Chapter 4
Induction and Recursion
Quy nạp và Đệ quy

Objectives

- Mathematical Induction
- Strong Induction and Well-Ordering
- Recursive Definitions and Structural Induction
- Recursive Algorithms
- Program Correctness

4.1- Mathematical Induction

- Introduction
- Mathematical Induction
- Examples of Proofs by Mathematical Induction

Principle of Mathematical Induction

<u>Principle of Mathematical Induction</u>

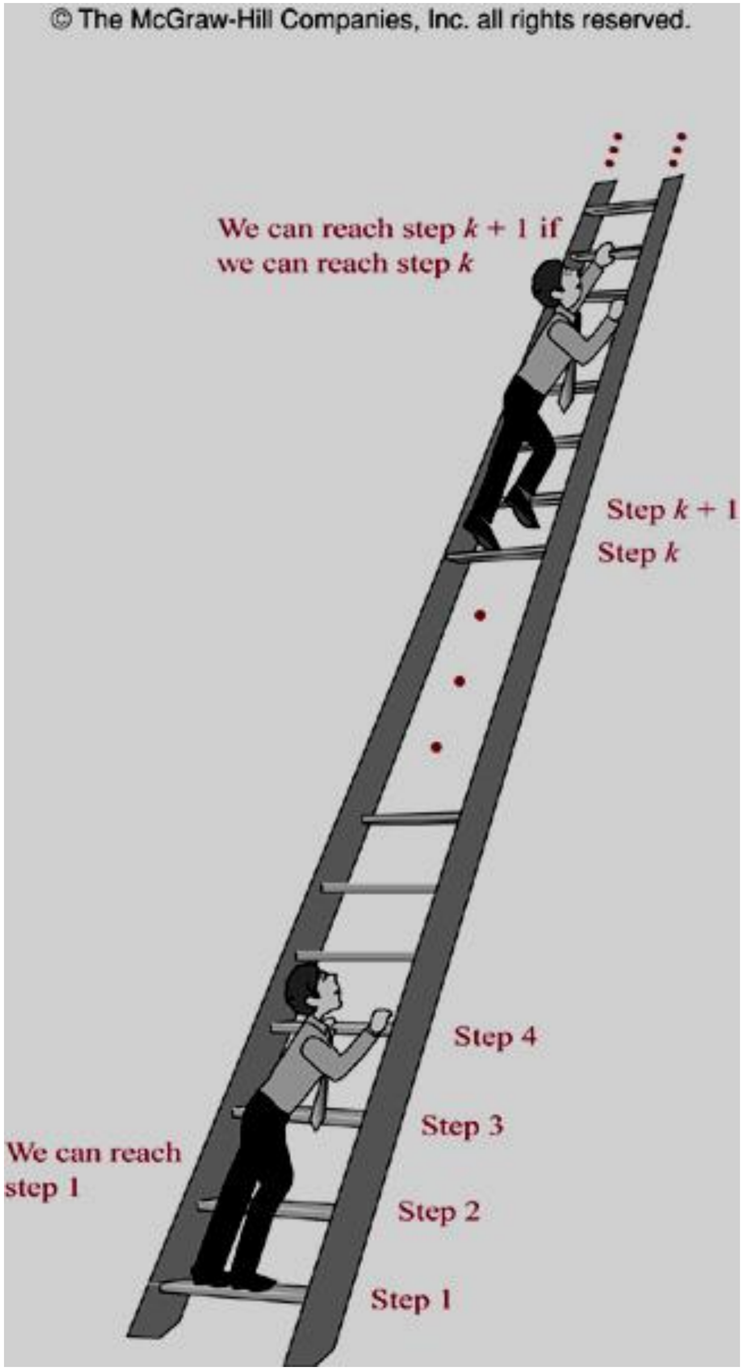
To prove P(n) is true for all possible integers n, where P(n) is a propositional function, we complete two step:

Basic step:

Verifying P(1) is true

Inductive step:

Show P(k) ⟶ P(k+1) is true for all k>0



Induction: Example 1

Prove that $1 + 2 + 3 + \ldots + n = n(n+1)/2$ for all integers $n>0$

Solution.

Let P(n)=" $1+2+3+\ldots+ n = n(n+1)/2$".

