Lecture 13

The Bayes Filter

CS 3630

Logistics Robots

Perception

In this chapter, the role of perception is to solve the ***localization*** problem, i.e., to determine an estimate of x_t, the robot's state at time t.

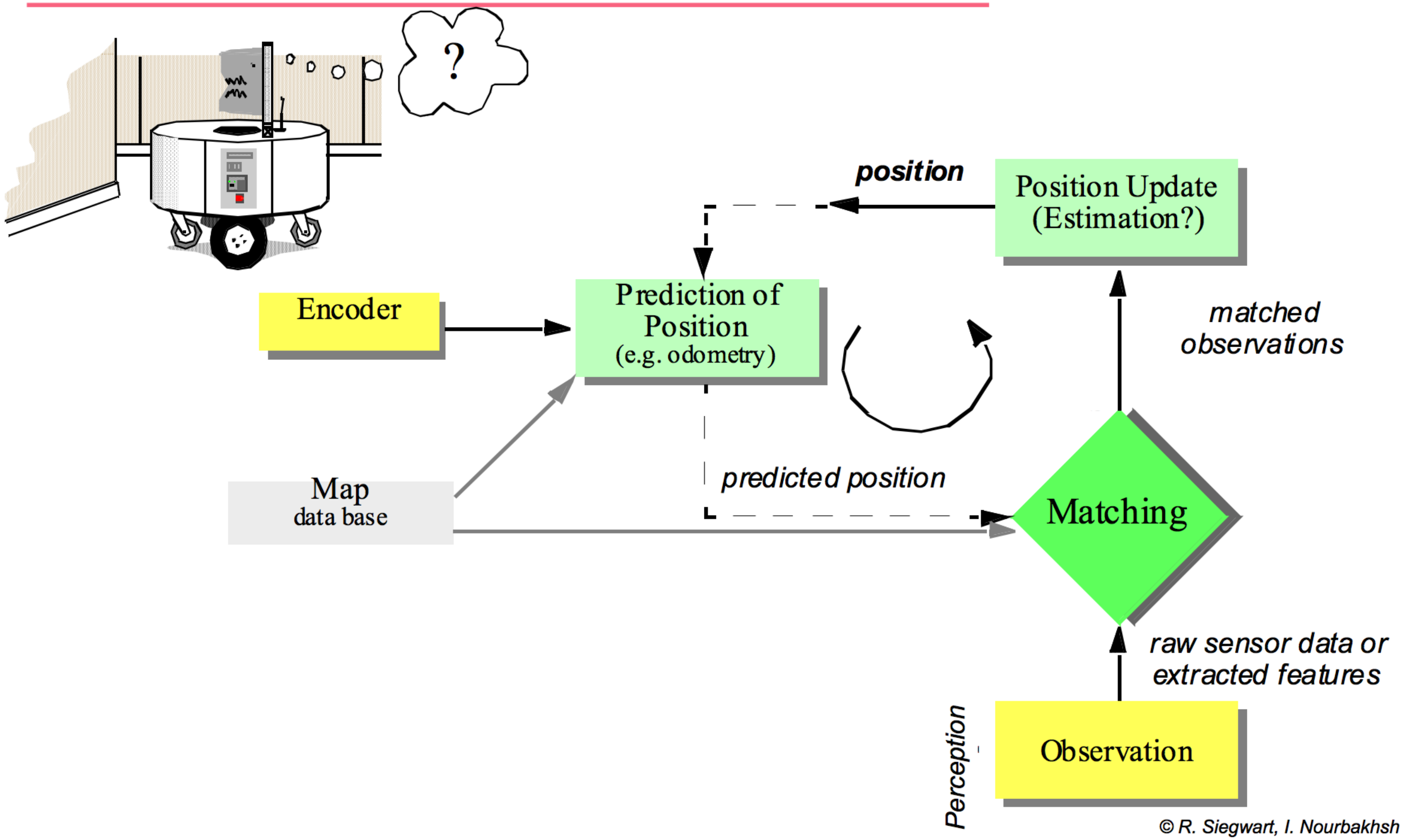
- Mathematically, the problem is to estimate the state x_t, given the action history $u_1 \dots u_n$ and sensing history $z_1 \dots z_n$

$$Bel(x_t) = P(x_t | u_1, z_1, u_2 \dots z_{t-1}, u_{t-1}, z_t)$$

- Computationally, this is a difficult problem.
- We'll see two approaches:
  - Particle Filtering
  - Markov Localization
- The Bayes filter is the workhorse in these.

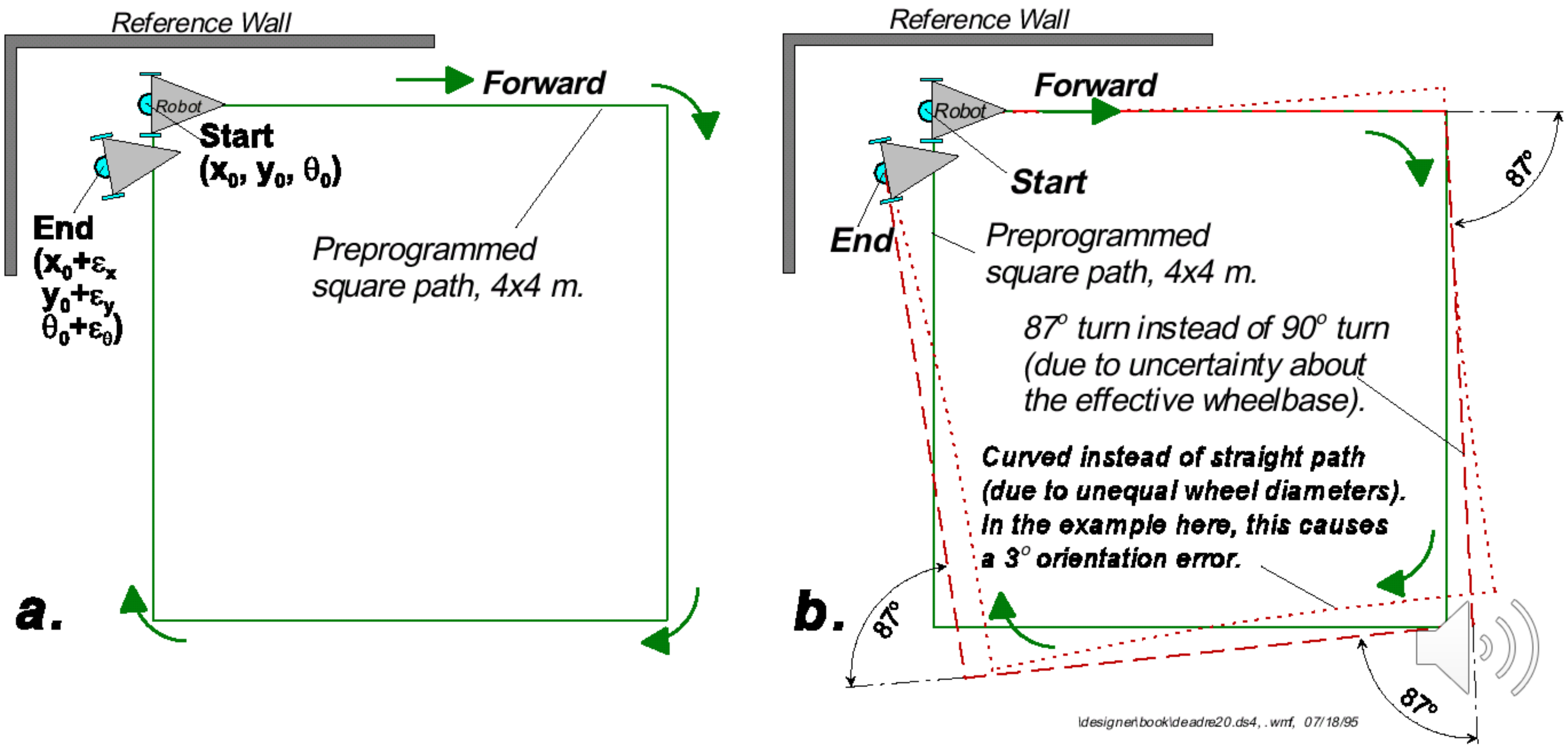
a.k.a. estimating the current *state* of the robot, or *state estimation*

Localization, Where am I?



Dead reckoning

Dead reckoning is the process of calculating vehicle's current position by using a previously determined position and estimated speeds over the elapsed time.



Example of warehouse robots using dead reckoning successfully, but only for 1m of travel distance

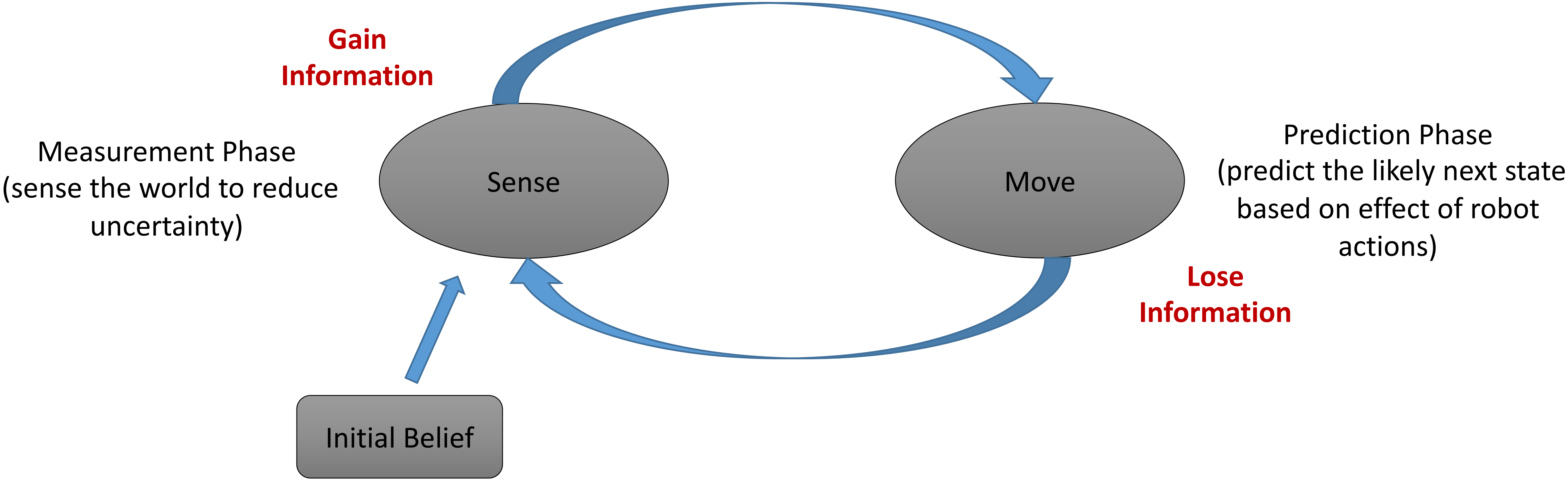
The Bayes Filter

The ***Bayes filter*** is the culmination of all the work we've done in applying probability theory to the representation of uncertainty in *state*, *actions*, and *sensing*.

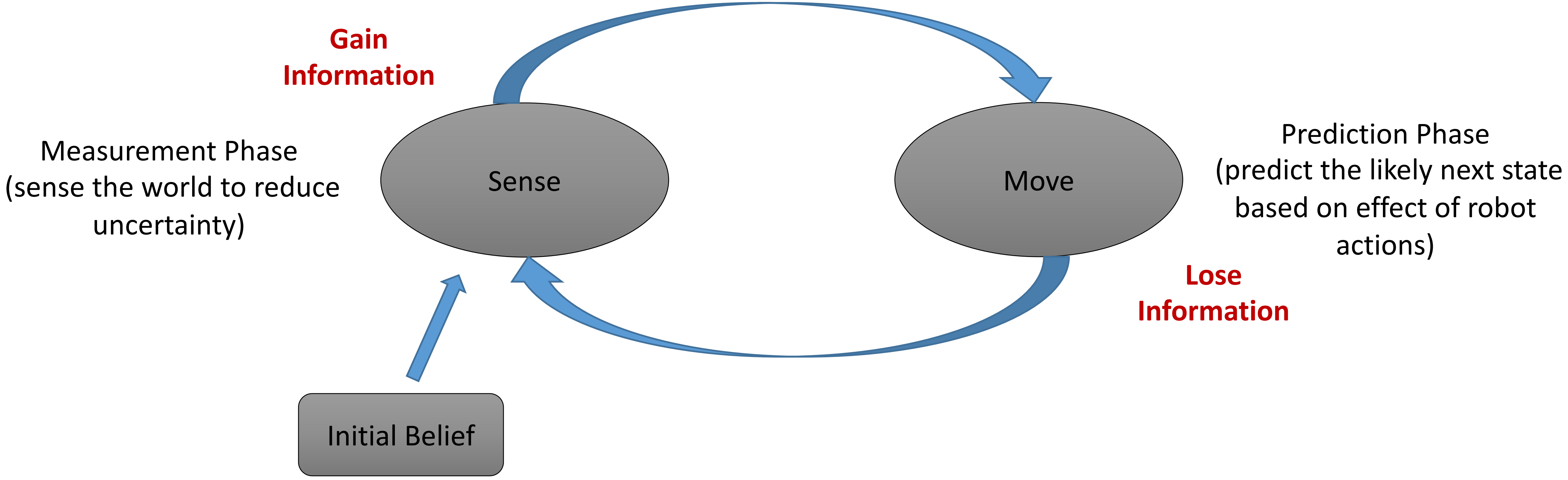
- ***Prior***: probabilistic description of *uncertainty in the state* (before acting or sensing at time t).
- ***Motion model***: conditional probability that describes *uncertainty in the actions.*
- ***Sensor model***: conditional probability model that describes *uncertainty in the sensor measurements*.

➢ The output of the Bayes filter at time t is $Bel(x_t)$.

The Bayes Filter



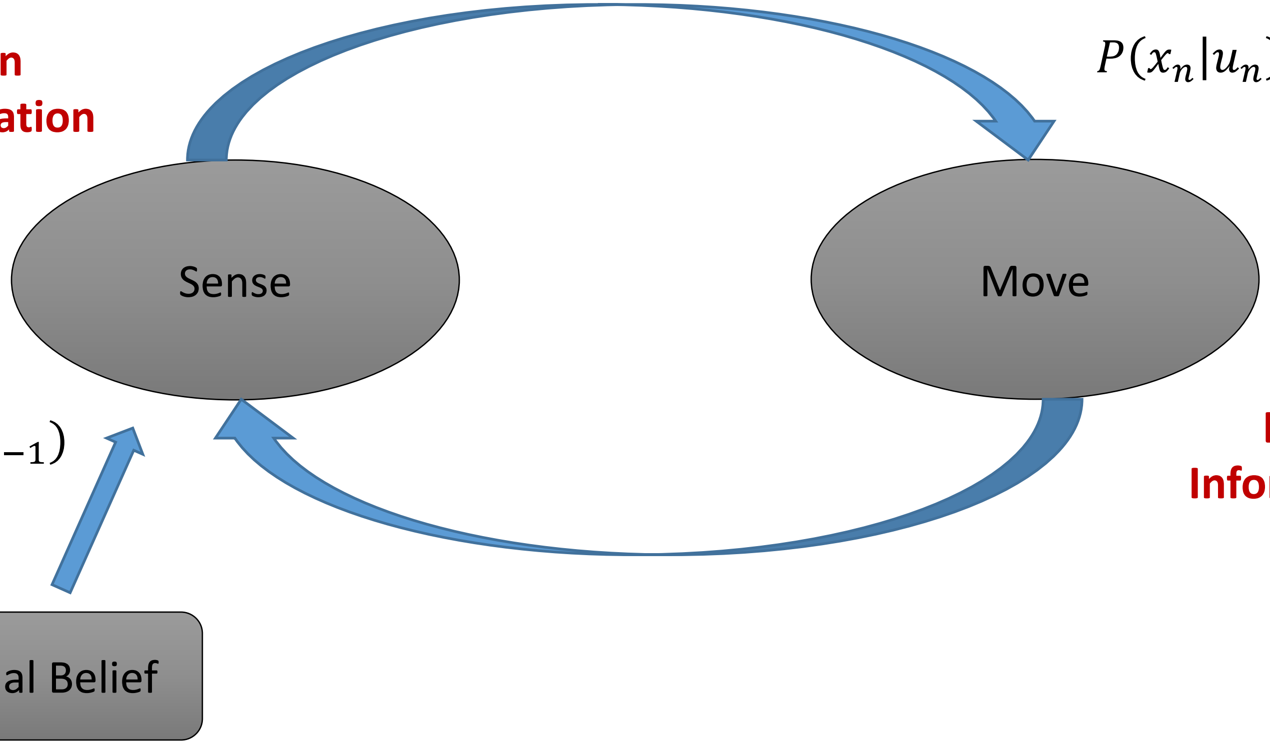
The Bayes Filter



The Bayes Filter

$$P(x_n|u_n) = \sum P(x_n|u_n, x_{n-1})P(x_{n-1})$$

Gain Information



Measurement Phase
(sense the world to reduce uncertainty)

$$P(x|z_{1:n}) = \alpha P(z_n|x)P(x|z_{1:n-1})$$

Prediction Phase
(predict the likely next state based on effect of robot actions)

Lose Information

Bayes Rule

$$P(x|z) = \frac{P(z|x)P(x)}{P(z)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

x is robot pose and z is sensor data

$p(x)$ → *Prior* probability distribution

$p(x|z)$ → *Posterior* (conditional) probability distribution

$p(z|x)$ → *Likelihood*, model of the characteristics of the event

$p(z)$ → *Evidence prior*, does not depend on x

Normalization Coefficient

$$P(x|z) = \frac{P(z|x)P(x)}{P(z)}$$

Note that the denominator is independent of x, and as a result will typically be the same for any value of x in the posterior $P(x|z)$.

