Lecture 3

Probability, actions, and expectation

CS 3630

Lecture 2 Recap

A Taxonomy of Robotics Topics

For each module in this class, we'll consider six distinct aspects of robotics:

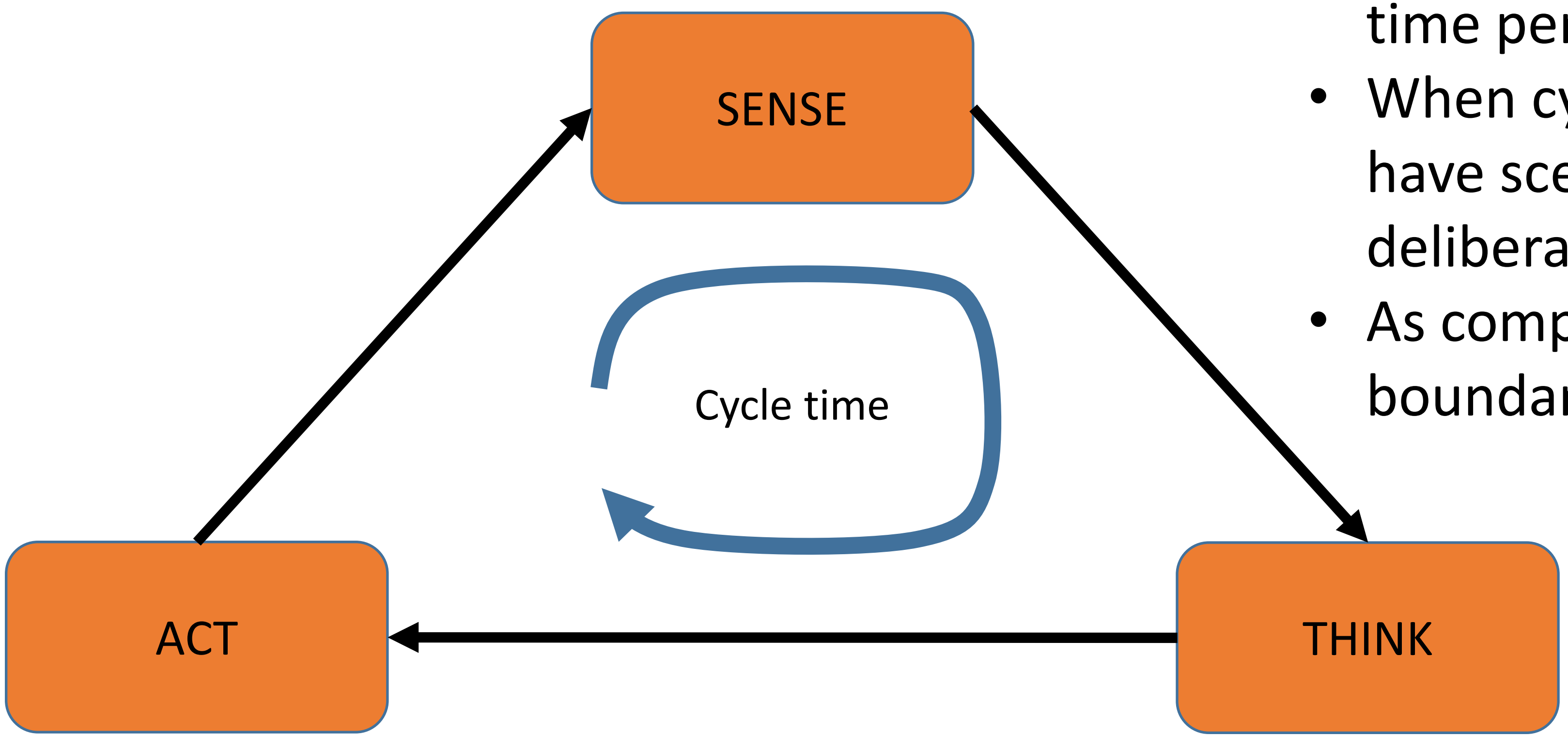
1. **State:** How does the robot represent its world, and itself?
2. **Actions:** What can the robot do, and how to represent this?
3. **Sensors:** What information about the world can be ascertained via sensing, and how do we model this process?
4. **Perception:** How can we combine sensor data with contextual knowledge to understand the current state?
5. **Planning:** What actions should the robot execute to transform the state of the world into a desired goal state?
6. **Learning:** How can the robot improve its knowledge over time, using information that it acquires during operation?

Each chapter of the book includes six sections, corresponding to these topics.

