# Part 3
# Attitude Dynamics and Control

# 5
# Rotational Kinematics

The formulation of spacecraft attitude dynamics and control problems involves considerations of kinematics. This chapter is concerned with rotational kinematics of a rigid body. In kinematics, we are primarily interested in describing the orientation of a body that is in rotational motion. The subject of rotational kinematics is somewhat mathematical in nature because it does not involve any forces associated with motion. Throughout this chapter, we will speak of the orientation of a reference frame fixed in a body to describe the orientation of the body itself.

## 5.1 Direction Cosine Matrix

Consider a reference frame $A$ with a right-hand set of three orthogonal unit vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ and a reference frame $B$ with another right-hand set of three orthogonal unit vectors $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$, as shown in Fig. 5.1. Basis vectors $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ of $B$ are expressed in terms of basis vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ of $A$ as follows:

$$\vec{b}_1 = C_{11}\vec{a}_1 + C_{12}\vec{a}_2 + C_{13}\vec{a}_3 \tag{5.1a}$$

$$\vec{b}_2 = C_{21}\vec{a}_1 + C_{22}\vec{a}_2 + C_{23}\vec{a}_3 \tag{5.1b}$$

$$\vec{b}_3 = C_{31}\vec{a}_1 + C_{32}\vec{a}_2 + C_{33}\vec{a}_3 \tag{5.1c}$$

where $C_{ij} \equiv \vec{b}_i \cdot \vec{a}_j$ is the cosine of the angle between $\vec{b}_i$ and $\vec{a}_j$, and $C_{ij}$ is simply called the *direction cosine.*

For convenience, we write Eqs. (5.1) in matrix (or vectrix) notation, as follows:

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} = \mathbf{C}^{B/A} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} \tag{5.2}$$

where $\mathbf{C}^{B/A} \equiv [C_{ij}]$ is called the direction cosine matrix, which describes the orientation of $B$ relative to $A$ and which can be written as

$$\mathbf{C}^{B/A} = \begin{bmatrix} \vec{b}_1 \cdot \vec{a}_1 & \vec{b}_1 \cdot \vec{a}_2 & \vec{b}_1 \cdot \vec{a}_3 \\ \vec{b}_2 \cdot \vec{a}_1 & \vec{b}_2 \cdot \vec{a}_2 & \vec{b}_2 \cdot \vec{a}_3 \\ \vec{b}_3 \cdot \vec{a}_1 & \vec{b}_3 \cdot \vec{a}_2 & \vec{b}_3 \cdot \vec{a}_3 \end{bmatrix} \equiv \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} \cdot [\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3] \tag{5.3}$$

The direction cosine matrix $\mathbf{C}^{B/A}$ is also called the *rotation matrix* or *coordinate transformation matrix* to $B$ from $A$. Such a coordinate transformation is symbolically represented as

$$\mathbf{C}^{B/A}: \quad B \leftarrow A$$

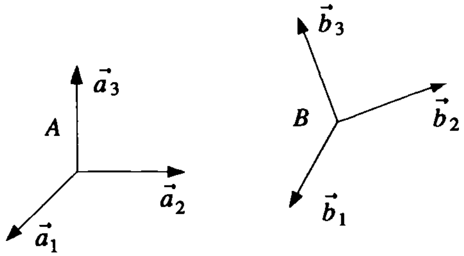


**Fig. 5.1 Two reference frames $A$ and $B$.**

For brevity, we often use $\mathbf{C}$ for $\mathbf{C}^{B/A}$. Because each set of basis vectors of $A$ and $B$ consists of orthogonal unit vectors, the direction cosine matrix $\mathbf{C}$ is an orthonormal matrix; thus, we have

$$\mathbf{C}^{-1} = \mathbf{C}^T \tag{5.4}$$

which is equivalent to

$$\mathbf{C}\mathbf{C}^T = \mathbf{I} = \mathbf{C}^T\mathbf{C} \tag{5.5}$$

In general, a square matrix $\mathbf{A}$ is called an orthogonal matrix if $\mathbf{A}\mathbf{A}^T$ is a diagonal matrix, and it is called an orthonormal matrix if $\mathbf{A}\mathbf{A}^T$ is an identity matrix. For an orthonormal matrix $\mathbf{A}$, we have $\mathbf{A}^{-1} = \mathbf{A}^T$ and $|\mathbf{A}| = \pm 1$.

We also use $\mathbf{C}^{A/B}$ to denote a coordinate transformation matrix to $A$ from $B$ or a direction cosine matrix of $A$ relative to $B$; i.e., we have

$$\mathbf{C}^{A/B} = \begin{bmatrix} \vec{a}_1 \cdot \vec{b}_1 & \vec{a}_1 \cdot \vec{b}_2 & \vec{a}_1 \cdot \vec{b}_3 \\ \vec{a}_2 \cdot \vec{b}_1 & \vec{a}_2 \cdot \vec{b}_2 & \vec{a}_2 \cdot \vec{b}_3 \\ \vec{a}_3 \cdot \vec{b}_1 & \vec{a}_3 \cdot \vec{b}_2 & \vec{a}_3 \cdot \vec{b}_3 \end{bmatrix} \equiv \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} \cdot \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$$

Consequently, we have the following intimate relationships between $\mathbf{C}^{A/B}$ and $\mathbf{C}^{B/A}$:

$$\left[\mathbf{C}^{A/B}\right]_{ij} = \vec{a}_i \cdot \vec{b}_j$$

$$\left[\mathbf{C}^{B/A}\right]_{ij} = \vec{b}_i \cdot \vec{a}_j$$

$$\left[\mathbf{C}^{A/B}\right]^{-1} = \left[\mathbf{C}^{A/B}\right]^T = \mathbf{C}^{B/A}$$

$$\left[\mathbf{C}^{B/A}\right]^{-1} = \left[\mathbf{C}^{B/A}\right]^T = \mathbf{C}^{A/B}$$

Given the two sets of reference frames $A$ and $B$, an arbitrary vector $\vec{H}$ can be expressed in terms of basis vectors of $A$ and $B$, as follows:

$$\begin{aligned} \vec{H} &= H_1\vec{a}_1 + H_2\vec{a}_2 + H_3\vec{a}_3 \\ &= H_1'\vec{b}_1 + H_2'\vec{b}_2 + H_3'\vec{b}_3 \end{aligned} \tag{5.6}$$

and we have

$$H_1' \equiv \vec{b}_1 \cdot \vec{H} = \vec{b}_1 \cdot (H_1\vec{a}_1 + H_2\vec{a}_2 + H_3\vec{a}_3) \tag{5.7a}$$

$$H_2' \equiv \vec{b}_2 \cdot \vec{H} = \vec{b}_2 \cdot (H_1\vec{a}_1 + H_2\vec{a}_2 + H_3\vec{a}_3) \tag{5.7b}$$

$$H_3' \equiv \vec{b}_3 \cdot \vec{H} = \vec{b}_3 \cdot (H_1\vec{a}_1 + H_2\vec{a}_2 + H_3\vec{a}_3) \tag{5.7c}$$

which can be written in matrix form, as follows:

$$\begin{bmatrix} H_1' \\ H_2' \\ H_3' \end{bmatrix} = \begin{bmatrix} \vec{b}_1 \cdot \vec{a}_1 & \vec{b}_1 \cdot \vec{a}_2 & \vec{b}_1 \cdot \vec{a}_3 \\ \vec{b}_2 \cdot \vec{a}_1 & \vec{b}_2 \cdot \vec{a}_2 & \vec{b}_2 \cdot \vec{a}_3 \\ \vec{b}_3 \cdot \vec{a}_1 & \vec{b}_3 \cdot \vec{a}_2 & \vec{b}_3 \cdot \vec{a}_3 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = \mathbf{C}^{B/A} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} \tag{5.8}$$

Thus, the components of a vector $\vec{H}$ are also transformed to $B$ from $A$ using the direction cosine matrix $\mathbf{C}^{B/A}$, which was defined in Eq. (5.2) for the transformation of orthogonal basis vectors.

Three elementary rotations respectively about the first, second, and third axes of the reference frame $A$ are described by the following rotation matrices:

$$\mathbf{C}_1(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \tag{5.9a}$$

$$\mathbf{C}_2(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \tag{5.9b}$$

$$\mathbf{C}_3(\theta_3) = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.9c}$$

where $\mathbf{C}_i(\theta_i)$ denotes the direction cosine matrix $\mathbf{C}$ of an elementary rotation about the $i$th axis of $A$ with an angle $\theta_i$.

