15

Fuzzy Decision Making

15.1 GENERAL DISCUSSION

Making decisions is undoubtedly one of the most fundamental activities of human beings. We all are faced in our daily life with varieties of alternative actions available to us and, at least in some instances, we have to decide which of the available actions to take. The beginnings of decision making, as a subject of study, can be traced, presumably, to the late 18th century, when various studies were made in France regarding methods of election and social choice. Since these initial studies, decision making has evolved into a respectable and rich field of study. The current literature on decision making, based largely on theories and methods developed in this century, is enormous.

The subject of decision making is, as the name suggests, the study of how decisions are actually made and how they can be made better or more successfully. That is, the field is concerned, in general, with both descriptive theories and normative theories. Much of the focus in developing the field has been in the area of management, in which the decision-making process is of key importance for functions such as inventory control, investment, personnel actions, new-product development, and allocation of resources, as well as many others. Decision making itself, however, is broadly defined to include any choice or selection of alternatives, and is therefore of importance in many fields in both the "soft" social sciences and the "hard" disciplines of natural sciences and engineering.

Applications of fuzzy sets within the field of decision making have, for the most part, consisted of fuzzifications of the classical theories of decision making. While decision making under conditions of risk have been modeled by probabilistic decision theories and game theories, fuzzy decision theories attempt to deal with the vagueness and nonspecificity inherent in human formulation of preferences, constraints, and goals. In this chapter, we overview the applicability of fuzzy set theory to the main classes of decision-making problems.

Classical decision making generally deals with a set of alternative states of nature (outcomes, results), a set of alternative actions that are available to the decision maker, a relation indicating the state or outcome to be expected from each alternative action, and, finally, a utility or objective function, which orders the outcomes according to their desirability. A decision is said to be made under conditions of *certainty* when the outcome for each action can be determined and ordered precisely. In this case, the alternative that leads to the outcome yielding the highest utility is chosen. That is, the decision-making problem becomes an optimization problem, the problem of maximizing the utility function. A decision is made under conditions of *risk*, on the other hand, when the only available knowledge concerning the outcomes consists of their conditional probability distributions, one for each action. In this case, the decision-making problem becomes an optimization problem of maximizing the expected utility. When probabilities of the outcomes are not known, or may not even be relevant, and outcomes for each action are characterized only approximately, we say that decisions are made under *uncertainty*. This is the prime domain for fuzzy decision making.

Decision making under uncertainty is perhaps the most important category of decision-making problems, as well characterized by the British economist Shackle [1961]:

> In a predestinate world, decision would be *illusory*; in a world of perfect foreknowledge, *empty*; in a world without natural order, *powerless*. Our intuitive attitude to life implies non-illusory, non-empty, non-powerless decision... Since decision in this sense excludes both perfect foresight and anarchy in nature, it must be defined as choice in face of bounded uncertainty.

This indicates the importance of fuzzy set theory in decision making.

Several classes of decision-making problems are usually recognized. According to one criterion, decision problems are classified as those involving a single decision maker and those which involve several decision makers. These problem classes are referred to as *individual decision making* and *multiperson decision making*, respectively. According to another criterion, we distinguish decision problems that involve a *simple optimization* of a utility function, an *optimization under constraints*, or an *optimization under multiple objective criteria*. Furthermore, decision making can be done in *one stage*, or it can be done iteratively, in *several stages*. This chapter is structured, by and large, according to these classifications.

We do not attempt to cover fuzzy decision making comprehensively. This would require a large book fully specialized on this subject. Instead, we want to convey the spirit of fuzzy decision making, as applied to the various classes of decision problems.

15.2 INDIVIDUAL DECISION MAKING

Fuzziness can be introduced into the existing models of decision models in various ways. In the first paper on fuzzy decision making, Bellman and Zadeh [1970] suggest a fuzzy model of decision making in which relevant goals and constraints are expressed in terms of fuzzy sets, and a decision is determined by an appropriate aggregation of these fuzzy sets. A decision situation in this model is characterized by the following components:

- a set A of *possible actions*;
- a set of *goals* $G_i (i \in \mathbb{N}_n)$, each of which is expressed in terms of a fuzzy set defined on A;
- a set of *constraints* $C_j (j \in \mathbb{N}_m)$, each of which is also expressed by a fuzzy set defined on A.

It is common that the fuzzy sets expressing goals and constraints in this formulation are not defined directly on the set of actions, but indirectly, through other sets that characterize relevant states of nature. Let G'_i and C'_j be fuzzy sets defined on sets X_i and Y_j, respectively, where $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_m$. Assume that these fuzzy sets represent goals and constraints expressed by the decision maker. Then, for each $i \in \mathbb{N}_n$ and each $j \in \mathbb{N}_m$, we describe the meanings of actions in set A in terms of sets X_i and Y_j by functions

$$g_i : A \to X_i,$$
$$c_j : A \to Y_j,$$

and express goals G_i and constraints C_j by the compositions of g_i with G'_i and the compositions of c_j and C'_j; that is,

$$G_i(a) = G'_i(g_i(a)), \tag{15.1}$$
$$C_j(a) = C'_j(c_j(a)) \tag{15.2}$$

for each $a \in A$.

Given a decision situation characterized by fuzzy sets A, $G_i (i \in \mathbb{N}_n)$, and $C_j (j \in \mathbb{N}_m)$, a *fuzzy decision*, D, is conceived as a fuzzy set on A that simultaneously satisfies the given goals G_i and constraints C_j. That is,

$$D(a) = \min[\inf_{i \in \mathbb{N}_n} G_i(a), \inf_{j \in \mathbb{N}_m} C_j(a)] \tag{15.3}$$

for all $a \in A$, provided that the standard operator of fuzzy intersection is employed.

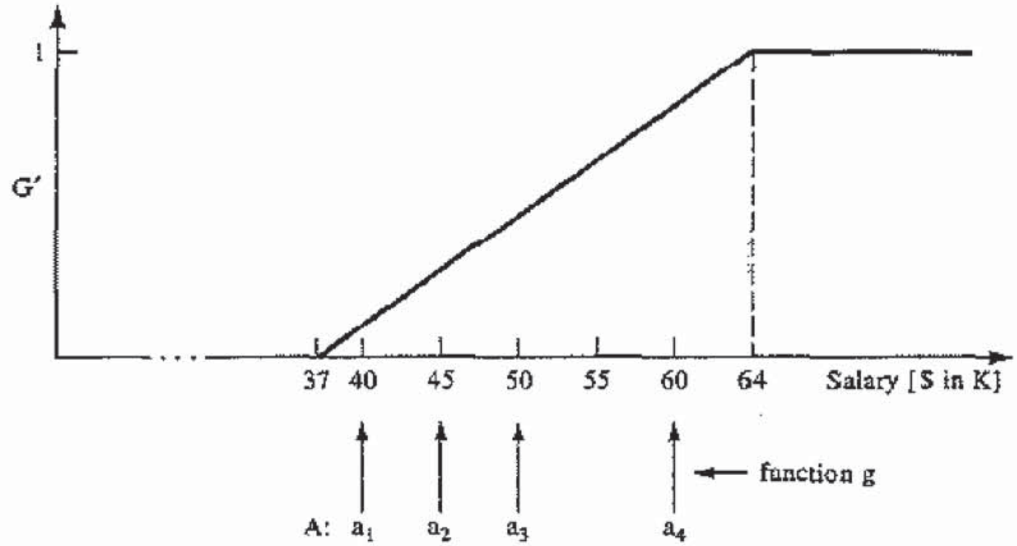
Once a fuzzy decision has been arrived at, it may be necessary to choose the "best" single crisp alternative from this fuzzy set. This may be accomplished in a straightforward manner by choosing an alternative $\hat{a} \in A$ that attains the maximum membership grade in D. Since this method ignores information concerning any of the other alternatives, it may not be desirable in all situations. When A is defined on $\mathbb{R}$, it is preferable to determine $\hat{a}$ by an appropriate defuzzification method (Sec. 12.2).

Before discussing the various features of this fuzzy decision model and its possible modifications or extensions, let us illustrate how it works by two simple examples.

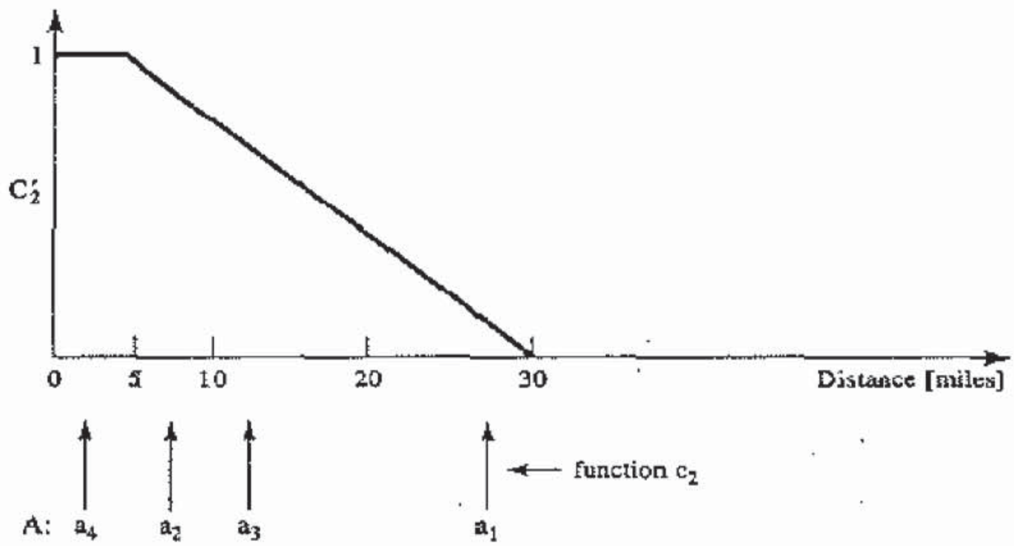
Example 15.1

Suppose that an individual needs to decide which of four possible jobs, a_1, a_2, a_3, a_4, to choose. His or her goal is to choose a job that offers a high salary under the constraints that the job is interesting and within close driving distance. In this case, $A = \{a_1, a_2, a_3, a_4\}$, and the fuzzy sets involved represent the concepts of *high salary*, *interesting job*, and *close driving distance*. These concepts are highly subjective and context-dependent, and must be defined by the individual in a given context. The goal is expressed in monetary terms, independent of the jobs available. Hence, according to our notation, we denote the fuzzy set expressing the goal by G'. A possible definition of G' is given in Fig. 15.1a, where we assume, for convenience, that the underlying universal set is $\mathbb{R}^+$. To express the goal in terms of set A, we need a function g: $A \to \mathbb{R}^+$, which assigns to each job the respective salary. Assume the following assignments:

$$g(a_1) = \$40{,}000,$$
$$g(a_2) = \$45{,}000,$$
$$g(a_3) = \$50{,}000,$$
$$g(a_4) = \$60{,}000.$$



(a) Goal G': High salary.



