Digital Image Processing

Image Restoration

Matrix Formulation

Matrix Formulation of Image Restoration Problem

1-D Case:

- We will consider the 1-D version first, for simplicity:

$$g(m) = f(m)*h(m) + \eta(m)$$

- We will assume that the arrays *f* and *h* have been zero-padded to be of size *M*, where $M \geq$ length(*f*) + length(*h*) - 1.
- Henceforth, we will not explicitly mention the zero-padding.
- The degradation equation: $$g(m) = \sum_{k=0}^{M-1} f(k)h(m-k) + \eta(m)$$

  can be written in matrix-vector form as follows:

$$\mathbf{g} = \mathbf{Hf} + \mathbf{n}, \text{ where}$$

$$\mathbf{g} = \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(M-1) \end{bmatrix} \quad \mathbf{f} = \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(M-1) \end{bmatrix} \quad \mathbf{n} = \begin{bmatrix} \eta(0) \\ \eta(1) \\ \vdots \\ \eta(M-1) \end{bmatrix}$$

Matrix Formulation of Image Restoration Problem

$$H = H_1 = \begin{bmatrix} h(0) & h(-1) & h(-2) & \cdots & h(-M+1) \\ h(1) & h(0) & h(-1) & \cdots & h(-M+2) \\ h(2) & h(1) & h(0) & \cdots & h(-M+3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h(M-1) & h(M-2) & h(M-3) & \cdots & h(0) \end{bmatrix}$$

$$= \begin{bmatrix} h(0) & 0 & 0 & \cdots & 0 \\ h(1) & h(0) & 0 & \cdots & 0 \\ h(2) & h(1) & h(0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h(M-1) & h(M-2) & h(M-3) & \cdots & h(0) \end{bmatrix}$$

Matrix Formulation of Image Restoration Problem

- However, since the arrays *f* and *h* are zero-padded, we can equivalently set:

$$\mathrm{H} = \mathrm{H}_2 = \begin{bmatrix} h(0) & h(M-1) & h(M-2) & \cdots & h(1) \\ h(1) & h(0) & h(M-1) & \cdots & h(2) \\ h(2) & h(1) & h(0) & \cdots & h(3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h(M-1) & h(M-2) & h(M-3) & \cdots & h(0) \end{bmatrix}$$

- Notice that the (second) matrix **H** is circulant; i.e., each row of **H** is a circular shift of the previous row.

Matrix Formulation of Image Restoration Problem

- **Example**:
  - A = length of array f = 3
  - B = length of array h = 2
  - $M \geq A + B - 1 = 4$, say $M = 4$.

$$\mathrm{f} = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ 0 \end{bmatrix} \qquad \mathrm{h} = \begin{bmatrix} h(0) \\ h(1) \\ 0 \\ 0 \end{bmatrix}$$

Matrix Formulation of Image Restoration Problem

$$H_1 = \begin{bmatrix} h(0) & h(-1) & h(-2) & h(-3) \\ h(1) & h(0) & h(-1) & h(-2) \\ h(2) & h(1) & h(0) & h(-1) \\ h(3) & h(2) & h(1) & h(0) \end{bmatrix} =$$

$$= \begin{bmatrix} h(0) & 0 & 0 & 0 \\ h(1) & h(0) & 0 & 0 \\ h(2) & h(1) & h(0) & 0 \\ h(3) & h(2) & h(1) & h(0) \end{bmatrix} = \begin{bmatrix} h(0) & 0 & 0 & 0 \\ h(1) & h(0) & 0 & 0 \\ 0 & h(1) & h(0) & 0 \\ 0 & 0 & h(1) & h(0) \end{bmatrix}$$

Matrix Formulation of Image Restoration Problem

$$H_2 = \begin{bmatrix} h(0) & h(M-1) & h(M-2) & h(M-3) \\ h(1) & h(0) & h(M-1) & h(M-2) \\ h(2) & h(1) & h(0) & h(M-1) \\ h(3) & h(2) & h(1) & h(0) \end{bmatrix} =$$

$$= \begin{bmatrix} h(0) & h(3) & h(2) & h(1) \\ h(1) & h(0) & h(3) & h(2) \\ h(2) & h(1) & h(0) & h(3) \\ h(3) & h(2) & h(1) & h(0) \end{bmatrix} = \begin{bmatrix} h(0) & 0 & 0 & h(1) \\ h(1) & h(0) & 0 & 0 \\ 0 & h(1) & h(0) & 0 \\ 0 & 0 & h(1) & h(0) \end{bmatrix}$$

Notice that $\mathbf{H}_1\mathbf{f} = \mathbf{H}_2\mathbf{f}$. Indeed

Matrix Formulation of Image Restoration Problem

$$\mathrm{H}_1\mathrm{f} = \begin{bmatrix} h(0) & 0 & 0 & 0 \\ h(1) & h(0) & 0 & 0 \\ 0 & h(1) & h(0) & 0 \\ 0 & 0 & h(1) & h(0) \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ 0 \end{bmatrix}$$

$$\mathrm{H}_2\mathrm{f} = \begin{bmatrix} h(0) & 0 & 0 & h(1) \\ h(1) & h(0) & 0 & 0 \\ 0 & h(1) & h(0) & 0 \\ 0 & 0 & h(1) & h(0) \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ 0 \end{bmatrix}$$

Henceforth, we will use $\mathbf{H} = \mathbf{H}_2$, so that we can apply properties of circulant matrices to **H**.

Matrix Formulation of Image Restoration Problem

2-D Case:

- Suppose *g*, *f*, *h*, are $M \times N$ arrays (after zero-padding). The degradation equation can be written in matrix-vector format as follows:

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \mathbf{n}, \text{ where}$$

Matrix Formulation of Image Restoration Problem

$$
\mathrm{g} = \begin{bmatrix} g(0,0) \\ \vdots \\ g(0,N-1) \\ g(1,0) \\ \vdots \\ g(1,N-1) \\ \vdots \\ g(M-1,0) \\ \vdots \\ g(M-1,N-1) \end{bmatrix}_{MN\times 1}
\quad
\mathrm{f} = \begin{bmatrix} f(0,0) \\ \vdots \\ f(0,N-1) \\ f(1,0) \\ \vdots \\ f(1,N-1) \\ \vdots \\ f(M-1,0) \\ \vdots \\ f(M-1,N-1) \end{bmatrix}_{MN\times 1}
\quad
\mathrm{n} = \begin{bmatrix} \eta(0,0) \\ \vdots \\ \eta(0,N-1) \\ \eta(1,0) \\ \vdots \\ \eta(1,N-1) \\ \vdots \\ \eta(M-1,0) \\ \vdots \\ \eta(M-1,N-1) \end{bmatrix}_{MN\times 1}
$$

Matrix Formulation of Image Restoration Problem

$$H = \begin{bmatrix} H_0 & H_{M-1} & H_{M-2} & \cdots & H_1 \\ H_1 & H_0 & H_{M-1} & \cdots & H_2 \\ H_2 & H_1 & H_0 & \cdots & H_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{M-1} & H_{M-2} & H_{M-3} & \cdots & H_0 \end{bmatrix}_{MN\times MN}$$

- Note that **H** is a $MN \times MN$ block-circulant matrix with $M \times M$ blocks.

Matrix Formulation of Image Restoration Problem

- Each block $\mathbf{H}_j$ is itself an $N \times N$ circulant matrix. Indeed, the matrix $\mathbf{H}_j$ is a circulant matrix formed from the j-th row of array $h(m,n)$:

$$\mathrm{H}_j = \begin{bmatrix} h(j,0) & h(j,N-1) & h(j,N-2) & \cdots & h(j,1) \\ h(j,1) & h(j,0) & h(j,N-1) & \cdots & h(j,2) \\ h(j,2) & h(j,1) & h(j,0) & \cdots & h(j,3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h(j,N-1) & h(j,N-2) & h(j,N-3) & \cdots & h(j,0) \end{bmatrix}$$

Matrix Formulation of Image Restoration Problem

- Given the degradation equation:

$$\mathbf{g} = \mathbf{Hf} + \mathbf{n},$$

  our objective is to recover **f** from observation **g**.
- We will assume that the array $h(m,n)$ (usually referred to as the blurring function) and statistics of the noise $\eta(m,n)$ are known. The problem becomes very complicated if array $h(m,n)$ is unknown and this case is usually referred to as blind restoration or blind deconvolution.

