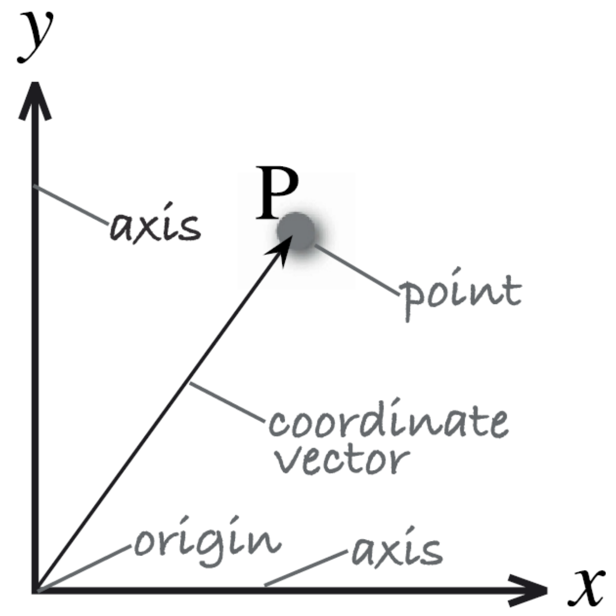
Lecture 10

Pose in two and three dimensions

CS 3630

Reference Frames

- Robotics is all about management of reference frames
  - **Perception** is about estimation of reference frames
  - **Planning** is how to move reference frames
  - **Control** is the implementation of trajectories for reference frames
- The relation between references frames is essential to a successful system



Why study planar (2D) reference frames?

- Ground robots operate in a plane (e.g., the floor they are on). As a result, fundamental navigation capabilities, such as localization and path planning, rely on being able to correctly represent transformations between 2D reference frames.

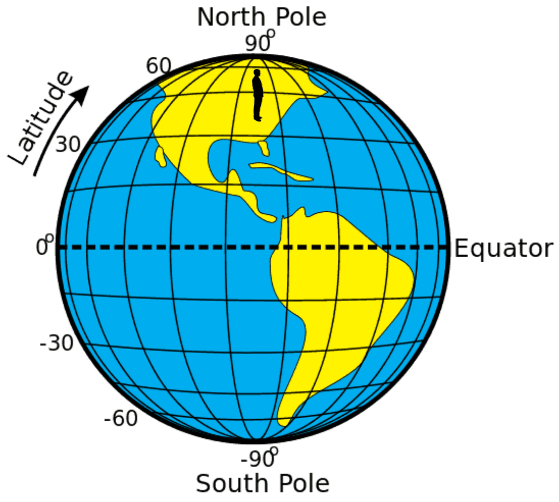
- Also, understanding 2D reference frames serves as a foundation for higher order transforms.

In the figure on the right, red represents the **global** reference frame, the (0,0) coordinate for this house. Purple represents the robot's **local** reference frame.

You can think of the red reference frame on the right, as latitude and longitude coordinates.

When a robot sees a couch in front of it, it first determines the couch's location in local coordinates. For example:

- 200 centimeters in front of me, or
- (200, 0)



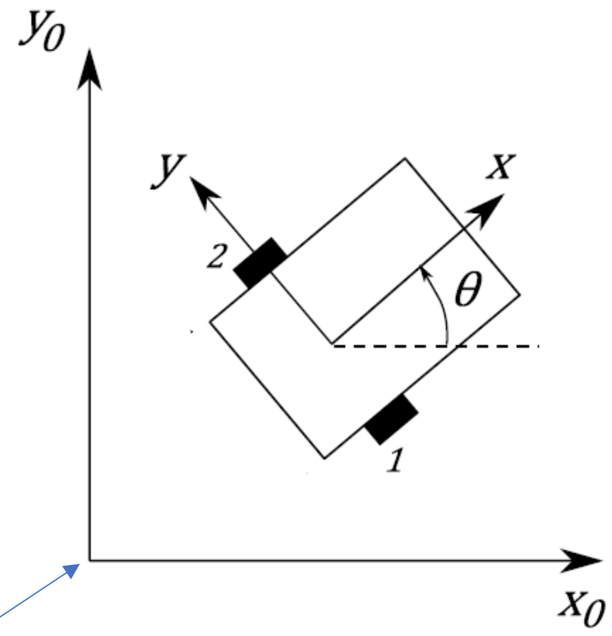
Similarly, a person standing in Atlanta would say the couch is in front of them.

But, if necessary, we could translate the location of the Atlanta couch as being at (33.777302, -84.398615) in latitude and longitude.

And similarly, the robot could say that the couch is located at (20, 250) in its house global coordinates.

Our goal is to learn to perform these transformations.

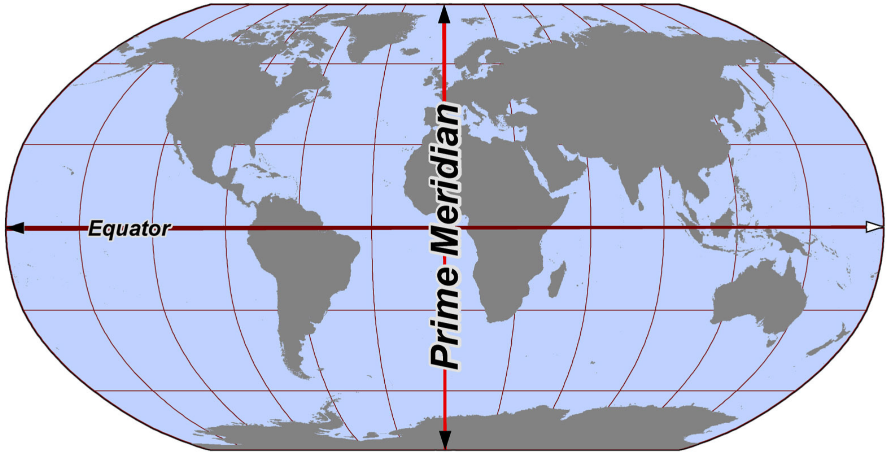
Reference Frames for Mobile Robots



Global Frame

How do we know where the global reference frame is?

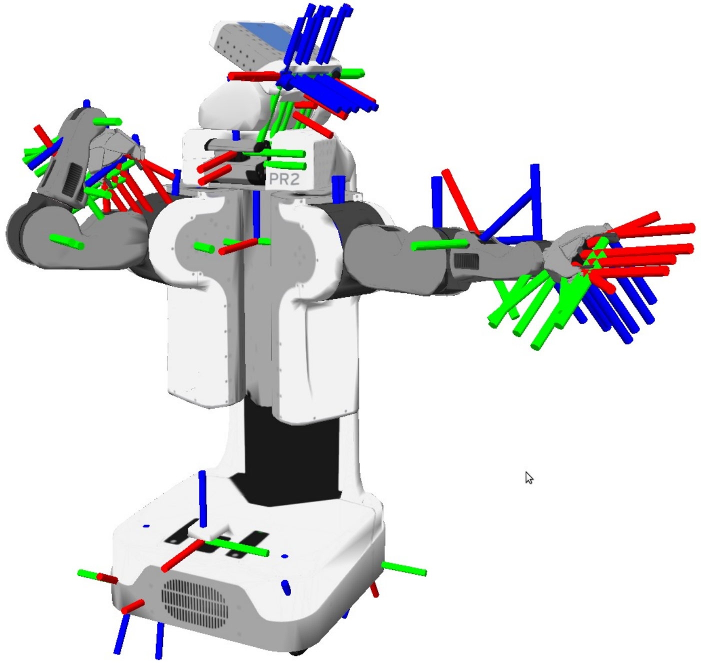
The first developer gets to define it as whatever makes sense to them.

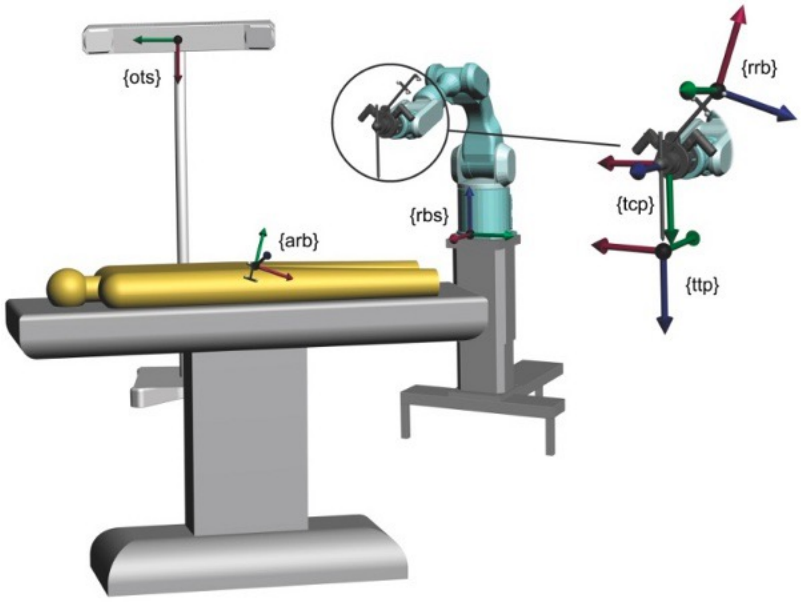


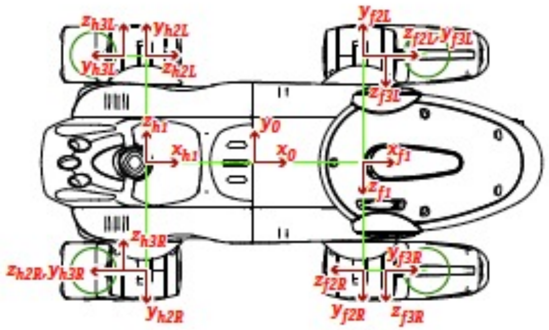
The Prime Meridian was defined at the Royal Observatory Greenwich by the legendary Airy Transit Circle telescope.

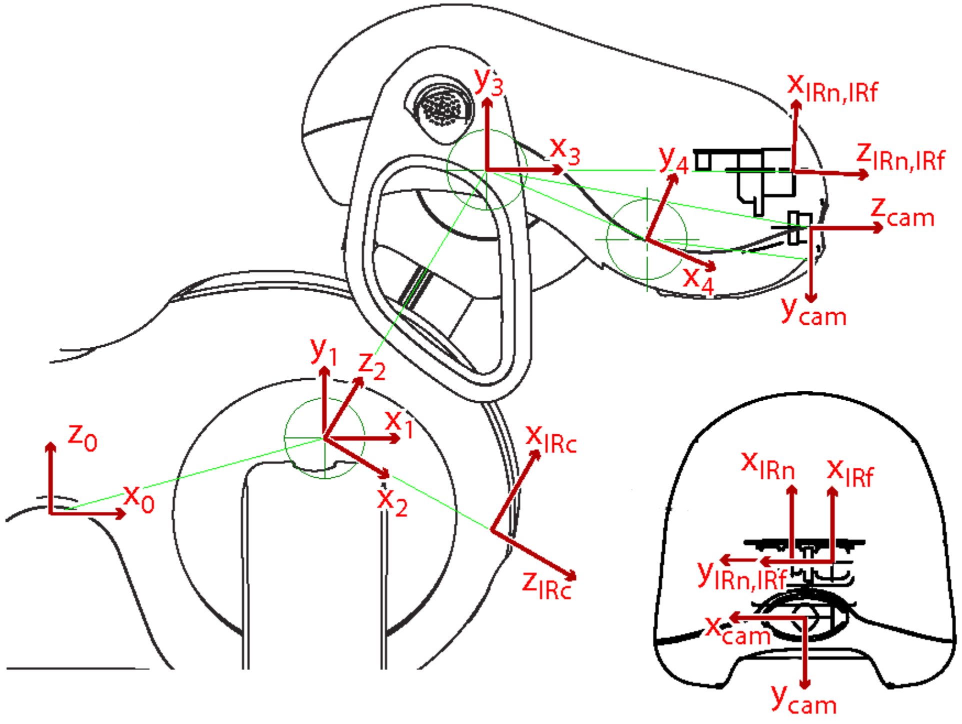
Examples of the types of reference frames we’re talking about

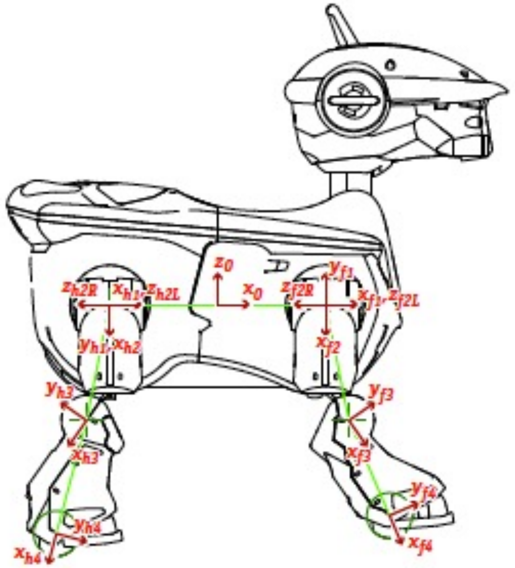
We rigidly attach coordinate frames to objects of interest. To specify the position and orientation of the object, we merely specify the position and orientation of the attached coordinate frame.

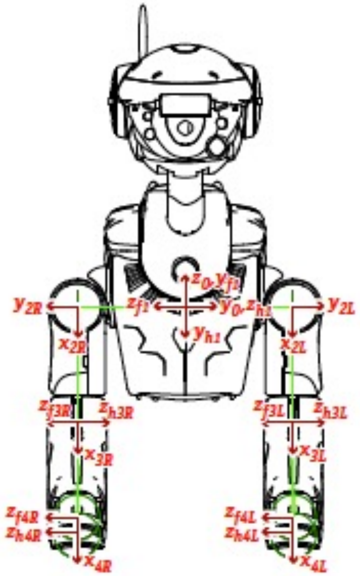










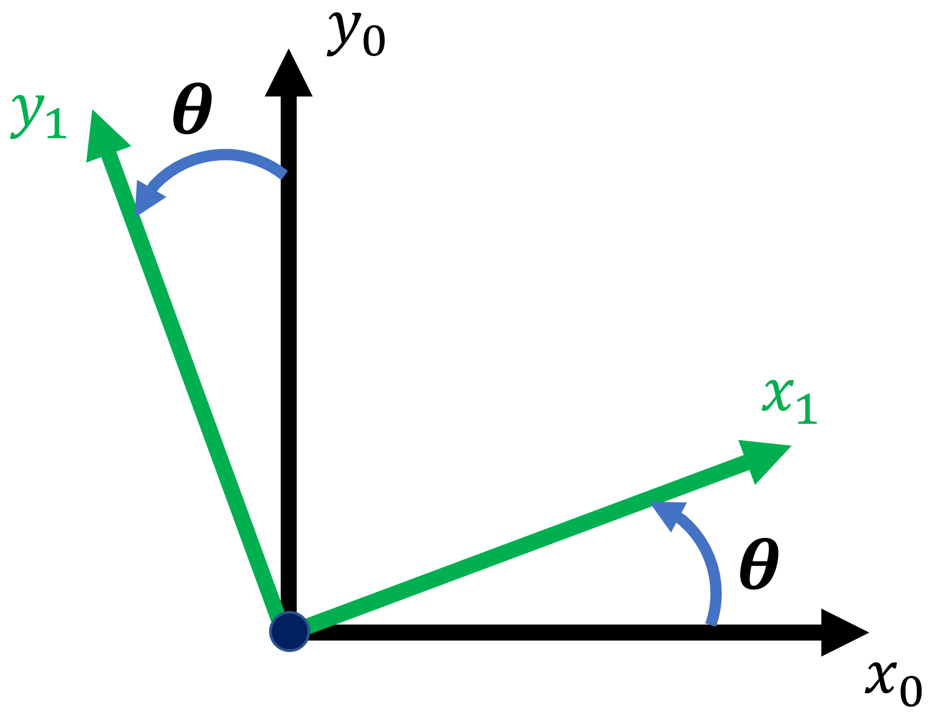


- The relationship between frames is often very simple to define, as in the case when two frames are related by the motion of a single joint/motor.
- For example, the upper and lower leg of the dog robot are related by a single motor at the knee.

Today – we consider only the case of 2D reference frames, corresponding to mobile robots moving in the plane.

Specifying Orientation in the Plane

Given two coordinate frames with a common origin, how should we describe the orientation of Frame 1 w.r.t. Frame 0?

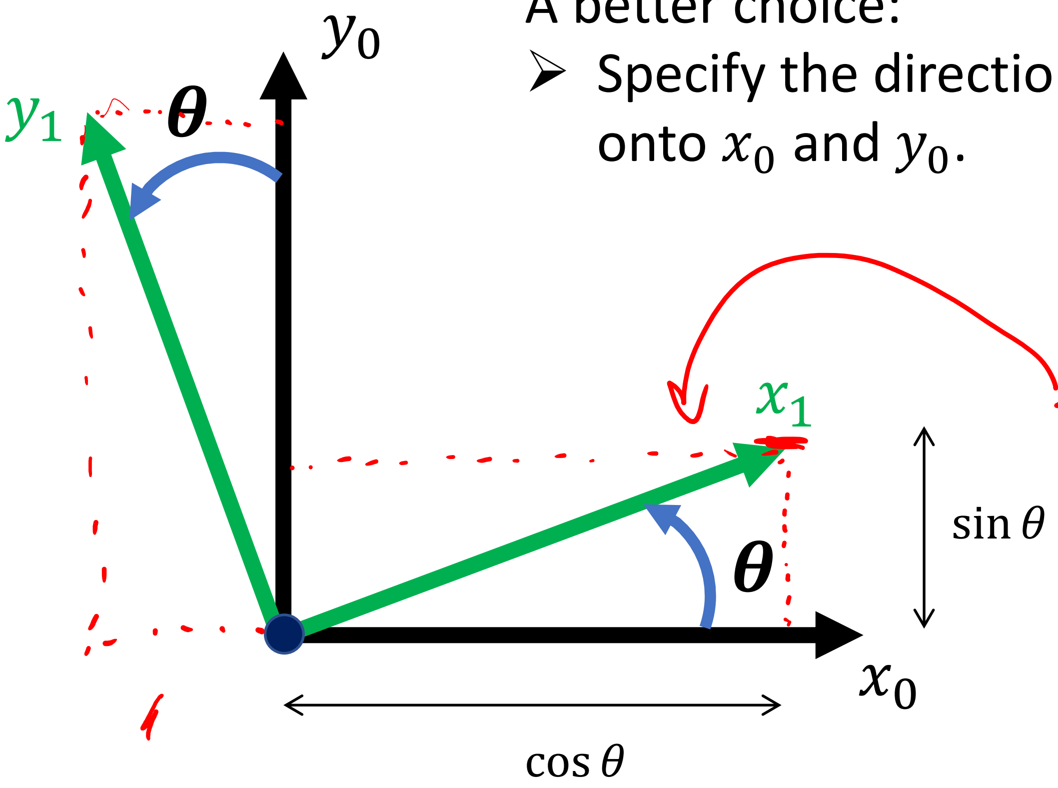


The obvious choice is to merely use the angle θ.
This isn't a great idea for two reasons:

- We have problems at $\theta = 2\,\pi - \epsilon$. For ϵ near 0, we approach a discontinuity: for small change in ϵ, we can have a large change in θ.
- This approach does not generalize to rotations in three dimensions (and not all robots live in the plane).

Specifying Orientation in the Plane

Given two coordinate frames with a common origin, how should we describe the orientation of Frame 1 w.r.t. Frame 0?



A better choice:

- Specify the directions of x_1 and y_1 with respect to Frame 0 by projecting onto x_0 and y_0.

$$x_1^0 = \begin{bmatrix} x_1 \cdot x_0 \\ x_1 \cdot y_0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Notation: x_1^0 denotes the x-axis of Frame 1, specified w.r.t Frame 0.

$$y_1^0 = \begin{bmatrix} y_1 \cdot x_0 \\ y_1 \cdot y_0 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

We obtain y_1^0 in the same way.

Rotation Matrices (rotation in the plane)

We combine these two vectors to obtain a ***rotation matrix***: $R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

All rotation matrices have certain properties:

1. The two columns are each unit vectors.
2. The two columns are orthogonal, i.e., $c_1 \cdot c_2 = 0$.
3. $\det R = +1$

For such matrices $R^{-1} = R^T$

➢ The first two properties imply that the matrix R is ***orthogonal***.

➢ The third property implies that the matrix is ***special***! (After all, there are plenty of orthogonal matrices whose determinant is -1, not at all special.)

The collection of 2×2 rotation matrices is called the ***Special Orthogonal Group of order 2***, or, more commonly $\boldsymbol{SO(2)}$.