(b) Constraint C_2': Close driving distance.

Figure 15.1 Fuzzy goal and constraint (Example 15.1): (a) goal G': high salary; (b) constraint C_2': close driving distance.

This assignment is also shown in Fig. 15.1a. Composing now functions g and G', according to (15.1), we obtain the fuzzy set

$$G = .11/a_1 + .3/a_2 + .48/a_3 + .8/a_4,$$

which expresses the goal in terms of the available jobs in set A.

The first constraint, requiring that the job be interesting, is expressed directly in terms of set A (i.e., c_1, in (15.2) is the identity function and $C_1 = C_1'$). Assume that the individual assigns to the four jobs in A the following membership grades in the fuzzy set of interesting jobs:

$$C_1 = .4/a_1 + .6/a_2 + .2/a_3 + .2/a_4.$$

The second constraint, requiring that the driving distance be close, is expressed in terms of the driving distance from home to work. Following our notation, we denote the fuzzy set expressing this constraint by C_2'. A possible definition of C_2' is given in Fig. 15.1b, where distances of the four jobs are also shown. Specifically,

$$c_2(a_1) = 27 \text{ miles},$$

$$c_2(a_2) = 7.5 \text{ miles},$$

$$c_2(a_3) = 12 \text{ miles},$$

$$c_2(a_4) = 2.5 \text{ miles}.$$

By composing functions c_2 and C_2', according to (15.2), we obtain the fuzzy set

$$C_2 = .1/a_1 + .9/a_2 + .7/a_3 + 1/a_4,$$

which expresses the constraint in terms of the set A.

Applying now formula (15.3), we obtain the fuzzy set

$$D = .1/a_1 + .3/a_2 + .2/a_3 + .2/a_4,$$

which represents a fuzzy characterization of the concept of *desirable job*. The job to be chosen is $\hat{a} = a_2$; this is the most desirable job among the four available jobs under the given goal G and constraints C_1, C_2, provided that we aggregate the goal and constraints as expressed by (15.3).

Example 15.2

In this very simple example, adopted from Zimmermann [1987], we illustrate a case in which A is not a discrete set. The board of directors of a company needs to determine the optimal dividend to be paid to the shareholders. For financial reasons, the dividend should be *attractive* (goal G); for reasons of wage negotiations, it should be *modest* (constraint C). The set of actions, A, is the set of possible dividends, assumed here to be the interval $[0, a_{max}]$ of real numbers, where a_{max} denotes the largest acceptable dividend. The goal as well as the constraint are expressed directly as fuzzy sets on $A = [0, a_{max}]$. A possible scenario is shown in Fig. 15.2, which is self-explanatory.

The described fuzzy decision model allows the decision maker to frame the goals and constraints in vague, linguistic terms, which may more accurately reflect practical problem solving situations. The membership functions of fuzzy goals in this model serve much the same purpose as utility or objective functions in classical decision making that order the outcomes according to preferability. Unlike the classical theory of decision making under constraints, however, the symmetry between the goals and constraints under this fuzzy model allows them to be treated in exactly the same manner.

Formula (15.3), based upon the standard operator of fuzzy intersection, does not allow, however, for any interdependence, interaction, or trade-off between the goals and constraints under consideration. For many decision applications, this lack of compensation may not be appropriate; the full compensation or trade-off offered by the union operation that corresponds to the logical "or" (the max operator) may be inappropriate as well. Therefore, an alternative fuzzy set intersection or an averaging operator may be used to reflect a situation in which some degree of positive compensation exists among the goals and constraints.

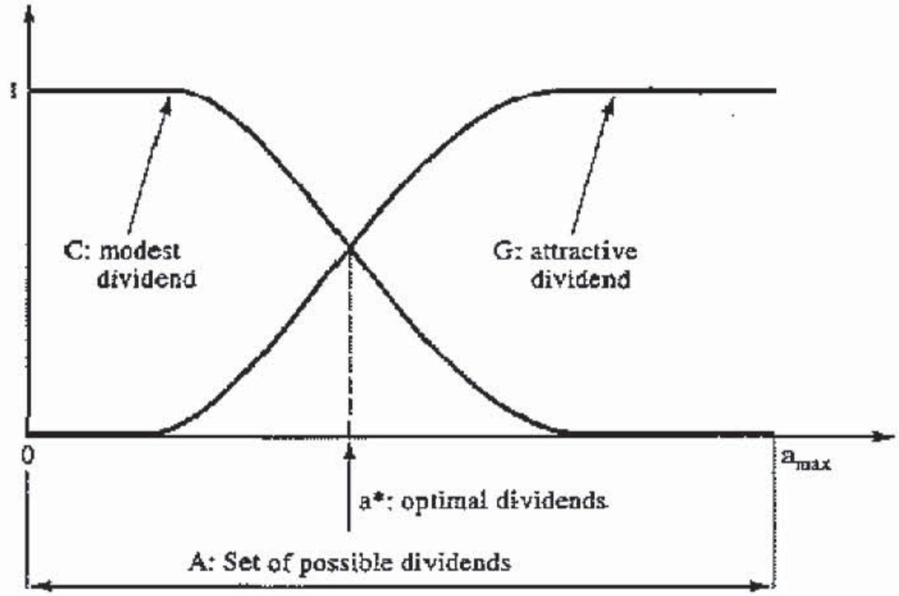


Figure 15.2 Illustration to Example 15.2.

This fuzzy model can be further extended to accommodate the relative importance of the various goals and constraints by the use of weighting coefficients. In this case, the fuzzy decision D can be arrived at by a convex combination of the n weighted goals and m weighted constraints of the form

$$D(a) = \sum_{i=1}^{n} u_i G_i(a) + \sum_{j=1}^{m} v_j C_j(a) \tag{15.4}$$

for all $a \in A$, where u_i and v_j are non-negative weights attached to each fuzzy goal $G_i (i \in \mathbb{N}_n)$ and each fuzzy constraint $C_j (j \in \mathbb{N}_m)$, respectively, such that

$$\sum_{i=1}^{n} u_i + \sum_{j=1}^{m} v_j = 1.$$

However, a direct extension of formula (15.3) may be used as well; that is,

$$D(a) = \min[\inf_{i \in \mathbb{N}_n} G_i^{u_i}(a), \inf_{j \in \mathbb{N}_m} C_j^{v_j}(a)], \tag{15.5}$$

where the weights u_i and v_j possess the above-specified properties.

15.3 MULTIPERSON DECISION MAKING

When decisions made by more than one person are modeled, two differences from the case of a single decision maker can be considered: first, the goals of the individual decision makers may differ such that each places a different ordering on the alternatives; second, the individual decision makers may have access to different information upon which to base their decision. Theories known as n-person game theories deal with both of these considerations, team theories of decision making deal only with the second, and group-decision theories deal only with the first.

A fuzzy model group decision was proposed by Blin [1974] and Blin and Whinston [1973]. Here, each member of a group of n individual decision makers is assumed to have a reflexive, antisymmetric, and transitive preference ordering $\mathbf{P}_k$, $k \in \mathbb{N}_n$, which totally or partially orders a set X of alternatives. A "social choice" function must then be found which, given the individual preference orderings, produces the most acceptable overall group preference ordering. Basically, this model allows for the individual decision makers to possess different aims and values while still assuming that the overall purpose is to reach a common, acceptable decision. In order to deal with the multiplicity of opinion evidenced in the group, the social preference S may be defined as a fuzzy binary relation with membership grade function

$$S : X \times X \to [0, 1],$$

which assigns the membership grade $S(x_i, x_j)$, indicating the degree of group preference of alternative x_i over x_j. The expression of this group preference requires some appropriate means of aggregating the individual preferences. One simple method computes the relative popularity of alternative x_i over x_j by dividing the number of persons preferring x_i to x_j, denoted by $N(x_i, x_j)$, by the total number of decision makers, n. This scheme corresponds to the simple majority vote. Thus,

$$S(x_i, x_j) = \frac{N(x_i, x_j)}{n}. \tag{15.6}$$

Other methods of aggregating the individual preferences may be used to accommodate different degrees of influence exercised by the individuals in the group. For instance, a dictatorial situation can be modeled by the group preference relation S for which

$$S(x_i, x_j) = \begin{cases} 1 & \text{if } x_i \overset{k}{>} x_j \text{ for some individual } k \\ 0 & \text{otherwise,} \end{cases}$$

where $\overset{k}{>}$ represents the preference ordering of the one individual k who exercises complete control over the group decision.

Once the fuzzy relationship S has been defined, the final nonfuzzy group preference can be determined by converting S into its resolution form

$$S = \bigcup_{\alpha \in [0,1]} \alpha\,{}^{\alpha}S,$$

which is the union of the crisp relations ${}^{\alpha}S$ comprising the α-cuts of the fuzzy relation S, each scaled by α. Each value α essentially represents the level of agreement between the individuals concerning the particular crisp ordering ${}^{\alpha}S$. One procedure that maximizes the final agreement level consists of intersecting the classes of crisp total orderings that are compatible with the pairs in the α-cuts ${}^{\alpha}S$ for increasingly smaller values of α until a single crisp total ordering is achieved. In this process, any pairs (x_i, x_j) that lead to an intransitivity are removed. The largest value α for which the unique compatible ordering on $X \times X$ is found represents the maximized agreement level of the group, and the crisp ordering itself represents the group decision. This procedure is illustrated in the following example.