- Basic step: P(1) = "$1 = 1(1+1)/2$" → true
- Inductive step: With arbitrary $k>0$,

P(k) =" $1+ 2+\ldots+ k = k(k+1)/2$" is true.

We have

$$\underline{1+2+3+\ldots+k}+(k+1) = k(k+1)/2 + (k+1)$$
$$= [k(k+1)+ 2(k+1)]/2$$
$$= (k+1)(k+2)/2$$
$$= (k+1)((k+1)+1)/2$$

P(k+1)="$1+2+3+\ldots+k+1 = (k+1)(k+2)/2$" is true.

P(k) → P(k+1): true

Proved.

Example 2 p.268

- Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.
- *Solution.*

  The sum of the first n positive odd integers for n=1, 2, 3, 4, 5 are:

  1=1, 1+3=4, 1+3+5=9,

  1+3+5+7=16, 1+3+5+7+9=25.
- *Conjecture*: **$1+3+5+\ldots+(2n-1)=n^2$**.
- *Proof.* Let P(n)="**$1+3+5+\ldots+(2n-1)=n^2$**."

  - Basic step. P(1)="1=1" is true.

  - Inductive step. $(P(k) \rightarrow P(k+1))$ is true.

  Suppose P(k) is true. That is, "$1+3+5+\ldots+(2k-1)=k^2$"

  We have, $\underline{1+3+5+\ldots+(2k-1)}+(2k+1)=\underline{k^2}+2k+1 = (k+1)^2$.

  So, P(k+1) is true.

Proved.

Induction: Examples 2..13 – pages: 268..278

- $1+3+5+\ldots+(2n-1) = n^2$
- $2^0+2^1+2^2+2^3+\ldots+2^n = \sum 2^n = 2^{n+1}-1$
- $\sum ar^j = a + ar + ar^2 + \ldots + ar^n = (ar^{n+1}-a)/(r-1)$
- $n < 2^n$
- $2^n < n!\ ,\ n>3$
- n^3-n is divisible by 3, n is positive integer
- The number of subsets of a finite set: a set with n elements has 2^n subsets.
- ….
- Let $H(j)= 1/1 + ½ + 1/3 + \ldots + 1/j$

  Prove that $H(2^n) \geq 1 + n/2$ for all $n \geq 0$

4.2- Strong Induction and Well-Ordering

Principle of Strong Induction

To prove P(n) is true for all positive integers n, where P(n) is a propositional function, two steps are performed:

Basic step:

Verifying P(1) is true

Inductive step:

Show $[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$ is true for all $k>0$

Strong Induction: Example 1

Prove that if n is an integer greater than 1, then n can be written as the product of primes

P(n) : n can be written as the product of primes

Basic steps: P(2) = true // 2=2 , product of 1 primes

P(4) = true // 4=2.2

Inductive step:

Assumption: P(j)=true for all positive j $\leq$ k

- Case k+1 is a prime ➔ P(k+1) =true
- Case k+1 is a composite ➔ k+1= ab, 2 $\leq$ a $\leq$ b<k+1

➔ P(k) is true

Strong Induction: Example 2

Prove that every amount of postage of 12 cents or more can be formed using just 4-cents and 5-cents stamps

P(n) : “ n cents can be formed using just 4-cent and 5-cent stamps”

