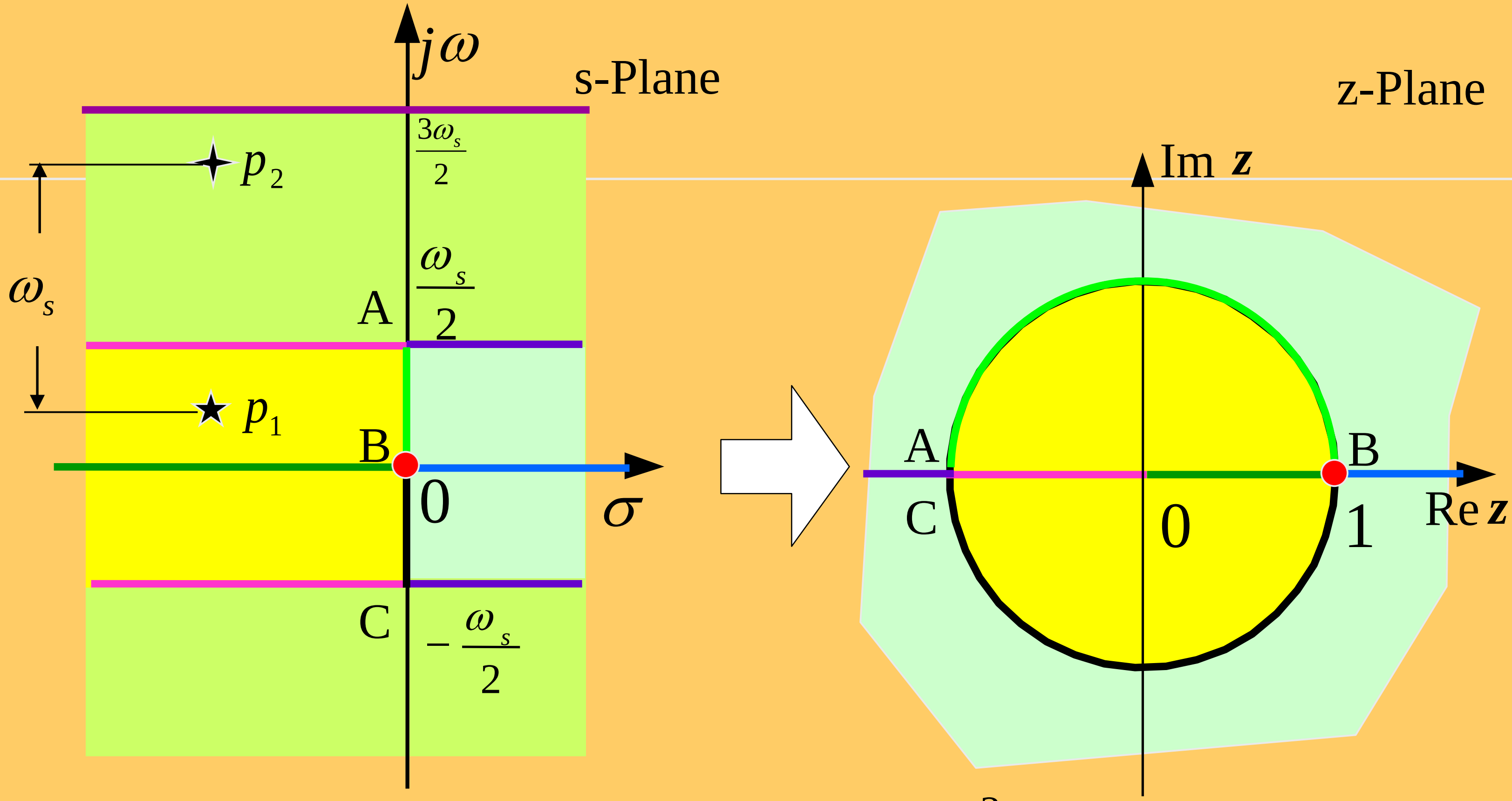
z-Plane




$$r = |z| = e^{\sigma T} = e^{\frac{2\pi}{\omega_s}\sigma}$$

$$\theta = \angle z = \omega T = 2\pi \frac{\omega}{\omega_s}$$




$$p_1 = \sigma_1 + j\omega_1$$

$$r = |z| = e^{\sigma T} = e^{\frac{2\pi}{\omega_s}\sigma}$$

$$\theta = \angle z = \omega T = 2\pi \frac{\omega}{\omega_s}$$

$$p_2 = \sigma_2 + j(\omega_1 + \omega_s)$$

$$r_1 = r_2$$

$$\theta_1 = \omega_1 T = 2\pi \frac{\omega_1}{\omega_s}$$

$$\sigma_1 = \sigma_2$$

$$\theta_2 = \omega_2 T = 2\pi \frac{\omega_1 + \omega_s}{\omega_s} = 2\pi + \theta_1$$

# Time Response of Discrete Systems

Look at the unit-step response of a general system:

$$Y(z)=G(z)R(z)=G(z)\cdot\frac{z}{z-1}$$

Assume $G(z)$ does not have any multiple poles

$$\frac{Y(z)}{z}=\frac{G(z)}{(z-1)}=\frac{A_0}{z-1}+\sum_{i=1}^{n}\frac{A_i}{z-p_i}$$

$$A_0=\left.\frac{(z-1)G(z)}{z-1}\right|_{z=1}=G(1),\quad A_i=\left.\frac{(z-p_i)G(z)}{(z-1)}\right|_{z=p_i}(i=1,2,\cdots,n)$$

$$y_{ss}=\lim_{z\to1}(1-z^{-1})G(z)\cdot\frac{1}{1-z^{-1}}=\lim_{z\to1}G(z)=G(1)=A_0$$