Therefore, we typically represent the normalization term by the coefficient $\eta = [P(z)]^{-1}$ and Bayes equation is written as

$$P(x|z) = \eta P(z|x)P(x)$$

Bayes Filters: Framework

- Let x be the state of the robot (e.g. its location)
- **Given:**
  - Stream of observations *z* and action data *u:* $\{u_1, z_1 \ldots, u_{t-1}, z_t\}$
  - Sensor model *P(z|x).*
  - Action model $P(x_t | u_{t-1}, x_{t-1})$.
  - Prior probability of the system state *P(x).*
- **Wanted:**
  - Estimate of the state *X* of a dynamical system.
  - The posterior of the state is also called **Belief**:

$$Bel(x_t) = P(x_t | u_1, z_1 \ \ldots, u_{t-1}, z_t)$$

Bayes Filters

z = observation
u = action
x = state

$$\boxed{Bel(x_t)} = P(x_t | u_1, z_1 \ \ldots, u_{t-1}, z_t)$$

Bayes $$= \eta \ P(z_t | x_t, u_1, z_1, \ \ldots, u_{t-1}) \ P(x_t | u_1, z_2, \ \ldots, u_{t-1})$$

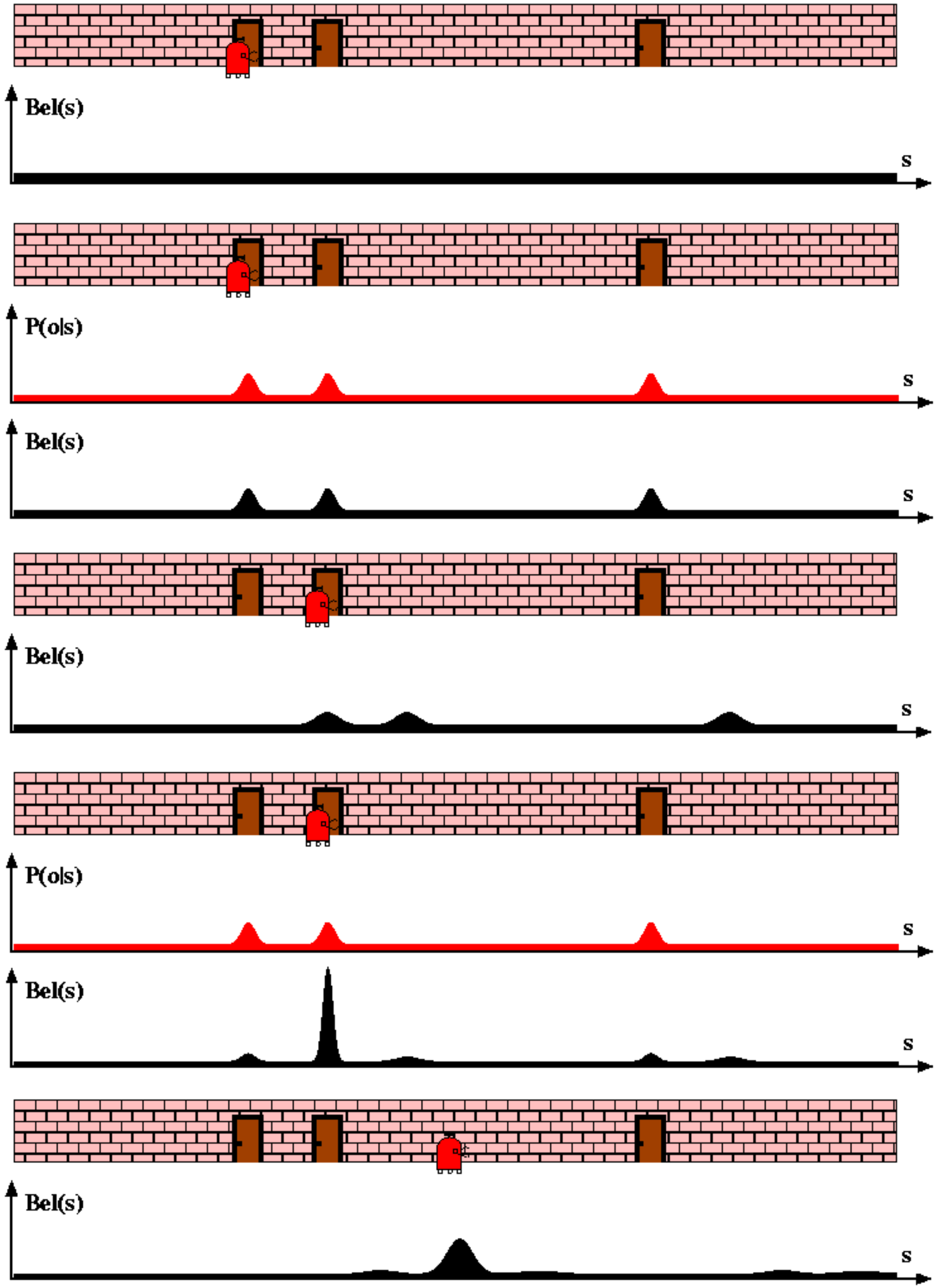
Markov $$= \eta \ P(z_t | x_t) \ P(x_t | u_1, z_1, \ \ldots, u_{t-1})$$

Total prob. $$= \eta \ P(z_t | x_t) \ \int P(x_t | u_1, z_1, \ \ldots, u_{t-1}, x_{t-1}) \, P(x_{t-1} | u_1, z_1, \ \ldots, u_{t-1}) \ dx_{t-1}$$

Markov $$= \eta \ P(z_t | x_t) \ \int P(x_t | u_{t-1}, x_{t-1}) \ P(x_{t-1} | u_1, z_1, \ \ldots, u_{t-1}) \ dx_{t-1}$$

$$\boxed{= \eta \ P(z_t \mid x_t) \int P(x_t \mid u_{t-1}, x_{t-1}) \ Bel(x_{t-1}) \ dx_{t-1}}$$

Bayes Filters for Robot Localization

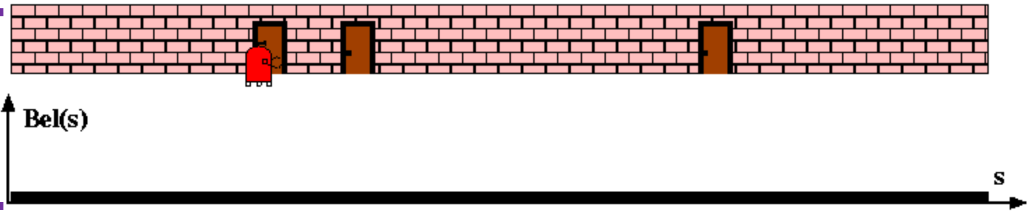


Let's see how it works using a simple example:

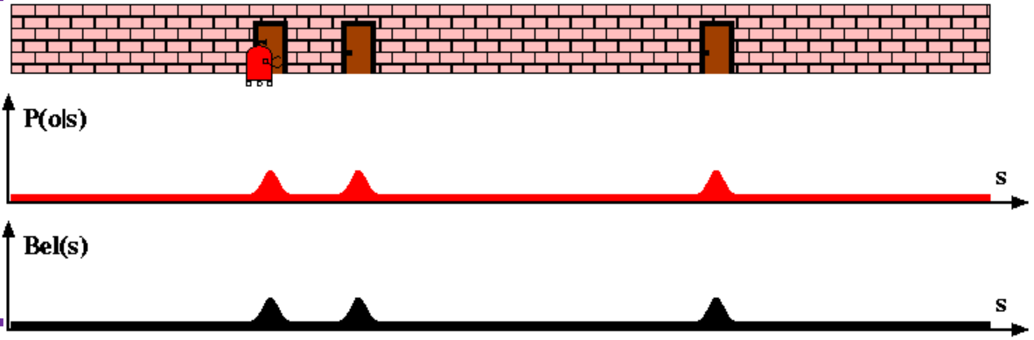
- The robot moves from left to right.
- From time to time, it takes a sensor reading.

How is the state estimate updated??

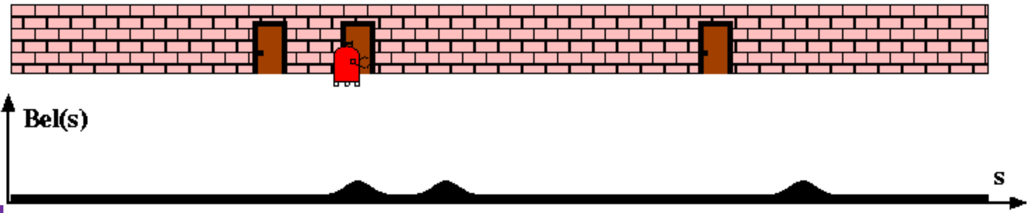
Initial Guess: Could be anywhere…



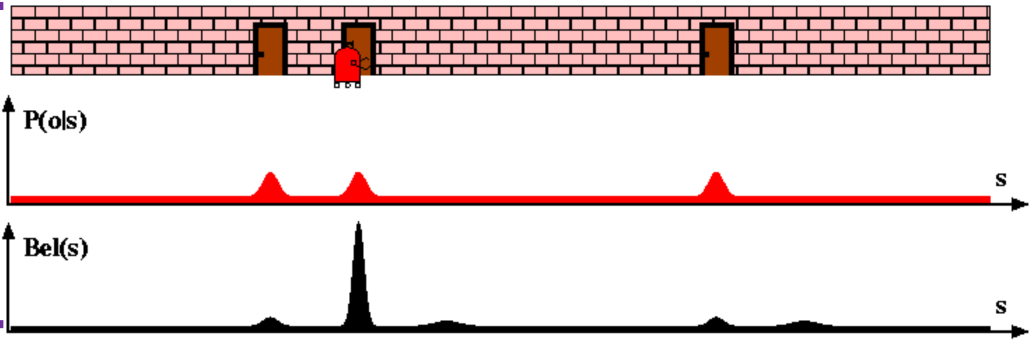
Take a measurement: we’re probably in front of a door…



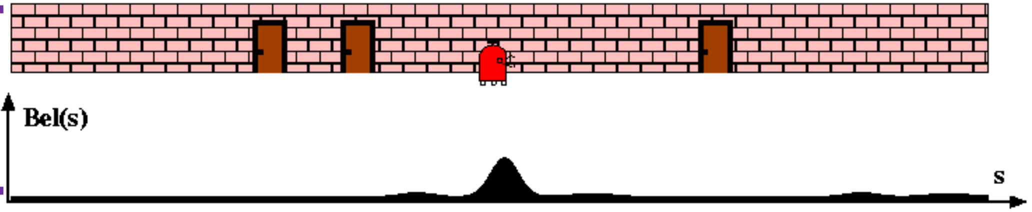
Execute an action – move to the right by about a meter… probability mass “spreads out”



Take another measurement. It seems we’re in front of a door again (red). Given what we believed before about position, the most likely place now is the second door.



Execute an action – move to the right by about a meter… probability mass “spreads out”



Bayes Filters

If I was in state x_{t-1} and I executed action u_{t-1} what is the probability that I arrive to state x_t

Weight this probability by the belief that I was actually in state x_{t-1}

Belief that robot is in state $X = x_t$ at time step t

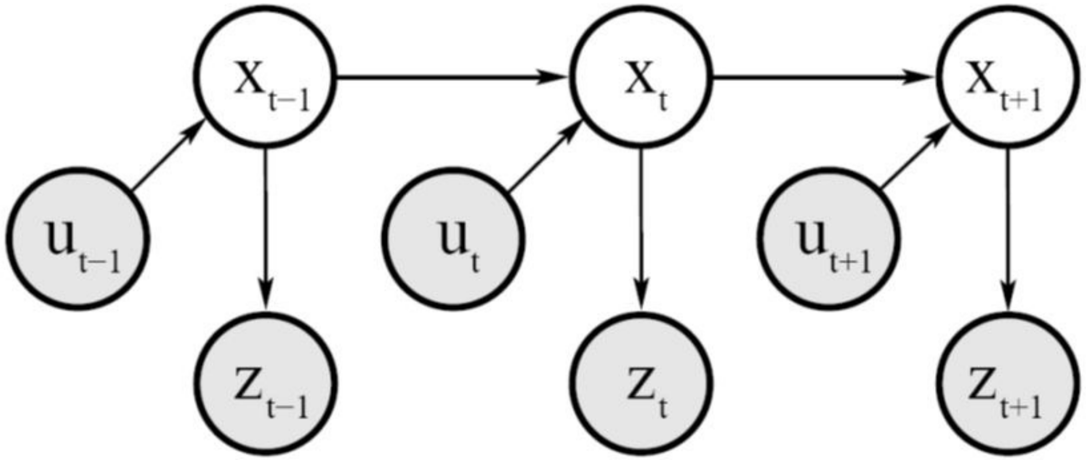
$$Bel(x_t) = \eta\, P(z_t|x_t) \int P(x_t|u_{t-1}, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

If I'm in state x_t, what is the probability I see observation z_t

Integrate over all possible previous states, x_{t-1}

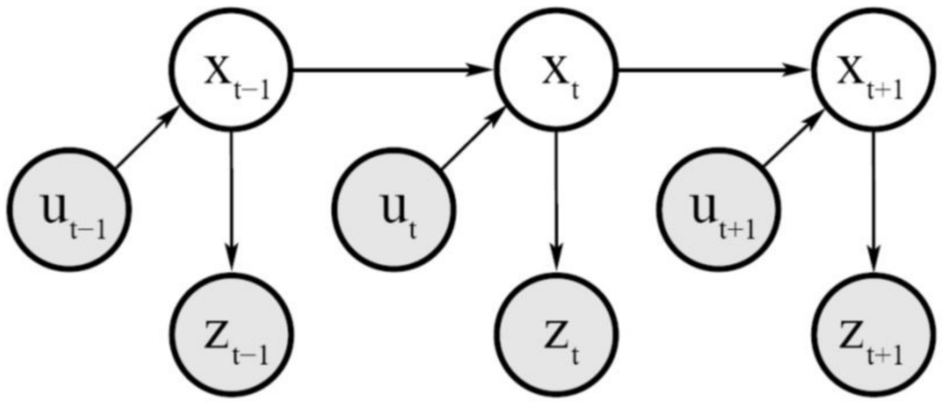
z = observation
u = action
x = state

- We can put all of this into our nice Bayes net formalism.
- The robot's state at time t is stochastically dependent on its state at time $t-1$ and the control input u_t. The measurement z_t depends stochastically on the state at time t.
- Gray elements are observable and white are hidden.



(This model is known as a hidden Markov model (HMM) or dynamic Bayesian network (DBN).)

Markov Assumption



$$p(z_t | x_{1:t}, z_{1:t-1}, u_{1:t}) = p(z_t | x_t)$$

$$p(x_t | x_{1:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t | x_{t-1}, u_t)$$

Underlying Assumptions

- Static world
- Independent noise

"The future is independent of the past given the present."

The Particle Filter

Particle filters represent a probability density function as a set of weighted samples.

The weighted samples are

1. Pushed through the motion model (including uncertainty)
2. Reweighted based on sensor measurements (using the sensor model)
3. Resampled using the new weights to define a probability distribution on the sample set.

- The approach is easy to implement, and has low computational overhead.
- Complexity does not grow exponentially with dimension of the state space.

Two localization problems

- "Global" localization
  - Figure out where the robot is, but we don't know where the robot started
  - Sometimes called the "kidnapped robot problem"

- "Position tracking"
  - Figure out where the robot is, given that we know where the robot started

➢ **To solve these problems at time t, we estimate**

$$\boldsymbol{Bel(x_t) = P(x_t | u_1, z_1, u_2 \;\; \ldots, z_t)}$$

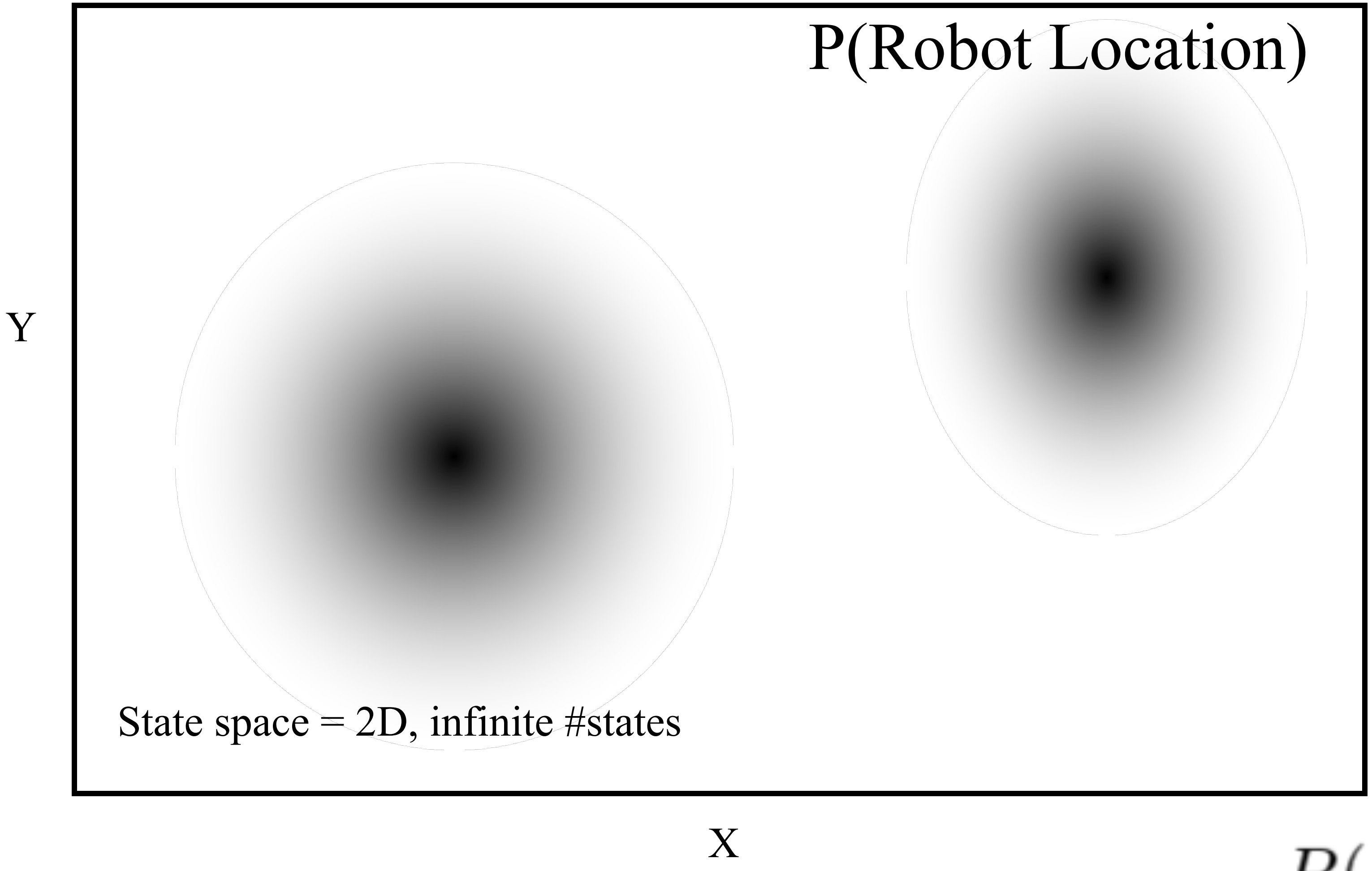
➢ ***The hard part: it's not feasible to exactly calculate or represent $Bel(x_t)$.***

Sampling to Approximate Densities

- Densities can become arbitrarily complex, even when noise models are Gaussian.
- One issue is nonlinear measurement and noise models.
- A second issue is the curse of dimensionality (for grid-based methods).