## Problem

**5.1.** Consider the direction cosine matrix, $\mathbf{C} \equiv [C_{ij}]$, between two sets of right-hand orthogonal unit vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ and $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$, defined as

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}$$

(a) Show that the direction cosine matrix $\mathbf{C}$ is an orthonormal matrix; i.e., $\mathbf{C}\mathbf{C}^T = \mathbf{I} = \mathbf{C}^T\mathbf{C}$.

*Hint:* $\left[\vec{b}_1\ \vec{b}_2\ \vec{b}_3\right] = \left[\vec{a}_1\ \vec{a}_2\ \vec{a}_3\right]\mathbf{C}^T$ and

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} \cdot \left[\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3\right] \equiv \begin{bmatrix} \vec{b}_1\cdot\vec{b}_1 & \vec{b}_1\cdot\vec{b}_2 & \vec{b}_1\cdot\vec{b}_3 \\ \vec{b}_2\cdot\vec{b}_1 & \vec{b}_2\cdot\vec{b}_2 & \vec{b}_2\cdot\vec{b}_3 \\ \vec{b}_3\cdot\vec{b}_1 & \vec{b}_3\cdot\vec{b}_2 & \vec{b}_3\cdot\vec{b}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Show that the $ij$th element of the direction cosine matrix $\mathbf{C}$ is equal to the $ij$th cofactor of $\mathbf{C}$; i.e., $C_{ij} = (-1)^{i+j}M_{ij}$ or adj $\mathbf{C} \equiv [(-1)^{i+j}M_{ij}]^T = [C_{ij}]^T = \mathbf{C}^T$.
*Hint:* Using $\vec{b}_1 = \vec{b}_2 \times \vec{b}_3$, show that

$$C_{11} = C_{22}C_{33} - C_{23}C_{32}$$

$$C_{12} = C_{23}C_{31} - C_{21}C_{33}$$

$$C_{13} = C_{21}C_{32} - C_{22}C_{31}$$

(c) Show that $|\mathbf{C}| = 1$.
*Hint:* $\mathbf{C}^{-1} \equiv \text{adj}\,\mathbf{C}/|\mathbf{C}|$.
(d) Find six independent equations for $C_{ij}$ using the row-orthonormality condition, $\mathbf{C}\mathbf{C}^T = \mathbf{I}$.
(e) Find six independent equations for $C_{ij}$ using the column-orthonormality condition, $\mathbf{C}^T\mathbf{C} = \mathbf{I}$. Are these six equations independent of those of part (d)?
(f) Finally, show that only three of the nine direction cosines are independent.
*Note:* Three direction cosines, however, do not uniquely define the orientation of two reference frames.

## 5.2 Euler Angles

One scheme for orienting a rigid body to a desired attitude is called a *body-axis rotation*; it involves successively rotating three times about the axes of the rotated, body-fixed reference frame. The first rotation is about any axis. The second rotation is about either of the two axes not used for the first rotation. The third rotation is then about either of the two axes not used for the second rotation. There are 12 sets of Euler angles for such successive rotations about the axes fixed in the body.*

It is also possible to bring a rigid body into an arbitrary orientation by performing three successive rotations that involve the axes fixed in an inertial reference frame. This scheme will then provide another 12 sets of Euler angles for the so-called *space-axis rotations.*[1] Because the coordinate transformation matrices for the body-axis rotation and the space-axis rotation are intimately related to each other, we often only consider the 12 sets of body-axis rotations.

Consider three successive body-axis rotations that describe the orientation of a reference frame $B$ relative to a reference frame $A$. A particular sequence chosen

*Leonard Euler (1707–1783), who was the leading mathematician and theoretical physicist of the 18th century, first introduced the concept of three successive rotations to describe the orientation of an orbit plane using the three angles $\Omega$, $i$, and $\omega$.

here is symbolically represented as

$$\mathbf{C}_3(\theta_3): \quad A' \leftarrow A \tag{5.10a}$$

$$\mathbf{C}_2(\theta_2): \quad A'' \leftarrow A' \tag{5.10b}$$

$$\mathbf{C}_1(\theta_1): \quad B \leftarrow A'' \tag{5.10c}$$

where each rotation is described as

$$\begin{bmatrix} \vec{a}_1' \\ \vec{a}_2' \\ \vec{a}_3' \end{bmatrix} = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} = \mathbf{C}_3(\theta_3) \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} \tag{5.11a}$$

$$\begin{bmatrix} \vec{a}_1'' \\ \vec{a}_2'' \\ \vec{a}_3'' \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \vec{a}_1' \\ \vec{a}_2' \\ \vec{a}_3' \end{bmatrix} = \mathbf{C}_2(\theta_2) \begin{bmatrix} \vec{a}_1' \\ \vec{a}_2' \\ \vec{a}_3' \end{bmatrix} \tag{5.11b}$$

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \vec{a}_1'' \\ \vec{a}_2'' \\ \vec{a}_3'' \end{bmatrix} = \mathbf{C}_1(\theta_1) \begin{bmatrix} \vec{a}_1'' \\ \vec{a}_2'' \\ \vec{a}_3'' \end{bmatrix} \tag{5.11c}$$

and $A'$ and $A''$ are two intermediate reference frames with basis vectors $\{\vec{a}_1', \vec{a}_2', \vec{a}_3'\}$ and $\{\vec{a}_1'', \vec{a}_2'', \vec{a}_3''\}$, respectively. The three angles $\theta_1$, $\theta_2$, and $\theta_3$ are called *Euler angles*.

By combining the preceding sequence of rotations, we obtain

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \mathbf{C}_1(\theta_1) \begin{bmatrix} \vec{a}_1'' \\ \vec{a}_2'' \\ \vec{a}_3'' \end{bmatrix} = \mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2) \begin{bmatrix} \vec{a}_1' \\ \vec{a}_2' \\ \vec{a}_3' \end{bmatrix} = \mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2)\mathbf{C}_3(\theta_3) \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} \tag{5.12}$$

The rotation matrix to $B$ from $A$, or the direction cosine matrix of $B$ relative to $A$, is then defined as

$$\begin{aligned} \mathbf{C}^{B/A} &\equiv \mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2)\mathbf{C}_3(\theta_3) \\ &= \begin{bmatrix} \mathrm{c}_2\,\mathrm{c}_3 & \mathrm{c}_2\,\mathrm{s}_3 & -\mathrm{s}_2 \\ \mathrm{s}_1\,\mathrm{s}_2\,\mathrm{c}_3 - \mathrm{c}_1\,\mathrm{s}_3 & \mathrm{s}_1\,\mathrm{s}_2\,\mathrm{s}_3 + \mathrm{c}_1\,\mathrm{c}_3 & \mathrm{s}_1\,\mathrm{c}_2 \\ \mathrm{c}_1\,\mathrm{s}_2\,\mathrm{c}_3 + \mathrm{s}_1\,\mathrm{s}_3 & \mathrm{c}_1\,\mathrm{s}_2\,\mathrm{s}_3 - \mathrm{s}_1\,\mathrm{c}_3 & \mathrm{c}_1\,\mathrm{c}_2 \end{bmatrix} \end{aligned} \tag{5.13}$$

where $\mathrm{c}_i \equiv \cos\theta_i$ and $\mathrm{s}_i \equiv \sin\theta_i$.

The preceding sequence of rotations to $B$ from $A$ is also symbolically denoted by*

$$\mathbf{C}_1(\theta_1) \leftarrow \mathbf{C}_2(\theta_2) \leftarrow \mathbf{C}_3(\theta_3)$$

*The notation $\mathbf{C}_1(\theta_1) \leftarrow \mathbf{C}_2(\theta_2) \leftarrow \mathbf{C}_3(\theta_3)$, which, in fact, denotes the same rotational sequence as denoted by the notation $\mathbf{C}_3(\theta_3) \rightarrow \mathbf{C}_2(\theta_2) \rightarrow \mathbf{C}_1(\theta_1)$, is introduced in this book to emphasize the resulting structure of the total rotation matrix $\mathbf{C}^{B/A} = \mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2)\mathbf{C}_3(\theta_3)$.

where $\mathbf{C}_i(\theta_i)$ indicates a rotation about the $i$th axis of the body-fixed frame with an angle $\theta_i$, or by

$$\theta_1 \vec{a}_1'' \leftarrow \theta_2 \vec{a}_2' \leftarrow \theta_3 \vec{a}_3$$

in which, for example, $\theta_3 \vec{a}_3$ denotes a rotation about the $\vec{a}_3$ axis with an angle $\theta_3$.