Example 15.3

Assume that each individual of a group of eight decision makers has a total preference ordering $\mathbf{P}_i (i \in \mathbb{N}_8)$ on a set of alternatives $X = \{w, x, y, z\}$ as follows:

$$\mathbf{P}_1 = \langle w, x, y, z\rangle$$
$$\mathbf{P}_2 = \mathbf{P}_5 = \langle z, y, x, w\rangle$$
$$\mathbf{P}_3 = \mathbf{P}_7 = \langle x, w, y, z\rangle$$
$$\mathbf{P}_4 = \mathbf{P}_8 = \langle w, z, x, y\rangle$$
$$\mathbf{P}_6 = \langle z, w, x, y\rangle$$

Using the membership function given in (15.6) for the fuzzy group preference ordering relation S (where $n = 8$), we arrive at the following fuzzy social preference relation:

$$\mathbf{S} = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & .5 & .75 & .625 \\ .5 & 0 & .75 & .375 \\ .25 & .25 & 0 & .375 \\ .375 & .625 & .625 & 0 \end{bmatrix} \end{array}$$

The α-cuts of this fuzzy relation S are:

$${}^{1}S = \varnothing$$
$${}^{.75}S = \{\langle w, y\rangle, \langle x, y\rangle\}$$
$${}^{.625}S = \{\langle w, z\rangle, \langle z, x\rangle, \langle z, y\rangle, \langle w, y\rangle, \langle x, y\rangle\}$$
$${}^{.5}S = \{\langle x, w\rangle, \langle w, x\rangle, \langle w, z\rangle, \langle z, x\rangle, \langle z, y\rangle, \langle w, y\rangle, \langle x, y\rangle\}$$
$${}^{.375}S = \{\langle z, w\rangle, \langle x, z\rangle, \langle y, z\rangle, \langle x, w\rangle, \langle w, x\rangle, \langle w, z\rangle, \langle z, x\rangle, \langle z, y\rangle, \langle w, y\rangle, \langle x, y\rangle\}$$
$${}^{.25}S = \{\langle y, w\rangle, \langle y, x\rangle, \langle z, w\rangle, \langle x, z\rangle, \langle y, z\rangle, \langle x, w\rangle, \langle w, x\rangle, \langle w, z\rangle, \langle z, x\rangle, \langle z, y\rangle, \langle w, y\rangle, \langle x, y\rangle\}$$

We can now apply the procedure to arrive at the unique crisp ordering that constitutes the group choice. All total orderings on $X \times X$ are, of course, compatible with the empty set of ${}^{1}S$. The total orderings ${}^{.75}O$ that are compatible with the pairs in the crisp relations ${}^{.75}S$ are

$${}^{.75}O = \{\langle z, w, x, y\rangle, \langle w, x, y, z\rangle, \langle w, z, x, y\rangle, \langle w, x, z, y\rangle,$$
$$\langle z, x, w, y\rangle, \langle x, w, y, z\rangle, \langle x, z, w, y\rangle, \langle x, w, z, y\rangle\}.$$

Thus,

$${}^{1}O \cap {}^{.75}O = {}^{.75}O.$$

The orderings compatible with ${}^{.625}S$ are

$${}^{.625}O = \{\langle w, z, x, y\rangle, \langle w, z, y, x\rangle\}$$

and

$${}^{1}O \cap {}^{.75}O \cap {}^{.625}O = \{\langle w, z, x, y\rangle\}.$$

Thus, the value .625 represents the group level of agreement concerning the social choice denoted by the total ordering $\langle w, z, x, y\rangle$.

In the described procedure of group decision making, it is required that each group member can order the given set of alternatives. This requirement may be too strong in some cases. However, it is relatively easy for each individual to make pairwise comparisons between the given alternatives. A simple method proposed by Shimura [1973] is designed to construct an ordering of all given alternatives on the basis of their pairwise comparisons. In this method, $f(x_i, x_j)$, denotes the attractiveness grade given by the individual to x_i with respect to x_j. These primitive evaluations, which are expressed by positive numbers in a given range, are made by the individual for all pairs of alternatives in the given set X. They are then converted to relative preference grades, $F(x_i, x_j)$, by the formula

$$\begin{aligned} F(x_i, x_j) &= \frac{f(x_i, x_j)}{\max[f(x_i, x_j), f(x_j, x_i)]} \\ &= \min[1, f(x_i, x_j)/f(x_j, x_i)] \end{aligned} \tag{15.7}$$

for each pair $\langle x_i, x_j \rangle \in X^2$. Clearly, $F(x_i, x_j) \in [0, 1]$ for all pairs $\langle x_i, x_j \rangle \in X^2$. When $F(x_i, x_j) = 1$, x_i is considered at least as attractive as x_j. Function F, which may be viewed as a membership function of a fuzzy relation on X, has for each pair $\langle x_i, y_j \rangle \in X^2$ the property

$$\max[F(x_i, x_j), F(x_j, x_i)] = 1.$$

The property means: for each pair of alternatives, at least one must be as attractive as the other.

For each $x_i \in X$, we can now calculate the overall relative preference grades, $p(x_i)$, of x_i with respect to all other alternatives in X by the formula

$$p(x_i) = \min_{x_j \in X} F(x_i, x_j). \tag{15.8}$$

The preference ordering of alternatives in X is then induced by the numerical ordering of these grades $p(x_i)$.

Example 15.4

To illustrate the described method, consider a group of people involved in a business partnership who intend to buy a common car for business purposes. To decide what car to buy is a multiperson decision problem. The method described in this section can be used, but each person in the group has to order the available alternatives first. To do that, the method based on the degrees of attractiveness can be used.

Assume, for the sake of simplicity, that only five car models are considered: Acclaim, Accord, Camry, Cutlass, and Sable. Assume further that, using the numbers suggested in Table 15.1a for specifying the attractiveness grades, the evaluation prepared by one person in the group is given in Table 15.1b. The corresponding relative preference grades (calculated by (15.7)) and the overall relative preference grades (calculated by (15.8)) are given in Table 15.1c. The latter induce the following preference ordering of the car models: Camry, Sable, Accord, Cutlass, Acclaim. Orderings expressing preference by the other members of the group can be determined in a similar way. Then, the method for multiperson decision making described in this section can be applied to these preference orderings to obtain a group decision.

TABLE 15.1 ILLUSTRATION TO EXAMPLE 15.4

(a) Suggested numbers for attractiveness grading

$f(x_i, x_j)$	Attractiveness of x_i with respect to x_j
1	Little attractive
3	Moderately attractive
5	Strongly attractive
7	Very strongly attractive
9	Extremely attractive
2, 4, 6, 8	Intermediate values between levels

(b) Given attractiveness grades

$f(x_i, x_j)$	Acclaim	Accord	Camry	Cutlass	Sable
Acclaim	1	7	9	3	8
Accord	3	1	3	2	4
Camry	1	1	1	3	5
Cutlass	2	7	7	1	7
Sable	2	6	8	3	1

(c) Relative preference grades and overall relative preference grades

$F(x_i, x_j)$	Acclaim	Accord	Camry	Cutlass	Sable	$p(x_i)$
Acclaim	1	0.43	0.11	0.67	0.25	0.11
Accord	1	1.00	0.33	1.00	1.00	0.33
Camry	1	1.00	1.00	1.00	1.00	1.00
Cutlass	1	0.29	0.43	1.00	0.43	0.29
Sable	1	0.66	0.625	1.00	1.00	0.63

15.4 MULTICRITERIA DECISION MAKING

In multicriteria decision problems, relevant alternatives are evaluated according to a number of criteria. Each criterion induces a particular ordering of the alternatives, and we need a procedure by which to construct one overall preference ordering. There is a visible similarity between these decision problems and problems of multiperson decision making. In both cases, multiple orderings of relevant alternatives are involved and have to be integrated into one global preference ordering. The difference is that the multiple orderings represent either preferences of different people or ratings based on different criteria.

The number of criteria in multicriteria decision making is virtually always assumed to be finite. In this section, we assume, in addition, that the number of considered alternatives is also finite. Decision situations with infinite sets of alternatives are considered in Sec. 15.7, which deals with fuzzy mathematical programming.

Let $X = \{x_1, x_2, \ldots, x_n\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ be, a set of alternatives and a set of criteria characterizing a decision situation, respectively. Then, the basic information involved in multicriteria decision making can be expressed by the matrix

$$\mathbf{R} = \begin{array}{c} \\ c_1 \\ c_2 \\ \vdots \\ c_m \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \dots & x_n \end{array} \\ \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ r_{m1} & r_{m2} & \dots & r_{mn} \end{bmatrix} \end{array}.$$

Assume first that all entries of this matrix are real numbers in [0, 1], and each entry r_{ij} expresses the degree to which criterion c_i is satisfied by alternative $x_j (i \in \mathbb{N}_m, j \in \mathbb{N}_n)$. Then $\mathbf{R}$ may be viewed as a matrix representation of a fuzzy relation on $C \times X$.

It may happen that, instead of matrix $\mathbf{R}$ with entries in [0, 1], an alternative matrix $\mathbf{R}' = [r'_{ij}]$, whose entries are arbitrary real numbers, is initially given. In this case, $\mathbf{R}'$ can be converted to the desired matrix $\mathbf{R}$ by the formula

$$r_{ij} = \frac{r'_{ij} - \min\limits_{j \in \mathbb{N}_n} r'_{ij}}{\max\limits_{j \in \mathbb{N}_n} r'_{ij} - \min\limits_{j \in \mathbb{N}_n} r'_{ij}} \tag{15.9}$$

for all $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$.

The most common approach to multicriteria decision problems is to convert them to single-criterion decision problems. This is done by finding a global criterion, $r_j = h(r_{1j}, r_{2j}, \dots, r_{mj})$, that for each $x_j \in X$ is an adequate aggregate of values $r_{1j}, r_{2j}, \dots, r_{mj}$ to which the individual criteria $c_1, c_2, \dots, c_m$ are satisfied.

An example of multicriteria decision problem is the problem of recruiting and selecting personnel. In this particular problem, the selection of conditions from a given set of individuals, say $x_1, x_2, \dots, x_n$, is guided by comparing candidates' profiles with a required profile in terms of given criteria $c_1, c_2, \dots, c_m$. This results in matrix $\mathbf{R}$ (or in a matrix that can be converted to matrix $\mathbf{R}$ by (15.9)). The entries r_{ij} of $\mathbf{R}$ express, for each $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$, the degree to which candidate x_j conforms to the required profile in terms of criterion c_i. Function h may be any of the aggregating operations examined in Chapter 3.

A frequently employed aggregating operator is the weighted average

$$r_j = \frac{\sum\limits_{i=1}^{m} w_i r_{ij}}{\sum\limits_{i=1}^{m} w_i} \quad (j \in \mathbb{N}_n), \tag{15.10}$$

where $w_1, w_2, \dots, w_m$ are weights that indicate the relative importance of criteria $c_1, c_2, \dots, c_m$. A class of possible weighted aggregations is given by the formula

$$r_j = h(r_{1j}^{w_1}, r_{2j}^{w_2}, \dots, r_{mj}^{w_m}),$$

where h is an aggregation operator and $w_1, w_2, \dots, w_m$ are weights.

Consider now a more general situation in which the entries of matrix $\mathbf{R}$ are fuzzy numbers $\tilde{r}_{ij}$ on $\mathbb{R}^+$, and weights are specified in terms of fuzzy numbers $\tilde{w}_i$ on [0, 1]. Then, using the operations of fuzzy addition and fuzzy multiplication, we can calculate the weighted average $\tilde{r}_j$ by the formula

$$\bar{r}_j = \sum_{i=1}^{m} \bar{w}_i \bar{r}_{ij}. \tag{15.11}$$

Since fuzzy numbers are not linearly ordered, a ranking method is needed to order the resulting fuzzy numbers $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n$. This issue is addressed in Sec. 15.6.