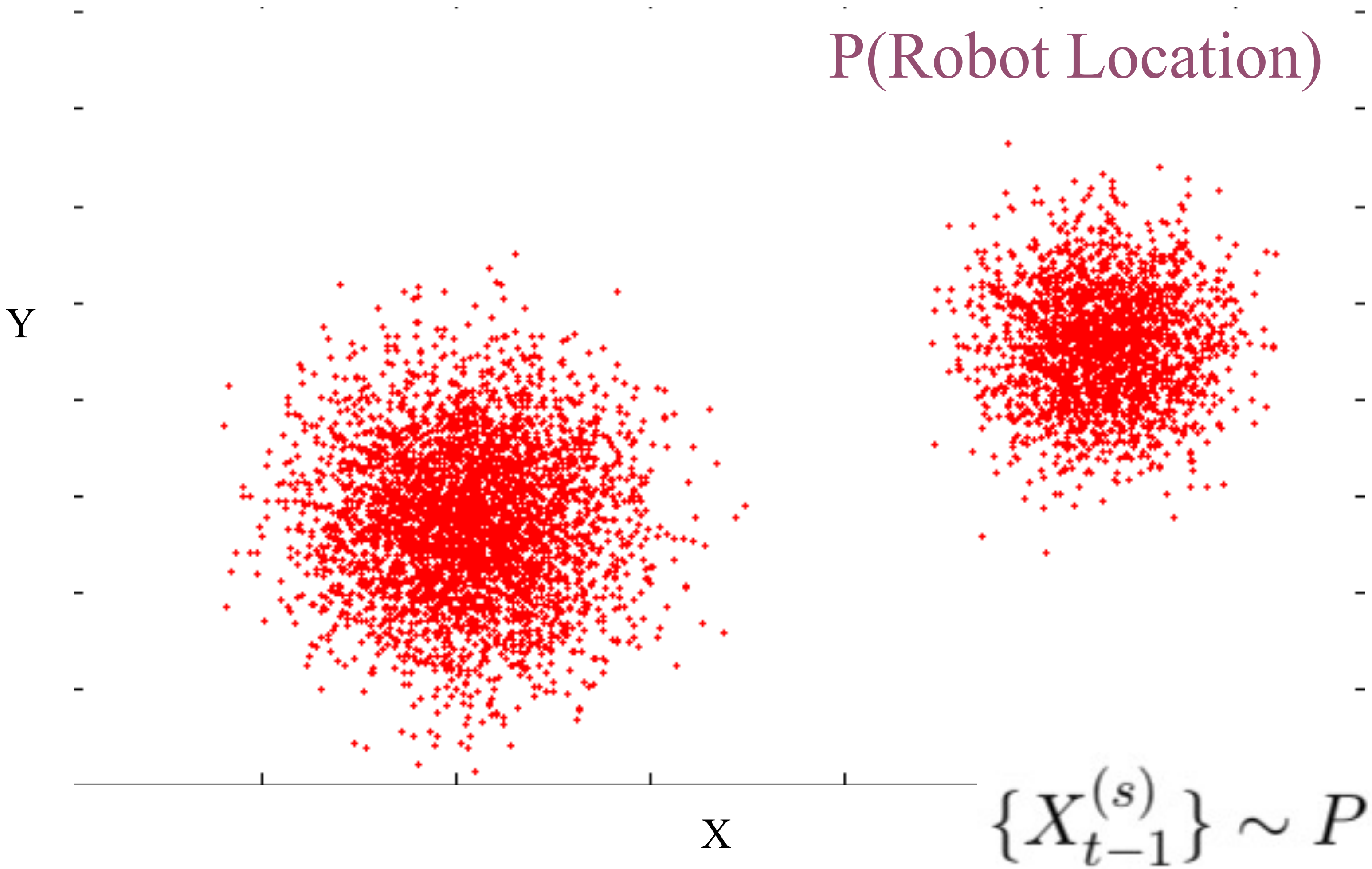
- One way out: ***sampling!***

Probability of Robot Location



$$P(X_{t-1}|Z^{t-1})$$

Sampling as Representation



$$\{X_{t-1}^{(s)}\} \sim P(X_{t-1}|Z^{t-1})$$

Particle Filter

- Represent *p*(*x*) by set of N weighted, random samples, called *particles,* of the form: $< (x_i, y_i), w_i >$

  (x_i, y_i) represents robot's pose

  w_i represents a weight, where $\sum w_i = 1$

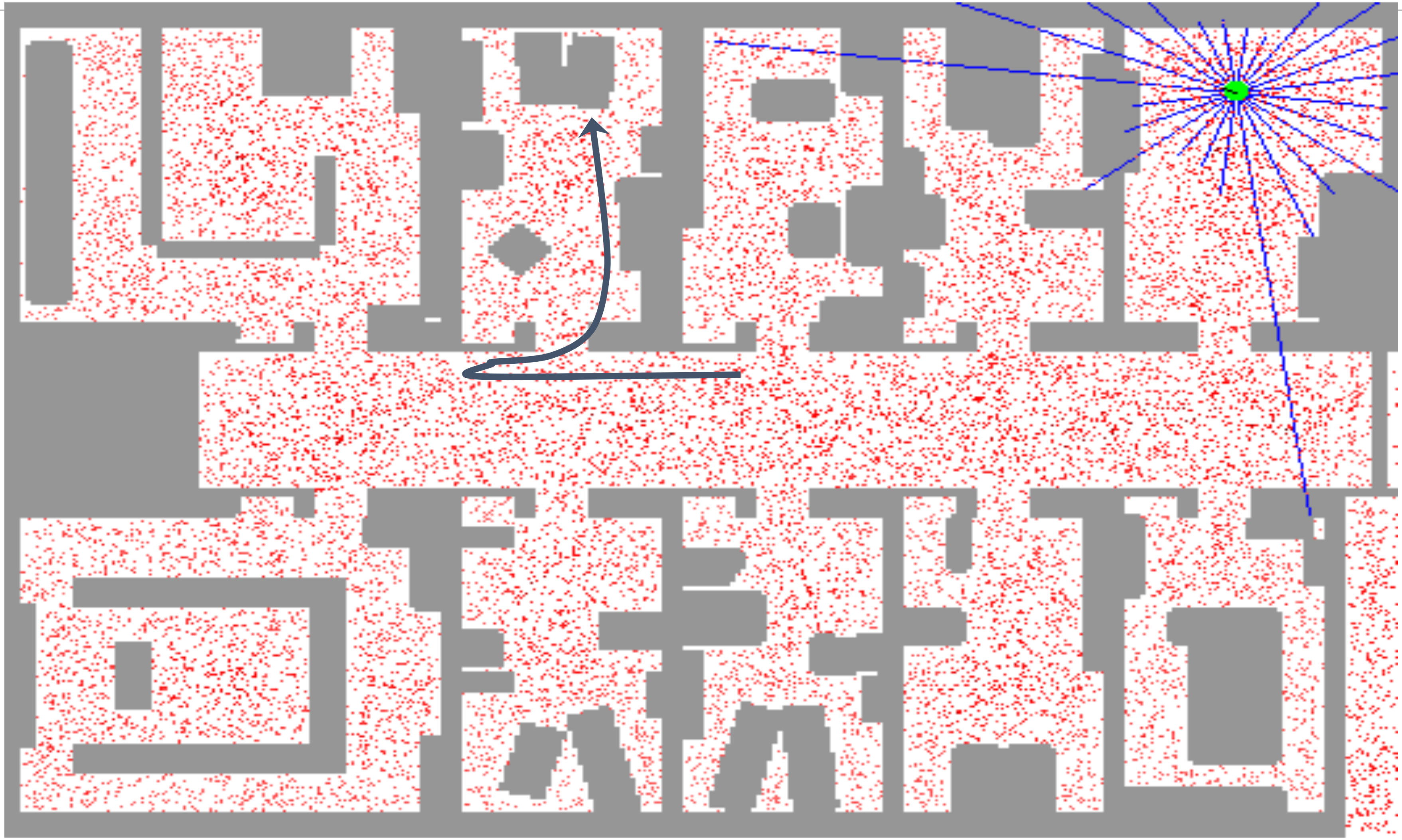
- A.K.A. Monte Carlo Localization (MCL)
  - Refers to techniques that are stochastic (random / non-deterministic)
  - Used in many modeling and simulation approaches

Sampling Advantages

- Arbitrary densities
- Memory = O(#samples)
- Only in "Typical Set"
- Great visualization tool!

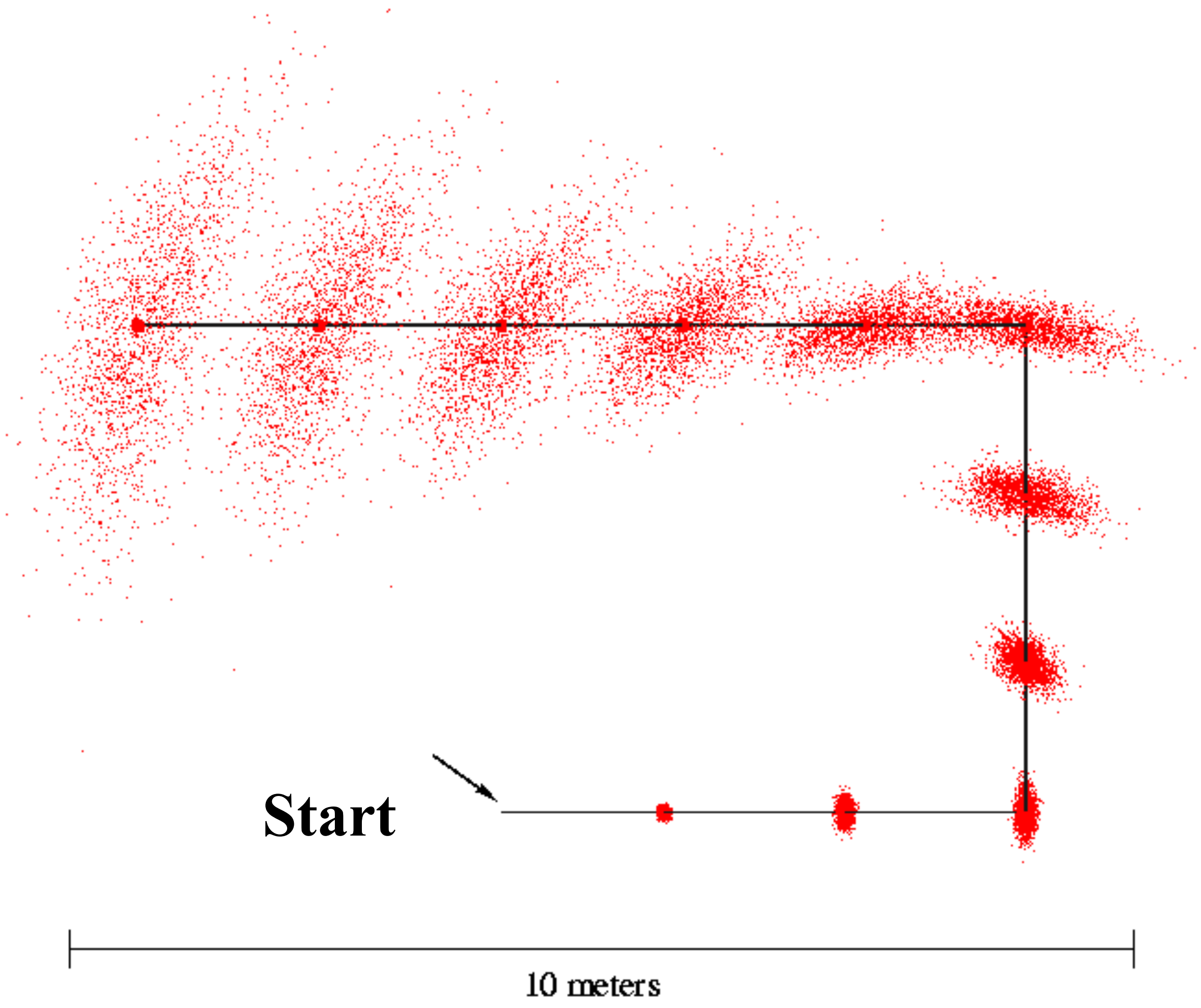
- Weakness: Approximate

Particle Filter Localization (using sonar)

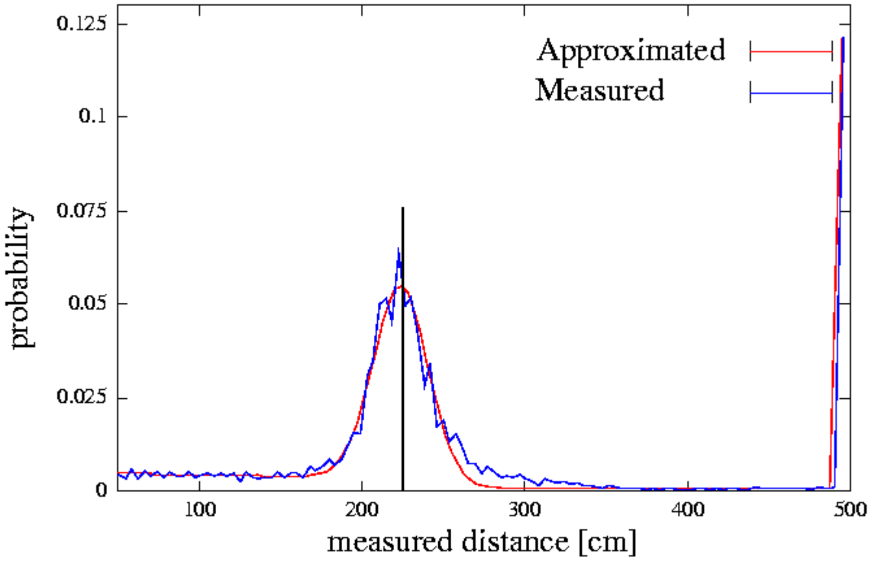


http://www.cs.washington.edu/ai/Mobile_Robotics//mcl/animations/global-floor-start.gif