In general, there are 12 sets of Euler angles, each resulting in a different form for the rotation matrix $\mathbf{C}^{B/A}$. For example, we may consider the sequence of $\mathbf{C}_1(\theta_1) \leftarrow \mathbf{C}_3(\theta_3) \leftarrow \mathbf{C}_2(\theta_2)$ to $B$ from $A$. For this case, the rotation matrix becomes

$$\begin{aligned}
\mathbf{C}^{B/A} &\equiv \mathbf{C}_1(\theta_1)\mathbf{C}_3(\theta_3)\mathbf{C}_2(\theta_2) \\
&= \begin{bmatrix} c_2\,c_3 & s_3 & -s_2\,c_3 \\ -c_1\,c_2\,s_3 + s_1\,s_2 & c_1\,c_3 & c_1\,s_2\,s_3 + s_1\,c_2 \\ s_1\,c_2\,s_3 + c_1\,s_2 & -s_1\,c_3 & -s_1\,s_2\,s_3 + c_1\,c_2 \end{bmatrix}
\end{aligned} \tag{5.14}$$

Note that for small (infinitesimal) Euler angles of $\theta_1$, $\theta_2$, and $\theta_3$, the direction cosine matrices in Eqs. (5.13) and (5.14) become

$$\mathbf{C} \approx \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix} \tag{5.15}$$

That is, the rotation sequence of Euler angles becomes unimportant for infinitesimal rotations, whereas rotation sequence is important for finite rotations.

For other sequences, such as a classical "3 $\leftarrow$ 1 $\leftarrow$ 3" rotational sequence in which the third axis is used twice, we use the following notational convention: $\mathbf{C}_3(\psi) \leftarrow \mathbf{C}_1(\theta) \leftarrow \mathbf{C}_3(\phi)$ to $B$ from $A$, in which, for example, $\mathbf{C}_3(\phi)$ indicates a rotation about the third axis with an angle $\phi$. Thus, for such a classical $\mathbf{C}_3(\psi) \leftarrow \mathbf{C}_1(\theta) \leftarrow \mathbf{C}_3(\phi)$ rotational sequence, the rotation matrix to $B$ from $A$ becomes

$$\begin{aligned}
\mathbf{C}^{B/A} &\equiv \mathbf{C}_3(\psi)\mathbf{C}_1(\theta)\mathbf{C}_3(\phi) \\
&= \begin{bmatrix} c\phi\,c\psi - s\phi\,c\theta\,s\psi & s\phi\,c\psi + c\phi\,c\theta\,s\psi & s\theta\,s\psi \\ -c\phi\,s\psi - s\phi\,c\theta\,c\psi & -s\phi\,s\psi + c\phi\,c\theta\,c\psi & s\theta\,c\psi \\ s\phi\,s\theta & -c\phi\,s\theta & c\theta \end{bmatrix}
\end{aligned}$$

where $c\,\phi \equiv \cos\phi$, $s\,\phi \equiv \sin\phi$, etc.

In general, Euler angles have an advantage over direction cosines in that three Euler angles determine a unique orientation, although there is no unique set of Euler angles for a given orientation.

## 5.3 Euler's Eigenaxis Rotation

In this section, we consider rotation of a rigid body (or a reference frame) about an arbitrary axis that is fixed to the body and stationary in an inertial reference frame. An intimate relationship between the body-axis and space-axis rotations is derived. Such a relationship provides insights into the understanding of Euler's eigenaxis rotation and the space-axis rotation.

### *5.3.1 Euler's Eigenaxis Rotation Theorem*

Euler's eigenaxis rotation theorem states that by rotating a rigid body about an axis that is fixed to the body and stationary in an inertial reference frame, the rigid-body attitude can be changed from any given orientation to any other orientation. Such an axis of rotation, whose orientation relative to both an inertial reference frame and the body remains unchanged throughout the motion, is called the *Euler axis* or *eigenaxis*.

Various different approaches can be used to develop several different parameterizations of the direction cosine matrix of the Euler axis rotation. Almost every formula can be derived in a variety of ways; however, the approach to be taken here is simple and it will provide insights into the understanding of the intimate relationship between the body-axis and space-axis rotations.

Suppose unit vectors $\vec{a}_i$ and $\vec{b}_i$ $(i = 1, 2, 3)$ are fixed in reference frames $A$ and $B$, respectively. The orientation of $B$ with respect to $A$ is characterized by a unit vector $\vec{e}$ along the Euler axis and the rotation angle $\theta$ about that axis, as follows:

$$\begin{aligned}\vec{e} &= e_1\vec{a}_1 + e_2\vec{a}_2 + e_3\vec{a}_3 \\ &= e_1\vec{b}_1 + e_2\vec{b}_2 + e_3\vec{b}_3\end{aligned} \tag{5.16}$$

where $e_i$ are the direction cosines of the Euler axis relative to both $A$ and $B$ and $e_1^2 + e_2^2 + e_3^2 = 1$. Let $\mathbf{C}^{B/A} = \mathbf{C} = [C_{ij}]$ be the direction cosine matrix of $B$ relative to $A$, then Euler's eigenaxis rotation is also characterized by

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \tag{5.17}$$

To parameterize the direction cosine matrix $\mathbf{C}$ in terms of $e_i$ and $\theta$, a sequence of Euler's successive rotations is used as follows:

1) Rotate the reference frame $A$, using a rotation matrix $\mathbf{R}$, to align the $\vec{a}_1$ axis of $A$ with the chosen direction $\vec{e}$. Let $A'$ be the new reference frame after this rotation and also let $A$ remain the original frame with basis vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ before this rotation; i.e., we have

$$\mathbf{C}^{A'/A} = \mathbf{R} = \begin{bmatrix} e_1 & e_2 & e_3 \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \tag{5.18}$$

2) Rotate both frames $A$ and $A'$ as a rigid body around direction $\vec{e}$ through an angle $\theta$. After this eigenaxis rotation, the frame $A$ will be aligned with the reference frame $B$ with basis vectors $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$, and $A'$ will become another reference frame $A''$ via the rotation matrix

$$\mathbf{C}^{A''/A'} = \mathbf{C}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{5.19}$$

The orientation of $B$ relative to $A$ is described by the direction cosine matrix $\mathbf{C}^{B/A}$. It is important to notice that the relative orientation of $A''$ and $B$ is the same as that of $A'$ and $A$; i.e., $\mathbf{C}^{A''/B} = \mathbf{C}^{A'/A} = \mathbf{R}$.

3) Rotate $A''$ through an inverse matrix $\mathbf{R}^{-1} = \mathbf{R}^T$, then the frame $A''$ will be aligned with $B$ since $\mathbf{C}^{B/A''} = \mathbf{R}^{-1}$.

These three successive rotations can be combined as

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \mathbf{C}^{B/A} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} \tag{5.20}$$

where

$$\mathbf{C}^{B/A} = \mathbf{C}^{B/A''}\mathbf{C}^{A''/A'}\mathbf{C}^{A'/A} = \mathbf{C}^{A/A'}\mathbf{C}_1(\theta)\mathbf{C}^{A'/A} = \mathbf{R}^T\mathbf{C}_1(\theta)\mathbf{R} \tag{5.21}$$

If the $\vec{a}_2$ or $\vec{a}_3$ axis, instead of $\vec{a}_1$ axis, is aligned with the chosen direction $\vec{e}$ for the first rotation, then the rotation matrix $\mathbf{C}_1(\theta)$ in Eq. (5.21) is replaced by $\mathbf{C}_2(\theta)$ or $\mathbf{C}_3(\theta)$, respectively, and the rotation matrix $\mathbf{R}$ is replaced, respectively, by

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ e_1 & e_2 & e_3 \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ e_1 & e_2 & e_3 \end{bmatrix}$$

Substituting Eqs. (5.18) and (5.19) into Eq. (5.21) and defining $\mathbf{C} = [C_{ij}] = \mathbf{C}^{B/A}$, we obtain

$$\begin{aligned}
C_{11} &= e_1^2 + \left(R_{21}^2 + R_{31}^2\right)\cos\theta \\
C_{12} &= e_1 e_2 + (R_{21}R_{22} + R_{31}R_{32})\cos\theta + (R_{21}R_{32} - R_{22}R_{31})\sin\theta \\
C_{13} &= e_1 e_3 + (R_{21}R_{23} + R_{31}R_{33})\cos\theta + (R_{21}R_{33} - R_{23}R_{31})\sin\theta \\
&\vdots \\
C_{33} &= e_3^2 + \left(R_{23}^2 + R_{33}^2\right)\cos\theta
\end{aligned}$$

The column-orthonormality condition of the rotation matrix gives

$$\begin{aligned}
e_1^2 + R_{21}^2 + R_{31}^2 &= 1 \\
e_2^2 + R_{22}^2 + R_{32}^2 &= 1 \\
e_3^2 + R_{23}^2 + R_{33}^2 &= 1 \\
e_1 e_2 + R_{21}R_{22} + R_{31}R_{32} &= 0 \\
e_2 e_3 + R_{22}R_{23} + R_{32}R_{33} &= 0 \\
e_1 e_3 + R_{21}R_{23} + R_{31}R_{33} &= 0
\end{aligned}$$