P(12) is true : 12 cents = 3. 4 cents

P(13) is true : 13= 2.4 + 1.5

P(14) is true: 14= 1.4 + 2.5

P(15) is true: 15= 3.5

Assumption: P(j) is true with 12 $\le$**j** $\le$ **k, k>15 ➔ P(k-3) is true**

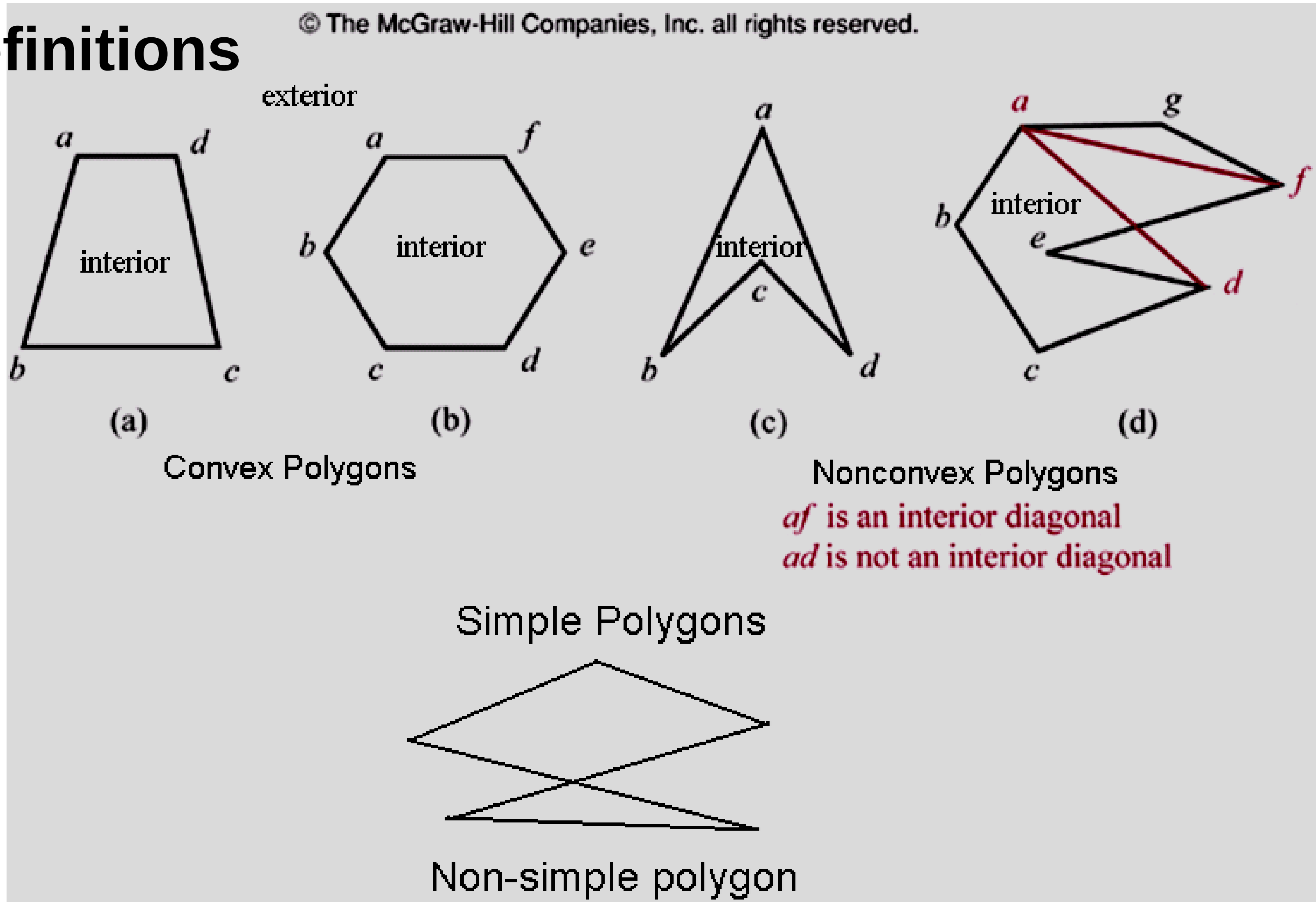
k+1= (k-3) + 4 , k>12

➔ **P(k+1) is true because k+1 is the result of adding a 4-cent stamp to the amount k-3**

➔ **Proved**

Using Strong Induction in Computational Geometry

- **Definitions**



Strong Induction in Computational Geometry

Lemma 1: Every simple polygon has an interior diagonal.