15.5 MULTISTAGE DECISION MAKING

Multistage decision making is a sort of dynamic process. A required goal is not achieved by solving a single decision problem, but by solving a sequence of decision-making problems. These decision-making problems, which represent stages in overall multistage decision making, are dependent on one another in the dynamic sense. Any task-oriented control, for example, is basically a multistage decision-making problem.

In general, theories of multistage decision making may be viewed as part of the theory of general dynamic systems (Sec. 12.7). The most important theory of multistage decision making, which is closely connected with dynamic systems, is that of *dynamic programming* [Bellman, 1957]. As can any mathematical theory, dynamic programming can be fuzzified. A fuzzification of dynamic programming extends its practical utility since it allows decision makers to express their goals, constraints, decisions, and so on in approximate, fuzzy terms, whenever desirable.

Fuzzy dynamic programming was formulated for the first time in the classical paper by Bellman and Zadeh [1970]. In this section, we explain basic ideas of this formulation, which is based on the concept of a finite-state fuzzy automaton introduced in Sec. 12.6. To see connections between the two sections, we adopt here the notation used in Sec. 12.6.

A decision problem conceived in terms of fuzzy dynamic programming is viewed as a decision problem regarding a fuzzy finite-state automaton. However, the automaton involved is a special version of the general fuzzy automaton examined in Sec. 12.6. One restriction of the automaton in dynamic programming is that the state-transition relation is crisp and, hence, characterized by the usual state-transition function of classical automata. Otherwise, the automaton operates with fuzzy input states and fuzzy internal states, and it is thus fuzzy in this sense. Another restriction is that no special output is needed. That is, the next internal state is also utilized as output and; consequently, the two need not be distinguished.

Under the mentioned restrictions, the automaton, $\mathcal{A}$, involved in fuzzy dynamic programming is defined by the triple

$$\mathcal{A} = (X, Z, f),$$

where X and Z are, respectively, the sets of input states and output states of $\mathcal{A}$, and

$$f : Z \times X \to Z$$

is the state-transition function of $\mathcal{A}$, whose meaning is to define, for each discrete time t ($t \in \mathbb{N}$), the next internal state, z^{t+1}, of the automaton in terms of its present internal state, z^t, and its present input state, x^t. That is,

$$z^{t+1} = f(z^t, x^t). \tag{15.12}$$

A scheme of the described automaton is shown in Fig. 15.3a. This type of automata are used

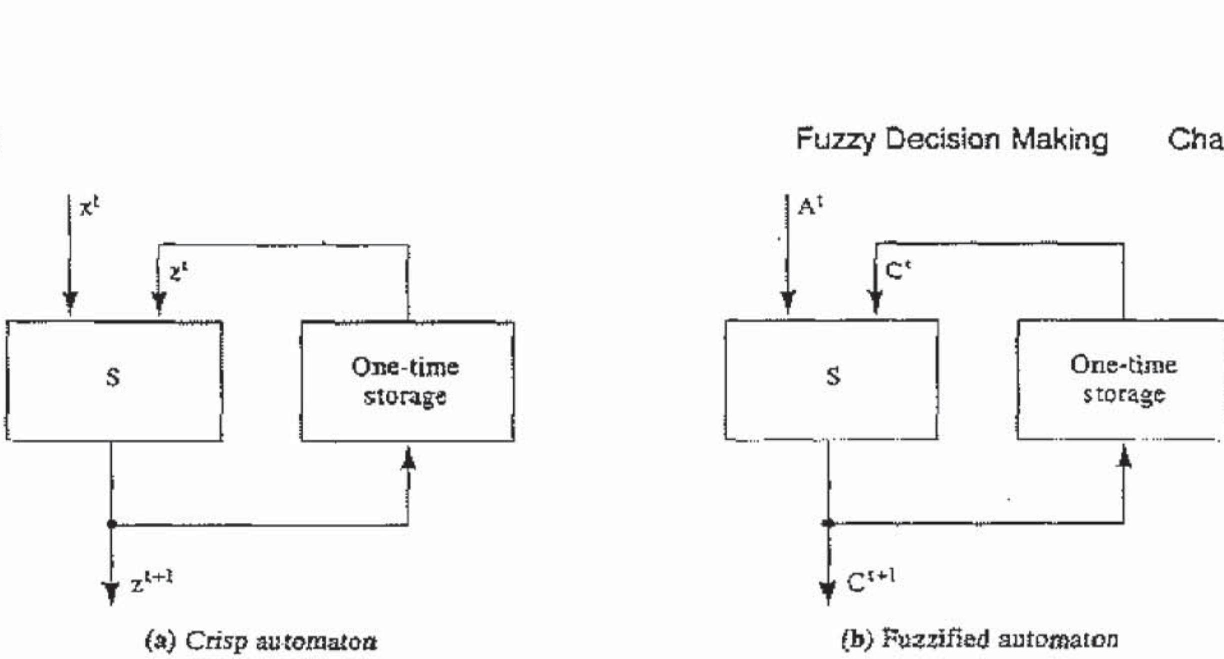


Figure 15.3 The automaton employed in crisp or fuzzy dynamic programming: (a) crisp automaton; (b) fuzzified automaton.

in classical dynamic programming. For fuzzy dynamic programming they must be fuzzified by using the extension principle. A scheme of the fuzzified version is shown in Fig. 15.3b, where A^t, C^t denote, respectively, the fuzzy input state and fuzzy internal state at time t, and C^{t+1} denotes the fuzzy internal state at time $t+1$. Clearly, A^t is a fuzzy set on X, while C^t and C^{t+1} are fuzzy sets on Z.

Employing the fuzzified automaton depicted in Fig. 15.3b, we can now proceed to a description of fuzzy dynamic programming. In this conception of decision making, the desired goal is expressed in terms of a fuzzy set C^N (the fuzzy internal state of A at time N), where N is the time of termination of the decision process. The value of N, which defines the number of stages in the decision process, is assumed to be given. It is also assumed that the input of A is expressed at each time t by a fuzzy state A^t and that a particular crisp initial internal state z^0 is given.

Considering fuzzy input states $A^0, A^1, \ldots, A^{N-1}$ as constraints and fuzzy internal state C^N as fuzzy goal in a fuzzy decision making, we may conceive of a fuzzy decision (in the sense discussed in Sec. 15.2) as a fuzzy set on X^N defined by

$$D = \bar{A}^0 \cap \bar{A}^1 \cap \ldots \bar{A}^{N-1} \cap \bar{C}^N,$$

where $\bar{A}^t$ is a cylindric extension of A^t from X to X^N for each $t = 0, 1, \ldots, N-1$, and $\bar{C}^N$ is the fuzzy set on X^N that induces C^N on Z. That is, for any sequence $x^0, x^1, \ldots, x^{N-1}$, viewed as a sequence of decisions, the membership grade of D is defined by

$$D(x^0, x^1, \ldots, x^{N-1}) = \min[A^0(x^0), A^1(x^1), \ldots, A^{N-1}(x^{N-1}), C^N(z^N)], \tag{15.13}$$

where z^N is uniquely determined by $x^0, x^1, \ldots, x^{N-1}$ and z^0 via (15.12); this definition assumes, of course, that we use the standard operator of intersection. The decision problem is to find a sequence $\hat{x}^0, \hat{x}^1 \ldots \hat{x}^{N-1}$ of input states such that

$$D(\hat{x}^0, \hat{x}^1, \ldots, \hat{x}^{N-1}) = \max_{x^0, \ldots, x^{N-1}} D(x^0, x^1, \ldots, x^{N-1}). \tag{15.14}$$

To solve this problem by fuzzy dynamic programming, we need to apply a principle known in dynamic programming as the *principle of optimality* [Bellman, 1957], which can be expressed as follows: An optimal decision sequence has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

Applying the principle of optimality and substituting for D from (15.13), we can write (15.14) in the form

$$D(\hat{x}^0, \hat{x}^1, \ldots, \hat{x}^{N-1}) = \max_{x^0,\ldots,x^{N-2}} \{\max_{x^{N-1}} \min[A^0(x^0), A^1(x^1), \ldots, A^{N-1}(x^{N-1}), C^N(f(z^{N-1}, x^{N-1}))]\}.$$

This equation can be rewritten as

$$\begin{aligned} D(\hat{x}^0, \hat{x}^1, \ldots, \hat{x}^{N-1}) &= \max_{x^0,\ldots,x^{N-2}} \{\min[A^0(x^0), A^1(x^1), \ldots, A^{N-2}(x^{N-2}), \\ &\quad \max_{x^{N-1}} \min[A^{N-1}(x^{N-1}), C^N(f(z^{N-1}, x^{N-1}))]]\} \\ &= \max_{x^0,\ldots,x^{N-2}} \{\min[A^0(x^0), A^1(x^1), \ldots, A^{N-2}(x^{N-2}), \\ &\quad \max_{x^{N-1}} \min[A^{N-1}(x^{N-1}), C^N(z^N)]]\} \\ &= \max_{x^0,\ldots,x^{N-2}} \{\min[A^0(x^0), A^1(x^1), \ldots, A^{N-2}(x^{N-2}), \\ &\quad C^{N-1}(z^{N-1})]\} \end{aligned}$$

where

$$C^{N-1}(z^{N-1}) = \max_{x^{N-1}} \min[A^{N-1}(x^{N-1}), C^N(z^N)].$$

Repeating this *backward iteration*, we obtain the set of N recurrence equations

$$C^{N-k}(z^{N-k}) = \max_{x^{N-k}} \min[A^{N-k}(x^{N-k}), C^{N-k+1}(z^{N-k+1})] \tag{15.15}$$

for $k = 1, 2, \ldots, N$, where

$$z^{N-k+1} = f(z^{N-k}, x^{N-k}).$$

Hence, the optimal sequence $\hat{x}^0, \hat{x}^1, \ldots, \hat{x}^{N-1}$ of decisions can be obtained by successively maximizing values x^{N-k} in (15.15) for $k = 1, 2, \ldots, N$. This results successively in values $\hat{x}^{N-1}, \hat{x}^{N-2}, \ldots, \hat{x}^0$.

