Tenth Edition

ERWIN KREYSZIG

ADVANCED ENGINEERING MATHEMATICS

Systems of Units. Some Important Conversion Factors

The most important systems of units are shown in the table below. The mks system is also known as the *International System of Units* (abbreviated *SI*), and the abbreviations sec (instead of s), gm (instead of g), and nt (instead of N) are also used.

System of units	Length	Mass	Time	Force
cgs system	centimeter (cm)	gram (g)	second (s)	dyne
mks system	meter (m)	kilogram (kg)	second (s)	newton (nt)
Engineering system	foot (ft)	slug	second (s)	pound (lb)

1 inch (in.) $=$ 2.540000 cm

1 foot (ft) $=$ 12 in. $=$ 30.480000 cm

1 yard (yd) $=$ 3 ft $=$ 91.440000 cm

1 statute mile (mi) $=$ 5280 ft $=$ 1.609344 km

1 nautical mile $=$ 6080 ft $=$ 1.853184 km

1 acre $= 4840\ \text{yd}^2 = 4046.8564\ \text{m}^2$

1 $\text{mi}^2 = 640$ acres $= 2.5899881\ \text{km}^2$

1 fluid ounce $=$ 1/128 U.S. gallon $= 231/128\ \text{in.}^3 = 29.573730\ \text{cm}^3$

1 U.S. gallon $=$ 4 quarts (liq) $=$ 8 pints (liq) $=$ 128 fl oz $= 3785.4118\ \text{cm}^3$

1 British Imperial and Canadian gallon $=$ 1.200949 U.S. gallons $= 4546.087\ \text{cm}^3$

1 slug $=$ 14.59390 kg

1 pound (lb) $=$ 4.448444 nt

1 newton (nt) $= 10^5$ dynes

1 British thermal unit (Btu) $=$ 1054.35 joules

1 joule $= 10^7$ ergs

1 calorie (cal) $=$ 4.1840 joules

1 kilowatt-hour (kWh) $=$ 3414.4 Btu $= 3.6 \cdot 10^6$ joules

1 horsepower (hp) $=$ 2542.48 Btu/h $=$ 178.298 cal/sec $=$ 0.74570 kW

1 kilowatt (kW) $=$ 1000 watts $=$ 3414.43 Btu/h $=$ 238.662 cal/s

$°\text{F} = °\text{C} \cdot 1.8 + 32$

$1° = 60' = 3600'' = 0.017453293$ radian

For further details see, for example, D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics.* 9th ed., Hoboken, N. J: Wiley, 2011. See also AN American National Standard, ASTM/IEEE Standard Metric Practice, Institute of Electrical and Electronics Engineers, Inc. (IEEE), 445 Hoes Lane, Piscataway, N. J. 08854, website at www.ieee.org.

Differentiation

$(cu)' = cu'$ $\quad$ (c constant)

$(u + v)' = u' + v'$

$(uv)' = u'v + uv'$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} \qquad \text{(Chain rule)}$$

$(x^n)' = nx^{n-1}$

$(e^x)' = e^x$

$(e^{ax})' = ae^{ax}$

$(a^x)' = a^x \ln a$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\cot x)' = -\csc^2 x$

$(\sinh x)' = \cosh x$

$(\cosh x)' = \sinh x$

$$(\ln x)' = \frac{1}{x}$$

$$(\log_a x)' = \frac{\log_a e}{x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}$$

$$(\arctan x)' = \frac{1}{1 + x^2}$$

$$(\text{arccot}\, x)' = -\frac{1}{1 + x^2}$$

Integration

$$\int uv'\, dx = uv - \int u'v\, dx \text{ (by parts)}$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)$$

$$\int \frac{1}{x}\, dx = \ln |x| + c$$

$$\int e^{ax}\, dx = \frac{1}{a} e^{ax} + c$$

$$\int \sin x\, dx = -\cos x + c$$

$$\int \cos x\, dx = \sin x + c$$

$$\int \tan x\, dx = -\ln |\cos x| + c$$

$$\int \cot x\, dx = \ln |\sin x| + c$$

$$\int \sec x\, dx = \ln |\sec x + \tan x| + c$$

$$\int \csc x\, dx = \ln |\csc x - \cot x| + c$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \text{arcsinh} \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \text{arccosh} \frac{x}{a} + c$$

$$\int \sin^2 x\, dx = \tfrac{1}{2}x - \tfrac{1}{4} \sin 2x + c$$

$$\int \cos^2 x\, dx = \tfrac{1}{2}x + \tfrac{1}{4} \sin 2x + c$$

$$\int \tan^2 x\, dx = \tan x - x + c$$

$$\int \cot^2 x\, dx = -\cot x - x + c$$

$$\int \ln x\, dx = x \ln x - x + c$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + c$$

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ADVANCED ENGINEERING MATHEMATICS

10TH EDITION

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WILEY JOHN WILEY & SONS, INC.

PUBLISHER	Laurie Rosatone
PROJECT EDITOR	Shannon Corliss
MARKETING MANAGER	Jonathan Cottrell
CONTENT MANAGER	Lucille Buonocore
PRODUCTION EDITOR	Barbara Russiello
MEDIA EDITOR	Melissa Edwards
MEDIA PRODUCTION SPECIALIST	Lisa Sabatini
TEXT AND COVER DESIGN	Madelyn Lesure
PHOTO RESEARCHER	Sheena Goldstein
COVER PHOTO	© Denis Jr. Tangney/iStockphoto Cover photo shows the Zakim Bunker Hill Memorial Bridge in Boston, MA.

This book was set in Times Roman. The book was composed by PreMedia Global, and printed and bound by RR Donnelley & Sons Company, Jefferson City, MO. The cover was printed by RR Donnelley & Sons Company, Jefferson City, MO.

This book is printed on acid free paper. ∞

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ISBN 978-0-470-45836-5

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

PREFACE

See also http://www.wiley.com/college/kreyszig

Purpose and Structure of the Book

This book provides a comprehensive, thorough, and up-to-date treatment of ***engineering mathematics***. It is intended to introduce students of engineering, physics, mathematics, computer science, and related fields to those areas of ***applied mathematics*** that are most relevant for solving practical problems. A course in elementary calculus is the sole ***prerequisite***. (However, a concise refresher of basic calculus for the student is included on the inside cover and in Appendix 3.)

The subject matter is arranged into seven parts as follows:

A. Ordinary Differential Equations (ODEs) in Chapters 1–6
B. Linear Algebra. Vector Calculus. See Chapters 7–10
C. Fourier Analysis. Partial Differential Equations (PDEs). See Chapters 11 and 12
D. Complex Analysis in Chapters 13–18
E. Numeric Analysis in Chapters 19–21
F. Optimization, Graphs in Chapters 22 and 23
G. Probability, Statistics in Chapters 24 and 25.

These are followed by five appendices: **1.** References, **2.** Answers to Odd-Numbered Problems, **3.** Auxiliary Materials (see also inside covers of book), **4.** Additional Proofs, **5.** Table of Functions. This is shown in a block diagram on the next page.

The parts of the book are kept independent. In addition, individual chapters are kept as independent as possible. (If so needed, any prerequisites—to the level of individual sections of prior chapters—are clearly stated at the opening of each chapter.) We give the instructor **maximum flexibility in selecting the material** and tailoring it to his or her need. *The book has helped to pave the way for the present development of engineering mathematics*. This new edition will prepare the student for the current tasks and the future by a modern approach to the areas listed above. We provide the material and learning tools for the students to get a good foundation of engineering mathematics that will help them in their careers and in further studies.

General Features of the Book Include:

- **Simplicity of examples** to make the book teachable—why choose complicated examples when simple ones are as instructive or even better?
- **Independence of parts and blocks of chapters** to provide flexibility in tailoring courses to specific needs.
- **Self-contained presentation**, except for a few clearly marked places where a proof would exceed the level of the book and a reference is given instead.
- **Gradual increase in difficulty of material with no jumps or gaps** to ensure an enjoyable teaching and learning experience.
- **Modern standard notation** to help students with other courses, modern books, and journals in mathematics, engineering, statistics, physics, computer science, and others.

Furthermore, we designed the book to be a **single, self-contained, authoritative, and convenient source** for studying and teaching applied mathematics, eliminating the need for time-consuming searches on the Internet or time-consuming trips to the library to get a particular reference book.

PARTS AND CHAPTERS OF THE BOOK

PART A

Chaps. 1–6
Ordinary Differential Equations (ODEs)

- Chaps. 1–4 Basic Material
  - Chap. 5 Series Solutions
  - Chap. 6 Laplace Transforms

PART B

Chaps. 7–10
Linear Algebra. Vector Calculus

- Chap. 7 Matrices, Linear Systems
  - Chap. 8 Eigenvalue Problems
- Chap. 9 Vector Differential Calculus
  - Chap. 10 Vector Integral Calculus

PART C

Chaps. 11–12
Fourier Analysis. Partial Differential Equations (PDEs)

- Chap. 11 Fourier Analysis
  - Chap. 12 Partial Differential Equations

PART D

Chaps. 13–18
Complex Analysis, Potential Theory

- Chaps. 13–17 Basic Material
  - Chap. 18 Potential Theory

PART E

Chaps. 19–21
Numeric Analysis

- Chap. 19 Numerics in General
- Chap. 20 Numeric Linear Algebra
- Chap. 21 Numerics for ODEs and PDEs

PART F

Chaps. 22–23
Optimization, Graphs

- Chap. 22 Linear Programming
- Chap. 23 Graphs, Optimization

PART G

Chaps. 24–25
Probability, Statistics

- Chap. 24 Data Analysis. Probability Theory
  - Chap. 25 Mathematical Statistics

GUIDES AND MANUALS

Maple Computer Guide
Mathematica Computer Guide

Student Solutions Manual and Study Guide

Instructor’s Manual

Four Underlying Themes of the Book

The driving force in engineering mathematics is the rapid growth of technology and the sciences. New areas—often drawing from several disciplines—come into existence. Electric cars, solar energy, wind energy, green manufacturing, nanotechnology, risk management, biotechnology, biomedical engineering, computer vision, robotics, space travel, communication systems, green logistics, transportation systems, financial engineering, economics, and many other areas are advancing rapidly. What does this mean for engineering mathematics? The engineer has to take a problem from any diverse area and be able to model it. This leads to the first of four underlying themes of the book.

