john pacetti/Alamy

5 Trigonometric Functions: Right Triangle Approach

Suppose we want to find the distance from the earth to the sun. Using a tape measure is obviously impractical, so we need something other than simple measurements to tackle this problem. Angles are easier to measure than distances. For example, we can find the angle formed by the sun, earth, and moon by simply pointing to the sun with one arm and to the moon with the other and estimating the angle between them. The key idea is to find relationships between angles and distances. So if we had a way of determining distances from angles, we would be able to find the distance to the sun without having to go there. The trigonometric functions that we study in this chapter provide us with just the tools we need.

The trigonometric functions can be defined in two different but equivalent ways: as functions of angles (Chapter 5) or as functions of real numbers (Chapter 6). The two approaches are independent of each other, so either Chapter 5 or Chapter 6 may be studied first. We study both approaches because the different approaches are required for different applications.

5.1 ANGLE MEASURE

■ Angle Measure ■ Angles in Standard Position ■ Length of a Circular Arc ■ Area of a Circular Sector ■ Circular Motion

An **angle** AOB consists of two rays R_1 and R_2 with a common vertex O (see Figure 1). We often interpret an angle as a rotation of the ray R_1 onto R_2. In this case R_1 is called the **initial side**, and R_2 is called the **terminal side** of the angle. If the rotation is counterclockwise, the angle is considered **positive**, and if the rotation is clockwise, the angle is considered **negative**.

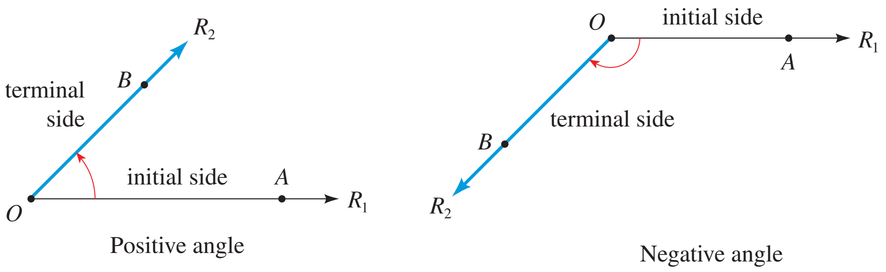


FIGURE 1

■ Angle Measure

The **measure** of an angle is the amount of rotation about the vertex required to move R_1 onto R_2. Intuitively, this is how much the angle "opens." One unit of measurement for angles is the **degree**. An angle of measure 1 degree is formed by rotating the initial side $\frac{1}{360}$ of a complete revolution. In calculus and other branches of mathematics a more natural method of measuring angles is used: *radian measure*. The amount an angle opens is measured along the arc of a circle of radius 1 with its center at the vertex of the angle.

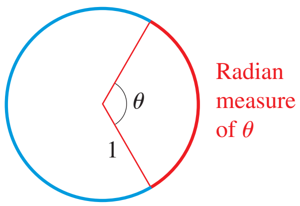


FIGURE 2

DEFINITION OF RADIAN MEASURE

If a circle of radius 1 is drawn with the vertex of an angle at its center, then the measure of this angle in **radians** (abbreviated **rad**) is the length of the arc that subtends the angle (see Figure 2).

The circumference of the circle of radius 1 is 2π, so a complete revolution has measure 2π rad, a straight angle has measure π rad, and a right angle has measure $\pi/2$ rad. An angle that is subtended by an arc of length 2 along the unit circle has radian measure 2 (see Figure 3).

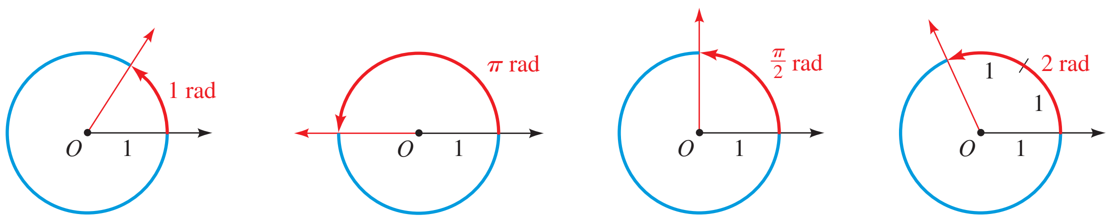


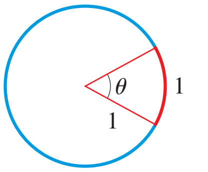
FIGURE 3 Radian measure

Since a complete revolution measured in degrees is 360° and measured in radians is 2π rad, we get the following simple relationship between these two methods of angle measurement.

RELATIONSHIP BETWEEN DEGREES AND RADIANS

$$180° = \pi \text{ rad} \qquad 1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ} \qquad 1° = \frac{\pi}{180} \text{ rad}$$

1. To convert degrees to radians, multiply by $\dfrac{\pi}{180}$.
2. To convert radians to degrees, multiply by $\dfrac{180}{\pi}$.



Measure of $\theta = 1$ rad
Measure of $\theta \approx 57.296°$

FIGURE 4

To get some idea of the size of a radian, notice that

$$1 \text{ rad} \approx 57.296° \qquad \text{and} \qquad 1° \approx 0.01745 \text{ rad}$$

An angle θ of measure 1 rad is shown in Figure 4.

EXAMPLE 1 ■ Converting Between Radians and Degrees

(a) Express 60° in radians. **(b)** Express $\dfrac{\pi}{6}$ rad in degrees.

SOLUTION The relationship between degrees and radians gives

(a) $60° = 60\left(\dfrac{\pi}{180}\right) \text{ rad} = \dfrac{\pi}{3} \text{ rad}$ **(b)** $\dfrac{\pi}{6} \text{ rad} = \left(\dfrac{\pi}{6}\right)\left(\dfrac{180}{\pi}\right) = 30°$

Now Try Exercises 5 and 17 ■

A note on terminology: We often use a phrase such as "a 30° angle" to mean *an angle whose measure is* 30°. Also, for an angle θ we write $\theta = 30°$ or $\theta = \pi/6$ to mean *the measure of* θ *is* 30° *or* $\pi/6$ *rad*. When no unit is given, the angle is assumed to be measured in radians.

■ Angles in Standard Position

An angle is in **standard position** if it is drawn in the xy-plane with its vertex at the origin and its initial side on the positive x-axis. Figure 5 gives examples of angles in standard position.

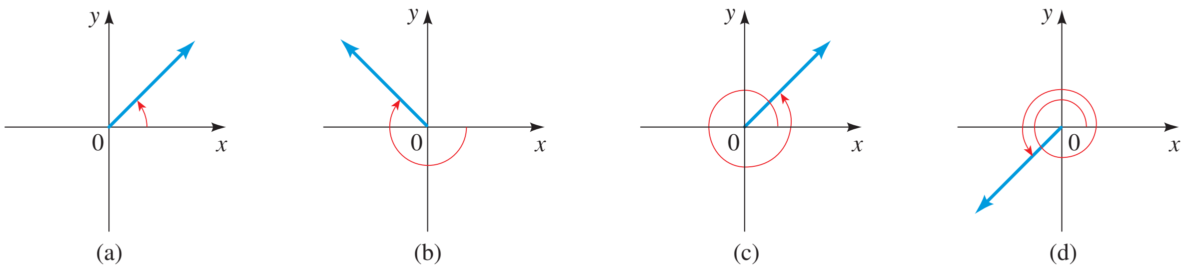


FIGURE 5 Angles in standard position

Two angles in standard position are **coterminal** if their sides coincide. In Figure 5 the angles in (a) and (c) are coterminal.

EXAMPLE 2 ■ Coterminal Angles

(a) Find angles that are coterminal with the angle $\theta = 30°$ in standard position.

(b) Find angles that are coterminal with the angle $\theta = \dfrac{\pi}{3}$ in standard position.

SOLUTION

(a) To find positive angles that are coterminal with θ, we add any multiple of $360°$. Thus

$$30° + 360° = 390° \qquad \text{and} \qquad 30° + 720° = 750°$$

are coterminal with $\theta = 30°$. To find negative angles that are coterminal with θ, we subtract any multiple of $360°$. Thus

$$30° - 360° = -330° \qquad \text{and} \qquad 30° - 720° = -690°$$

are coterminal with θ. (See Figure 6.)

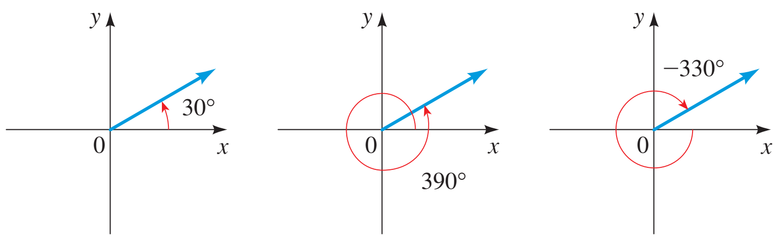


FIGURE 6

(b) To find positive angles that are coterminal with θ, we add any multiple of 2π. Thus

$$\frac{\pi}{3} + 2\pi = \frac{7\pi}{3} \qquad \text{and} \qquad \frac{\pi}{3} + 4\pi = \frac{13\pi}{3}$$

are coterminal with $\theta = \pi/3$. To find negative angles that are coterminal with θ, we subtract any multiple of 2π. Thus

$$\frac{\pi}{3} - 2\pi = -\frac{5\pi}{3} \qquad \text{and} \qquad \frac{\pi}{3} - 4\pi = -\frac{11\pi}{3}$$

are coterminal with θ. (See Figure 7.)

