Haefner
Ch 4

Model development - Quantitative

Finite Difference equations & Differential equations

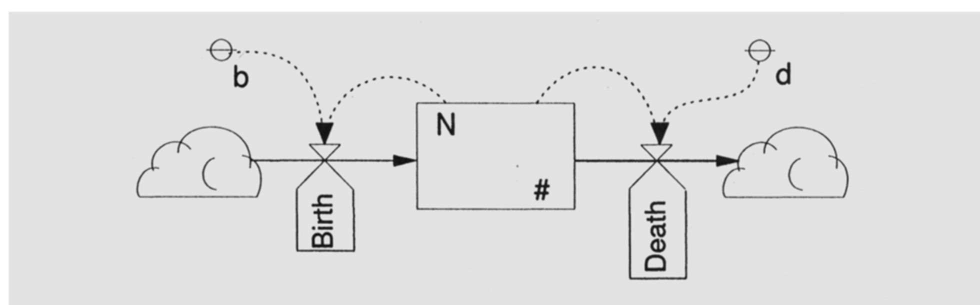
$$N_{t+1} = f(N_t)$$

$$N_{t+1} = N_t + f(N_t)$$

Population grows in discrete steps of time (months, years)

Lab 1 – Simple exponential growth model :

$$N_{t+1} = N_t + rN_t$$



Differential equation models

see Haefner, Sec 4.2.2

$$\frac{dN}{dt} = rN$$

DE version

$$N_{t+\Delta t} = N_t + rN_t\Delta t$$

FDE version

Analytical Solution of Simple DE Population Model

$$\frac{dN}{dt} = rN$$

$$\frac{dN}{N} = r\,dt$$

$$\int \frac{1}{N}\,dN = \int r\,dt$$

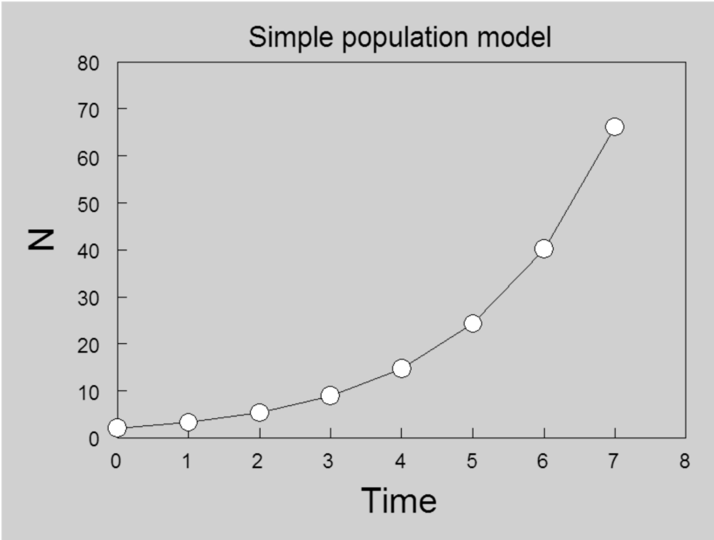
$$\ln N = rt + C$$

$$N = e^{rt+C}$$

$$N = e^{c}e^{rt} = N_0 e^{rt}$$

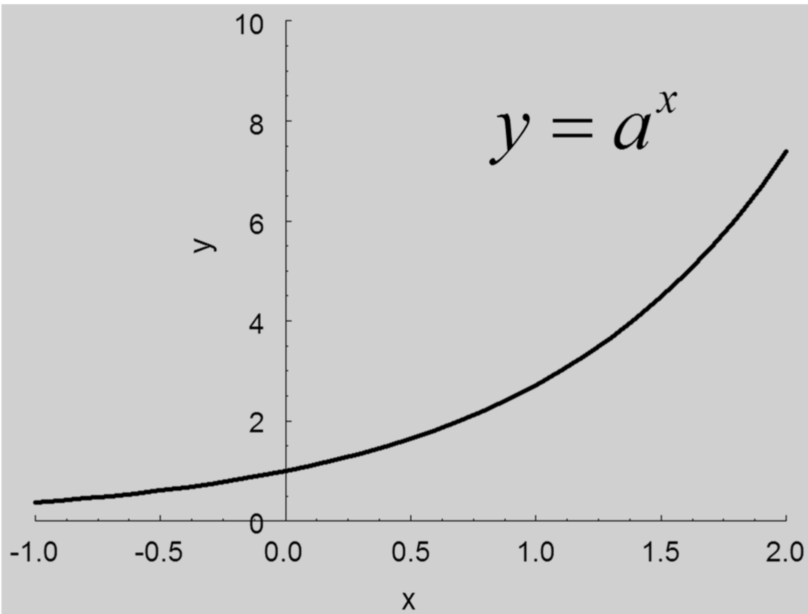
Solution :