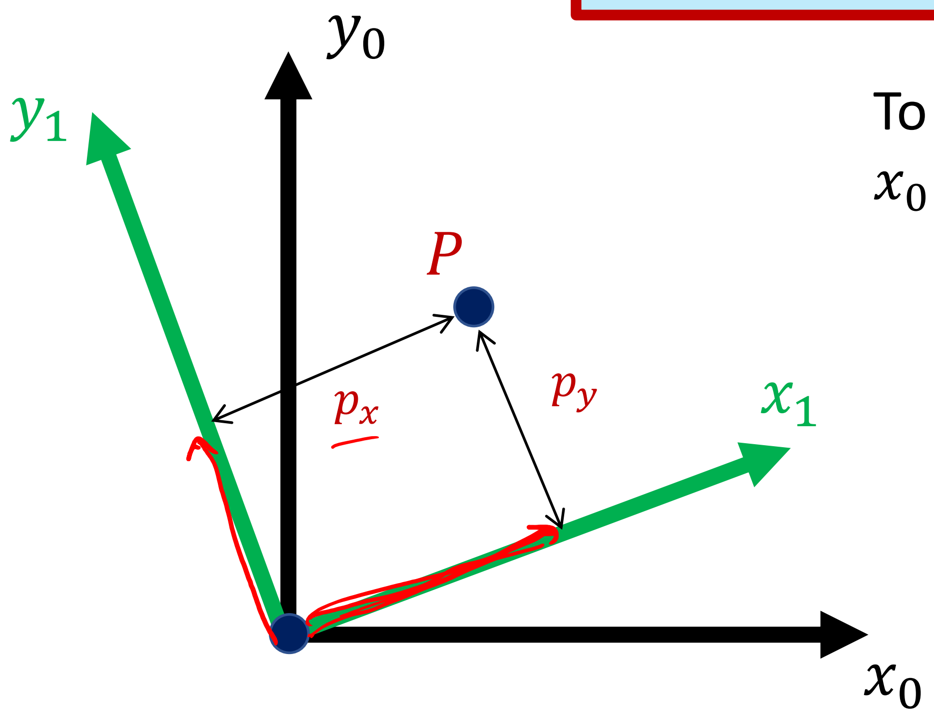
This concept generalizes to $\boldsymbol{SO(n)}$ for $n \times n$ rotation matrices.

Coordinate Transformations (rotation only)

Suppose a point P is rigidly attached to coordinate Frame 1, with coordinates given by $P^1 = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$.

We can express the location of the point P in terms of its coordinates

$$P = p_x x_1 + p_y y_1$$



To obtain the coordinates of P w.r.t. Frame 0, we project P onto the x_0 and y_0 axes:

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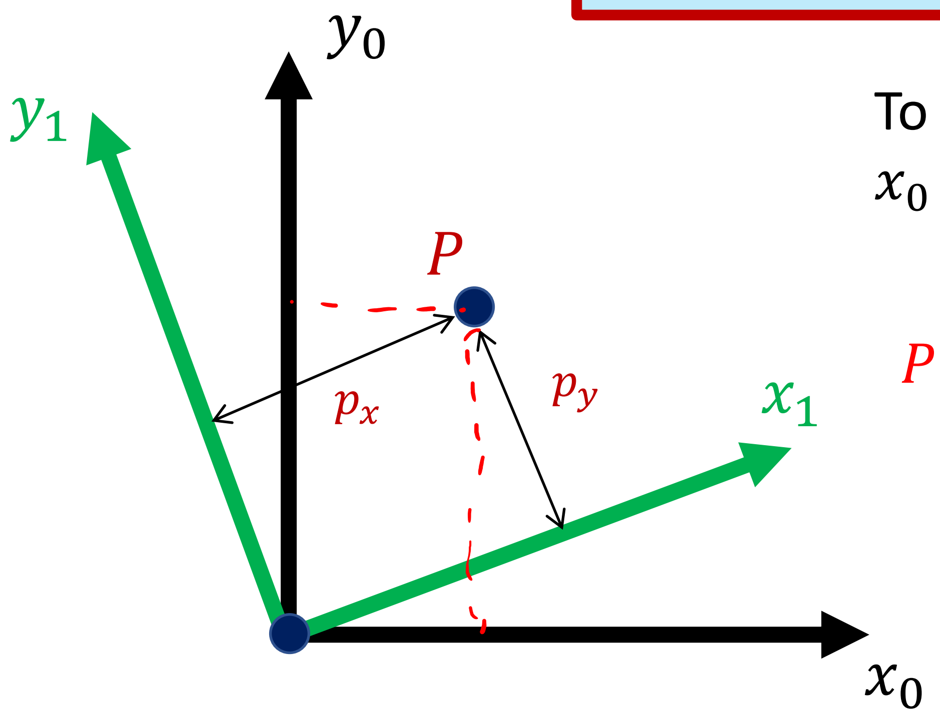

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$$P^0 = \begin{bmatrix} P \cdot x_0 \\ P \cdot y_0 \end{bmatrix} =$$

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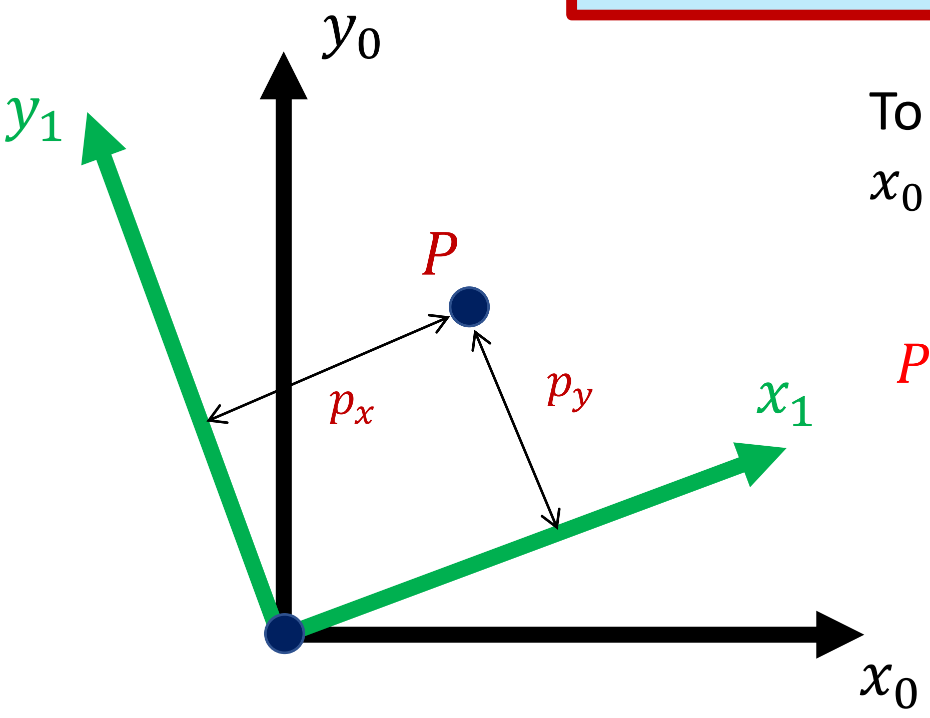

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$$P^0 = \begin{bmatrix} P \cdot x_0 \\ P \cdot y_0 \end{bmatrix} = \begin{bmatrix} (p_x x_1 + p_y y_1) \cdot x_0 \\ (p_x x_1 + p_y y_1) \cdot y_0 \end{bmatrix} =$$

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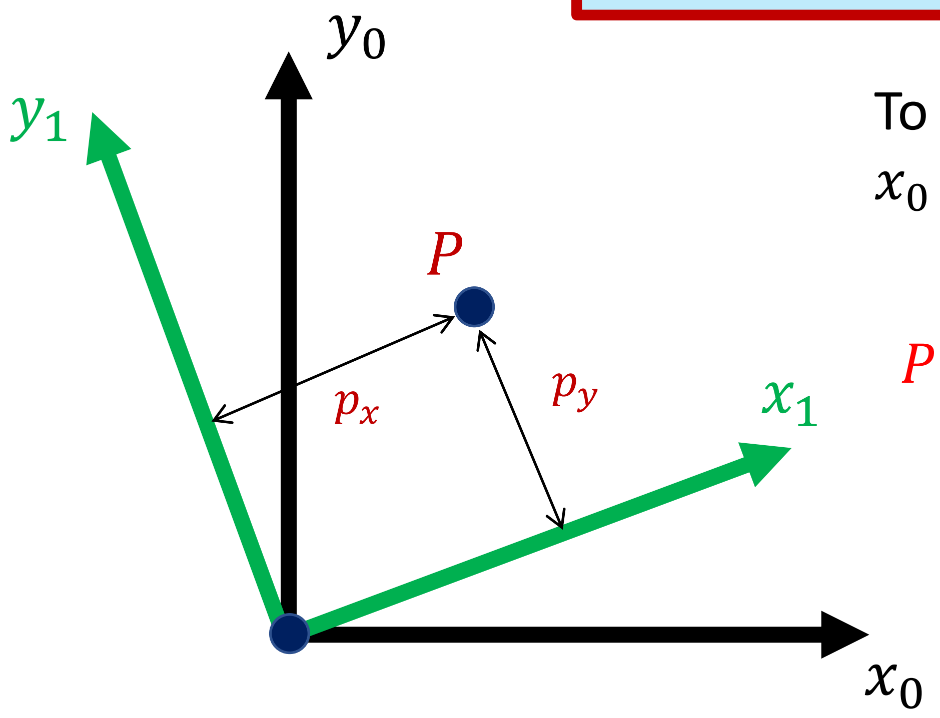

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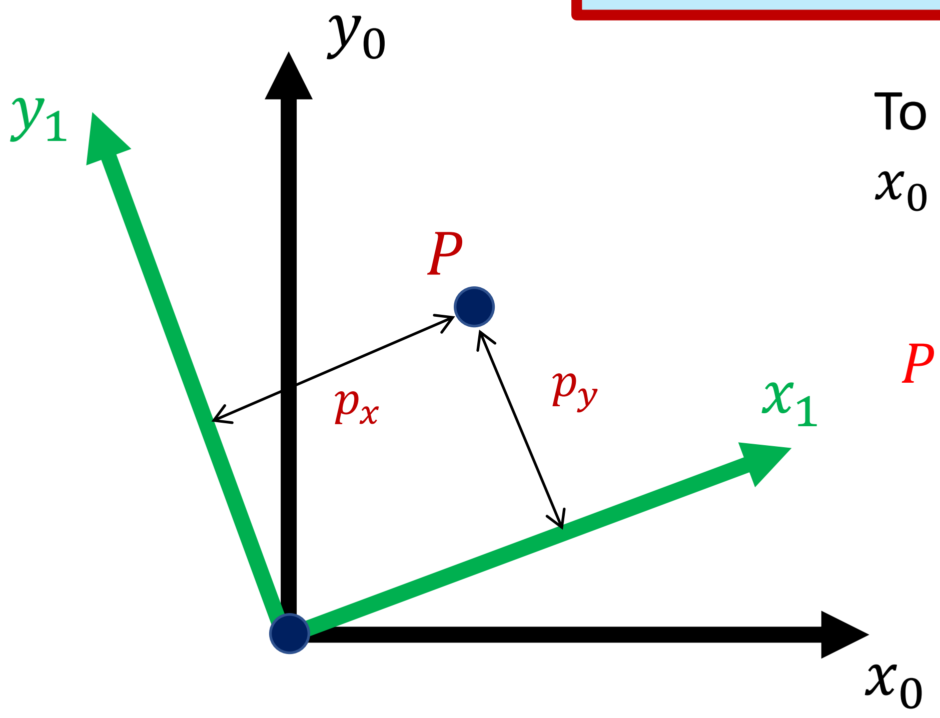

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$$= \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

Coordinate Transformations (rotation only)

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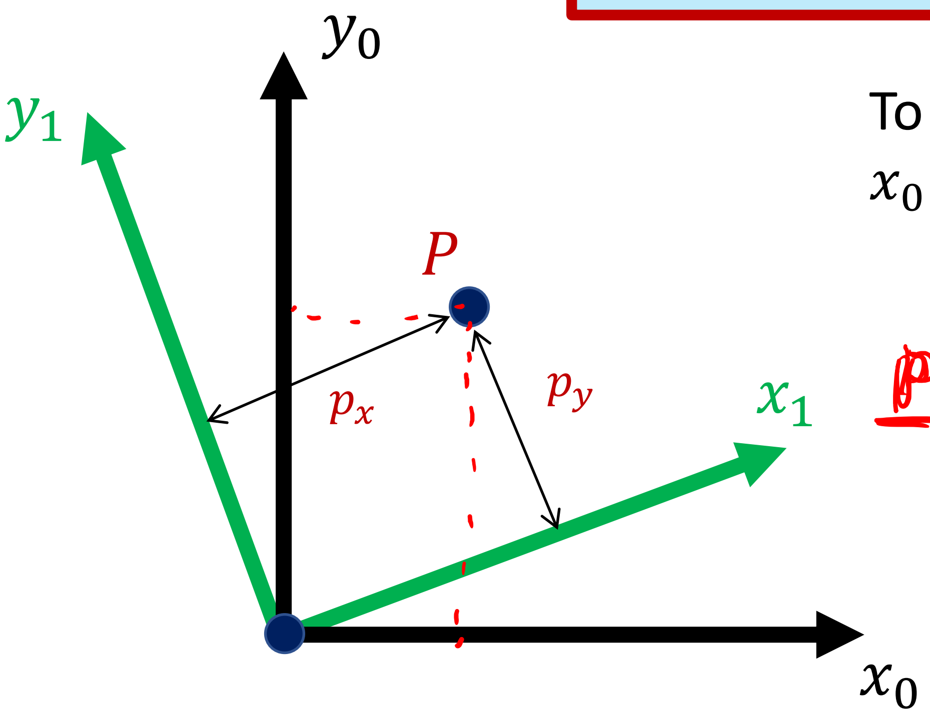
$$\underline{P = p_x x_1 + p_y y_1}$$



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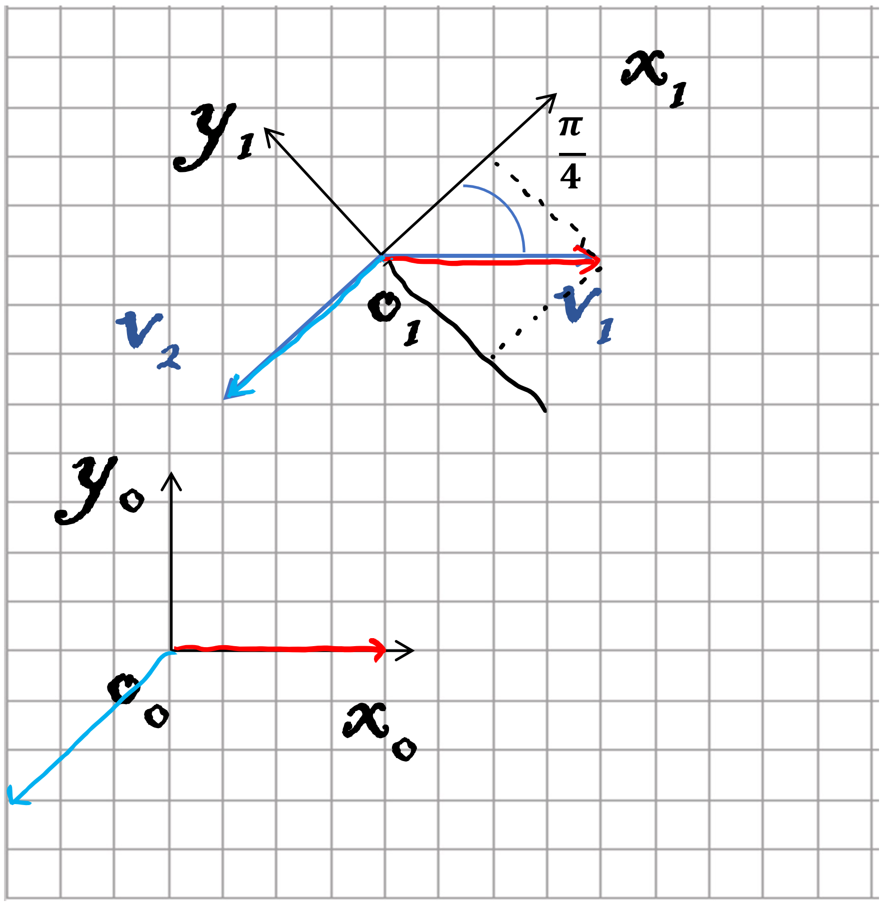
$$= \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \boldsymbol{R_1^0 \, P^1}$$

$$\boxed{\boldsymbol{P^0 = R_1^0 \, P^1}}$$

Let's practice...

Note: $\|v_1\| = 4, \ \|v_2\| = 3\sqrt{2},$



- Two coordinate frames: o_0 and o_1
- Two free vectors: v_1 and v_2

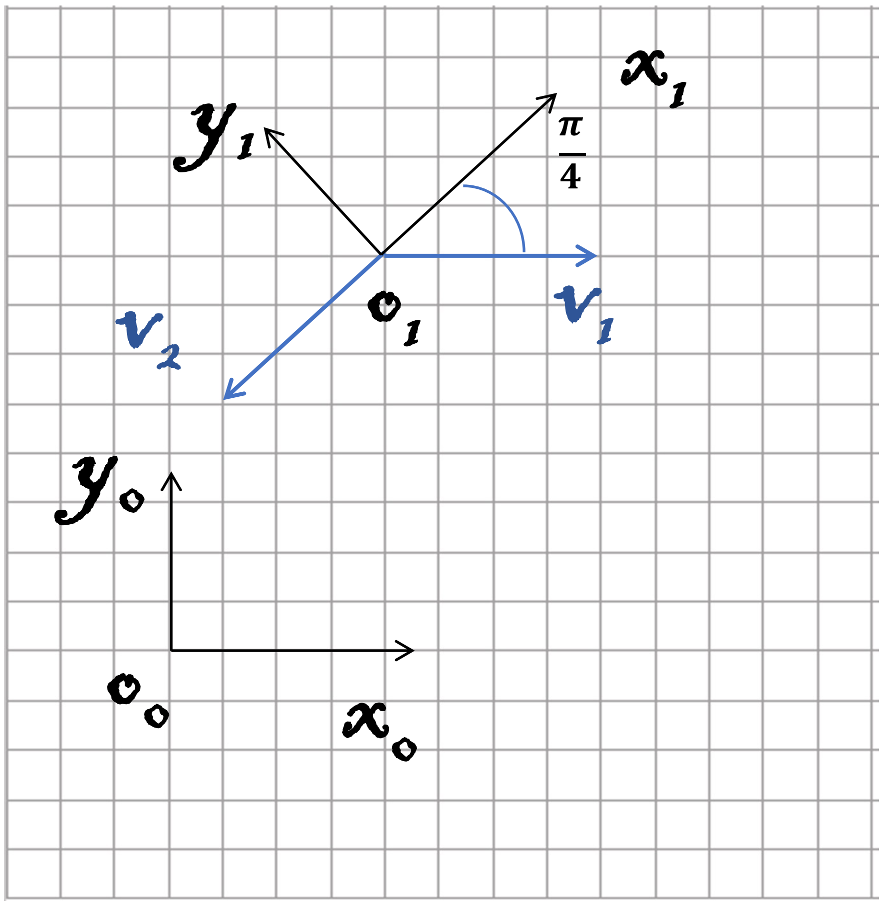
$$v_1^0 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \qquad v_1^1 = \begin{bmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix}$$

$$v_2^0 = \begin{bmatrix} -3 \\ -3 \end{bmatrix} \qquad v_2^1 = \begin{bmatrix} \ \end{bmatrix}$$

Recall: $\cos\frac{\pi}{4} = 0.5\sqrt{2}, \qquad \sin\frac{\pi}{4} = 0.5\sqrt{2}$

Let's practice…

Note: $\|v_1\| = 4, \ \|v_2\| = 3\sqrt{2},$



- Two coordinate frames: o_0 and o_1
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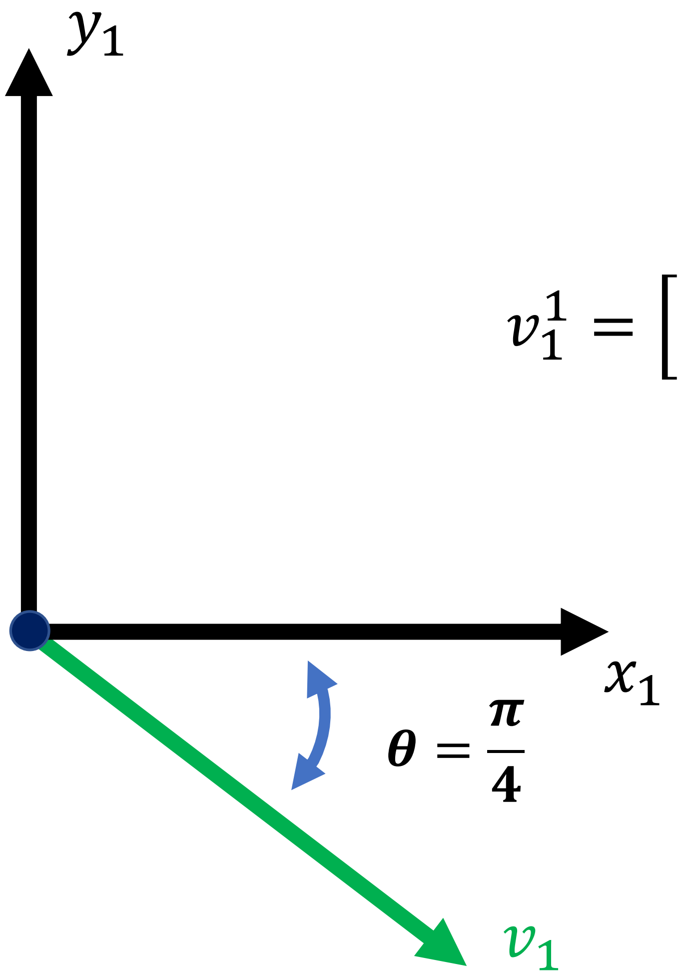
Recall: $\cos\frac{\pi}{4} = 0.5\sqrt{2}, \qquad \sin\frac{\pi}{4} = 0.5\sqrt{2}$

We can do this with simple geometry.