$$Y(z) == \frac{A_0 z}{z-1} + \sum_{i=1}^{n} \frac{A_i z}{z-p_i}$$

$$1(k) \rightarrow \frac{z}{1-z^{-1}}$$

$$y(k) = A_0 \cdot 1(k) + \sum_{i=1}^{n} A_i \cdot (p_i)^k$$

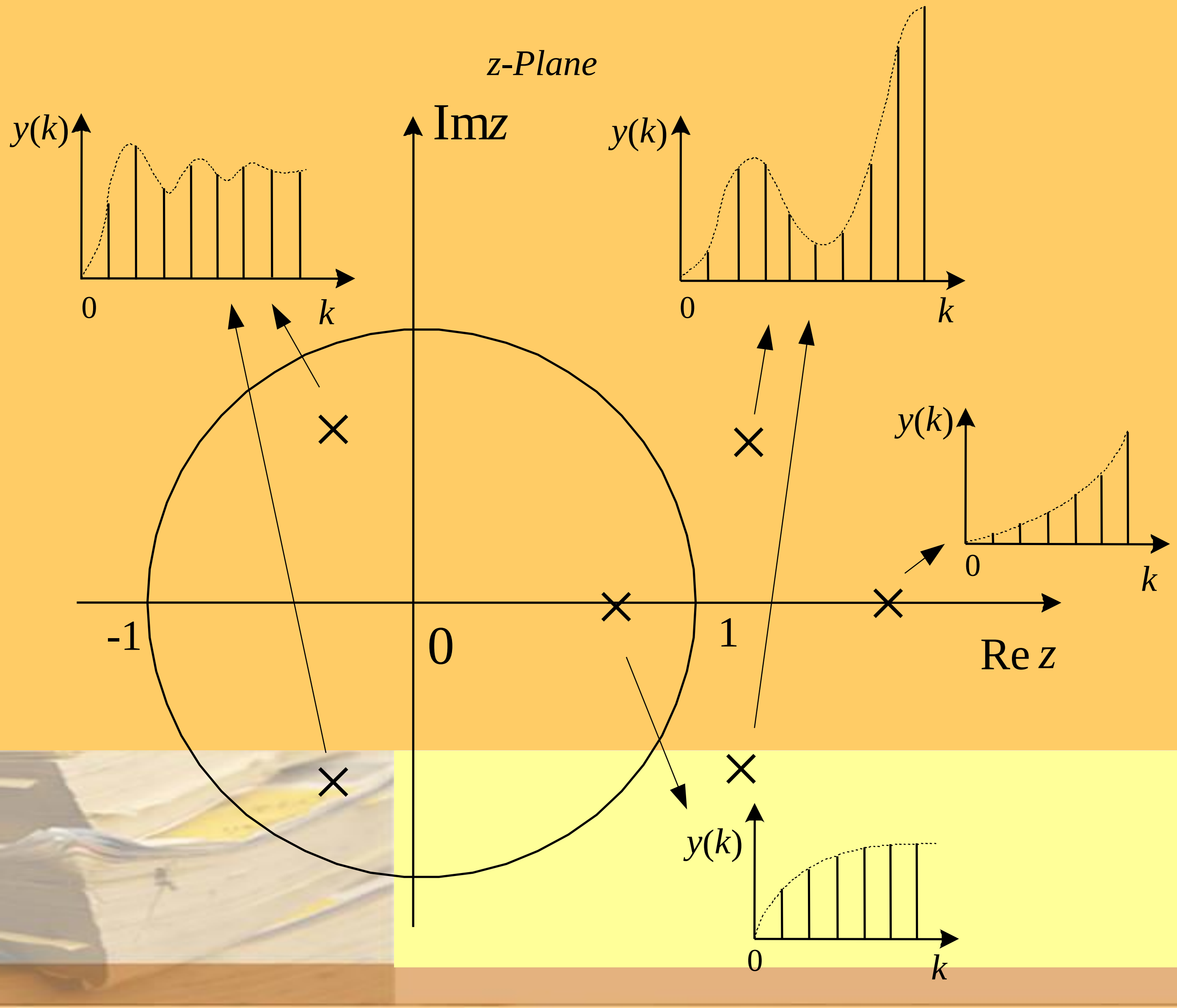
$$p^k \rightarrow \frac{z}{z-p}$$

If $p_i$ is a complex number

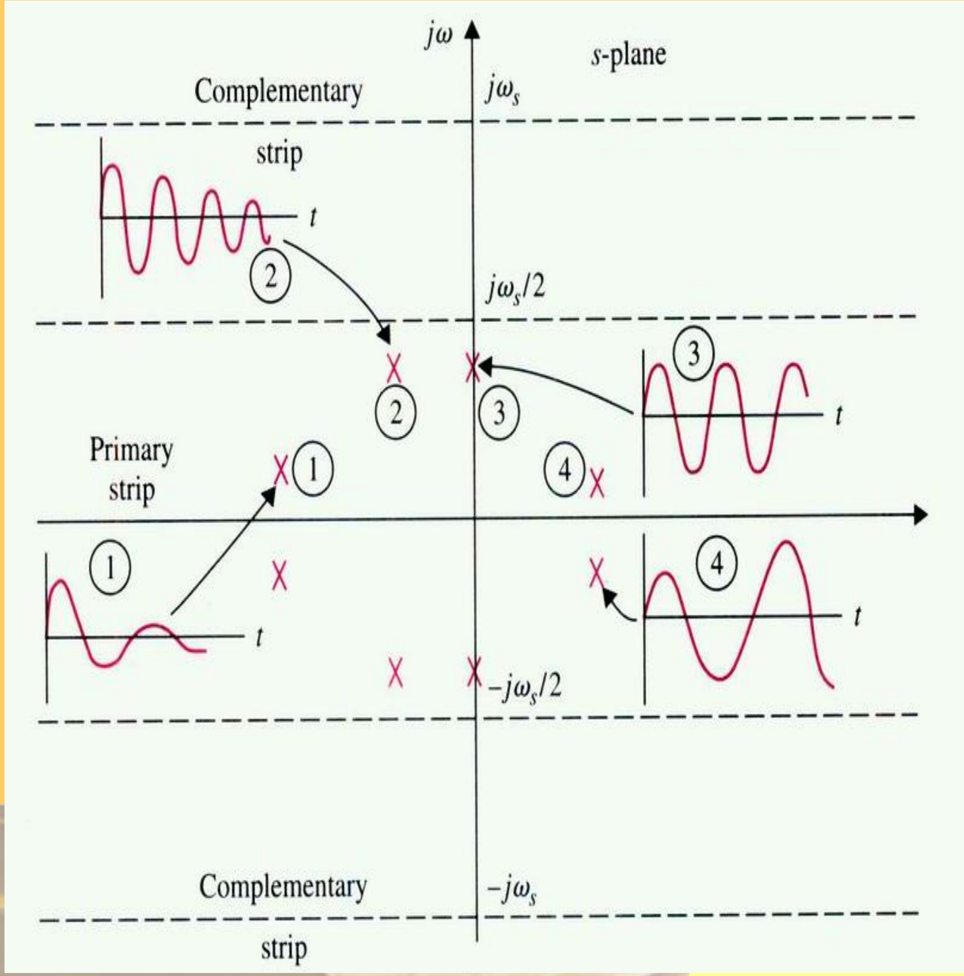
$$p_i = r_i e^{j\theta} = r_i(\cos\theta_i + j\sin\theta_i)$$

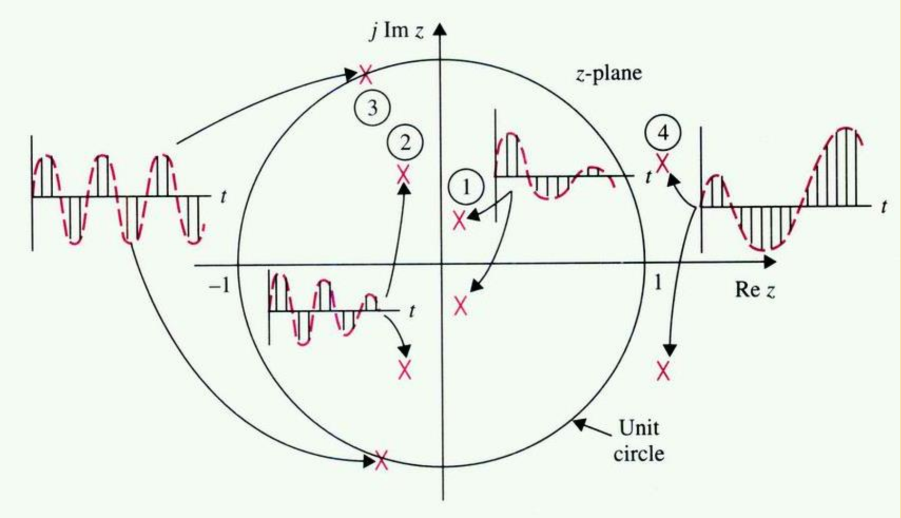
$$y(k) = A_0 \cdot 1\left(k\right) + \sum_{i=1}^{n} A_i \cdot r_i^{\,k}[\cos(k\theta_i) + j\sin(k\theta_i)]$$

$$y(k) = A_0 \cdot 1(k) + \sum_{i=1}^{n} A_i \cdot r_i^{\,k}[\cos(k\theta_i) + j\sin(k\theta_i)]$$

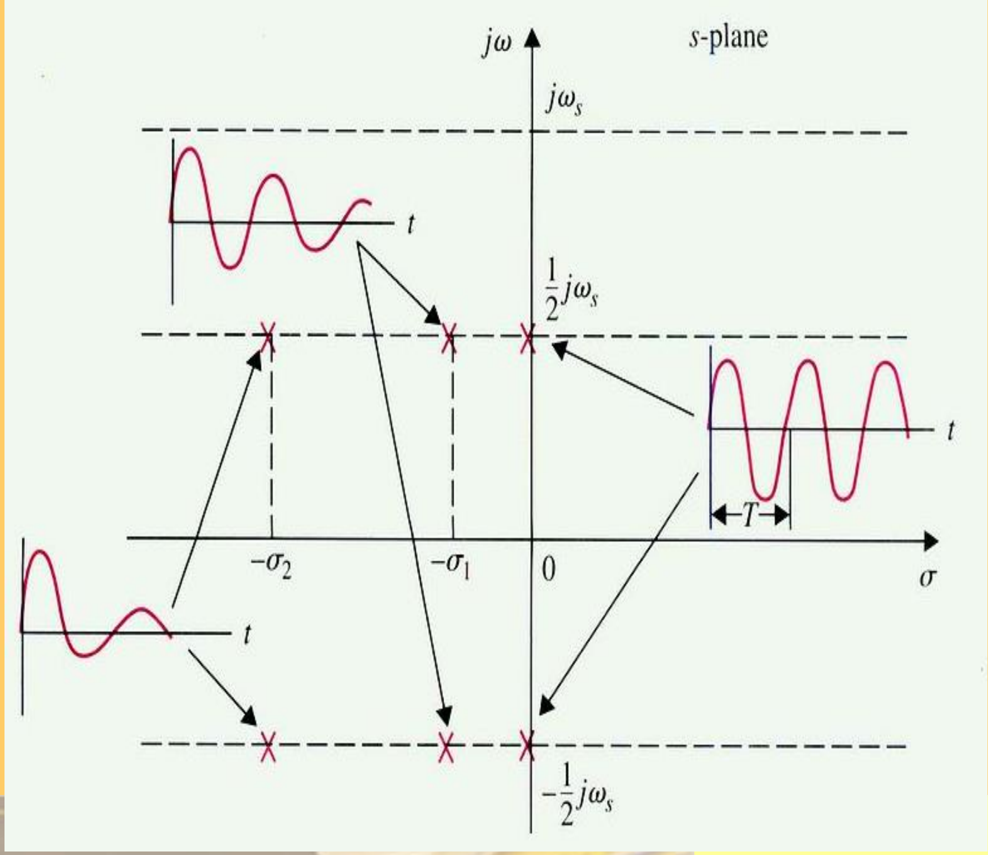


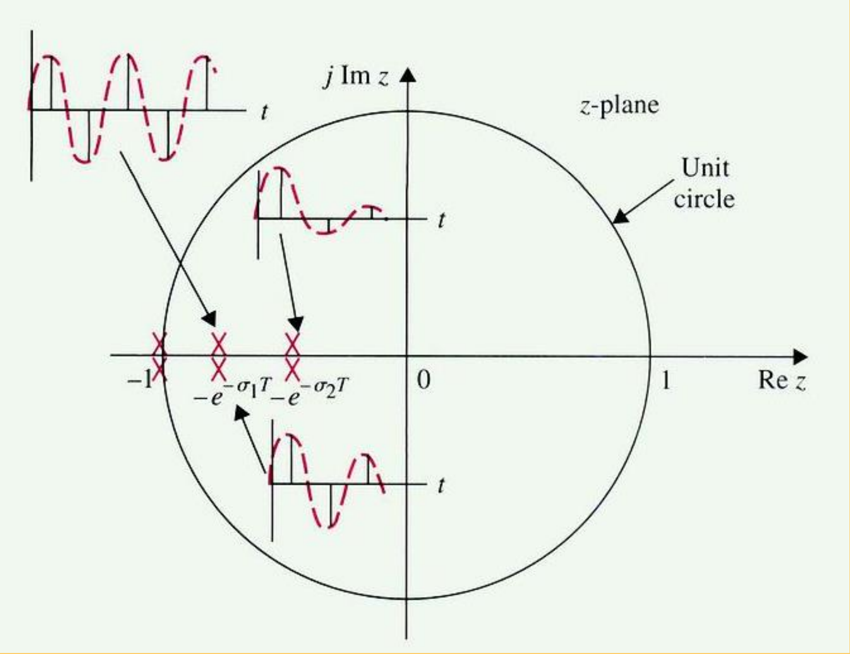
# Transient Responses Corresponding to various pole locations



# Transient Responses Corresponding to various pole locations



# Controllability and Observability of Systems with Discrete State Equation Models

$$X(k+1) = GX(k) + HU(k)$$

$$Y(k) = CX(k)$$

Controllability condition:

$$\operatorname{rank}\begin{bmatrix} H & GH & \cdots & G^{n-1}H \end{bmatrix} = n$$

Observability condition:

$$\operatorname{rank}\begin{bmatrix} C \\ CG \\ \vdots \\ CG^{n-1} \end{bmatrix} = n$$

# Design of Discrete-Data Control

Use methods for continuous systems to design an anolog controller first, then discretize the controller to a digital controller.