Sense, Think, Act at Different Time Scales

The time to complete one cycle of this loop depends on the task:

- Playing chess: minutes
- Hand-eye coordination: 30 Hz
- Force controlled robot: Order of KHz



- When cycle time is very fast, we use tools from control theory, and model systems using differential equations (continuous time performance).
- When cycle time is very slow, we might have scene understanding and deliberative planning.
- As computers become faster, the boundary between these begins to blur.

Modeling Uncertainty

We assume that there is uncertainty in sensing, and therefore, it is not possible to know with certainty the world state.

We consider the state to be a random quantity, with five possible outcomes:

$$\Omega = \{\text{cardboard, paper, cans, scrap metal, bottle}\}$$

In probability theory,

- The set Ω is called the sample space.
- Each $\omega \in \Omega$ is called an outcome.
- A subset $A \subset \Omega$ is called an event.

Denote by $\mathfrak{B} = \{A | A \subset \Omega\}$ the set of all events.

Probability distributions map events to probabilities, $\boldsymbol{P}: \boldsymbol{\mathfrak{B}} \rightarrow [\mathbf{0}, \mathbf{1}]$

Some properties of probability distributions

Three Axioms of Probability Theory:

1. For A $\subset \Omega$, P(A) ≥ 0
   - *There's no such thing as negative probability.*
2. P(Ω) =1
   - *The probability that something happened is 1.*
3. For $\mathrm{A_i}, \mathrm{A_j} \subset \Omega$, if $\mathrm{A_i} \cap \mathrm{A_j} = \emptyset$, then $P\left(A_i \cup A_j\right) = P(A_i) + P\left(A_j\right)$
   - *If two events are disjoint (aka mutually exclusive), then the probability that one of the two events occurred equals the sum of the probabilities for the two events.*
   - *The second and third axiom immediately imply that* P($\emptyset$) =0.

Borel’s law of large numbers

- Let $A \subset \Omega$ be an event with probability $\mathrm{P}(A) = p$.
- Suppose we run our experiment n times, and we observe that event A occurs $N_n(A)$ times.
- Then, with probability one

$$\frac{N_n(A)}{n} \to p \text{ as } n \to \infty$$

➢ ***As the number of trials goes to infinity, the proportion of times that an event occurs approaches the probability of that event.***

➢ ***If we make enough observations, we can start to trust that we have good estimates of prior probabilities!***

Shamelessly taken from Wikipedia.

Random Variables, Discrete Probability Distributions, Expectation

Random Variables

A ***random variable*** is a mapping from outcomes to real numbers, $X: \Omega \rightarrow \mathbb{R}$.

For example, we can map our categories to integers:

- Cardboard $\longrightarrow$ 0
- Paper $\longrightarrow$ 1
- Can $\longrightarrow$ 2
- Scrap Metal $\longrightarrow$ 3
- Bottle $\longrightarrow$ 4

- We typically use upper case letters, e.g., X, to denote a random variable, and lower-case letters, e.g., x_i, to denote the values taken by X.
- In our example, $X \in \{0,1,2,3,4\}$ indicates that X is a random variable that can take values from the set $\{0,1,2,3,4\}$.

Probability Mass Functions (pmf's)

- When a random variable takes its values from a finite (or possibly countably infinite) set, it is called a ***discrete random variable***.
- The probability distribution for a discrete random variable is typically defined as a ***probability mass function (pmf)***.
- For random variable X, the pmf is defined as

$$p_X(x) \triangleq P(X = x)$$

For our example,

- $p_X(0) = 0.20$ $\quad$ *cardboard*
- $p_X(1) = 0.30$ $\quad$ *paper*
- $p_X(2) = 0.25$ $\quad$ *can*
- $p_X(3) = 0.20$ $\quad$ *scrap metal*
- $p_X(4) = 0.05$ $\quad$ *bottle*

As we will soon see, random variables can be very useful for outcomes that are naturally associated to real numbers, e.g., roll of a die, weight of a person, or, in our case, cost of applying an action.

Using pmf's

Even for this example, where categories don't naturally have numerical semantics, we can use the pmf to answer interesting questions.

For example, what is the probability that an object is a paper product?

- Paper products correspond to paper and cardboard, $X \in \{0,1\}$:

$$P(X \in \{0,1\}) = p_X(0) + p_X(1) = 0.5$$

Alternatively, we could write:

$$P(X \in \{0,1\}) = P(X \leq 1)$$

This form, $P(X \leq \alpha)$ turns out to be very useful.