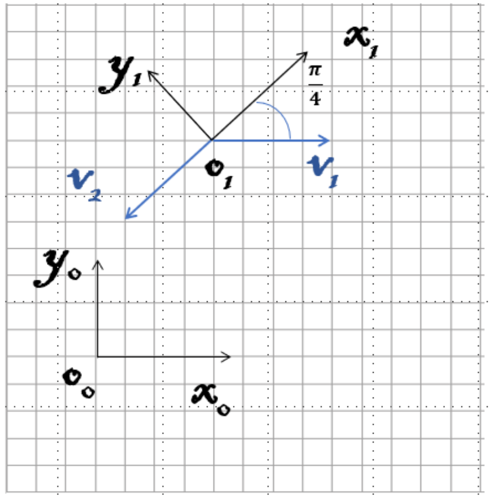
A Close look at v_1^1

Note: $\|v_1\| = 4$

$$\cos\frac{\pi}{4} = 0.5\sqrt{2}, \qquad \sin\frac{\pi}{4} = 0.5\sqrt{2}$$

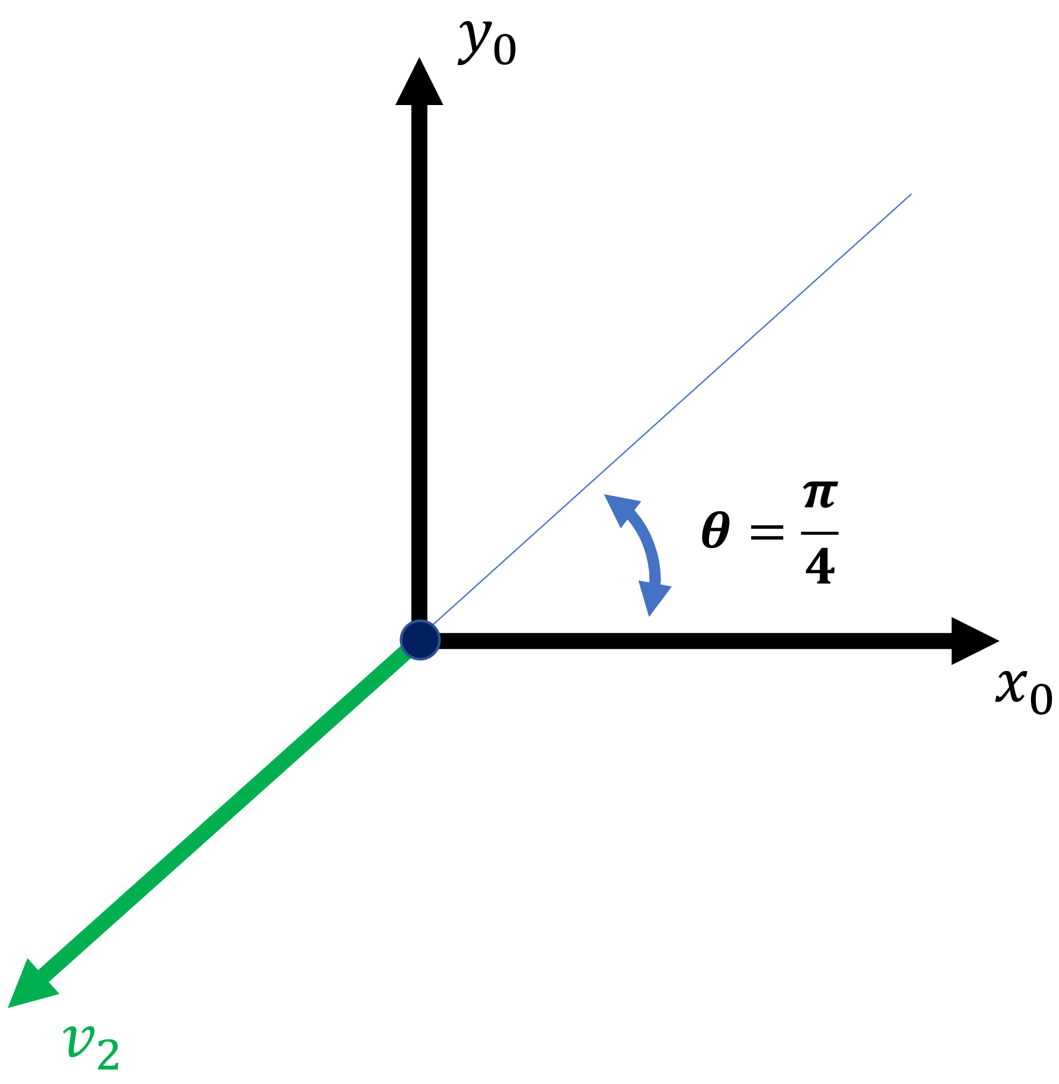


$$v_1^1 = \begin{bmatrix} 4\cos\theta \\ -4\sin\theta \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix}$$



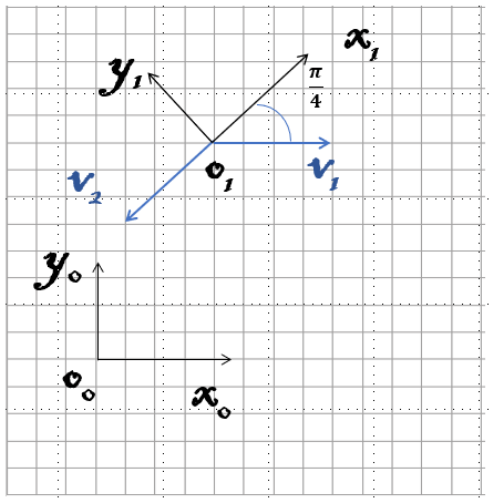
A Close look at v_2^0

Note: $\|v_2\| = 3\sqrt{2}$



$$\cos\frac{\pi}{4} = 0.5\sqrt{2}, \qquad \sin\frac{\pi}{4} = 0.5\sqrt{2}$$

$$v_2^0 = \begin{bmatrix} -3\sqrt{2}\cos\theta \\ -3\sqrt{2}\sin\theta \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

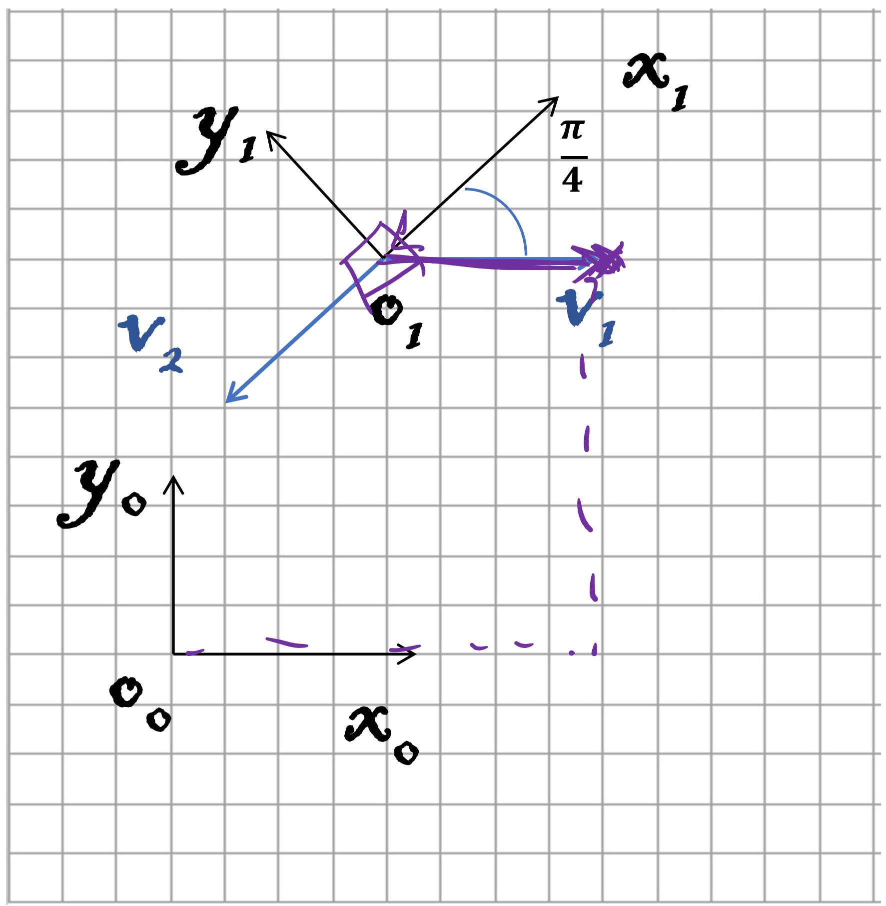


More Practice…

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad v_1^0 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad v_2^0 = \begin{bmatrix} -3 \\ -3 \end{bmatrix} \quad v_1^1 = \begin{bmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix} \quad v_2^1 = \begin{bmatrix} -3\sqrt{2} \\ 0 \end{bmatrix}$$

$\theta = \pi/4$

$$R_1^0 = \begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} \\ 0.5\sqrt{2} & 0.5\sqrt{2} \end{bmatrix} \qquad R_0^1 = \begin{bmatrix} 0.5\sqrt{2} & 0.5\sqrt{2} \\ -0.5\sqrt{2} & 0.5\sqrt{2} \end{bmatrix}$$



$R_a^b = R_b^{a\,-1} = R_b^{a\,T}$

$$v_1^0 = R_1^0 v_1^1 = \begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} \\ 0.5\sqrt{2} & 0.5\sqrt{2} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$v_2^0 = R_1^0 v_2^1 = \begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} \\ 0.5\sqrt{2} & 0.5\sqrt{2} \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

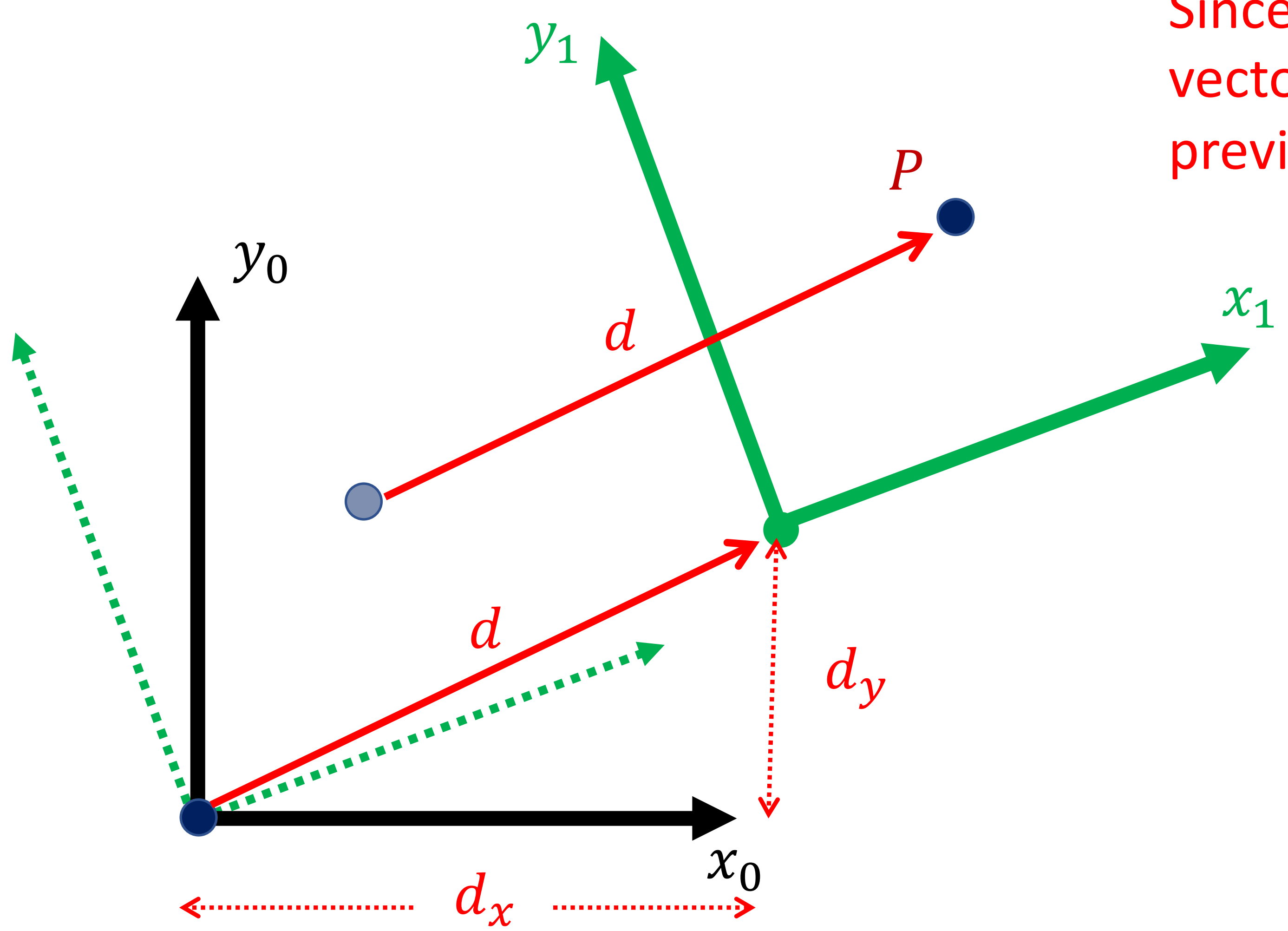
$$v_1^1 = R_0^1 v_1^0 = \begin{bmatrix} 0.5\sqrt{2} & 0.5\sqrt{2} \\ -0.5\sqrt{2} & 0.5\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix}$$

$$v_2^1 = R_0^1 v_2^0 = \begin{bmatrix} 0.5\sqrt{2} & 0.5\sqrt{2} \\ -0.5\sqrt{2} & 0.5\sqrt{2} \end{bmatrix} \begin{bmatrix} -3 \\ -3 \end{bmatrix} = \begin{bmatrix} -3\sqrt{2} \\ 0 \end{bmatrix}$$

OR, we can use coordinate transformations!

Specifying Pose in the Plane

Suppose we now translate Frame 1 (*no new rotatation*).
What are the coordinates of P w.r.t. Frame 0?



Since we merely translated P by a fixed vector d, simply add the offset to our previous result!

$$\boldsymbol{P^0} = \boldsymbol{R_1^0}\,\boldsymbol{P^1} + \boldsymbol{d^0}$$

$$\boldsymbol{d^0} = \begin{bmatrix} \boldsymbol{d_x} \\ \boldsymbol{d_y} \end{bmatrix}$$

Homogeneous Transformations

We can simplify the equation for coordinate transformations by augmenting the vectors and matrices with an extra row:

This is just our eqn from the previous page

$$\boxed{\begin{bmatrix} \boldsymbol{P^0} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R_1^0 P^1 + d^0} \\ 1 \end{bmatrix}} = \begin{bmatrix} \boldsymbol{R_1^0} & \boldsymbol{d^0} \\ 0_2 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{P^1} \\ 1 \end{bmatrix}$$

in which $0_2 = [0 \quad 0]$

rotation

displacement

The set of matrices of the form $\begin{bmatrix} R & d \\ 0_n & 1 \end{bmatrix}$, where $R \in SO(n)$ and $d \in \mathbb{R}^n$ is called

n 0s

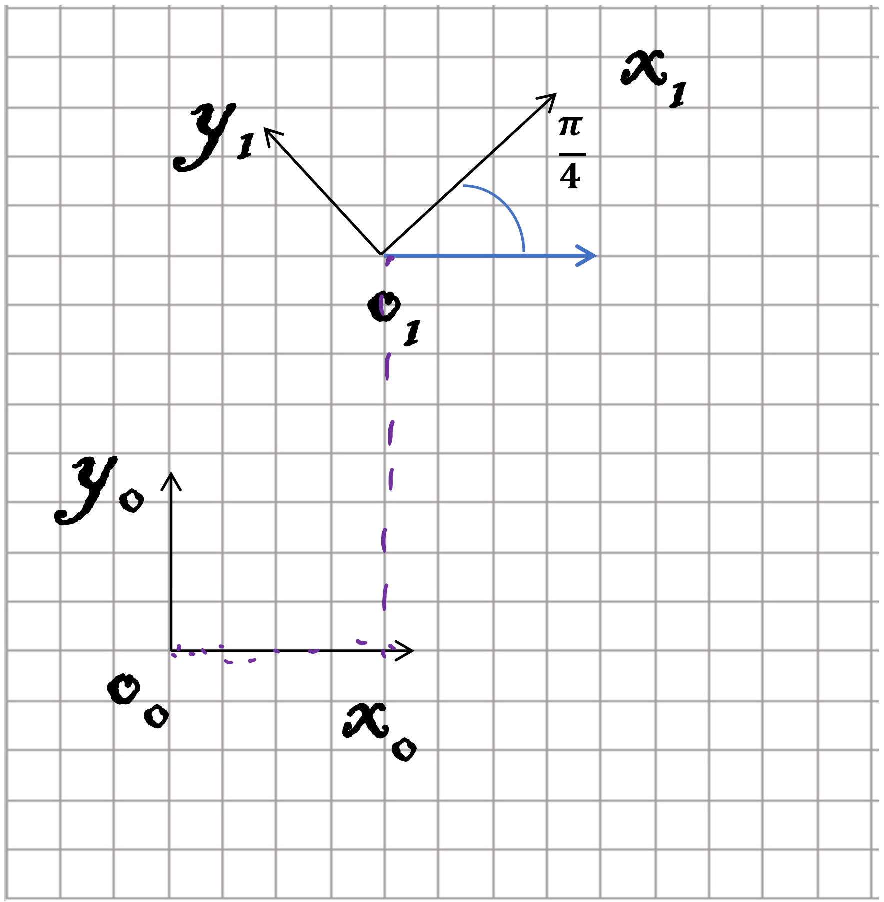
1

the ***Special Euclidean Group of order n***, or $SE(n)$.

Lets practice…

$$T_1^0 = \begin{bmatrix} R_1^0 & d \\ 0_2 & 1 \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

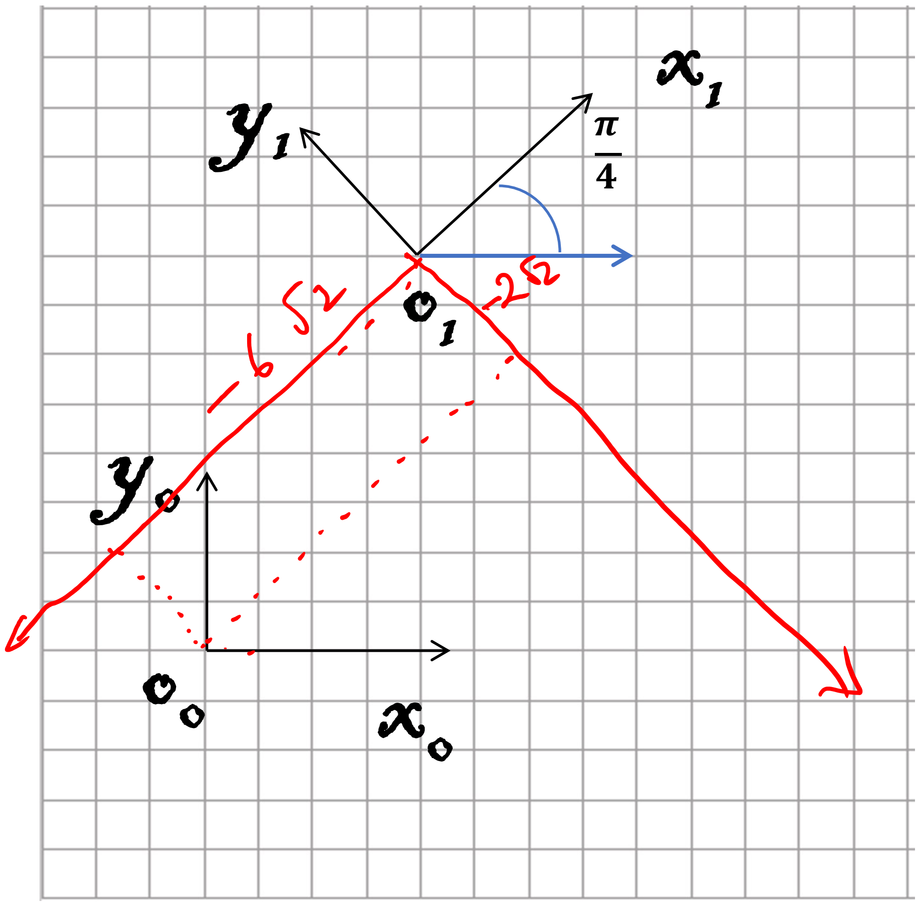


$$T_1^0 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 4 \\ \sqrt{2}/2 & \sqrt{2}/2 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_0^1 = \begin{bmatrix} \quad \end{bmatrix}$$

Recall: $\cos\frac{\pi}{4} = 0.5\sqrt{2}$, $\sin\frac{\pi}{4} = 0.5\sqrt{2}$

Lets practice…



$$T_1^0 = \begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} & 4 \\ 0.5\sqrt{2} & 0.5\sqrt{2} & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_0^1 = \begin{bmatrix} 0.5\sqrt{2} & 0.5\sqrt{2} & -6\sqrt{2} \\ -0.5\sqrt{2} & 0.5\sqrt{2} & -2\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Recall: $\cos\frac{\pi}{4} = 0.5\sqrt{2}$, $\sin\frac{\pi}{4} = 0.5\sqrt{2}$

Inverse of a Homogeneous Transformation

What is the relationship between T_1^0 and T_0^1?