Example 15.5 [Bellman and Zadeh, 1970]

Let us consider an automaton with $X = \{x_1, x_2\}$, $Z = \{z_1, z_2, z_3\}$, and the state-transition function expressed by the matrix

$$\begin{array}{c c} & \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} & \begin{bmatrix} z_1 & z_2 \\ z_3 & z_1 \\ z_1 & z_3 \end{bmatrix} \end{array}$$

whose entries are next internal states for any given present internal and output states. Assume that $N = 2$, and the fuzzy goal at $t = 2$ is

$$C^2 = .3/z_1 + 1/z_2 + .8/z_3.$$

Assume further that the fuzzy constraints at input at times $t = 0$ and $t = 1$ are

$$A^0 = .7/x_1 + 1/x_2,$$
$$A^1 = 1/x_1 + .6/x_2.$$

To solve this decision problem, we need to find a sequence $\hat{x}^0, \hat{x}^1$ of input states for which the maximum,

$$\max_{x^0, x^1} \min[A^0(x^0), A^1(x^1), C^2(f(z^1, x^1))],$$

is obtained. Applying the first backward iteration for $t = 1$, we obtain

$$\begin{aligned}
C^1(z_1) &= \max\{\min[A^1(x_1), C^2(f(z_1, x_1))], \min[A^1(x_2), C^2(f(z_1, x_2))]\} \\
&= \max\{\min[A^1(x_1), C^2(z_1)], \min[A^1(x_2), C^2(z_2)]\} \\
&= \max\{\min[1, .3], \min[.6, 1]\} \\
&= .6 \\
C^1(z_2) &= \max\{\min[A^1(x_1), C^2(f(z_2, x_1))], \min[A^1(x_2), C^2(f(z_2, x_2))]\} \\
&= \max\{\min[A^1(x_1), C^2(z_3)], \min[A^1(x_2), C^2(z_1)]\} \\
&= \max\{\min[1, .8], \min[.6, .3]\} \\
&= .8 \\
C^1(z_3) &= \max\{\min[A^1(x_1), C^2(f(z_3, x_1))], \min[A^1(x_2), C^2(f(z_3, x_2))]\} \\
&= \max\{\min[A^1(x_1), C^2(z_1)], \min[A^1(x_2), C^2(z_3)]\} \\
&= \max\{\min[1, .3], \min[.6, .8]\} \\
&= .6
\end{aligned}$$

Hence,

$$C^1 = .6/z_1 + .8/z_2 + .6/z_3.$$

By maximizing the expression

$$\min[A^1(x^1), C^2(f(z^1, x^1))],$$

we find the following best decision $\hat{x}^1$ for each state $z^1 \in Z$ at time $t = 1$:

z^1	z_1	z_2	z_3
$\hat{x}^1$	x_2	x_1	x_2

Applying now the second backward iteration for $t = 0$, we obtain

$$\begin{aligned}
C^0(z_1) &= \max\{\min[A^0(x_1), C^1(f(z_1, x_1))], \min[A^0(x_2), C^1(f(z_1, x_2))]\} \\
&= \max\{\min[A^0(x_1), C^1(z_1)], \min[A^0(x_2), C^1(z_2)]\} \\
&= \max\{\min[.7, .6], \min[1, .8]\} \\
&= .8 \\
C^0(z_2) &= \max\{\min[A^0(x_1), C^1(f(z_2, x_1))], \min[A^0(x_2), C^1(f(z_2, x_2))]\} \\
&= \max\{\min[A^0(x_1), C^1(z_3)], \min[A^0(x_2), C^1(z_1)]\} \\
&= \max\{\min[.7, .6], \min[1, .6]\} \\
&= .6
\end{aligned}$$

$$\begin{aligned} C^0(z_3) &= \max\{\min[A^0(x_1), C^1(f(z_3, x_1))], \min[A^0(x_2), C^1(f(z_3, x_2))]\} \\ &= \max\{\min[A^0(x_1), C^1(z_1)], \min[A^0(x_2), C^1(z_3)]\} \\ &= \max\{\min[.7, .6], \min[1, .6]\} \\ &= .6 \end{aligned}$$

Hence,

$$C^0 = .8/z_1 + .6/z_2 + .6/z_3.$$

By maximizing the expression

$$\min[A^0(x^0), C^1(f(z^0, x^0))],$$

we find the following best decision $\hat{x}^0$ for each state $z^0 \in Z$ at time $t = 0$:

z^0	z_1	z_2	z_3
$\hat{x}^0$	x_2	x_1 or x_2	x_1 or x_2

The maximizing decisions for different initial states z^0 are summarized in Fig. 15.4. For example, when the initial state is z_1, the maximizing decision is to apply action x_2 followed by x_1. In this case, the goal is satisfied to the degree

$$\begin{aligned} C^0(z_1) &= \min[A^0(x_2), C^1(z_2)] \\ &= \min[A^0(x_2), \min[A^1(x_1), C^2(z_3)]] \\ &= \min[A^0(x_2), A^1(x_1), C^2(z_3)] \\ &= \min[1, 1, .8] \\ &= .8. \end{aligned}$$

That is, the degree to which the goal is satisfied is expressed in terms of $C^0(z_1)$, where z_1 is the initial state. When the initial state is z_2, we have two maximizing decisions (Fig. 15.4); hence, there are two ways of calculating $C^0(z_2)$:

$$\begin{aligned} C^0(z_2) &= \min[A^0(x_1), A^1(x_2), C^2(z_3)] \\ &= \min[.7, .6, .8] = .6 \\ C^0(z_2) &= \min[A^0(x_2), A^1(x_2), C^2(z_2)] \\ &= \min[1, .6, 1] = .6. \end{aligned}$$

That is, this goal is satisfied to the degree .6 when the initial state is z_2, regardless of which of the two maximizing decisions is used. We can easily find the same result for the initial state z_3.

15.6 FUZZY RANKING METHODS

In many fuzzy decision problems, the final scores of alternatives are represented in terms of fuzzy numbers. In order to express a crisp preference of alternatives, we need a method for constructing a crisp total ordering from fuzzy numbers. Unfortunately, the lattice of fuzzy numbers, $\langle \mathcal{R}, \text{MIN}, \text{MAX}\rangle$, is not linearly ordered, as discussed in Sec. 4.5. Thus, some fuzzy numbers are not directly comparable.

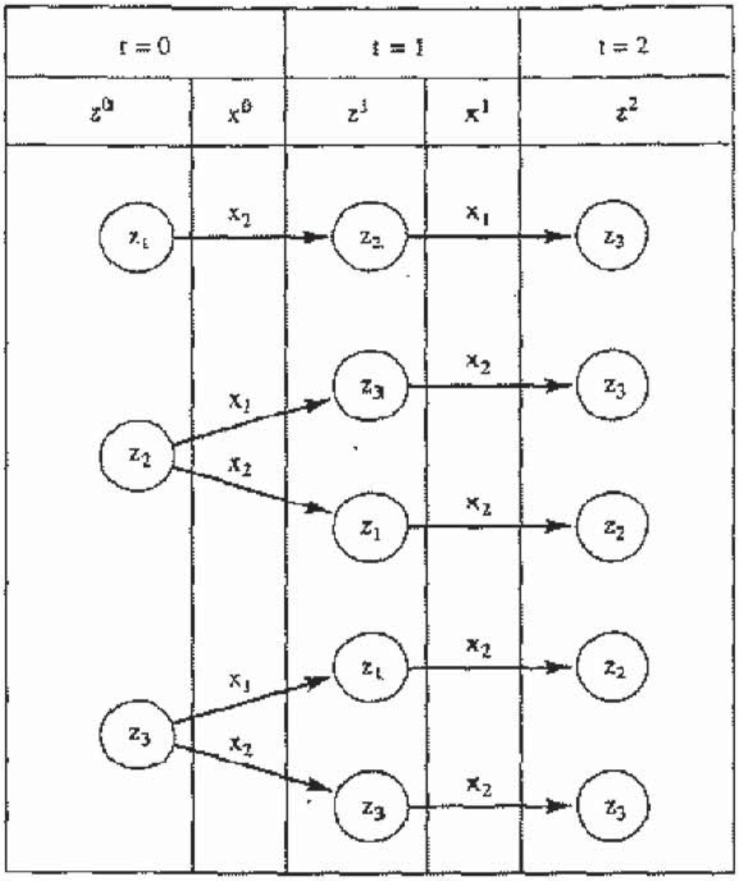


Figure 15.4 Maximizing decisions in Example 15.5 for different initial states z^0.

Numerous methods for total ordering of fuzzy numbers have been suggested in the literature. Each method appears to have some advantages as well as disadvantages. In the context of each application, some methods seem more appropriate than others. However, the issue of choosing a proper ordering method in a given context is still a subject of active research. To illustrate the problem of total ordering of fuzzy numbers, we describe three simple methods and illustrate them by examples.

The first method is based upon defining the *Hamming distance* on the set $\mathcal{R}$ of all fuzzy numbers. For any given fuzzy numbers A and B, the Hamming distance, $d(A, B)$, is defined by the formula

$$d(A, B) = \int_{\mathbb{R}} |A(x) - B(x)| dx. \tag{15.16}$$

For any given fuzzy numbers A and B, which we want to compare, we first determine their least upper bound, $\text{MAX}(A, B)$, in the lattice. Then, we calculate the Hamming distances $d(\text{MAX}(A, B), A)$ and $d(\text{MAX}(A, B), B)$, and define

$$A \leq B \text{ if } d(\text{MAX}(A, B), A) \geq d(\text{MAX}(A, B), B).$$

If $A \preceq B$ (i.e., fuzzy numbers are directly comparable), then $\text{MAX}(A, B) = B$ and, hence, $A \leq B$. That is, the ordering defined by the Hamming distance is compatible with the ordering of comparable fuzzy numbers in $\mathcal{R}$. Observe that we can also define a similar ordering of fuzzy numbers A and B via the greatest lower bound $\text{MIN}(A, B)$.

The second method is based on α-cuts. In fact, a number of variations of this method have been suggested in the literature. A simple variation of this methods proceeds as follows. Given fuzzy numbers A and B to be compared, we select a particular value of $\alpha \in [0, 1]$ and determine the α-cuts ${}^\alpha A = [a_1, a_2]$ and ${}^\alpha B = [b_1, b_2]$. Then, we define

$$A \leq B \text{ if } a_2 \leq b_2.$$

This definition is, of course, dependent on the chosen value of α. It is usually required that $\alpha > 0.5$. More sophisticated methods based on α-cuts, such as the one developed by Mabuchi [1988], aggregate appropriately defined degrees expressing the dominance of one fuzzy number over the other one for all α-cuts.