Matrix Formulation of Image Restoration Problem

- Notice that, even when there is no noise; i.e. $\eta(m,n) = 0$, or the values of $\eta(m,n)$ were exactly known, and matrix H is invertible, computing

$$\hat{f} = \mathbf{H}^{-1}(\mathbf{g}-\mathbf{n})$$

directly would not be practical.

- **Example**: Suppose $M = N = 256$. Therefore $MN = 65536$ and **H** would be a 65536 by 65536 matrix to be inverted!

Matrix Formulation of Image Restoration Problem

- Naturally, direct inversion of **H** would not be feasible.
- But **H** has several useful properties; in particular:
  - **H** is block circulant.
  - **H** is usually sparse (has very few non-zero entries).

  We will exploit these properties to obtain $\hat{\mathbf{f}}$ more efficiently.
- In particular, we will derive the theoretical solutions to the restoration problem using matrix algebra. However, when it comes to implementing the solution, we can resort to the Fourier domain, thanks to the properties of circulant matrices.

Constrained least squares filtering (restoration)

- Recall that the knowledge of blur function $h(m, n)$ is essential to obtain a meaningful solution to the restoration problem.
- Often, knowledge of $h(m, n)$ is not perfect and subject to errors.
- One way to alleviate sensitivity of the result to errors in $h(m, n)$ is to base optimality of restoration on a measure of smoothness, such as the second derivative of the image.

Constrained least squares filtering (restoration)

- We will approximate the second derivative (Laplacian) by a matrix *Q*. Indeed, we will first formulate the constrained restoration problem and obtain its solution in terms of a general matrix *Q*.

Constrained least squares filtering (restoration)

- Later different choices of matrix Q will be considered, each giving rise to a different restoration filter.
- Suppose Q is any matrix (of appropriate dimension). In constrained image restoration, we choose $\hat{\mathbf{f}}$ to minimize $||Q\hat{\mathbf{f}}||^2$, subject to the constraint,

$$||g-H\hat{\mathbf{f}}||^2 = ||n||^2.$$

(Recall the degradation equation

$$g=H\hat{\mathbf{f}} +n \Rightarrow g-H\hat{\mathbf{f}} = \mathbf{n}.)$$

Constrained least squares filtering (restoration)

- Introduction of matrix *Q* allows considerable flexibility in the design of appropriate restoration filters (we will discuss specific choices of *Q* later). So our problem is formulated as follows:

  min $\|\mathbf{Q}\hat{\mathbf{f}}\|^2$

  subject to $\|\mathbf{g}-\mathbf{H}\hat{\mathbf{f}}\|^2 = \|\mathbf{n}\|^2$ or $\|\mathbf{g}-\mathbf{H}\hat{\mathbf{f}}\|^2 - \|\mathbf{n}\|^2 = 0$

A brief review of matrix differentiation

- Suppose

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and $f(x_1,x_2)$ is a function of two variables. Then

$$\frac{\partial f\left(x_1, x_2\right)}{\partial \mathrm{x}} = \begin{bmatrix} \dfrac{\partial f\left(x_1, x_2\right)}{\partial x_1} \\ \dfrac{\partial f\left(x_1, x_2\right)}{\partial x_2} \end{bmatrix}$$

A brief review of matrix differentiation

- If

$$f(x_1,x_2) = ||\mathbf{Ax}-\mathbf{b}||^2 = (\mathbf{Ax}-\mathbf{b})^T(\mathbf{Ax}-\mathbf{b})$$

for some matrix **A** and some vector **b** , then

$$\frac{\partial f(x_1, x_2)}{\partial \mathrm{x}} = 2\mathrm{A}^T(\mathrm{Ax}-\mathrm{b})$$

where superscript *T* denotes matrix transpose.

A brief review of matrix differentiation

- Recall from calculus that such a constrained minimization problem can be solved by means of Lagrange multipliers. We need to minimize the augmented objective function $J(\hat{\mathbf{f}})$:

  $$J(\hat{\mathbf{f}}) = \|\mathbf{Q}\hat{\mathbf{f}}\|^2 + \alpha(\|\mathbf{g}-\mathbf{H}\hat{\mathbf{f}}\|^2 - \|\mathbf{n}\|^2),$$

  where α is a Lagrange multiplier.
- We set the derivative of $J(\hat{\mathbf{f}})$ with respect to $\hat{\mathbf{f}}$ to zero.

$$\nabla J(\hat{\mathbf{f}}) = 2\mathbf{Q}^T\mathbf{Q}\hat{\mathbf{f}} - 2\alpha\mathbf{H}^T(\mathbf{g}-\mathbf{H}\hat{\mathbf{f}}) = 0$$

$$\Rightarrow (\mathbf{Q}^T\mathbf{Q} + \alpha\mathbf{H}^T\mathbf{H})\hat{\mathbf{f}} = \alpha\mathbf{H}^T\mathbf{g}$$

A brief review of matrix differentiation

- Therefore,

$$\hat{f} = (Q^T Q + \alpha H^T H)^{-1} \alpha H^T g$$
$$= (Q^T Q/\alpha + H^T H)^{-1} H^T g$$
$$= (\gamma Q^T Q + H^T H)^{-1} H^T g$$

where $\gamma = 1/\alpha$ is chosen to satisfy the constraint $||g-H\hat{f}||^2 = ||n||^2$.

- **We will now use the above formulation to derive a number of restoration filters.**

Pseudo-inverse Filtering

- The pseudo-inverse filter tries to avoid the pitfalls of applying an inverse filter in the presence of noise.
- Consider the constrained restoration solution,

$$\hat{f} = (\gamma Q^T Q + H^T H)^{-1} H^T g$$

with $Q = I$. This gives,

$$\hat{f} = (\gamma I + H^T H)^{-1} H^T g$$

Pseudo-inverse Filtering

- It can be implemented in the Fourier domain by the following equation:

  ^*F(u,v)* = *R(u,v)G(u,v)*, where

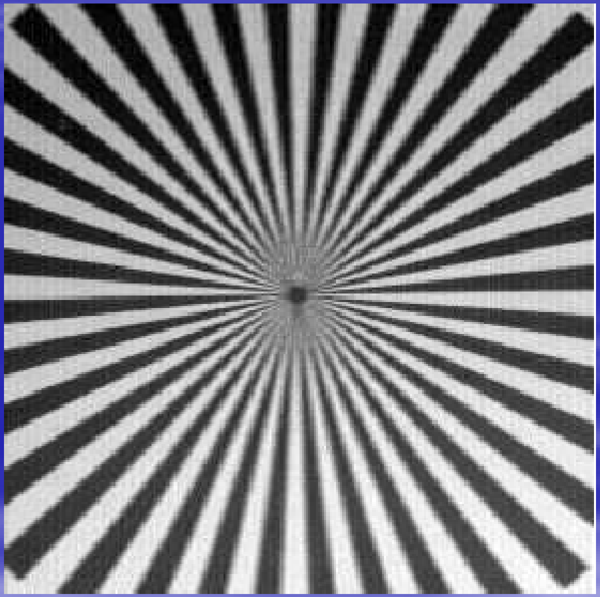
$$R(u,v)=\frac{H^*(u,v)}{|H(u,v)|^2+\gamma}=\frac{1}{H(u,v)}\left[\frac{|H(u,v)|^2}{|H(u,v)|^2+\gamma}\right]$$

- The parameter γ is a constant to be chosen.
- Note that $\gamma = 0$ gives us back the inverse filter. For $\gamma > 0$, the denominator of *R*(*u*, *v*) is strictly positive and the pseudo-inverse filter is well defined.