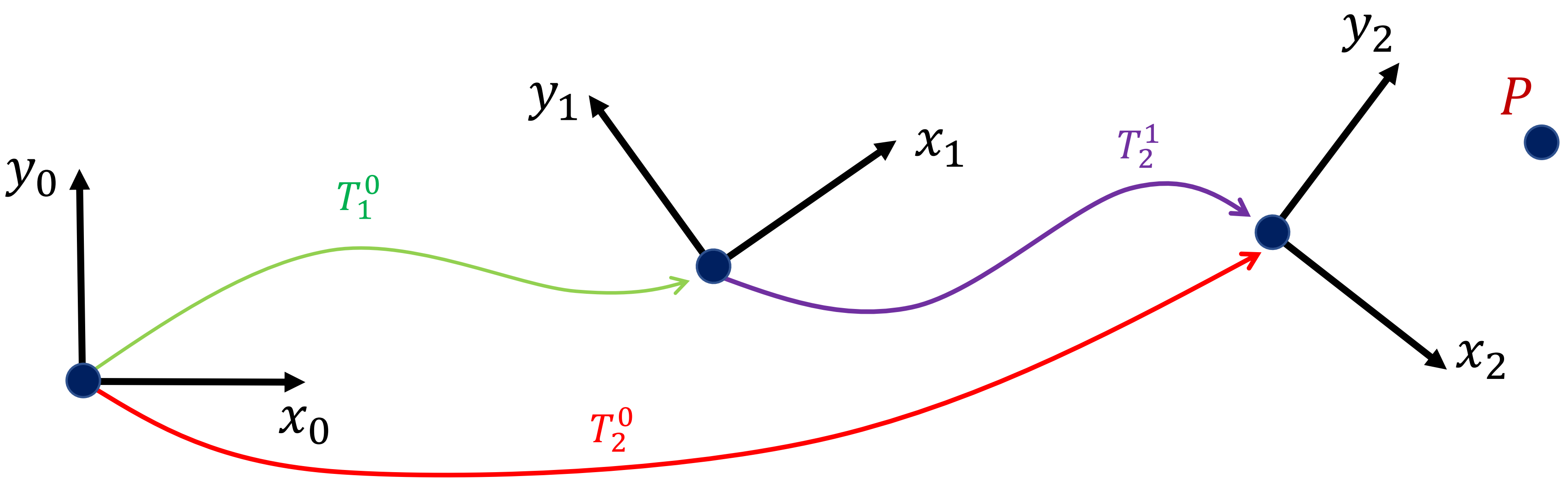
$$T_1^0 T_0^1 = \begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} & 4 \\ 0.5\sqrt{2} & 0.5\sqrt{2} & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5\sqrt{2} & 0.5\sqrt{2} & -6\sqrt{2} \\ -0.5\sqrt{2} & 0.5\sqrt{2} & -2\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In general, $T_k^j = \left(T_j^k\right)^{-1}$ and $\begin{bmatrix} \boldsymbol{R} & \boldsymbol{d} \\ 0_n & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{d} \\ 0_n & 1 \end{bmatrix}$

This is easy to verify:

$$\begin{bmatrix} \boldsymbol{R} & \boldsymbol{d} \\ 0_n & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{d} \\ 0_n & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}\boldsymbol{R}^T & -\boldsymbol{R}\boldsymbol{R}^T\boldsymbol{d} + \boldsymbol{d} \\ 0_n & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_{n\times n} & \boldsymbol{0}_n \\ 0_n & 1 \end{bmatrix} = I_{(n+1)\times(n+1)}$$

Composition of Transformations



From our previous results, we know:

$$P^0 = T_1^0 P^1$$
$$P^1 = T_2^1 P^2$$

$$P^0 = T_1^0 T_2^1 P^2$$

But we also know: $P^0 = T_2^0 P^2$

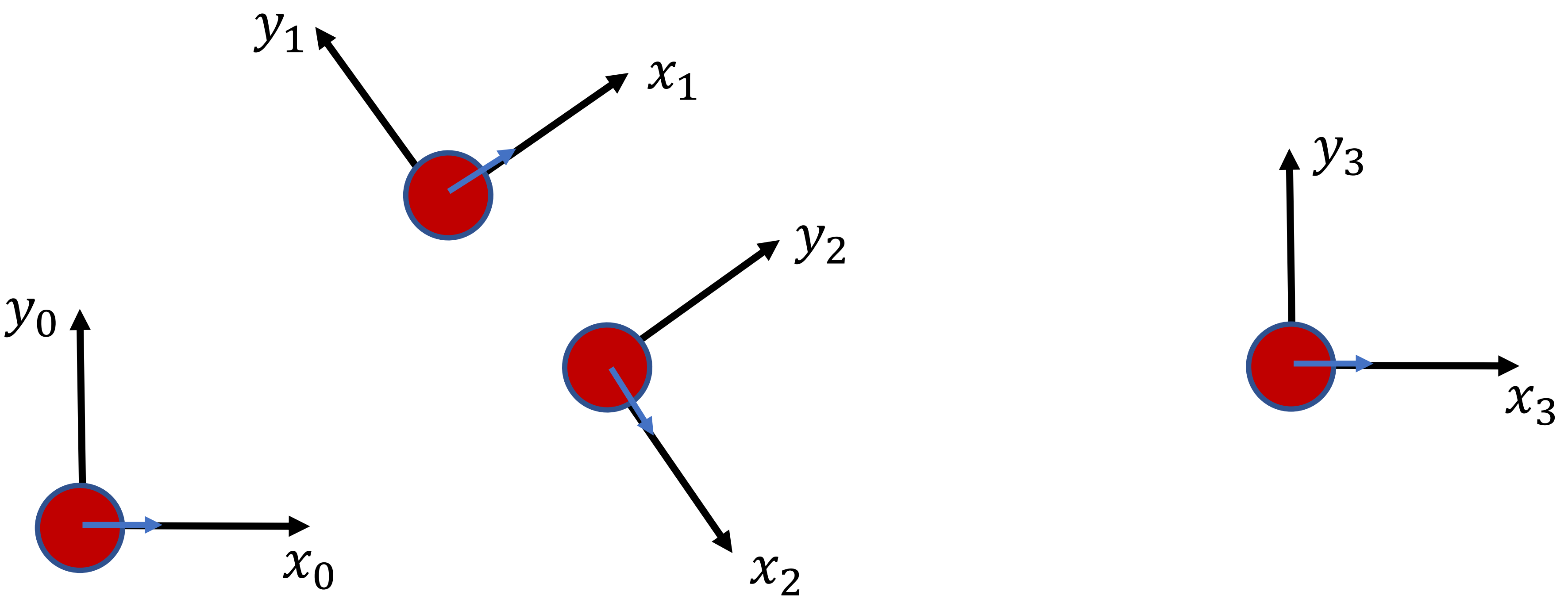
$$T_2^0 = T_1^0 T_2^1$$

<u>This is the composition law for homogeneous transformations.</u>

Example

1. Robot rotates by $\frac{\pi}{4}$ and then moves forward $2\sqrt{2}$

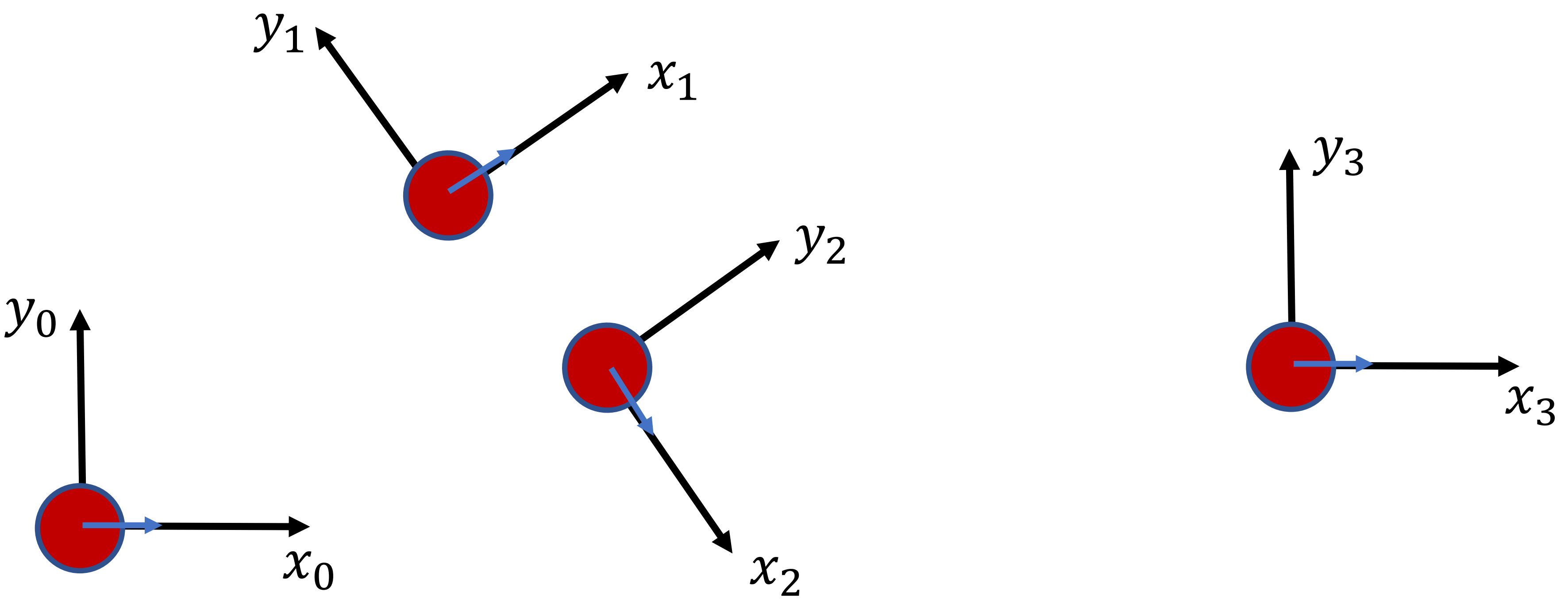
$$T_1^0 = \underbrace{\begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} & 0 \\ 0.5\sqrt{2} & 0.5\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Rot}} \underbrace{\begin{bmatrix} 1 & 0 & 2\sqrt{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Transl}} = \begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} & 2 \\ 0.5\sqrt{2} & 0.5\sqrt{2} & 2 \\ 0 & 0 & 1 \end{bmatrix}$$



Example

2. Robot rotates by $-\frac{\pi}{2}$ and then moves forward $\sqrt{2}$

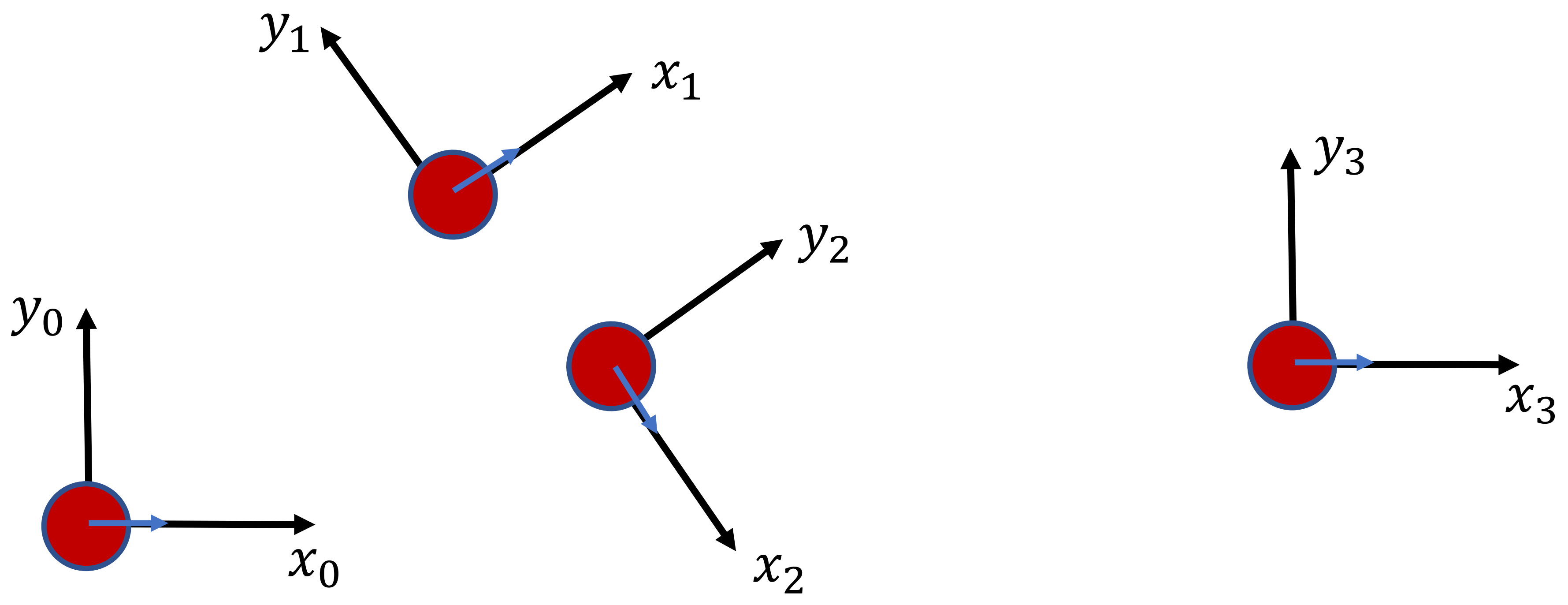
$$T_2^1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$



Example

$$T_2^0 = T_1^0 T_2^1 = \begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} & 2 \\ 0.5\sqrt{2} & 0.5\sqrt{2} & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

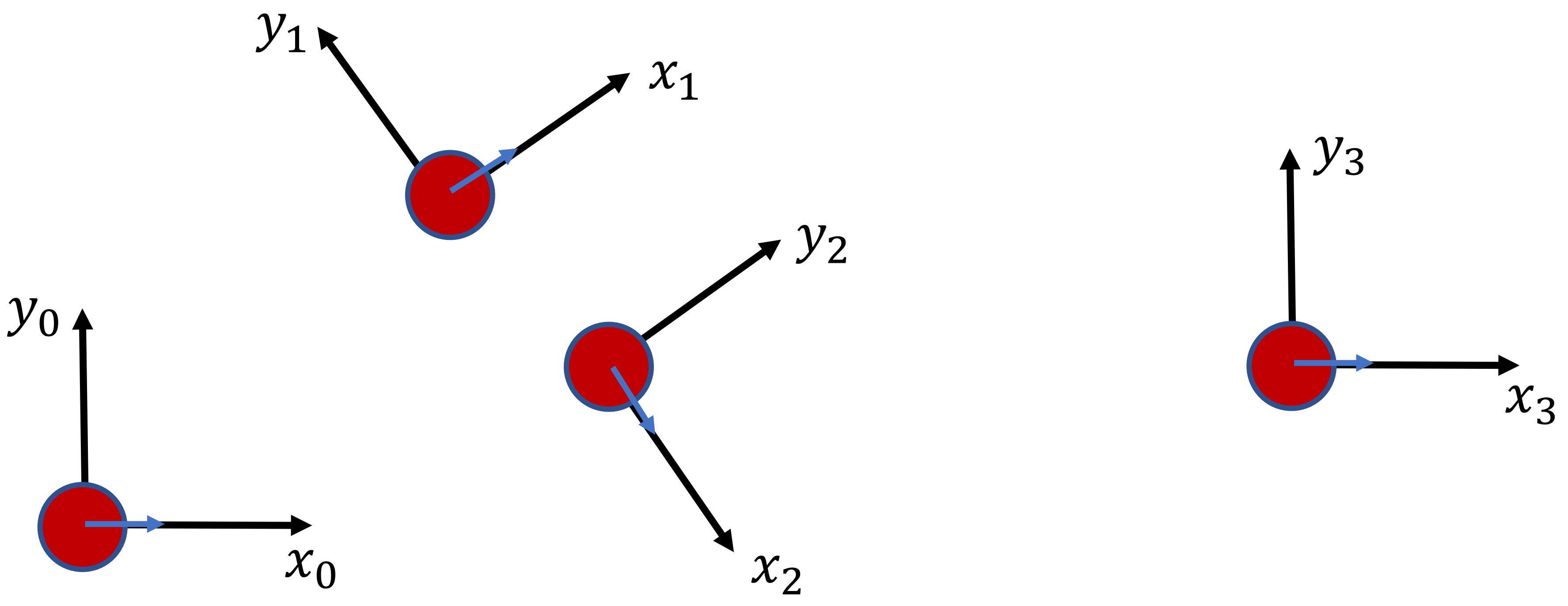
$$= \begin{bmatrix} 0.5\sqrt{2} & 0.5\sqrt{2} & (-0.5\sqrt{2})(-\sqrt{2}) + 2 \\ -0.5\sqrt{2} & 0.5\sqrt{2} & (0.5\sqrt{2})(-\sqrt{2}) + 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5\sqrt{2} & 0.5\sqrt{2} & 3 \\ -0.5\sqrt{2} & 0.5\sqrt{2} & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



Example

3. Robot rotates by $\frac{\pi}{4}$ and then moves forward 4

$$T_3^2 = \begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} & 0 \\ 0.5\sqrt{2} & 0.5\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} & 2\sqrt{2} \\ 0.5\sqrt{2} & 0.5\sqrt{2} & 2\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$