To do this, the sampling frequency must be much greater than the gain crossover frequency of the controller.

There are two commonly used discretization methods, one is the matched zero-pole method, the other is the bilinear approximation method.

# Matched Zero-Pole Method

$$s=-a \quad \Rightarrow \quad z=e^{-aT}$$

$$s=\infty \quad \Rightarrow \quad z=-1$$

$$G_c(0)=D_c(1) \quad \Rightarrow$$ Determine the gain

Example:

① $G_c(s)=\dfrac{s+b}{s+a}$ $\qquad D_c(z)=k\dfrac{z-e^{-bT}}{z-e^{-aT}}$

② $G_c(s)=\dfrac{b}{s+a}$ $\qquad D_c(z)=k\dfrac{z+1}{z-e^{-aT}}$

③ $G_c(s)=s+a$ $\qquad D_c(z)=k\dfrac{z-e^{-aT}}{z+1}$

# Bilinear Approximation Method

$$z = e^{sT} = \frac{e^{\frac{sT}{2}}}{e^{-\frac{sT}{2}}}$$

Perform Taylor series expansion and neglect items equal to and higher than the second order

$$e^{-\frac{sT}{2}} \approx 1 - \frac{T}{2}s \qquad e^{\frac{sT}{2}} \approx 1 + \frac{T}{2}s$$

$$z \approx \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s} \qquad s = \frac{2}{T} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}$$

# Physical Significance of Bilinear Transform Method

$$\frac{dy(t)}{dt} + ay(t) = u(t)$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s+a}$$

$$s = \frac{2}{T} \cdot \frac{1-z^{-1}}{1+z^{-1}}$$

$$\frac{Y(z)}{U(z)} = \frac{1}{\frac{2}{T} \cdot \frac{1-z^{-1}}{1+z^{-1}} + a}$$

# Physical Significance of Bilinear Transform Method

$$\frac{dy(t)}{dt} + ay(t) = u(t)$$

$$\frac{y(k+1) - y(k)}{T} + a\frac{\left[y(k+1) + y(k)\right]}{2} = \frac{\left[u(k+1) + u(k)\right]}{2}$$

z transformation

$$\frac{zY(z) - Y(z)}{T} + a\frac{\left[zY(z) + Y(z)\right]}{2} = \frac{\left[zU(z) + U(z)\right]}{2}$$

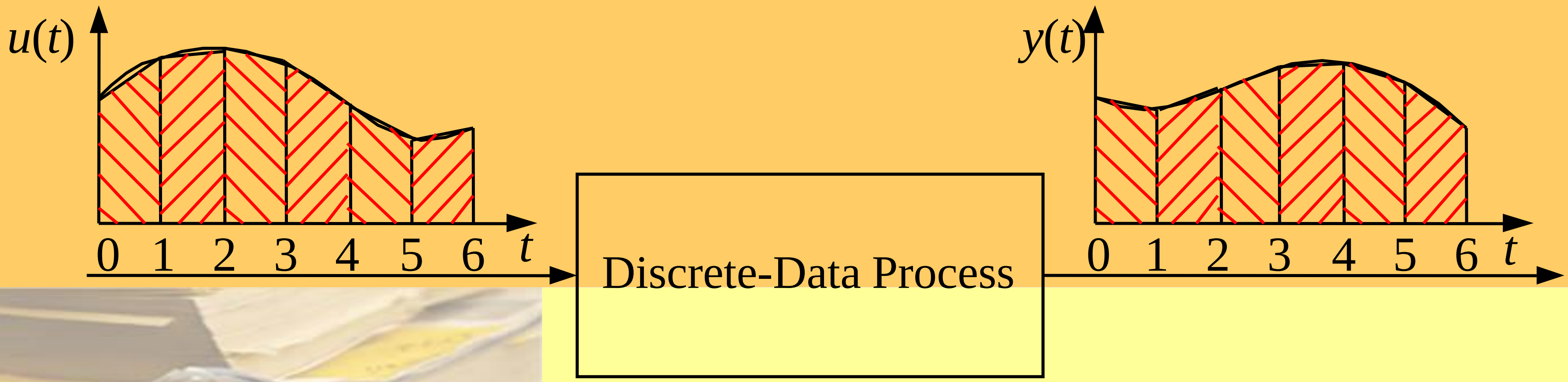
$$\left[\frac{z-1}{T} + a\frac{(z+1)}{2}\right]Y(z) = \frac{\left[z+1\right]}{2}U(z)$$

$$\frac{Y(z)}{U(z)} = \frac{1}{\frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}} + a}$$

# Physical Significance of Bilinear Transform Method

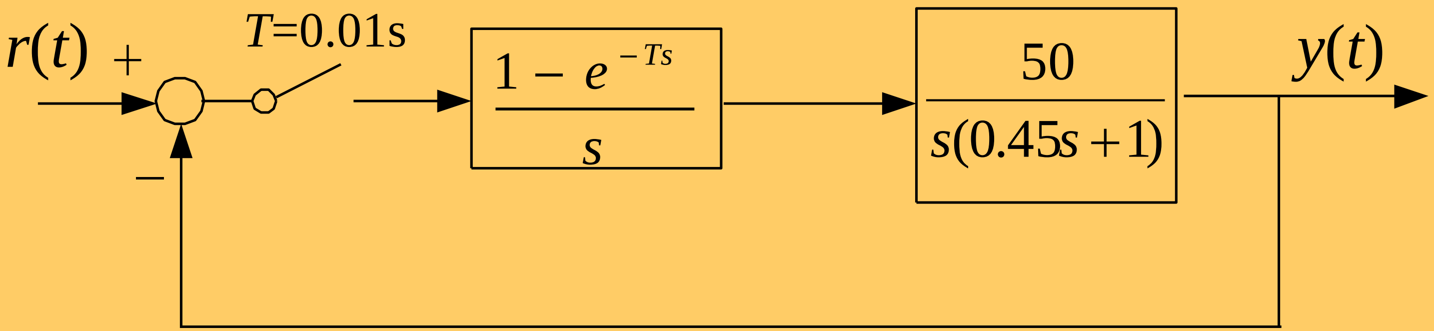
$$\frac{dy(t)}{dt} + ay(t) = u(t)$$

$$\frac{y(k+1) - y(k)}{T} + a\frac{\left[y(k+1) + y(k)\right]}{2} = \frac{\left[u(k+1) + u(k)\right]}{2}$$



# Example – 7.20

Q: please design a digital controller to make the gain crossover frequency of the open-loop transfer function $\omega_c = 18$ and the phase margin $\gamma \geq 45^\circ$




A: find the continuous transfer function of the above system

Transfer function of the ZOH

$$G_h(s) = \frac{1-e^{-Ts}}{s} = \frac{1}{s}\left[1-(1-Ts+\frac{1}{2}T^2s^2-\cdots)\right] \approx T(1-\frac{1}{2}Ts) \approx \frac{T}{\frac{T}{2}s+1}$$