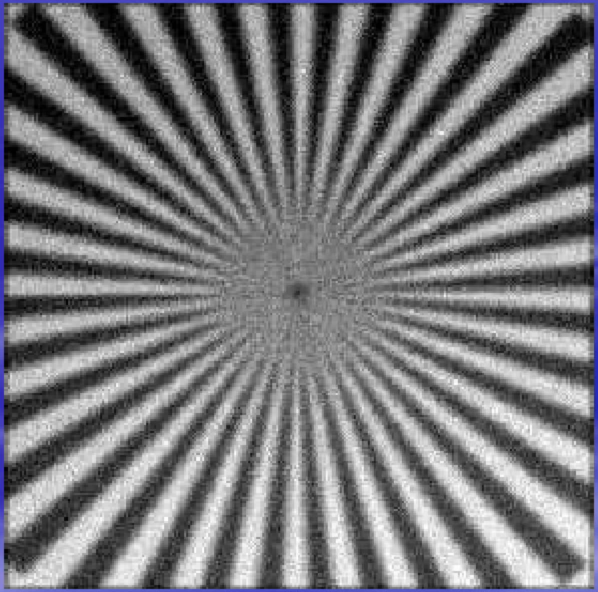
Pseudo-inverse Filtering example

Added Zero-mean Gaussian noise
with variance $\sigma^2 = 0.003$

$$h = \frac{1}{25}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



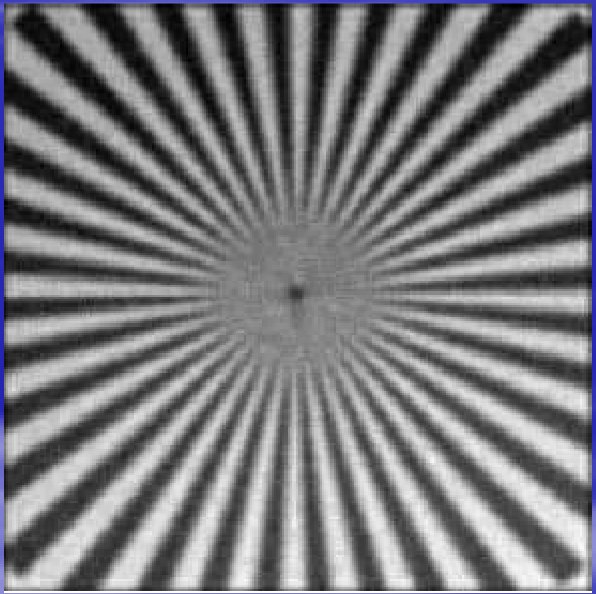
f(m,n)



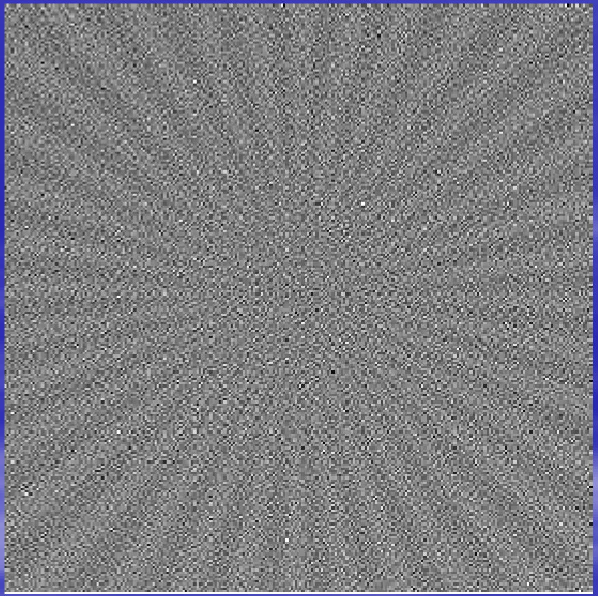
g(m,n) MSE = 0.01

Pseudo-inverse Filtering

^f(x,y)



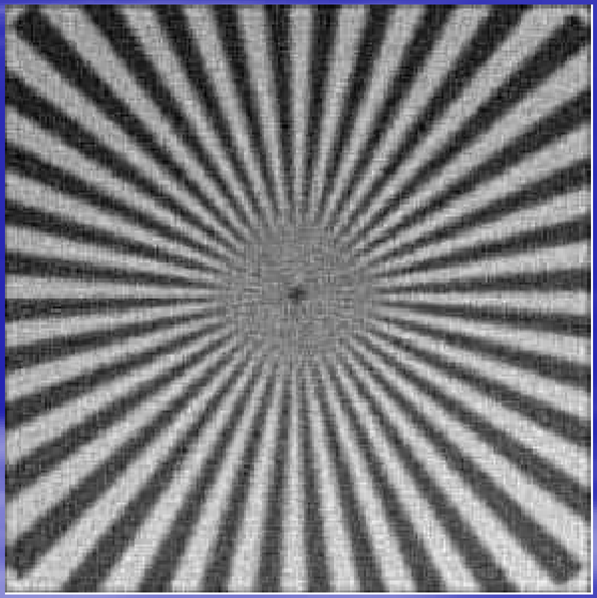
$\gamma = 0.0076$ MSE = 0.075



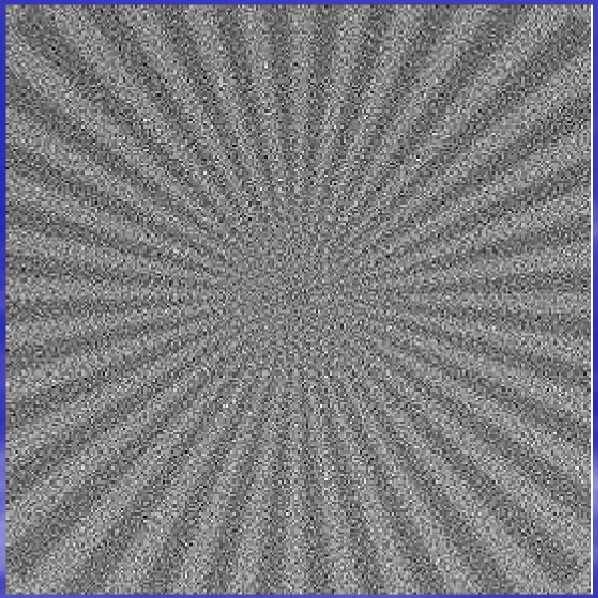
$\gamma = 0.001$ MSE = 0.3308

Pseudo-inverse Filtering

^f(x,y)



$\gamma = 0.05$ MSE = 0.0447



$\gamma = 0$ (Inv. Filter) MSE = 2.7

Minimum Mean Square Error (Wiener) Filter

- This is a restoration technique based on the statistics (mean and correlation) of the image and noise.
- We consider each element of **f** and **n** as random variables. Define the correlation matrices

$$\mathrm{R}_{\mathrm{f}} = E\left\{\mathrm{ff}^{T}\right\} = \begin{bmatrix} E\left(f_0 f_0\right) & E\left(f_0 f_1\right) & \cdots & E\left(f_0 f_{MN-1}\right) \\ E\left(f_1 f_0\right) & E\left(f_1 f_1\right) & & E\left(f_1 f_{MN-1}\right) \\ \vdots & \vdots & \ddots & \vdots \\ E\left(f_{MN-1} f_0\right) & E\left(f_{MN-1} f_1\right) & \cdots & E\left(f_{MN-1} f_{MN-1}\right) \end{bmatrix}$$

Minimum Mean Square Error (Wiener) Filter

$$\mathrm{R}_{\mathrm{n}} = E\left\{\mathrm{nn}^{T}\right\} = \begin{bmatrix} E(n_0 n_0) & E(n_0 n_1) & \cdots & E(n_0 n_{MN-1}) \\ E(n_1 n_0) & E(n_1 n_1) & & E(n_1 n_{MN-1}) \\ \vdots & \vdots & \ddots & \vdots \\ E(n_{MN-1} n_0) & E(n_{MN-1} n_1) & \cdots & E(n_{MN-1} n_{MN-1}) \end{bmatrix}$$

- The matrices $\mathbf{R_f}$ and $\mathbf{R_n}$ are real and symmetric, with all eigenvalues being non-negative.
- The 2D-DFT of the correlations $\mathbf{R_f}$ and $\mathbf{R_n}$ are called the power spectra and are denoted by $S_f(u,v)$ and $S_n(u,v)$ respectively.