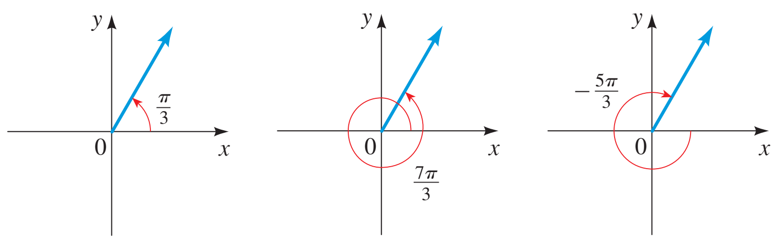


FIGURE 7

Now Try Exercises 29 and 31

EXAMPLE 3 ■ Coterminal Angles

Find an angle with measure between $0°$ and $360°$ that is coterminal with the angle of measure $1290°$ in standard position.

SOLUTION We can subtract $360°$ as many times as we wish from $1290°$, and the resulting angle will be coterminal with $1290°$. Thus $1290° - 360° = 930°$ is coterminal with $1290°$, and so is the angle $1290° - 2(360)° = 570°$.

To find the angle we want between $0°$ and $360°$, we subtract $360°$ from $1290°$ as many times as necessary. An efficient way to do this is to determine how many times $360°$ goes into $1290°$, that is, divide 1290 by 360, and the remainder will be the angle we are looking for. We see that 360 goes into 1290 three times with a remainder of 210. Thus 210° is the desired angle (see Figure 8).

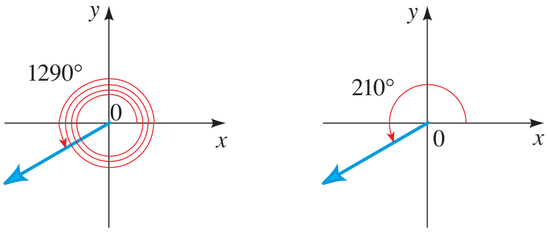


FIGURE 8

Now Try Exercise 41

Length of a Circular Arc

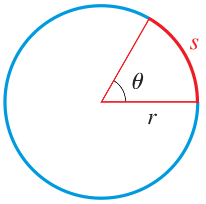


FIGURE 9 $s = \theta r$

An angle whose radian measure is θ is subtended by an arc that is the fraction $\theta/(2\pi)$ of the circumference of a circle. Thus in a circle of radius r the length s of an arc that subtends the angle θ (see Figure 9) is

$$s = \frac{\theta}{2\pi} \times \text{circumference of circle}$$

$$= \frac{\theta}{2\pi}(2\pi r) = \theta r$$

LENGTH OF A CIRCULAR ARC

In a circle of radius r the length s of an arc that subtends a central angle of θ radians is

$$s = r\theta$$

Solving for θ, we get the important formula

$$\boxed{\theta = \frac{s}{r}}$$

This formula allows us to define radian measure using a circle of any radius r: The radian measure of an angle θ is s/r, where s is the length of the circular arc that subtends θ in a circle of radius r (see Figure 10).

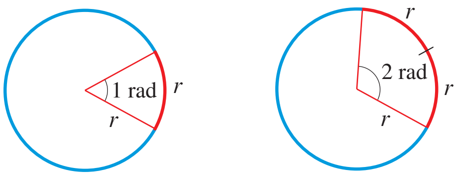


FIGURE 10 The radian measure of θ is the number of "radiuses" that can fit in the arc that subtends θ; hence the term *radian*.

EXAMPLE 4 ■ Arc Length and Angle Measure

(a) Find the length of an arc of a circle with radius 10 m that subtends a central angle of 30°.

(b) A central angle θ in a circle of radius 4 m is subtended by an arc of length 6 m. Find the measure of θ in radians.

SOLUTION

(a) From Example 1(b) we see that $30° = \pi/6$ rad. So the length of the arc is

$$s = r\theta = (10)\frac{\pi}{6} = \frac{5\pi}{3} \text{ m}$$

The formula $s = r\theta$ is true only when θ is measured in radians.

(b) By the formula $\theta = s/r$ we have

$$\theta = \frac{s}{r} = \frac{6}{4} = \frac{3}{2} \text{ rad}$$

Now Try Exercises 57 and 59 ■

Area of a Circular Sector

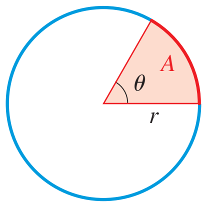


FIGURE 11
$A = \frac{1}{2}r^2\theta$

The area of a circle of radius r is $A = \pi r^2$. A sector of this circle with central angle θ has an area that is the fraction $\theta/(2\pi)$ of the area of the entire circle (see Figure 11). So the area of this sector is

$$A = \frac{\theta}{2\pi} \times \text{area of circle}$$

$$= \frac{\theta}{2\pi}(\pi r^2) = \frac{1}{2}r^2\theta$$

AREA OF A CIRCULAR SECTOR

In a circle of radius r the area A of a sector with a central angle of θ radians is

$$A = \frac{1}{2}r^2\theta$$

EXAMPLE 5 ■ Area of a Sector

Find the area of a sector of a circle with central angle 60° if the radius of the circle is 3 m.

SOLUTION To use the formula for the area of a circular sector, we must find the central angle of the sector in radians: $60° = 60(\pi/180)$ rad $= \pi/3$ rad. Thus the area of the sector is

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}(3)^2\left(\frac{\pi}{3}\right) = \frac{3\pi}{2} \text{ m}^2$$

The formula $A = \frac{1}{2}r^2\theta$ is true only when θ is measured in radians.

Now Try Exercise 63 ■

Circular Motion

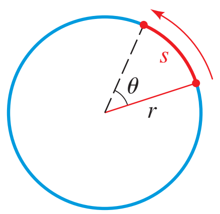


FIGURE 12

Suppose a point moves along a circle as shown in Figure 12. There are two ways to describe the motion of the point: linear speed and angular speed. **Linear speed** is the rate at which the distance traveled is changing, so linear speed is the distance traveled divided by the time elapsed. **Angular speed** is the rate at which the central angle θ is changing, so angular speed is the number of radians this angle changes divided by the time elapsed.

LINEAR SPEED AND ANGULAR SPEED

Suppose a point moves along a circle of radius r and the ray from the center of the circle to the point traverses θ radians in time t. Let $s = r\theta$ be the distance the point travels in time t. Then the speed of the object is given by

$$\textbf{Angular speed} \qquad \omega = \frac{\theta}{t}$$

$$\textbf{Linear speed} \qquad v = \frac{s}{t}$$

The symbol ω is the Greek letter "omega."

EXAMPLE 6 ■ Finding Linear and Angular Speed

A boy rotates a stone in a 3-ft-long sling at the rate of 15 revolutions every 10 seconds. Find the angular and linear velocities of the stone.

SOLUTION In 10 s the angle θ changes by $15 \cdot 2\pi = 30\pi$ rad. So the *angular speed* of the stone is

$$\omega = \frac{\theta}{t} = \frac{30\pi \text{ rad}}{10 \text{ s}} = 3\pi \text{ rad/s}$$

The distance traveled by the stone in 10 s is $s = 15 \cdot 2\pi r = 15 \cdot 2\pi \cdot 3 = 90\pi$ ft. So the *linear speed* of the stone is

$$v = \frac{s}{t} = \frac{90\pi \text{ ft}}{10 \text{ s}} = 9\pi \text{ ft/s}$$

Now Try Exercise 85 ■

Notice that angular speed does *not* depend on the radius of the circle; it depends only on the angle θ. However, if we know the angular speed ω and the radius r, we can find linear speed as follows: $v = s/t = r\theta/t = r(\theta/t) = r\omega$.

RELATIONSHIP BETWEEN LINEAR AND ANGULAR SPEED

If a point moves along a circle of radius r with angular speed ω, then its linear speed v is given by

$$v = r\omega$$

EXAMPLE 7 ■ Finding Linear Speed from Angular Speed

A woman is riding a bicycle whose wheels are 26 in. in diameter. If the wheels rotate at 125 revolutions per minute (rpm), find the speed (in mi/h) at which she is traveling.

SOLUTION The angular speed of the wheels is $2\pi \cdot 125 = 250\pi$ rad/min. Since the wheels have radius 13 in. (half the diameter), the linear speed is

$$v = r\omega = 13 \cdot 250\pi \approx 10{,}210.2 \text{ in./min}$$

Since there are 12 inches per foot, 5280 feet per mile, and 60 minutes per hour, her speed in miles per hour is

$$\frac{10{,}210.2 \text{ in./min} \times 60 \text{ min/h}}{12 \text{ in./ft} \times 5280 \text{ ft/mi}} = \frac{612{,}612 \text{ in./h}}{63{,}360 \text{ in./mi}}$$
$$\approx 9.7 \text{ mi/h}$$

Now Try Exercise 87 ■

5.1 EXERCISES

CONCEPTS

1. **(a)** The radian measure of an angle θ is the length of the ________ that subtends the angle in a circle of radius ________.

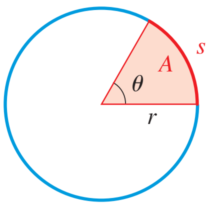
(b) To convert degrees to radians, we multiply by ________.

(c) To convert radians to degrees, we multiply by ________.