Because each element of the rotation matrix $\mathbf{R}$ of Eq. (5.18) is equal to its cofactor, we also have

$$\begin{aligned}
e_1 &= R_{22}R_{33} - R_{23}R_{32} \\
e_2 &= R_{23}R_{31} - R_{21}R_{33} \\
e_3 &= R_{21}R_{32} - R_{22}R_{31}
\end{aligned}$$

Using these relationships, we obtain $\mathbf{C} = \mathbf{R}^T\mathbf{C}_1(\theta)\mathbf{R}$ as

$$\mathbf{C} = \begin{bmatrix} \mathrm{c}\,\theta + e_1^2(1-\mathrm{c}\,\theta) & e_1e_2(1-\mathrm{c}\,\theta)+e_3\,\mathrm{s}\,\theta & e_1e_3(1-\mathrm{c}\,\theta)-e_2\,\mathrm{s}\,\theta \\ e_2e_1(1-\mathrm{c}\,\theta)-e_3\,\mathrm{s}\,\theta & \mathrm{c}\,\theta+e_2^2(1-\mathrm{c}\,\theta) & e_2e_3(1-\mathrm{c}\,\theta)+e_1\,\mathrm{s}\,\theta \\ e_3e_1(1-\mathrm{c}\,\theta)+e_2\,\mathrm{s}\,\theta & e_3e_2(1-\mathrm{c}\,\theta)-e_1\,\mathrm{s}\,\theta & \mathrm{c}\,\theta+e_3^2(1-\mathrm{c}\,\theta) \end{bmatrix} \tag{5.22}$$

where $\mathrm{c}\,\theta \equiv \cos\theta$ and $\mathrm{s}\,\theta \equiv \sin\theta$. This is the parameterization of the direction cosine matrix $\mathbf{C}$ in terms of $e_i$ and $\theta$. Note that $e_1$, $e_2$, and $e_3$ are not independent of each other, but constrained by the relationship $e_1^2 + e_2^2 + e_3^2 = 1$.

By defining

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \quad \text{and} \quad \mathbf{E} = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix} \tag{5.23}$$

we can also express the direction cosine matrix $\mathbf{C}$ in Eq. (5.22) as

$$\mathbf{C} = \cos\theta\mathbf{I} + (1-\cos\theta)\mathbf{e}\mathbf{e}^T - \sin\theta\mathbf{E} \tag{5.24}$$

where $\mathbf{I}$ is the identity matrix; i.e, $C_{ij} = \delta_{ij}\cos\theta + (1-\cos\theta)e_ie_j - \sin\theta E_{ij}$.

Given a direction cosine matrix $\mathbf{C} = [C_{ij}]$, $\theta$ can be found from

$$\cos\theta = \tfrac{1}{2}(C_{11} + C_{22} + C_{33} - 1) \tag{5.25}$$

From Eq. (5.24), we obtain

$$\mathbf{E} = \frac{1}{2\sin\theta}(\mathbf{C}^T - \mathbf{C}) \qquad \text{if } \theta \neq 0,\ \pm\pi,\ \pm 2\pi, \ldots \tag{5.26}$$

from which the eigenaxis $\mathbf{e}$ can be found as

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \frac{1}{2\sin\theta}\begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix} \tag{5.27}$$

## Problems

**5.2.** Verify that $\vec{b}_i$ can be expressed as

$$\begin{aligned} \vec{b}_i &= \vec{a}_i\cos\theta + \vec{e}(\vec{a}_i\cdot\vec{e})(1-\cos\theta) - \vec{a}_i\times\vec{e}\sin\theta \\ &= \vec{a}_i + \vec{e}\times(\vec{e}\times\vec{a}_i)(1-\cos\theta) - \vec{a}_i\times\vec{e}\sin\theta, \qquad i = 1,2,3 \end{aligned}$$

*Hint:* $\vec{e} = e_1\vec{a}_1 + e_2\vec{a}_2 + e_3\vec{a}_3$ and $\vec{b}_i = C_{i1}\vec{a}_1 + C_{i2}\vec{a}_2 + C_{i3}\vec{a}_3$ where $C_{ij}$ are given by Eq. (5.22).

**5.3.** Consider two successive eigenaxis rotations to $A''$ from $A$ represented by

$$\begin{aligned} &\mathbf{C}(\mathbf{e}_1, \theta_1): \quad A' \leftarrow A \\ &\mathbf{C}(\mathbf{e}_2, \theta_2): \quad A'' \leftarrow A' \end{aligned}$$

where

$$\mathbf{C}(\mathbf{e}_1, \theta_1) = \left[\cos\theta_1 \mathbf{I} + (1 - \cos\theta_1)\mathbf{e}_1\mathbf{e}_1^T - \sin\theta_1 \mathbf{E}_1\right]$$

$$\mathbf{C}(\mathbf{e}_2, \theta_2) = \left[\cos\theta_2 \mathbf{I} + (1 - \cos\theta_2)\mathbf{e}_2\mathbf{e}_2^T - \sin\theta_2 \mathbf{E}_2\right]$$

and $\mathbf{e}_1$ and $\mathbf{e}_2$ are the eigenaxes associated with the first and second eigenaxis rotations, respectively.

These successive rotations are also represented by an equivalent single eigenaxis rotation to $A''$ directly from $A$, as follows:

$$\mathbf{C}(\mathbf{e}, \theta): \quad A'' \leftarrow A$$

where

$$\mathbf{C}(\mathbf{e}, \theta) = [\cos\theta \mathbf{I} + (1 - \cos\theta)\mathbf{e}\mathbf{e}^T - \sin\theta \mathbf{E}]$$

and we have $\mathbf{C}(\mathbf{e}, \theta) = \mathbf{C}(\mathbf{e}_2, \theta_2)\mathbf{C}(\mathbf{e}_1, \theta_1)$.

(a) Show that the equivalent eigenangle $\theta$ can be determined as

$$\cos\theta = \tfrac{1}{2}(\mathrm{tr}\, \mathbf{C} - 1)$$

and

$$\begin{aligned}\mathrm{tr}\, \mathbf{C} = {} & \cos\theta_1 + \cos\theta_2 + \cos\theta_1 \cos\theta_2 \\ & + (1 - \cos\theta_1)(1 - \cos\theta_2)\cos^2\gamma - 2\sin\theta_1 \sin\theta_2 \cos\gamma\end{aligned}$$

and $\gamma$ is the angle between the two eigenaxes $\mathbf{e}_1$ and $\mathbf{e}_2$; i.e., $\cos\gamma = \mathbf{e}_1^T \mathbf{e}_2$.

(b) Show that $\cos\theta$ obtained in (a) can be expressed in terms of half-angles:

$$\cos\frac{\theta}{2} = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\gamma$$

*Hint:* $\sin^2(\theta/2) = (1 - \cos\theta)/2$ and $\cos^2(\theta/2) = (1 + \cos\theta)/2$.

(c) Show that the equivalent eigenaxis of rotation $\mathbf{e}$ can be found as

$$\begin{aligned}2\sin\theta\, \mathbf{e} = {} & \mathbf{e}_1\{\sin\theta_1(1 + \cos\theta_2) - \sin\theta_2(1 - \cos\theta_1)\cos\gamma\} \\ & + \mathbf{e}_2\{\sin\theta_2(1 + \cos\theta_1) - \sin\theta_1(1 - \cos\theta_2)\cos\gamma\} \\ & + (\mathbf{e}_1 \times \mathbf{e}_2)\{\sin\theta_1 \sin\theta_2 - (1 - \cos\theta_1)(1 - \cos\theta_2)\cos\gamma\}\end{aligned}$$

which can be rewritten as

$$\mathbf{e}\sin\frac{\theta}{2} = \mathbf{e}_1 \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \mathbf{e}_2 \sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + (\mathbf{e}_1 \times \mathbf{e}_2)\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}$$

where $\mathbf{e}_1 \times \mathbf{e}_2 \equiv \mathbf{E}_1\mathbf{e}_2$.

*Note:* See Ref. 2 for additional information pertaining to Problem 5.3.