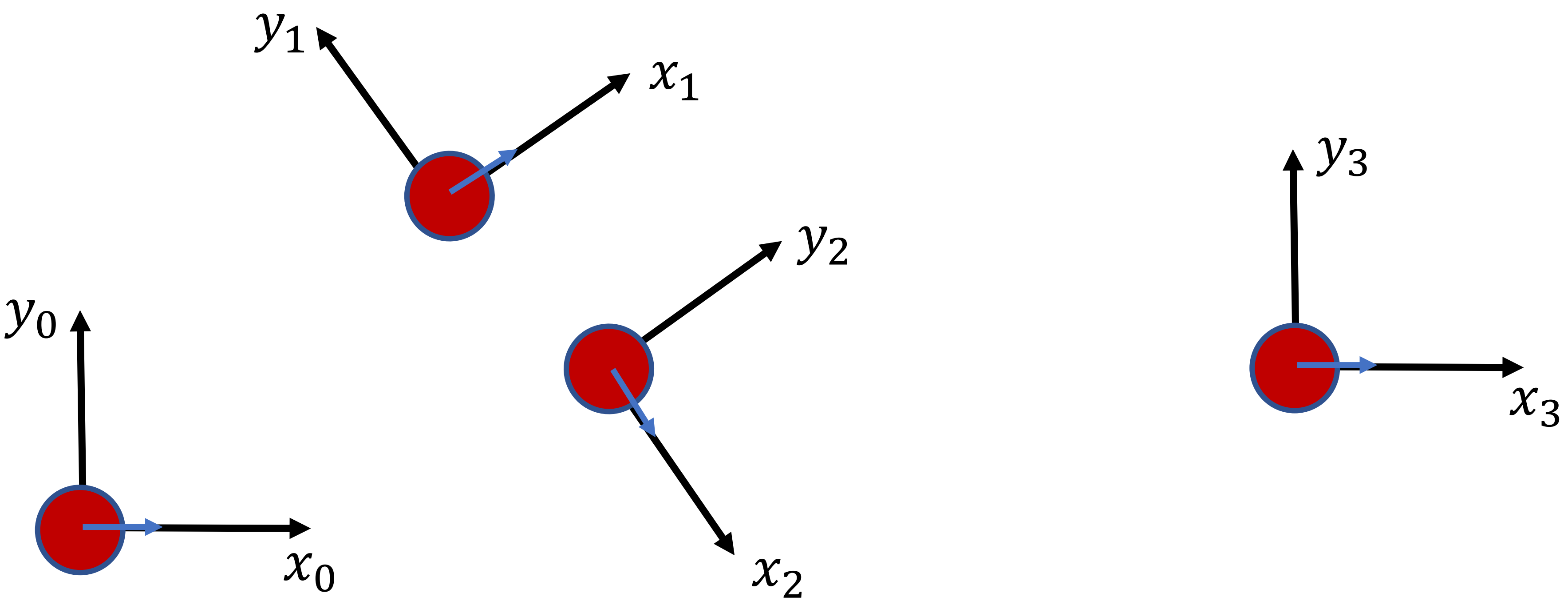
Example

$$T_3^0 = T_2^0 T_3^2 = \begin{bmatrix} 0.5\sqrt{2} & 0.5\sqrt{2} & 3 \\ -0.5\sqrt{2} & 0.5\sqrt{2} & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5\sqrt{2} & -0.5\sqrt{2} & 2\sqrt{2} \\ 0.5\sqrt{2} & 0.5\sqrt{2} & 2\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & (0.5\sqrt{2})(2\sqrt{2}) + (0.5\sqrt{2})(2\sqrt{2}) + 3 \\ 0 & 1 & (-0.5\sqrt{2})(2\sqrt{2}) + (0.5\sqrt{2})(2\sqrt{2}) + 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

displacement

$$T_3^0 P^3 = P^0$$



Solving Transformation Equations

- It's easy to solve for any individual T_j^i given the equation$T_1^0 \ldots T_n^{n-1} = T_n^0$
- Simply premultiply each side by the appropriate inverse transformations.
- Example: find T_3^2 given $T_1^0 T_2^1 T_3^2 T_4^3 T_5^4 = T_5^0$

$$T_1^0 T_2^1 T_3^2 T_4^3 T_5^4 = T_5^0$$

$$(T_1^0 T_2^1)^{-1}(T_1^0 T_2^1) T_3^2 (T_4^3 T_5^4)(T_4^3 T_5^4)^{-1} = (T_1^0 T_2^1)^{-1} T_5^0 (T_4^3 T_5^4)^{-1}$$

$$T_3^2 = (T_1^0 T_2^1)^{-1} T_5^0 (T_4^3 T_5^4)^{-1}$$

$$T_3^2 = (T_2^0)^{-1} T_5^0 \left(T_5^3\right)^{-1}$$

$$T_0^1 = (T_1^0)^{-1}$$

Solving Transformation Equations

- It's easy to solve for any individual T_j^i given the equation$T_1^0 \dots T_n^{n-1} = T_n^0$
- Simply premultiply each side by the appropriate inverse transformations.
- Example: find T_4^2 given $T_1^0 T_2^1 T_3^2 T_4^3 T_5^4 = T_5^0$

$$T_1^0 T_2^1 (T_3^2 T_4^3) T_5^4 = T_5^0$$

$$(T_1^0 T_2^1)^{-1} (T_1^0 T_2^1) T_4^2 T_5^4 (T_5^4)^{-1} = (T_1^0 T_2^1)^{-1} T_5^0 (T_5^4)^{-1}$$

$$T_4^2 = (T_1^0 T_2^1)^{-1} T_5^0 (T_5^4)^{-1}$$

$$T_4^2 = (T_2^0)^{-1} T_5^0 (T_5^4)^{-1}$$

Rotation Matrices (3D)

All of the properties of SO(2) apply as well to SO(3)!

All rotation matrices have certain properties:

1. The two columns are each unit vectors.
2. The two columns are orthogonal, e.g., $c_1 \cdot c_2 = 0.$
3. $\det R = +1$

For such matrices $R^{-1} = R^T$

- The first two properties imply that the matrix R is ***orthogonal***.
- The third property implies that the matrix is ***special***! (After all, there are plenty of orthogonal matrices whose determinant is -1, not at all special.)

The collection of 3×3 rotation matrices is called the ***Special Orthogonal Group of order 3***, or, more commonly $\boldsymbol{SO(3)}$.

Rotation Matrices for 3D rotations

To build a rotation matrix, say R_1^0: project the axes of Frame 1 onto Frame 0. Each column of R_1^0 corresponds to the projection of one axis of Frame 1 onto Frame 0.

$$R_1^0 = [x_1 \cdot F_0 \mid y_1 \cdot F_0 \mid z_1 \cdot F_0] = \left[\begin{array}{c|c|c} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{array}\right]$$

Rotation Matrices for 3D rotations

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$$R_1^0 = \begin{bmatrix} \boxed{x_1 \cdot F_0} & y_1 \cdot F_0 & z_1 \cdot F_0 \end{bmatrix} = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

Project the x-axis of Frame 1
onto the axes of Frame 0

Rotation Matrices for 3D rotations

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$$R_1^0 = \begin{bmatrix} x_1 \cdot F_0 & \boxed{y_1 \cdot F_0} & z_1 \cdot F_0 \end{bmatrix} = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

Project the y-axis of Frame 1 onto the axes of Frame 0

Rotation Matrices for 3D rotations

To build a rotation matrix, say R_1^0: project the axes of Frame 1 onto Frame 0. Each column of R_1^0 corresponds to the projection of one axis of Frame 1 onto Frame 0.

$$R_1^0 = \begin{bmatrix} x_1 \cdot F_0 & y_1 \cdot F_0 & \boxed{z_1 \cdot F_0} \end{bmatrix} = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

Project the z-axis of Frame 1 onto the axes of Frame 0

Rotation Matrices for 3D rotations

To build a rotation matrix, say R_1^0: project the axes of Frame 1 onto Frame 0. Each column of R_1^0 corresponds to the projection of one axis of Frame 1 onto Frame 0.

$$R_1^0 = \begin{bmatrix} \boxed{x_1 \cdot F_0} & \boxed{y_1 \cdot F_0} & \boxed{z_1 \cdot F_0} \end{bmatrix} = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

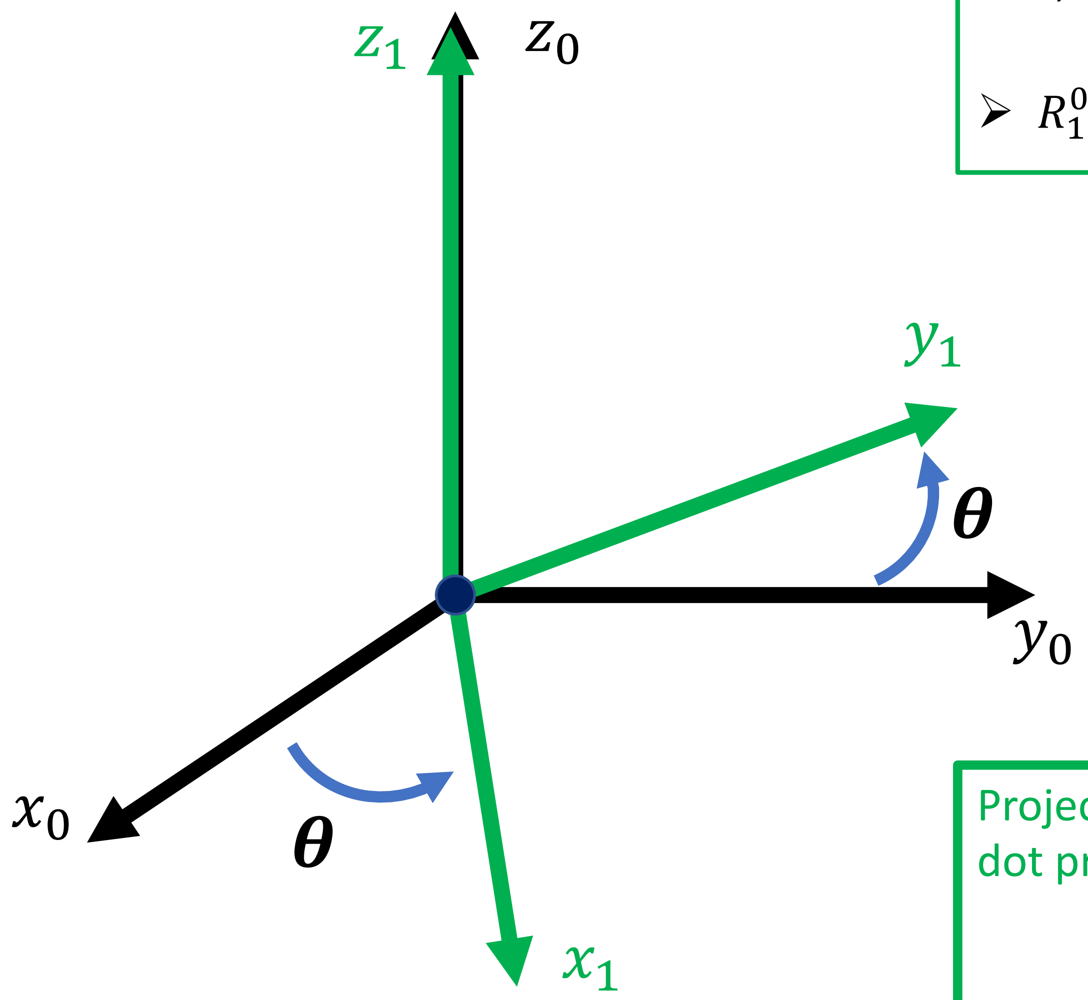
Project the x-axis of Frame 1 onto the axes of Frame 0

Project the y-axis of Frame 1 onto the axes of Frame 0

Project the z-axis of Frame 1 onto the axes of Frame 0

This process is exactly the same as the process for building rotation matrices in SO(2), even though it can be more difficult to visualize in 3D for rotation matrices in SO(3).

The simplest example: rotation about the z axis



Recall: for rotation in the plane, we built a rotation matrix as a function of θ, the angle between x_1 and x_0 (and also between y_1 and y_0):

- $R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ FOR ROTATION IN THE PLANE

This is *easily* extended to the case of rotation in 3D about the z-axis, since all of the interesting action is in the x-y plane (the two z-axes are the same)!

In fact, you'll see that the 2D rotation matrix shows up in the 3D rotation matrix:

- $R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ FOR ROTATION IN 3D

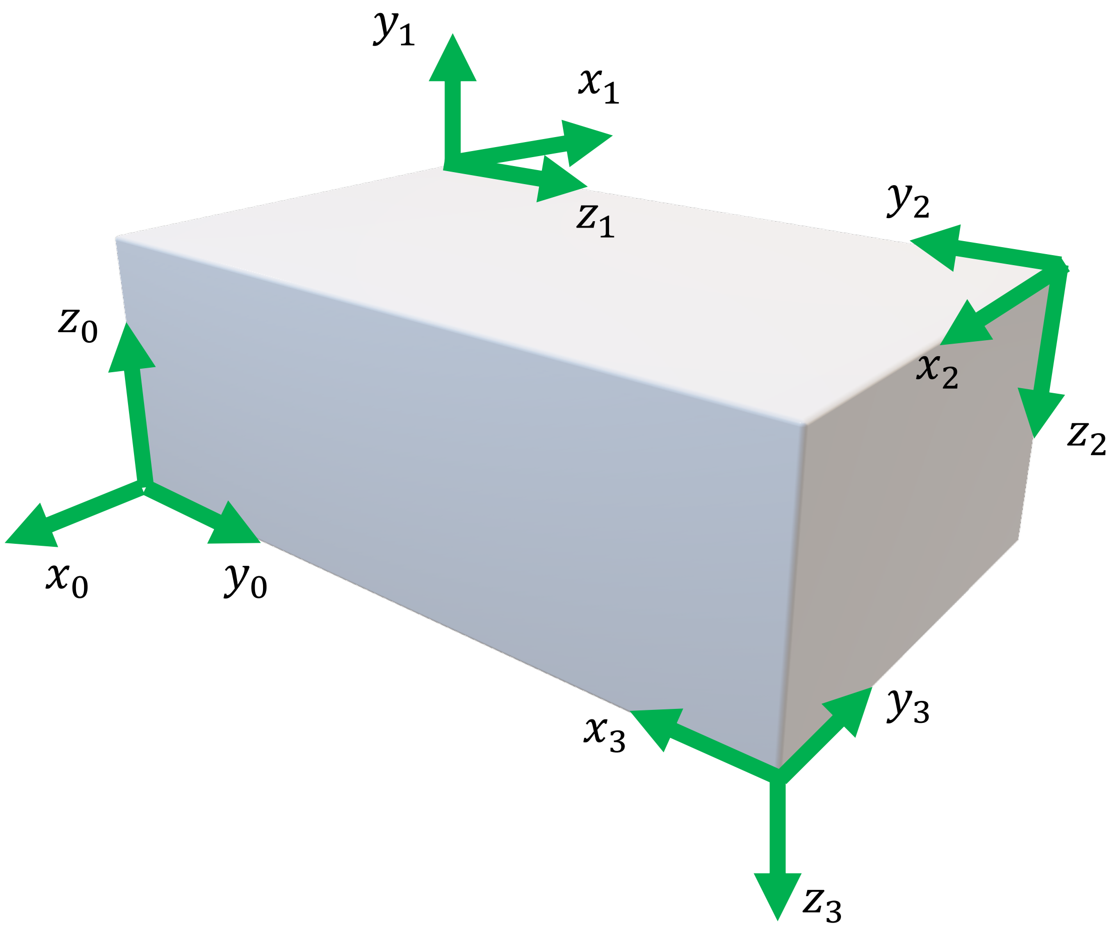
Projecting z_1 onto Frame 0 involves three dot products:

$$z_1 \cdot x_0 = 0$$
$$z_1 \cdot y_0 = 0$$
$$z_1 \cdot z_0 = 1$$

A bunch of examples:

A rectangular solid: all angles are multiples of $\pi/2$.

$$R_j^i = \begin{bmatrix} x_j \cdot x_i & y_j \cdot x_i & z_j \cdot x_i \\ x_j \cdot y_i & y_j \cdot y_i & z_j \cdot y_i \\ x_j \cdot z_i & y_j \cdot z_i & z_j \cdot z_i \end{bmatrix}$$



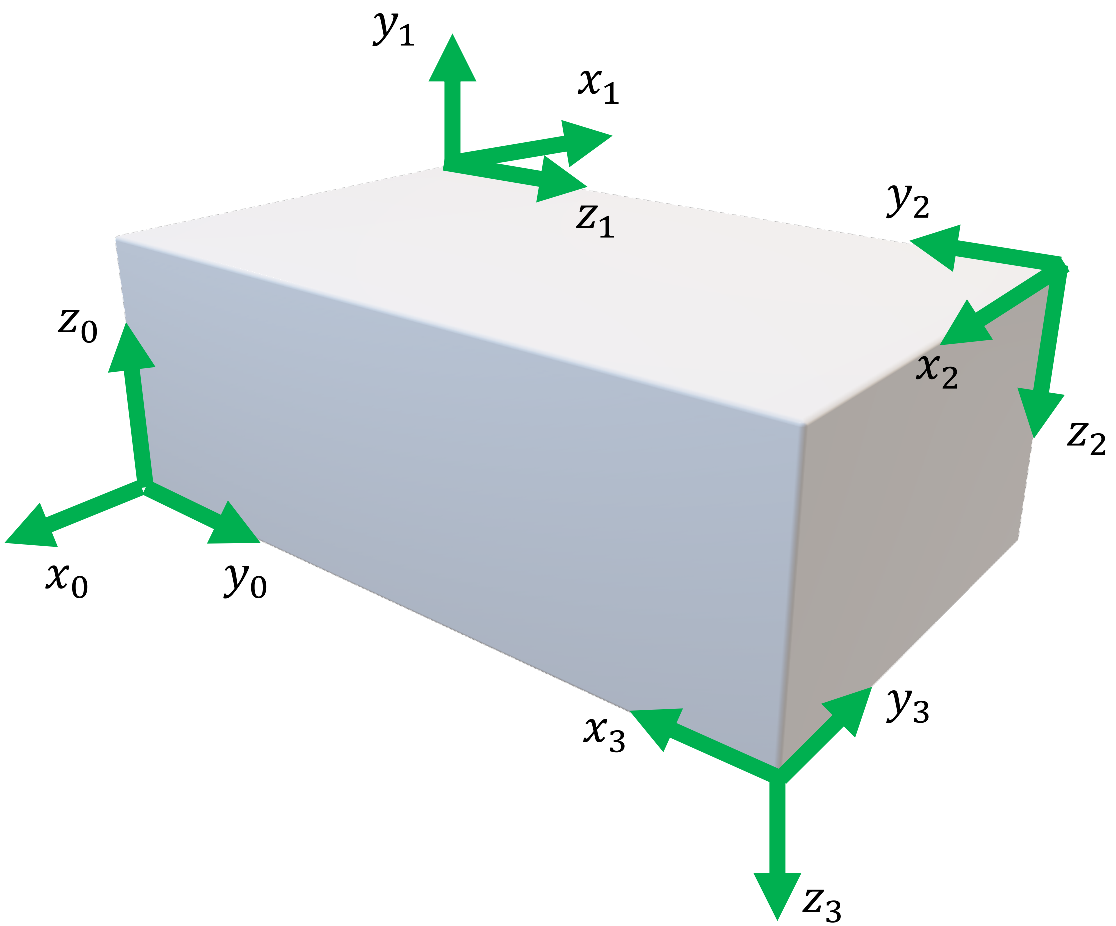
$$R_1^0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_0^1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

A bunch of examples:

A rectangular solid: all angles are multiples of $\pi/2$.

$$R_j^i = \begin{bmatrix} x_j \cdot x_i & y_j \cdot x_i & z_j \cdot x_i \\ x_j \cdot y_i & y_j \cdot y_i & z_j \cdot y_i \\ x_j \cdot z_i & y_j \cdot z_i & z_j \cdot z_i \end{bmatrix}$$



$$R_1^0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad R_0^1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

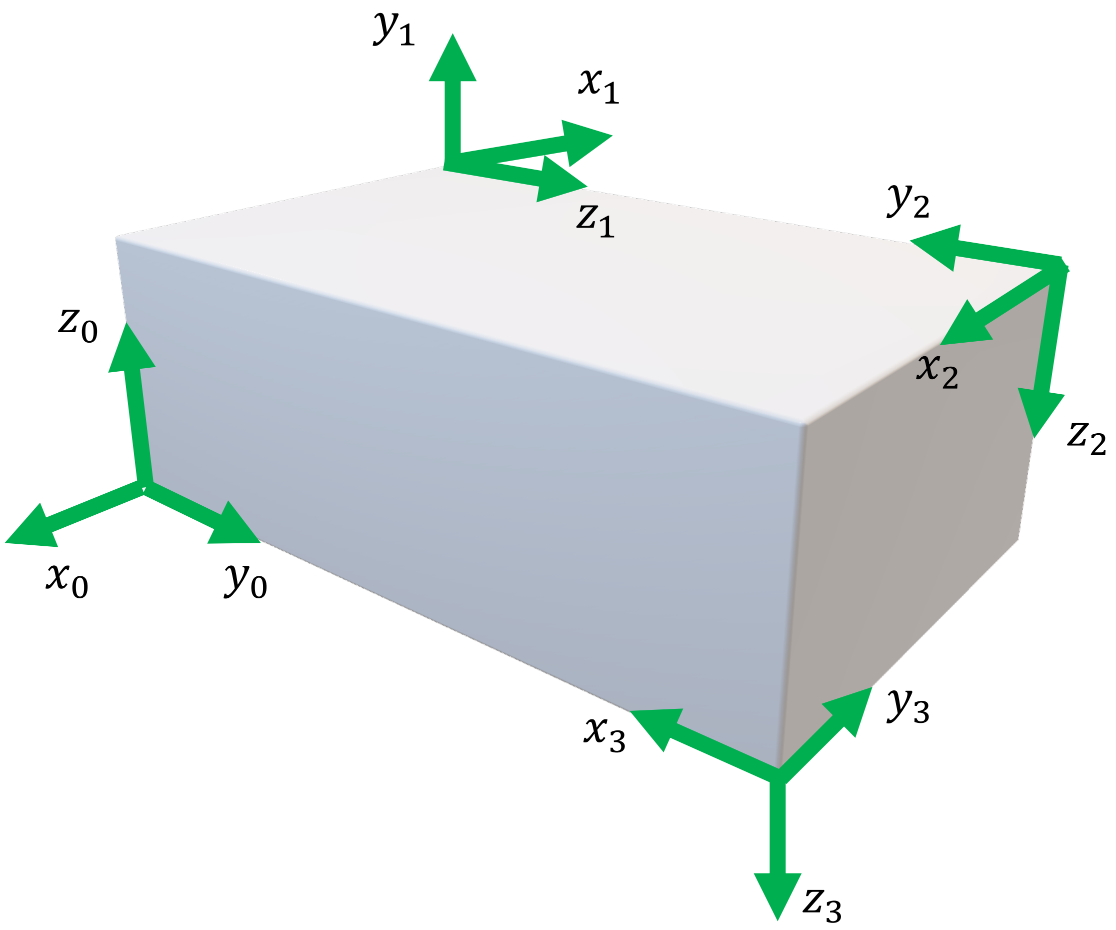
$$R_1^0 R_0^1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{(R_1^0)^{-1} = R_0^1 = (R_1^0)^T}$$