1. Modeling is the process in engineering, physics, computer science, biology, chemistry, environmental science, economics, and other fields whereby a physical situation or some other observation is translated into a mathematical model. This mathematical model could be a system of differential equations, such as in population control (Sec. 4.5), a probabilistic model (Chap. 24), such as in risk management, a linear programming problem (Secs. 22.2–22.4) in minimizing environmental damage due to pollutants, a financial problem of valuing a bond leading to an algebraic equation that has to be solved by Newton's method (Sec. 19.2), and many others.

The next step is **solving the mathematical problem** obtained by one of the many techniques covered in *Advanced Engineering Mathematics*.

The third step is **interpreting the mathematical result** in physical or other terms to see what it means in practice and any implications.

Finally, we may have to **make a decision** that may be of an industrial nature or **recommend a public policy**. For example, the population control model may imply the policy to stop fishing for 3 years. Or the valuation of the bond may lead to a recommendation to buy. The variety is endless, but the underlying mathematics is surprisingly powerful and able to provide advice leading to the achievement of goals toward the betterment of society, for example, by recommending wise policies concerning global warming, better allocation of resources in a manufacturing process, or making statistical decisions (such as in Sec. 25.4 whether a drug is effective in treating a disease).

While we cannot predict what the future holds, we do know that the student has to practice modeling by being given problems from many different applications as is done in this book. We teach modeling from scratch, right in Sec. 1.1, and give many examples in Sec. 1.3, and continue to reinforce the modeling process throughout the book.

2. Judicious use of powerful software for numerics (listed in the beginning of Part E) and statistics (Part G) is of growing importance. Projects in engineering and industrial companies may involve large problems of modeling very complex systems with hundreds of thousands of equations or even more. They require the use of such software. However, our policy has always been to leave it up to the instructor to determine the degree of use of computers, from none or little use to extensive use. More on this below.

3. The beauty of engineering mathematics. ***Engineering mathematics relies on relatively few basic concepts and involves powerful unifying principles***. We point them out whenever they are clearly visible, such as in Sec. 4.1 where we "grow" a mixing problem from one tank to two tanks and a circuit problem from one circuit to two circuits, thereby also increasing the number of ODEs from one ODE to two ODEs. This is an example of an attractive mathematical model because the "growth" in the problem is reflected by an "increase" in ODEs.

4. To clearly identify the conceptual structure of subject matters. For example, complex analysis (in Part D) is a field that is not monolithic in structure but was formed by three distinct schools of mathematics. Each gave a different approach, which we clearly mark. The first approach is solving complex integrals by Cauchy's integral formula (Chaps. 13 and 14), the second approach is to use the Laurent series and solve complex integrals by residue integration (Chaps. 15 and 16), and finally we use a geometric approach of conformal mapping to solve boundary value problems (Chaps. 17 and 18). Learning the conceptual structure and terminology of the different areas of engineering mathematics is very important for three reasons:

a. It allows the student to *identify a new problem and put it into the right group of problems*. The areas of engineering mathematics are growing but most often retain their conceptual structure.

b. The student can *absorb new information more rapidly* by being able to fit it into the conceptual structure.

c. Knowledge of the conceptual structure and terminology is also important when *using the Internet to search for mathematical information*. Since the search proceeds by putting in key words (i.e., terms) into the search engine, the student has to remember the important concepts (or be able to look them up in the book) that identify the application and area of engineering mathematics.

Big Changes in This Edition

1 Problem Sets Changed

The problem sets have been revised and rebalanced with some problem sets having more problems and some less, reflecting changes in engineering mathematics. There is a greater emphasis on modeling. Now there are also problems on the discrete Fourier transform (in Sec. 11.9).

2 Series Solutions of ODEs, Special Functions and Fourier Analysis Reorganized

Chap. 5, on series solutions of ODEs and special functions, has been shortened. Chap. 11 on Fourier Analysis now contains Sturm–Liouville problems, orthogonal functions, and orthogonal eigenfunction expansions (Secs. 11.5, 11.6), where they fit better conceptually (rather than in Chap. 5), being extensions of Fourier's idea of using orthogonal functions.

3 Openings of Parts and Chapters Rewritten As Well As Parts of Sections

In order to give the student a better idea of the structure of the material (see Underlying Theme 4 above), we have entirely rewritten the openings of parts and chapters. Furthermore, large parts or individual paragraphs of sections have been rewritten or new sentences inserted into the text. This should give the students a better intuitive understanding of the material (see Theme 3 above), let them draw conclusions on their own, and be able to tackle more advanced material. Overall, we feel that the book has become more detailed and leisurely written.

4 Student Solutions Manual and Study Guide Enlarged

Upon the explicit request of the users, the answers provided are more detailed and complete. More explanations are given on how to learn the material effectively by pointing out what is most important.

5 More Historical Footnotes, Some Enlarged

Historical footnotes are there to show the student that many people from different countries working in different professions, such as surveyors, researchers in industry, etc., contributed to the field of engineering mathematics. It should encourage the students to be creative in their own interests and careers and perhaps also to make contributions to engineering mathematics.

Further Changes and New Features

- Parts of Chap. 1 on first-order ODEs are rewritten. More emphasis on modeling, also new block diagram explaining this concept in Sec. 1.1. Early introduction of Euler's method in Sec. 1.2 to familiarize student with basic numerics. More examples of separable ODEs in Sec. 1.3.
- For Chap. 2, on second-order ODEs, note the following changes: For ease of reading, the first part of Sec. 2.4, which deals with setting up the mass-spring system, has been rewritten; also some rewriting in Sec. 2.5 on the Euler–Cauchy equation.
- Substantially shortened Chap. 5, Series Solutions of ODEs. Special Functions: combined Secs. 5.1 and 5.2 into one section called "Power Series Method," shortened material in Sec. 5.4 Bessel's Equation (of the first kind), removed Sec. 5.7 (Sturm–Liouville Problems) and Sec. 5.8 (Orthogonal Eigenfunction Expansions) and moved material into Chap. 11 (see "Major Changes" above).
- New equivalent definition of ***basis*** (Sec. 7.4).
- In Sec. 7.9, completely new part on **composition of linear transformations** with two new examples. Also, more detailed explanation of the role of axioms, in connection with the definition of vector space.
- New table of orientation (opening of Chap. 8 "Linear Algebra: Matrix Eigenvalue Problems") where eigenvalue problems occur in the book. More intuitive explanation of what an eigenvalue is at the begining of Sec. 8.1.
- Better definition of ***cross product*** (in vector differential calculus) by properly identifying the degenerate case (in Sec. 9.3).
- **Chap. 11 on Fourier Analysis extensively rearranged:** Secs. 11.2 and 11.3 combined into one section (Sec. 11.2), old Sec. 11.4 on complex Fourier Series removed and new Secs. 11.5 (Sturm–Liouville Problems) and 11.6 (Orthogonal Series) put in (see "Major Changes" above). New problems (new!) in problem set 11.9 on **discrete Fourier transform**.
- **New section 12.5** on modeling heat flow from a body in space by setting up the heat equation. Modeling PDEs is more difficult so we separated the modeling process from the solving process (in Sec. 12.6).
- **Introduction to Numerics** rewritten for greater clarity and better presentation; new Example 1 on how to round a number. Sec. 19.3 on interpolation shortened by removing the less important central difference formula and giving a reference instead.
- Large new footnote with historical details in Sec. 22.3, honoring George Dantzig, the inventor of the **simplex method**.
- **Traveling salesman problem** now described better as a "difficult" problem, typical of combinatorial optimization (in Sec. 23.2). More careful explanation on how to compute the capacity of a cut set in Sec. 23.6 (Flows on Networks).
- In Chap. 24, material on data representation and characterization restructured in terms of five examples and enlarged to include empirical rule on distribution of

data, outliers, and the z-score (Sec. 24.1). Furthermore, new example on encription (Sec. 24.4).

- Lists of **software** for numerics (Part E) and statistics (Part G) updated.
- References in **Appendix 1** updated to include new editions and some references to websites.

Use of Computers

The presentation in this book is ***adaptable to various degrees of use of software, Computer Algebra Systems (CAS's), or programmable graphic calculators***, ranging from no use, very little use, medium use, to intensive use of such technology. The choice of how much computer content the course should have is left up to the instructor, thereby exhibiting our philosophy of maximum flexibility and adaptability. And, no matter what the instructor decides, there will be no gaps or jumps in the text or problem set. Some problems are clearly designed as routine and drill exercises and should be solved by hand (paper and pencil, or typing on your computer). Other problems require more thinking and can also be solved without computers. Then there are problems where the computer can give the student a hand. And finally, the book has ***CAS projects***, ***CAS problems*** and ***CAS experiments***, which ***do require*** a computer, and show its power in solving problems that are difficult or impossible to access otherwise. Here our goal is to combine intelligent computer use with high-quality mathematics. The computer invites visualization, experimentation, and independent discovery work. In summary, the high degree of flexibility of computer use for the book is possible since there are plenty of problems to choose from and the CAS problems can be omitted if desired.

Note that ***information on software*** (what is available and where to order it) is at the beginning of Part E on Numeric Analysis and Part G on Probability and Statistics. Since ***Maple*** and ***Mathematica*** are popular Computer Algebra Systems, there are two computer guides available that are specifically tailored to *Advanced Engineering Mathematics:* E. Kreyszig and E.J. Norminton, *Maple Computer Guide*, 10*th Edition* and *Mathematica Computer Guide*, 10*th Edition*. Their use is completely optional as the text in the book is written without the guides in mind.