Cumulative Distribution Function

The Cumulative Distribution Function (CDF) is defined as

$$F_X(\alpha) = P(X \leq \alpha) = \sum_{x_i \leq \alpha} p_X(x_i)$$

If we order the x_i's, such that $x_0 < x_2 \ldots < x_n$ we can write this as:

$$F_X(\alpha) = P(X \leq \alpha) = \sum_{i=0}^{k-1} p_X(x_i)$$

when we choose k such that $x_{k-1} \leq \alpha < x_k$.

CDF for our trash categories

It is straightforward to compute the CDF for the r.v. associated to various trash categories:

$$F_X(\alpha) = P(X \leq \alpha) = \sum_{i=0}^{k-1} p_X(x_i)$$

r.v. x	$\boldsymbol{p_X(x)}$
0	0.20
1	0.30
2	0.25
3	0.20
4	0.05

Category (ω)	**r.v. x**	$\boldsymbol{F_X(\alpha)}$
Cardboard	0	
Paper	1	
Cans	2	
Scrap Metal	3	
Bottle	4	

Work out these five values

CDF for our trash categories

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$$F_X(\alpha) = P(X \le \alpha) = \sum_{i=0}^{k-1} p_X(x_i)$$

r.v. $\boldsymbol{x}$	$\boldsymbol{p_X(x)}$
0	0.20
1	0.30
2	0.25
3	0.20
4	0.05

Category ($\boldsymbol{\omega}$)	**r.v.** $\boldsymbol{x}$	$\boldsymbol{F_X(\alpha)}$
Cardboard	0	$P(X \le 0) = 0.20$, $\boldsymbol{\alpha} = 0$
Paper	1	$P(X \le 1) = 0.50$, $\boldsymbol{\alpha} = 1$
Cans	2	$P(X \le 2) = 0.75$, $\boldsymbol{\alpha} = 2$
Scrap Metal	3	$P(X \le 3) = 0.90$, $\boldsymbol{\alpha} = 3$
Bottle	4	$P(X \le 4) = 1.00$, $\boldsymbol{\alpha} = 4$

Simulation by sampling

- It is often useful to simulate robotic systems. In our case, we might like to simulate the arrival of trash to our sorting system, such that it accurately reflects the prior distribution?
- How can we generate a sequence of samples, say $\omega_1, \omega_2, \dots, \omega_n$, such that $\omega_i = cardboard$ for approximately 20% of the samples, $\omega_i = paper$ for approximately 30% of the samples, etc.?
- Sadly, most programming languages do not include library functions to sample from arbitrary probability distributions.
- Happily, the is almost always a random number generator that generates a random sample from the unit interval, $x \sim U(0,1)$.
- The notation $x \sim U(0,1)$ indicates that x is a number chosen at random from the interval $[0,1]$, and that all possible outcomes are equally likely.

➢ Let's see how to use this…

Simulation by sampling

Suppose we generate the samples $s_1 = 0.97$ and $s_2 = 0.29$

r.v. x	$p_X(x)$	$F_X(\alpha)$, $\alpha = 0, 1, 2, 3, 4$
0	0.20	0.20
1	0.30	0.50
2	0.25	0.75
3	0.20	0.95
4	0.05	1.00

- Note that $\boldsymbol{F_X(x_3)} = 0.95 < (s_1 = 0.97) \leq 1 = \boldsymbol{F_X(x_4)}$.
- The probability that this occurs is exactly 0.05, since the probability of $x \in [a, b] = (b - a)$ for the uniform distribution on $[0,1]$.
- $P(bottle) = 0.05$ …. Return category bottle.

- Similarly, $\boldsymbol{F_X(x_0)} = 0.20 < (s_2 = 0.29) \leq 0.50 = \boldsymbol{F_X(x_1)}$
- The probability that this occurs is exactly 0.30.
- $P(paper) = 0.30$ … Return category paper.

We can generalize this to develop an algorithm that draws a sample from an arbitrary distribution.

1. Generate a sample $x \sim U(0,1)$.
2. Determine k such that $F_X(x_{k-1}) < x \leq F_X(x_k)$.
3. Select category ω_k

Use the book!

Now is the time to visit the online book, explore the concepts, and play with the code to ensure that you understand what we have just discussed.

- *Try different prior distributions, and build the corresponding CDF.*
- *Be sure that your hand calculations match the results from the code.*
- *Generate many samples. Compare the sample distribution (i.e., the proportion of occurrences of each category) to the true prior.*
- *Increase the number of samples. You should notice that the sampling distribution becomes increasingly similar to the true prior as you increase the number of samples.*

Actions

For this problem, the robot either places an item of trash into one of three bins, or lets the item pass through the work cell.