Transfer function of the ideal sampler

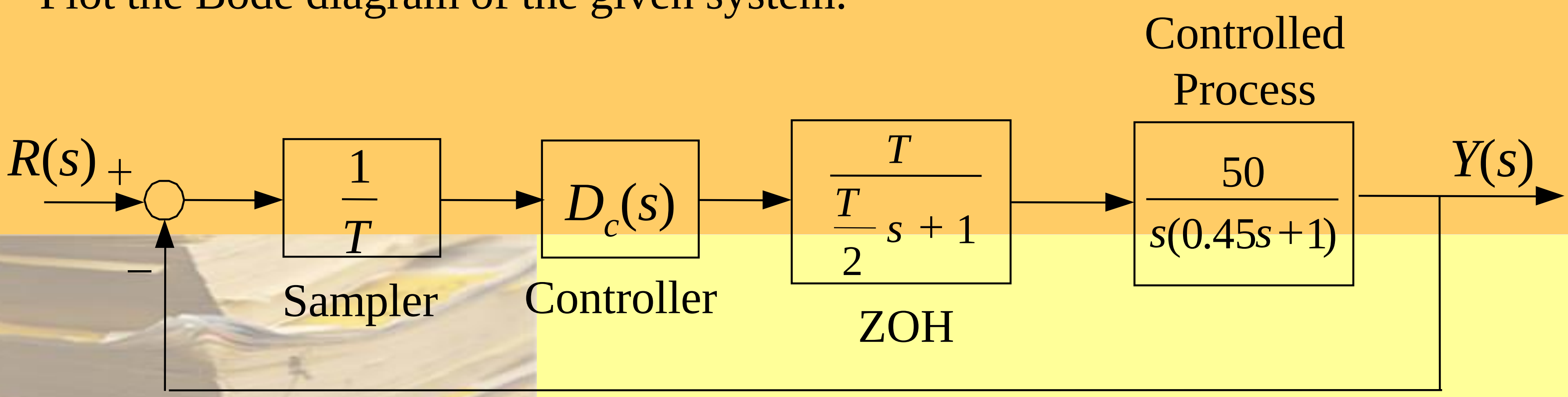
$$G_s(s) = \frac{1}{T}$$

# Example – 7.20

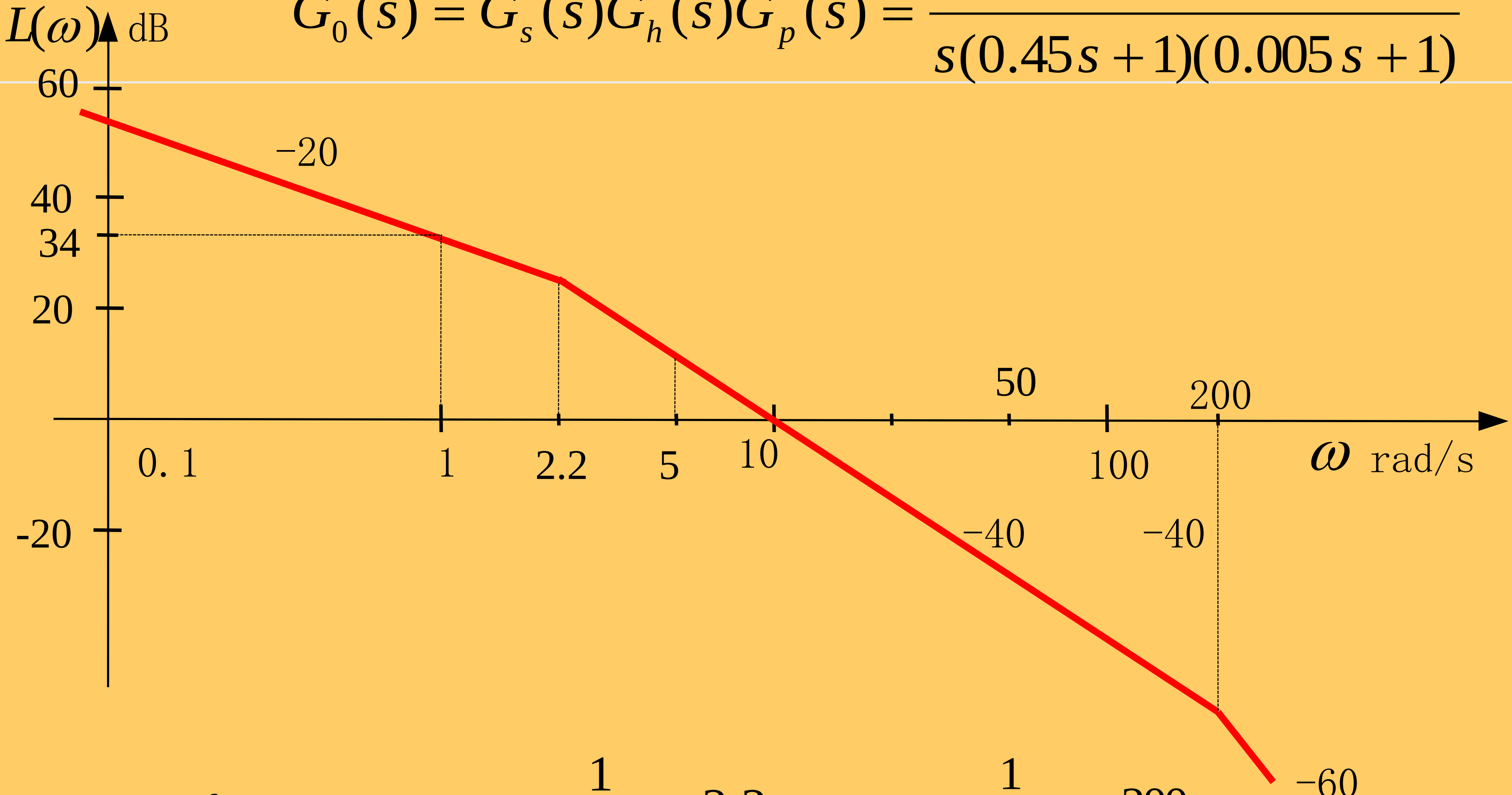
$$G_s(s)G_h(s) = \frac{1}{Ts/2 + 1}$$

When $T = 0.01s$, $\omega_s = \dfrac{2\pi}{T} = \dfrac{2\pi}{0.01} = 628$ , which is much greater than the gain crossover gain $\omega_c = 18$. The discretization condition is satisfied.

Plot the Bode diagram of the given system:



$$G_0(s) = G_s(s)G_h(s)G_p(s) = \frac{50}{s(0.45s+1)(0.005s+1)}$$

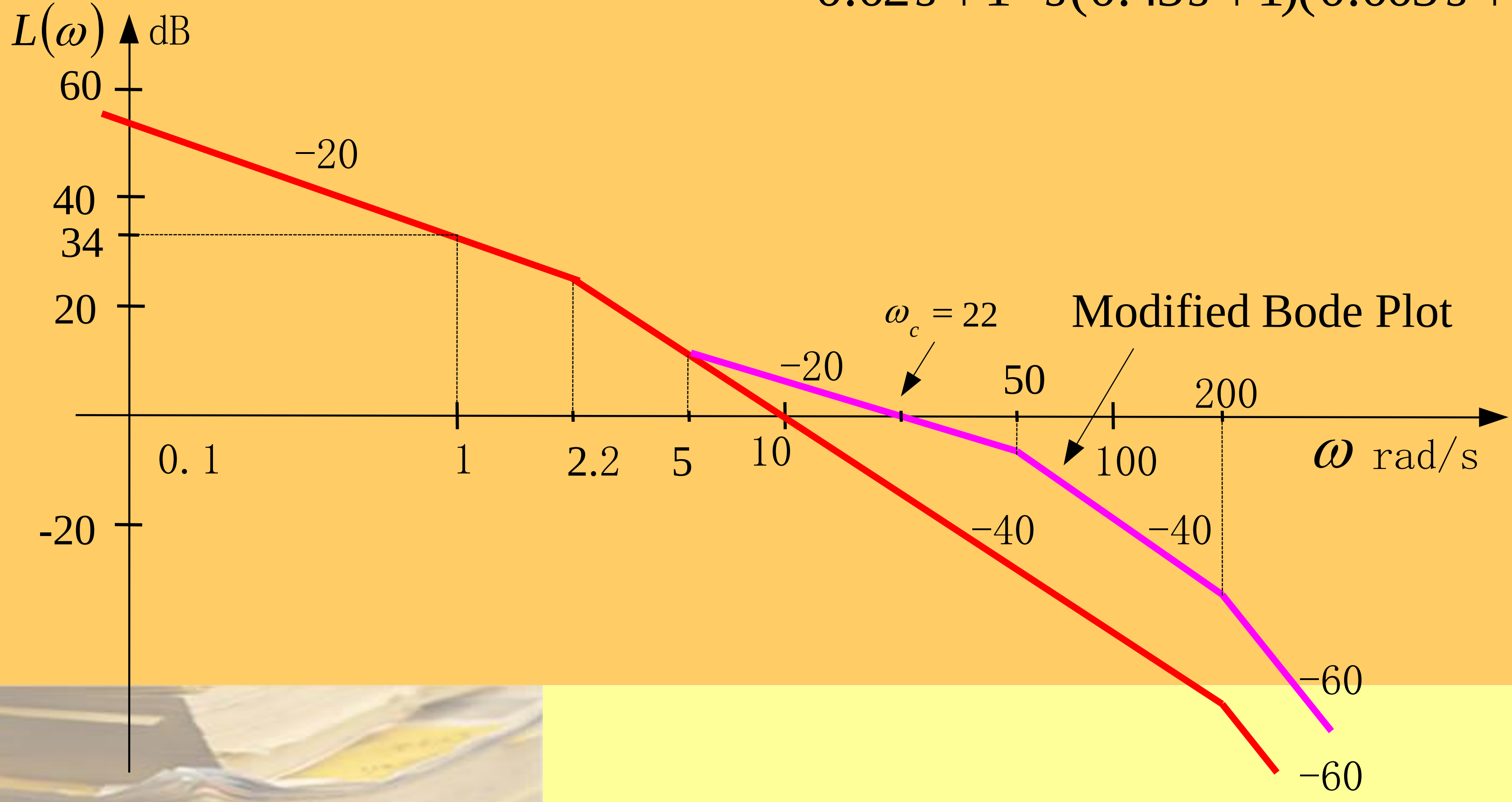



Corner frequency: $\omega_1 = \frac{1}{0.45} = 2.2$ $\omega_2 = \frac{1}{0.005} = 200$