The third method is based on the extension principle. This method can be employed for ordering several fuzzy numbers, say $A_1, A_2, \ldots, A_n$. The basic idea is to construct a fuzzy set P on $\{A_1, A_2, \ldots, A_n\}$, called a *priority set*, such as $P(A_i)$ is the degree to which A_i is ranked as the greatest fuzzy number. Using the extension principle, P is defined for each $i \in \mathbb{N}_n$ by the formula

$$P(A_i) = \sup \min_{k \in \mathbb{N}_n} A_k(r_k), \tag{15.17}$$

where the supremum is taken over all vectors $\langle r_1, r_2, \ldots, r_n \rangle \in \mathbb{R}^n$ such that $r_i \geq r_j$ for all $j \in \mathbb{N}_n$.

Example 15.6

In this example, we illustrate and compare the three fuzzy ranking methods. Let A and B be fuzzy numbers whose triangular-type membership functions are given in Fig. 15.5a. Then, $\mathrm{MAX}(A, B)$ is the fuzzy number whose membership function is indicated in the figure in bold. We can see that the Hamming distances $d(\mathrm{MAX}(A, B), A)$ and $d(\mathrm{MAX}(A, B), B)$ are expressed by the areas in the figure that are hatched horizontally and vertically, respectively. Using (15.16), we obtain

$$\begin{aligned} d(\mathrm{MAX}(A, B), A) &= \int_{1.5}^{2} [x - 1 - \frac{x}{3}]dx + \int_{2}^{2.25} [-x + 3 - \frac{x}{3}]dx \\ &\quad + \int_{2.25}^{3} [\frac{x}{3} + x - 3]dx + \int_{3}^{4} [4 - x]dx \\ &= \frac{1}{12} + \frac{1}{24} + \frac{3}{8} + \frac{1}{2} = 1 \\ d(\mathrm{MAX}(A, B), B) &= \int_{0}^{1.5} \frac{x}{3}dx - \int_{1}^{1.5} [x - 1]dx \\ &= \frac{3}{8} - \frac{1}{8} = 0.25. \end{aligned}$$

Since $d(\mathrm{MAX}(A, B), A) > d(\mathrm{MAX}(A, B), B)$, we may conclude that, according to the first ranking method, $A \leq B$. When applying the second method to the same example, we can easily find, from Fig. 15.5a, that $A \leq B$ for any $\alpha \in [0, 1]$. According to the third method, we construct the priority fuzzy set P on $\{A, B\}$ as follows:

$$P(A) = \sup_{r_1 \geq r_2} \min[A(r_1), B(r_2)] = 0.75,$$

$$P(B) = \sup_{r_2 \geq r_1} \min[A(r_1), B(r_2)] = 1.$$

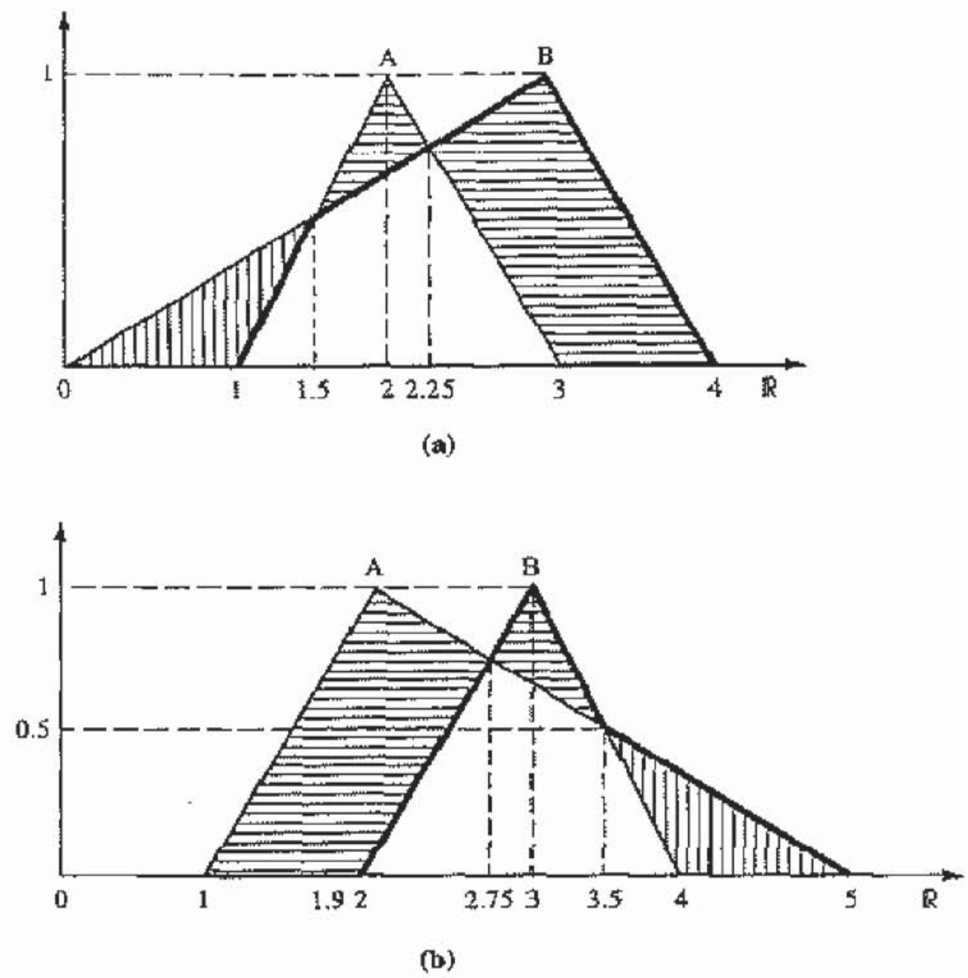


Figure 15.5 Ranking of fuzzy members (Example 15.6).

Hence, again, we conclude that $A \leq B$.

Consider now the fuzzy numbers A and B whose membership functions are given in Fig. 15.5b. The horizontally and vertically hatched areas have the same meaning as before. We can easily find that

$$d(\text{MAX}(A, B), A) = 1, d(\text{MAX}(A, B), B) = 0.25.$$

Hence, $A \leq B$ *according to the first method. The second method gives the same result only for* $\alpha > 0.5$. This shows that the method is inconsistent. According to the third method, we again obtain $P(A) = 0.75$ and $P(B) = 1$; hence, $A \leq B$.

15.7 FUZZY LINEAR PROGRAMMING

The *classical linear programming problem* is to find the minimum or maximum values of a linear function under constraints represented by linear inequalities or equations. The most typical linear programming problem is:

$$\begin{aligned}
&\text{Minimize (or maximize) } c_1x_1 + c_2x_2 + \ldots + c_nx_n\\
&\text{Subject to} \qquad a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1\\
&\qquad\qquad\qquad a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2\\
&\qquad\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots\\
&\qquad\qquad\qquad a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m\\
&\qquad\qquad\qquad x_1, x_2, \ldots, x_n \ge 0.
\end{aligned}$$

The function to be minimized (or maximized) is called an *objective function*; let us denote it by z. The numbers c_i ($i \in \mathbb{N}_n$) are called cost coefficients, and the vector $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$ is called a *cost vector*. The matrix $\mathbf{A} = [a_{ij}]$, where $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$, is called a *constraint matrix*, and the vector $\mathbf{b} = \langle b_1, b_2, \ldots, b_m \rangle^T$ is called a *right-hand-side vector*. Using this notation, the formulation of the problem can be simplified as

$$\begin{aligned}
\text{Min} \quad & z = \mathbf{cx}\\
\text{s.t.} \quad & \mathbf{Ax} \le \mathbf{b}\\
& \mathbf{x} \ge 0,
\end{aligned} \tag{15.18}$$

where $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle^T$ is a *vector of variables*, and s.t. stands for "subject to." The set of vectors $\mathbf{x}$ that satisfy all given constraints is called a *feasible set*. As is well known, many practical problems can be formulated as linear programming problems.

Example 15.7

To illustrate the spirit of classical linear programming, let us consider a simple example:

$$\begin{aligned}
\text{Min} \quad & z = x_1 - 2x_2\\
\text{s.t.} \quad & 3x_1 - x_2 \ge 1\\
& 2x_1 + x_2 \le 6\\
& 0 \le x_2 \le 2\\
& 0 \le x_1
\end{aligned}$$

Using Fig. 15.6 as a guide, we can show graphically how the solution of this linear programming problem can be obtained. First, we need to determine the feasible set. Employing an obvious geometrical interpretation, the feasible set is obtained in Fig. 15.6 by drawing straight lines representing the equations $x_1 = 0$, $x_2 = 0$, $x_2 = 2$, $3x_1 - x_2 = 1$, and $2x_1 + x_2 = 6$. These straight lines, each of which constrains the whole plane into a half-plane, express the five inequalities in our example. When we take the intersection of the five-half planes, we obtain the shaded area in Fig. 15.6, which represents the feasible set. This area is always a convex polygon.

To find the minimum of the objective function z within the feasible set, we can draw a family of parallel straight lines representing the equation $x_1 - 2x_2 = p$, where p is a parameter, and observe the direction in which p decreases. Then, we can imagine a straight line parallel to the others moving in that direction until it touches either an edge or a vertex of the convex polygon. At that point, the value of parameter p is the minimum value of the objective function z. If the requirement were to maximize the objective function, we would move the line in the opposite direction, the direction in which p increases.

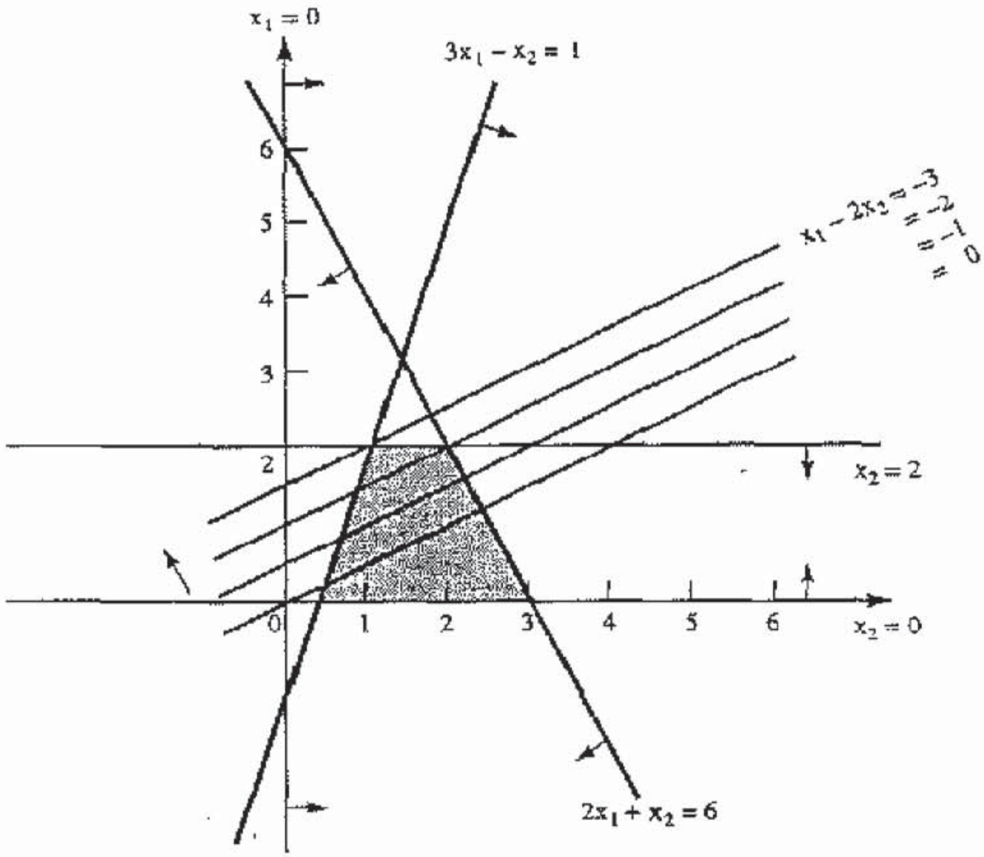


Figure 15.6 An example of a classical linear programming problem.