Suggestions for Courses: A Four-Semester Sequence

The material, when taken in sequence, is suitable for four consecutive semester courses, meeting 3 to 4 hours a week:

1st Semester	ODEs (Chaps. 1–5 or 1–6)
2nd Semester	Linear Algebra. Vector Analysis (Chaps. 7–10)
3rd Semester	Complex Analysis (Chaps. 13–18)
4th Semester	Numeric Methods (Chaps. 19–21)

Suggestions for Independent One-Semester Courses

The book is also suitable for various independent one-semester courses meeting 3 hours a week. For instance,

Introduction to ODEs (Chaps. 1–2, 21.1)
Laplace Transforms (Chap. 6)
Matrices and Linear Systems (Chaps. 7–8)

Vector Algebra and Calculus (Chaps. 9–10)
Fourier Series and PDEs (Chaps. 11–12, Secs. 21.4–21.7)
Introduction to Complex Analysis (Chaps. 13–17)
Numeric Analysis (Chaps. 19, 21)
Numeric Linear Algebra (Chap. 20)
Optimization (Chaps. 22–23)
Graphs and Combinatorial Optimization (Chap. 23)
Probability and Statistics (Chaps. 24–25)

Acknowledgments

We are indebted to former teachers, colleagues, and students who helped us directly or indirectly in preparing this book, in particular this new edition. We profited greatly from discussions with engineers, physicists, mathematicians, computer scientists, and others, and from their written comments. We would like to mention in particular Professors Y. A. Antipov, R. Belinski, S. L. Campbell, R. Carr, P. L. Chambré, Isabel F. Cruz, Z. Davis, D. Dicker, L. D. Drager, D. Ellis, W. Fox, A. Goriely, R. B. Guenther, J. B. Handley, N. Harbertson, A. Hassen, V. W. Howe, H. Kuhn, K. Millet, J. D. Moore, W. D. Munroe, A. Nadim, B. S. Ng, J. N. Ong, P. J. Pritchard, W. O. Ray, L. F. Shampine, H. L. Smith, Roberto Tamassia, A. L. Villone, H. J. Weiss, A. Wilansky, Neil M. Wigley, and L. Ying; Maria E. and Jorge A. Miranda, JD, all from the United States; Professors Wayne H. Enright, Francis. L. Lemire, James J. Little, David G. Lowe, Gerry McPhail, Theodore S. Norvell, and R. Vaillancourt; Jeff Seiler and David Stanley, all from Canada; and Professor Eugen Eichhorn, Gisela Heckler, Dr. Gunnar Schroeder, and Wiltrud Stiefenhofer from Europe. Furthermore, we would like to thank Professors John B. Donaldson, Bruce C. N. Greenwald, Jonathan L. Gross, Morris B. Holbrook, John R. Kender, and Bernd Schmitt; and Nicholaiv Villalobos, all from Columbia University, New York; as well as Dr. Pearl Chang, Chris Gee, Mike Hale, Joshua Jayasingh, MD, David Kahr, Mike Lee, R. Richard Royce, Elaine Schattner, MD, Raheel Siddiqui, Robert Sullivan, MD, Nancy Veit, and Ana M. Kreyszig, JD, all from New York City. We would also like to gratefully acknowledge the use of facilities at Carleton University, Ottawa, and Columbia University, New York.

Furthermore we wish to thank John Wiley and Sons, in particular Publisher Laurie Rosatone, Editor Shannon Corliss, Production Editor Barbara Russiello, Media Editor Melissa Edwards, Text and Cover Designer Madelyn Lesure, and Photo Editor Sheena Goldstein for their great care and dedication in preparing this edition. In the same vein, we would also like to thank Beatrice Ruberto, copy editor and proofreader, WordCo, for the Index, and Joyce Franzen of PreMedia and those of PreMedia Global who typeset this edition.

Suggestions of many readers worldwide were evaluated in preparing this edition. Further comments and suggestions for improving the book will be gratefully received.

KREYSZIG

CONTENTS

PART A

Ordinary Differential Equations (ODEs)

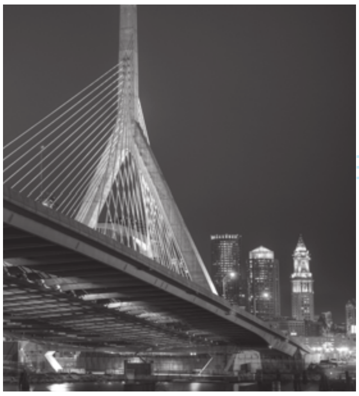
Many physical laws and relations can be expressed mathematically in the form of differential equations. Thus it is natural that this book opens with the study of differential equations and their solutions. Indeed, many engineering problems appear as differential equations.

The main objectives of Part A are twofold: the study of ordinary differential equations and their most important methods for solving them and the study of modeling.

Ordinary differential equations **(ODEs)** are differential equations that depend on a single variable. The more difficult study of partial differential equations (PDEs), that is, differential equations that depend on several variables, is covered in Part C.

Modeling is a crucial general process in engineering, physics, computer science, biology, medicine, environmental science, chemistry, economics, and other fields that translates a physical situation or some other observations into a "mathematical model." Numerous examples from engineering (e.g., mixing problem), physics (e.g., Newton's law of cooling), biology (e.g., Gompertz model), chemistry (e.g., radiocarbon dating), environmental science (e.g., population control), etc. shall be given, whereby this process is explained in detail, that is, how to set up the problems correctly in terms of differential equations.

For those interested in solving ODEs numerically on the computer, look at Secs. 21.1–21.3 of Chapter 21 of Part F, that is, ***numeric methods for ODEs***. These sections are kept independent by design of the other sections on numerics. ***This allows for the study of numerics for ODEs directly after Chap. 1 or 2.***



CHAPTER 1

First-Order ODEs

Chapter 1 begins the study of ordinary differential equations (ODEs) by deriving them from physical or other problems (***modeling***), solving them by standard mathematical methods, and interpreting solutions and their graphs in terms of a given problem. The simplest ODEs to be discussed are ODEs ***of the first order*** because they involve only the first derivative of the unknown function and no higher derivatives. These unknown functions will usually be denoted by $y(x)$ or $y(t)$ when the independent variable denotes time t. The chapter ends with a study of the existence and uniqueness of solutions of ODEs in Sec. 1.7.

Understanding the basics of ODEs requires solving problems by hand (paper and pencil, or typing on your computer, but first without the aid of a CAS). In doing so, you will gain an important conceptual understanding and feel for the basic terms, such as ODEs, direction field, and initial value problem. If you wish, you can use your **Computer Algebra System (CAS)** for checking solutions.

COMMENT. ***Numerics for first-order ODEs can be studied immediately after this chapter.*** See Secs. 21.1–21.2, which are independent of other sections on numerics.

Prerequisite: Integral calculus.
Sections that may be omitted in a shorter course: 1.6, 1.7.
References and Answers to Problems: App. 1 Part A, and App. 2.

1.1 Basic Concepts. Modeling

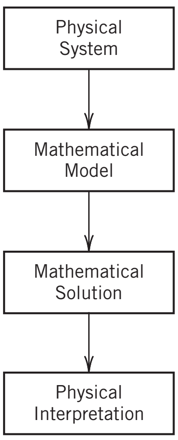


Fig. 1. Modeling, solving, interpreting

If we want to solve an engineering problem (usually of a physical nature), we first have to formulate the problem as a mathematical expression in terms of variables, functions, and equations. Such an expression is known as a mathematical **model** of the given problem. The process of setting up a model, solving it mathematically, and interpreting the result in physical or other terms is called *mathematical modeling* or, briefly, **modeling**.

Modeling needs experience, which we shall gain by discussing various examples and problems. (Your computer may often help you in *solving* but rarely in *setting up* models.)

Now many physical concepts, such as velocity and acceleration, are derivatives. Hence a model is very often an equation containing derivatives of an unknown function. Such a model is called a **differential equation**. Of course, we then want to find a solution (a function that satisfies the equation), explore its properties, graph it, find values of it, and interpret it in physical terms so that we can understand the behavior of the physical system in our given problem. However, before we can turn to methods of solution, we must first define some basic concepts needed throughout this chapter.

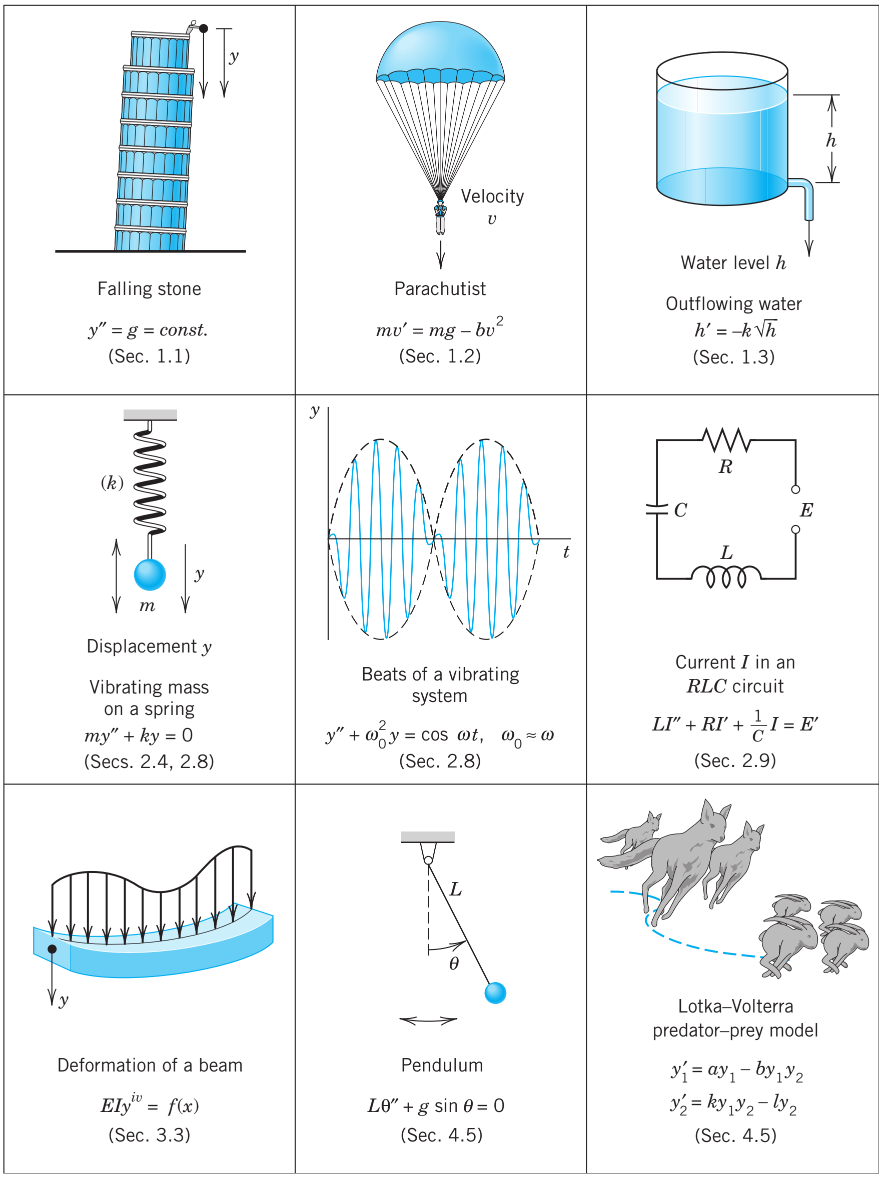


Fig. 2. Some applications of differential equations

An **ordinary differential equation (ODE)** is an equation that contains one or several derivatives of an unknown function, which we usually call $y(x)$ (or sometimes $y(t)$ if the independent variable is time t). The equation may also contain y itself, known functions of x (or t), and constants. For example,