When $\omega = 1$ $20\lg 50 = 34\ dB$ Slope: $-20\,dB/Decade$

Design a phase-lead controller: $G_c(s) = \dfrac{0.2s+1}{0.02s+1}$

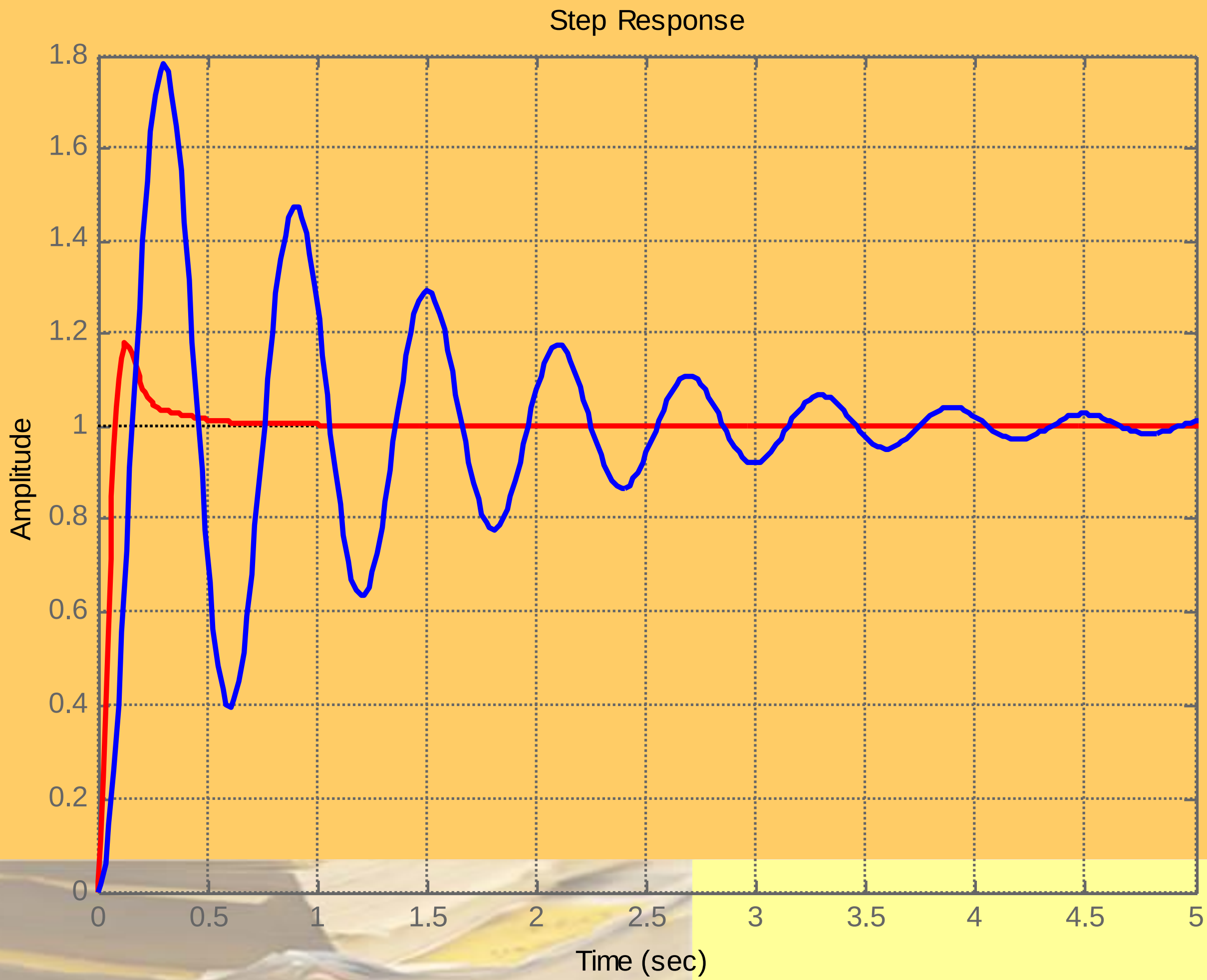
The compensated system: $G_0{}'(s) = \dfrac{0.2s+1}{0.02s+1} \cdot \dfrac{50}{s(0.45s+1)(0.005s+1)}$




$$\gamma = 180^0 - 90^0 + \mathrm{tg}^{-1}(0.2\times 22) - \mathrm{tg}^{-1}(0.02\times 22) - \mathrm{tg}^{-1}(0.45\times 22) - \mathrm{tg}^{-1}(0.005\times 22)$$

$$=52.9$$

# Matlab simulation of the anolog controller




```
s=tf('s');
g=50/(s*(0.45*s+1)*(0.005*s+1));
sys=feedback(g,1);
step(sys,’b’);
grid on;
hold on;
c= (0.2*s+1)/(0.02*s+1);
g1=g*c;
sys1=feedback(g1,1);
step(sys1,’r’);
```

Discretize the analog controller

Use zero-pole matching method

$$G_c(s) = \frac{0.2s+1}{0.02s+1} \qquad s = -\frac{1}{0.2} = -5 \qquad s = -\frac{1}{0.02} = -50$$

$$D_c(z) = k\,\frac{z-e^{-5T}}{z-e^{-50T}} = k\,\frac{z-e^{-0.05}}{z-e^{-0.5}} = k\,\frac{z-0.951}{z-0.607}$$

$$D_c(1) = k\,\frac{1-0.951}{1-0.607} = k\,\frac{0.049}{0.393} \qquad G_c(0) = D_c(1) \qquad G_c(0) = 1$$

$$k\,\frac{0.049}{0.393} = 1 \qquad k = 8.02$$

$$D_c(z) = \frac{8.02(z-0.951)}{z-0.607} = \frac{8.02-7.63z^{-1}}{1-0.607\,z^{-1}}$$

Use bilinear transform method:

$$s = \frac{2}{T} \cdot \frac{1 - z^{-1}}{1 + z^{-1}} = 200 \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$D_c(z) = \left. \frac{0.2s + 1}{0.02s + 1} \right|_{s = 200 \cdot \frac{1 - z^{-1}}{1 + z^{-1}}} = \frac{41 - 39z^{-1}}{5 - 3z^{-1}} = \frac{8.2 - 7.79z^{-1}}{1 - 0.6z^{-1}}$$

Perform inverse z-transform:

$$D_c(z) = \frac{U(z)}{E(z)} = \frac{8.2 - 7.79z^{-1}}{1 - 0.6z^{-1}}$$

$$U(z) - 0.6z^{-1}U(z) = 8.2E(z) - 7.79z^{-1}E(z)$$

$$u(k) = 0.6u(k-1) + 8.2e(k) - 7.79e(k-1)$$

# Matlab simulation of the discretization

## Zero-Pole Matching

```
s=tf('s');
c=(0.2*s+1)/(0.02*s+1);
c1=c2d(c,0.01,'matched')
```

```
Transfer function:
8.068 z - 7.674
---------------
  z - 0.6065
Sampling time: 0.01
```

$$D_c(z) = \frac{8.02 - 7.63z^{-1}}{1 - 0.607\, z^{-1}}$$

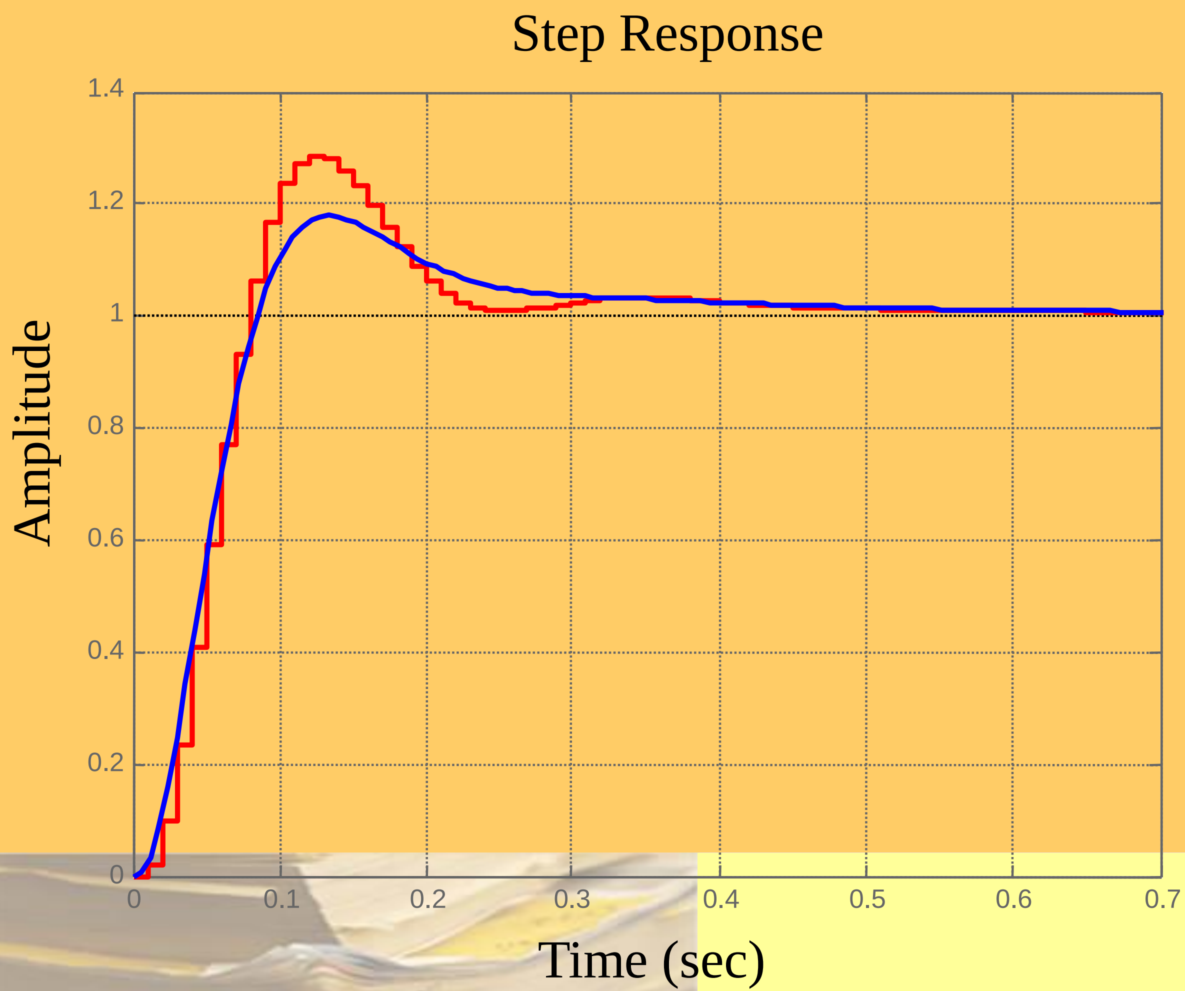
## Bilinear transform

```
s=tf('s');
c=(0.2*s+1)/(0.02*s+1);
c1=c2d(c,0.01,'tustin')
```

```
Transfer function:
8.2 z - 7.8
-----------
  z - 0.6
Sampling time: 0.01
```