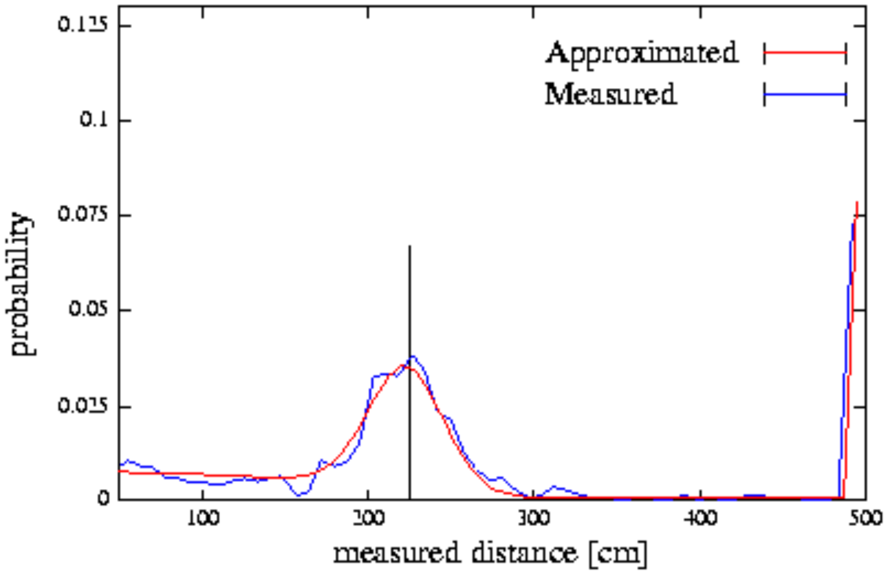
Motion Model for a Car-Like Robot



Sensor Model



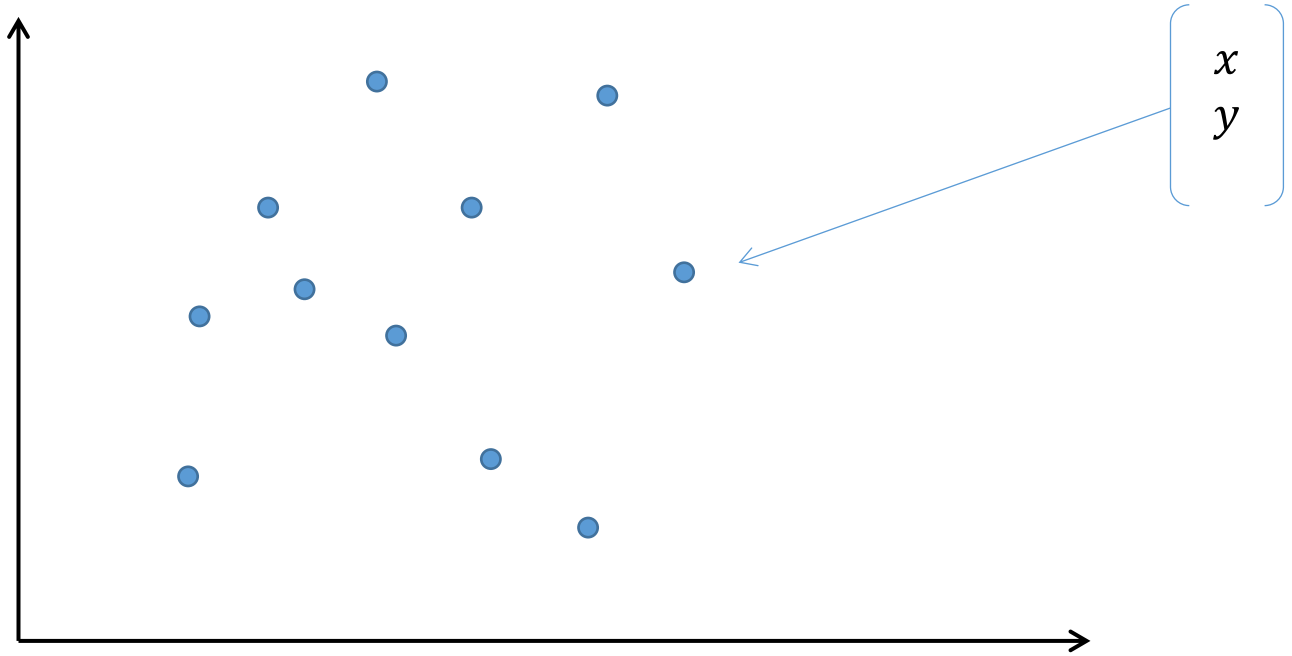
Laser sensor



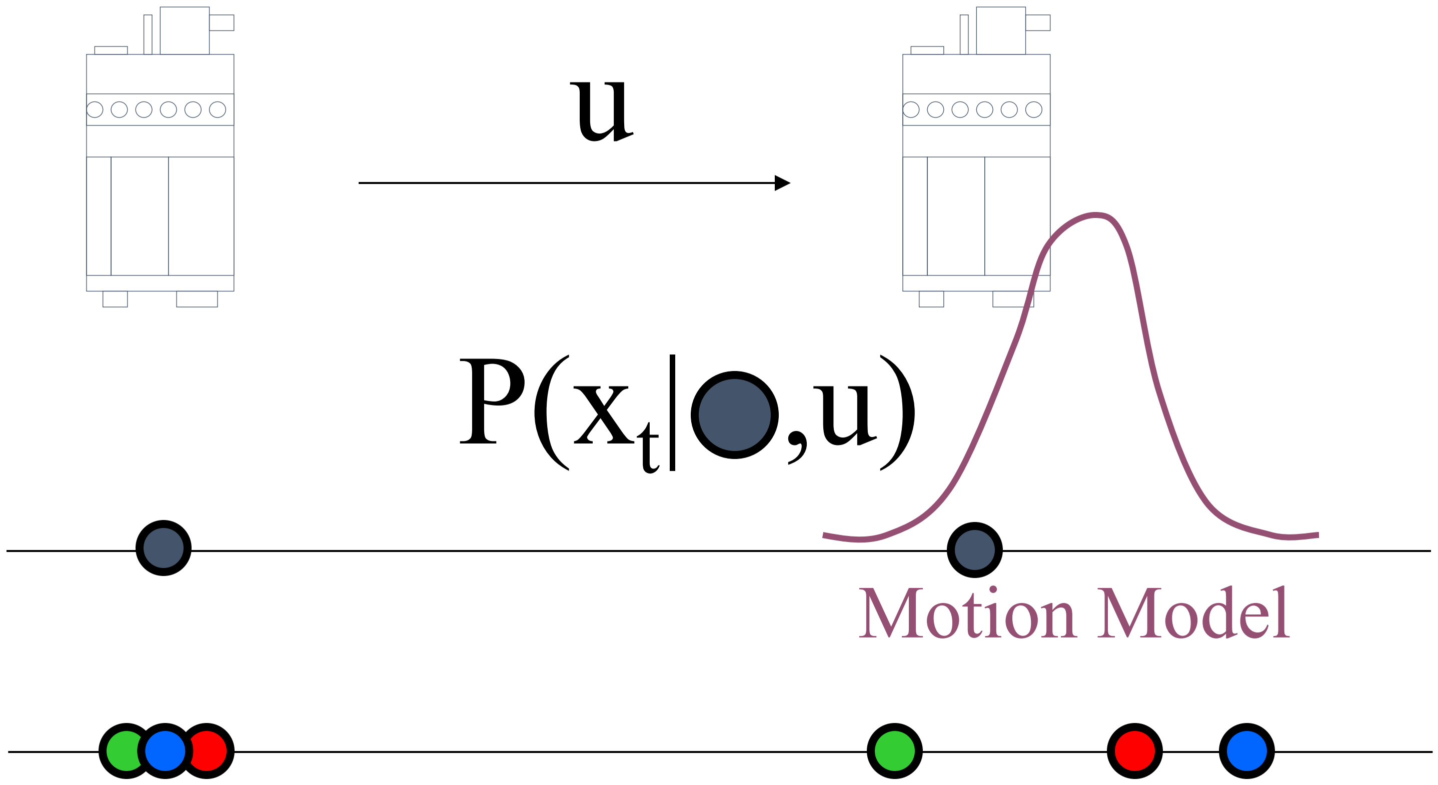
Sonar sensor

Particles

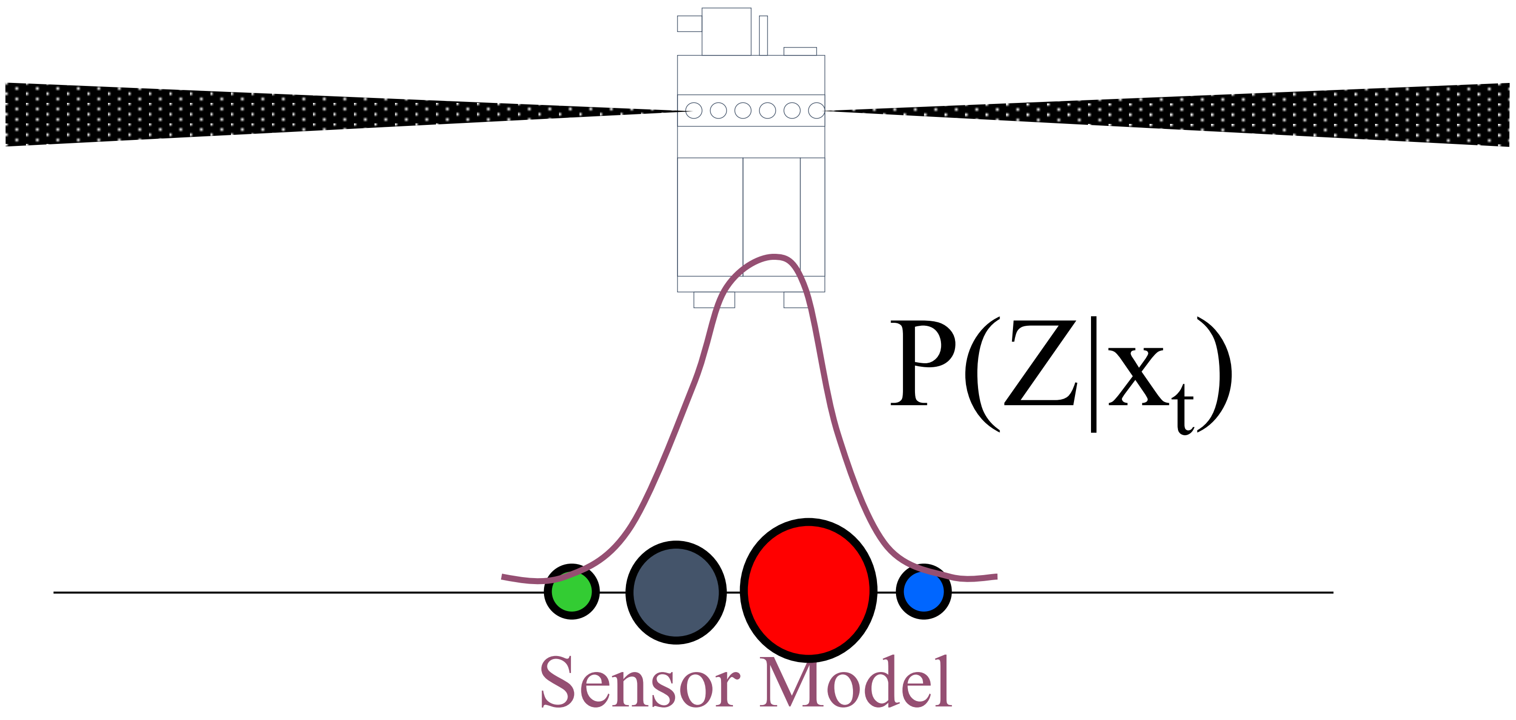
- Each particle is a guess about where the robot might be