$$y' = \cos x \tag{1}$$

$$y'' + 9y = e^{-2x} \tag{2}$$

$$y'y''' - \tfrac{3}{2}y'^2 = 0 \tag{3}$$

are ordinary differential equations (ODEs). Here, as in calculus, y' denotes dy/dx, $y'' = d^2y/dx^2$, etc. The term *ordinary* distinguishes them from *partial differential equations* (PDEs), which involve partial derivatives of an unknown function of *two or more* variables. For instance, a PDE with unknown function u of two variables x and y is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

PDEs have important engineering applications, but they are more complicated than ODEs; they will be considered in Chap. 12.

An ODE is said to be of **order** n if the nth derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, and so on. Thus, (1) is of first order, (2) of second order, and (3) of third order.

In this chapter we shall consider **first-order ODEs**. Such equations contain only the first derivative y' and may contain y and any given functions of x. Hence we can write them as

$$F(x, y, y') = 0 \tag{4}$$

or often in the form

$$y' = f(x, y).$$

This is called the *explicit form*, in contrast to the *implicit form* (4). For instance, the implicit ODE $x^{-3}y' - 4y^2 = 0$ (where $x \neq 0$) can be written explicitly as $y' = 4x^3y^2$.

Concept of Solution

A function

$$y = h(x)$$

is called a **solution** of a given ODE (4) on some open interval $a < x < b$ if $h(x)$ is defined and differentiable throughout the interval and is such that the equation becomes an identity if y and y' are replaced with h and h', respectively. The curve (the graph) of h is called a **solution curve**.

Here, **open interval** $a < x < b$ means that the endpoints a and b are not regarded as points belonging to the interval. Also, $a < x < b$ includes *infinite intervals* $-\infty < x < b$, $a < x < \infty$, $-\infty < x < \infty$ (the real line) as special cases.

EXAMPLE 1 **Verification of Solution**

Verify that $y = c/x$ (c an arbitrary constant) is a solution of the ODE $xy' = -y$ for all $x \neq 0$. Indeed, differentiate $y = c/x$ to get $y' = -c/x^2$. Multiply this by x, obtaining $xy' = -c/x$; thus, $xy' = -y$, the given ODE. ■

EXAMPLE 2 **Solution by Calculus. Solution Curves**

The ODE $y' = dy/dx = \cos x$ can be solved directly by integration on both sides. Indeed, using calculus, we obtain $y = \int \cos x \, dx = \sin x + c$, where c is an arbitrary constant. This is a ***family of solutions***. Each value of c, for instance, 2.75 or 0 or -8, gives one of these curves. Figure 3 shows some of them, for $c = -3, -2, -1, 0, 1, 2, 3, 4$. ■

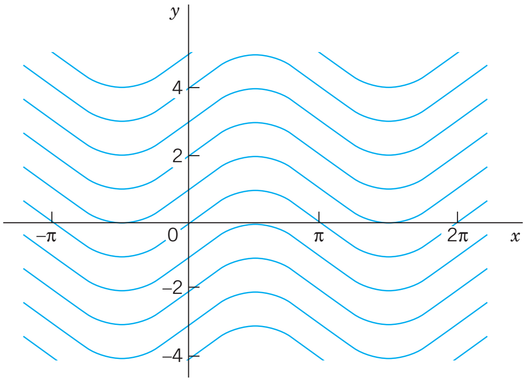


Fig. 3. Solutions $y = \sin x + c$ of the ODE $y' = \cos x$

EXAMPLE 3 **(A) Exponential Growth. (B) Exponential Decay**

From calculus we know that $y = ce^{0.2t}$ has the derivative

$$y' = \frac{dy}{dt} = 0.2e^{0.2t} = 0.2y.$$

Hence y is a solution of $y' = 0.2y$ (Fig. 4A). This ODE is of the form $y' = ky$. With positive-constant k it can model exponential growth, for instance, of colonies of bacteria or populations of animals. It also applies to humans for small populations in a large country (e.g., the United States in early times) and is then known as **Malthus's law**.[1] We shall say more about this topic in Sec. 1.5.

(B) Similarly, $y' = -0.2$ (with a minus on the right) has the solution $y = ce^{-0.2t}$, (Fig. 4B) modeling **exponential decay**, as, for instance, of a radioactive substance (see Example 5). ■

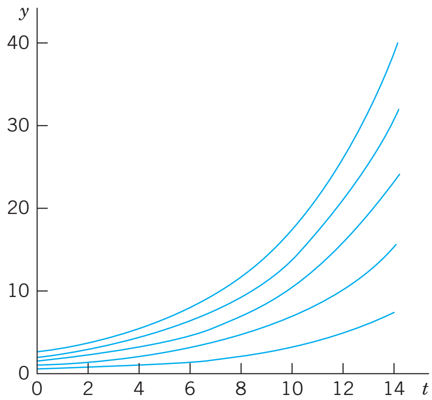


Fig. 4A. Solutions of $y' = 0.2y$ in Example 3 **(exponential growth)**

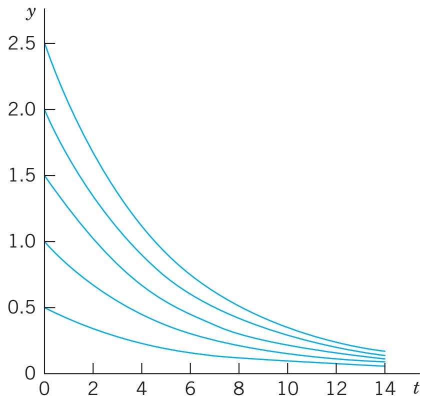


Fig. 4B. Solutions of $y' = -0.2y$ in Example 3 **(exponential decay)**

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[1]Named after the English pioneer in classic economics, THOMAS ROBERT MALTHUS (1766–1834).

We see that each ODE in these examples has a solution that contains an arbitrary constant c. Such a solution containing an arbitrary constant c is called a **general solution** of the ODE.

(We shall see that c is sometimes not completely arbitrary but must be restricted to some interval to avoid complex expressions in the solution.)

We shall develop methods that will give general solutions *uniquely* (perhaps except for notation). Hence we shall say ***the*** general solution of a given ODE (instead of ***a*** general solution).

Geometrically, the general solution of an ODE is a family of infinitely many solution curves, one for each value of the constant c. If we choose a specific c (e.g., $c = 6.45$ or 0 or -2.01) we obtain what is called a **particular solution** of the ODE. A particular solution does not contain any arbitrary constants.

In most cases, general solutions exist, and every solution not containing an arbitrary constant is obtained as a particular solution by assigning a suitable value to c. Exceptions to these rules occur but are of minor interest in applications; see Prob. 16 in Problem Set 1.1.

Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an **initial condition** $y(x_0) = y_0$, with given values x_0 and y_0, that is used to determine a value of the arbitrary constant c. Geometrically this condition means that the solution curve should pass through the point (x_0, y_0) in the xy-plane. An ODE, together with an initial condition, is called an **initial value problem**. Thus, if the ODE is explicit, $y' = f(x, y)$, the initial value problem is of the form

$$y' = f(x, y), \qquad y(x_0) = y_0. \tag{5}$$

EXAMPLE 4 **Initial Value Problem**

Solve the initial value problem

$$y' = \frac{dy}{dx} = 3y, \qquad y(0) = 5.7.$$

Solution. The general solution is $y(x) = ce^{3x}$; see Example 3. From this solution and the initial condition we obtain $y(0) = ce^0 = c = 5.7$. Hence the initial value problem has the solution $y(x) = 5.7e^{3x}$. This is a particular solution. ■

More on Modeling

The general importance of modeling to the engineer and physicist was emphasized at the beginning of this section. We shall now consider a basic physical problem that will show the details of the typical steps of modeling. Step 1: the transition from the physical situation (the physical system) to its mathematical formulation (its mathematical model); Step 2: the solution by a mathematical method; and Step 3: the physical interpretation of the result. This may be the easiest way to obtain a first idea of the nature and purpose of differential equations and their applications. Realize at the outset that your ***computer*** (your ***CAS***) may perhaps give you a hand in Step 2, but Steps 1 and 3 are basically your work.

And Step 2 requires a solid knowledge and good understanding of solution methods available to you—*you* have to choose the method for your work by hand or by the computer. Keep this in mind, and always check computer results for errors (which may arise, for instance, from false inputs).

EXAMPLE 5 Radioactivity. Exponential Decay

Given an amount of a radioactive substance, say, 0.5 g (gram), find the amount present at any later time.

Physical Information. Experiments show that at each instant a radioactive substance decomposes—and is thus decaying in time—proportional to the amount of substance present.

Step 1. Setting up a mathematical model of the physical process. Denote by $y(t)$ the amount of substance still present at any time t. By the physical law, the time rate of change $y'(t) = dy/dt$ is proportional to $y(t)$. This gives the ***first-order ODE***

$$\frac{dy}{dt} = -ky \tag{6}$$

where the constant k is positive, so that, because of the minus, we do get decay (as in [B] of Example 3). The value of k is known from experiments for various radioactive substances (e.g., $k = 1.4 \cdot 10^{-11}\ \text{sec}^{-1}$, approximately, for radium ${}^{226}_{88}\text{Ra}$).