2. A central angle θ is drawn in a circle of radius r, as in the figure below.

(a) The length of the arc subtended by θ is $s =$ ________.

(b) The area of the sector with central angle θ is $A =$ ________.

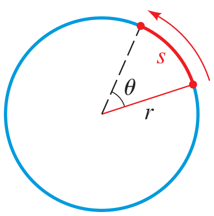


3. Suppose a point moves along a circle with radius r as shown in the figure below. The point travels a distance s along the circle in time t.

(a) The angular speed of the point is $\omega = \dfrac{\quad}{\quad}$.

(b) The linear speed of the point is $v = \dfrac{\quad}{\quad}$.

(c) The linear speed v and the angular speed ω are related by the equation $v =$ ________.



4. Object A is traveling along a circle of radius 2, and Object B is traveling along a circle of radius 5. The objects have the same angular speed. Do the objects have the same linear speed? If not, which object has the greater linear speed?

SKILLS

5–16 ■ From Degrees to Radians Find the radian measure of the angle with the given degree measure. Round your answer to three decimal places.

5. $15°$ **6.** $36°$ **7.** $54°$ **8.** $75°$

9. $-45°$ **10.** $-30°$ **11.** $100°$ **12.** $200°$

13. $1000°$ **14.** $3600°$ **15.** $-70°$ **16.** $-150°$

17–28 ■ From Radians to Degrees Find the degree measure of the angle with the given radian measure.

17. $\dfrac{5\pi}{3}$ **18.** $\dfrac{3\pi}{4}$ **19.** $\dfrac{5\pi}{6}$

20. $-\dfrac{3\pi}{2}$ **21.** 3 **22.** -2

23. -1.2 **24.** 3.4 **25.** $\dfrac{\pi}{10}$

26. $\dfrac{5\pi}{18}$ **27.** $-\dfrac{2\pi}{15}$ **28.** $-\dfrac{13\pi}{12}$

29–34 ■ Coterminal Angles The measure of an angle in standard position is given. Find two positive angles and two negative angles that are coterminal with the given angle.

29. $50°$ **30.** $135°$ **31.** $\dfrac{3\pi}{4}$

32. $\dfrac{11\pi}{6}$ **33.** $-\dfrac{\pi}{4}$ **34.** $-45°$

35–40 ■ Coterminal Angles? The measures of two angles in standard position are given. Determine whether the angles are coterminal.

35. $70°,\ 430°$ **36.** $-30°,\ 330°$

37. $\dfrac{5\pi}{6},\ \dfrac{17\pi}{6}$ **38.** $\dfrac{32\pi}{3},\ \dfrac{11\pi}{3}$

39. $155°,\ 875°$ **40.** $50°,\ 340°$

41–46 ■ Finding a Coterminal Angle Find an angle between $0°$ and $360°$ that is coterminal with the given angle.

41. $400°$ **42.** $375°$

43. $780°$ **44.** $-100°$

45. $-800°$ **46.** $1270°$

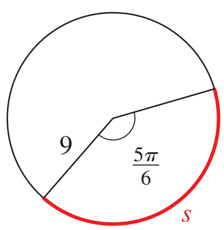
47–52 ■ Finding a Coterminal Angle Find an angle between 0 and 2π that is coterminal with the given angle.

47. $\dfrac{19\pi}{6}$ **48.** $-\dfrac{5\pi}{3}$ **49.** 25π

50. 10 **51.** $\dfrac{17\pi}{4}$ **52.** $\dfrac{51\pi}{2}$

53–62 ■ Circular Arcs Find the length s of the circular arc, the radius r of the circle, or the central angle θ, as indicated.

53.



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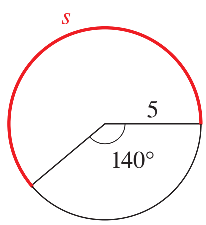

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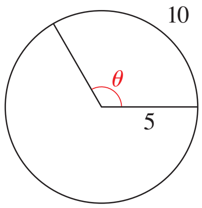

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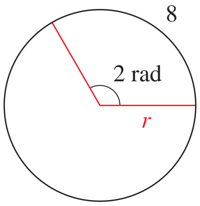


57. Find the length s of the arc that subtends a central angle of measure 3 rad in a circle of radius 5 cm.

58. Find the length s of the arc that subtends a central angle of measure 40° in a circle of radius 12 m.

59. A central angle θ in a circle of radius 9 m is subtended by an arc of length 14 m. Find the measure of θ in degrees and radians.

60. An arc of length 15 ft subtends a central angle θ in a circle of radius 9 ft. Find the measure of θ in degrees and radians.

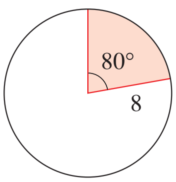
61. Find the radius r of the circle if an arc of length 15 m on the circle subtends a central angle of $5\pi/6$.

62. Find the radius r of the circle if an arc of length 20 cm on the circle subtends a central angle of 50°.

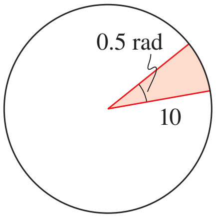
63–70 ■ Area of a Circular Sector These exercises involve the formula for the area of a circular sector.

63. Find the area of the sector shown in each figure.

(a)

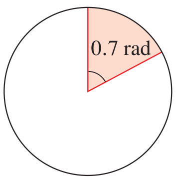


(b)



64. Find the radius of each circle if the area of the sector is 12.

(a)



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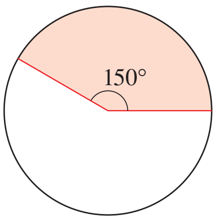


65. Find the area of a sector with central angle $2\pi/3$ rad in a circle of radius 10 m.

66. A sector of a circle has a central angle of 145°. Find the area of the sector if the radius of the circle is 6 ft.

67. The area of a sector of a circle with a central angle of 140° is 70 m^2. Find the radius of the circle.

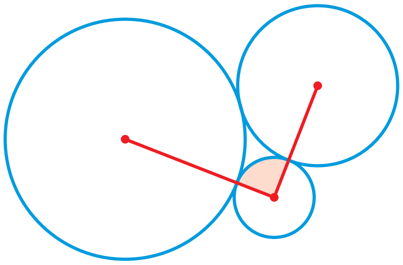
68. The area of a sector of a circle with a central angle of $5\pi/12$ rad is 20 m^2. Find the radius of the circle.

69. A sector of a circle of radius 80 mi has an area of 1600 mi^2. Find the central angle (in radians) of the sector.

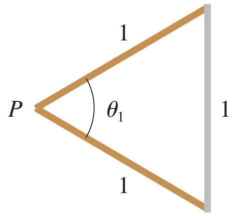
70. The area of a circle is 600 m^2. Find the area of a sector of this circle that subtends a central angle of 3 rad.

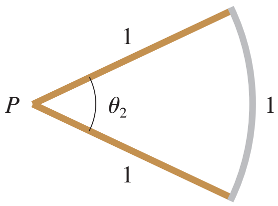
SKILLS Plus

71. Area of a Sector of a Circle Three circles with radii 1, 2, and 3 ft are externally tangent to one another, as shown in the figure. Find the area of the sector of the circle of radius 1 that is cut off by the line segments joining the center of that circle to the centers of the other two circles.



72. Comparing a Triangle and a Sector of a Circle Two wood sticks and a metal rod, each of length 1, are connected to form a triangle with angle θ_1 at the point P, as shown in the first figure below. The rod is then bent to form an arc of a circle with center P, resulting in a smaller angle θ_2 at the point P, as shown in the second figure. Find θ_1, θ_2, and $\theta_1 - \theta_2$.





73–74 ■ Clocks and Angles In 1 h the minute hand on a clock moves through a complete circle, and the hour hand moves through $\frac{1}{12}$ of a circle.

73. Through how many radians do the minute hand and the hour hand move between 1:00 P.M. and 1:45 P.M. (on the same day)?

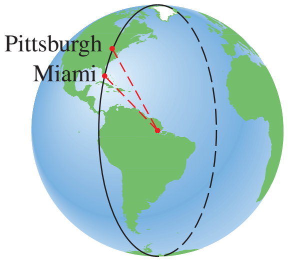
74. Through how many radians do the minute hand and the hour hand move between 1:00 P.M. and 6:45 P.M. (on the same day)?

APPLICATIONS

75. Travel Distance A car's wheels are 28 in. in diameter. How far (in mi.) will the car travel if its wheels revolve 10,000 times without slipping?

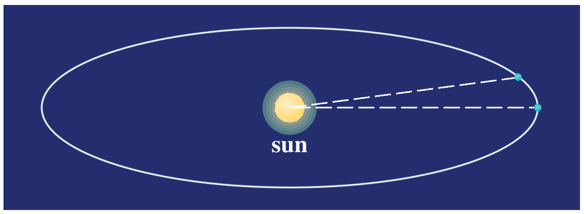
76. Wheel Revolutions How many revolutions will a car wheel of diameter 30 in. make as the car travels a distance of one mile?

77. Latitudes Pittsburgh, Pennsylvania, and Miami, Florida, lie approximately on the same meridian. Pittsburgh has a latitude of 40.5° N, and Miami has a latitude of 25.5° N. Find the distance between these two cities. (The radius of the earth is 3960 mi.)