### 5.3.2 Space-Axis Rotation

The space-axis and body-axis rotations are defined as a successive rotation about the space-fixed axes and body-fixed axes, respectively. An interesting relationship between these different schemes of successive rotations exists.[1,3,4]

Consider a space-axis rotation in the sequence of $\theta_3\vec{a}_3 \leftarrow \theta_2\vec{a}_2 \leftarrow \theta_1\vec{a}_1$, in which $\theta_i\vec{a}_i$ means an $\vec{a}_i$ axis rotation through an angle $\theta_i$. Its total rotation matrix is defined as

$$\mathbf{C}^{B/A} = \mathbf{C}^{B/A''}\mathbf{C}^{A''/A'}\mathbf{C}^{A'/A} \tag{5.28}$$

in which the first rotation matrix is simply

$$\mathbf{C}^{A'/A} = \mathbf{C}_1(\theta_1) \tag{5.29}$$

Next, to construct a matrix $\mathbf{C}^{A''/A'}$ that characterizes the $\vec{a}_2$ axis rotation with an angle $\theta_2$, we use the approach discussed in the preceding section, as follows:

$$\begin{aligned}\mathbf{C}^{A''/A'} &= \mathbf{C}^{A''/A'''}\mathbf{C}^{A'''/A}\mathbf{C}^{A/A'} \\ &= \mathbf{C}^{A'/A}\mathbf{C}_2(\theta_2)\mathbf{C}^{A/A'}\end{aligned} \tag{5.30}$$

Combining Eqs. (5.29) and (5.30), we obtain

$$\begin{aligned}\mathbf{C}^{A''/A} &= \mathbf{C}^{A''/A'}\mathbf{C}^{A'/A} \\ &= \mathbf{C}^{A'/A}\mathbf{C}_2(\theta_2)\mathbf{C}^{A/A'}\mathbf{C}^{A'/A} \\ &= \mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2)\end{aligned}$$

Similarly, for the $\vec{a}_3$ axis rotation through an angle $\theta_3$, we have

$$\mathbf{C}^{B/A''} = \mathbf{C}^{A''/A}\mathbf{C}_3(\theta_3)\mathbf{C}^{A/A''} \tag{5.31}$$

Finally, the total rotation matrix becomes

$$\begin{aligned}\mathbf{C}^{B/A} &= \mathbf{C}^{B/A''}\mathbf{C}^{A''/A} \\ &= \mathbf{C}^{A''/A}\mathbf{C}_3(\theta_3)\mathbf{C}^{A/A''}\mathbf{C}^{A''/A} \\ &= \mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2)\mathbf{C}_3(\theta_3)\end{aligned} \tag{5.32}$$

Thus, the total rotation matrix for the space-axis rotation of $\theta_3\vec{a}_3 \leftarrow \theta_2\vec{a}_2 \leftarrow \theta_1\vec{a}_1$ is identical to the total rotation matrix for the body-axis rotation of $\theta_1\vec{a}_1'' \leftarrow \theta_2\vec{a}_2' \leftarrow \theta_3\vec{a}_3$, which has been denoted by $\mathbf{C}_1(\theta_1) \leftarrow \mathbf{C}_2(\theta_2) \leftarrow \mathbf{C}_3(\theta_3)$ in preceding sections.

Although the total rotation matrix for the space-axis rotation has a simple form as Eq. (5.32), each intermediate rotation matrix is rather complicated as can be seen from Eqs. (5.30) and (5.31); however, the approach used here does not require an explicit determination of these intermediate matrices to find the total rotation matrix. Indeed, an intimate relationship between the two different rotation schemes has been obtained directly.[4]

## 5.4 Quaternions

### 5.4.1 Euler Parameters or Quaternions

Consider again Euler's eigenaxis rotation about an arbitrary axis fixed both in a body-fixed reference frame $B$ and in an inertial reference frame $A$. In Sec. 5.3, a unit vector $\vec{e}$ along the Euler axis was defined as

$$\begin{aligned}\vec{e} &= e_1\vec{a}_1 + e_2\vec{a}_2 + e_3\vec{a}_3 \\ &= e_1\vec{b}_1 + e_2\vec{b}_2 + e_3\vec{b}_3\end{aligned}$$

where $e_i$ are the direction cosines of the Euler axis relative to both $A$ and $B$, and $e_1^2 + e_2^2 + e_3^2 = 1$.

Then we define the four *Euler parameters* as follows:

$$q_1 = e_1 \sin(\theta/2) \tag{5.33a}$$

$$q_2 = e_2 \sin(\theta/2) \tag{5.33b}$$

$$q_3 = e_3 \sin(\theta/2) \tag{5.33c}$$

$$q_4 = \cos(\theta/2) \tag{5.33d}$$

where $\theta$ is the rotation angle about the Euler axis. Like the eigenaxis vector $\mathbf{e} = (e_1, e_2, e_3)$, we define a vector $\mathbf{q} = (q_1, q_2, q_3)$ such that

$$\mathbf{q} = \mathbf{e}\sin(\theta/2) \tag{5.34}$$

Note that the Euler parameters are not independent of each other, but constrained by the relationship

$$\mathbf{q}^T\mathbf{q} + q_4^2 = q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \tag{5.35}$$

because $e_1^2 + e_2^2 + e_3^2 = 1$.

The Euler parameters are also called *quaternions*. Hamilton invented quaternions as a result of searching for hypercomplex numbers that could be represented by points in three-dimensional space.* Although the historical importance of quaternions is significant, we will not discuss quaternion algebra here. Instead, we simply use the terms *quaternions* and *Euler parameters* interchangeably.

The direction cosine matrix parameterized as Eq. (5.22) can also be parameterized in terms of quaternions, as follows:

$$\mathbf{C}^{B/A} = \mathbf{C}(\mathbf{q}, q_4) = \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_2q_1 - q_3q_4) & 1 - 2(q_1^2 + q_3^2) & 2(q_2q_3 + q_1q_4) \\ 2(q_3q_1 + q_2q_4) & 2(q_3q_2 - q_1q_4) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix} \tag{5.36}$$

where $(q_1, q_2, q_3, q_4)$ is the quaternion associated with the direction cosine matrix $\mathbf{C}^{B/A}$. Note that $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$ and $\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2) = 2\cos^2(\theta/2) - 1 = 1 - 2\sin^2(\theta/2)$.

*William Hamilton (1805–1865) regarded his discovery of quaternions as his greatest achievement, whereas we may consider his contributions to analytical dynamics as his greatest achievement.

In terms of the quaternion vector $\mathbf{q}$ and a skew-symmetric matrix $\mathbf{Q}$ defined, respectively, as

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \qquad \mathbf{Q} = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{5.37}$$

the direction cosine matrix (5.36) becomes

$$\mathbf{C} = (q_4^2 - \mathbf{q}^T\mathbf{q})\mathbf{I} + 2\mathbf{q}\mathbf{q}^T - 2q_4\mathbf{Q} \tag{5.38}$$

Given a direction cosine matrix $\mathbf{C}$, we can determine $q_4$ and $\mathbf{q}$ as follows:

$$q_4 = (1 + C_{11} + C_{22} + C_{33})^{\frac{1}{2}} \qquad \text{for } 0 \le \theta \le \pi \tag{5.39}$$

$$\mathbf{q} = \frac{1}{4q_4}\begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix} \qquad \text{if } q_4 \neq 0 \tag{5.40}$$

Consider two successive rotations to $A''$ from $A$ represented by

$$\mathbf{C}(\mathbf{q}', q_4'): \quad A' \leftarrow A \tag{5.41a}$$

$$\mathbf{C}(\mathbf{q}'', q_4''): \quad A'' \leftarrow A' \tag{5.41b}$$

where $(\mathbf{q}', q_4')$ is the quaternion associated with the coordinate transformation $A' \leftarrow A$, and $(\mathbf{q}'', q_4'')$ is the quaternion associated with the coordinate transformation $A'' \leftarrow A'$. These successive rotations are also represented by a single rotation to $A''$ directly from $A$, as follows:

$$\mathbf{C}(\mathbf{q}, q_4): \quad A'' \leftarrow A \tag{5.42}$$

where $(\mathbf{q}, q_4)$ is the quaternion associated with the coordinate transformation $A'' \leftarrow A$, and we have

$$\mathbf{C}(\mathbf{q}, q_4) = \mathbf{C}(\mathbf{q}'', q_4'')\mathbf{C}(\mathbf{q}', q_4') \tag{5.43}$$

Note that Eq. (5.43) can also be represented as

$$\mathbf{C}(\mathbf{e}, \theta) = \mathbf{C}(\mathbf{e}_2, \theta_2)\mathbf{C}(\mathbf{e}_1, \theta_1)$$

where $(\mathbf{e}_1, \theta_1)$ and $(\mathbf{e}_2, \theta_2)$ are the eigenaxes and angles associated with the first and second eigenaxis rotations, respectively, and $(\mathbf{e}, \theta)$ are the eigenaxis and angle associated with the equivalent single eigenaxis rotation (see Problem 5.3).