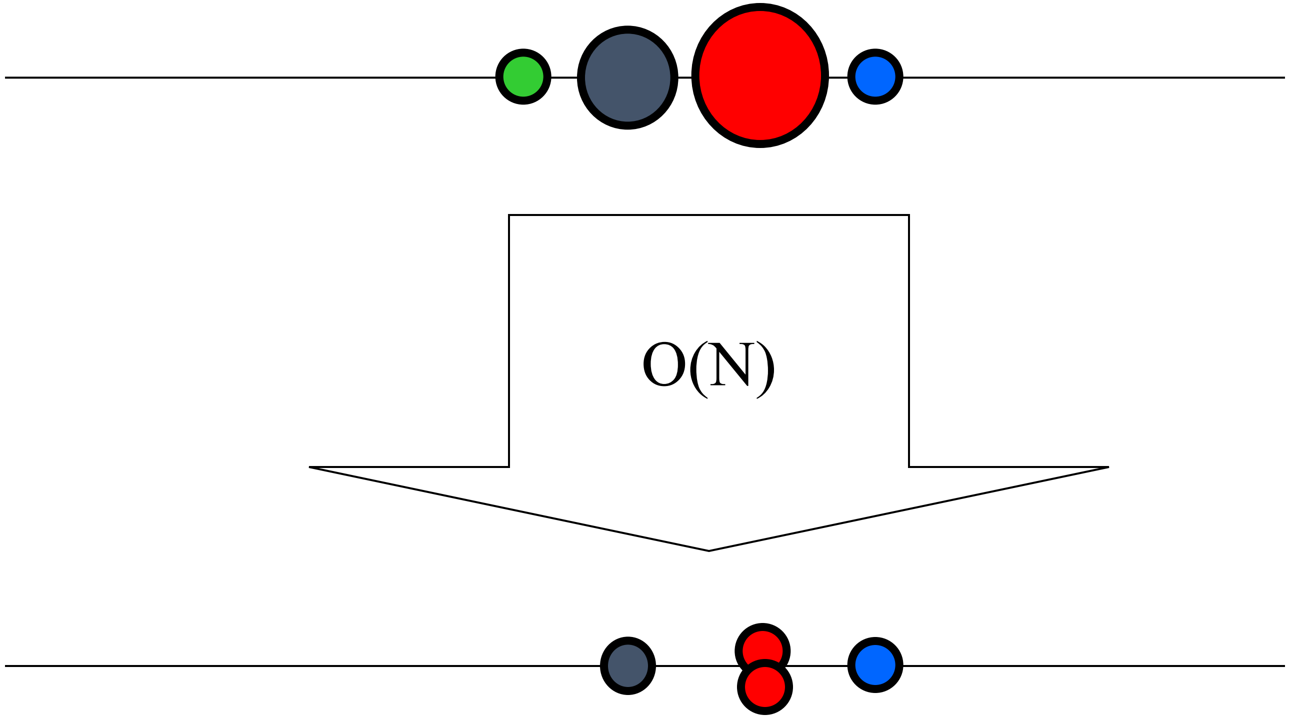
1. Prediction Phase

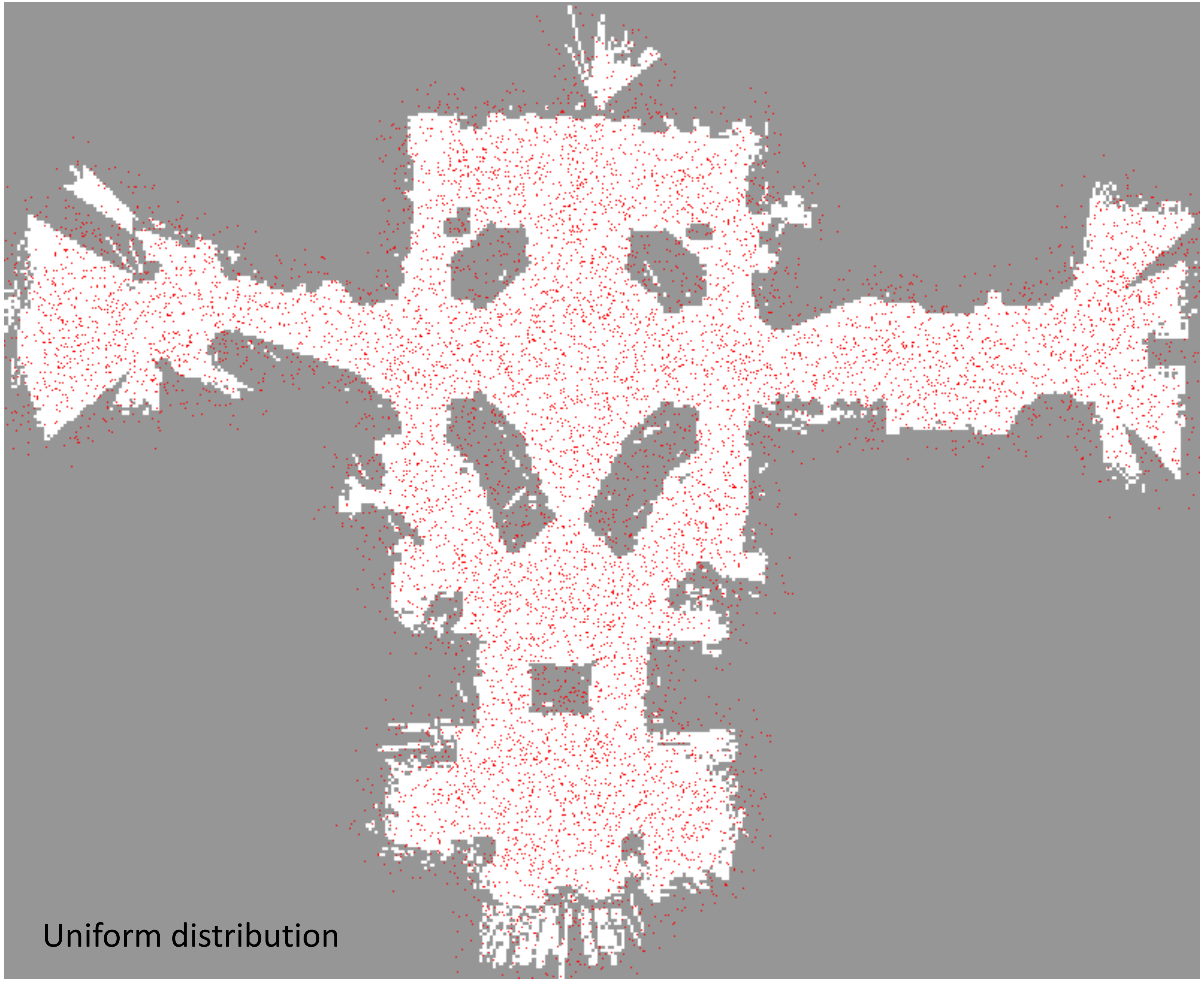


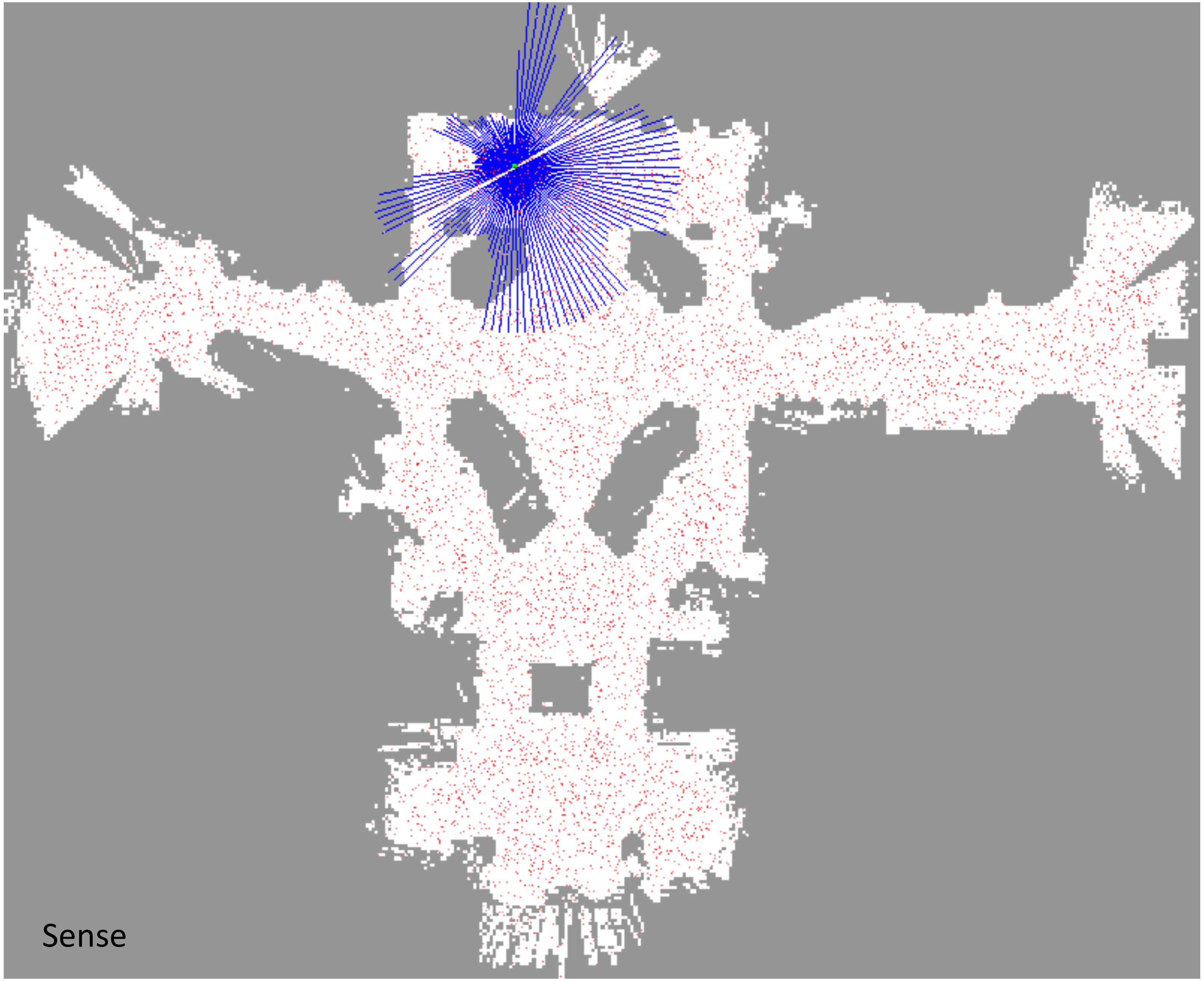
2. Measurement Phase

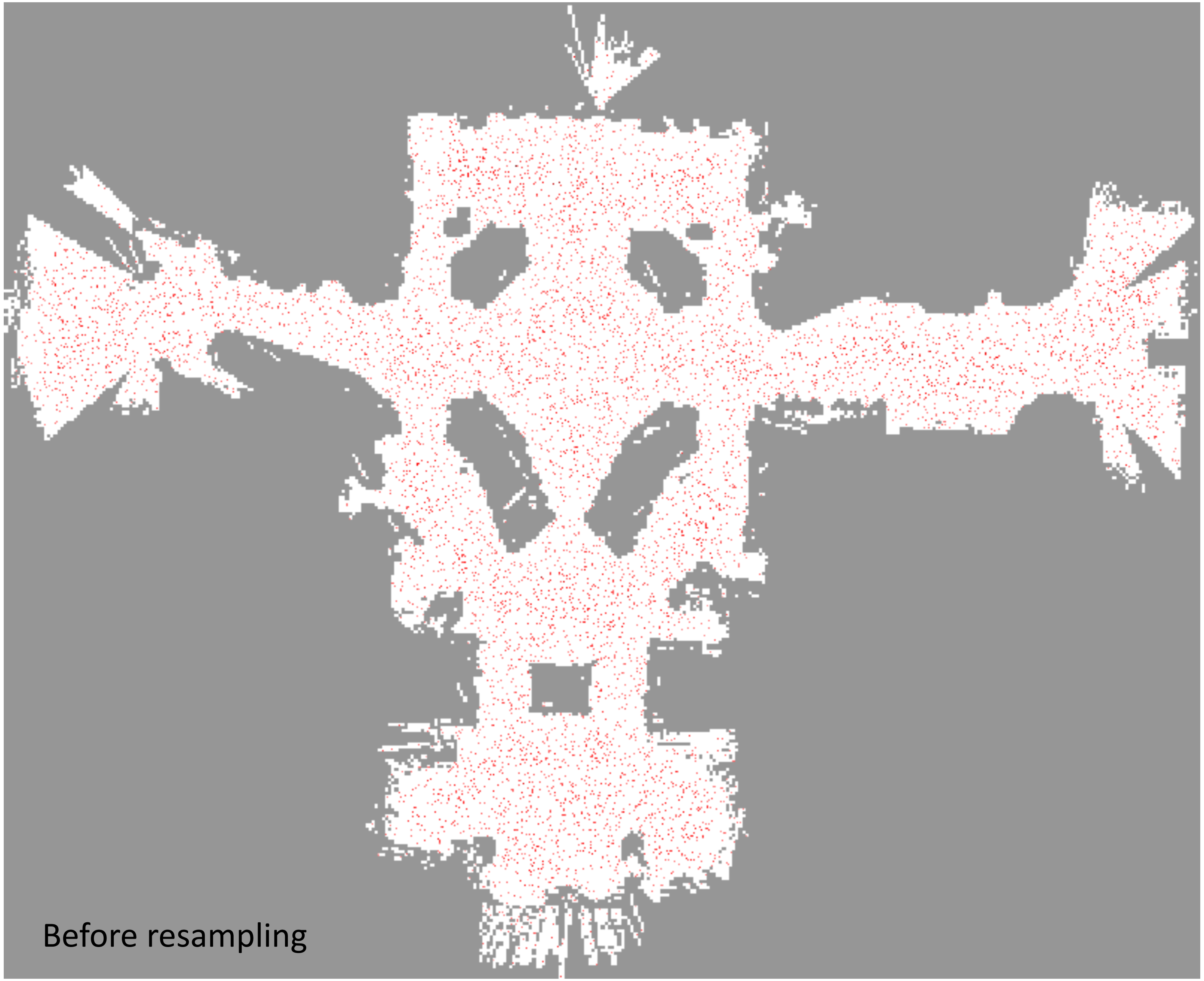


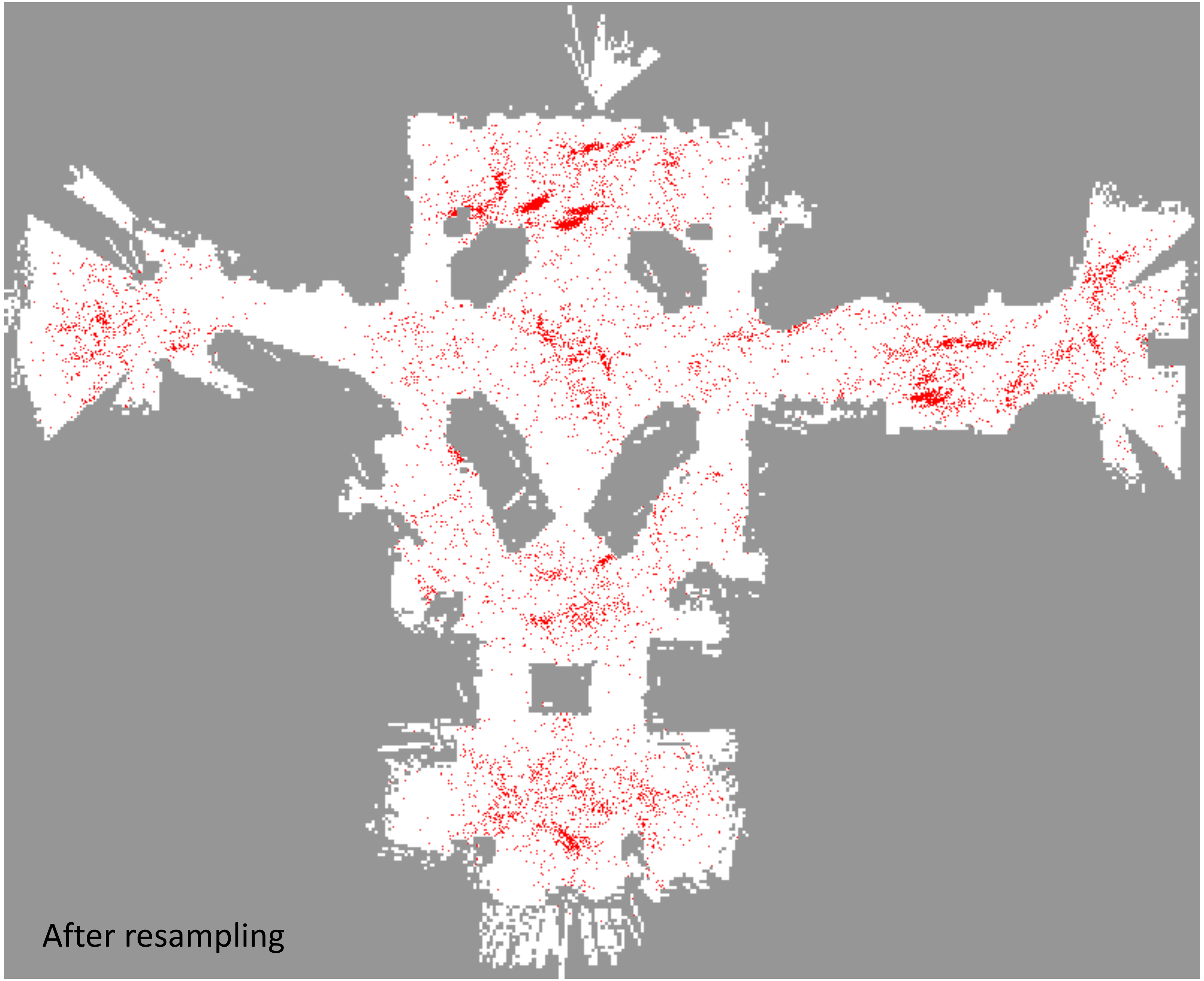
3. Resampling Step

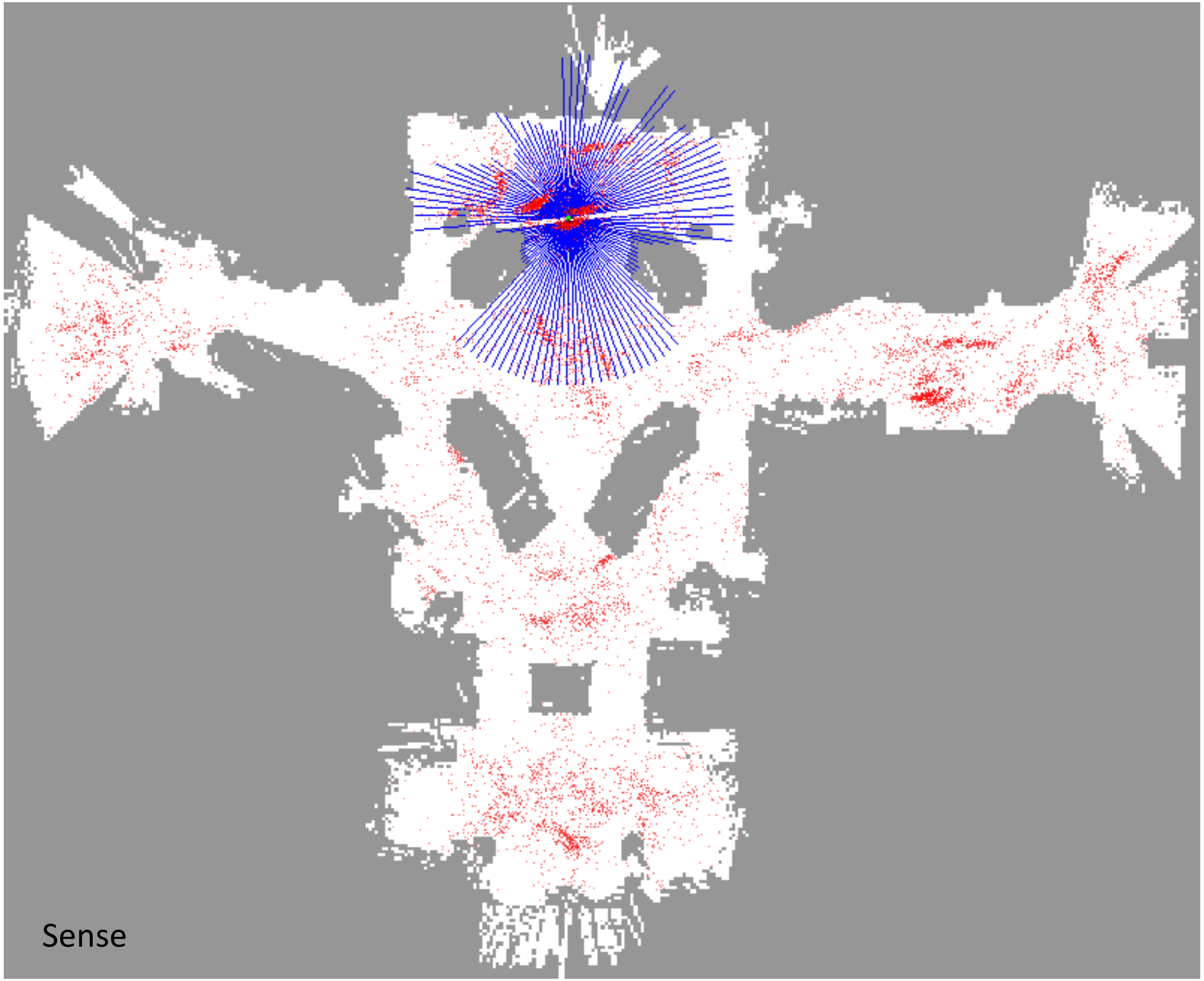


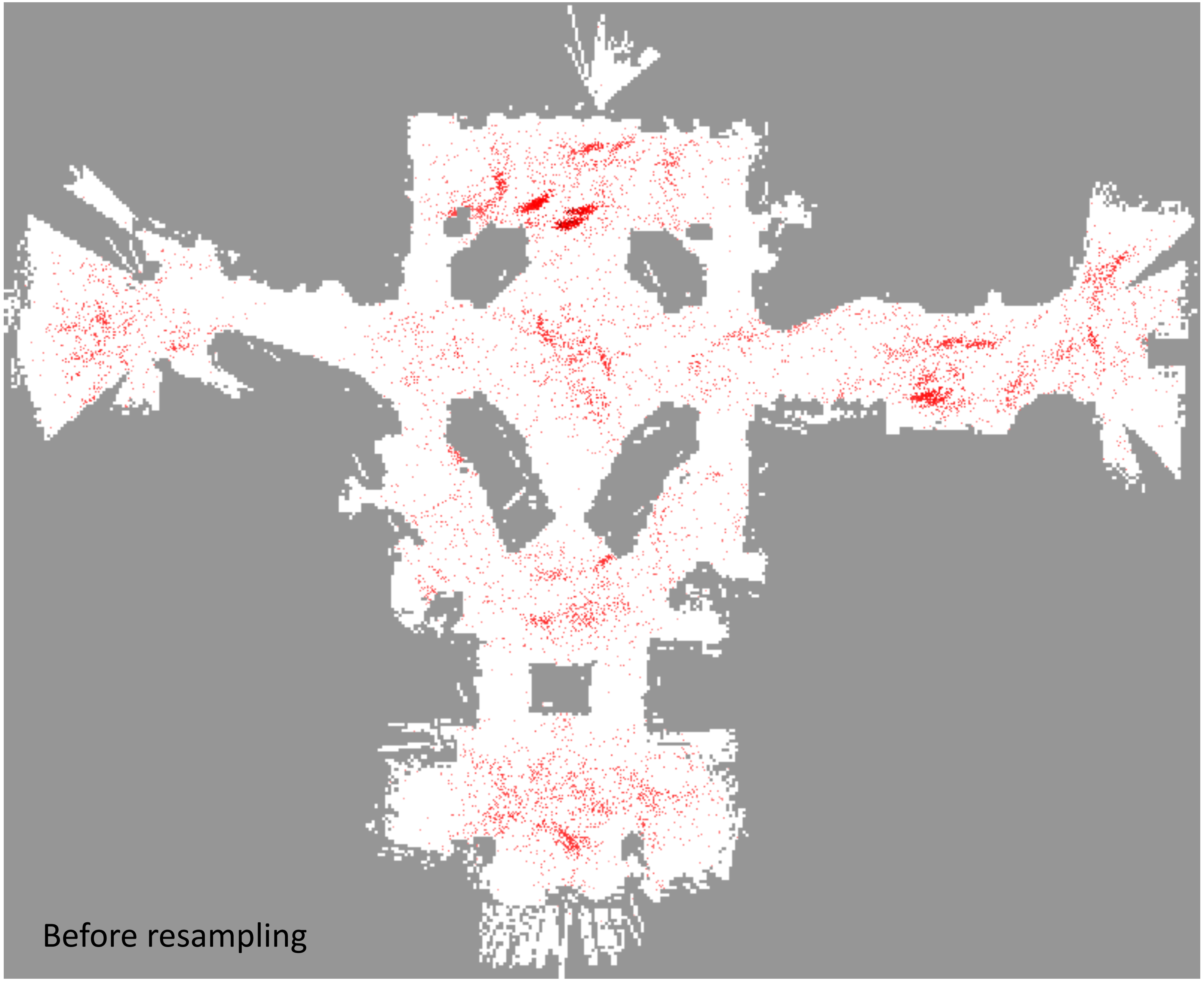

Uniform distribution

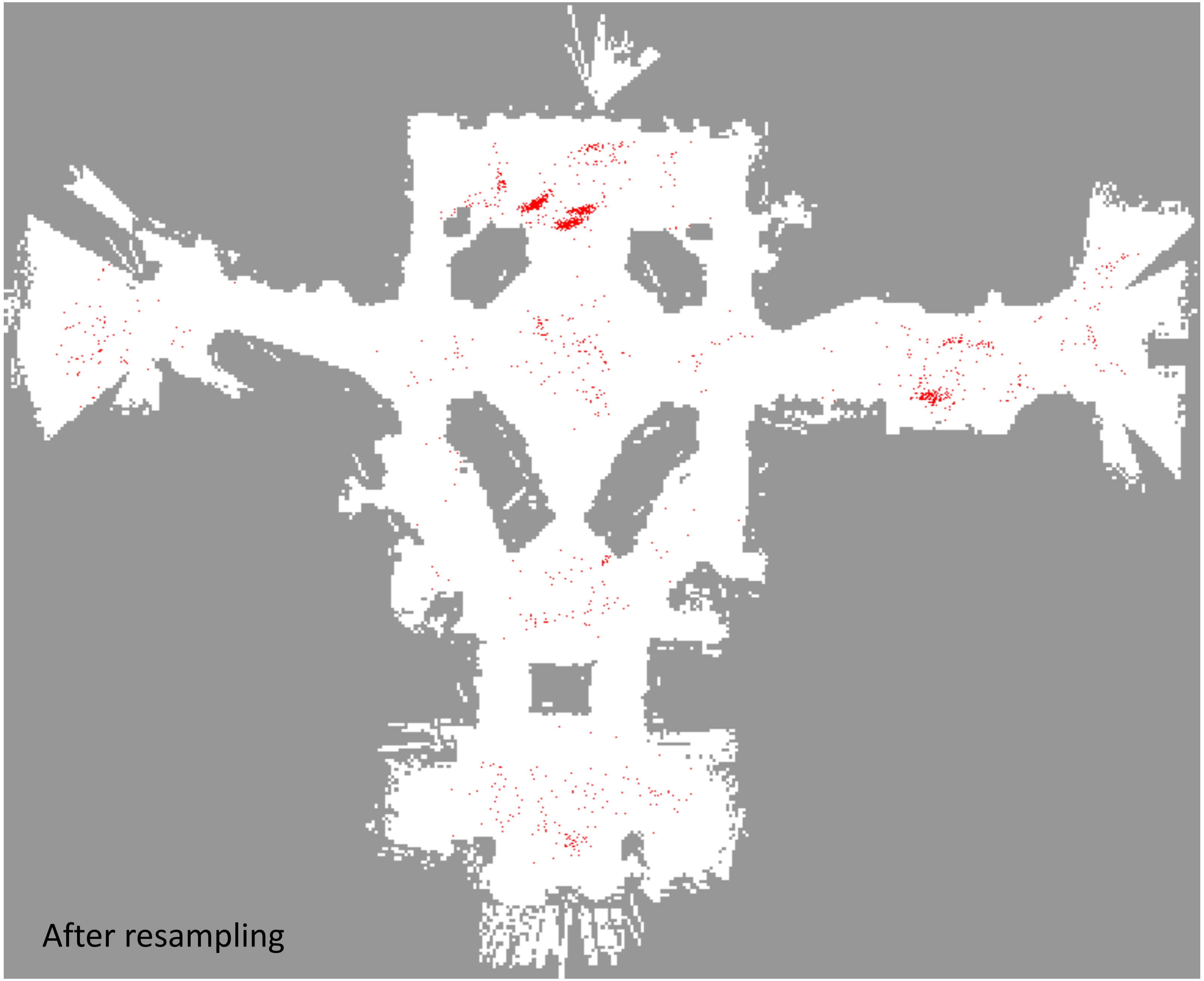

Sense


Before resampling

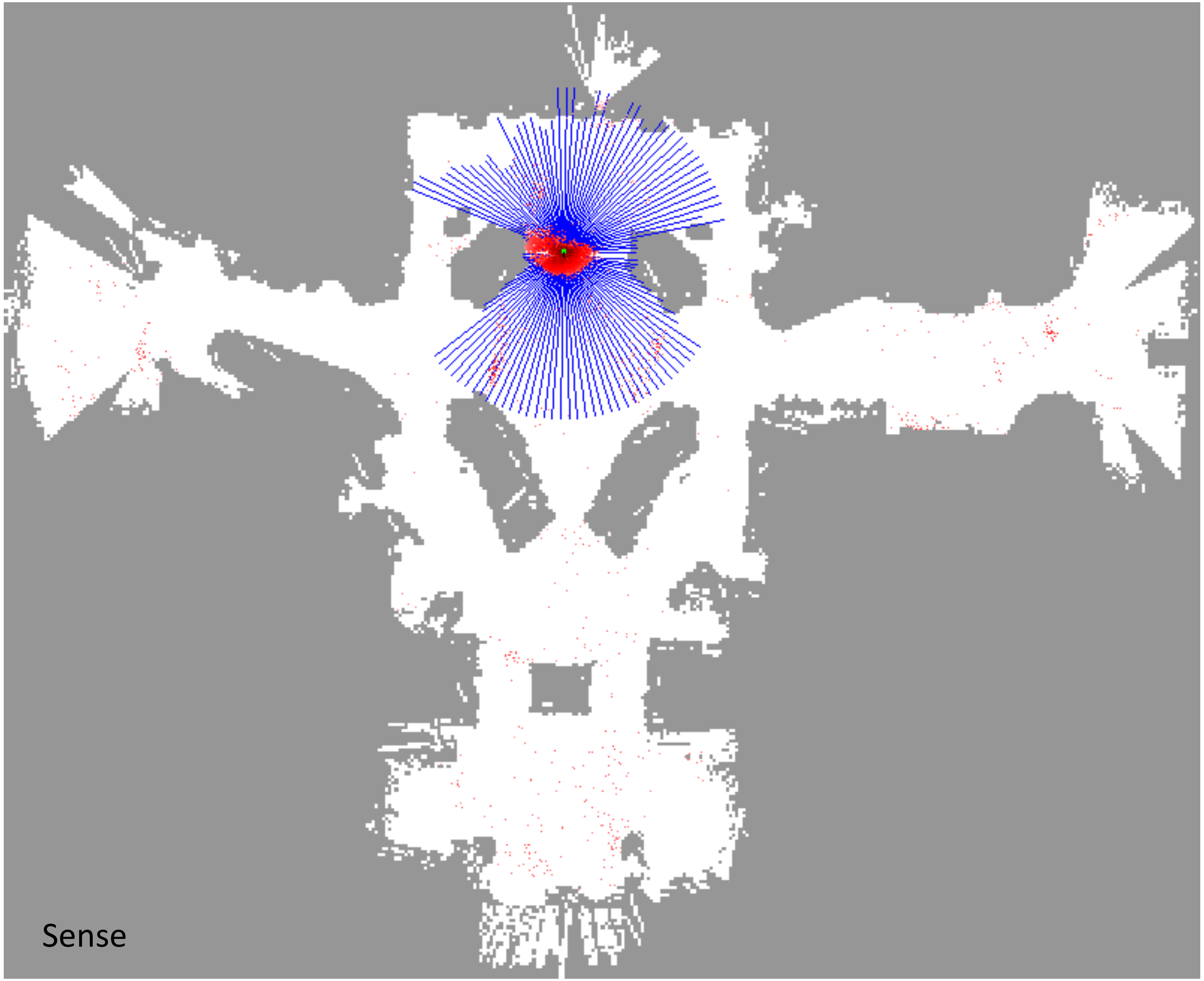

After resampling


Sense