This gives four possible actions:

- a_1: Glass Bin
- a_2: Metal Bin
- a_3: Paper Bin
- a_4: Nop (let object pass through the workcell)

For this chapter, we assume that actions are executed without error, every time.

However, since we don't know with certainty the category for an item of trash in the work cell, the efficacy of an action is also uncertain.

Assessing Risk

- Because there is uncertainty in the category of a piece of trash, the robot risks making mistakes when choosing actions.
- Different mistakes have different costs.
  - Placing metal in the paper bin might seriously damage paper processing equipment.
  - Placing paper in the metal bin is unlikely to cause much harm.

COST	cardboard	paper	can	scrap metal	bottle
glass bin	2	2	4	6	0
metal bin	1	1	0	0	2
paper bin	0	0	5	10	3
nop	1	1	1	1	1

To account for these variations, we can define a table of costs for applying each action (rows) to each category (columns).

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We assign zero costs to correct actions.

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We assign zero costs to correct actions.

The cost of Nop is due to the need for human labor to sort the item of trash.

Cost as a Random Variable

Since we only have probabilistic knowledge of an item's category, we can regard the cost of executing an action as a discrete random variable.

Consider action a_3, *place the item in the mixed paper bin*.

- Let X be the r.v. that denotes the cost of applying action a_3.
- From the table of costs, we see that $X \in \{0,3,5,10\}$, since these are the only possible costs for this action.

➢ What can we say about the probability distribution for X?

Computing pmf's

To compute the pmf, recall that the random variable is a mapping from outcomes to real numbers.

There are five possible outcomes. The object must be from one of five categories, each of which has a cost.

➢ Compute $p_X(x)$ for each x.

Category	P(C)	Cost
cardboard	0.20	0
paper	0.30	0
can	0.25	5
scrap metal	0.20	10
bottle	0.05	3

- $X = 0$ for cardboard and paper.
- $P(\{cardboard, paper\}) = P(\{cardboard\}) + P(\{paper\}) = 0.5$

➢ $p_X(0) = 0.5$

- $X = 5$ for can.
- $P(\{can\}) = 0.25$

➢ $p_X(5) = 0.25$

- $X = 3$ for bottle.
- $P(\{bottle\}) = 0.05$

➢ $p_X(3) = 0.05$

- $X = 10$ for scrap metal.
- $P(\{scrap\ metal\}) = 0.20$

➢ $p_X(10) = 0.20$

If we apply action a_3, *place the item in the mixed paper bin,* this is the pmf for cost!

Expectation

- Probabilities tell us something about a single outcome, but this isn't really very useful. Gamblers who make one-time bets based on probabilities can lose a lot of money.
- Most robots operate for prolonged periods of time.
- The notion of average cost over many trials seems like a useful thing to know.

➢ ***This is exactly the concept of expectation in probability theory.***

Expectation

If a r.v. X takes its values from a finite set, $X \in \{x_1, \dots, x_n\}$, the ***expected value*** of X, denoted $\boldsymbol{E[X]}$, is defined by:

$$E[X] = \sum_{i=1}^{n} x_i \, p_X(x_i)$$

- Expectation is a *property of a probability distribution*.
- **$E[X]$ is *not* the value you should expect to see for any specific outcome!!**

Examples

Let $X \in \{x_1, \dots, x_n\}$ be a discrete r.v. that corresponds to the number of dots shown on a fair die.

- $X \in \{1,2,3,4,5,6\}$ and $p_X(x_i) = \frac{1}{6}$ for all i

➢ Compute $E[X]$.

$$E[X] = \sum_{i=1}^{n} x_i\, p_X(x_i) = \sum_{i=1}^{6} \frac{1}{6} i = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{21}{6} = 3.5$$

Examples

Let $X \in \{0,5\}$ be a discrete r.v.

$$X = \begin{cases} 0 \text{ for an even roll} \\ 5 \text{ for an odd roll} \end{cases}$$

➢ Compute $E[X]$.

$$E[X] = \sum_{i=1}^{2} x_i \, p_X(x_i) = \frac{1}{2} \times 0 + \frac{1}{2} \times 5 = 2.5$$

Expectation

If a r.v. X takes its values from a finite set, $X \in \{x_1, \dots, x_n\}$, the ***expected value*** of $f(X)$, denoted $E[f(X)]$, is defined by:

$$E[f(X)] = \sum_{i=1}^{n} f(x_i)p_X(x_i)$$

- Expectation is a *property of a probability distribution*.
- **$E[f(X)]$ is *not* the value you should expect to see for any specific outcome!!**

Examples

Let $X \in \{x_1, \ldots, x_n\}$ be a discrete r.v. that corresponds to the number of dots shown on a fair die.