Now the given initial amount is 0.5 g, and we can call the corresponding instant $t = 0$. Then we have the ***initial condition*** $y(0) = 0.5$. This is the instant at which our observation of the process begins. It motivates the term *initial condition* (which, however, is also used when the independent variable is not time or when we choose a t other than $t = 0$). Hence the mathematical model of the physical process is the ***initial value problem***

$$\frac{dy}{dt} = -ky, \qquad y(0) = 0.5. \tag{7}$$

Step 2. Mathematical solution. As in (B) of Example 3 we conclude that the ODE (6) models exponential decay and has the general solution (with arbitrary constant c but definite given k)

$$y(t) = ce^{-kt}. \tag{8}$$

We now determine c by using the initial condition. Since $y(0) = c$ from (8), this gives $y(0) = c = 0.5$. Hence the particular solution governing our process is (cf. Fig. 5)

$$y(t) = 0.5e^{-kt} \qquad (k > 0). \tag{9}$$

Always check your result—it may involve human or computer errors! Verify by differentiation (chain rule!) that your solution (9) satisfies (7) as well as $y(0) = 0.5$:

$$\frac{dy}{dt} = -0.5ke^{-kt} = -k \cdot 0.5e^{-kt} = -ky, \qquad y(0) = 0.5e^{0} = 0.5.$$

Step 3. Interpretation of result. Formula (9) gives the amount of radioactive substance at time t. It starts from the correct initial amount and decreases with time because k is positive. The limit of y as $t \to \infty$ is zero. ■

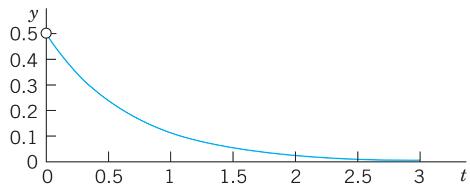

Fig. 5. Radioactivity (Exponential decay, $y = 0.5e^{-kt}$, with $k = 1.5$ as an example)

PROBLEM SET 1.1

1–8 CALCULUS

Solve the ODE by integration or by remembering a differentiation formula.

1. $y' + 2 \sin 2\pi x = 0$
2. $y' + xe^{-x^2/2} = 0$
3. $y' = y$
4. $y' = -1.5y$
5. $y' = 4e^{-x} \cos x$
6. $y'' = -y$
7. $y' = \cosh 5.13x$
8. $y''' = e^{-0.2x}$

9–15 VERIFICATION. INITIAL VALUE PROBLEM (IVP)

(a) Verify that y is a solution of the ODE. **(b)** Determine from y the particular solution of the IVP. **(c)** Graph the solution of the IVP.

9. $y' + 4y = 1.4, \quad y = ce^{-4x} + 0.35, \quad y(0) = 2$
10. $y' + 5xy = 0, \quad y = ce^{-2.5x^2}, \quad y(0) = \pi$
11. $y' = y + e^x, \quad y = (x + c)e^x, \quad y(0) = \frac{1}{2}$
12. $yy' = 4x, \quad y^2 - 4x^2 = c \ (y > 0), \quad y(1) = 4$
13. $y' = y - y^2, \quad y = \dfrac{1}{1 + ce^{-x}}, \quad y(0) = 0.25$
14. $y' \tan x = 2y - 8, \quad y = c \sin^2 x + 4, \quad y(\frac{1}{2}\pi) = 0$
15. Find two constant solutions of the ODE in Prob. 13 by inspection.
16. **Singular solution.** An ODE may sometimes have an additional solution that cannot be obtained from the general solution and is then called a *singular solution.* The ODE $y'^2 - xy' + y = 0$ is of this kind. Show by differentiation and substitution that it has the general solution $y = cx - c^2$ and the singular solution $y = x^2/4$. Explain Fig. 6.

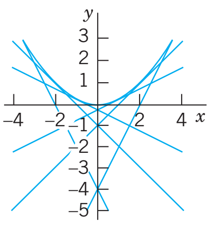

Fig. 6. Particular solutions and singular solution in Problem 16

17–20 MODELING, APPLICATIONS

These problems will give you a first impression of modeling. Many more problems on modeling follow throughout this chapter.

17. **Half-life.** The *half-life* measures exponential decay. It is the time in which half of the given amount of radioactive substance will disappear. What is the half-life of ${}^{226}_{88}\mathrm{Ra}$ (in years) in Example 5?
18. **Half-life.** Radium ${}^{224}_{88}\mathrm{Ra}$ has a half-life of about 3.6 days.
    (a) Given 1 gram, how much will still be present after 1 day?
    (b) After 1 year?
19. **Free fall.** In dropping a stone or an iron ball, air resistance is practically negligible. Experiments show that the acceleration of the motion is constant (equal to $g = 9.80 \text{ m/sec}^2 = 32 \text{ ft/sec}^2$, called the **acceleration of gravity**). Model this as an ODE for $y(t)$, the distance fallen as a function of time t. If the motion starts at time $t = 0$ from rest (i.e., with velocity $v = y' = 0$), show that you obtain the familiar law of free fall
$$y = \tfrac{1}{2}gt^2.$$
20. **Exponential decay. Subsonic flight.** The efficiency of the engines of subsonic airplanes depends on air pressure and is usually maximum near 35,000 ft. Find the air pressure $y(x)$ at this height. *Physical information.* The rate of change $y'(x)$ is proportional to the pressure. At 18,000 ft it is half its value $y_0 = y(0)$ at sea level. *Hint.* Remember from calculus that if $y = e^{kx}$, then $y' = ke^{kx} = ky$. Can you see without calculation that the answer should be close to $y_0/4$?

1.2 Geometric Meaning of $y' = f(x, y)$. Direction Fields, Euler's Method

A first-order ODE

$$y' = f(x, y) \tag{1}$$

has a simple geometric interpretation. From calculus you know that the derivative $y'(x)$ of $y(x)$ is the slope of $y(x)$. Hence a solution curve of (1) that passes through a point (x_0, y_0) must have, at that point, the slope $y'(x_0)$ equal to the value of f at that point; that is,

$$y'(x_0) = f(x_0, y_0).$$

Using this fact, we can develop graphic or numeric methods for obtaining approximate solutions of ODEs (1). This will lead to a better conceptual understanding of an ODE (1). Moreover, such methods are of practical importance since many ODEs have complicated solution formulas or no solution formulas at all, whereby numeric methods are needed.

Graphic Method of Direction Fields. Practical Example Illustrated in Fig. 7. We can show directions of solution curves of a given ODE (1) by drawing short straight-line segments (lineal elements) in the xy-plane. This gives a **direction field** (or *slope field*) into which you can then fit (approximate) solution curves. This may reveal typical properties of the whole family of solutions.

Figure 7 shows a direction field for the ODE

$$y' = y + x \tag{2}$$

obtained by a CAS (Computer Algebra System) and some approximate solution curves fitted in.

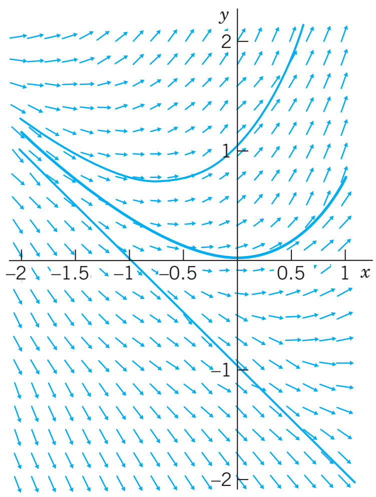

Fig. 7. Direction field of $y' = y + x$, with three approximate solution curves passing through (0, 1), (0, 0), (0, −1), respectively

If you have no CAS, first draw a few *level curves* $f(x, y) = \text{const}$ of $f(x, y)$, then parallel lineal elements along each such curve (which is also called an **isocline**, meaning a curve of equal inclination), and finally draw approximation curves fit to the lineal elements.

We shall now illustrate how numeric methods work by applying the simplest numeric method, that is Euler's method, to an initial value problem involving ODE (2). First we give a brief description of Euler's method.

Numeric Method by Euler

Given an ODE (1) and an initial value $y(x_0) = y_0$, **Euler's method** yields approximate solution values at equidistant x-values $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \cdots$, namely,

$$y_1 = y_0 + hf(x_0, y_0) \qquad \text{(Fig. 8)}$$

$$y_2 = y_1 + hf(x_1, y_1), \qquad \text{etc.}$$

In general,

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

where the step h equals, e.g., 0.1 or 0.2 (as in Table 1.1) or a smaller value for greater accuracy.

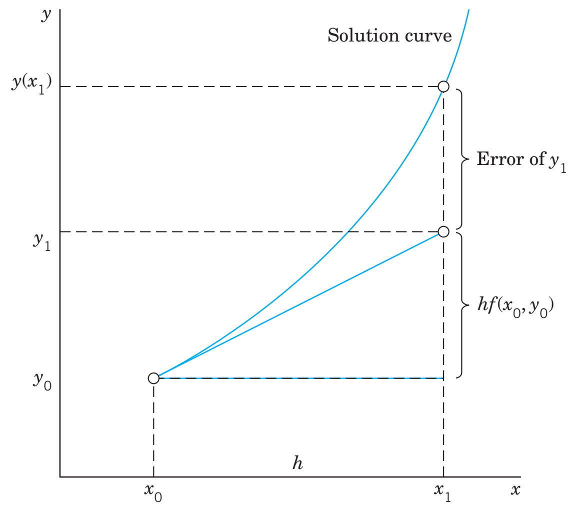

Fig. 8. First Euler step, showing a solution curve, its tangent at (x_0, y_0), step h and increment $hf(x_0, y_0)$ in the formula for y_1

Table 1.1 shows the computation of $n = 5$ steps with step $h = 0.2$ for the ODE (2) and initial condition $y(0) = 0$, corresponding to the middle curve in the direction field. We shall solve the ODE exactly in Sec. 1.5. For the time being, verify that the initial value problem has the solution $y = e^x - x - 1$. The solution curve and the values in Table 1.1 are shown in Fig. 9. These values are rather inaccurate. The errors $y(x_n) - y_n$ are shown in Table 1.1 as well as in Fig. 9. Decreasing h would improve the values, but would soon require an impractical amount of computation. Much better methods of a similar nature will be discussed in Sec. 21.1.