Minimum Mean Square Error (Wiener) Filter

- Recall the constrained restoration solution given by

$$\hat{f} = (\gamma Q^T Q + H^T H)^{-1} H^T g$$

- Choose matrix **Q** such that

$$Q^T Q = R_f^{-1} R_n$$

- In a sense, we are trying to minimize the noise-to-signal ratio.
- The constrained restoration is then given by

$$\hat{f} = (H^T H + \gamma\, R_f^{-1} R_n)^{-1} H^T g$$

Minimum Mean Square Error (Wiener) Filter

- This can be implemented using DFT as

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \gamma\left[S_\eta(u,v)/S_f(u,v)\right]} \right] G(u,v) = R(u,v)G(u,v)$$

$$\text{where } R(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \gamma\left[S_\eta(u,v)/S_f(u,v)\right]} \right]$$

$$= \frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + \gamma\left[S_\eta(u,v)/S_f(u,v)\right]}$$

Minimum Mean Square Error (Wiener) Filter

- Here $S_f(u,v) = E(|F(u,v)|^2)$ is the power spectral density of the image $f(m, n)$ and $S_\eta(u,v) = E(|N(u,v)|^2)$ is the power spectral density of the noise h(m, n).
- The restoration filter $R(u,v)$ is called the parametric Wiener filter, with parameter γ
- *Special cases:*
  - $\gamma = 1$: *Wiener Filter*
  - $\gamma = 0$: *Inverse Filter*
  - $\gamma \neq 0,1$: *Parametric Wiener Filter*

Minimum Mean Square Error (Wiener) Filter

- According to the constrained restoration filter derived earlier, parameter g should be chosen to satisfy $||\mathbf{g}-\mathbf{H}\hat{\mathbf{f}}||^2 = ||\mathbf{n}||^2$.
- However, choice of $\gamma = 1$ yields an optimal filter in the sense of minimizing the error function $e^2 = E\{[f(m,n)-\hat{f}(m,n)]^2\}$. In other words, setting $\gamma = 1$ yields a statistically optimal restoration.
- Implementation of the parametric Wiener filter requires knowledge of the image and noise power spectra $S_f(u,v)$ and $S_n(u,v)$. In particular, we need the so called signal-to-noise ratio (SNR) $\rho(u,v) = S_f(u,v)/S_n(u,v)$

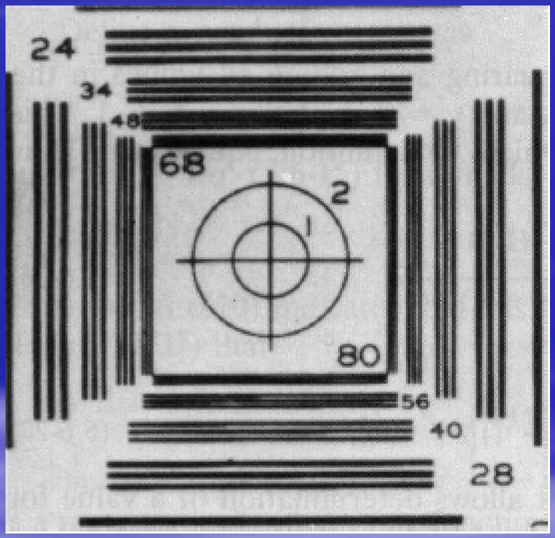
Minimum Mean Square Error (Wiener) Filter

- This is not always available and a simple approximation is to replace $\rho(u,v)$ by a constant ρ. In this case, the Wiener filter is given by

$$R(u,v) = \left[\frac{H^*(u,v)}{\left|H(u,v)\right|^2 + \dfrac{\gamma}{\rho}} \right]$$

- Note that as $\rho \to \infty$ (no noise), the Wiener filter tends to the inverse filter.

Wiener Filter example



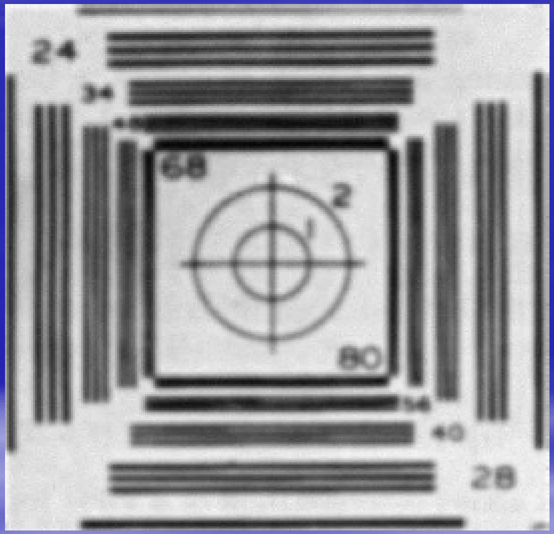
f(m,n)

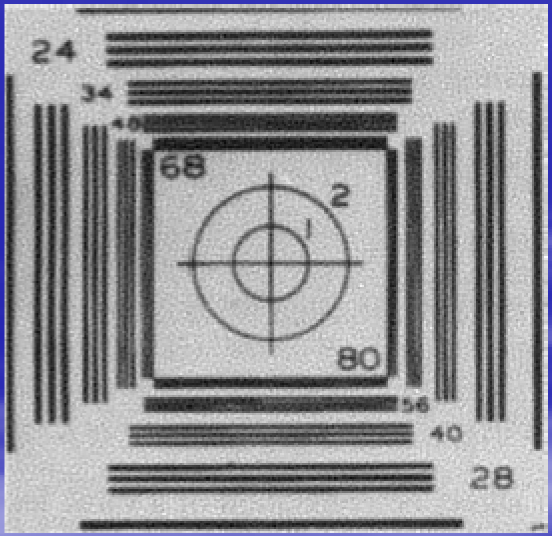
$$H(u,v) = \frac{1}{1 + \left[\frac{\sqrt{u^2 + v^2}}{r_0} \right]^2}$$

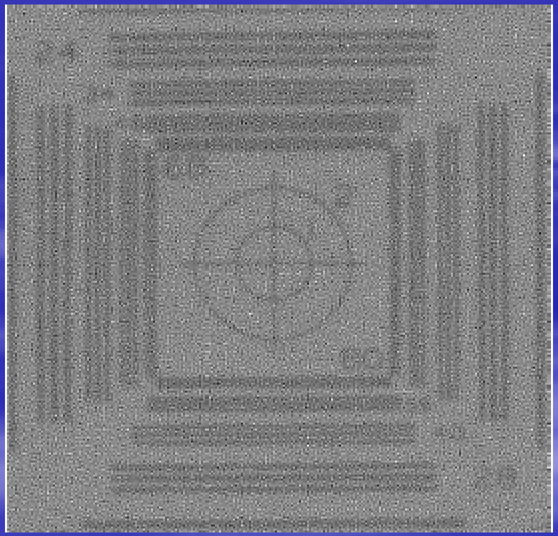
$$\rho = \frac{\sigma_f^2}{\sigma_n^2} \text{ or } 10\log\left(\frac{\sigma_f^2}{\sigma_n^2}\right) dB$$