Using the result of Problem 5.3(c), and defining

$$q_4' = \cos\frac{\theta_1}{2}, \qquad q_4'' = \cos\frac{\theta_2}{2}$$

and

$$\mathbf{q}' = \begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} = \mathbf{e}_1 \sin\frac{\theta_1}{2}, \qquad \mathbf{q}'' = \begin{bmatrix} q_1'' \\ q_2'' \\ q_3'' \end{bmatrix} = \mathbf{e}_2 \sin\frac{\theta_2}{2}$$

we obtain

$$\mathbf{q} = q_4''\mathbf{q}' + q_4'\mathbf{q}'' + \mathbf{q}' \times \mathbf{q}'' \tag{5.44}$$

$$q_4 = q_4'q_4'' - (\mathbf{q}')^T\mathbf{q}'' \tag{5.45}$$

These equations can be combined as

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} q_4'' & q_3'' & -q_2'' & q_1'' \\ -q_3'' & q_4'' & q_1'' & q_2'' \\ q_2'' & -q_1'' & q_4'' & q_3'' \\ -q_1'' & -q_2'' & -q_3'' & q_4'' \end{bmatrix} \begin{bmatrix} q_1' \\ q_2' \\ q_3' \\ q_4' \end{bmatrix} \tag{5.46}$$

which is known as the quaternion multiplication rule in matrix form. The $4 \times 4$ orthonormal matrix in Eq. (5.46) is called the quaternion matrix. Equation (5.46) can also be written as

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} q_4' & -q_3' & q_2' & q_1' \\ q_3' & q_4' & -q_1' & q_2' \\ -q_2' & q_1' & q_4' & q_3' \\ -q_1' & -q_2' & -q_3' & q_4' \end{bmatrix} \begin{bmatrix} q_1'' \\ q_2'' \\ q_3'' \\ q_4'' \end{bmatrix} \tag{5.47}$$

The $4 \times 4$ matrix in Eq. (5.47) is also orthonormal and is called the quaternion transmuted matrix.[5]

### 5.4.2 Gibbs Parameters

The direction cosine matrix can also be parameterized in terms of the *Gibbs vector*, which is defined as

$$\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} q_1/q_4 \\ q_2/q_4 \\ q_3/q_4 \end{bmatrix} = \mathbf{e} \tan\frac{\theta}{2} \tag{5.48}$$

The components of the Gibbs vector, called the *Gibbs parameters*, are also referred to as the *Rodrigues parameters* in the literature, and the direction cosine matrix can be parameterized in terms of them as follows:

$$\mathbf{C} = \frac{1}{1 + g_1^2 + g_2^2 + g_3^2} \times \begin{bmatrix} 1 + g_1^2 - g_2^2 - g_3^2 & 2(g_1g_2 + g_3) & 2(g_1g_3 - g_2) \\ 2(g_2g_1 - g_3) & 1 - g_1^2 + g_2^2 - g_3^2 & 2(g_2g_3 + g_1) \\ 2(g_3g_1 + g_2) & 2(g_3g_2 - g_1) & 1 - g_1^2 - g_2^2 + g_3^2 \end{bmatrix} \tag{5.49}$$

which can be rewritten as

$$\mathbf{C} = \frac{(1 - \mathbf{g}^T\mathbf{g})\mathbf{I} + 2\mathbf{g}\mathbf{g}^T - 2\mathbf{G}}{1 + \mathbf{g}^T\mathbf{g}} \equiv [\mathbf{I} - \mathbf{G}][\mathbf{I} + \mathbf{G}]^{-1} \tag{5.50}$$

where

$$\mathbf{G} = \begin{bmatrix} 0 & -g_3 & g_2 \\ g_3 & 0 & -g_1 \\ -g_2 & g_1 & 0 \end{bmatrix}$$

For a given direction cosine matrix $\mathbf{C}$, the Gibbs vector can be determined as

$$\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \frac{1}{1 + C_{11} + C_{22} + C_{33}} \begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix} \tag{5.51}$$

## Problems

**5.4.** Show that

$$\vec{b}_i = \vec{a}_i + 2\{q_4\vec{q} \times \vec{a}_i + \vec{q} \times (\vec{q} \times \vec{a}_i)\}, \qquad i = 1, 2, 3$$

where $\vec{q} = \vec{e}\sin(\theta/2)$.

**5.5.** Consider the body-fixed rotational sequence to $B$ from $A$: $\mathbf{C}_1(\theta_1) \leftarrow \mathbf{C}_2(\theta_2) \leftarrow \mathbf{C}_3(\theta_3)$.

(a) Show that the three Euler angles of this rotational sequence are related to quaternions, as follows:

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \mathrm{s}_1\,\mathrm{c}_2\,\mathrm{c}_3 - \mathrm{c}_1\,\mathrm{s}_2\,\mathrm{s}_3 \\ \mathrm{c}_1\,\mathrm{s}_2\,\mathrm{c}_3 + \mathrm{s}_1\,\mathrm{c}_2\,\mathrm{s}_3 \\ \mathrm{c}_1\,\mathrm{c}_2\,\mathrm{s}_3 - \mathrm{s}_1\,\mathrm{s}_2\,\mathrm{c}_3 \\ \mathrm{c}_1\,\mathrm{c}_2\,\mathrm{c}_3 + \mathrm{s}_1\,\mathrm{s}_2\,\mathrm{s}_3 \end{bmatrix}$$

where $\mathrm{s}_i = \sin(\theta_i/2)$, $\mathrm{c}_i = \cos(\theta_i/2)$, and $(q_1, q_2, q_3, q_4)$ is the quaternion associated with the coordinate transformation $B \leftarrow A$.

*Hint:* Use Eq. (5.46). The quaternions associated with $\mathbf{C}_1(\theta_1) \leftarrow \mathbf{C}_2(\theta_2) \leftarrow \mathbf{C}_3(\theta_3)$ are represented as

$$\begin{bmatrix} \sin(\theta_1/2) \\ 0 \\ 0 \\ \cos(\theta_1/2) \end{bmatrix} \leftarrow \begin{bmatrix} 0 \\ \sin(\theta_2/2) \\ 0 \\ \cos(\theta_2/2) \end{bmatrix} \leftarrow \begin{bmatrix} 0 \\ 0 \\ \sin(\theta_3/2) \\ \cos(\theta_3/2) \end{bmatrix}$$

(b) Also verify that, for small (infinitesimal) rotational angles of $\theta_1$, $\theta_2$, and $\theta_3$, we simply have

$$q_1 \approx \theta_1/2$$

$$q_2 \approx \theta_2/2$$

$$q_3 \approx \theta_3/2$$

$$q_4 \approx 1$$

## 5.5 Kinematic Differential Equations

In preceding sections, we have studied the problem of describing the orientation of a reference frame (or a rigid body) in terms of the direction cosine matrix, Euler angles, and quaternions. In this section, we treat *kinematics* in which the relative orientation between two reference frames is time dependent. The time-dependent relationship between two reference frames is described by the so-called *kinematic differential equations*. In this section, we derive the kinematic differential equations for the direction cosine matrix, Euler angles, and quaternions.

### 5.5.1 Direction Cosine Matrix

Consider two reference frames $A$ and $B$, shown in Fig. 5.1, which are moving relative to each other. The angular velocity vector of a reference frame $B$ with respect to a reference frame $A$ is denoted by $\vec{\omega} \equiv \vec{\omega}^{B/A}$, and it is expressed in terms of basis vectors of $B$ as follows:

$$\vec{\omega} = \omega_1 \vec{b}_1 + \omega_2 \vec{b}_2 + \omega_3 \vec{b}_3 \tag{5.52}$$

where the angular velocity vector $\vec{\omega}$ is time dependent.