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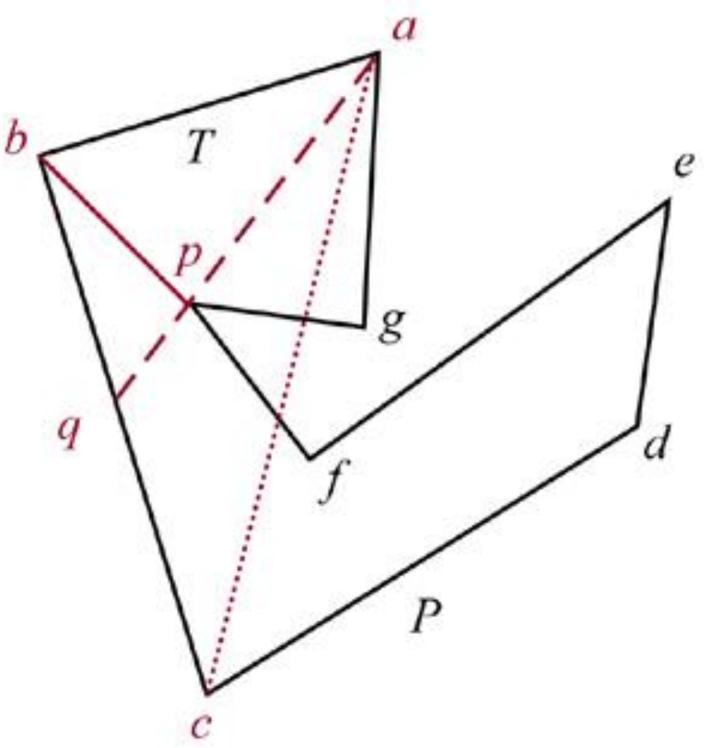

T is the triangle abc

p is the vertex of P inside T such that the $\angle bap$ is smallest

bp must be an interior diagonal of P

Proof: page 290

Strong Induction in Computational Geometry

Theorem 1:

A simple polygon with n sides, where n is integer with $n \geq 3$, can be triangulated into n-2 triangles

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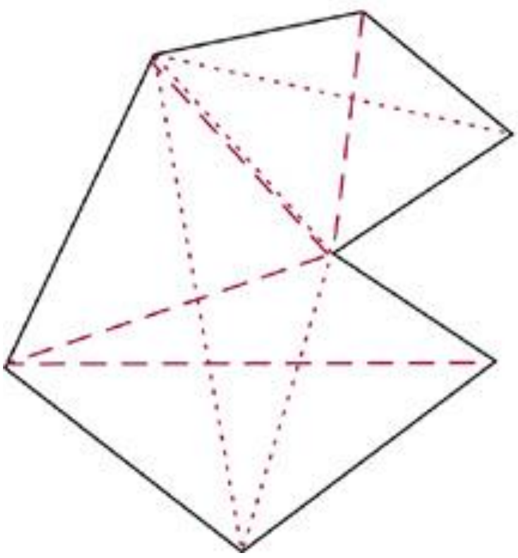

Two different triangulations of a simple polygon with seven sides into five triangles, shown with dotted lines and with dashed lines, respectively

Proof: page 289

4.3- Recursive Definition and Structural Induction

- Introduction
- Recursively Defined Functions
- Recursively Defined Sets and Structures
- Structural Induction
- Generalized Induction
- Recursive Algorithms

Recursion: Introduction

- Objects/ functions may be difficultly defined.
- Define an object/function in terms of itself
- Examples:

$$2^n = \begin{cases} 1, n = 0 \\ 2.2^{n-1}, n > 0 \end{cases}$$

$$\sum_{i=0}^{n} i = \begin{cases} 0, n = 0 \\ n + \sum_{i=0}^{n-1} i, n > 0 \end{cases}$$

```
#include <stdio.h>
#include <conio.h>
double compute2n( int n)
{ if (n==0) return 1;
  return 2*compute2n(n-1);
}
void main()
{ clrscr();
  int n;
  scanf("%d",&n);
  double r=compute2n(n);
  printf("%lf",r);
  getch();
}
```

Recursively Defined Functions

- Recursive (inductive) function

  Two steps to define a function with the set of nonnegative integers as its domain:
- Basis step: Specify the value of the function at zero.
- Recursive step: Give a rule for finding its value at an integer from its values at smaller integers
- Example: Find f(1), f(2), f(4),f(6) of the following function:

$$f(n) = \begin{cases} n, n < 3 \\ 3n + f(n-1), n \geq 3 \end{cases}$$

Recursively Defined Functions

- Example: Give the recursive definition of $\sum a_i$, i=0..k
- Basis step: $\sum a_i = a_0$, i=0
- Inductive step:

$$\boxed{a_0 + a_1 + \ldots + a_{k-1}} + a_k$$

$(\sum a_i$, i=0..k-1)

$\sum a_i = a_k + (\sum a_i$, i=0..k-1)

Recursively Defined Functions

Definition 1: **Fibonacci numbers** $f(n) = \begin{cases} 1, n = 0,1 \\ f(n-1) + f(n-2), n > 1 \end{cases}$

Theorem 1: Lamé's theorem: Let a,b be integers, a ≥ b. Then the number of divisions used by the Euclidean algorithm to find gcd(a,b) is less than or equal to five times the number of decimal digits in b.