In many practical situations, it is not reasonable to require that the constraints or the objective function in linear programming problems be specified in precise, crisp terms. In such situations, it is desirable to use some type of fuzzy linear programming.

The most general type of fuzzy linear programming is formulated as follows:

$$\max \sum_{j=1}^{n} C_j X_j$$

$$\text{s.t.} \sum_{j=1}^{n} A_{ij} X_j \leq B_i \quad (i \in \mathbb{N}_m) \tag{15.19}$$

$$X_j \geq 0 \quad (j \in \mathbb{N}_n),$$

where A_{ij}, B_i, C_j are fuzzy numbers, and X_j are variables whose states are fuzzy numbers ($i \in \mathbb{N}_m$, $j \in \mathbb{N}_n$); the operations of addition and multiplication are operations of fuzzy arithmetic, and $\leq$ denotes the ordering of fuzzy numbers. Instead of discussing this general type, we exemplify the issues involved by two special cases of fuzzy linear programming problems.

Case 1. Fuzzy linear programming problems in which only the right-hand-side numbers B_i are fuzzy numbers:

$$\begin{aligned} \max \ & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \ & \sum_{j=1}^{n} a_{ij} x_j \le B_i \quad (i \in \mathbb{N}_m) \\ & x_j \ge 0 \quad (j \in \mathbb{N}_n). \end{aligned} \tag{15.20}$$

Case 2. Fuzzy linear programming problems in which the right-hand-side numbers B_i and the coefficients A_{ij} of the constraint matrix are fuzzy numbers:

$$\begin{aligned} \max \ & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \ & \sum_{j=1}^{n} A_{ij} x_j \le B_i \quad (i \in \mathbb{N}_m) \\ & x_j \ge 0 \quad (j \in \mathbb{N}_n). \end{aligned} \tag{15.21}$$

In general, fuzzy linear programming problems are first converted into equivalent crisp linear or nonlinear problems, which are then solved by standard methods. The final results of a fuzzy linear programming problem are thus real numbers, which represent a compromise in terms of the fuzzy numbers involved.

Let us discuss now fuzzy linear programming problems of type (15.20). In this case, fuzzy numbers $B_i (i \in \mathbb{N}_m)$ typically have the form

$$B_i(x) = \begin{cases} 1 & \text{when } x \le b_i \\ \dfrac{b_i + p_i - x}{p_i} & \text{when } b_i < x < b_i + p_i \\ 0 & \text{when } b_i + p_i \le x, \end{cases}$$

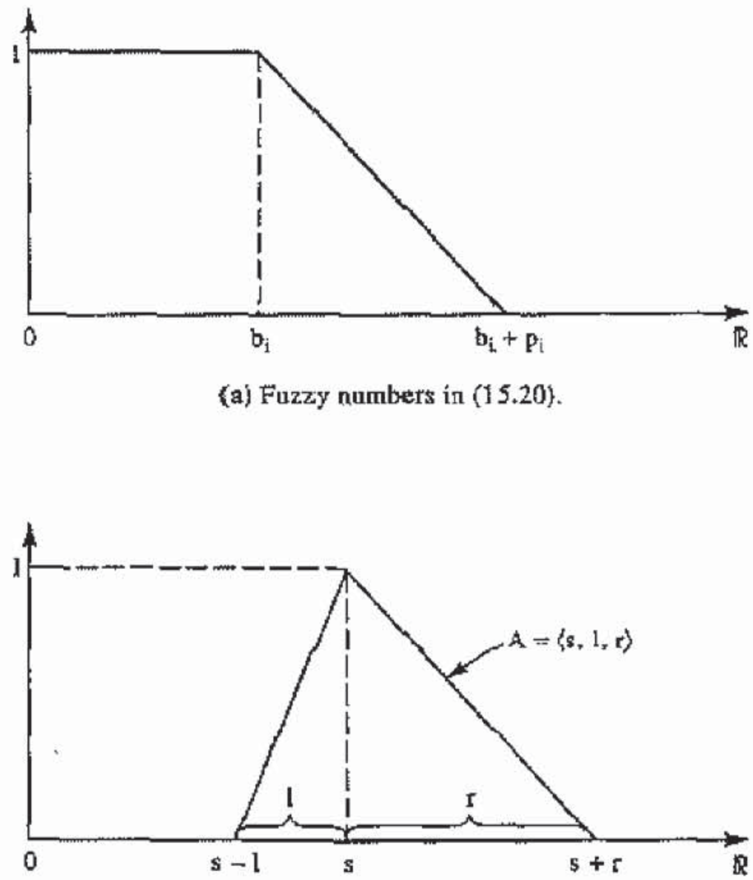
where $x \in \mathbb{R}$ (Fig. 15.7a). For each vector $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we first calculate the degree, $D_i(\mathbf{x})$, to which $\mathbf{x}$ satisfies the ith constraint $(i \in \mathbb{N}_m)$ by the formula

$$D_i(\mathbf{x}) = B_i(\sum_{j=1}^{n} a_{ij} x_j).$$

These degrees are fuzzy sets on $\mathbb{R}^n$, and their intersection, $\bigcap_{i=1}^{m} D_i$, is a *fuzzy feasible set.*

Next, we determine the fuzzy set of optimal values. This is done by calculating the lower and upper bounds of the optimal values first. The lower bound of the optimal values, z_l, is obtained by solving the standard linear programming problem:

$$\begin{aligned} \max \ & z = \mathbf{cx} \\ \text{s.t.} \ & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad (i \in \mathbb{N}_m) \\ & x_j \ge 0 \quad (j \in \mathbb{N}_n); \end{aligned}$$



(a) Fuzzy numbers in (15.20).

(b) Triangular fuzzy numbers employed in (15.21).

Figure 15.7 Types of fuzzy numbers employed in fuzzy linear programming problem: (a) fuzzy number in (15.20); (b) triangular fuzzy numbers employed in (15.21).

the upper bound of the optimal values, z_u, is obtained by a similar linear programming problem in which each b_i is replaced with $b_i + p_i$:

$$\begin{aligned} &\max \ \ z = \mathbf{cx} \\ &\text{s.t.} \ \ \sum_{j=1}^{n} a_{ij}x_j \leq b_i + p_i \quad (i \in \mathbb{N}_m) \\ &\qquad x_j \geq 0 \quad (j \in \mathbb{N}_n). \end{aligned}$$

Then, the fuzzy set of optimal values, G, which is a fuzzy subset of $\mathbb{R}^n$, is defined by

$$G(\mathbf{x}) = \begin{cases} 1 & \text{when } z_u \leq \mathbf{cx} \\ \dfrac{\mathbf{cx} - z_l}{z_u - z_l} & \text{when } z_l \leq \mathbf{cx} \leq z_u \\ 0 & \text{when } \mathbf{cx} \leq z_l. \end{cases}$$

Now, the problem (15.20) becomes the following classical optimization problem:

$$\begin{aligned} &\max \ \ \lambda \\ &\text{s.t.} \ \ \lambda(z_u - z_l) - \mathbf{cx} \leq -z_l \\ &\qquad \lambda p_i + \sum_{j=1}^{n} a_{ij}x_j \leq b_i + p_i \quad (i \in \mathbb{N}_m) \\ &\qquad \lambda, x_j \geq 0 \quad (j \in \mathbb{N}_n). \end{aligned}$$

The above problem is actually a problem of finding $\mathbf{x} \in \mathbb{R}^n$ such that

$$[(\bigcap_{i=1}^{m} D_i) \cap G](\mathbf{x})$$

reaches the maximum value; that is, a problem of finding a point which satisfies the constraints and goal with the maximum degree. As discussed in Sec. 15.2, this idea is due to Bellman and Zadeh [1970]. The method employed here is called a *symmetric method* (i.e., the constraints and the goal are treated symmetrically). There are also nonsymmetric methods. The following example illustrates the described method.

Example 15.8

Assume that a company makes two products. Product P_1 has a \$0.40 per unit profit and product P_2 has a \$0.30 per unit profit. Each unit of product P_1 requires twice as many labor hours as each product P_2. The total available labor hours are at least 500 hours per day, and may possibly be extended to 600 hours per day, due to special arrangements for overtime work. The supply of material is at least sufficient for 400 units of both products, P_1 and P_2, per day, but may possibly be extended to 500 units per day according to previous experience. The problem is, how many units of products P_1 and P_2 should be made per day to maximize the total profit?