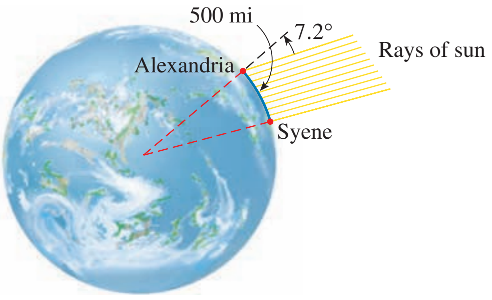


78. Latitudes Memphis, Tennessee, and New Orleans, Louisiana, lie approximately on the same meridian. Memphis has a latitude of 35° N, and New Orleans has a latitude of 30° N. Find the distance between these two cities. (The radius of the earth is 3960 mi.)

79. Orbit of the Earth Find the distance that the earth travels in one day in its path around the sun. Assume that a year has 365 days and that the path of the earth around the sun is a circle of radius 93 million miles. [*Note:* The path of the earth around the sun is actually an *ellipse* with the sun at one focus (see Section 12.2). This ellipse, however, has very small eccentricity, so it is nearly circular.]

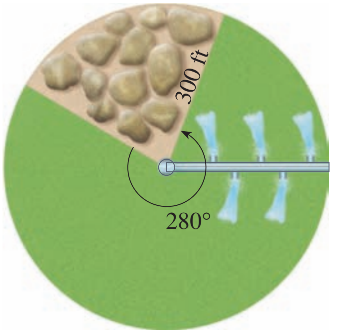


80. Circumference of the Earth The Greek mathematician Eratosthenes (ca. 276–195 B.C.) measured the circumference of the earth from the following observations. He noticed that on a certain day the sun shone directly down a deep well in Syene (modern Aswan). At the same time in Alexandria, 500 miles north (on the same meridian), the rays of the sun shone at an angle of 7.2° to the zenith. Use this information and the figure to find the radius and circumference of the earth.

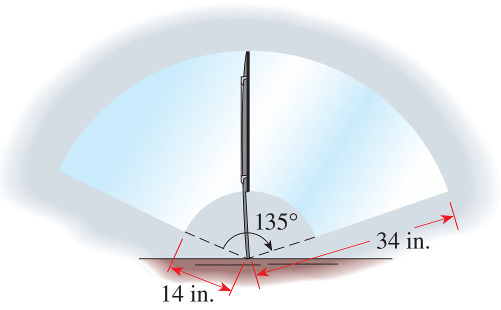


81. Nautical Miles Find the distance along an arc on the surface of the earth that subtends a central angle of 1 minute $\left(1 \text{ minute} = \frac{1}{60} \text{ degree}\right)$. This distance is called a *nautical mile*. (The radius of the earth is 3960 mi.)

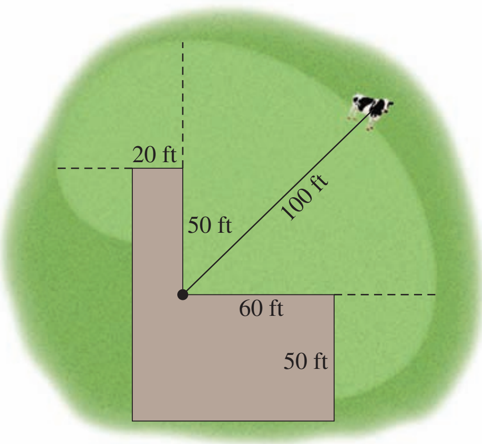
82. Irrigation An irrigation system uses a straight sprinkler pipe 300 ft long that pivots around a central point as shown. Because of an obstacle the pipe is allowed to pivot through 280° only. Find the area irrigated by this system.



83. Windshield Wipers The top and bottom ends of a windshield wiper blade are 34 in. and 14 in., respectively, from the pivot point. While in operation, the wiper sweeps through 135°. Find the area swept by the blade.



84. The Tethered Cow A cow is tethered by a 100-ft rope to the inside corner of an L-shaped building, as shown in the figure. Find the area that the cow can graze.



85. Fan A ceiling fan with 16-in. blades rotates at 45 rpm.

(a) Find the angular speed of the fan in rad/min.

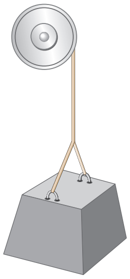
(b) Find the linear speed of the tips of the blades in in./min.

86. Radial Saw A radial saw has a blade with a 6-in. radius. Suppose that the blade spins at 1000 rpm.

(a) Find the angular speed of the blade in rad/min.

(b) Find the linear speed of the sawteeth in ft/s.

87. Winch A winch of radius 2 ft is used to lift heavy loads. If the winch makes 8 revolutions every 15 s, find the speed at which the load is rising.



88. Speed of a Car The wheels of a car have radius 11 in. and are rotating at 600 rpm. Find the speed of the car in mi/h.

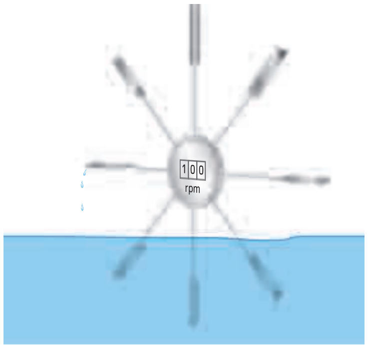
89. Speed at the Equator The earth rotates about its axis once every 23 h 56 min 4 s, and the radius of the earth is 3960 mi. Find the linear speed of a point on the equator in mi/h.

90. Truck Wheels A truck with 48-in.-diameter wheels is traveling at 50 mi/h.

(a) Find the angular speed of the wheels in rad/min.

(b) How many revolutions per minute do the wheels make?

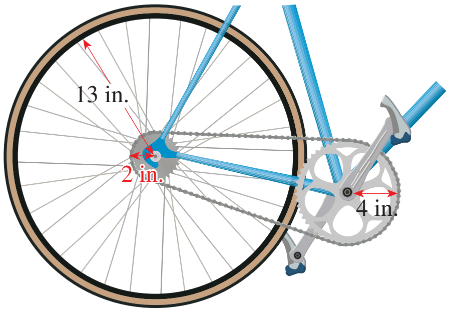
91. Speed of a Current To measure the speed of a current, scientists place a paddle wheel in the stream and observe the rate at which it rotates. If the paddle wheel has radius 0.20 m and rotates at 100 rpm, find the speed of the current in m/s.



92. Bicycle Wheel The sprockets and chain of a bicycle are shown in the figure. The pedal sprocket has a radius of 4 in., the wheel sprocket a radius of 2 in., and the wheel a radius of 13 in. The cyclist pedals at 40 rpm.

(a) Find the angular speed of the wheel sprocket.

(b) Find the speed of the bicycle. (Assume that the wheel turns at the same rate as the wheel sprocket.)



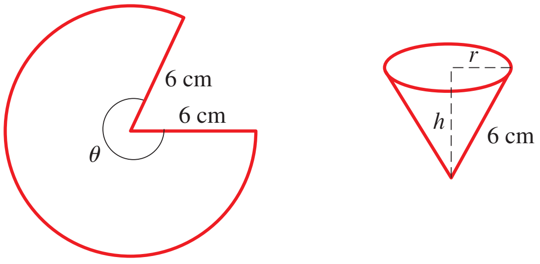
93. Conical Cup A conical cup is made from a circular piece of paper with radius 6 cm by cutting out a sector and joining the edges as shown below. Suppose $\theta = 5\pi/3$.

(a) Find the circumference C of the opening of the cup.

(b) Find the radius r of the opening of the cup. [*Hint:* Use $C = 2\pi r$.]

(c) Find the height h of the cup. [*Hint:* Use the Pythagorean Theorem.]

(d) Find the volume of the cup.



94. Conical Cup In this exercise we find the volume of the conical cup in Exercise 93 for any angle θ.

(a) Follow the steps in Exercise 93 to show that the volume of the cup as a function of θ is

$$V(\theta) = \frac{9}{\pi^2}\theta^2\sqrt{4\pi^2 - \theta^2}, \qquad 0 < \theta < 2\pi$$

(b) Graph the function V.

(c) For what angle θ is the volume of the cup a maximum?

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

95. WRITE: Different Ways of Measuring Angles The custom of measuring angles using degrees, with 360° in a circle, dates back to the ancient Babylonians, who used a number system based on groups of 60. Another system of measuring angles divides the circle into 400 units, called *grads*. In this system a right angle is 100 grad, so this fits in with our base 10 number system.

Write a short essay comparing the advantages and disadvantages of these two systems and the radian system of measuring angles. Which system do you prefer? Why?

5.2 TRIGONOMETRY OF RIGHT TRIANGLES

■ Trigonometric Ratios ■ Special Triangles; Calculators ■ Applications of Trigonometry of Right Triangles

In this section we study certain ratios of the sides of right triangles, called trigonometric ratios, and give several applications.

■ Trigonometric Ratios

Consider a right triangle with θ as one of its acute angles. The trigonometric ratios are defined as follows (see Figure 1).