Wiener Filter example

$\rho = 25.9dB$

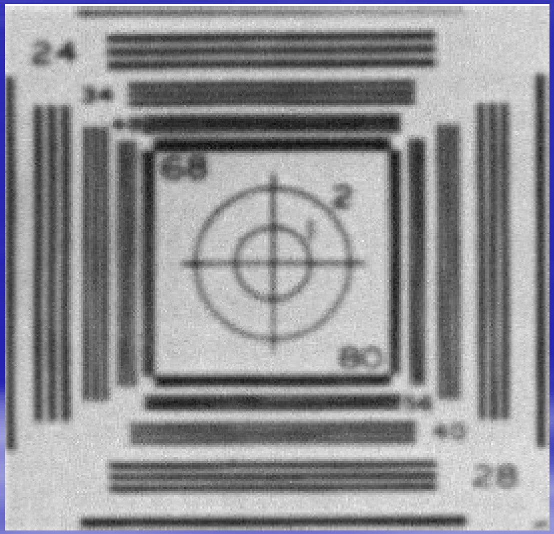


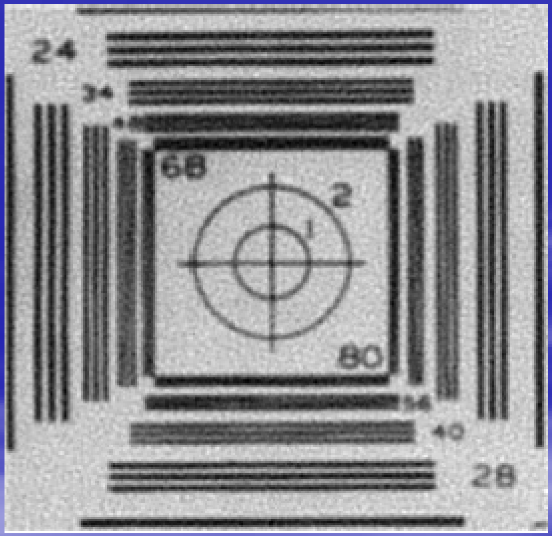


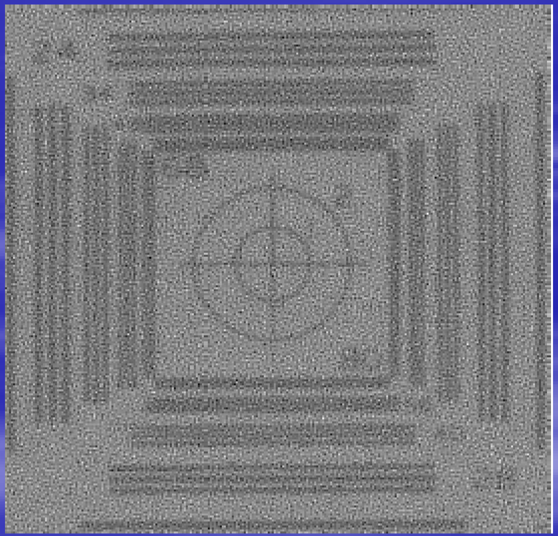


Wiener Filter example

$\rho = 15.9dB$

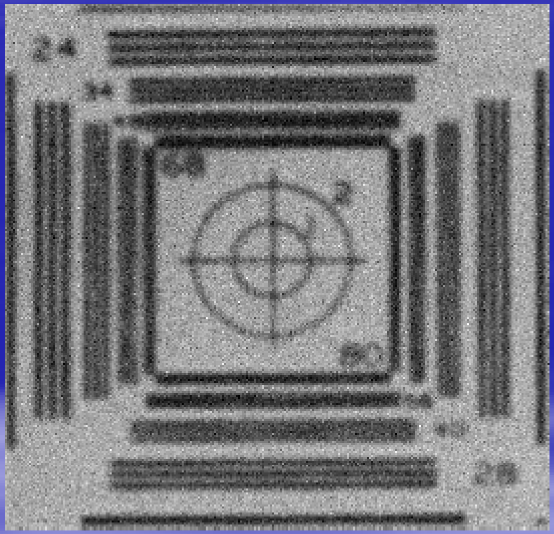


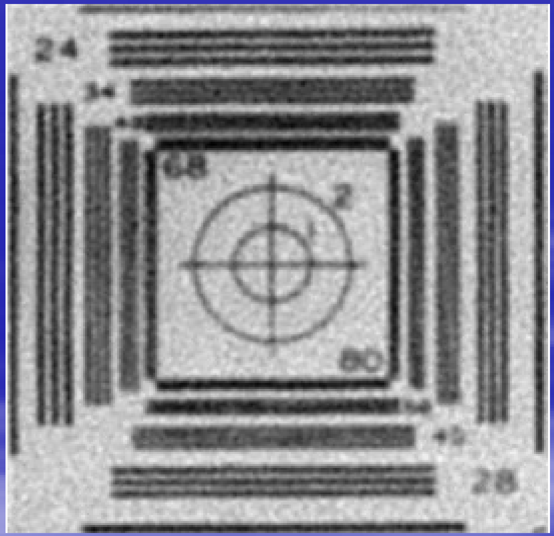


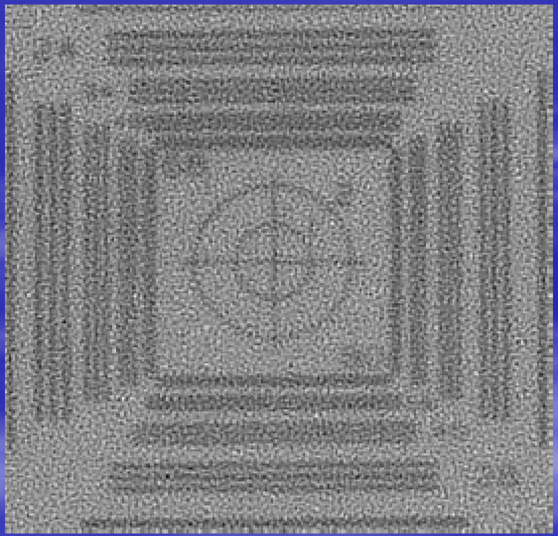


Wiener Filter example

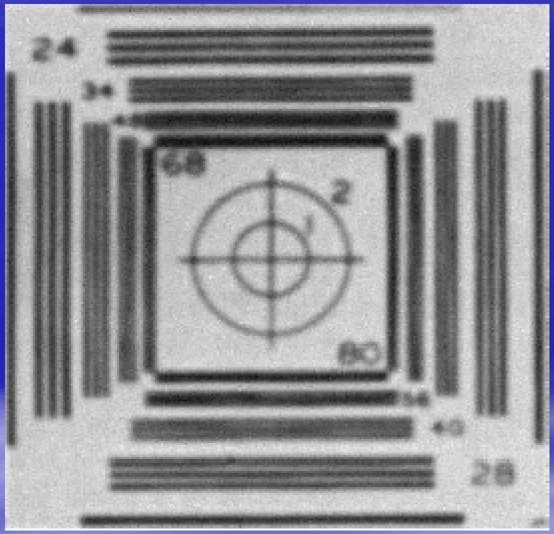
$\rho = 5.9dB$



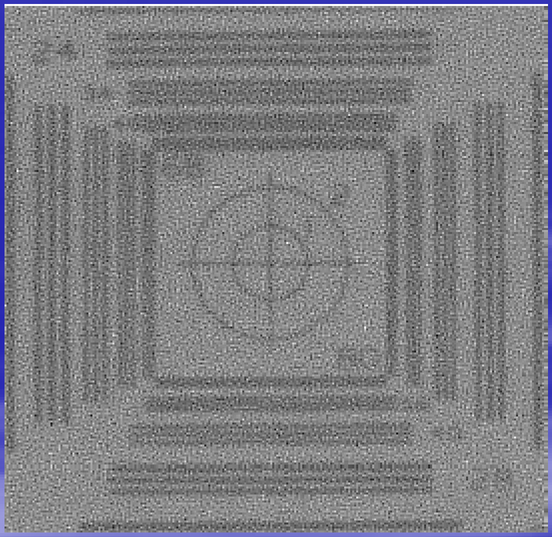




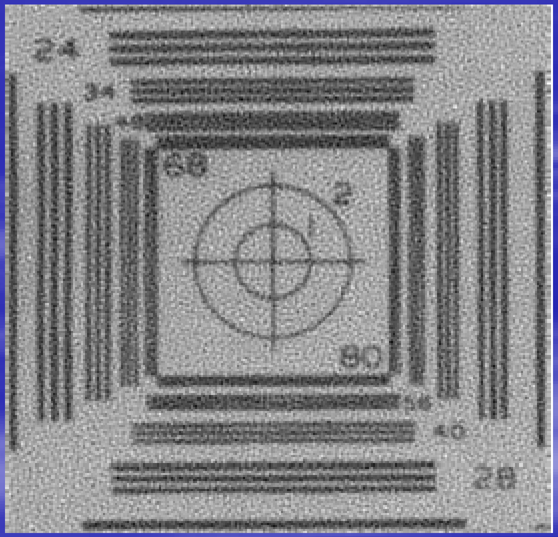
Parametric Wiener Filter example (effect of parameter γ)



g(m,n)