$$N = N_0 e^{rt}$$



So what is this e ? e = 2.71828174591…

Exponential functions



$$if \;\; a = e$$

$$at \;\; x = 0, \;\; \frac{dy}{dx} = 1$$

$$y = e^x$$

$$\frac{dy}{dx} = e^x$$

Bank account interest

i=0.12
M_0=10

Annual

$$M_1 = M_0 + iM_0$$
$$M_1 = 11.2$$

Semi-annual

$$M_1 = M_0 + \frac{0.12}{2} M_0$$
$$M_1 = 10.6$$
$$M_2 = M_1 + 0.06 M_1$$
$$M_2 = 11.236$$

Quarterly

$$M_4 = 11.255$$

Exponential

$$M = M_0 \, e^{0.12t}$$
$$M = 11.27497$$

Solving the DE – Euler's Method

$$\frac{dN}{dt} = rN$$

$$\frac{\Delta N}{\Delta t} \cong rN$$

$$\Delta N \cong rN\Delta t$$

$$\Delta N = N_{t+1} - N_t$$

$$N_{t+1} = N_t + \Delta N$$

$$N_{t+\Delta t} = N_t + r N_t \Delta t$$

Does this look like our first model ?

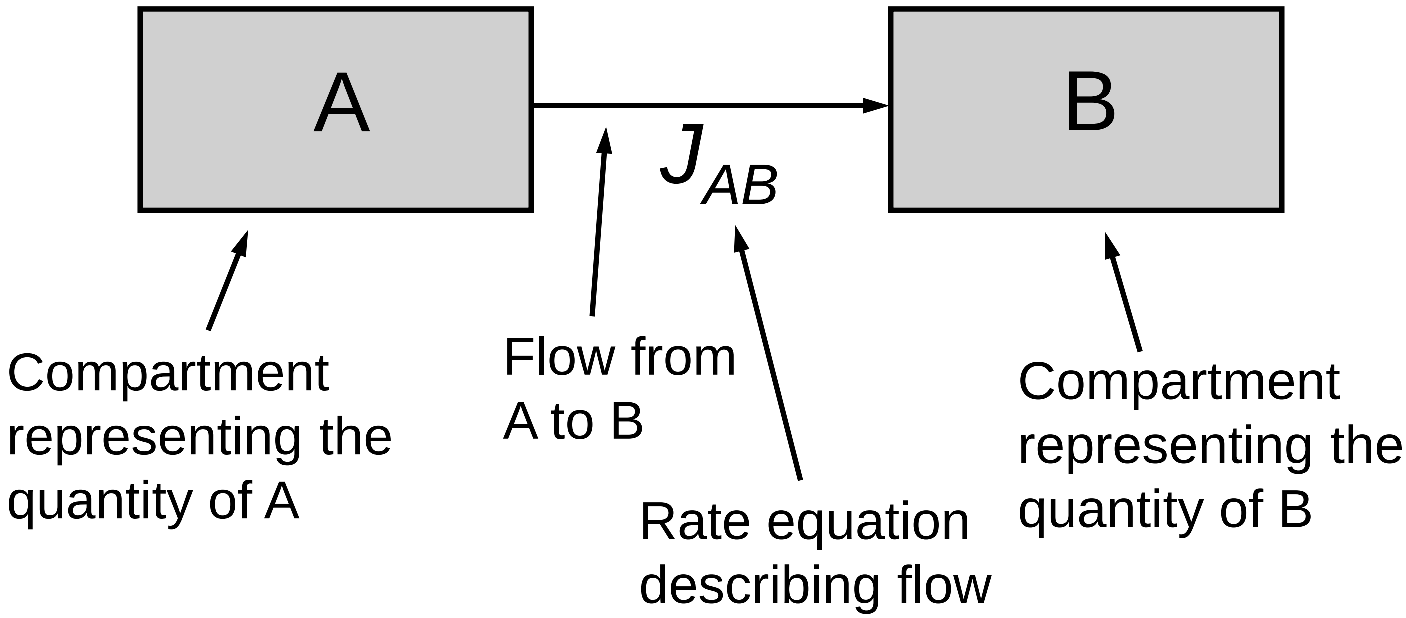
Numerical Solution (Euler's Method)