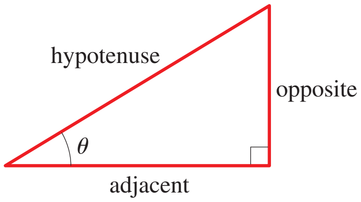


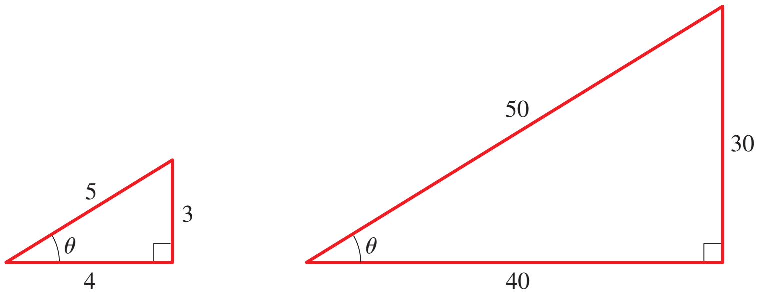
FIGURE 1

THE TRIGONOMETRIC RATIOS

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

The symbols we use for these ratios are abbreviations for their full names: **sine**, **cosine**, **tangent**, **cosecant**, **secant**, **cotangent**. Since any two right triangles with angle θ are similar, these ratios are the same, regardless of the size of the triangle; the trigonometric ratios depend only on the angle θ (see Figure 2).



$\sin\theta = \frac{3}{5}$ $\qquad$ $\sin\theta = \frac{30}{50} = \frac{3}{5}$

FIGURE 2

EXAMPLE 1 ■ Finding Trigonometric Ratios

Find the six trigonometric ratios of the angle θ in Figure 3.

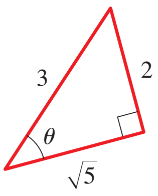


FIGURE 3

SOLUTION By the definition of trigonometric ratios, we get

$$\sin\theta = \frac{2}{3} \qquad \cos\theta = \frac{\sqrt{5}}{3} \qquad \tan\theta = \frac{2}{\sqrt{5}}$$

$$\csc\theta = \frac{3}{2} \qquad \sec\theta = \frac{3}{\sqrt{5}} \qquad \cot\theta = \frac{\sqrt{5}}{2}$$

Now Try Exercise 3 ■

EXAMPLE 2 ■ Finding Trigonometric Ratios

If $\cos\alpha = \frac{3}{4}$, sketch a right triangle with acute angle α, and find the other five trigonometric ratios of α.

SOLUTION Since $\cos \alpha$ is defined as the ratio of the adjacent side to the hypotenuse, we sketch a triangle with hypotenuse of length 4 and a side of length 3 adjacent to α. If the opposite side is x, then by the Pythagorean Theorem, $3^2 + x^2 = 4^2$ or $x^2 = 7$, so $x = \sqrt{7}$. We then use the triangle in Figure 4 to find the ratios.

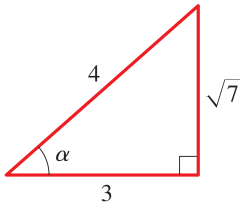


FIGURE 4

$$\sin \alpha = \frac{\sqrt{7}}{4} \qquad \cos \alpha = \frac{3}{4} \qquad \tan \alpha = \frac{\sqrt{7}}{3}$$

$$\csc \alpha = \frac{4}{\sqrt{7}} \qquad \sec \alpha = \frac{4}{3} \qquad \cot \alpha = \frac{3}{\sqrt{7}}$$

Now Try Exercise 23

Special Triangles; Calculators

There are special trigonometric ratios that can be calculated from certain triangles (which we call special triangles). We can also use a calculator to find trigonometric ratios.

Special Ratios Certain right triangles have ratios that can be calculated easily from the Pythagorean Theorem. Since they are used frequently, we mention them here.

The first triangle is obtained by drawing a diagonal in a square of side 1 (see Figure 5). By the Pythagorean Theorem this diagonal has length $\sqrt{2}$. The resulting triangle has angles 45°, 45°, and 90° (or $\pi/4$, $\pi/4$, and $\pi/2$). To get the second triangle, we start with an equilateral triangle ABC of side 2 and draw the perpendicular bisector DB of the base, as in Figure 6. By the Pythagorean Theorem the length of DB is $\sqrt{3}$. Since DB bisects angle ABC, we obtain a triangle with angles 30°, 60°, and 90° (or $\pi/6$, $\pi/3$, and $\pi/2$).

HIPPARCHUS (circa 140 B.C.) is considered the founder of trigonometry. He constructed tables for a function closely related to the modern sine function and evaluated for angles at half-degree intervals. These are considered the first trigonometric tables. He used his tables mainly to calculate the paths of the planets through the heavens.

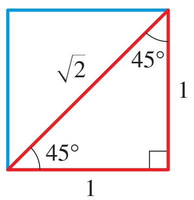


FIGURE 5

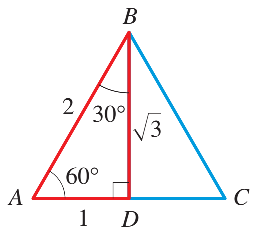


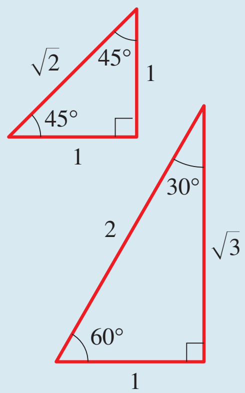
FIGURE 6

We can now use the special triangles in Figures 5 and 6 to calculate the trigonometric ratios for angles with measures 30°, 45°, and 60° (or $\pi/6$, $\pi/4$, and $\pi/3$). These are listed in the table below.

SPECIAL VALUES OF THE TRIGONOMETRIC FUNCTIONS

The following values of the trigonometric functions are obtained from the special triangles.

θ in degrees	θ in radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	0	0	1	0	—	1	—
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	—	1	—	0



It's useful to remember these special trigonometric ratios because they occur often. Of course, they can be recalled easily if we remember the triangles from which they are obtained.

For an explanation of numerical methods, see the margin note on page 535.

Using a Calculator To find the values of the trigonometric ratios for other angles, we use a calculator. Mathematical methods (called *numerical methods*) used in finding the trigonometric ratios are programmed directly into scientific calculators. For instance, when the $\boxed{\text{SIN}}$ key is pressed, the calculator computes an approximation to the value of the sine of the given angle. Calculators give the values of sine, cosine, and tangent; the other ratios can be easily calculated from these by using the following *reciprocal relations*:

$$\csc t = \frac{1}{\sin t} \qquad \sec t = \frac{1}{\cos t} \qquad \cot t = \frac{1}{\tan t}$$

You should check that these relations follow immediately from the definitions of the trigonometric ratios.

We follow the convention that when we write $\sin t$, *we mean the sine of the angle whose radian measure is* t. For instance, $\sin 1$ means the sine of the angle whose radian measure is 1. When using a calculator to find an approximate value for this number, set your calculator to radian mode; you will find that $\sin 1 \approx 0.841471$. If you want to find the sine of the angle whose measure is 1°, set your calculator to degree mode; you will find that $\sin 1° \approx 0.0174524$.

EXAMPLE 3 ■ Using a Calculator

Using a calculator, find the following.

(a) $\tan 40°$ **(b)** $\cos 20°$ **(c)** $\cot 14°$ **(d)** $\csc 80°$

SOLUTION Making sure our calculator is set in degree mode and rounding the results to six decimal places, we get the following:

(a) $\tan 40° \approx 0.839100$

(b) $\cos 20° \approx 0.939693$

(c) $\cot 14° = \dfrac{1}{\tan 14°} \approx 4.010781$

(d) $\csc 80° = \dfrac{1}{\sin 80°} \approx 1.015427$

Now Try Exercise 11 ■

■ Applications of Trigonometry of Right Triangles

A triangle has six parts: three angles and three sides. To **solve a triangle** means to determine all of its parts from the information known about the triangle, that is, to determine the lengths of the three sides and the measures of the three angles.

Hulton Archive/Moviepix/Getty Images

DISCOVERY PROJECT

Similarity

Similarity of triangles is the basic concept underlying the definition of the trigonometric functions. The ratios of the sides of a triangle are the same as the corresponding ratios in any similar triangle. But the concept of similarity of figures applies to all shapes, not just triangles. In this project we explore how areas and volumes of similar figures are related. These relationships allow us to determine whether an ape the size of King Kong (that is, an ape similar to, but much larger than, a real ape) can actually exist. You can find the project at **www.stewartmath.com**.

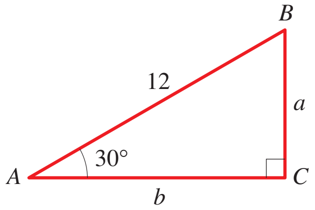


FIGURE 7

EXAMPLE 4 ■ Solving a Right Triangle

Solve triangle ABC, shown in Figure 7.

SOLUTION It's clear that $\angle B = 60°$. From Figure 7 we have

$$\begin{aligned} \sin 30° &= \frac{a}{12} && \text{Definition of sine} \\ a &= 12 \sin 30° && \text{Multiply by 12} \\ &= 12\left(\tfrac{1}{2}\right) = 6 && \text{Evaluate} \end{aligned}$$

Also from Figure 7 we have

$$\begin{aligned} \cos 30° &= \frac{b}{12} && \text{Definition of cosine} \\ b &= 12 \cos 30° && \text{Multiply by 12} \\ &= 12\left(\frac{\sqrt{3}}{2}\right) = 6\sqrt{3} && \text{Evaluate} \end{aligned}$$

Now Try Exercise 37 ■

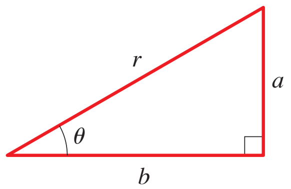


FIGURE 8
$a = r \sin \theta, \quad b = r \cos \theta$

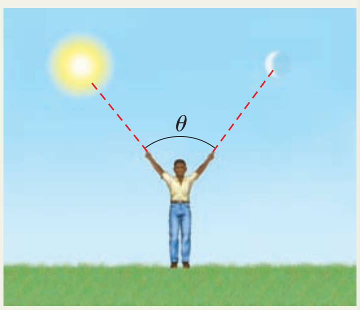
Figure 8 shows that if we know the hypotenuse r and an acute angle θ in a right triangle, then the legs a and b are given by

$$a = r \sin \theta \qquad \text{and} \qquad b = r \cos \theta$$

The ability to solve right triangles by using the trigonometric ratios is fundamental to many problems in navigation, surveying, astronomy, and the measurement of distances. The applications we consider in this section always involve right triangles, but as we will see in the next three sections, trigonometry is also useful in solving triangles that are not right triangles.