Table 1.1. **Euler method for $y' = y + x, y(0) = 0$ for $x = 0, \cdots, 1.0$ with step $h = 0.2$**

n	x_n	y_n	$y(x_n)$	Error
0	0.0	0.000	0.000	0.000
1	0.2	0.000	0.021	0.021
2	0.4	0.04	0.092	0.052
3	0.6	0.128	0.222	0.094
4	0.8	0.274	0.426	0.152
5	1.0	0.488	0.718	0.230

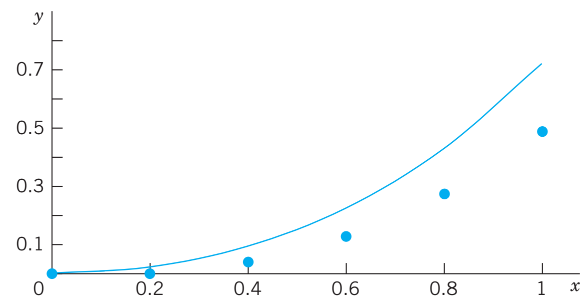

Fig. 9. Euler method: Approximate values in Table 1.1 and solution curve

PROBLEM SET 1.2

1–8 DIRECTION FIELDS, SOLUTION CURVES

Graph a direction field (by a CAS or by hand). In the field graph several solution curves by hand, particularly those passing through the given points (x, y).

1. $y' = 1 + y^2, \quad (\frac{1}{4}\pi, 1)$

2. $yy' + 4x = 0, \quad (1, 1), (0, 2)$

3. $y' = 1 - y^2, \quad (0, 0), (2, \frac{1}{2})$

4. $y' = 2y - y^2, \quad (0, 0), (0, 1), (0, 2), (0, 3)$

5. $y' = x - 1/y, \quad (1, \frac{1}{2})$

6. $y' = \sin^2 y, \quad (0, -0.4), (0, 1)$

7. $y' = e^{y/x}, \quad (2, 2), (3, 3)$

8. $y' = -2xy, \quad (0, \frac{1}{2}), (0, 1), (0, 2)$

9–10 ACCURACY OF DIRECTION FIELDS

Direction fields are very useful because they can give you an impression of all solutions without solving the ODE, which may be difficult or even impossible. To get a feel for the accuracy of the method, graph a field, sketch solution curves in it, and compare them with the exact solutions.

9. $y' = \cos \pi x$

10. $y' = -5y^{1/2}$ (Sol. $\sqrt{y} + \frac{5}{2}x = c$)

11. Autonomous ODE. This means an ODE not showing x (the independent variable) *explicitly*. (The ODEs in Probs. 6 and 10 are autonomous.) What will the level curves $f(x, y) = \text{const}$ (also called *isoclines* = curves of equal inclination) of an autonomous ODE look like? Give reason.

12–15 MOTIONS

Model the motion of a body B on a straight line with velocity as given, $y(t)$ being the distance of B from a point $y = 0$ at time t. Graph a direction field of the model (the ODE). In the field sketch the solution curve satisfying the given initial condition.

12. Product of velocity times distance constant, equal to 2, $y(0) = 2$.

13. Distance = Velocity × Time, $\quad y(1) = 1$

14. Square of the distance plus square of the velocity equal to 1, initial distance $1/\sqrt{2}$

15. Parachutist. Two forces act on a parachutist, the attraction by the earth mg (m = mass of person plus equipment, $g = 9.8 \text{ m/sec}^2$ the acceleration of gravity) and the air resistance, assumed to be proportional to the square of the velocity $v(t)$. Using **Newton's second law** of motion (mass × acceleration = resultant of the forces), set up a model (an ODE for $v(t)$). Graph a direction field (choosing m and the constant of proportionality equal to 1). Assume that the parachute opens when $v = 10$ m/sec. Graph the corresponding solution in the field. What is the limiting velocity? Would the parachute still be sufficient if the air resistance were only proportional to $v(t)$?

16. CAS PROJECT. Direction Fields. Discuss direction fields as follows.

(a) Graph portions of the direction field of the ODE (2) (see Fig. 7), for instance, $-5 \leqq x \leqq 2$, $-1 \leqq y \leqq 5$. Explain what you have gained by this enlargement of the portion of the field.

(b) Using implicit differentiation, find an ODE with the general solution $x^2 + 9y^2 = c\ (y > 0)$. Graph its direction field. Does the field give the impression that the solution curves may be semi-ellipses? Can you do similar work for circles? Hyperbolas? Parabolas? Other curves?

(c) Make a conjecture about the solutions of $y' = -x/y$ from the direction field.

(d) Graph the direction field of $y' = -\frac{1}{2}y$ and some solutions of your choice. How do they behave? Why do they decrease for $y > 0$?

17–20 EULER'S METHOD

This is the simplest method to explain numerically solving an ODE, more precisely, an initial value problem (IVP). (More accurate methods based on the same principle are explained in Sec. 21.1.) Using the method, to get a feel for numerics as well as for the nature of IVPs, solve the IVP numerically with a PC or a calculator, 10 steps. Graph the computed values and the solution curve on the same coordinate axes.

17. $y' = y, \quad y(0) = 1, \quad h = 0.1$

18. $y' = y, \quad y(0) = 1, \quad h = 0.01$

19. $y' = (y - x)^2, \quad y(0) = 0, \quad h = 0.1$
Sol. $y = x - \tanh x$

20. $y' = -5x^4y^2, \quad y(0) = 1, \quad h = 0.2$
Sol. $y = 1/(1 + x)^5$

1.3 Separable ODEs. Modeling

Many practically useful ODEs can be reduced to the form

$$g(y)\, y' = f(x) \tag{1}$$

by purely algebraic manipulations. Then we can integrate on both sides with respect to x, obtaining

$$\int g(y)\, y'\, dx = \int f(x)\, dx + c. \tag{2}$$

On the left we can switch to y as the variable of integration. By calculus, $y'\,dx = dy$, so that

$$\int g(y)\, dy = \int f(x)\, dx + c. \tag{3}$$

If f and g are continuous functions, the integrals in (3) exist, and by evaluating them we obtain a general solution of (1). This method of solving ODEs is called the **method of separating variables**, and (1) is called a **separable equation**, because in (3) the variables are now separated: x appears only on the right and y only on the left.

EXAMPLE 1 Separable ODE

The ODE $y' = 1 + y^2$ is separable because it can be written

$$\frac{dy}{1 + y^2} = dx. \qquad \text{By integration,} \qquad \arctan y = x + c \qquad \text{or} \qquad y = \tan(x + c).$$

It is very important to introduce the constant of integration immediately when the integration is performed. If we wrote $\arctan y = x$, then $y = \tan x$, and *then* introduced c, we would have obtained $y = \tan x + c$, which is not a solution (when $c \neq 0$). Verify this. ■

EXAMPLE 2 **Separable ODE**

The ODE $y' = (x + 1)e^{-x}y^2$ is separable; we obtain $y^{-2}\,dy = (x + 1)e^{-x}\,dx$.

By integration, $\quad -y^{-1} = -(x + 2)e^{-x} + c, \qquad y = \dfrac{1}{(x + 2)e^{-x} - c}.$ ■

EXAMPLE 3 **Initial Value Problem (IVP). Bell-Shaped Curve**

Solve $y' = -2xy$, $y(0) = 1.8$.

Solution. By separation and integration,

$$\frac{dy}{y} = -2x\,dx, \qquad \ln y = -x^2 + \tilde{c}, \qquad y = ce^{-x^2}.$$

This is the general solution. From it and the initial condition, $y(0) = ce^0 = c = 1.8$. Hence the IVP has the solution $y = 1.8e^{-x^2}$. This is a particular solution, representing a bell-shaped curve (Fig. 10). ■

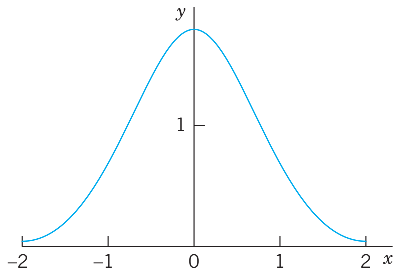

Fig. 10. Solution in Example 3 (bell-shaped curve)

Modeling

The importance of modeling was emphasized in Sec. 1.1, and separable equations yield various useful models. Let us discuss this in terms of some typical examples.

EXAMPLE 4 **Radiocarbon Dating**[2]

In September 1991 the famous Iceman (Oetzi), a mummy from the Neolithic period of the Stone Age found in the ice of the Oetztal Alps (hence the name "Oetzi") in Southern Tyrolia near the Austrian–Italian border, caused a scientific sensation. When did Oetzi approximately live and die if the ratio of carbon ${}^{14}_{6}C$ to carbon ${}^{12}_{6}C$ in this mummy is 52.5% of that of a living organism?

Physical Information. In the atmosphere and in living organisms, the ratio of radioactive carbon ${}^{14}_{6}C$ (made radioactive by cosmic rays) to ordinary carbon ${}^{12}_{6}C$ is constant. When an organism dies, its absorption of ${}^{14}_{6}C$ by breathing and eating terminates. Hence one can estimate the age of a fossil by comparing the radioactive carbon ratio in the fossil with that in the atmosphere. To do this, one needs to know the half-life of ${}^{14}_{6}C$, which is 5715 years (*CRC Handbook of Chemistry and Physics,* 83rd ed., Boca Raton: CRC Press, 2002, page 11–52, line 9).

Solution. ***Modeling.*** Radioactive decay is governed by the ODE $y' = ky$ (see Sec. 1.1, Example 5). By separation and integration (where t is time and y_0 is the initial ratio of ${}^{14}_{6}C$ to ${}^{12}_{6}C$)

$$\frac{dy}{y} = k\,dt, \qquad \ln|y| = kt + c, \qquad y = y_0 e^{kt} \qquad (y_0 = e^c).$$

[2]Method by WILLARD FRANK LIBBY (1908–1980), American chemist, who was awarded for this work the 1960 Nobel Prize in chemistry.