A bunch of examples:

A rectangular solid: all angles are multiples of $\pi/2$.

$$R_j^i = \begin{bmatrix} x_j \cdot x_i & y_j \cdot x_i & z_j \cdot x_i \\ x_j \cdot y_i & y_j \cdot y_i & z_j \cdot y_i \\ x_j \cdot z_i & y_j \cdot z_i & z_j \cdot z_i \end{bmatrix}$$



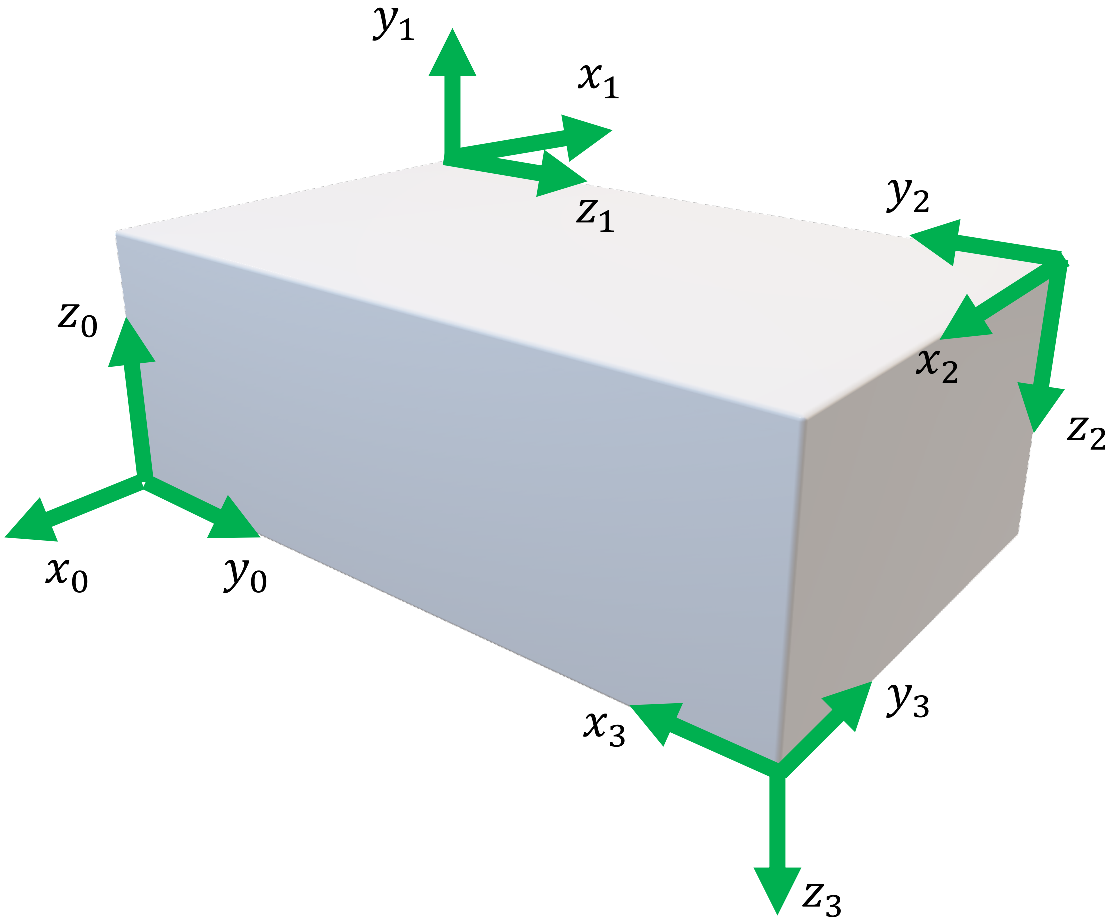
$$R_2^1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$R_2^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

A bunch of examples:

A rectangular solid: all angles are multiples of $\pi/2$.

$$R_j^i = \begin{bmatrix} x_j \cdot x_i & y_j \cdot x_i & z_j \cdot x_i \\ x_j \cdot y_i & y_j \cdot y_i & z_j \cdot y_i \\ x_j \cdot z_i & y_j \cdot z_i & z_j \cdot z_i \end{bmatrix}$$



$$R_3^0 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

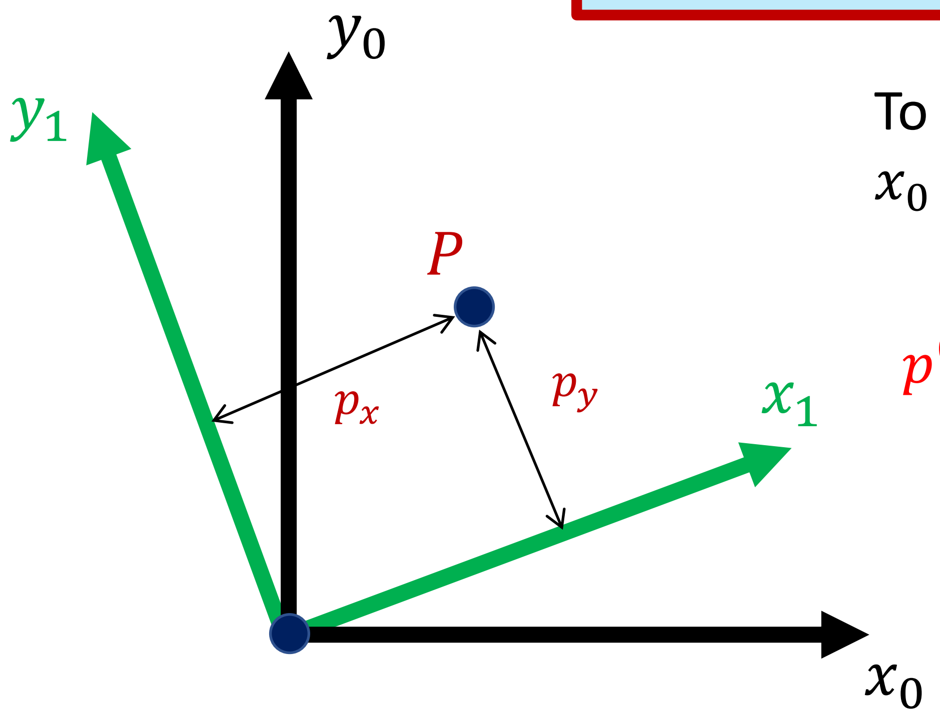
Let’s extend this to 3D rotational coordinate transformations.

Coordinate Transformations (rotation only)

Suppose a point P is rigidly attached to coordinate Frame 1, with coordinates given by $P^1 = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$.

We can express the location of the point P in terms of its coordinates

$$P = p_x x_1 + p_y y_1$$



To obtain the coordinates of P w.r.t. Frame 0, we project P onto the x_0 and y_0 axes:

$$p^0 = \begin{bmatrix} P \cdot x_0 \\ P \cdot y_0 \end{bmatrix} = \begin{bmatrix} (p_x x_1 + p_y y_1) \cdot x_0 \\ (p_x x_1 + p_y y_1) \cdot y_0 \end{bmatrix} = \begin{bmatrix} p_x (x_1 \cdot x_0) + p_y (y_1 \cdot x_0) \\ p_x (x_1 \cdot y_0) + p_y (y_1 \cdot y_0) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \boldsymbol{R_1^0}\, \boldsymbol{P^1}$$

$$\boxed{\boldsymbol{P^0} = \boldsymbol{R_1^0}\, \boldsymbol{P^1}}$$

3D Coordinate Transformations (rotation only)

Suppose a point P is rigidly attached to coordinate Frame 1, with coordinates given by $P^1 = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$.

We can express the location of the point P in terms of its coordinates

$$P = p_x x_1 + p_y y_1 + p_z z_1$$

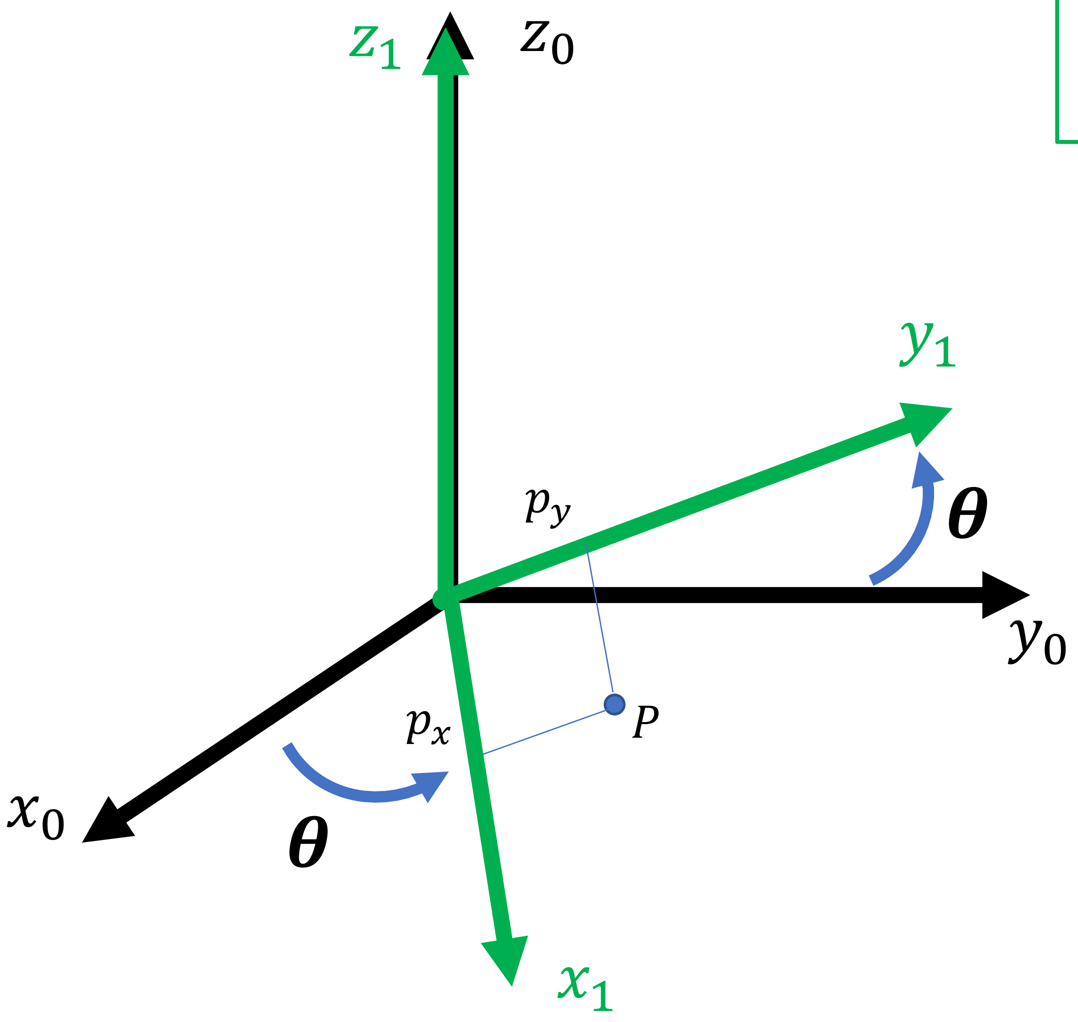
To obtain the coordinates of P w.r.t. Frame 0, we project P onto the x_0 and y_0 axes:

$$p^0 = \begin{bmatrix} P \cdot x_0 \\ P \cdot y_0 \\ P \cdot z_0 \end{bmatrix} = \begin{bmatrix} (p_x x_1 + p_y y_1 + p_z z_1) \cdot x_0 \\ (p_x x_1 + p_y y_1 + p_z z_1) \cdot y_0 \\ (p_x x_1 + p_y y_1 + p_z z_1) \cdot z_0 \end{bmatrix} = \begin{bmatrix} p_x (x_1 \cdot x_0) + p_y (y_1 \cdot x_0) + p_z (z_1 \cdot x_0) \\ p_x (x_1 \cdot y_0) + p_y (y_1 \cdot y_0) + p_z (z_1 \cdot z_0) \\ p_x (x_1 \cdot z_0) + p_y (y_1 \cdot z_0) + p_z (z_1 \cdot z_0) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot z_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \boldsymbol{R_1^0 \, P^1}$$

$$\boxed{\boldsymbol{P^0 = R_1^0 \, P^1}}$$

The simplest example: rotation about the z axis



As we saw above:

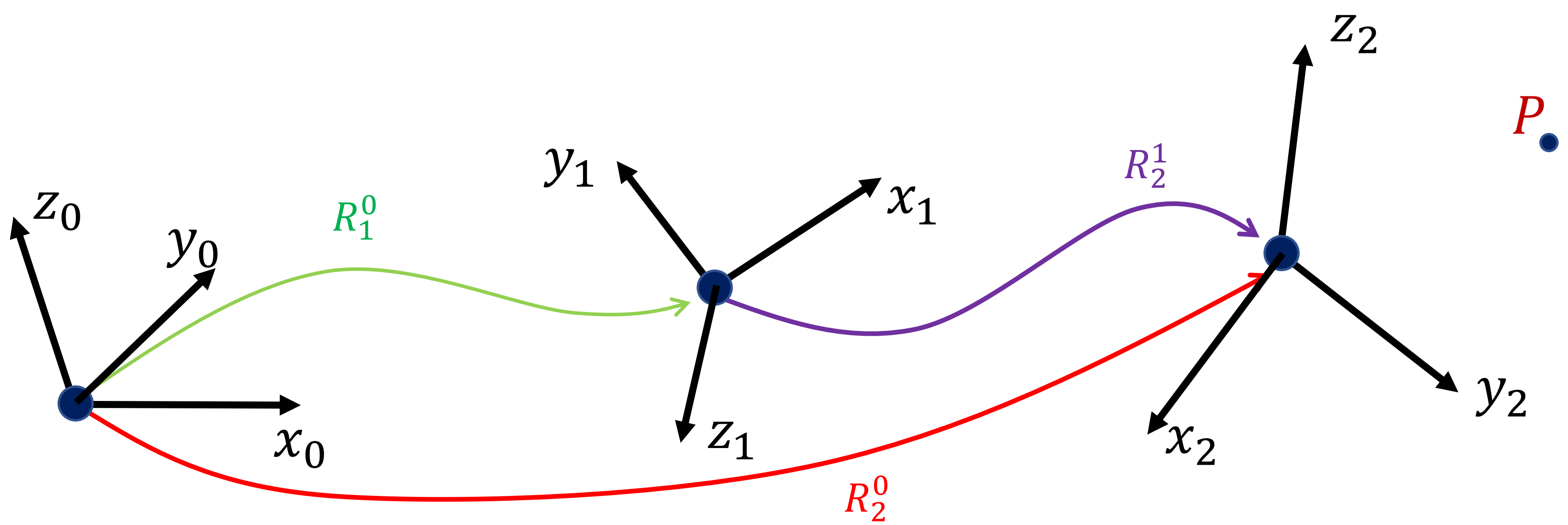
$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The equation for rotational coordinate transformations generalizes immediately to the 3D case!

$$\boldsymbol{P}^{\mathbf{0}} = \boldsymbol{R}_{\mathbf{1}}^{\mathbf{0}}\, \boldsymbol{P}^{\mathbf{1}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix}$$

Composition of Rotations

For now, only consider the rotation, *not the translation*!
This is an "exploded" view of three coordinate frames that share the same origin.



From our previous results, we know:

$$P^0 = R_1^0 P^1$$
$$P^1 = R_2^1 P^2$$

$$P^0 = R_1^0 R_2^1 P^2$$

But we also know: $P^0 = R_2^0 P^2$

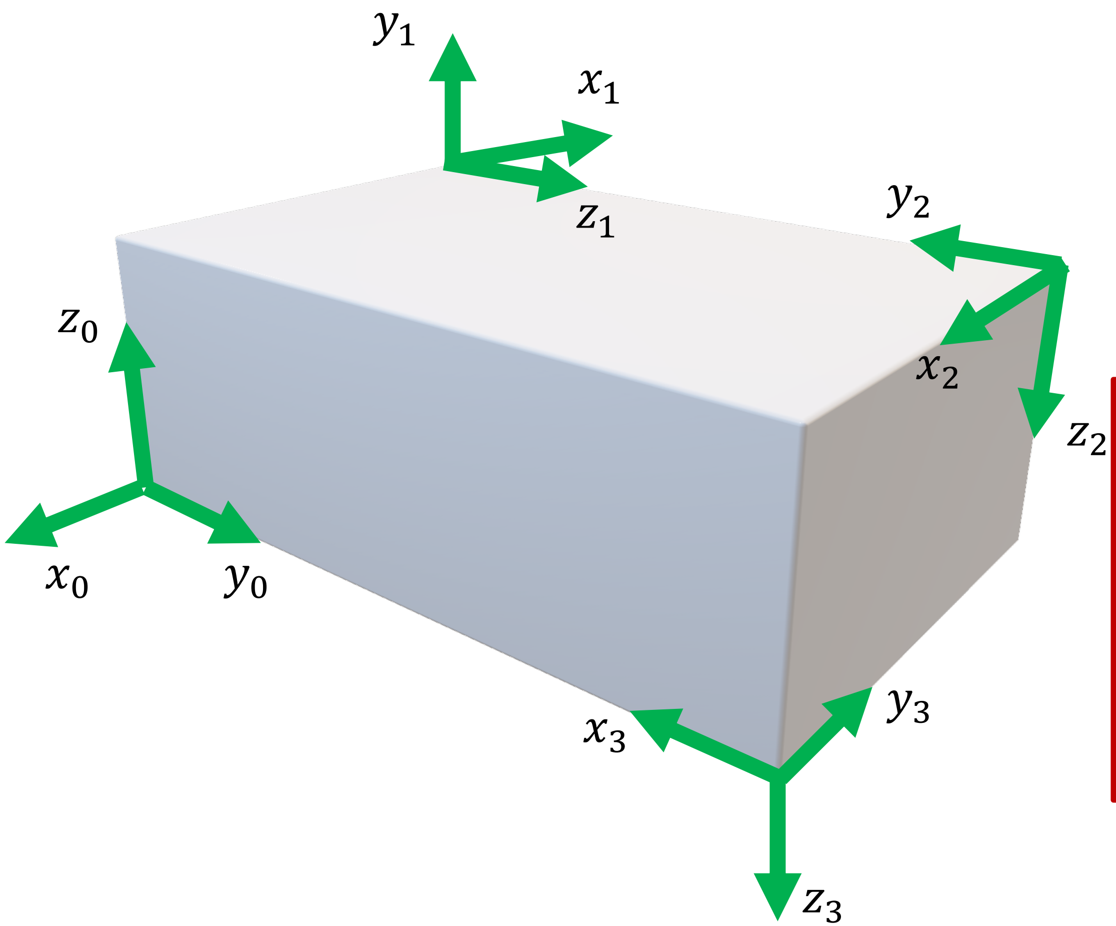
$$R_2^0 = R_1^0 R_2^1$$

This is the composition law for rotation transformations.

A bunch of examples:

A rectangular solid: all angles are multiples of $\pi/2$.

$$R_j^i = \begin{bmatrix} x_j \cdot x_i & y_j \cdot x_i & z_j \cdot x_i \\ x_j \cdot y_i & y_j \cdot y_i & z_j \cdot y_i \\ x_j \cdot z_i & y_j \cdot z_i & z_j \cdot z_i \end{bmatrix}$$



$$R_1^0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad R_2^1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$R_2^0 = R_1^0 R_2^1$$

$$\boxed{R_2^0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}$$

This agrees with our earlier result!