In Sec. 5.1, we have defined the direction cosine matrix $\mathbf{C} \equiv \mathbf{C}^{B/A}$ such that

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \mathbf{C} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} \tag{5.53}$$

which can be rewritten as

$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} = \mathbf{C}^{-1} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \mathbf{C}^T \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} \tag{5.54}$$

Because the two reference frames are rotating relative to each other, the direction cosine matrix and its elements $C_{ij}$ are functions of time. Taking the time derivative of Eq. (5.54) in $A$ and denoting it by an overdot, we obtain

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} &= \dot{\mathbf{C}}^T \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} + \mathbf{C}^T \begin{bmatrix} \dot{\vec{b}}_1 \\ \dot{\vec{b}}_2 \\ \dot{\vec{b}}_3 \end{bmatrix} \\
&= \dot{\mathbf{C}}^T \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} + \mathbf{C}^T \begin{bmatrix} \vec{\omega} \times \vec{b}_1 \\ \vec{\omega} \times \vec{b}_2 \\ \vec{\omega} \times \vec{b}_3 \end{bmatrix} \\
&= \dot{\mathbf{C}}^T \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} - \mathbf{C}^T \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix}
\end{aligned} \tag{5.55}$$

where

$$\dot{\mathbf{C}} \equiv \begin{bmatrix} \dot{C}_{11} & \dot{C}_{12} & \dot{C}_{13} \\ \dot{C}_{21} & \dot{C}_{22} & \dot{C}_{23} \\ \dot{C}_{31} & \dot{C}_{32} & \dot{C}_{33} \end{bmatrix} \tag{5.56}$$

By defining the skew-symmetric matrix in Eq. (5.55) as

$$\boldsymbol{\Omega} \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{5.57}$$

we obtain

$$[\dot{\mathbf{C}}^T - \mathbf{C}^T\boldsymbol{\Omega}] \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

from which we obtain

$$\dot{\mathbf{C}}^T - \mathbf{C}^T\boldsymbol{\Omega} = 0 \tag{5.58}$$

Taking the transpose of Eq. (5.58) and using the relationship $\boldsymbol{\Omega}^T = -\boldsymbol{\Omega}$, we obtain

$$\dot{\mathbf{C}} + \boldsymbol{\Omega}\mathbf{C} = 0 \tag{5.59}$$

which is called the kinematic differential equation for the direction cosine matrix $\mathbf{C}$. Differential equations for each element of $\mathbf{C}$ can be written as

$$\begin{aligned}
\dot{C}_{11} &= \omega_3 C_{21} - \omega_2 C_{31} \\
\dot{C}_{12} &= \omega_3 C_{22} - \omega_2 C_{32} \\
\dot{C}_{13} &= \omega_3 C_{23} - \omega_2 C_{33} \\
\dot{C}_{21} &= \omega_1 C_{31} - \omega_3 C_{11} \\
\dot{C}_{22} &= \omega_1 C_{32} - \omega_3 C_{12} \\
\dot{C}_{23} &= \omega_1 C_{33} - \omega_3 C_{13} \\
\dot{C}_{31} &= \omega_2 C_{11} - \omega_1 C_{21} \\
\dot{C}_{32} &= \omega_2 C_{12} - \omega_1 C_{22} \\
\dot{C}_{33} &= \omega_2 C_{13} - \omega_1 C_{23}
\end{aligned}$$

If $\omega_1$, $\omega_2$, and $\omega_3$ are known as functions of time, then the orientation of $B$ relative to $A$ as a function of time can be determined by solving Eq. (5.59). In general, it is difficult to solve Eq. (5.59) analytically in closed form except in special cases; hence, in most cases, Eq. (5.59) is integrated numerically using a digital computer. It can be shown that the orthonormality condition $\mathbf{C}\mathbf{C}^T = \mathbf{I} = \mathbf{C}^T\mathbf{C}$ is a constant integral of Eq. (5.59); that is, if the orthonormality condition is satisfied at $t = 0$, then any (exact) solution of Eq. (5.59) automatically satisfies the orthonormality condition of $\mathbf{C}$ for all $t > 0$. However, the orthonormality condition is often used to check the accuracy of numerical integration on a digital computer.

## Problem

**5.6.** Given the kinematic differential equation (5.59), show that

$$\omega_1 = \dot{C}_{21}C_{31} + \dot{C}_{22}C_{32} + \dot{C}_{23}C_{33}$$
$$\omega_2 = \dot{C}_{31}C_{11} + \dot{C}_{32}C_{12} + \dot{C}_{33}C_{13}$$
$$\omega_3 = \dot{C}_{11}C_{21} + \dot{C}_{12}C_{22} + \dot{C}_{13}C_{23}$$

### *5.5.2 Euler Angles*

Like the kinematic differential equation for the direction cosine matrix **C**, the orientation of a reference frame $B$ relative to a reference frame $A$ can also be described by introducing the time dependence of Euler angles.

Consider the rotational sequence of $\mathbf{C}_1(\theta_1) \leftarrow \mathbf{C}_2(\theta_2) \leftarrow \mathbf{C}_3(\theta_3)$ to $B$ from $A$, which is symbolically represented as

$$\mathbf{C}_3(\theta_3): \quad A' \leftarrow A \tag{5.60a}$$
$$\mathbf{C}_2(\theta_2): \quad A'' \leftarrow A' \tag{5.60b}$$
$$\mathbf{C}_1(\theta_1): \quad B \leftarrow A'' \tag{5.60c}$$

The time derivatives of Euler angles, called Euler rates, are denoted by $\dot{\theta}_3$, $\dot{\theta}_2$, and $\dot{\theta}_1$. These successive rotations are also represented as

$$\vec{\omega}^{A'/A}: \quad A' \leftarrow A \tag{5.61a}$$
$$\vec{\omega}^{A''/A'}: \quad A'' \leftarrow A' \tag{5.61b}$$
$$\vec{\omega}^{B/A''}: \quad B \leftarrow A'' \tag{5.61c}$$

and the angular velocity vectors $\vec{\omega}^{A'/A}$, $\vec{\omega}^{A''/A'}$, and $\vec{\omega}^{B/A''}$ are expressed as

$$\vec{\omega}^{A'/A} = \dot{\theta}_3 \vec{a}_3 = \dot{\theta}_3 \vec{a}_3' \tag{5.62a}$$
$$\vec{\omega}^{A''/A'} = \dot{\theta}_2 \vec{a}_2' = \dot{\theta}_2 \vec{a}_2'' \tag{5.62b}$$
$$\vec{\omega}^{B/A''} = \dot{\theta}_1 \vec{a}_1'' = \dot{\theta}_1 \vec{b}_1 \tag{5.62c}$$

The angular velocity vector $\vec{\omega}^{B/A}$ then becomes

$$\vec{\omega}^{B/A} = \vec{\omega}^{B/A''} + \vec{\omega}^{A''/A'} + \vec{\omega}^{A'/A} = \dot{\theta}_1 \vec{b}_1 + \dot{\theta}_2 \vec{a}_2'' + \dot{\theta}_3 \vec{a}_3' \tag{5.63}$$

which can be rewritten as

$$\vec{\omega}^{B/A} = [\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3] \begin{bmatrix} \dot{\theta}_1 \\ 0 \\ 0 \end{bmatrix} + [\vec{a}_1'' \quad \vec{a}_2'' \quad \vec{a}_3''] \begin{bmatrix} 0 \\ \dot{\theta}_2 \\ 0 \end{bmatrix} + [\vec{a}_1' \quad \vec{a}_2' \quad \vec{a}_3'] \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} \tag{5.64}$$

and we have

$$[\vec{a}_1'' \quad \vec{a}_2'' \quad \vec{a}_3''] = [\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3]\mathbf{C}_1(\theta_1)$$
$$[\vec{a}_1' \quad \vec{a}_2' \quad \vec{a}_3'] = [\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3]\mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2)$$

Because the angular velocity vector $\vec{\omega} \equiv \vec{\omega}^{B/A}$ can also be represented as

$$\vec{\omega} = \omega_1 \vec{b}_1 + \omega_2 \vec{b}_2 + \omega_3 \vec{b}_3 = [\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3] \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{5.65}$$

we obtain

$$\begin{aligned} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} &= \begin{bmatrix} \dot{\theta}_1 \\ 0 \\ 0 \end{bmatrix} + \mathbf{C}_1(\theta_1) \begin{bmatrix} 0 \\ \dot{\theta}_2 \\ 0 \end{bmatrix} + \mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2) \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -\sin\theta_2 \\ 0 & \cos\theta_1 & \sin\theta_1 \cos\theta_2 \\ 0 & -\sin\theta_1 & \cos\theta_1 \cos\theta_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \end{aligned} \tag{5.66}$$

Note that the $3 \times 3$ matrix in Eq. (5.66) is not an orthogonal matrix because $\vec{b}_1$, $\vec{a}''_2$, and $\vec{a}'_3$ do not constitute a set of orthogonal unit vectors. The inverse relationship can be found by inverting the $3 \times 3$ nonorthogonal matrix in Eq. (5.66), as follows:

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{\cos\theta_2} \begin{bmatrix} \cos\theta_2 & \sin\theta_1 \sin\theta_2 & \cos\theta_1 \sin\theta_2 \\ 0 & \cos\theta_1 \cos\theta_2 & -\sin\theta_1 \cos\theta_2 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{5.67}$$

which is the kinematic differential equation for the sequence of $\mathbf{C}_1(\theta_1) \leftarrow \mathbf{C}_2(\theta_2) \leftarrow \mathbf{C}_3(\theta_3)$.