Next we use the half-life $H = 5715$ to determine k. When $t = H$, half of the original substance is still present. Thus,

$$y_0 e^{kH} = 0.5 y_0, \qquad e^{kH} = 0.5, \qquad k = \frac{\ln 0.5}{H} = -\frac{0.693}{5715} = -0.0001213.$$

Finally, we use the ratio 52.5% for determining the time t when Oetzi died (actually, was killed),

$$e^{kt} = e^{-0.0001213t} = 0.525, \qquad t = \frac{\ln 0.525}{-0.0001213} = 5312. \qquad \textit{Answer:} \qquad \text{About 5300 years ago.}$$

Other methods show that radiocarbon dating values are usually too small. According to recent research, this is due to a variation in that carbon ratio because of industrial pollution and other factors, such as nuclear testing. ■

EXAMPLE 5 **Mixing Problem**

Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank in Fig. 11 contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissoved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the amount of salt in the tank at any time t.

Solution. ***Step 1. Setting up a model.*** Let $y(t)$ denote the amount of salt in the tank at time t. Its time rate of change is

$$y' = \text{Salt inflow rate} - \text{Salt outflow rate} \qquad \textbf{Balance law.}$$

5 lb times 10 gal gives an inflow of 50 lb of salt. Now, the outflow is 10 gal of brine. This is $10/1000 = 0.01$ $(= 1\%)$ of the total brine content in the tank, hence 0.01 of the salt content $y(t)$, that is, 0.01 $y(t)$. Thus the model is the ODE

$$y' = 50 - 0.01y = -0.01(y - 5000). \tag{4}$$

Step 2. Solution of the model. The ODE (4) is separable. Separation, integration, and taking exponents on both sides gives

$$\frac{dy}{y - 5000} = -0.01\,dt, \qquad \ln|y - 5000| = -0.01t + c^*, \qquad y - 5000 = ce^{-0.01t}.$$

Initially the tank contains 100 lb of salt. Hence $y(0) = 100$ is the initial condition that will give the unique solution. Substituting $y = 100$ and $t = 0$ in the last equation gives $100 - 5000 = ce^0 = c$. Hence $c = -4900$. Hence the amount of salt in the tank at time t is

$$y(t) = 5000 - 4900e^{-0.01t}. \tag{5}$$

This function shows an exponential approach to the limit 5000 lb; see Fig. 11. Can you explain physically that $y(t)$ should increase with time? That its limit is 5000 lb? Can you see the limit directly from the ODE?

The model discussed becomes more realistic in problems on pollutants in lakes (see Problem Set 1.5, Prob. 35) or drugs in organs. These types of problems are more difficult because the mixing may be imperfect and the flow rates (in and out) may be different and known only very roughly. ■

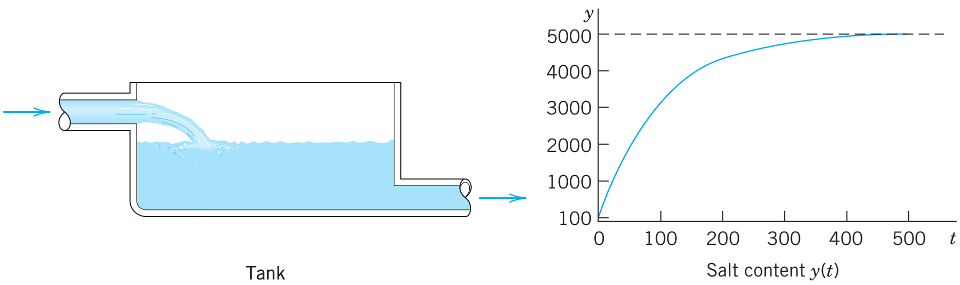

Fig. 11. Mixing problem in Example 5

EXAMPLE 6 **Heating an Office Building (Newton's Law of Cooling[3])**

Suppose that in winter the daytime temperature in a certain office building is maintained at 70°F. The heating is shut off at 10 P.M. and turned on again at 6 A.M. On a certain day the temperature inside the building at 2 A.M. was found to be 65°F. The outside temperature was 50°F at 10 P.M. and had dropped to 40°F by 6 A.M. What was the temperature inside the building when the heat was turned on at 6 A.M.?

Physical information. Experiments show that the time rate of change of the temperature T of a body B (which conducts heat well, for example, as a copper ball does) is proportional to the difference between T and the temperature of the surrounding medium **(Newton's law of cooling).**

Solution. ***Step 1. Setting up a model.*** Let $T(t)$ be the temperature inside the building and T_A the outside temperature (assumed to be constant in Newton's law). Then by Newton's law,

$$\frac{dT}{dt} = k(T - T_A). \tag{6}$$

Such experimental laws are derived under idealized assumptions that rarely hold exactly. However, even if a model seems to fit the reality only poorly (as in the present case), it may still give valuable qualitative information. To see how good a model is, the engineer will collect experimental data and compare them with calculations from the model.

Step 2. General solution. We cannot solve (6) because we do not know T_A, just that it varied between 50°F and 40°F, so we follow the ***Golden Rule:*** *If you cannot solve your problem, try to solve a simpler one.* We solve (6) with the unknown function T_A replaced with the average of the two known values, or 45°F. For physical reasons we may expect that this will give us a reasonable approximate value of T in the building at 6 A.M.

For constant $T_A = 45$ (or any other *constant* value) the ODE (6) is separable. Separation, integration, and taking exponents gives the general solution

$$\frac{dT}{T-45} = k\,dt, \qquad \ln|T - 45| = kt + c^*, \qquad T(t) = 45 + ce^{kt} \qquad (c = e^{c^*}).$$

Step 3. Particular solution. We choose 10 P.M. to be $t = 0$. Then the given initial condition is $T(0) = 70$ and yields a particular solution, call it T_p. By substitution,

$$T(0) = 45 + ce^0 = 70, \qquad c = 70 - 45 = 25, \qquad T_p(t) = 45 + 25e^{kt}.$$

Step 4. Determination of k. We use $T(4) = 65$, where $t = 4$ is 2 A.M. Solving algebraically for k and inserting k into $T_p(t)$ gives (Fig. 12)

$$T_p(4) = 45 + 25e^{4k} = 65, \qquad e^{4k} = 0.8, \qquad k = \tfrac{1}{4}\ln 0.8 = -0.056, \qquad T_p(t) = 45 + 25e^{-0.056t}.$$

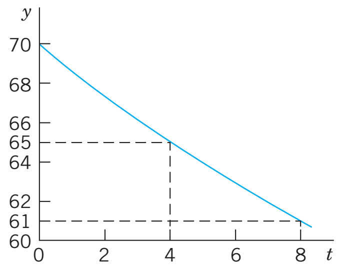

Fig. 12. Particular solution (temperature) in Example 6

[3]Sir ISAAC NEWTON (1642–1727), great English physicist and mathematician, became a professor at Cambridge in 1669 and Master of the Mint in 1699. He and the German mathematician and philosopher GOTTFRIED WILHELM LEIBNIZ (1646–1716) invented (independently) the differential and integral calculus. Newton discovered many basic physical laws and created the method of investigating physical problems by means of calculus. His *Philosophiae naturalis principia mathematica* (*Mathematical Principles of Natural Philosophy,* 1687) contains the development of classical mechanics. His work is of greatest importance to both mathematics and physics.

Step 5. Answer and interpretation. 6 A.M. is $t = 8$ (namely, 8 hours after 10 P.M.), and

$$T_p(8) = 45 + 25e^{-0.056 \cdot 8} = 61[°\text{F}].$$

Hence the temperature in the building dropped 9°F, a result that looks reasonable. ■

EXAMPLE 7 Leaking Tank. Outflow of Water Through a Hole (Torricelli's Law)

This is another prototype engineering problem that leads to an ODE. It concerns the outflow of water from a cylindrical tank with a hole at the bottom (Fig. 13). You are asked to find the height of the water in the tank at any time if the tank has diameter 2 m, the hole has diameter 1 cm, and the initial height of the water when the hole is opened is 2.25 m. When will the tank be empty?

Physical information. Under the influence of gravity the outflowing water has velocity

$$v(t) = 0.600\sqrt{2gh(t)} \qquad \textbf{(Torricelli's law}^4\textbf{)}, \tag{7}$$

where $h(t)$ is the height of the water above the hole at time t, and $g = 980 \text{ cm/sec}^2 = 32.17 \text{ ft/sec}^2$ is the acceleration of gravity at the surface of the earth.

Solution. ***Step 1. Setting up the model.*** To get an equation, we relate the decrease in water level $h(t)$ to the outflow. The volume ΔV of the outflow during a short time Δt is

$$\Delta V = Av\,\Delta t \qquad (A = \text{Area of hole}).$$

ΔV must equal the change ΔV^* of the volume of the water in the tank. Now

$$\Delta V^* = -B\,\Delta h \qquad (B = \text{Cross-sectional area of tank})$$

where $\Delta h\ (> 0)$ is the decrease of the height $h(t)$ of the water. The minus sign appears because the volume of the water in the tank decreases. Equating ΔV and ΔV^* gives

$$-B\,\Delta h = Av\,\Delta t.$$

We now express v according to Torricelli's law and then let Δt (the length of the time interval considered) approach 0—this is a ***standard way*** of obtaining an ODE as a model. That is, we have

$$\frac{\Delta h}{\Delta t} = -\frac{A}{B}v = -\frac{A}{B}0.600\sqrt{2gh(t)}$$

and by letting $\Delta t \to 0$ we obtain the ODE

$$\frac{dh}{dt} = -26.56\frac{A}{B}\sqrt{h},$$

where $26.56 = 0.600\sqrt{2 \cdot 980}$. This is our model, a first-order ODE.

Step 2. General solution. Our ODE is separable. A/B is constant. Separation and integration gives

$$\frac{dh}{\sqrt{h}} = -26.56\frac{A}{B}dt \qquad \text{and} \qquad 2\sqrt{h} = c^* - 26.56\frac{A}{B}t.$$

Dividing by 2 and squaring gives $h = (c - 13.28At/B)^2$. Inserting $13.28A/B = 13.28 \cdot 0.5^2\pi/100^2\pi = 0.000\,332$ yields the general solution

$$h(t) = (c - 0.000\,332t)^2.$$

[4] EVANGELISTA TORRICELLI (1608–1647), Italian physicist, pupil and successor of GALILEO GALILEI (1564–1642) at Florence. The "contraction factor" 0.600 was introduced by J. C. BORDA in 1766 because the stream has a smaller cross section than the area of the hole.