Proof: page 298

Example gcd(25,7), b= 7 , 1 digit

x	y	r
25	7	25 mod 7=4
7	4	7 mod 4=3
4	3	4 mod 3=1
3	1	3 mod 1=0 (4 divisions)
1	0	Stop

```
procedure  gcd(a,b)
 x:=a;   y:=b
 while y ≠ 0
  begin
    r := x mod y
    x:=y
    y:= r
   end { gcd(a,b) is x}
```

Recursively Defined Sets and Structures

- Example $S = \{ 3,6,9,12,15, 18,21,\ldots\}$

Step 1: $3 \in S$

Step 2: If $x \in S$ and $y \in S$ then $x+y \in S$

- **Definition 2**: The set $\sum^*$ of string over alphabet $\sum$ can be defined recursively by:

Basis step: $\lambda \in \sum^*$, λ is the empty string with no symbols

Recursive step: If $w \in \sum^*$ and $x \in \sum$ then $wx \in \sum^*$

Example: $\sum = \{0,1\}$ ➔ $\sum^*$ is the set of string made by 0 and 1 with arbitrary length and arbitrary order of symbols 0 and 1

Recursively Defined Sets and Structures

- **Definition 3**: String Concatenation

Basis step: If w $\in \Sigma^*$ then w.λ=w, λ is the empty string

Recursive step: If $w_1 \in \Sigma^*$ and $w_2 \in \Sigma^*$ and $x \in \Sigma$ then $w_1.(w_2x) = (w_1.w_2)x$

Example: $\Sigma = \{0,1\} \rightarrow \Sigma^*$ is the set of string made by 0 and 1 with arbitrary length and arbitrary order of symbols 0 and 1

4.4- Recursive Algorithms

- **Definition 1**: An algorithm is called recursive if it solves a problem by reducing it to an instance of the same problem with smaller input.

Example: Recursive algorithm for computing n!

procedure factorial (n: nonnegative integer)
  if n=0 **then** factorial(n) :=1
  else factorial(n) = n.factorial(n-1)

n!= 1 , n=0
n!= 1.2.3.4…n = n.(n-1)!, n>0

Recursive Algorithms…

Example: Recursive algorithm for computing a^n

procedure power (a: nonzero real number
n: nonnegative integer)
if n=0 **then** power(a,n) :=1
else power(a,n)=a.power(a,n-1)

$a^n = 1$, $n=0$
$a^n = a.a.a...a = a.a^{n-1}$, $n>0$

Recursive Algorithms…

Example: Recursive algorithm for computing b^n mod m
$m \geq 2$, $n \geq 0$, $1 \leq b < m$.

$b^n \bmod m = (b.(b^{n-1} \bmod m) \bmod m$

$b^0 \bmod m = 1$

Using division to improve performance: (n steps backward to 0 faster)

If n is even → $b^n = b^{n/2}.b^{n/2}$

→ $b^n \bmod m = ((b^{n/2} \bmod m). (b^{n/2} \bmod m)) \bmod m$

→ $b^n \bmod m = (b^{n/2} \bmod m)^2 \bmod m$

If n is odd→ $b^n = b.b^{\lfloor n/2 \rfloor}.b^{\lfloor n/2 \rfloor}$

→ $b^n \bmod m = ([(b^{\lfloor n/2 \rfloor} \bmod m)^2 \bmod m].(b \bmod m))\bmod m$

Algorithm: page 313

Recursive Algorithms…

Example: Recursive algorithm for computing gcd(a,b)

a,b: non negative integer, a < b

If a>b then swap a,b
gcd(a,b)=b , a=0
gcd (a,b) = gcd(b mod a, a)

Algorithm: page 313

Recursive Algorithms…

Example: Recursive algorithm for linear searching the value x in the sequence

a_i, a_{i+1},…, a_j, sub-sequence of a_n.

$1 \le i \le n$, $1 \le j \le n$

$i>j \rightarrow$ location $=0$

$a_i=x \rightarrow$ location $= i$

location (i, j, x) = location (i+1, j, x)

Algorithm: page 314 – You should modify it.