Let x_1, x_2 denote the number of units of products P_1, P_2 made in one day, respectively. Then the problem can be formulated as the following fuzzy linear programming problem:

$$\begin{aligned} \max \quad & z = .4x_1 + .3x_2 \ \text{(profit)} \\ \text{s.t.} \quad & x_1 + x_2 \leq B_1 \quad \text{(material)} \\ & 2x_1 + x_2 \leq B_2 \quad \text{(labor hours)} \\ & x_1, x_2 \geq 0, \end{aligned}$$

where B_1 is defined by

$$B_1(x) = \begin{cases} 1 & \text{when } x \leq 400 \\ \dfrac{500 - x}{100} & \text{when } 400 < x \leq 500 \\ 0 & \text{when } 500 < x, \end{cases}$$

and B_2 is defined by

$$B_2(x) = \begin{cases} 1 & \text{when } x \leq 500 \\ \dfrac{600 - x}{100} & \text{when } 500 < x \leq 600 \\ 0 & \text{when } 600 < x. \end{cases}$$

First we need to calculate the lower and upper bounds of the objective function. By solving the following two classical linear programming problems, we obtain $z_l = 130$ and $z_u = 160$.

$$\begin{aligned} (P_1) \ \max \quad & z = .4x_1 + .3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 400 \\ & 2x_1 + x_2 \leq 500 \\ & x_1, x_2 \geq 0. \end{aligned}$$

$$\begin{aligned} (P_2) \ \max \quad & z = .4x_1 + .3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 500 \\ & 2x_1 + x_2 \leq 600 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Then, the fuzzy linear programming problem becomes:

$$\begin{aligned} \max\ & \lambda \\ \text{s.t.}\ & 30\lambda - (.4x_1 + .3x_2) \leq -130 \\ & 100\lambda + x_1 + x_2 \leq 500 \\ & 100\lambda + 2x_1 + x_2 \leq 600 \\ & x_1, x_2, \lambda \geq 0. \end{aligned}$$

Solving this classical optimization problem, we find that the maximum, $\lambda = 0.5$, is obtained for $\hat{x}_1 = 100, \hat{x}_2 = 350$. The maximum profit, $\hat{z}$, is then calculated by

$$\hat{z} = .4\hat{x}_1 + .3\hat{x}_2 = 145.$$

Let us consider now the more general problem of fuzzy linear programming defined by (15.21). In this case, we assume that all fuzzy numbers are triangular. Any triangular fuzzy number A can be represented by three real numbers, s, l, r, whose meanings are defined in Fig. 15.7b. Using this representation, we write $A = \langle s, l, r\rangle$. Problem (15.21) can then be rewritten as

$$\begin{aligned} \max\ & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.}\ & \sum_{j=1}^{n} \langle s_{ij}, l_{ij}, r_j\rangle x_{ij} \leq \langle t_i, u_i, v_i\rangle \quad (i \in \mathbb{N}_m) \\ & x_j \geq 0 \quad (j \in \mathbb{N}_n), \end{aligned}$$

where $A_{ij} = \langle s_{ij}, l_{ij}, r_{ij}\rangle$ and $B_i = \langle t_i, u_i, v_i\rangle$ are fuzzy numbers. Summation and multiplication are operations on fuzzy numbers, and the partial order $\leq$ is defined by $A \leq B$ iff $\mathrm{MAX}(A, B) = B$. It is easy to prove that for any two triangular fuzzy numbers $A = \langle s_1, l_1, r_1\rangle$ and $B = \langle s_2, l_2, r_2\rangle$, $A \leq B$ iff $s_1 \leq s_2, s_1 - l_1 \leq s_2 - l_2$ and $s_1 + r_1 \leq s_2 + r_2$. Moreover, $\langle s_1, l_1, r_1\rangle + \langle s_2, l_2, r_2\rangle = \langle s_1+s_2, l_1+l_2, r_1+r_2\rangle$ and $\langle s_1, l_1, r_1\rangle x = \langle s_1x, l_1x, r_1x\rangle$ for any nonnegative real number x. Then, the problem can be rewritten as

$$\begin{aligned} \max\ & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.}\ & \sum_{j=1}^{n} s_{ij} x_j \leq t_i \\ & \sum_{j=1}^{n} (s_{ij} - l_{ij}) x_j \leq t_i - u_i \\ & \sum_{j=1}^{n} (s_{ij} + r_{ij}) x_j \leq t_i + v_i \quad (i \in \mathbb{N}_m) \\ & x_j \geq 0 \quad (j \in \mathbb{N}_n). \end{aligned}$$

However, since all numbers involved are real numbers, this is a classical linear programming problem.

Example 15.9

Consider the following fuzzy linear programming problem:

$$\begin{aligned} \max\ & z = 5x_1 + 4x_2 \\ \text{s.t.}\ & \langle 4, 2, 1\rangle x_1 + \langle 5, 3, 1\rangle x_2 \leq \langle 24, 5, 8\rangle \\ & \langle 4, 1, 2\rangle x_1 + \langle 1, .5, 1\rangle x_2 \leq \langle 12, 6, 3\rangle \\ & x_1, x_2 \geq 0. \end{aligned}$$

We can rewrite it as

$$\begin{aligned} \max\ & z = 5x_1 + 4x_2 \\ \text{s.t.}\ & 4x_1 + 5x_2 \leq 24 \\ & 4x_1 + x_2 \leq 12 \\ & 2x_1 + 2x_2 \leq 19 \\ & 3x_1 + 0.5x_2 \leq 6 \\ & 5x_1 + 6x_2 \leq 32 \\ & 6x_1 + 2x_2 \leq 15 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Solving this problem, we obtain $\hat{x}_1 = 1.5$, $\hat{x}_2 = 3$, $\hat{z} = 19.5$.

Notice that if we defuzzified the fuzzy numbers in the constraints of the original problem by the maximum method, we would obtain another classical linear programming problem:

$$\begin{aligned} \max\ & z = 5x_1 + 4x_2 \\ \text{s.t.}\ & 4x_1 + 5x_2 \leq 24 \\ & 4x_1 + x_2 \leq 12 \\ & x_1, x_2 \geq 0. \end{aligned}$$

We can see that this is a classical linear programming problem with a smaller number of constraints than the one converted from a fuzzy linear programming problem. Therefore, fuzziness in (15.21) results in stronger constraints, while fuzziness in (15.20) results in weaker constraints.

NOTES

15.1. The classical paper by Bellman and Zadeh [1970] is a rich source of ideas regarding fuzzy decision making and certainly worth reading; it is also reprinted in [Yager *et al.*, 1987]. An early book on fuzzy decision making was written by Kickert [1978]; although it is not fully up to date, this book is still pedagogically the best comprehensive introduction to the subject. A deeper and more up-to-date introduction to fuzzy decision making was written by Zimmermann [1987]. A rich source of information on various aspects of fuzzy decision making, which consists of 30 properly selected articles, was prepared by Zimmermann *et al.* [1984]. An approach to decision making (both fuzzy and crisp) based on binary relations that encode pairwise preferences is systematically investigated by Kitainik [1993].

15.2. For information on the various issues of multiperson fuzzy decision making, we recommend the book edited by Kacprzyk and Fedrizzi [1990].

15.3. Literature on multicriteria fuzzy decision making is extensive. Two important monographs on the subject, written by Hwang and Yoon [1981] and Chen and Hwang [1992], are now available. They cover the subject in great detail and provide the reader with a comprehensive overview of relevant literature. For a systematic study, they should be read in the given order. Another important source on the subject, focusing on the comparison between fuzzy and stochastic approaches, is the book edited by Slowinski and Teghem [1990].

15.4. Literature dealing with multistage fuzzy decision making is rather restricted. The book written by Kacprzyk [1983] is undoubtedly the most important source. Another notable reference is the paper by Baldwin and Pilsworth [1982].

15.5. The problem of fuzzy ranking or, more generally, ordering of fuzzy sets defined on $\mathbb{R}$ has been discussed in the literature quite extensively. The discussion is still ongoing. The following are some major representative references or this subject: Baas and Kwakernaak [1977], Efstathiou, and Tong [1982], Dubois and Prade [1983], Bortolan and Degani [1985], and Saade and Schwarzlander [1992]. A good overview of fuzzy ranking methods is in Chapter IV of the book by Chen and Hwang [1992].

15.6. Fuzzy linear programming is covered in the literature quite extensively. The book by Lai and Hwang [1992] is a comprehensive overview of this subject as well as relevant literature.

EXERCISES

15.1. Consider five travel packages a_1, a_2, a_3, a_4, a_5, from which we want to choose one. Their costs are \$1,000, \$3,000, \$10,000, \$5,000, and \$7,000, respectively. Their travel times in hours are 15, 10, 28, 10, and 15, respectively. Assume that they are viewed as interesting with the degrees 0.4, 0.3, 1, 0.6, 0.5, respectively. Define your own fuzzy set of acceptable costs and your own fuzzy set of acceptable travel times. Then, determine the fuzzy set of interesting travel packages whose costs and travel times are acceptable, and use this set to choose one of the five travel packages.

15.2. Repeat Exercise 15.1 under the assumption that the importance of cost, travel time, and interest are expressed by the weights of .6, .1, and .3, respectively.

15.3. Assume that each individual of a group of five judges has a total preference ordering $\mathbf{P}_i (i \in \mathbb{N}_5)$ on four figure skaters a, b, c, d. The orderings are: $\mathbf{P}_1 = \langle a, b, d, c\rangle, \mathbf{P}_2 = \langle a, c, d, b\rangle, \mathbf{P}_3 = \langle b, a, c, d\rangle, \mathbf{P}_4 = \langle a, d, b, c\rangle$. Use fuzzy multiperson decision making to determine the group decision.

15.4. Employ the fuzzy multicriteria decision-making method described in Sec. 15.4 for solving Exercise 15.2.

15.5. Repeat Example 15.5 for the fuzzy goal

$$C^2 = .8/z_1 + 1/z_2 + .9/z_3$$

and the fuzzy constraints

$$A^0 = .8/x_1 + 1/x_2 \text{ and } A^1 = 1/x_1 + .7/x_2.$$

15.6. Let A be a symmetric trapezoidal-type fuzzy number with ${}^{0+}A = [0, 4]$ and ${}^{1}A = [1, 3]$, and let B, C be symmetric triangular-type fuzzy numbers with centers $c_B = 4$, $c_C = 5$, and spreads $s_B = s_C = 2$. Rank these fuzzy numbers by each of the three ranking methods described in Sec. 15.6.

15.7. Solve the following fuzzy linear programming problems.

(a)

$$\begin{aligned} \max \quad & z = .5x_1 + .2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le B_1 \\ & 2x_1 + x_2 \le B_2 \\ & x_1, x_2 \ge 0, \end{aligned}$$

where

$$B_1(x) = \begin{cases} 1 & \text{for } x \le 300 \\ \dfrac{400 - x}{100} & \text{for } 300 < x \le 400 \\ 0 & \text{for } x > 400 \end{cases}$$

and

$$B_2(x) = \begin{cases} 1 & \text{for } x \le 400 \\ \dfrac{500 - x}{100} & \text{for } 400 < x \le 500 \\ 0 & \text{for } x > 500 \end{cases}.$$

(b)

$$\begin{aligned} \max \quad & z = 6x_1 + 5x_2 \\ \text{s.t.} \quad & \langle 5, 3, 2\rangle x_1 + \langle 6, 4, 2\rangle x_2 \le \langle 25, 6, 9\rangle \\ & \langle 5, 2, 3\rangle x_1 + \langle 2, 1.5, 1\rangle x_2 \le \langle 13, 7, 4\rangle \\ & x_1, x_2 \ge 0. \end{aligned}$$