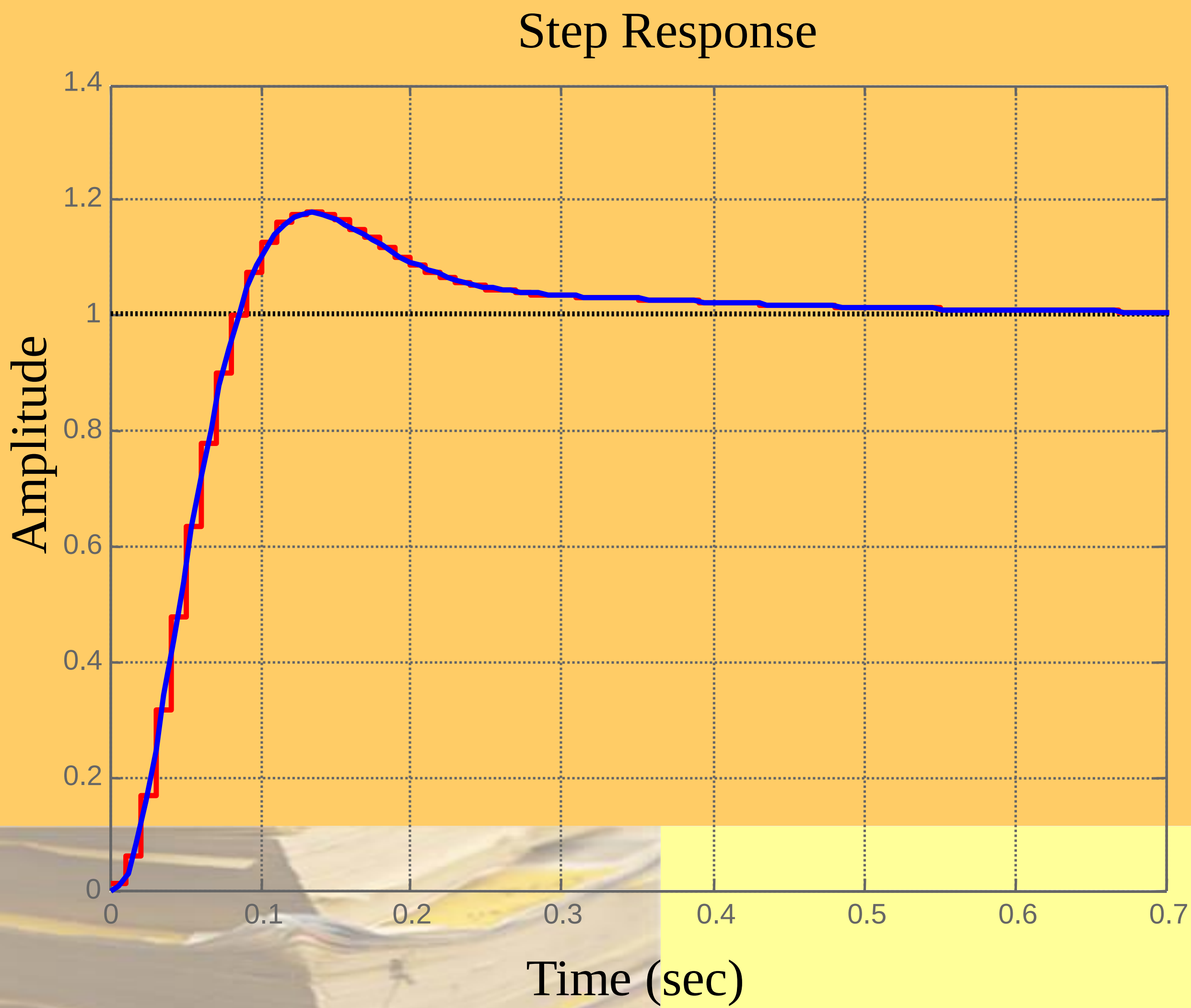
$$D_c(z) = \frac{8.2 - 7.79z^{-1}}{1 - 0.6z^{-1}}$$

# Matlab simulation of the digital controller obtained by zero-pole matching method




```
g2=c2d(g,0.01,'matched');
c2=c2d(c,0.01,'matched');
g3=c2*g2;
sys2=feedback(g3,1);
step(sys2);
```

# Matlab simulation of the digital controller obtained by bilinear transform



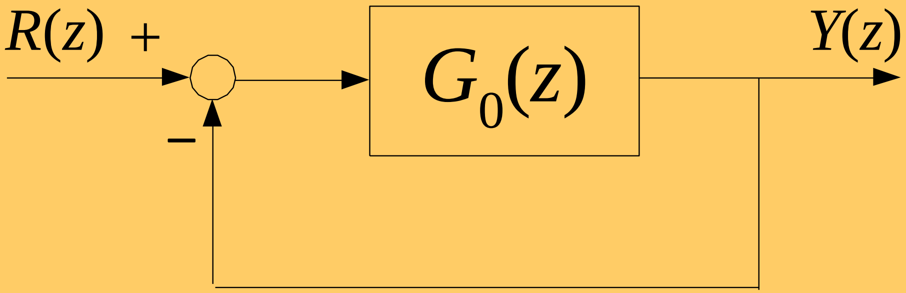

```
g2=c2d(g,0.01,'tustin');
c2=c2d(c,0.01,'tustin');
g3=c2*g2;
sys2=feedback(g3,1);
step(sys2)
```

# Design of Discrete-Data Control

Another way of design a digital controller is to discretize the controlled continuous system first, then use methods for discrete-data systems to design a digital controller.

Root locus method in z-plane and zero-pole placement in z-plane are two commonly used methods to design digital controllers. They will be illustrated by the following examples.

# Root Locus Method




$$\frac{Y(z)}{R(z)} = \frac{G_0(z)}{1+G_0(z)}$$

Characteristic equation: $1+G_0(z)=0$

$$G_0(z) = -1$$

Magnitude condition: $|G_0(z)| = 1$

Angle condition: $\angle G_0(z) = -180^0 \pm k360^0$

投票 最多可选1项

设置

Are the eight rules for constructing the root loci of continuous systems still applicable?

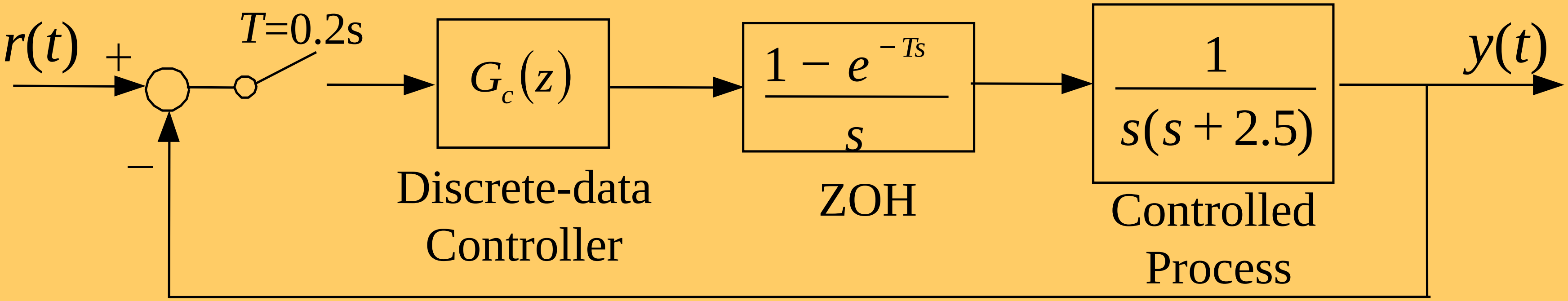
A Yes

B No

提交

# Example – 7.12

Q: please design a discrete-data controller, which makes the modified system satisfy the following performance specifications: $\zeta = 0.5$ , $t_s = 2s$