Recursive Algorithms…

Example: Recursive algorithm for binary searching the value x in the increasingly ordered sequence $a_i, a_{i+1}, \ldots, a_{j-1}$, sub-sequence of a_n. $1 \leq i \leq n, 1 \leq j \leq n$

```
procedure  binary-search(x, i, j)
if  i>j then location=0
m= ⌊(i+j)/2⌋
 if x= a_m then location =m
else if  x< a_m then location= binary-search(x, i, m-1)
else  location= binary-search(x, m+1, j)
```

Algorithm: page 314 – You should modify it.

Proving Recursive Algorithms Correct

- Using mathematical induction.
- Example: prove the algorithm that computes n! is correct.

```
procedure f (n: nonnegative integer)
  if n=0 then f(n) :=1
  else f(n) = n.f(n-1)
```

If n=0, first step of the algorithm tells us f(0)=1 ➔ true

Assuming f(n) is true for all n≥0

f(n)= 1.2.3….(n)

(n+1).f(n)= 1.2.3…n.(n+1) = (n+1)!

f(n+1) = (n+1)!

Conclusion: f(n) is true for all integer n, n≥0

More examples: Page 315

Recursion and Iteration

procedure rfibo (n: nonnegative integer)
If n=0 **then** rFibo(0)=0
Else if n=1 **then** rFibo(1)=1
Else rFibo(n) := rFibo(n-2) + rFibo(n-1)

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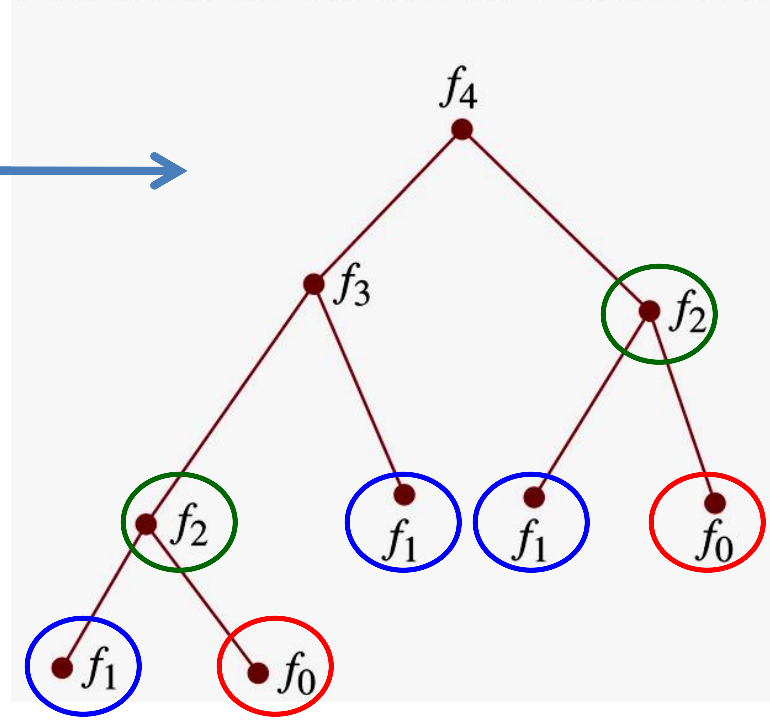

procedure iFibo (n: nonnegative integer)
If n=0 **then** y:=0
Else if n=1 **then** y:=1
Else Begin
x:=0 ; y:=1
for i:= 2 **to** n
Begin
z:= x+y; x:= y; y:=z
End
End { iFibo(n) = z}