To discuss the next examples, we need some terminology. If an observer is looking at an object, then the line from the eye of the observer to the object is called the **line of sight** (Figure 9). If the object being observed is above the horizontal, then the angle between the line of sight and the horizontal is called the **angle of elevation**. If the object is below the horizontal, then the angle between the line of sight and the horizontal is called the **angle of depression**. In many of the examples and exercises in this chapter, angles of elevation and depression will be given for a hypothetical observer at ground level. If the line of sight follows a physical object, such as an inclined plane or a hillside, we use the term **angle of inclination**.

ARISTARCHUS OF SAMOS (310–230 B.C.) was a famous Greek scientist, musician, astronomer, and geometer. He observed that the angle between the sun and moon can be measured directly (see the figure below). In his book *On the Sizes and Distances of the Sun and the Moon* he estimated the distance to the sun by observing that when the moon is exactly half full, the triangle formed by the sun, the moon, and the earth has a right angle at the moon. His method was similar to the one described in Exercise 67 in this section. Aristarchus was the first to advance the theory that the earth and planets move around the sun, an idea that did not gain full acceptance until after the time of Copernicus, 1800 years later. For this reason Aristarchus is often called "the Copernicus of antiquity."



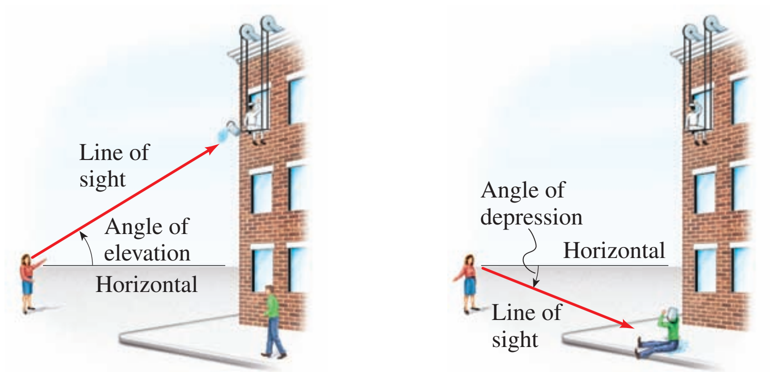


FIGURE 9

The next example gives an important application of trigonometry to the problem of measurement: We measure the height of a tall tree without having to climb it! Although the example is simple, the result is fundamental to understanding how the trigonometric ratios are applied to such problems.

THALES OF MILETUS (circa 625–547 B.C.) is the legendary founder of Greek geometry. It is said that he calculated the height of a Greek column by comparing the length of the shadow of his staff with that of the column. Using properties of similar triangles, he argued that the ratio of the height h of the column to the height h' of his staff was equal to the ratio of the length s of the column's shadow to the length s' of the staff's shadow:

$$\frac{h}{h'} = \frac{s}{s'}$$

Since three of these quantities are known, Thales was able to calculate the height of the column.

According to legend, Thales used a similar method to find the height of the Great Pyramid in Egypt, a feat that impressed Egypt's king. Plutarch wrote that "although he [the king of Egypt] admired you [Thales] for other things, yet he particularly liked the manner by which you measured the height of the pyramid without any trouble or instrument." The principle Thales used, the fact that ratios of corresponding sides of similar triangles are equal, is the foundation of the subject of trigonometry.

EXAMPLE 5 ■ Finding the Height of a Tree

A giant redwood tree casts a shadow 532 ft long. Find the height of the tree if the angle of elevation of the sun is 25.7°.

SOLUTION Let the height of the tree be h. From Figure 10 we see that

$$\frac{h}{532} = \tan 25.7° \qquad \text{Definition of tangent}$$

$$h = 532 \tan 25.7° \qquad \text{Multiply by 532}$$

$$\approx 532(0.48127) \approx 256 \qquad \text{Use a calculator}$$

Therefore the height of the tree is about 256 ft.

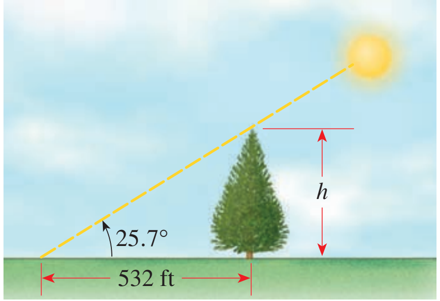


FIGURE 10

Now Try Exercise 53 ■

EXAMPLE 6 ■ A Problem Involving Right Triangles

From a point on the ground 500 ft from the base of a building, an observer finds that the angle of elevation to the top of the building is 24° and that the angle of elevation to the top of a flagpole atop the building is 27°. Find the height of the building and the length of the flagpole.

SOLUTION Figure 11 illustrates the situation. The height of the building is found in the same way that we found the height of the tree in Example 4.

$$\frac{h}{500} = \tan 24° \qquad \text{Definition of tangent}$$

$$h = 500 \tan 24° \qquad \text{Multiply by 500}$$

$$\approx 500(0.4452) \approx 223 \qquad \text{Use a calculator}$$

The height of the building is approximately 223 ft.

To find the length of the flagpole, let's first find the height from the ground to the top of the pole.

$$\frac{k}{500} = \tan 27° \qquad \text{Definition of tangent}$$

$$k = 500 \tan 27° \qquad \text{Multiply by 500}$$

$$\approx 500(0.5095) \qquad \text{Use a calculator}$$

$$\approx 255$$

To find the length of the flagpole, we subtract h from k. So the length of the pole is approximately $255 - 223 = 32$ ft.

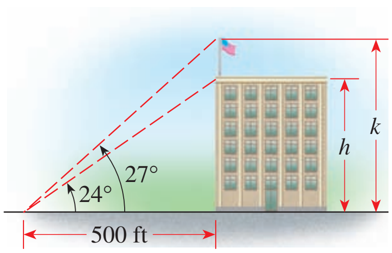


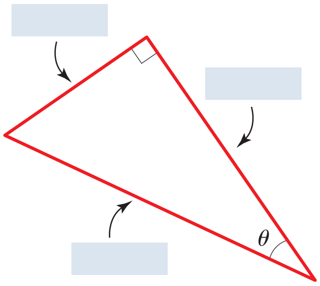
FIGURE 11

Now Try Exercise 61 ■

5.2 EXERCISES

CONCEPTS

1. A right triangle with an angle θ is shown in the figure.



(a) Label the "opposite" and "adjacent" sides of θ and the hypotenuse of the triangle.

(b) The trigonometric functions of the angle θ are defined as follows:

$$\sin\theta = \frac{\quad}{\quad} \qquad \cos\theta = \frac{\quad}{\quad} \qquad \tan\theta = \frac{\quad}{\quad}$$

(c) The trigonometric ratios do not depend on the size of the triangle. This is because all right triangles with the same acute angle θ are ________________.

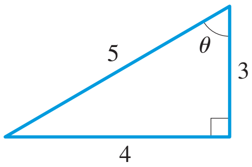
2. The reciprocal identities state that

$$\csc\theta = \frac{1}{\quad} \qquad \sec\theta = \frac{1}{\quad} \qquad \cot\theta = \frac{1}{\quad}$$

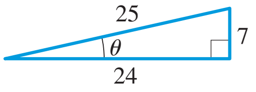
SKILLS

3–8 ■ Trigonometric Ratios Find the exact values of the six trigonometric ratios of the angle θ in the triangle.

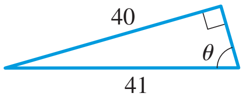
3.



4.



5.



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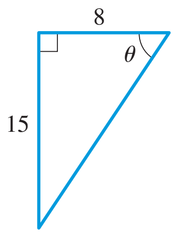

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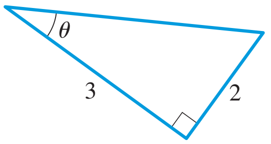

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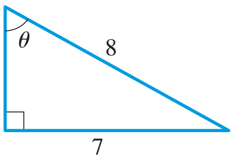
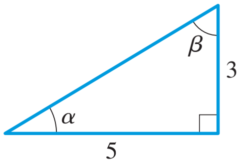


9–10 ■ Trigonometric Ratios Find **(a)** $\sin\alpha$ and $\cos\beta$, **(b)** $\tan\alpha$ and $\cot\beta$, and **(c)** $\sec\alpha$ and $\csc\beta$.

9.



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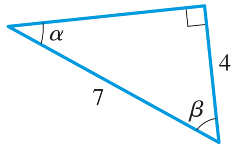


11–14 ■ Using a Calculator Use a calculator to evaluate the expression. Round your answer to five decimal places.