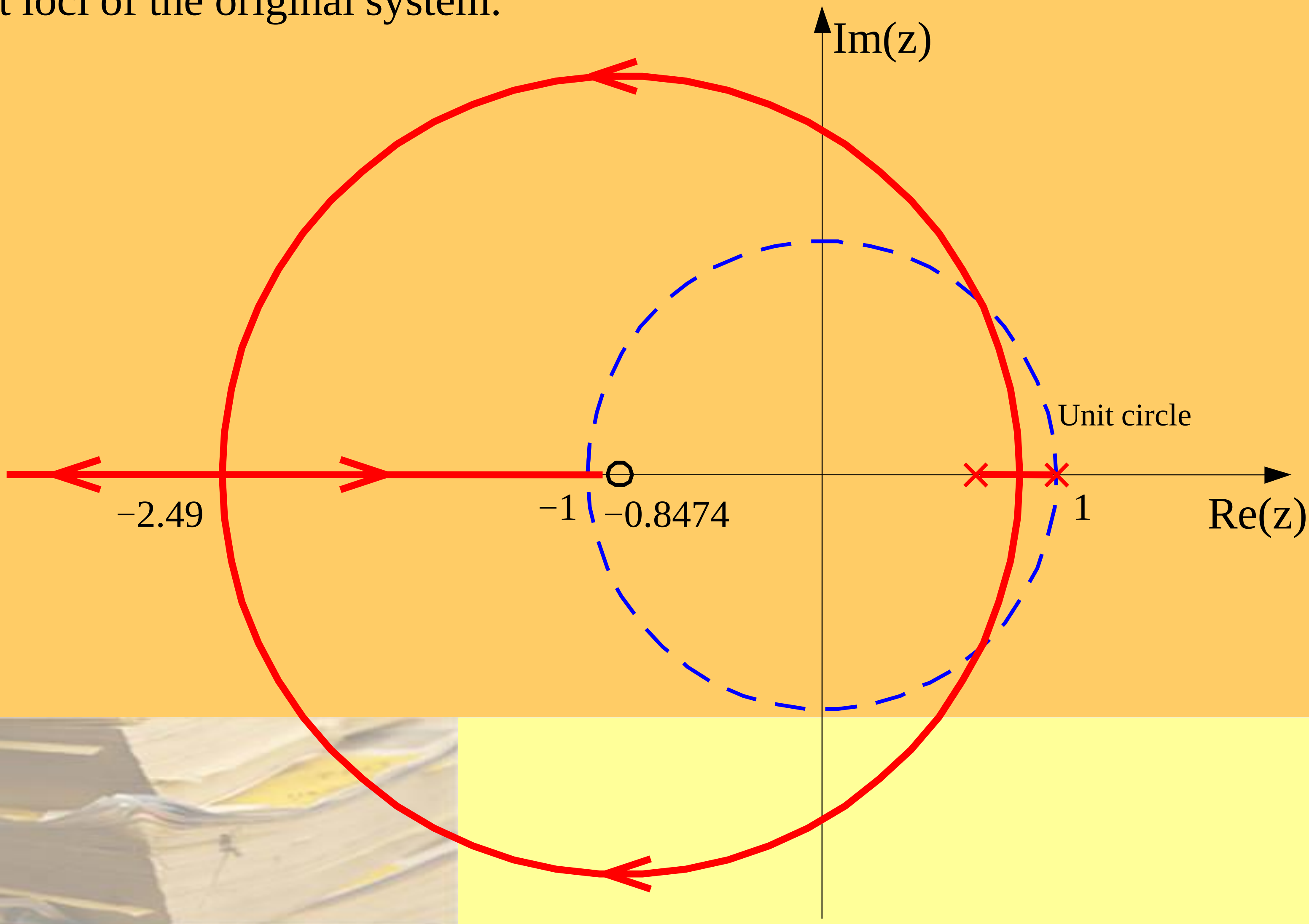

A: find the z-transfer function of the original system:

$$G_0(z) = Z\left[(1-e^{-Ts})\frac{1}{s^2(s+2.5)}\right] = Z(1-e^{-Ts})\cdot Z\left[\left(\frac{0.4}{s^2}-\frac{0.16}{s}+\frac{0.16}{s+2.5}\right)\right]$$

$$= (1-z^{-1})\left[\frac{0.4Tz}{(z-1)^2}-\frac{0.16z}{(z-1)}+\frac{0.16z}{(z-e^{-2.5T})}\right]$$

Substitute T=0.2s into the above function: $$G_0(z) = \frac{0.01704\ (z + 0.8474)}{(z-1)(z-0.6065)}$$

Root loci of the original system:



The poles' locations of the desired system:

$$\zeta = 0.5 \qquad t_s = 2s$$

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.5\omega_n} = 2 \qquad \omega_n = 4$$

The damped oscillatory frequency: $\omega_d = \omega_n\sqrt{1-\zeta^2} = 4\sqrt{1-0.5^2} = 3.464$

The sampling frequency: $\omega_s = \frac{2\pi}{T} = \frac{2\pi}{0.2} = 10\pi = 31.42$

The coordinates of the desired dominant poles on the s-plane:

$$-\zeta\omega_n = -0.5\times 4 = -2$$

$$\pm j\omega_d = \pm j3.464$$

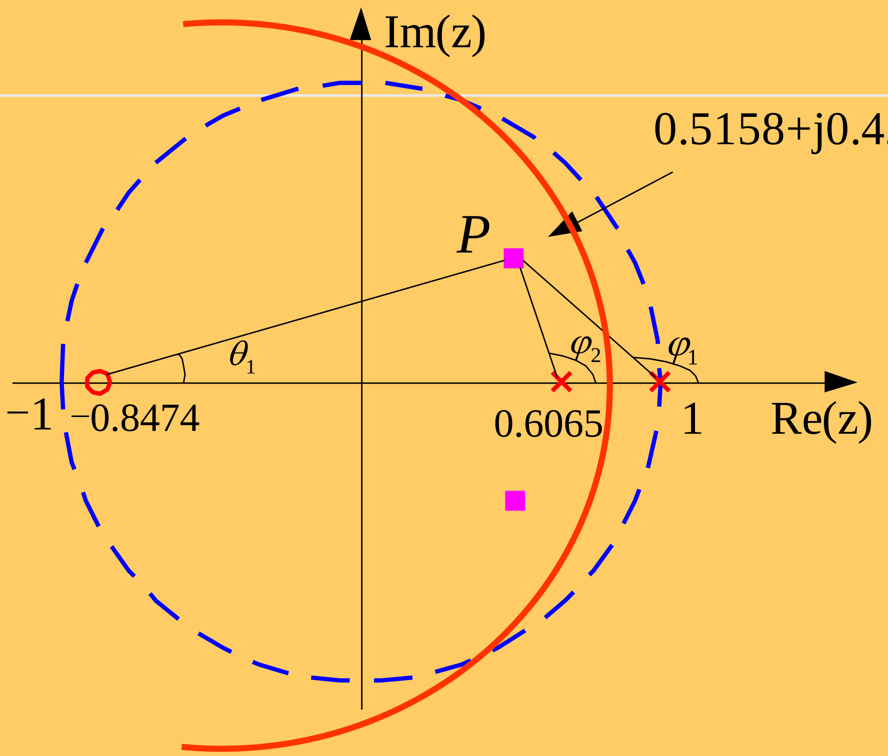
The coordinates of the desired dominant poles on the z-plane:

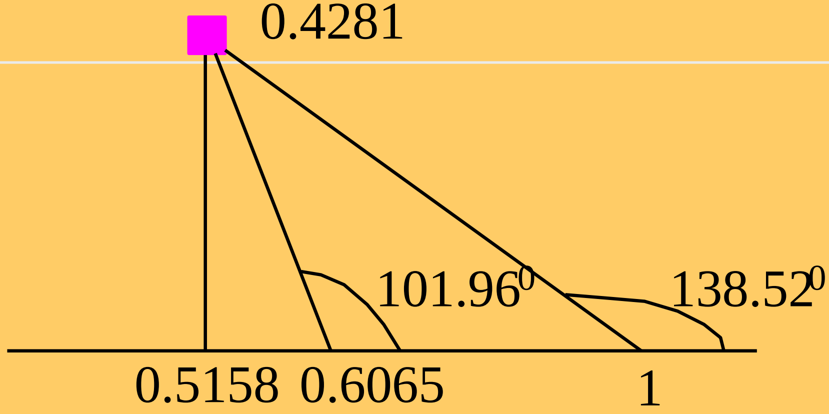
$$z = e^{Ts} \qquad T = \frac{2\pi}{\omega_s} \qquad s = -\zeta\omega_n \pm j\omega_d$$

$$|z| = e^{T(-\zeta\omega_n)} = \exp(-\frac{2\pi\zeta\omega_n}{\omega_s}) = \exp(-\frac{2\pi \times 0.5 \times 4}{31.42}) = e^{-0.4} = 0.6703$$

$$\angle z = T\omega_d = \frac{2\pi\omega_d}{\omega_s} = 2\pi \frac{3.464}{31.42} = 0.6927 \ rad = 39.69^0$$

$$z = 0.6703 \angle 39.69^0 = 0.5158 + j0.4281$$




$$\theta_1 - \varphi_1 - \varphi_2 = 17.43^0 - 138.52^0 - 101.96^0 = -223.05^0$$

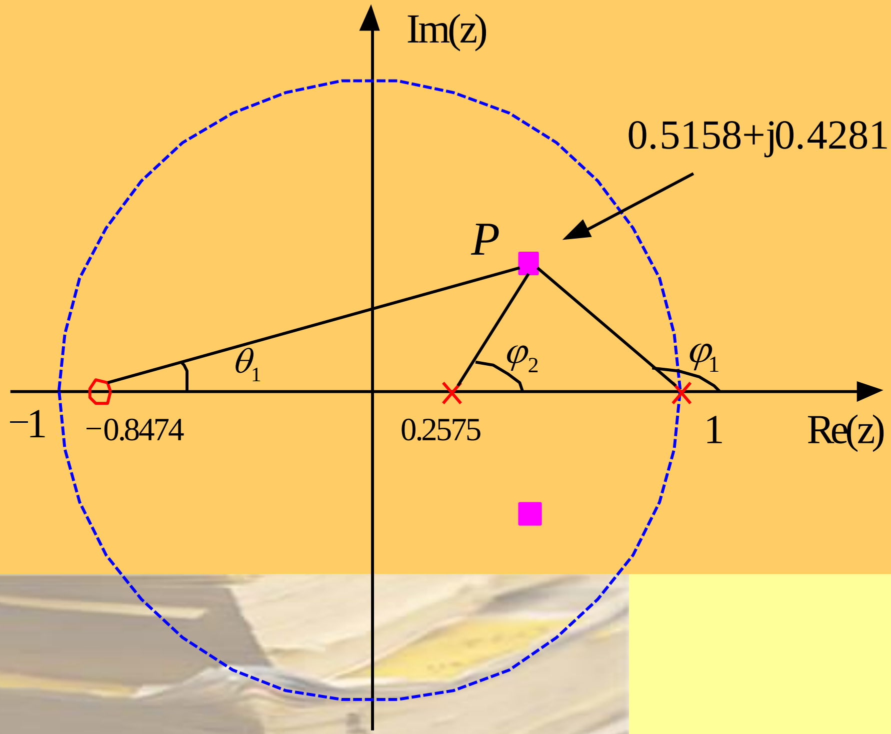
To satisfy the angle condition, the angle needs to be compensated is:

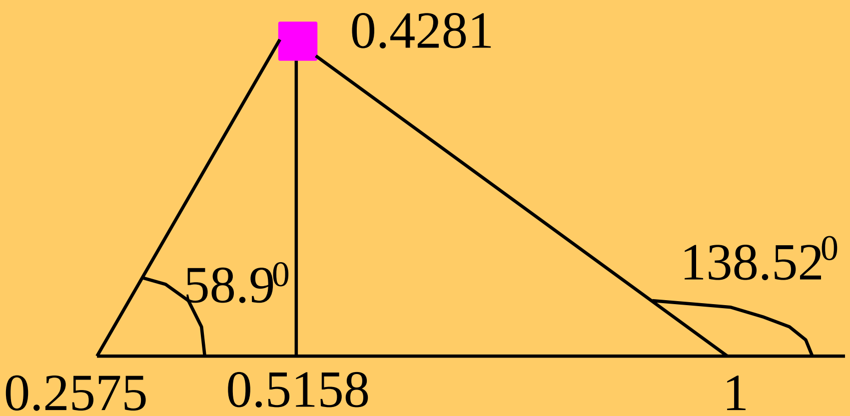
$$-223.05^0 + 180^0 = -43.05^0$$

Design a phase-lead controller with a z-transfer function of: $G_c(z) = k_c \dfrac{z+a}{z+b}$

Use zero-pole cancellation method to determine the value of a, then use angle condition to determine b.

$$\theta_1 - \varphi_1 - \varphi_2 = 17.43^0 - 138.52^0 - \mathbf{58.91^0} = -180^0$$

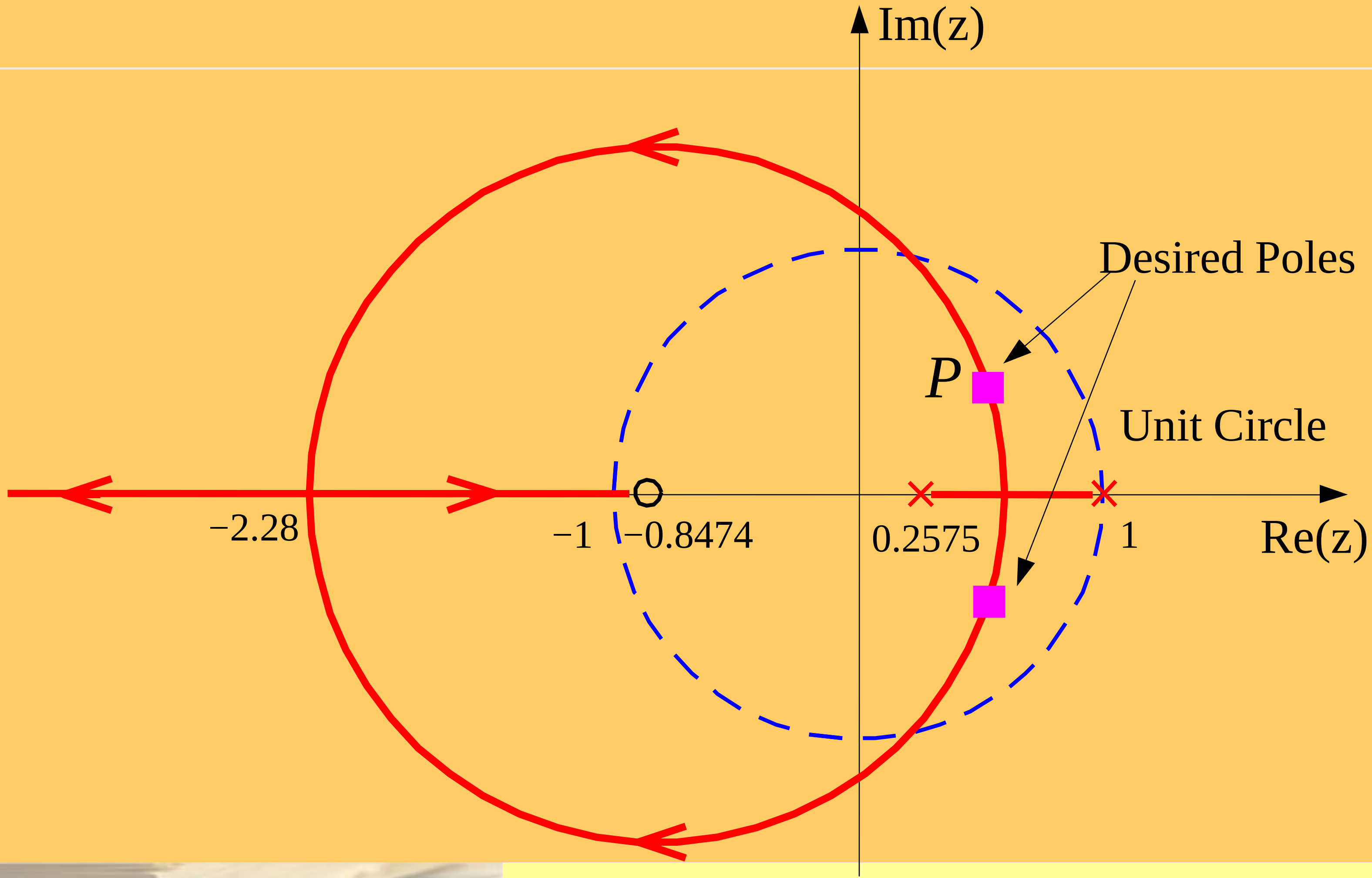



0.5158-0.2575=0.2583

$$G_c(z) = k_c \frac{z - 0.6065}{z - 0.2575}$$

The root loci of the compensated system:




$$G_c(z)G(z) = k_c \frac{z-0.6065}{z-0.2575} \cdot \frac{0.01704\,(z+0.8474\,)}{(z-1)(z-0.6065\,)} = k_c \frac{0.01704\,(z+0.8474\,)}{(z-1)(z-0.2575\,)}$$

Use magnitude condition to determine the open-loop gain:

$$\left|G_c(z)G(z)\right|_{z=0.5158+j0.4281}=1$$

$$k_c=13.27 \qquad G_c(z)=13.27\frac{z-0.6065}{z-0.2575}$$

Apply the inverse z-transform:

$$G_c(z)=\frac{U(z)}{E(z)}=13.27\frac{z-0.6065}{z-0.2575}=\frac{13.27-8.048z^{-1}}{1-0.2575z^{-1}}$$

$$U(z)-0.2575\,z^{-1}U(z)=13.27\,E(z)-8.048\,z^{-1}E(z)$$

$$u(k)=0.2575\,u(k-1)+13.27\,e(k)-8.048\,e(k-1)$$

# Matlab Simulation


```
z=tf('z',0.2);
g=0.01704*(z+0.8474)
/((z-1)*(z-0.6065));
sys=feedback(g,1);
step(sys,’b’);
hold on; grid on;
c=13.27*(z-0.6065)/(z-
0.2575);
g1=c*g;
sys1=feedback(g1,1);
step(sys1,’r’);
```

# Pole Placement through State Feedback

Discrete-date state equation $$X(k+1) = GX(k) + HU(k)$$

State feedback $$u(k) = K^T X(k)$$

Characteristic equation after state feedback $$\left| zI - (G - HK^T) \right| = 0$$

Example: please place the poles at $0.5 \pm j0.5$ for the following system.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$G = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

A: check the controllability of the system:

$$S=[H \quad GH]=\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix} \qquad rank(S)=2$$

system is controllable

Characteristic equation of the desired system:

$$(z-0.5-j0.5)(z-0.5+j0.5)=z^2-z+0.5$$

Characteristic equation of the system after introducing state feedback:

$$G-HK^T=\begin{bmatrix}0 & 1\\ -0.16 & -1\end{bmatrix}-\begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -0.16-k_1 & -1-k_2\end{bmatrix}$$

$$\left|zI-(G-HK^T)\right|=\begin{vmatrix}z & -1\\ 0.16+k_1 & z+1+k_2\end{vmatrix}=z^2+(k_2+1)s+k_1+0.16$$

$$k_1=0.34, \quad k_2=-2$$

# Wrap-Up

- Stability assessment of discrete-data systems
- Steady-state error of discrete-data systems
- Transient response of a discrete-data system
- Mapping between s-plane and z-plane trajectory
- Design a discrete-data controller through discretizing a continuous-data controller
- Design a discrete-data controller in the z-plane

# Assignment

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- 7
- 10， (1)
- 11， (2)