Recursive algorithm uses far more computation than iterative one

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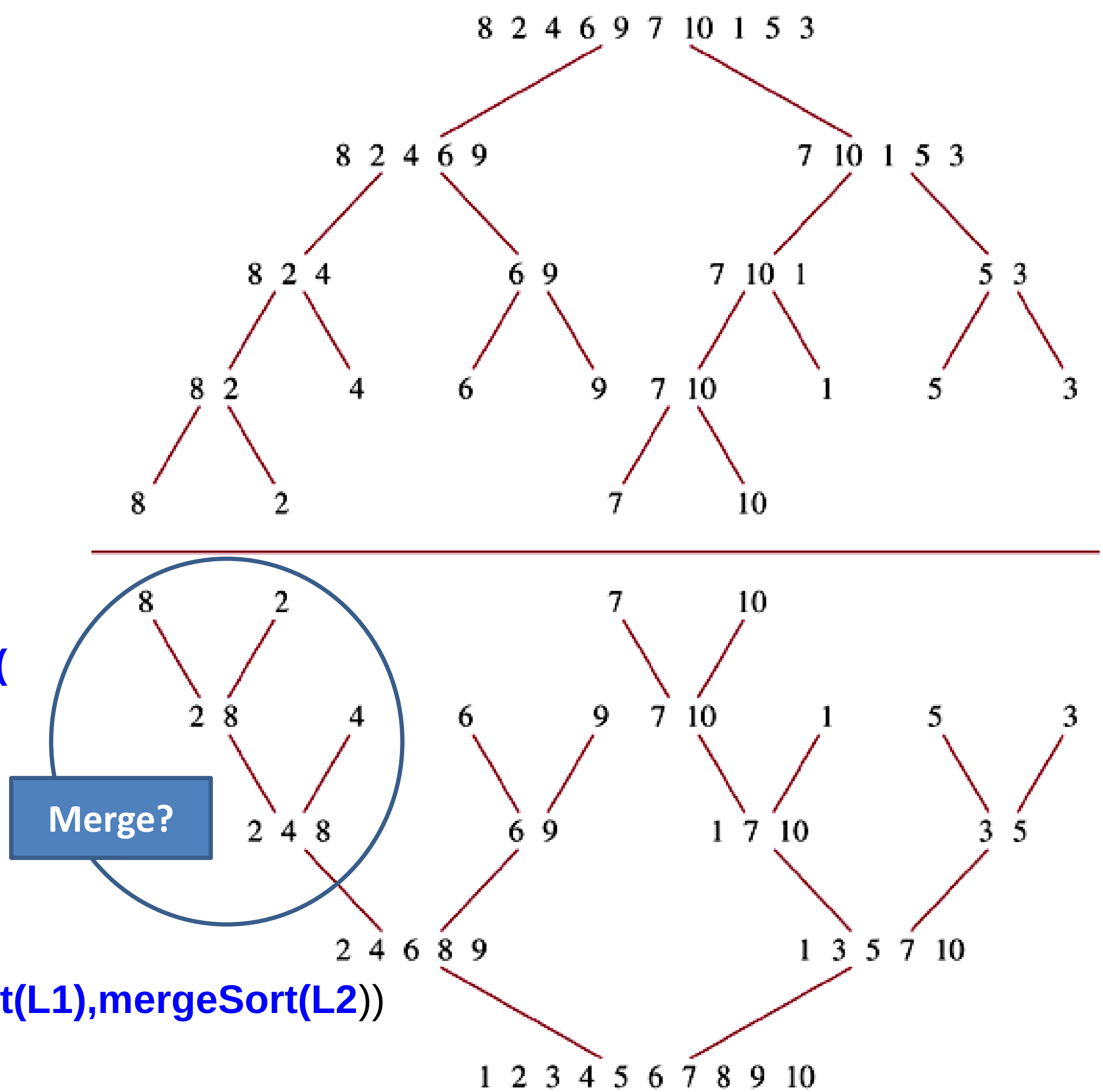
Merge Sort



Procedure mergeSort ($L=a_1 \ldots a_n$)

If $n>1$ **then**

Begin

  $m := \lfloor n/2 \rfloor$

  $L1 := a_1 \ldots a_m$

  $L2 := a_{m+1} \ldots a_n$

  L:=merge(mergeSort(L1),mergeSort(L2))

End

{L is sorted}

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Merge Sort

L1: 1 2 2 5 7 9 12 15 17 19

L2: 3 5 8 9 11 15

$\leq$

L : 1 2 2 3 5 5 7 8 9 9 11 12 15 15 17 19

- Merge two sorted lists L1, L2 to list L, an increasing ordered list.

procedure Merge (L1, L2: sorted list)
L:= empty list
While L1 and L2 are both no empty
Begin
remove smaller of first element of L1 and L2 and put it to the right end of L
if removal of this element makes one list empty then remove all elements from the other list and append them to L
End { L has increasing order }

Theorem 1: The number of comparisons needed to merge sort a list with n elements is O(nlog n)

Thanks