11. (a) $\sin 22°$ **(b)** $\cot 23°$

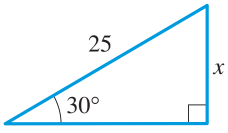
12. (a) $\cos 37°$ **(b)** $\csc 48°$

13. (a) $\sec 13°$ **(b)** $\tan 51°$

14. (a) $\csc 10°$ **(b)** $\sin 46°$

15–20 ■ Finding an Unknown Side Find the side labeled x. In Exercises 17 and 18 state your answer rounded to five decimal places.

15.



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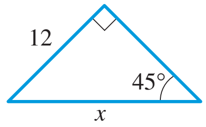

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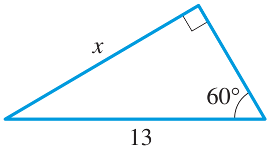

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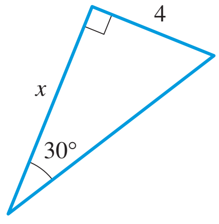

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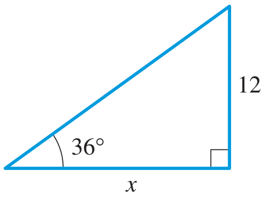

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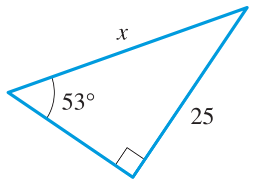
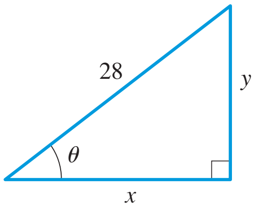


21–22 ■ Trigonometric Ratios Express x and y in terms of trigonometric ratios of θ.

21.



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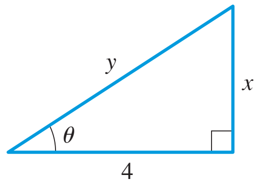


23–28 ■ Trigonometric Ratios Sketch a triangle that has acute angle θ, and find the other five trigonometric ratios of θ.

23. $\tan\theta = \frac{5}{6}$ **24.** $\cos\theta = \frac{12}{13}$ **25.** $\cot\theta = 1$

26. $\tan\theta = \sqrt{3}$ **27.** $\csc\theta = \frac{11}{6}$ **28.** $\cot\theta = \frac{5}{3}$

29–36 ■ Evaluating an Expression Evaluate the expression without using a calculator.

29. $\sin \dfrac{\pi}{6} + \cos \dfrac{\pi}{6}$

30. $\sin 30° \csc 30°$

31. $\sin 30° \cos 60° + \sin 60° \cos 30°$

32. $(\sin 60°)^2 + (\cos 60°)^2$

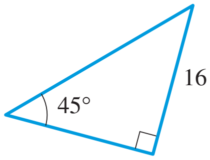
33. $(\cos 30°)^2 - (\sin 30°)^2$

34. $\left(\sin \dfrac{\pi}{3} \cos \dfrac{\pi}{4} - \sin \dfrac{\pi}{4} \cos \dfrac{\pi}{3}\right)^2$

35. $\left(\cos \dfrac{\pi}{4} + \sin \dfrac{\pi}{6}\right)^2$

36. $\left(\sin \dfrac{\pi}{3} \tan \dfrac{\pi}{6} + \csc \dfrac{\pi}{4}\right)^2$

37–44 ■ Solving a Right Triangle Solve the right triangle.

37.


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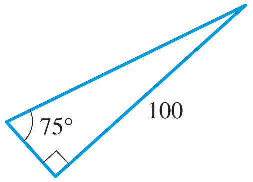

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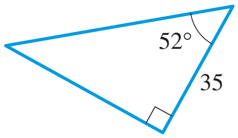

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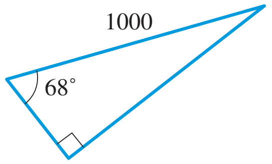

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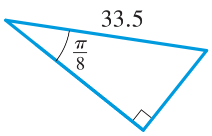

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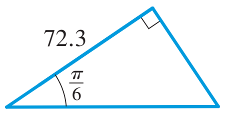

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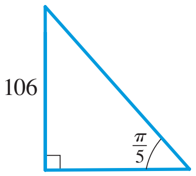

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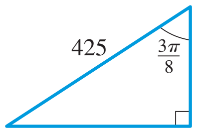
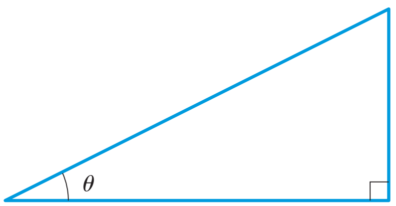


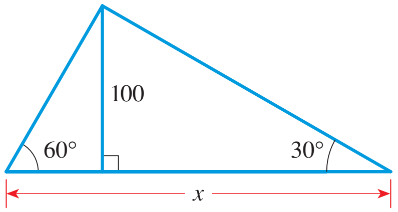
SKILLS Plus

45. Using a Ruler to Estimate Trigonometric Ratios Use a ruler to carefully measure the sides of the triangle, and then use your measurements to estimate the six trigonometric ratios of θ.



46. Using a Protractor to Estimate Trigonometric Ratios Using a protractor, sketch a right triangle that has the acute angle 40°. Measure the sides carefully, and use your results to estimate the six trigonometric ratios of 40°.

47–50 ■ Finding an Unknown Side Find x rounded to one decimal place.

47.


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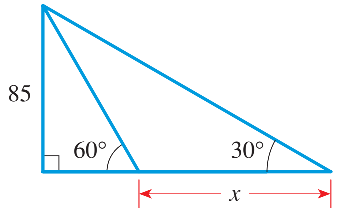

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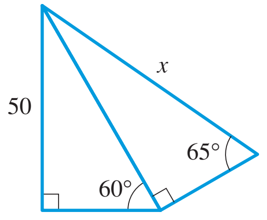

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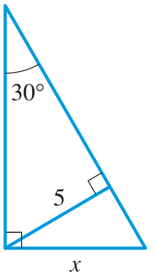
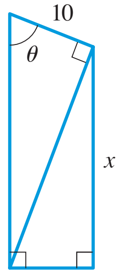
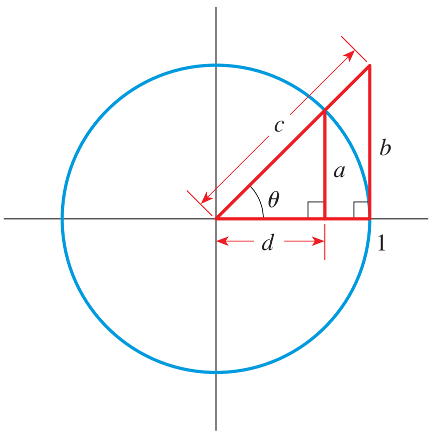


51. Trigonometric Ratios Express the length x in terms of the trigonometric ratios of θ.



52. Trigonometric Ratios Express the lengths a, b, c, and d in the figure in terms of the trigonometric ratios of θ.



APPLICATIONS

53. Height of a Building The angle of elevation to the top of the Empire State Building in New York is found to be 11° from the ground at a distance of 1 mi from the base of the building. Using this information, find the height of the Empire State Building.

54. Gateway Arch A plane is flying within sight of the Gateway Arch in St. Louis, Missouri, at an elevation of 35,000 ft. The pilot would like to estimate her distance from the Gateway Arch. She finds that the angle of depression to a point on the ground below the arch is 22°.

(a) What is the distance between the plane and the arch?

(b) What is the distance between a point on the ground directly below the plane and the arch?

55. Deviation of a Laser Beam A laser beam is to be directed toward the center of the moon, but the beam strays 0.5° from its intended path.

(a) How far has the beam diverged from its assigned target when it reaches the moon? (The distance from the earth to the moon is 240,000 mi.)

(b) The radius of the moon is about 1000 mi. Will the beam strike the moon?

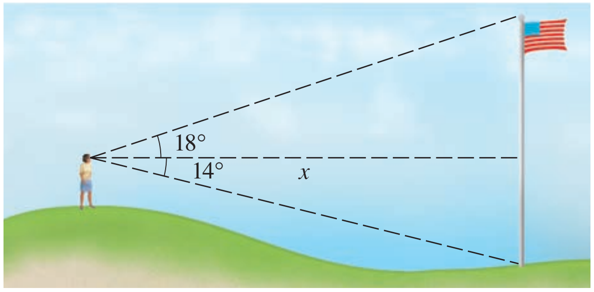
56. Distance at Sea From the top of a 200-ft lighthouse, the angle of depression to a ship in the ocean is 23°. How far is the ship from the base of the lighthouse?

57. Leaning Ladder A 20-ft ladder leans against a building so that the angle between the ground and the ladder is 72°. How high does the ladder reach on the building?

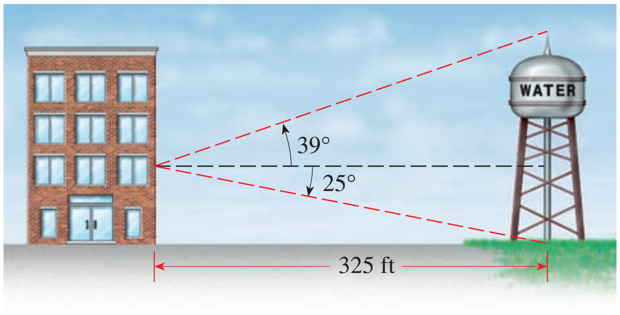
58. Height of a Tower A 600-ft guy wire is attached to the top of a communications tower. If the wire makes an angle of 65° with the ground, how tall is the communications tower?