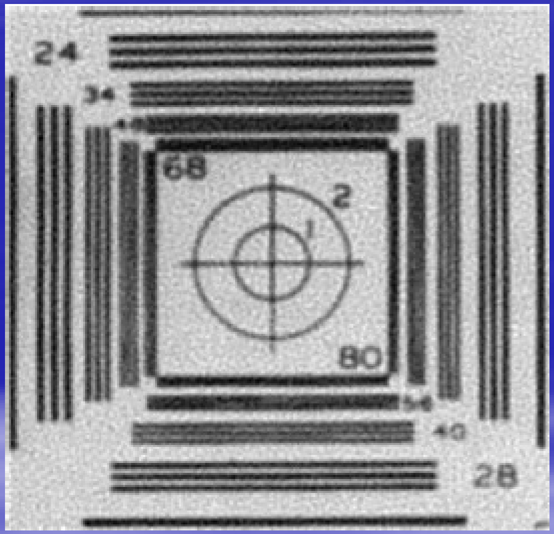


^f(m,n), g = 0.01

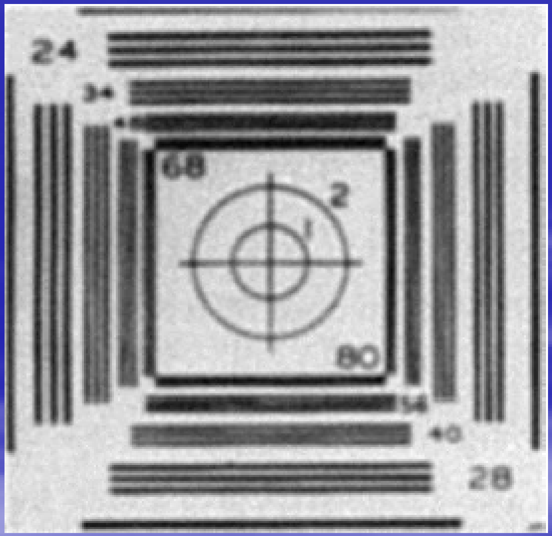


^f(m,n), g = 0.1

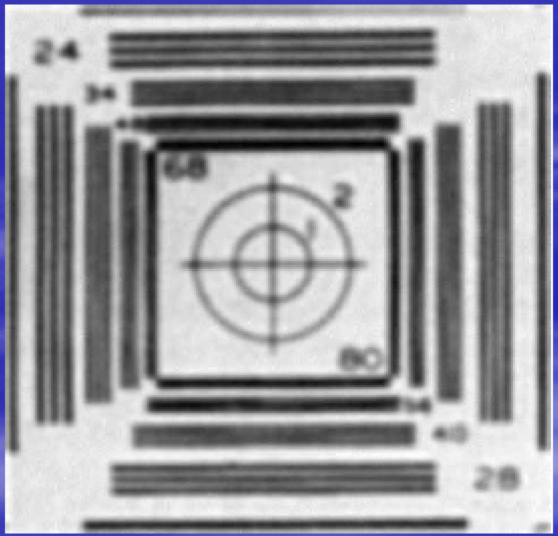
Parametric Wiener Filter example (effect of parameter γ)



$\hat{f}(m,n),\ \gamma = 1$



$\hat{f}(m,n),\ \gamma = 5$



$\hat{f}(m,n),\ \gamma = 50$

Parametric Wiener Filter example (effect of parameter γ)

- Small values of γ result in better "blur removal" and poor noise filtering.
- Large values of γ result in poor "blur removal" and better noise filtering.

Constrained Least Squares Restoration

- Recall the constrained restoration problem:

  min $||Q\hat{f}||^2$

  subject to $||\mathbf{g}-\mathbf{H}\hat{\mathbf{f}}||^2 - ||\mathbf{n}||^2 = 0$ or $||\mathbf{g}-\mathbf{H}\hat{\mathbf{f}}||^2 = ||\mathbf{n}||^2$

- Its solution

$$\hat{\mathbf{f}} = (\mathbf{H}^T\mathbf{H} + \gamma\mathbf{Q}^T\mathbf{Q})^{-1}\mathbf{H}^T\mathbf{g}$$

  depends on the choice of **Q**.

Constrained Least Squares Restoration

- One possibility is to formulate a criterion of optimality (choice of **Q**) that is based on a measure of smoothness (minimize "roughness" or oscillatory behavior of the solution).
- This is normally done by choosing **Q** to represent a second derivative of the image.

Constrained Least Squares Restoration

- Consider the 1-D case: A discrete approximation of the second derivative at a point $x = m\Delta x$ can be obtained as follows:

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=m\Delta x} \approx \frac{1}{\Delta x}\left[\left.\frac{\partial f}{\partial x}\right|_{x=m\Delta x} - \left.\frac{\partial f}{\partial x}\right|_{x=(m-1)\Delta x}\right]$$

$$\approx \frac{1}{\Delta x}\left[\frac{f((m+1)\Delta x) - f(m\Delta x)}{\Delta x} - \frac{f(m\Delta x) - f((m-1)\Delta x)}{\Delta x}\right]$$

$$\approx \left[\frac{f((m+1)\Delta x) - 2f(m\Delta x) + f((m-1)\Delta x)}{(\Delta x)^2}\right]$$

Constrained Least Squares Restoration

- In a discrete formulation with $\Delta x = 1$, this can be written as

$$\left\{ \sum_m \left[f(m+1) - 2f(m) + f(m-1) \right]^2 \right\} = \sum_m \left[f(m) * p(m) \right]^2,$$

$$\text{where } \; p(m) = \left[1 \;\; \text{-}2 \;\; 1 \right]$$

- Therefore, we seek an estimate $\hat{\mathbf{f}}$ of **f** which is smooth in the sense that it minimizes the above “roughness measure.”

Constrained Least Squares Restoration

- This can be formulated in our standard matrix notation as follows:

  min $\|C\hat{f}\|^2 = \{\hat{f}^T C^T C \hat{f}\}$

  subject to $\|\mathbf{g}-\mathbf{H}\hat{\mathbf{f}}\|^2 - \|\mathbf{n}\|^2 = 0$ or
  $\|\mathbf{g}-\mathbf{H}\hat{\mathbf{f}}\|^2 = \|\mathbf{n}\|^2$

  where (recall zero-padding)

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

Constrained Least Squares Restoration

- or equivalently

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 \\ -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \end{bmatrix}$$

is a “smoothing matrix” and ^**f** is a vector representing the restored image.

Constrained Least Squares Restoration

- In the 2D case (with $\Delta x = \Delta y = 1$), we have

$$\frac{\partial^2 f}{\partial x^2} \approx f(m+1,n) - 2f(m,n) + f(m-1,n)$$

$$\frac{\partial^2 f}{\partial y^2} \approx f(m,n+1) - 2f(m,n) + f(m,n-1)$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \approx \left[f(m+1,n) + f(m-1,n) + f(m,n+1) + f(m,n-1) \right] - 4f(m,n)$$

Constrained Least Squares Restoration

- The roughness measure can then be written as

$$\sum_{n}\sum_{m}\left[4f(m,n)-\left[f(m+1,n)+f(m-1,n)+f(m,n+1)+f(m,n-1)\right]\right]^2$$

$$=\sum_{n}\sum_{m}\left[f(m,n)*p(m,n)\right]^2$$

$$\text{where }\ p(m,n)=\begin{bmatrix}0 & -1 & 0\\ -1 & 4 & -1\\ 0 & -1 & 0\end{bmatrix}$$

Constrained Least Squares Restoration

- This can be formulated in our standard matrix notation as follows:

  $\min ||C\hat{f}||^2 = \{\hat{f}^T C^T C \hat{f}\}$

  subject to $||\mathbf{g}-\mathbf{H}\hat{\mathbf{f}}||^2 - ||\mathbf{n}||^2 = 0$ or $||\mathbf{g}-\mathbf{H}\hat{\mathbf{f}}||^2 = ||\mathbf{n}||^2$

  where (recall zero-padding):

Constrained Least Squares Restoration

$$C = \begin{bmatrix} C_0 & C_{M\text{-}1} & C_{M\text{-}2} & \cdots & C_1 \\ C_1 & C_0 & C_{M-1} & \cdots & C_2 \\ C_2 & C_1 & C_0 & \cdots & C_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{M\text{-}1} & C_{M\text{-}2} & C_{M\text{-}3} & \cdots & C_0 \end{bmatrix}$$

$$C_i = \begin{bmatrix} p(i,0) & p(i,N-1) & \cdots & p(i,1) \\ p(i,1) & p(i,0) & \cdots & p(i,2) \\ \vdots & \vdots & \ddots & \vdots \\ p(i,N-1) & p(i,N-2) & \cdots & p(i,0) \end{bmatrix}$$

is a “smoothing matrix” and ˆ**f** is a vector representing the restored image.

Constrained Least Squares Restoration

- Notice that **C** is a block circulant matrix.
- As before, the solution to the above optimization problem is given by

$$\hat{\mathbf{f}} = (\mathbf{H}^T\mathbf{H} + \gamma \mathbf{C}^T\mathbf{C})^{-1}\mathbf{H}^T\mathbf{g}$$

Constrained Least Squares Restoration

- Using properties of the block circulant matrix **C**, we get the following implementation of this filter:

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \gamma |P(u,v)|^2} \right] G(u,v) = R(u,v) G(u,v)$$

$$\text{where } R(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \gamma |P(u,v)|^2} \right]$$

Constrained Least Squares Restoration

- Here *P*(*u*,*v*) is the 2D-DFT of matrix *p*(*m*,*n*), after appropriate zeropadding.
- Compare this with the parametric Wiener filter:

$$R(u,v) = \left[\frac{H^*(u,v)}{\left|H(u,v)\right|^2 + \gamma \left[S_\eta(u,v) / S_f(u,v) \right]} \right]$$

no power spectrum information is required in the constrained leastsquares restoration!