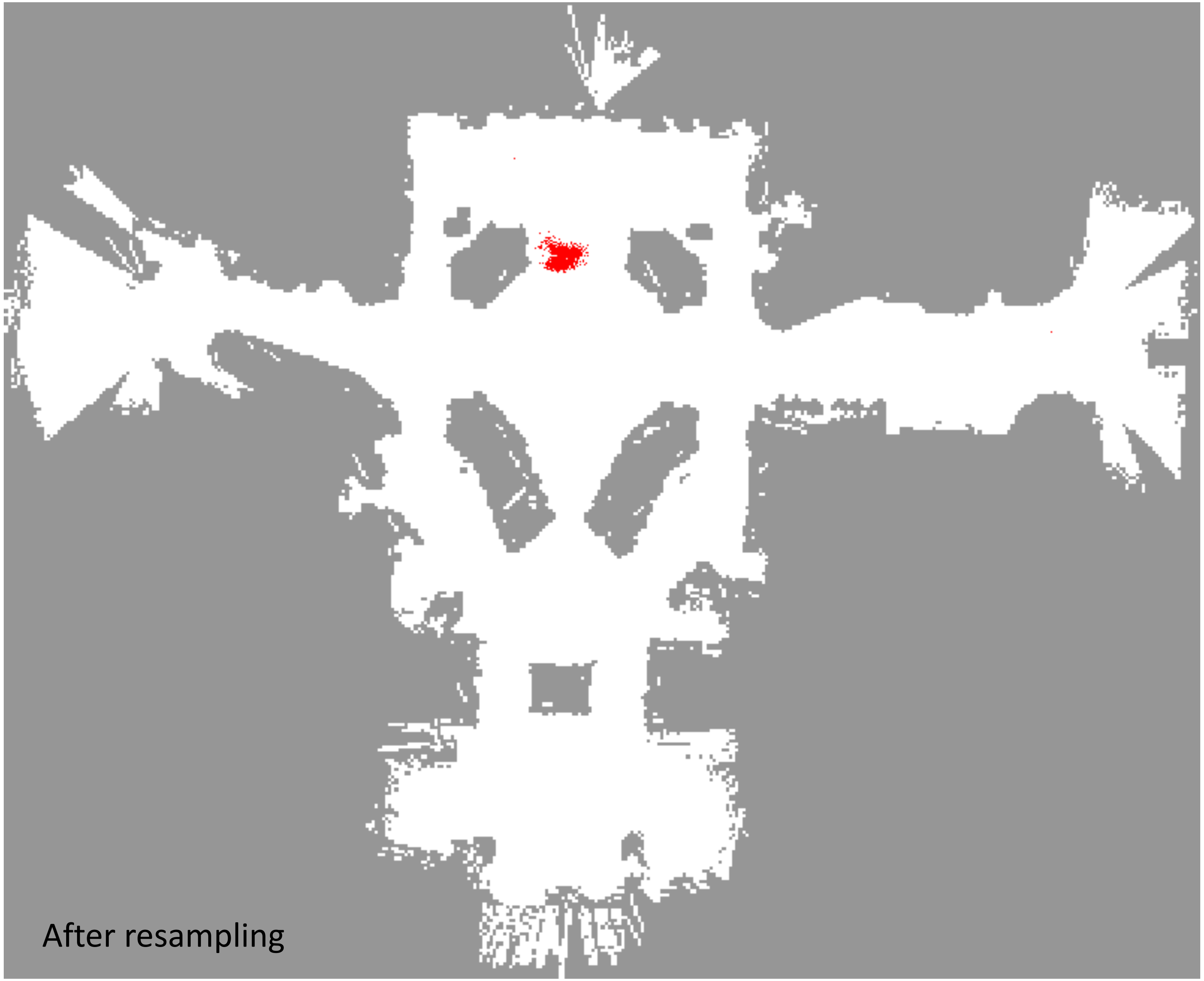

Before resampling

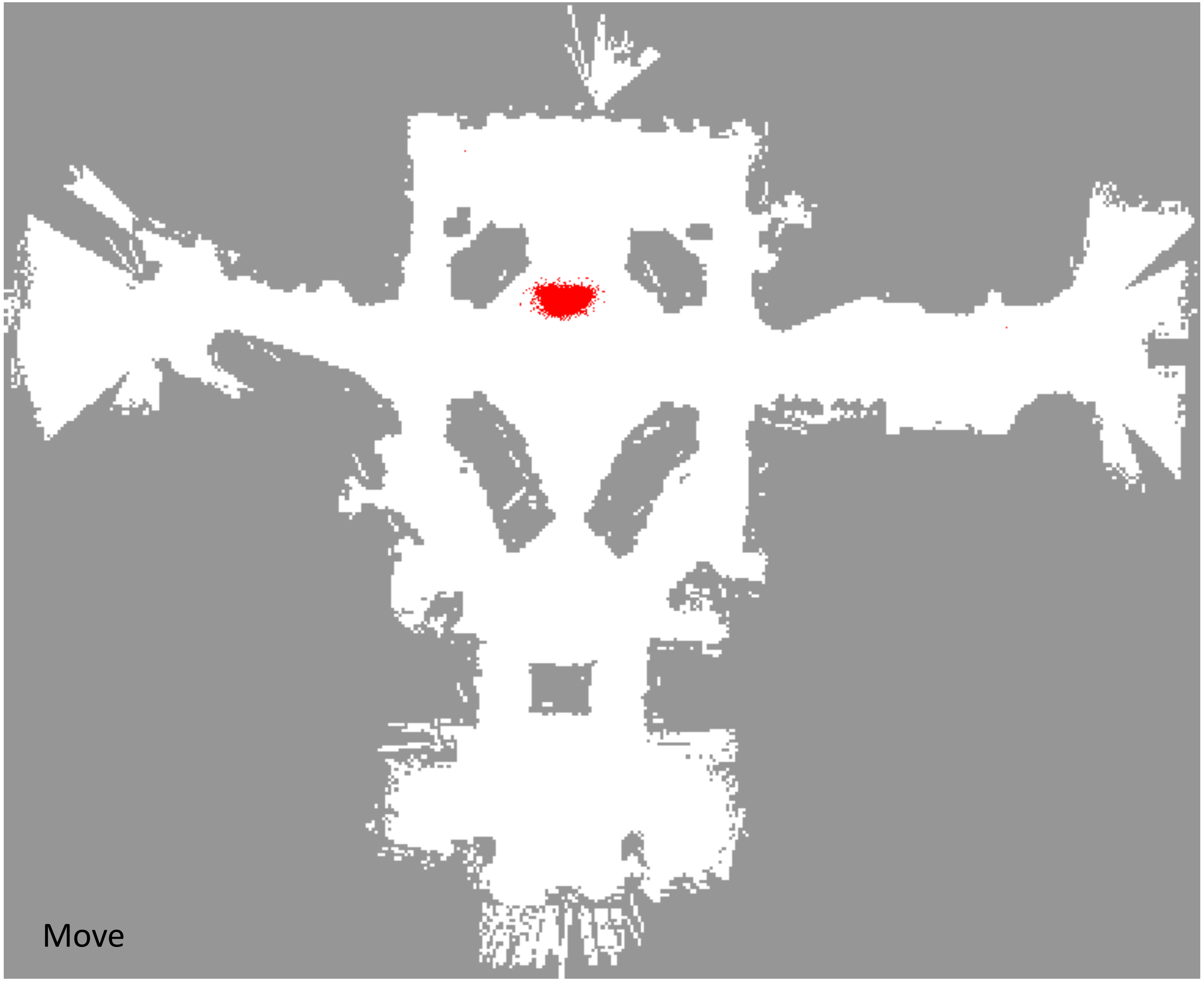

After resampling

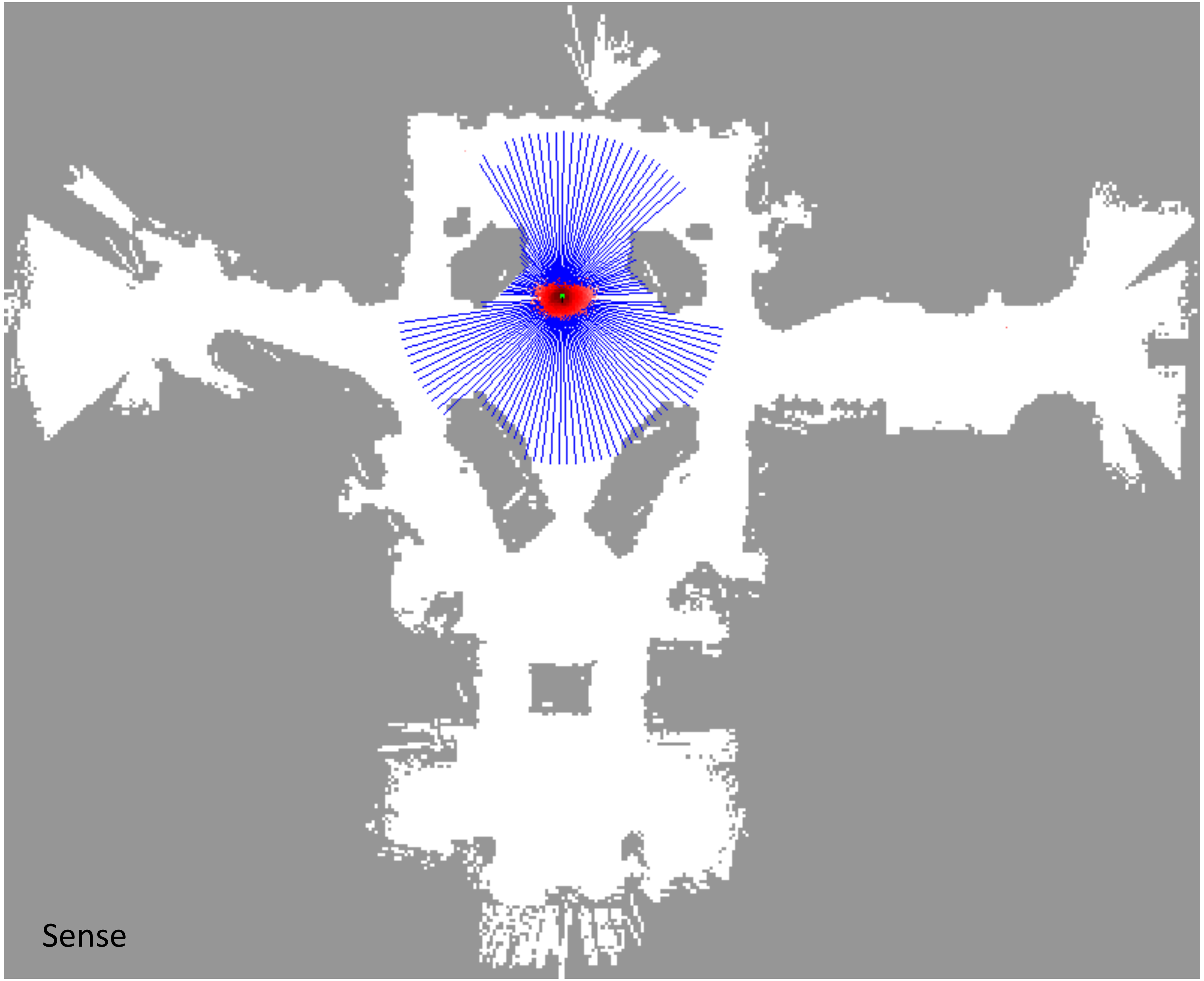

Move

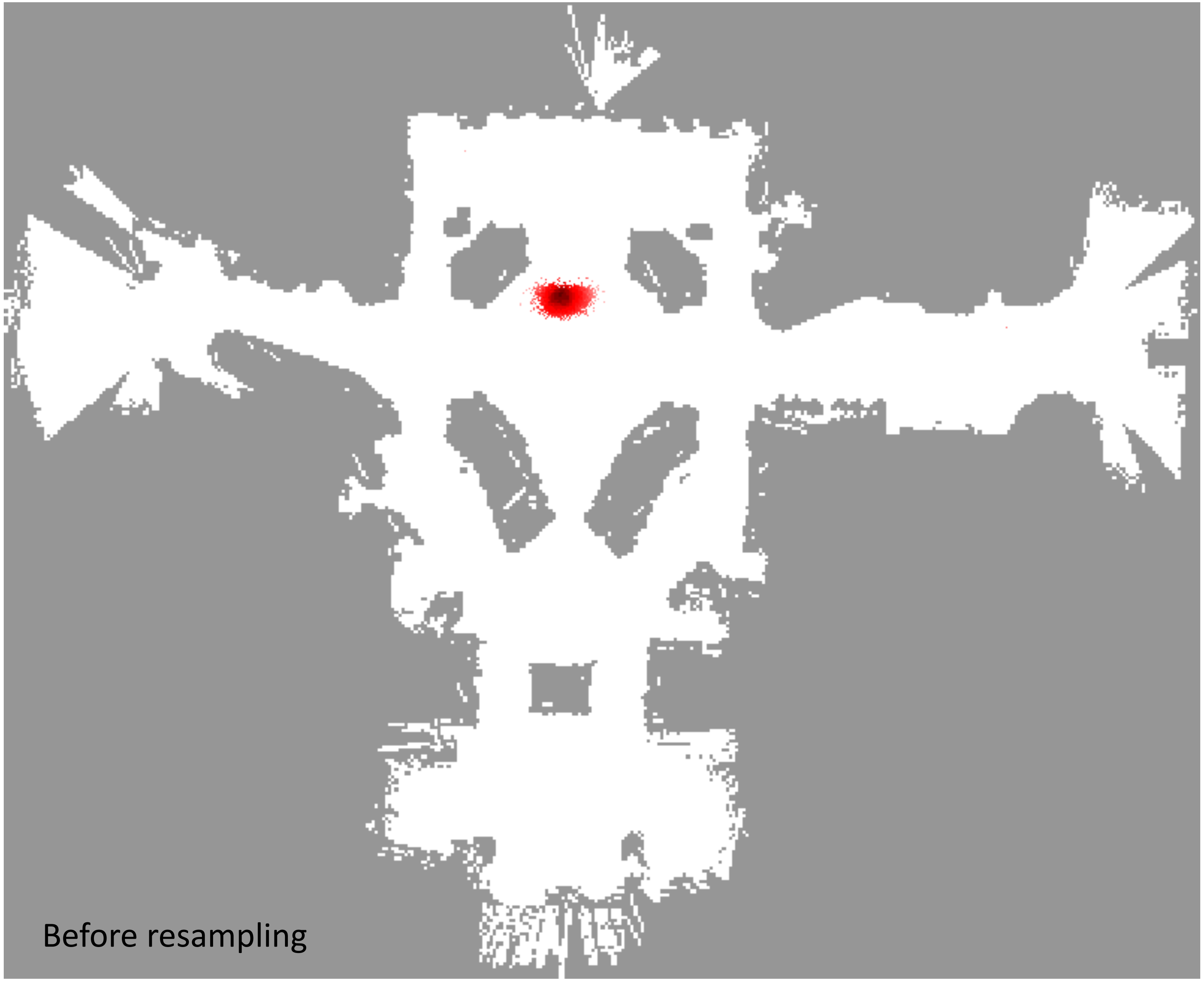

Sense

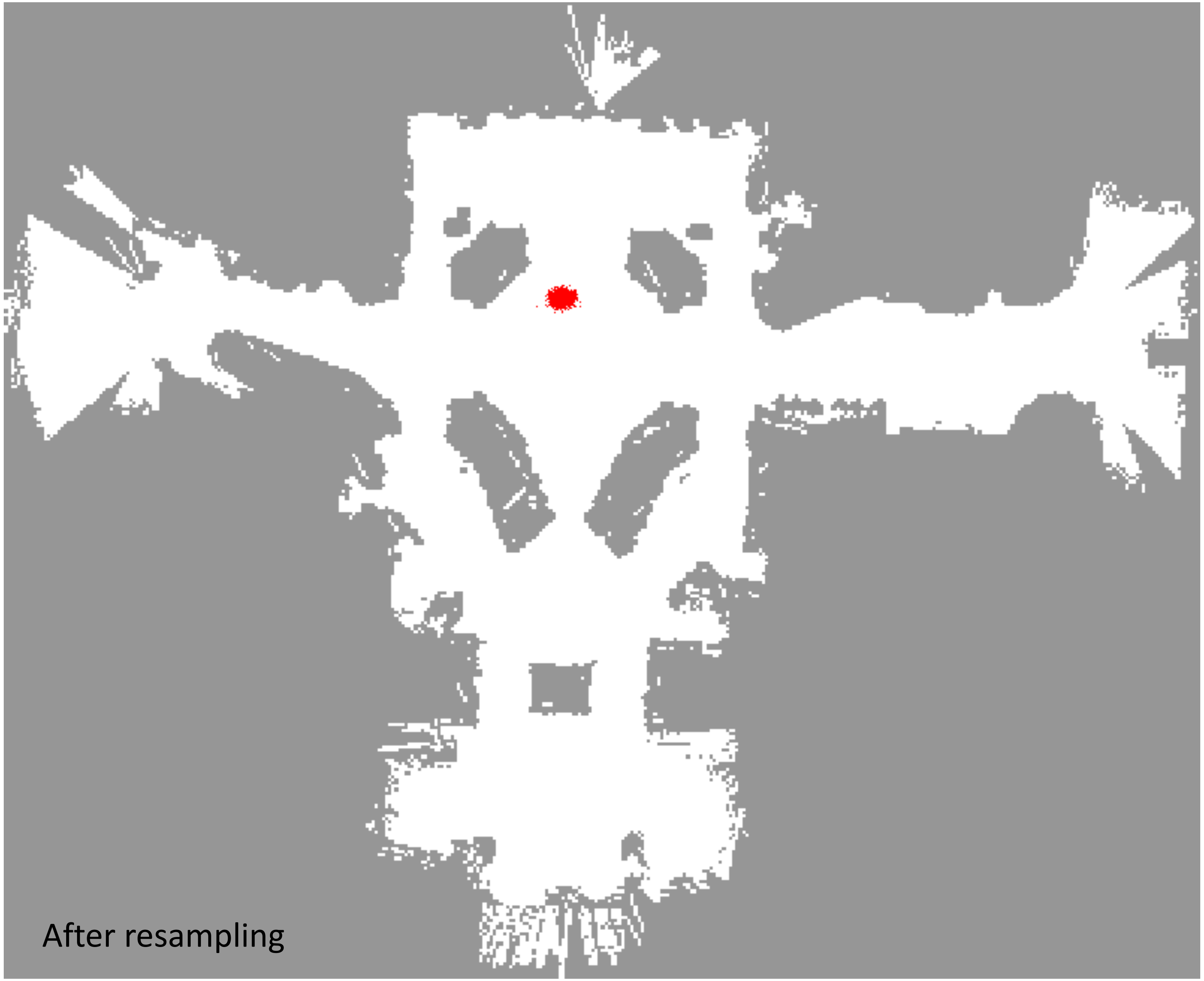

Before resampling

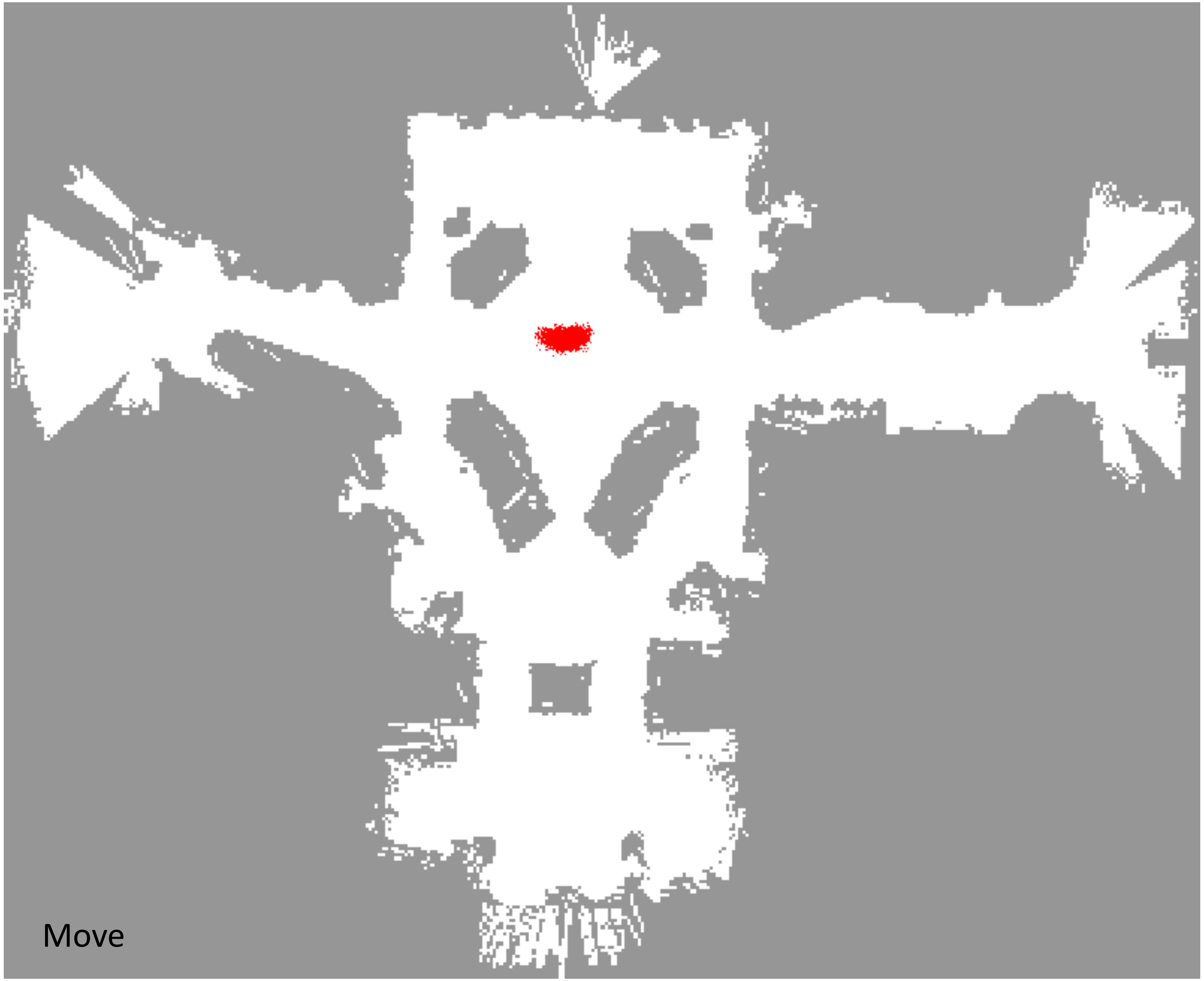

After resampling


Move


Sense

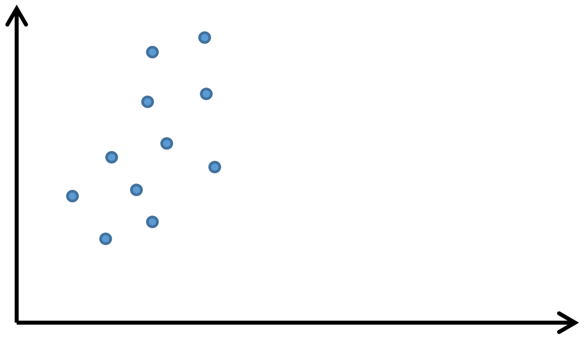

Before resampling


After resampling

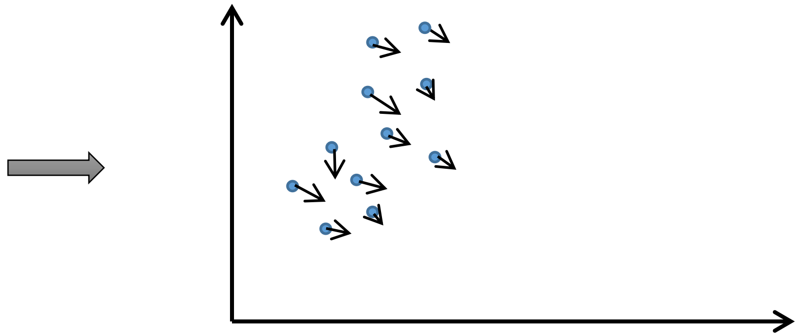

Move

Motion Model