➢ Compute $E[X^2]$.

$$E[X] = \sum_{i=1}^{6} x_i^2 p_X(x_i)$$

$$= \sum_{i=1}^{6} \frac{1}{6} i^2 = \frac{1}{6}1 + \frac{1}{6}4 + \frac{1}{6}9 + \frac{1}{6}16 + \frac{1}{6}25 + \frac{1}{6}36 = \frac{91}{6} = 15.1666$$

Trash Sorting...

We can now easily evaluate the expected cost for each action under the prior probability distribution.

COST	Card board	paper	can	scrap metal	bottle	Expected Cost
glass bin	2	2	4	6	0	3.2
metal bin	1	1	0	0	2	0.6
paper bin	0	0	5	10	3	3.4
nop	1	1	1	1	1	1.0
$P(\omega)$	0.20	0.30	0.25	0.20	0.05	

$$2 \times 0.5 + 4 \times 0.25 + 6 \times 0.2 = 3.2$$

$$1 \times 0.5 + 2 \times 0.05 = 0.6$$

$$5 \times 0.25 + 10 \times 0.2 + 3 \times 0.05 = 3.4$$

$$1 \times 0.5 + 1 \times 0.25 + 1 \times 0.2 + 1 \times 0.05 = 1.0$$

Acting Randomly

Suppose we choose an action at random:

$$P(a_1) = P(a_2) = P(a_3) = P(a_4) = 0.25$$

What is the expected cost?

Action	Expected Cost, x_i	$p_X(x_i)$
glass bin	3.2	0.25
metal bin	0.6	0.25
paper bin	3.4	0.25
nop	1.0	0.25

Let the random variable $\in \{0.6, 1.0, 3.2, 3.4\}$ denote the expected cost for applying action a_i.

Then $p_X(x_i) = 0.25$ for each action a_i.

$$E[X]=\sum_{i=1}^{4} x_i\, p_X(x_i) = 0.25(3.2 + 0.6 + 3.4 + 1.0) = 2.05$$

Always using the metal bin (action a_2) would be a better choice than randomly choosing actions.

Simulation by sampling

Earlier, we simulated our trash sorting system using a sampling algorithm. Let's apply those ideas here.

1. Generate N samples from the prior distribution on categories.
2. Compute the cost c_i for each sample for action a_k.
3. Compute the average cost as:

$$\overline{cost_k} = \frac{1}{N}\sum_{i=1}^{N} c_i$$

4. Compare $\overline{cost_k}$ to $E[X]$for action a_k (where X is the r.v. for cost).

Probability *vs* Statistics

- **Probability theory** is the study of a certain class of mathematical functions (probability distributions).

- A **statistic** is any function of data (including the identity function), and statistics is the study of such functions.

$$E[X] = \sum_{i=1}^{n} x_i \, p_X(x_i)$$

$E[X]$ is a property of $p_X(x_i)$

➢ ***Probability Theory***

$$\overline{cost_k} = \frac{1}{N} \sum_{i=1}^{N} c_i$$

$\overline{cost_k}$ is a function of data, c_i

➢ ***Statistics***

Probability Theory *and* Statistics

If it happens that certain probability distributions do a good job of describing how the world behaves, then probability theory can provide a rigorous basis for a system of inference about data.

The Weak Law of Large Numbers:

Consider a data set drawn from probability distribution p_X, with expected value $E[X] = \mu$.
For any $\epsilon > 0$, if $\bar{x}_N$ denotes the average of a data set of size N , then

$$\lim_{n \to \infty} P(|\bar{x}_N - \mu| < \epsilon) = 1$$

As the size of the data set increases, with probability one the average is arbitrarily close to the mean.

Probability Theory and Statistics

The connections between probability theory and statistics are often formalized by theorems that express variations on a simple concept:

As the size of a data set becomes large, the statistics of that data set will become increasingly good approximations for various properties of the underlying probability distribution from which the data set was generated.

- *This is one of the reasons simulation by sampling works.*
- *These theorems are important for statistical inference, machine learning, and many other problems that involve data drawn from stochastic systems.*

Next Lecture: Sensing and Perception

- Conditional probability:
  - How do sensor observations affect our beliefs about the world?
  - A key tool for data-based inference

- Continuous random variables:
  - Unlike our five categories of trash, some things are best described along a continuum.
  - Things like weight, distance are described using continuous measurements.
  - Gaussian Distributions

- Maximum likelihood inference
  - Making decisions using conditional probabilities
  - Combining information from multiple sensors