Constrained Least Squares Restoration

- However, for the new filter to be optimal, the parameter γ must be chosen to satisfy the constraint $||\mathbf{g}-\mathbf{H}\hat{\mathbf{f}}||= ||\mathbf{n}||$.
- Define the residual vector

$$\mathbf{r} = \mathbf{g}-\mathbf{H}\hat{\mathbf{f}} = \mathbf{g} - \mathbf{H}\,(\mathbf{H}^T\mathbf{H} + \gamma\mathbf{C}^T\mathbf{C}\,)^{-1}\mathbf{H}^T\mathbf{g}$$

- Therefore, we need to choose γ such that $||r|| = ||n||$
- It can be shown that the function

$$\phi(\gamma)= r^Tr = ||r||^2$$

is a monotonically increasing function of γ.

- We want to adjust γ so that

$$\phi(\gamma) = ||r||^2 = ||n||^2 \pm a$$

for some accuracy factor *a*.

Constrained Least Squares Restoration

- Since $\phi(\gamma)$ is monotonically increasing, this can be accomplished by the following procedure:
  1. Specify an initial value of $\gamma = \gamma_0$
  2. For $k = 1, 2, \ldots,$
     - compute $\hat{\mathbf{f}}_k = (\mathbf{H}^T\mathbf{H} + \gamma_k \mathbf{C}^T\mathbf{C})^{-1}\mathbf{H}^T\mathbf{g}$. This can be done using a frequency domain implementation: $\hat{F}_k(u,v) = R_k(u,v)G(u,v)$, where

$$R_k(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \gamma_k |P(u,v)|^2} \right]$$

Constrained Least Squares Restoration

- Compute

$$\phi(\gamma_k) = \left\| \mathrm{g} - \mathrm{H}\hat{\mathrm{f}}_k \right\|^2 = \frac{1}{MN} \sum_v \sum_u \left| G(u,v) - H(u,v)\hat{F}_k(u,v) \right|^2$$

3. If $\phi(\gamma_k) < ||n||^2 - a$, $\gamma_{k+1} = \gamma_k + b$, set k=k+1, return to step 2

If $\phi(\gamma_k) > ||n||^2 + a$, $\gamma_{k+1} = \gamma_k - b$, set k=k+1, return to step 2

Otherwise, STOP (current $\hat{f}_k$ or $\hat{F}_k(u,v)$ is the restored image and γ_k is the optimal choice of parameter γ).

Constrained Least Squares Restoration

- Implementation of this procedure requires knowledge of $||n||^2$, which denotes the strength of noise.
- If

$$\bar{\eta} = E[\eta(m,n)] \text{ and } \sigma_\eta^2 = E\left[[\eta(m,n) - \bar{\eta}]^2\right] = Var[\eta(m,n)]$$

are the noise mean and variance, respectively, then

$$\bar{\eta} \approx \frac{1}{MN}\sum_{m,n}\eta(m,n) \text{ and } \sigma_\eta^2 = \frac{1}{MN}\sum_{m,n}[\eta(m,n) - \bar{\eta}]^2 = \frac{1}{MN}\|\mathrm{n}\|^2 - \bar{\eta}^2$$

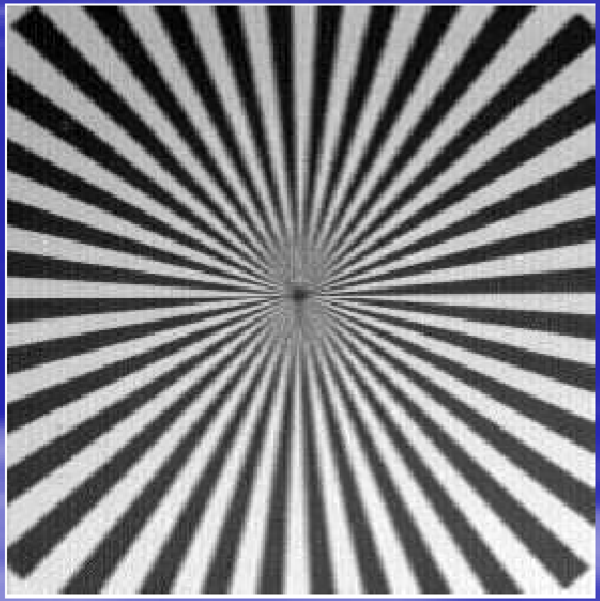
Constrained Least Squares Restoration

- Therefore

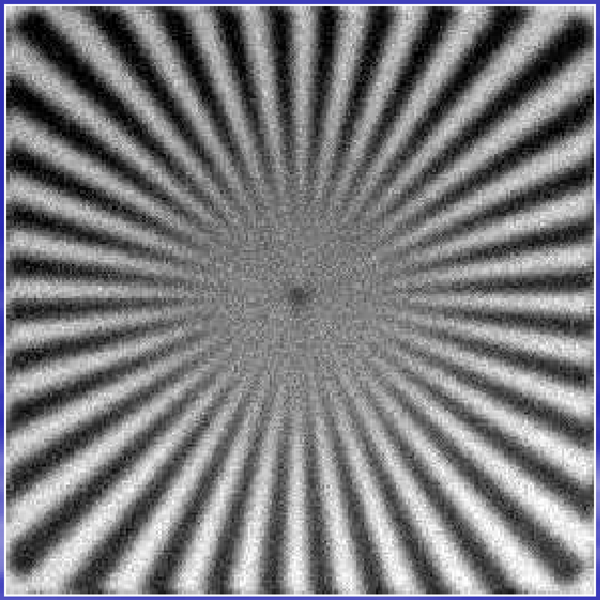
$$\|\mathrm{n}\|^2 = MN\left(\sigma_\eta^2 + \overline{\eta}^2\right)$$

which can be computed from knowledge of the mean and variance of noise.