- When the command u_{t-1} is executed, each particle is updated to approximate the robot's movement by ***sampling*** from $p(x_t|x_{t-1}, u_{t-1})$.
- At this stage, typically all particles have equal weight ($w = \frac{1}{N}$).



n particles



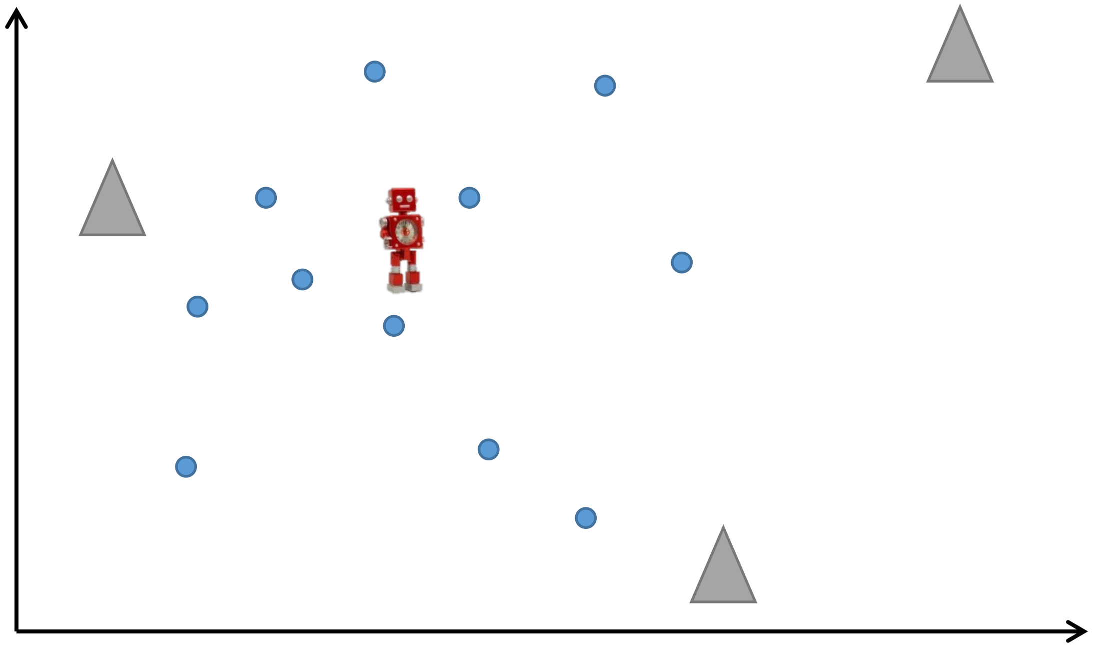
n particles

Sensing Model

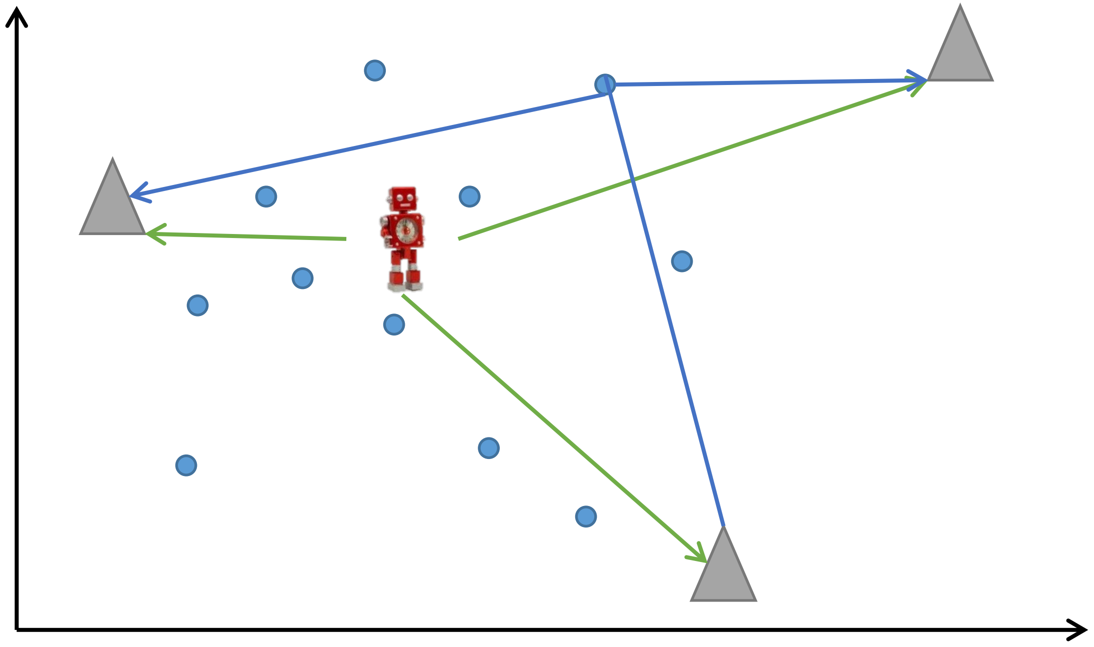
- ***Re-weight sample set***, according to the likelihood that robot's current sensors match what would be seen at a given location
  - Let $< x, w >$ be a sample.
  - Then, $w \leftarrow \eta P(z|x)$