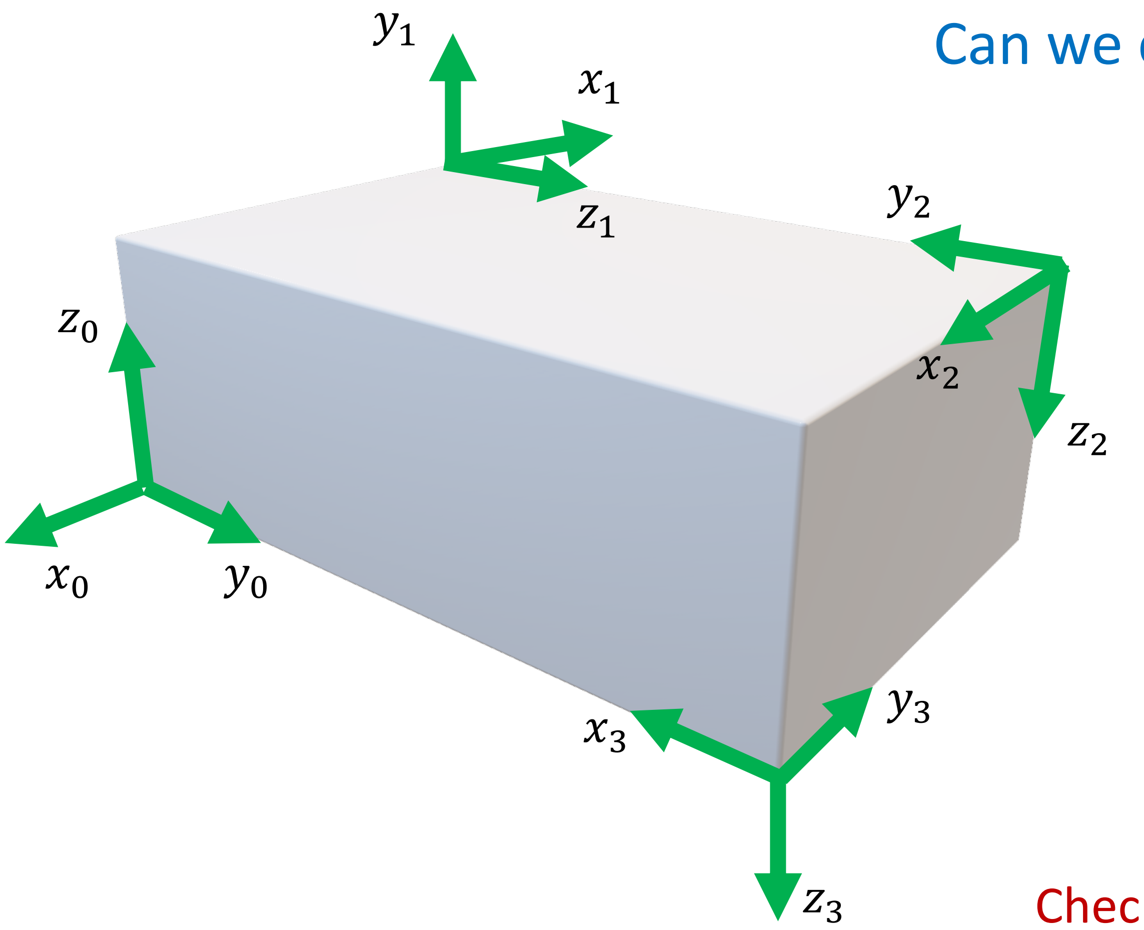
A bunch of examples:

$$R_j^i = \begin{bmatrix} x_j \cdot x_i & y_j \cdot x_i & z_j \cdot x_i \\ x_j \cdot y_i & y_j \cdot y_i & z_j \cdot y_i \\ x_j \cdot z_i & y_j \cdot z_i & z_j \cdot z_i \end{bmatrix}$$

A rectangular solid: all angles are multiples of $\pi/2$.

In preceding examples, we have computed R_1^0, R_2^0, R_3^0.

Can we compute R_3^2?



$$\boxed{\begin{aligned} R_3^0 &= R_2^0 R_3^2 \\ (R_2^0)^{-1}\, R_3^0 &= R_3^2 \\ (R_2^0)^T\, R_3^0 &= R_3^2 \\ R_0^2 R_3^0 &= R_3^2 \end{aligned}}$$

$$R_3^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

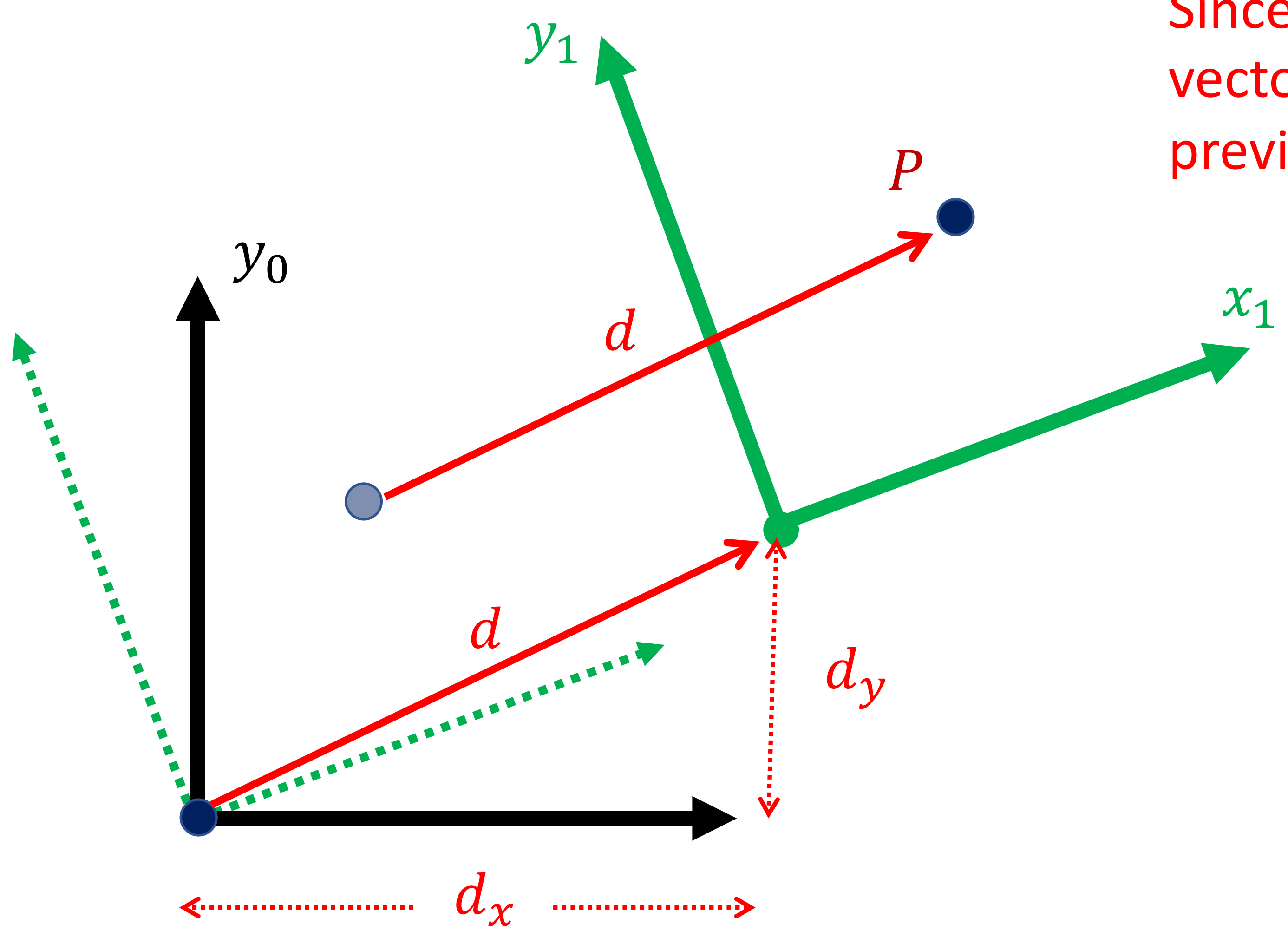
Check this against the figure by directly determining R_3^2… it works!

Now let’s add translation…

Specifying Pose in the Plane

Suppose we now translate Frame 1 (*no new rotatation*).
What are the coordinates of P w.r.t. Frame 0?

Since we merely translated P by a fixed vector d, simply add the offset to our previous result!

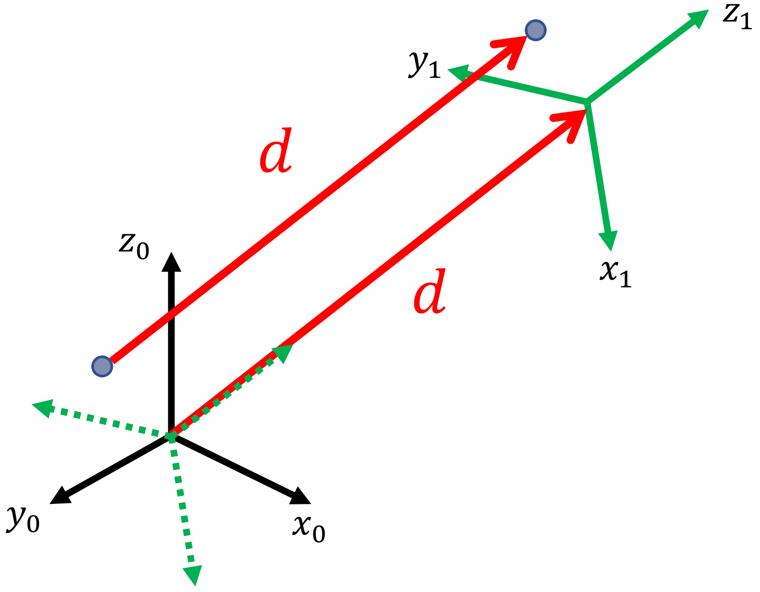


$$\boldsymbol{P^0 = R_1^0\, P^1 + d^0}$$

$$\boldsymbol{d^0 = \begin{bmatrix} d_x \\ d_y \end{bmatrix}}$$

Specifying Pose in 3D

Suppose, after rotation, we now translate Frame 1 (*no new rotatation*).
What are the coordinates of P w.r.t. Frame 0?



Since we merely translated P by a fixed vector d, simply add the offset to our previous result!

$$\boldsymbol{P^0} = \boldsymbol{R_1^0}\,\boldsymbol{P^1} + \boldsymbol{d^0}$$

$$\boldsymbol{d^0} = \begin{bmatrix} \boldsymbol{d_x} \\ \boldsymbol{d_y} \\ \boldsymbol{d_z} \end{bmatrix}$$

Homogeneous Transformations

We can simplify the equation for coordinate transformations by augmenting the vectors and matrices with an extra row:

This is just our eqn from the previous page

$$\boxed{\begin{bmatrix} \boldsymbol{P}^0 \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_1^0 \boldsymbol{P}^1 + \boldsymbol{d}^0 \\ 1 \end{bmatrix}} = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{d}^0 \\ 0_n & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^1 \\ 1 \end{bmatrix}$$

in which $0_n = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}$

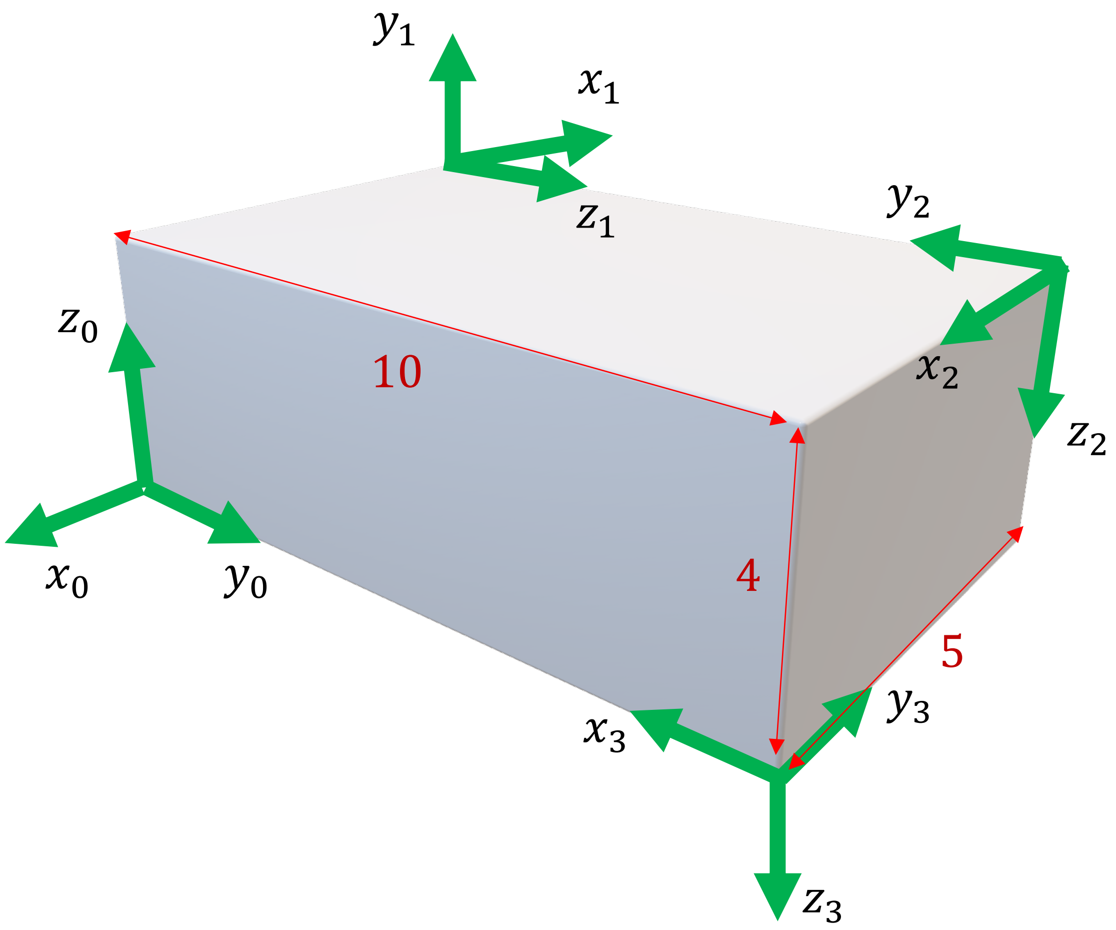
The set of matrices of the form $\begin{bmatrix} R & d \\ 0_n & 1 \end{bmatrix}$, where $R \in SO(n)$ and $d \in \mathbb{R}^n$ is called

the ***Special Euclidean Group of order n***, or $SE(n)$.

A bunch of examples:

$$R_j^i = \begin{bmatrix} x_j \cdot x_i & y_j \cdot x_i & z_j \cdot x_i \\ x_j \cdot y_i & y_j \cdot y_i & z_j \cdot y_i \\ x_j \cdot z_i & y_j \cdot z_i & z_j \cdot z_i \end{bmatrix}$$

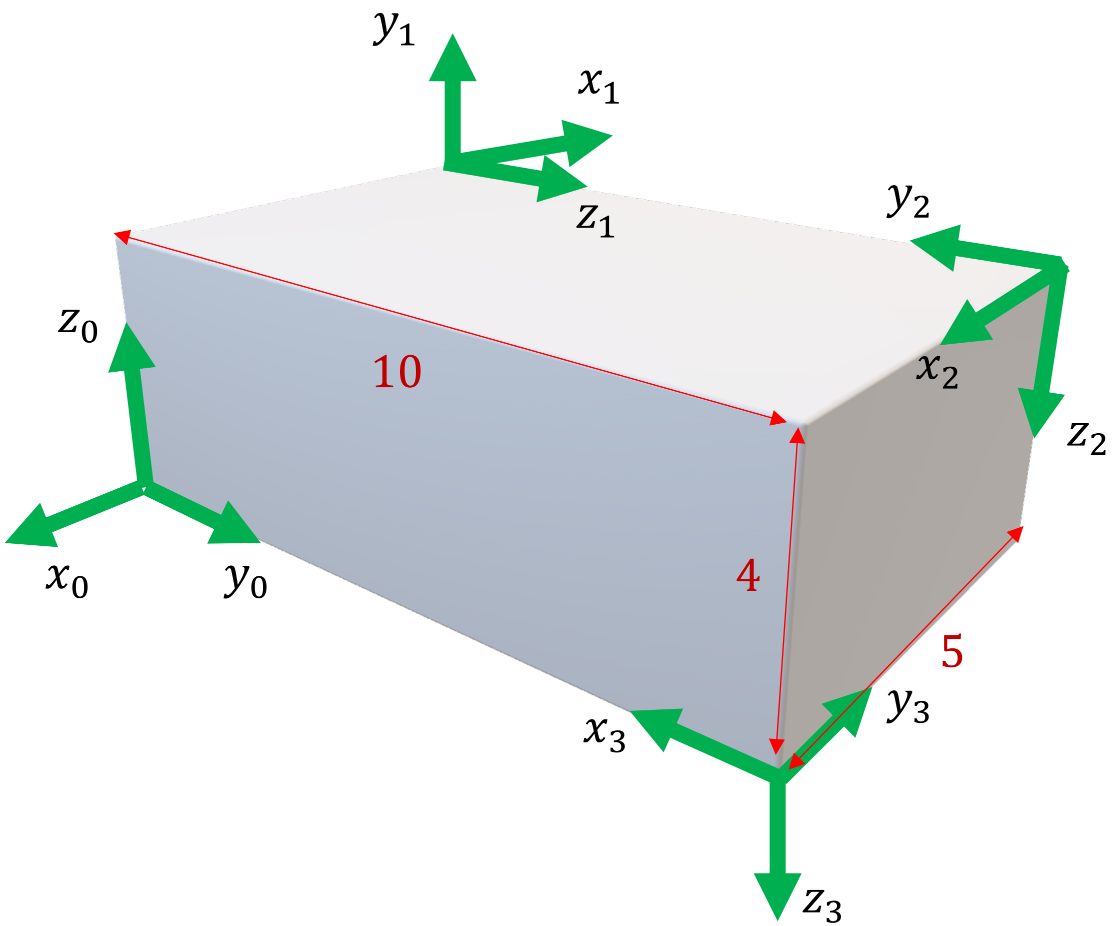
A rectangular solid: all angles are multiples of $\pi/2$.



Now let's look at both the relative orientation and relative position of frames.

A bunch of examples:

A rectangular solid: all angles are multiples of $\pi/2$.

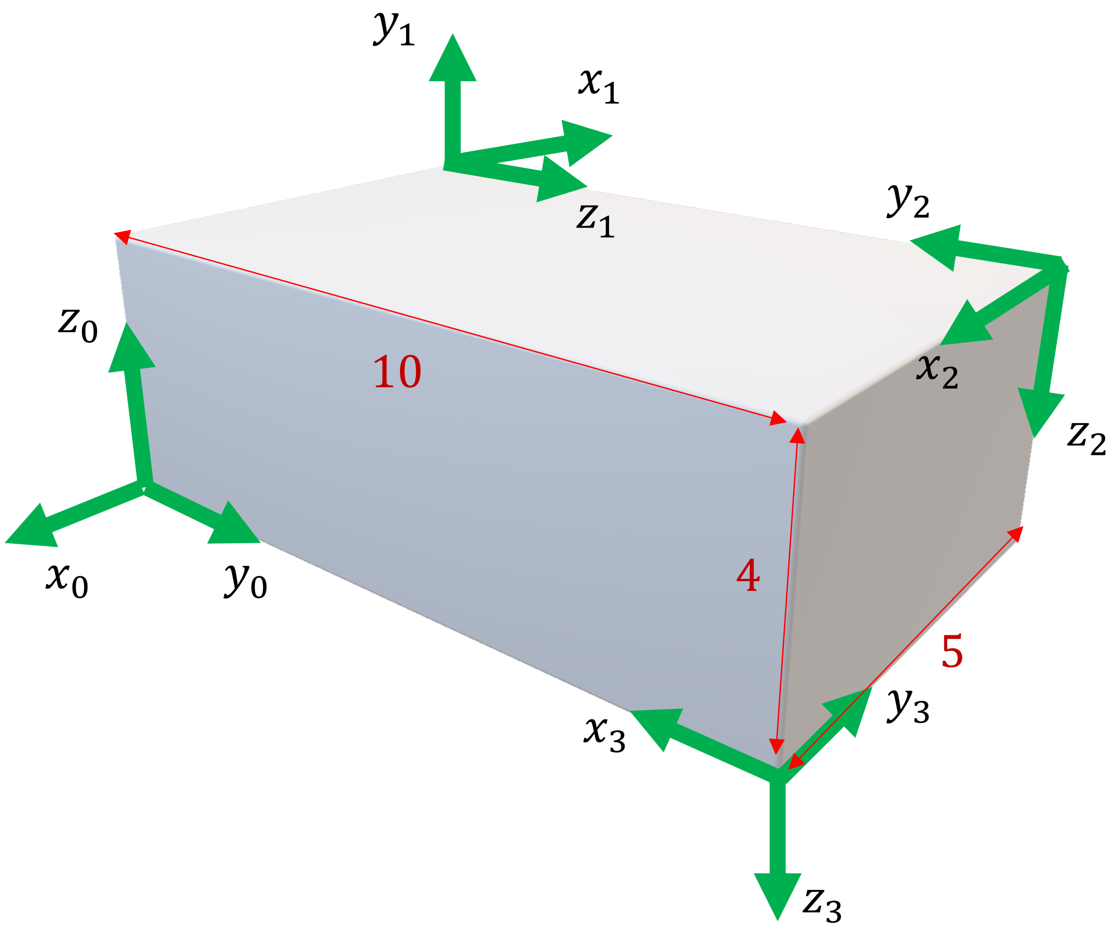


$$R_1^0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$T_1^0 = \begin{bmatrix} -1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A bunch of examples:

A rectangular solid: all angles are multiples of $\pi/2$.