If $\omega_1$, $\omega_2$, and $\omega_3$ are known as functions of time, then the orientation of $B$ relative to $A$ as a function of time can be determined by solving Eq. (5.67). Numerical integration of Eq. (5.67), however, involves the computation of trigonometric functions of the angles. Also note that Eq. (5.67) becomes singular when $\theta_2 = \pi/2$. Such a mathematical singularity problem for a certain orientation angle can be avoided by selecting a different set of Euler angles, but it is an inherent property of all different sets of Euler angles.

Similarly, for the sequence of $\mathbf{C}_3(\psi) \leftarrow \mathbf{C}_1(\theta) \leftarrow \mathbf{C}_3(\phi)$, we have

$$\begin{aligned} \vec{\omega} \equiv \vec{\omega}^{B/A} &= \omega_1 \vec{b}_1 + \omega_2 \vec{b}_2 + \omega_3 \vec{b}_3 \\ &= \dot{\psi} \vec{b}_3 + \dot{\theta} \vec{a}''_1 + \dot{\phi} \vec{a}'_3 \end{aligned} \tag{5.68}$$

$$\begin{aligned} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + \mathbf{C}_3(\psi) \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} + \mathbf{C}_3(\psi)\mathbf{C}_1(\theta) \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} \\ &= \begin{bmatrix} \sin\theta \sin\psi & \cos\psi & 0 \\ \sin\theta \cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \end{aligned} \tag{5.69}$$

and

$$
\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \frac{1}{\sin\theta} \begin{bmatrix} \sin\psi & \cos\psi & 0 \\ \cos\psi\sin\theta & -\sin\psi\sin\theta & 0 \\ -\sin\psi\cos\theta & -\cos\psi\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{5.70}
$$

which is the kinematic differential equation for the $\mathbf{C}_3(\psi) \leftarrow \mathbf{C}_1(\theta) \leftarrow \mathbf{C}_3(\phi)$ sequence.

## Problems

**5.7.** For the sequence of $\mathbf{C}_1(\theta_1) \leftarrow \mathbf{C}_3(\theta_3) \leftarrow \mathbf{C}_2(\theta_2)$, derive the following kinematic differential equation:

$$
\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{\cos\theta_3} \begin{bmatrix} \cos\theta_3 & -\cos\theta_1\sin\theta_3 & \sin\theta_1\sin\theta_3 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1\cos\theta_3 & \cos\theta_1\cos\theta_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}
$$

**5.8.** For the sequence of $\mathbf{C}_3(\theta_3) \leftarrow \mathbf{C}_2(\theta_2) \leftarrow \mathbf{C}_1(\theta_1)$, derive the following kinematic differential equation:

$$
\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{\cos\theta_2} \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \cos\theta_2\sin\theta_3 & \cos\theta_2\cos\theta_3 & 0 \\ -\sin\theta_2\cos\theta_3 & \sin\theta_2\sin\theta_3 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}
$$

### 5.5.3 Quaternions

Substituting $C_{ij}$ of Eq. (5.36) into the equations derived in Problem 5.6, we obtain

$$
\omega_1 = 2(\dot{q}_1 q_4 + \dot{q}_2 q_3 - \dot{q}_3 q_2 - \dot{q}_4 q_1) \tag{5.71a}
$$

$$
\omega_2 = 2(\dot{q}_2 q_4 + \dot{q}_3 q_1 - \dot{q}_1 q_3 - \dot{q}_4 q_2) \tag{5.71b}
$$

$$
\omega_3 = 2(\dot{q}_3 q_4 + \dot{q}_1 q_2 - \dot{q}_2 q_1 - \dot{q}_4 q_3) \tag{5.71c}
$$

Differentiating Eq. (5.35) gives

$$
0 = 2(\dot{q}_1 q_1 + \dot{q}_2 q_2 + \dot{q}_3 q_3 + \dot{q}_4 q_4) \tag{5.72}
$$

These four equations can be combined into matrix form, as follows:

$$
\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} q_4 & q_3 & -q_2 & -q_1 \\ -q_3 & q_4 & q_1 & -q_2 \\ q_2 & -q_1 & q_4 & -q_3 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} \tag{5.73}
$$

Because the $4 \times 4$ matrix in this equation is orthonormal, we simply obtain the kinematic differential equation for quaternions, as follows:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} q_4 & -q_3 & q_2 & q_1 \\ q_3 & q_4 & -q_1 & q_2 \\ -q_2 & q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{bmatrix} \tag{5.74}$$

which can be rewritten as

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \tag{5.75}$$

In terms of $\mathbf{q}$ and $\boldsymbol{\omega}$ defined as

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

we can rewrite the kinematic differential equation (5.75) as follows:

$$\dot{\mathbf{q}} = \tfrac{1}{2}(q_4\boldsymbol{\omega} - \boldsymbol{\omega} \times \mathbf{q}) \tag{5.76a}$$

$$\dot{q}_4 = -\tfrac{1}{2}\boldsymbol{\omega}^T\mathbf{q} \tag{5.76b}$$

where

$$\boldsymbol{\omega} \times \mathbf{q} \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

It is historically interesting to note that Eq. (5.75) was first published by Robinson[6] in 1958 and derived independently by Harding,[7] Mortenson,[8] and Margulies (see Ref. 9) in the mid-1960s.

In *strapdown inertial reference systems* of aerospace vehicles, the body rates, $\omega_1$, $\omega_2$, and $\omega_3$ are measured by rate gyros that are "strapped down" to the vehicles. The kinematic differential equation (5.75) is then integrated numerically using an onboard flight computer to determine the orientation of the vehicles in terms of quaternions. Quaternions have no inherent geometric singularity as do Euler angles. Moreover, quaternions are well suited for onboard real-time computation because only products and no trigonometric relations exist in the quaternion kinematic differential equations. Thus, spacecraft orientation is now commonly described in terms of quaternions.

There are a number of numerical methods available for solving Eq. (5.75). Methods that can be applied to the strapdown attitude algorithms include Taylor series expansion, the rotation vector concept, Runge–Kutta algorithms, and the state transition matrix. Of these methods, the Taylor series expansion lends itself well to the use of an incremental angle output from the digital rate integrating gyros. A tradeoff between algorithm complexity vs algorithm truncation and roundoff errors is generally required (see, e.g., Ref. 10).

For further details of rotational kinematics and spacecraft attitude determination, the reader is referred to Refs. 11–15.

Currently, spacecraft attitude determination using the Global Positioning System (GPS) is also being considered for near-Earth satellites. The GPS, consisting of a constellation of 24 satellites and a ground monitoring and control network, is widely used for positioning vehicles near the surface of the Earth and for orbit determination of near-Earth satellites.[16] The GPS is also capable of providing vehicle attitude using L-band carrier phase interferometry between multiple antennas. Consequently, GPS-based attitude determination is of current research interest for near-Earth satellites because of the potential for reducing the number of onboard navigation and attitude sensors.[17]

## Problems

**5.9.** Show that the constraint $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$ is a constant integral of Eq. (5.75). If this constraint is satisfied at $t = 0$, then any exact solution of Eq. (5.75) automatically satisfies the constraint for all $t > 0$; however, this constraint is often used to check the accuracy of numerical integration on a digital computer.

**5.10.** Consider the Gibbs vector defined as

$$\mathbf{g} = \mathbf{e} \tan \frac{\theta}{2}$$

where $\mathbf{g} = (g_1, g_2, g_3)$, $\mathbf{e} = (e_1, e_2, e_3)$ is Euler's eigenaxis vector, and $\theta$ is the angle associated with Euler's eigenaxis rotation.

(a) Show that the kinematic differential equation for the Gibbs vector can be found as

$$\begin{bmatrix} \dot{g}_1 \\ \dot{g}_2 \\ \dot{g}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 + g_1^2 & g_1 g_2 - g_3 & g_1 g_3 + g_2 \\ g_2 g_1 + g_3 & 1 + g_2^2 & g_2 g_3 - g_1 \\ g_3 g_1 - g_2 & g_3 g_2 + g_1 & 1 + g_3^2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

or

$$\dot{\mathbf{g}} = \tfrac{1}{2}[\boldsymbol{\omega} - \boldsymbol{\omega} \times \mathbf{g} + (\boldsymbol{\omega}^T \mathbf{g})\mathbf{g}] = \tfrac{1}{2}[\mathbf{I} + \mathbf{G} + \mathbf{g}\mathbf{g}^T]\boldsymbol{\omega}$$

where

$$\mathbf{G} = \begin{bmatrix} 0 & -g_3 & g_2 \\ g_3 & 0 & -g_1 \\ -g_2 & g_1 & 0 \end{bmatrix}$$

(b) Also show that for infinitesimal rotations, we have

$$\dot{\mathbf{g}} \approx \tfrac{1}{2}\boldsymbol{\omega}$$

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