    - z is the sensor measurement;
    - η a normalization constant to enforce the sum of w's equaling 1

Incorporating Sensing



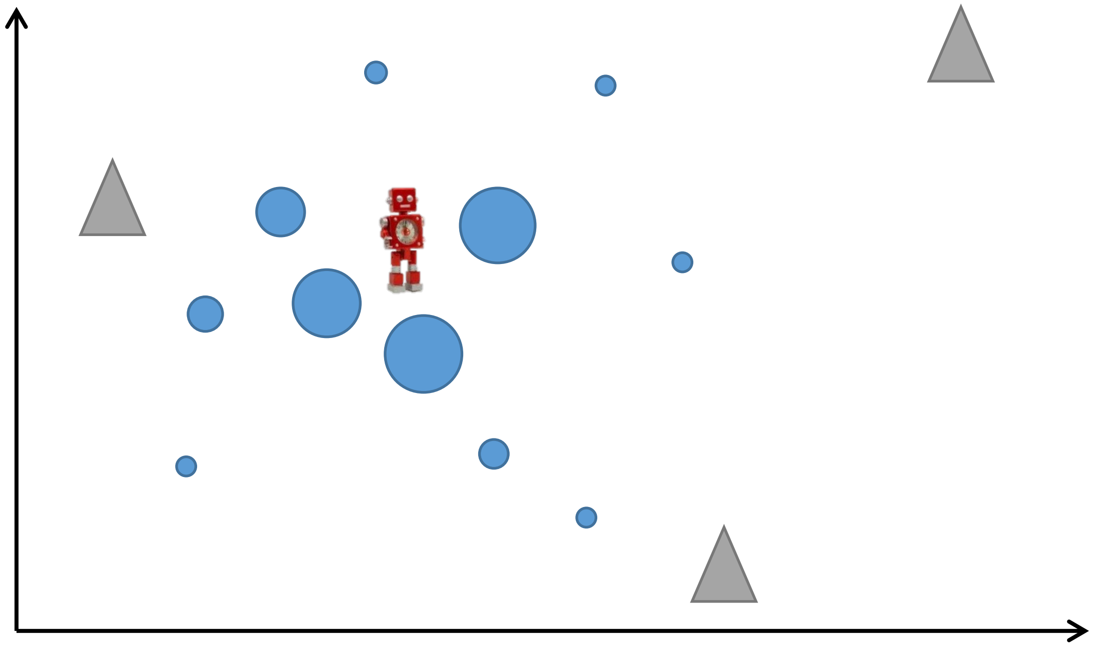
Incorporating Sensing



Difference between the
actual measurement
and the
estimated measurement

Importance weight

Incorporating Sensing



Resampling

- After applying the motion update and sensing update, we end up with new positions and weights for particles
- We want to eliminate particles that have very low weight (unlikely to represent robot position) and generate more particles in the more likely areas of the state space.
- ***Resample***, according to latest weights
- Add a few uniformly distributed, **random samples**
    - Very helpful in case robot completely loses track of its location

Resampling

n original particles	Importance Weight $w(x_i)$
	0.2
	0.6
	0.2
	0.8
	0.8
	0.2

$$\sum = 2.8$$

$$\eta = \frac{1}{2.8}$$

Resampling

n original particles	Importance Weight $w(x_i)$	Normalized Probability $p(x_i)$
	0.2	0.07
	0.6	0.21
	0.2	0.07
	0.8	0.29
	0.8	0.29
	0.2	0.07

$$\sum = 2.8$$

Resampling

n original particles	Importance Weight $w(x_i)$	Normalized Probability $p(x_i)$
	0.2	0.07
	0.6	0.21
	0.2	0.07
	0.8	0.29
	0.8	0.29
	0.2	0.07

$$\sum = 2.8$$

Sample n new particles from the previous set.

- Each particle is chosen with probability $p(x_i)$, <u>with replacement</u>. Add a little random noise to each resampled particle to avoid identical duplicates.

Resampling

Is it possible that one of the particles is never chosen?

Yes!

Is it possible that one of the particles is chosen more than once?

Yes!

n original particles	Importance Weight $w(x_i)$	Normalized Probability $p(x_i)$
	0.2	0.07
	0.6	0.21
	0.2	0.07
	0.8	0.29
	0.8	0.29
	0.2	0.07

$$\sum = 2.8$$

Sample n new particles from the previous set.

- Each particle is chosen with probability $p(x_i)$, <u>with replacement.</u>

Resampling

What is the probability that this particle is not chosen during the resampling of the six new particles?

$$(0.71)^6 = 0.13$$

n original particles	Importance Weight $w(x_i)$	Normalized Probability $p(x_i)$
	0.2	0.07
	0.6	0.21
	0.2	0.07
	0.8	0.29
	0.8	0.29
	0.2	0.07

$$\sum = 2.8$$

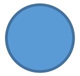
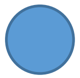
Sample n new particles from the previous set.

- Each particle is chosen with probability $p(x_i)$, with replacement.

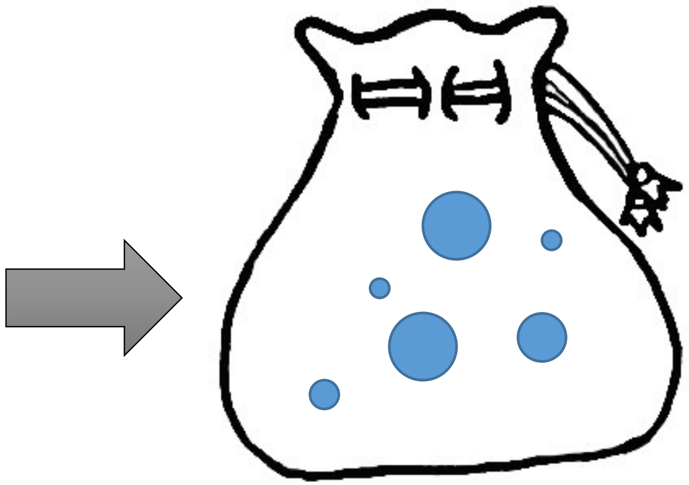
Resampling

What is the probability that this particle is not chosen during the resampling of the six new particles?

$$(0.93)^6 = .65$$

n original particles	Importance Weight $w(x_i)$	Normalized Probability $p(x_i)$
	0.2	0.07
	0.6	0.21
	0.2	0.07
	0.8	0.29
	0.8	0.29
	0.2	0.07

$$\sum = 2.8$$



Sample n new particles from the previous set.

- Each particle is chosen with probability $p(x_i)$, <u>with replacement.</u>

Particle Filter Algorithm

1. Algorithm **particle_filter**(X_{t-1}, u_{t-1}, z_t):
2. $X_t = \emptyset, \eta = 0$

Input:

- u_{t-1} is the action that was executed at time $t-1$
- $X_{t-1} = \left\{< x_{t-1}^j, w_j >\right\}_{j=1\ldots N}$ is the set of weighted particles at time $t-1$
- z_t is the sensor measurement at time $t-1$

Output:

- $X_t = \left\{< x_t^j, w_j >\right\}_{j=1\ldots N}$ is a set of weighted particles at time t

Particle Filter Algorithm

1. Algorithm **particle_filter**(X_{t-1}, u_{t-1}, z_t):
2. $X_t = \emptyset, \eta = 0$
3. **For $j = 1 \ldots N$** ***Generate new samples***
4. Sample index j from discrete index set $\{1, \ldots N\}$ based on w_{t-1}

   Sample x_t^j from $p\left(x_t^j \middle|, x_{t-1}^j, u_{t-1}\right)$

NOTE: j indicates a randomly chosen particle based on weights at time t − 1
x_t^j is determined using only the motion model for specific
action, u_{t-1} applied in state x_{t-1}^j

Particle Filter Algorithm

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2. $X_t = \emptyset, \eta = 0$
3. **For** $\boldsymbol{j = 1 \ldots N}$ ***Generate new samples***
4. Sample index j from discrete index set $\{1, \ldots N\}$ based on w_{t-1}

   Sample x_t^j from $p\left(x_t^j \middle|, x_{t-1}^j, u_{t-1}\right)$
5. $w_t^j = p(z_t | x_t^j)$ ***Compute importance weight***
6. $\eta = \eta + w_t^j$ ***Update normalization factor***
7. $X_t = X_t \cup \{< x_t^j, w_t^j >\}$ ***Add to set of new particles***

Particle Filter Algorithm

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6. $\eta = \eta + w_t^j$ ***Update normalization factor***
7. $X_t = X_t \cup \{< x_t^j, w_t^j >\}$ ***Add to set of new particles***
8. **For $j = 1 \ldots N$**
9. $w_t^j = w_t^j / \eta$ ***Normalize weights***

Markov Localization

Markov localization approximates the state space using a discrete grid.

At time t, the value in the grid cell x^{ij} represent the probability that $x_t = x^{ij}$.

At time $t + 1$, every grid cell updates its probability value based on:

- Prediction from the motion model
- Observation from sensors

This is a grid-cell-centric view of probability updating.

Instead of keeping track of moving probability mass (e.g., particles), each grid cell pays attention to the probability mass that arrives to its specific location.

Markov Localization

- Perception (or sensing) model: represents likelihood that robot senses a particular reading at a particular position.

$$P(x) \leftarrow \eta P(z|x)P(x)$$

Probability that robot will perceive z, given that the robot is in position *x,* times the probability the robot is in position *x*

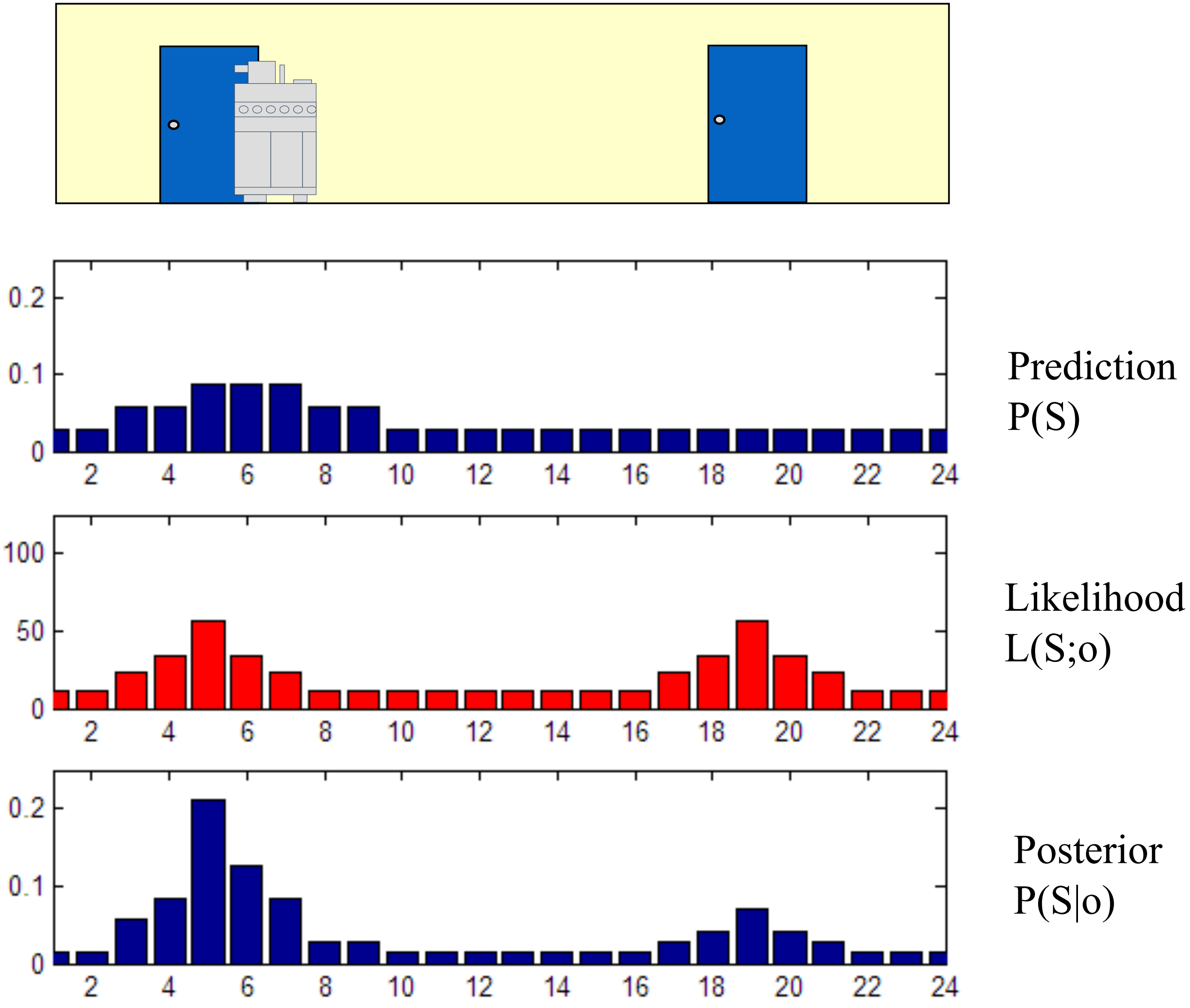
- Action (or motion) model: represents movements of robot

$$P(x) \leftarrow \sum P(x|u, x')P(x')$$

Probability that action u from position x' moves the robot to position x, weighted by the probability that the robot is in position x', summed over all possible x' where the robot might have been.

➢ ***Perform these computations at every grid cell, at each time t.***

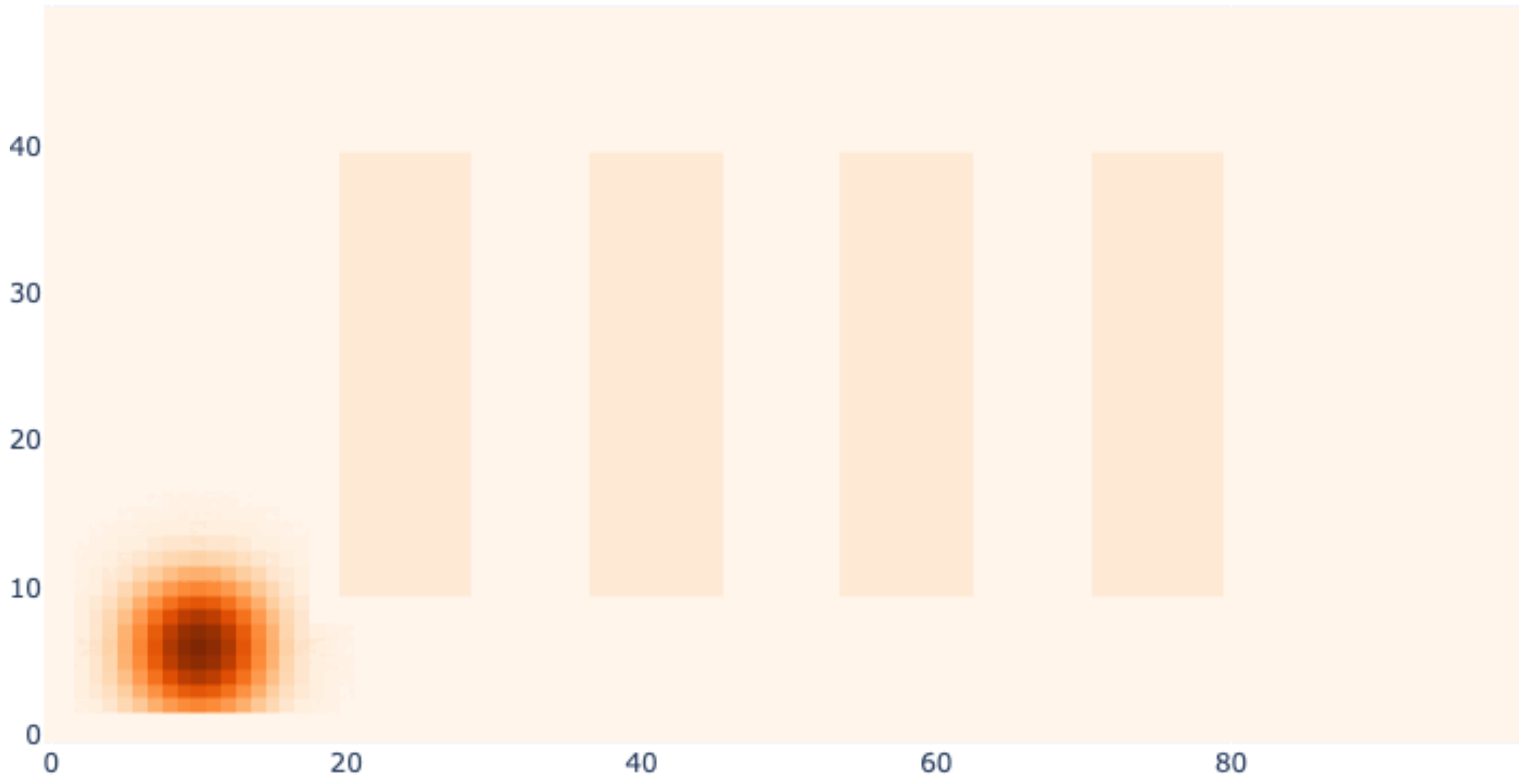
Markov Localization: a 1D Example



Each bin in the histogram is updated in each step.

Example

- The robot moves through the world, and each cell in the grid updates its probability estimate at each time step:



Implementing Markov Localization

- In practice, many grid cells have very small probability values.
- We can speed computation by ignoring these cells, with little risk of going astray in our state estimation.
- If we care about the robot's orientation, then we need to add a θ dimension to our grid.