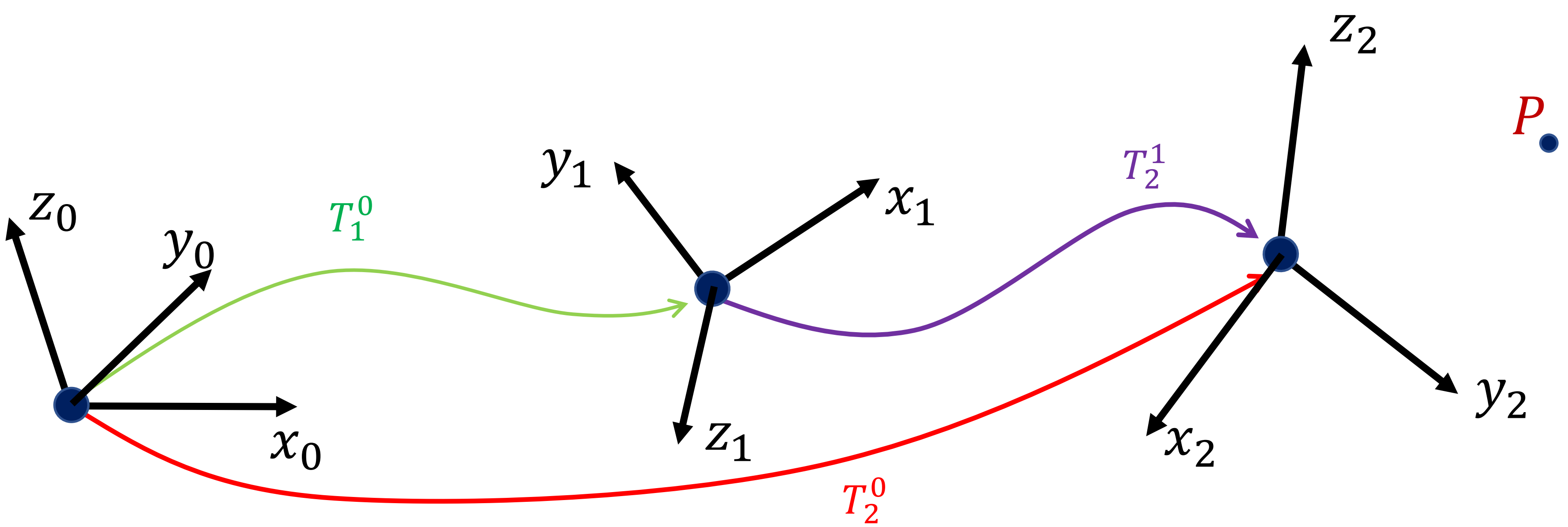


$$R_2^1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$T_2^1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Composition of Transformations

Now, consider the rotation *and the translation!*



$$\boxed{\tilde{P} = \begin{bmatrix} p_x \\ P_y \\ p_z \\ 1 \end{bmatrix}}$$

From our previous results, we know:

$$\left.\begin{aligned} \tilde{P}^0 &= T_1^0 P^1 \\ \tilde{P}^1 &= T_2^1 \tilde{P}^2 \end{aligned}\right\} \Longrightarrow \tilde{P}^0 = T_1^0 T_2^1 \tilde{P}^2$$

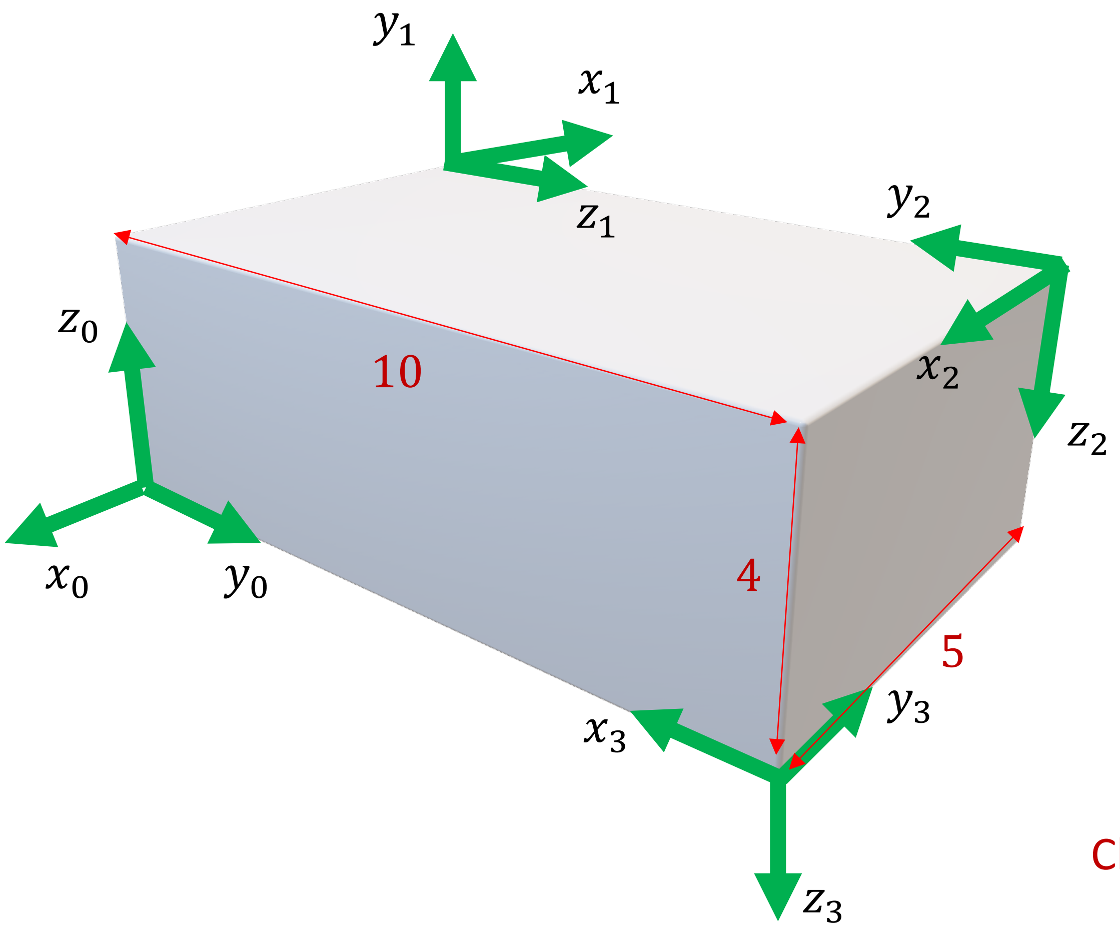
But we also know: $\tilde{P}^0 = T_2^0 \tilde{P}^2$

$$\left.\begin{aligned} \tilde{P}^0 &= T_1^0 T_2^1 \tilde{P}^2 \\ \tilde{P}^0 &= T_2^0 \tilde{P}^2 \end{aligned}\right\} \Longrightarrow T_2^0 = T_1^0 T_2^1$$

This is the composition law for homogeneous transformations.

A bunch of examples:

A rectangular solid: all angles are multiples of $\pi/2$.



$$T_1^0 = \begin{bmatrix} -1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 = \begin{bmatrix} -1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & -1 & 0 & 10 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Check this by directly determining T_2^0 from the figure… it works!

Inverse of a Homogeneous Transformation

What is the relationship between T_j^i and T_i^j?

$$\text{In general, } \quad T_k^j = \left(T_j^k\right)^{-1} \text{ and } \begin{bmatrix} \boldsymbol{R} & \boldsymbol{d} \\ 0_n & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{d} \\ 0_n & 1 \end{bmatrix}$$

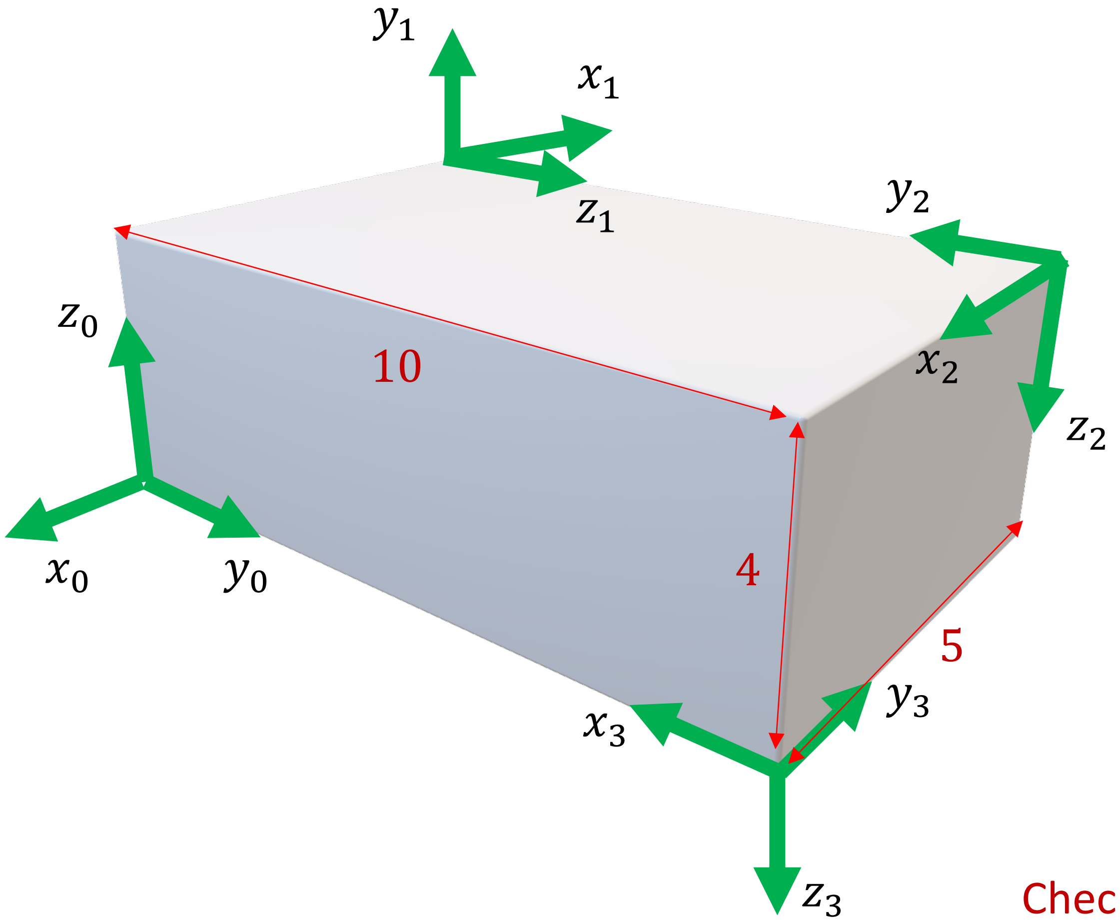
This is easy to verify:

$$\begin{bmatrix} \boldsymbol{R} & \boldsymbol{d} \\ 0_n & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{d} \\ 0_n & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}\boldsymbol{R}^T & -\boldsymbol{R}\boldsymbol{R}^T\boldsymbol{d} + \boldsymbol{d} \\ 0_n & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_{n\times n} & \boldsymbol{0}_n \\ 0_n & 1 \end{bmatrix} = I_{(n+1)\times(n+1)}$$

A bunch of examples:

A rectangular solid: all angles are multiples of $\pi/2$.

$$T_2^0 = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & -1 & 0 & 10 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

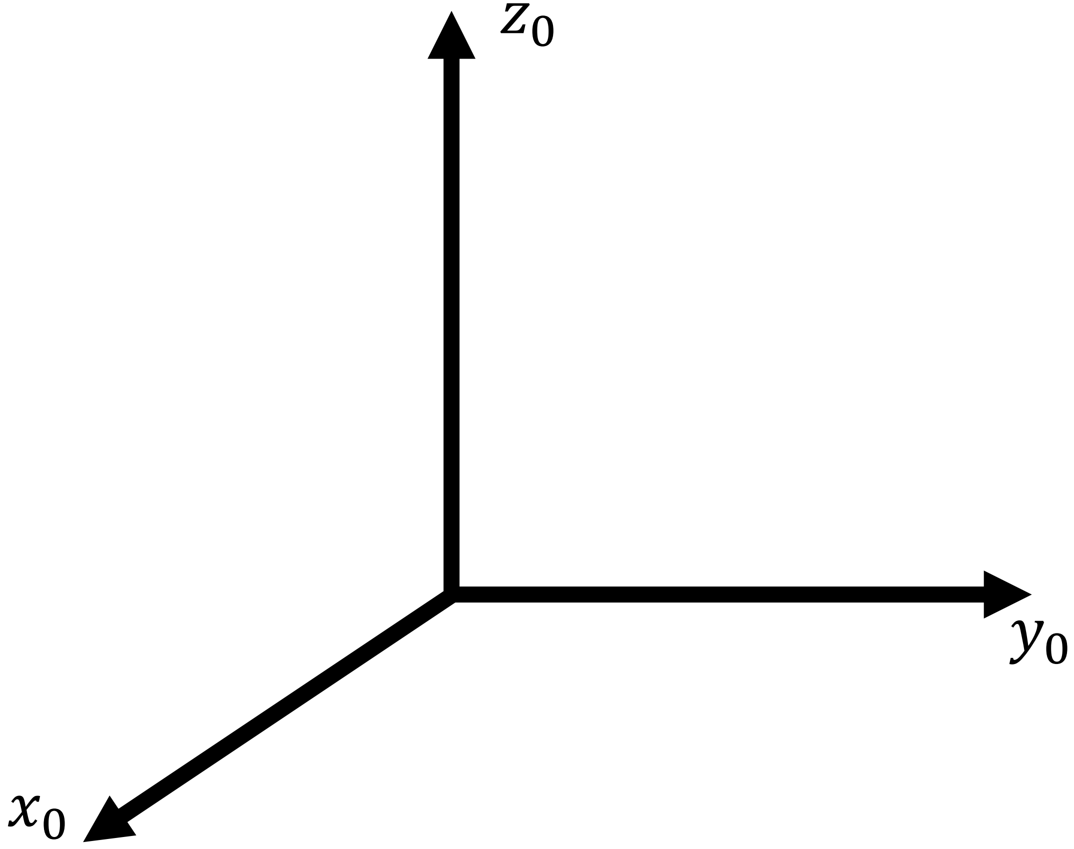


$$(T_2^0)^{-1} = \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{d} \\ 0_n & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 10 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 (T_2^0)^{-1} = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & -1 & 0 & 10 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 10 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Check this by directly determining T_0^2 from the figure… it works!

Rotations about Coordinate Axes



$$R_{x,\psi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\phi} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Parameterization of 3D Rotations

- Consider the three successive rotations: $R = R_{z,\phi} R_{y,\theta} R_{x,\psi}$

$$R_{z,\phi} R_{y,\theta} R_{x,\psi} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

$$= \begin{bmatrix} C_\phi C_\theta & -S_\phi C_\psi + C_\phi S_\theta S_\psi & -S_\phi S_\psi + C_\phi S_\theta C_\psi \\ S_\phi C_\theta & C_\phi C_\psi + S_\phi S_\theta S_\psi & -C_\phi S_\psi + S_\phi S_\theta C_\psi \\ S_\theta & C_\theta S_\psi & C_\theta C_\psi \end{bmatrix}$$

Parameterization of 3D Rotations

Any rotation matrix can be expressed in this form!

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} C_\phi C_\theta & -S_\phi C_\psi + C_\phi S_\theta S_\psi & S_\phi S_\psi + C_\phi S_\theta C_\psi \\ S_\phi C_\theta & C_\phi C_\psi + S_\phi S_\theta S_\psi & -C_\phi S_\psi + S_\phi S_\theta C_\psi \\ -S_\theta & C_\theta S_\psi & C_\theta C_\psi \end{bmatrix}$$

1. Solve for θ using $\mathrm{r}_{31} = S_\theta$
2. Solve for ψ using $\mathrm{r}_{32} = C_\theta S_\psi, r_{33} = C_\theta C_\psi$
3. Solve for ϕ using $\mathrm{r}_{11} = C_\phi C_\theta, \mathrm{r}_{21} = S_\phi C_\theta$

The function $ATAN2(y, x)$ returns the angle whose tangent is y/x, in the appropriate quadrant. Thus:

$$\psi = ATAN2(r_{32}, r_{33})$$
$$\phi = ATAN2(r_{21}, r_{11})$$

We can parameterize SO(3) using these three angles, ϕ, θ, ψ.

Singularities for Roll,Pitch,Yaw

- For Roll, Pitch, and Yaw, when $S_\theta = 1$, we have:

$$R_{z,\phi}R_{y,\theta}R_{x,\psi} = \begin{bmatrix} C_\phi C_\theta & -S_\phi C_\psi + C_\phi S_\theta S_\psi & S_\phi S_\psi + C_\phi S_\theta C_\psi \\ S_\phi C_\theta & C_\phi C_\psi + S_\phi S_\theta S_\psi & -C_\phi S_\psi + S_\phi S_\theta C_\psi \\ -S_\theta & C_\theta S_\psi & C_\theta C_\psi \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -S_\phi C_\psi + C_\phi S_\psi & -S_\phi S_\psi + C_\phi C_\psi \\ 0 & C_\phi C_\psi + S_\phi S_\psi & -C_\phi S_\psi + S_\phi C_\psi \\ 1 & 0 & 0 \end{bmatrix}$$

When $S_\theta = 1$, there are infinitely many solutions for ψ and ϕ.
In this case, we have, e.g.,

$$r_{22} = C_\phi C_\psi + S_\phi S_\psi = C_{\phi-\psi}$$

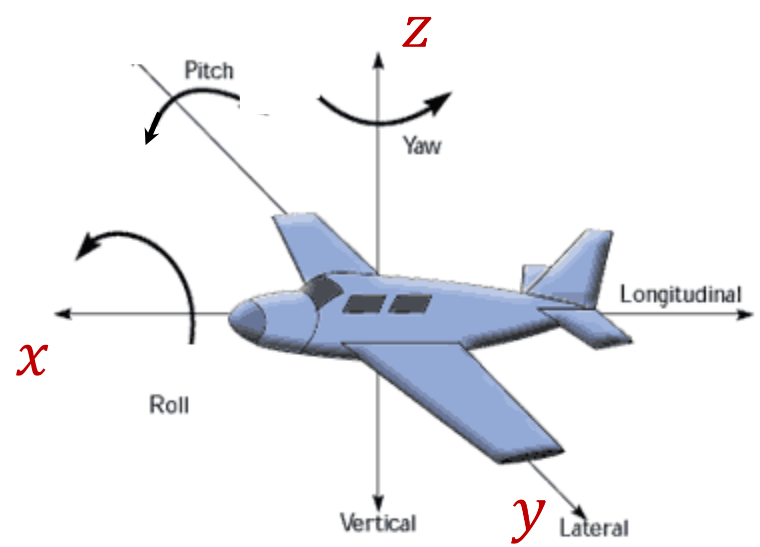
So, only the difference $\phi - \psi$ is uniquely determined.

So, this is only a local parameterization.
It works when $r_{31} \neq \pm 1$.

Roll, Pitch, and Yaw

When we parameterize the rotation matrices using $R = R_{z,\phi}R_{y,\theta}R_{x,\psi}$, the angles are called roll, pitch, and yaw:

- Yaw is a rotation about the world's z-axis.
- Pitch is a rotation about the plane's y-axis (note, this axis moves as a function of the yaw angle).
- Roll is a rotation about the plane's x-axis (note, this axis moves as a function of the yaw angle and the pitch angle).



➢ Remember, things break down when $\theta = \pm\frac{\pi}{2}$ --- but this makes sense!

If $\theta = \pm\frac{\pi}{2}$ then the planes x-axis is aligned with the world's z-axis.

In this case, roll and yaw are rotations about the same axis!

- Roll, Pitch, and Yaw are useful when the plane is roughly horizontal.
- If the plane tips completely up or completely down (i.e., $\theta = \pm\frac{\pi}{2}$), things have already gone very wrong, so it's not such a big deal that the parameterization breaks down for this case.

This coordinate frame assignment is known as *Forward-Left-Up (FLU).*

- The x-axis is the Forward direction.
- The y-axis points to the Left.
- The z-axis points Up.

Other Parameterization of 3D Rotations

Consider the three successive rotations: $R = R_{a,\phi} R_{b,\theta} R_{c,\psi}$

where a, b, c denote coordinate axes and $a \neq b, b \neq c$.

There are lots of possibilities!

$R_{z,\phi} R_{y,\theta} R_{z,\psi}$

$R_{x,\phi} R_{y,\theta} R_{z,\psi}$

$R_{x,\phi} R_{y,\theta} R_{x,\psi}$

$R_{y,\phi} R_{z,\theta} R_{x,\psi}$

$\vdots$

$R_{z,\phi} R_{x,\theta} R_{z,\psi}$

- These are generically referred to as ***Euler angles***.
- Each set of Euler angles admits a local parameterization of $SO(3)$.
- Like Roll-Pitch-Yaw, each Euler angle parameterization is a *local* parameterization, and has problems for certain configurations.
- The configurations where things break down are referred to as ***singularities***.
- These singularities are the reason Oculus was successful in the VR headset market. Previous designs often used Euler angle parameterizations of rotation, and looking directly up caused the display to spin wildly.
- ***The z-y-z Euler angles are commonly used to parameterize the orientation of the wrist mechanism of robot manipulators.***