$$\frac{dN}{dt} = rN$$

$$N_{t+\Delta t} = N_t + rN_t\Delta t$$

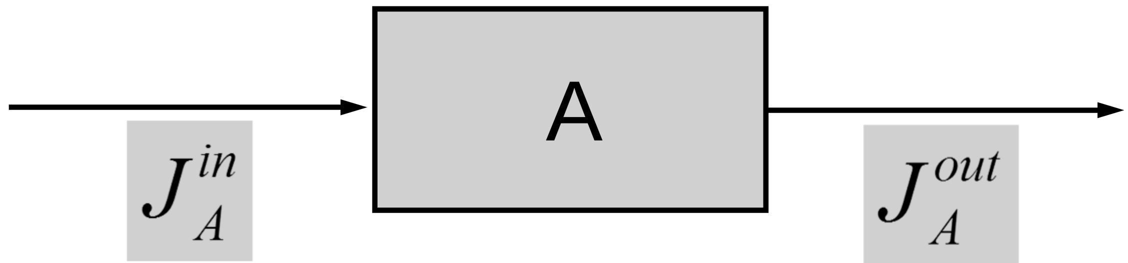
Many (most!) DE cannot be solved analytically, so we will need to use numerical solutions most of the time

Quantitative Development – Compartment models



State variables are conserved quantities –

- What leaves one box enters another
- If material enters faster than it leaves, quantity must be increasing

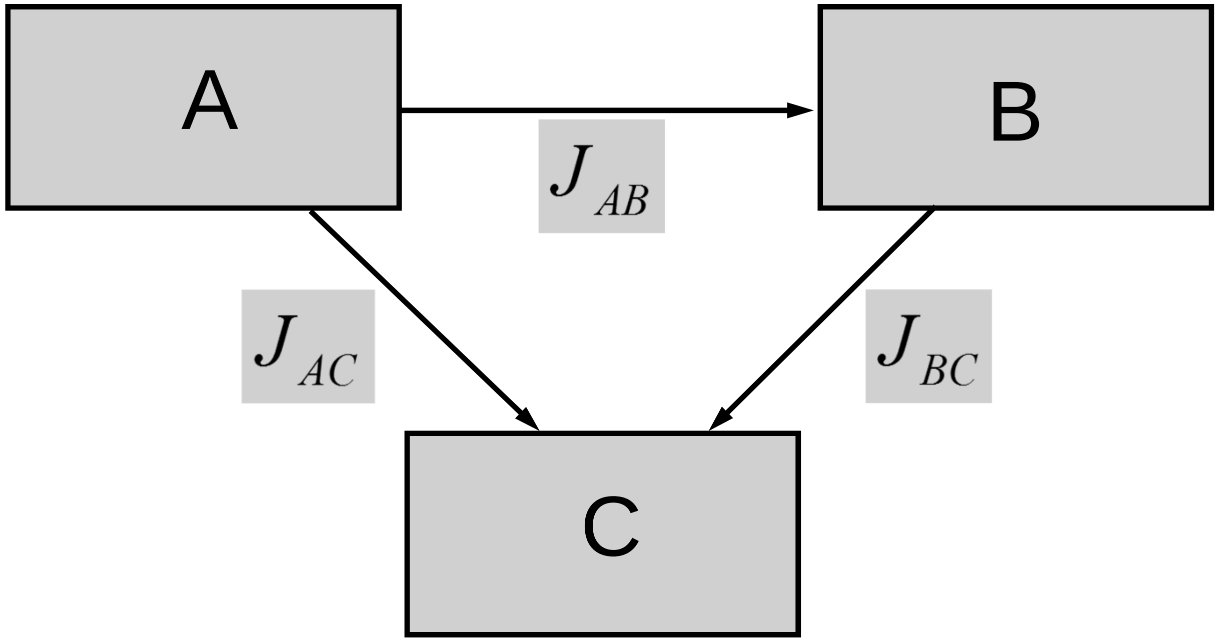


If in > out, A must increase
If out > in, A must decrease

$$\frac{dA}{dt} = J_A^{in} - J_A^{out}$$

Multiple boxes
multiple flows

$$\frac{dA}{dt} = \sum inflows - \sum outflows$$



$$\frac{d}{dt} = inflows - outflows$$

$$\frac{dA}{dt} = -J_{AB} - J_{AC}$$

$$\frac{dB}{dt} = J_{AB} - J_{BC}$$

$$\frac{dC}{dt} = J_{AC} + J_{BC}$$