Step 3. Particular solution. The initial height (the initial condition) is $h(0) = 225$ cm. Substitution of $t = 0$ and $h = 225$ gives from the general solution $c^2 = 225$, $c = 15.00$ and thus the particular solution (Fig. 13)

$$h_p(t) = (15.00 - 0.000\,332t)^2.$$

Step 4. Tank empty. $h_p(t) = 0$ if $t = 15.00/0.000\,332 = 45{,}181\ [\text{sec}] = 12.6$ [hours].

Here you see distinctly the ***importance of the choice of units***—we have been working with the cgs system, in which time is measured in seconds! We used $g = 980\ \text{cm/sec}^2$.

Step 5. Checking. Check the result. ■

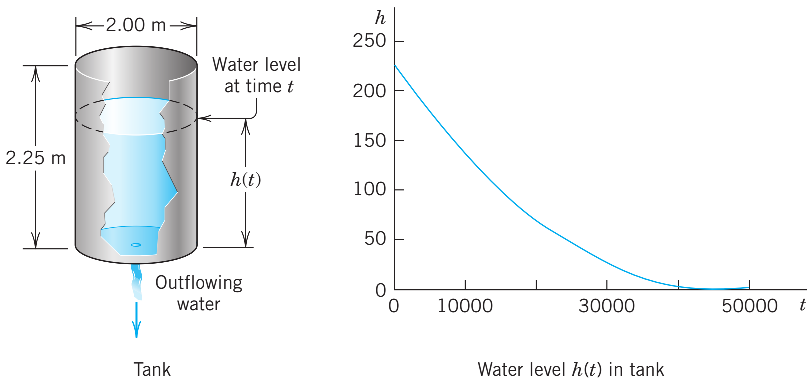

Fig. 13. Example 7. Outflow from a cylindrical tank ("leaking tank"). Torricelli's law

Extended Method: Reduction to Separable Form

Certain nonseparable ODEs can be made separable by transformations that introduce for y a new unknown function. We discuss this technique for a class of ODEs of practical importance, namely, for equations

$$y' = f\left(\frac{y}{x}\right). \tag{8}$$

Here, f is any (differentiable) function of y/x, such as $\sin(y/x)$, $(y/x)^4$, and so on. (Such an ODE is sometimes called a *homogeneous ODE*, a term we shall not use but reserve for a more important purpose in Sec. 1.5.)

The form of such an ODE suggests that we set $y/x = u$; thus,

$$y = ux \qquad \text{and by product differentiation} \qquad y' = u'x + u. \tag{9}$$

Substitution into $y' = f(y/x)$ then gives $u'x + u = f(u)$ or $u'x = f(u) - u$. We see that if $f(u) - u \neq 0$, this can be separated:

$$\frac{du}{f(u) - u} = \frac{dx}{x}. \tag{10}$$

EXAMPLE 8 **Reduction to Separable Form**

Solve

$$2xyy' = y^2 - x^2.$$

Solution. To get the usual explicit form, divide the given equation by $2xy$,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substitute y and y' from (9) and then simplify by subtracting u on both sides,

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

You see that in the last equation you can now separate the variables,

$$\frac{2u\,du}{1 + u^2} = -\frac{dx}{x}. \qquad \text{By integration,} \qquad \ln(1 + u^2) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

Take exponents on both sides to get $1 + u^2 = c/x$ or $1 + (y/x)^2 = c/x$. Multiply the last equation by x^2 to obtain (Fig. 14)

$$x^2 + y^2 = cx. \qquad \text{Thus} \qquad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

This general solution represents a family of circles passing through the origin with centers on the x-axis. ■

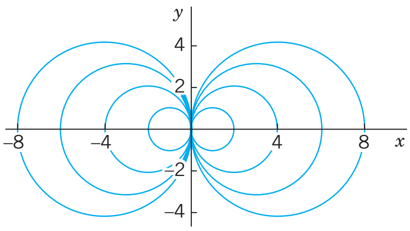

Fig. 14. General solution (family of circles) in Example 8

PROBLEM SET 1.3

1. **CAUTION! Constant of integration.** Why is it important to introduce the constant of integration immediately when you integrate?

2–10 GENERAL SOLUTION

Find a general solution. Show the steps of derivation. Check your answer by substitution.

2. $y^3y' + x^3 = 0$
3. $y' = \sec^2 y$
4. $y' \sin 2\pi x = \pi y \cos 2\pi x$
5. $yy' + 36x = 0$
6. $y' = e^{2x-1}y^2$
7. $xy' = y + 2x^3 \sin^2 \dfrac{y}{x}$ (Set $y/x = u$)
8. $y' = (y + 4x)^2$ (Set $y + 4x = v$)
9. $xy' = y^2 + y$ (Set $y/x = u$)
10. $xy' = x + y$ (Set $y/x = u$)

11–17 INITIAL VALUE PROBLEMS (IVPs)

Solve the IVP. Show the steps of derivation, beginning with the general solution.

11. $xy' + y = 0$, $y(4) = 6$
12. $y' = 1 + 4y^2$, $y(1) = 0$
13. $y' \cosh^2 x = \sin^2 y$, $y(0) = \frac{1}{2}\pi$
14. $dr/dt = -2tr$, $r(0) = r_0$
15. $y' = -4x/y$, $y(2) = 3$
16. $y' = (x + y - 2)^2$, $y(0) = 2$ (Set $v = x + y - 2$)
17. $xy' = y + 3x^4 \cos^2(y/x)$, $y(1) = 0$ (Set $y/x = u$)
18. **Particular solution.** Introduce limits of integration in (3) such that y obtained from (3) satisfies the initial condition $y(x_0) = y_0$.

19–36 MODELING, APPLICATIONS

19. Exponential growth. If the growth rate of the number of bacteria at any time t is proportional to the number present at t and doubles in 1 week, how many bacteria can be expected after 2 weeks? After 4 weeks?

20. Another population model.

(a) If the birth rate and death rate of the number of bacteria are proportional to the number of bacteria present, what is the population as a function of time.

(b) What is the limiting situation for increasing time? Interpret it.

21. Radiocarbon dating. What should be the ${}^{14}_{6}C$ content (in percent of y_0) of a fossilized tree that is claimed to be 3000 years old? (See Example 4.)

22. Linear accelerators are used in physics for accelerating charged particles. Suppose that an alpha particle enters an accelerator and undergoes a constant acceleration that increases the speed of the particle from 10^3 m/sec to 10^4 m/sec in 10^{-3} sec. Find the acceleration a and the distance traveled during that period of 10^{-3} sec.

23. Boyle–Mariotte's law for ideal gases.[5] Experiments show for a gas at low pressure p (and constant temperature) the rate of change of the volume $V(p)$ equals $-V/p$. Solve the model.

24. Mixing problem. A tank contains 400 gal of brine in which 100 lb of salt are dissolved. Fresh water runs into the tank at a rate of 2 gal/min.The mixture, kept practically uniform by stirring, runs out at the same rate. How much salt will there be in the tank at the end of 1 hour?

25. Newton's law of cooling. A thermometer, reading 5°C, is brought into a room whose temperature is 22°C. One minute later the thermometer reading is 12°C. How long does it take until the reading is practically 22°C, say, 21.9°C?

26. Gompertz growth in tumors. The Gompertz model is $y' = -Ay \ln y$ $(A > 0)$, where $y(t)$ is the mass of tumor cells at time t. The model agrees well with clinical observations. The declining growth rate with increasing $y > 1$ corresponds to the fact that cells in the interior of a tumor may die because of insufficient oxygen and nutrients. Use the ODE to discuss the growth and decline of solutions (tumors) and to find constant solutions. Then solve the ODE.

27. Dryer. If a wet sheet in a dryer loses its moisture at a rate proportional to its moisture content, and if it loses half of its moisture during the first 10 min of drying, when will it be practically dry, say, when will it have lost 99% of its moisture? First guess, then calculate.

28. Estimation. Could you see, practically without calculation, that the answer in Prob. 27 must lie between 60 and 70 min? Explain.

29. Alibi? Jack, arrested when leaving a bar, claims that he has been inside for at least half an hour (which would provide him with an alibi). The police check the water temperature of his car (parked near the entrance of the bar) at the instant of arrest and again 30 min later, obtaining the values 190°F and 110°F, respectively. Do these results give Jack an alibi? (Solve by inspection.)

30. Rocket. A rocket is shot straight up from the earth, with a net acceleration (= acceleration by the rocket engine minus gravitational pullback) of $7t$ m/sec^2 during the initial stage of flight until the engine cut out at $t = 10$ sec. How high will it go, air resistance neglected?

31. Solution curves of $y' = g(y/x)$. Show that any (nonvertical) straight line through the origin of the xy-plane intersects all these curves of a given ODE at the same angle.

32. Friction. If a body slides on a surface, it experiences friction F (a force against the direction of motion). Experiments show that $|F| = \mu|N|$ (*Coulomb's*[6] *law of kinetic friction without lubrication*), where N is the normal force (force that holds the two surfaces together; see Fig. 15) and the constant of proportionality μ is called the *coefficient of kinetic friction*. In Fig. 15 assume that the body weighs 45 nt (about 10 lb; see front cover for conversion). $\mu = 0.20$ (corresponding to steel on steel), $a = 30°$, the slide is 10 m long, the initial velocity is zero, and air resistance is negligible. Find the velocity of the body at the end of the slide.

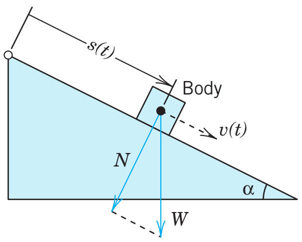

Fig. 15. Problem 32

[5]ROBERT BOYLE (1627–1691), English physicist and chemist, one of the founders of the Royal Society. EDME MARIOTTE (about 1620–1684), French physicist and prior of a monastry near Dijon. They found the law experimentally in 1662 and 1676, respectively.

[6]CHARLES AUGUSTIN DE COULOMB (1736–1806), French physicist and engineer.