Constrained LS Example

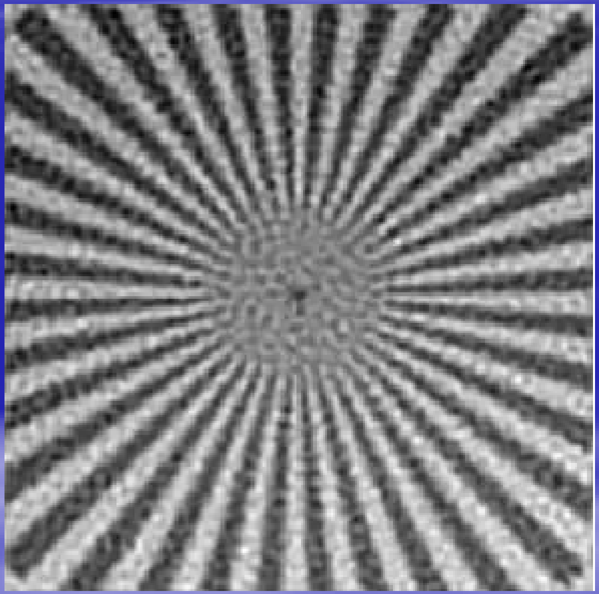


$f(m,n)$

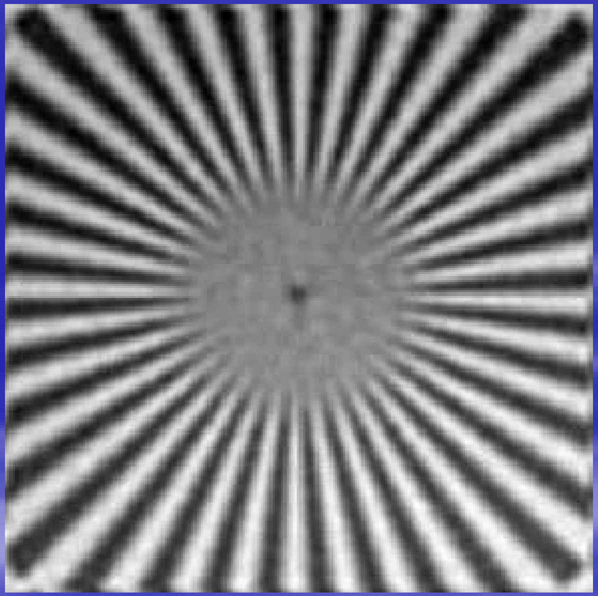


$g(m,n)$

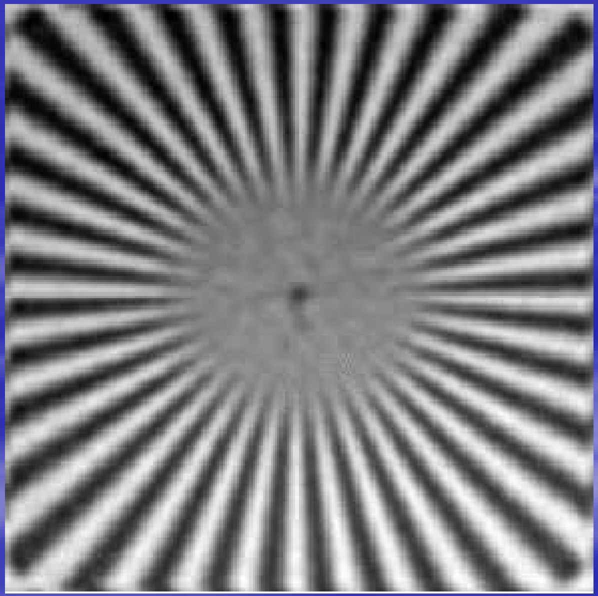
Constrained LS Example



$\hat{}f(m,n)$, $\gamma = 0.01$



$\hat{}f(m,n)$, $\gamma = 0.442$



$\hat{}f(m,n)$, $\gamma = 1$

Geometric Distortion

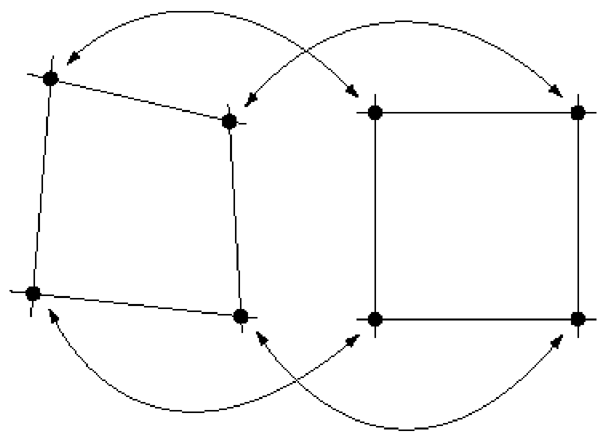


FIGURE 5.32 Corresponding tiepoints in two image segments.

Gray-level Interpolation

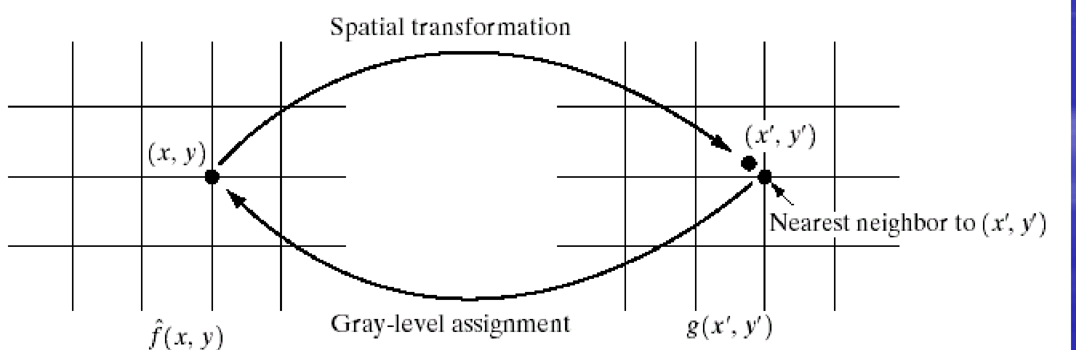
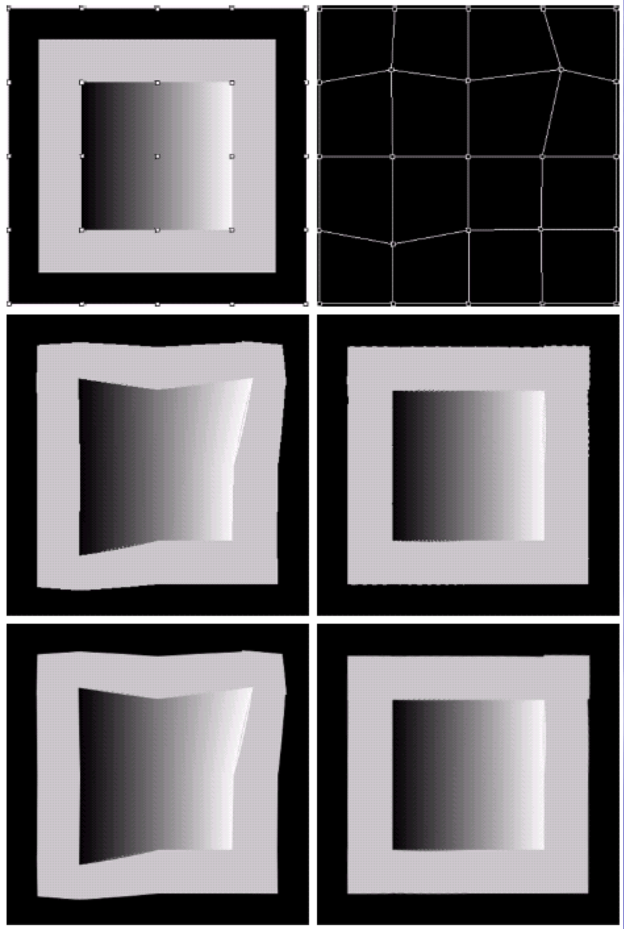


FIGURE 5.33 Gray-level interpolation based on the nearest neighbor concept.

Example



a b
c d
e f

FIGURE 5.34 (a) Image showing tiepoints. (b) Tiepoints after geometric distortion. (c) Geometrically distorted image, using nearest neighbor interpolation. (d) Restored result. (e) Image distorted using bilinear interpolation. (f) Restored image.

Example

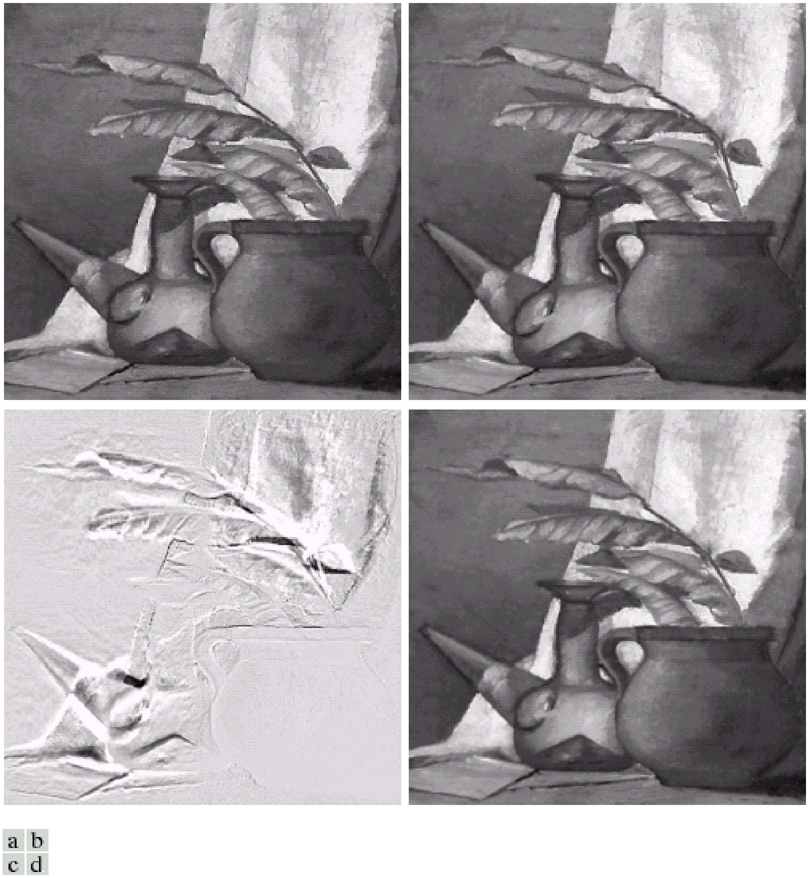


FIGURE 5.35 (a) An image before geometric distortion. (b) Image geometrically distorted using the same parameters as in Fig. 5.34(e). (c) Difference between (a) and (b). (d) Geometrically restored image.