59. Elevation of a Kite A man is lying on the beach, flying a kite. He holds the end of the kite string at ground level and estimates the angle of elevation of the kite to be 50°. If the string is 450 ft long, how high is the kite above the ground?

60. Determining a Distance A woman standing on a hill sees a flagpole that she knows is 60 ft tall. The angle of depression to the bottom of the pole is 14°, and the angle of elevation to the top of the pole is 18°. Find her distance x from the pole.



61. Height of a Tower A water tower is located 325 ft from a building (see the figure). From a window in the building, an observer notes that the angle of elevation to the top of the tower is 39° and that the angle of depression to the bottom of the tower is 25°. How tall is the tower? How high is the window?



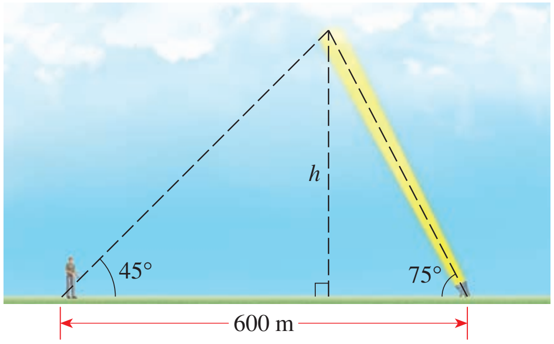
62. Determining a Distance An airplane is flying at an elevation of 5150 ft, directly above a straight highway. Two motorists are driving cars on the highway on opposite sides of the plane. The angle of depression to one car is 35°, and that to the other is 52°. How far apart are the cars?

63. Determining a Distance If both cars in Exercise 62 are on one side of the plane and if the angle of depression to one car is 38° and that to the other car is 52°, how far apart are the cars?

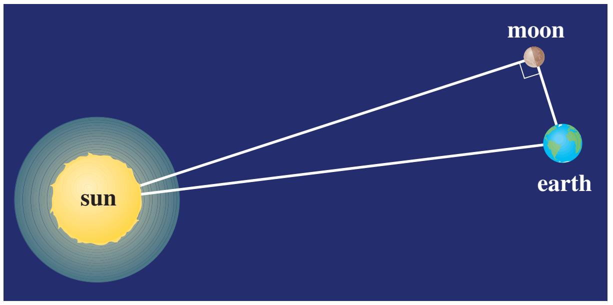
64. Height of a Balloon A hot-air balloon is floating above a straight road. To estimate their height above the ground, the balloonists simultaneously measure the angle of depression to two consecutive mileposts on the road on the same side of the balloon. The angles of depression are found to be 20° and 22°. How high is the balloon?

65. Height of a Mountain To estimate the height of a mountain above a level plain, the angle of elevation to the top of the mountain is measured to be 32°. One thousand feet closer to the mountain along the plain, it is found that the angle of elevation is 35°. Estimate the height of the mountain.

66. Height of Cloud Cover To measure the height of the cloud cover at an airport, a worker shines a spotlight upward at an angle 75° from the horizontal. An observer 600 m away measures the angle of elevation to the spot of light to be 45°. Find the height h of the cloud cover.



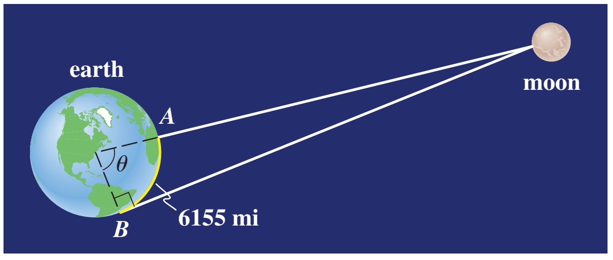
67. Distance to the Sun When the moon is exactly half full, the earth, moon, and sun form a right angle (see the figure). At that time the angle formed by the sun, earth, and moon is measured to be 89.85°. If the distance from the earth to the moon is 240,000 mi, estimate the distance from the earth to the sun.



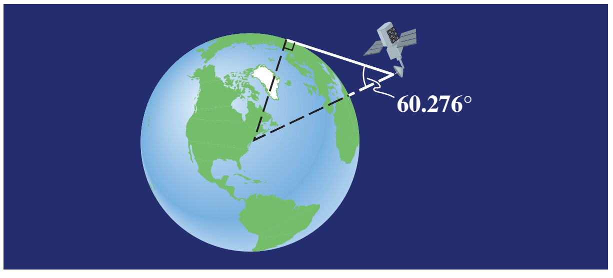
68. Distance to the Moon To find the distance to the sun as in Exercise 67, we needed to know the distance to the moon. Here is a way to estimate that distance: When the moon is seen at its zenith at a point A on the earth, it is observed to be at the horizon from point B (see the following figure). Points A and B are 6155 mi apart, and the radius of the earth is 3960 mi.

(a) Find the angle θ in degrees.

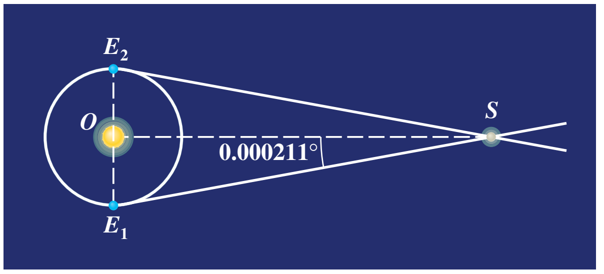
(b) Estimate the distance from point A to the moon.



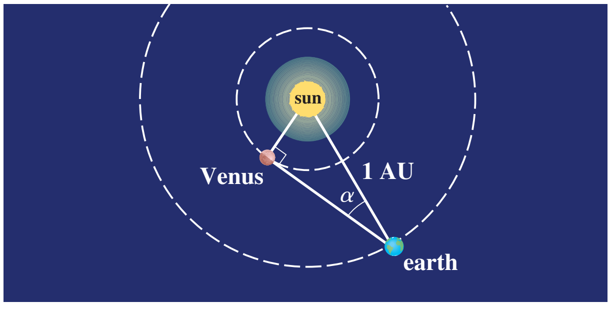
69. Radius of the Earth In Exercise 80 of Section 5.1 a method was given for finding the radius of the earth. Here is a more modern method: From a satellite 600 mi above the earth it is observed that the angle formed by the vertical and the line of sight to the horizon is 60.276°. Use this information to find the radius of the earth.



70. Parallax To find the distance to nearby stars, the method of parallax is used. The idea is to find a triangle with the star at one vertex and with a base as large as possible. To do this, the star is observed at two different times exactly 6 months apart, and its apparent change in position is recorded. From these two observations $\angle E_1SE_2$ can be calculated. (The times are chosen so that $\angle E_1SE_2$ is as large as possible, which guarantees that $\angle E_1OS$ is 90°.) The angle E_1SO is called the *parallax* of the star. Alpha Centauri, the star nearest the earth, has a parallax of 0.000211°. Estimate the distance to this star. (Take the distance from the earth to the sun to be 9.3×10^7 mi.)



71. Distance from Venus to the Sun The **elongation** α of a planet is the angle formed by the planet, earth, and sun (see the figure). When Venus achieves its maximum elongation of 46.3°, the earth, Venus, and the sun form a triangle with a right angle at Venus. Find the distance between Venus and the sun in astronomical units (AU). (By definition the distance between the earth and the sun is 1 AU.)



DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

72. DISCUSS: Similar Triangles If two triangles are similar, what properties do they share? Explain how these properties make it possible to define the trigonometric ratios without regard to the size of the triangle.

5.3 TRIGONOMETRIC FUNCTIONS OF ANGLES

■ Trigonometric Functions of Angles ■ Evaluating Trigonometric Functions at Any Angle
■ Trigonometric Identities ■ Areas of Triangles

In Section 5.2 we defined the trigonometric ratios for acute angles. Here we extend the trigonometric ratios to all angles by defining the trigonometric functions of angles. With these functions we can solve practical problems that involve angles that are not necessarily acute.

Trigonometric Functions of Angles

Let POQ be a right triangle with acute angle θ as shown in Figure 1(a). Place θ in standard position as shown in Figure 1(b).

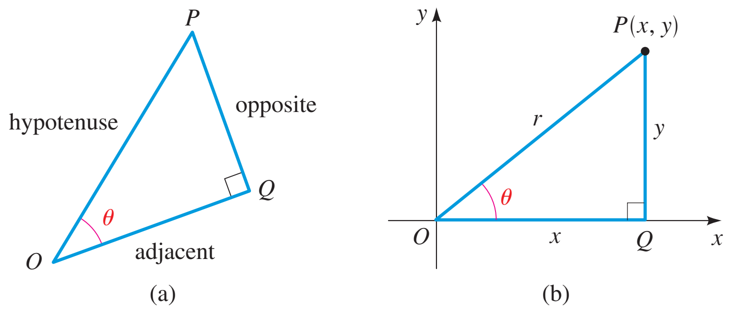


FIGURE 1

Then $P = P(x, y)$ is a point on the terminal side of θ. In triangle POQ the opposite side has length y and the adjacent side has length x. Using the Pythagorean Theorem, we see that the hypotenuse has length $r = \sqrt{x^2 + y^2}$. So

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

The other trigonometric ratios can be found in the same way.

These observations allow us to extend the trigonometric ratios to any angle. We define the trigonometric functions of angles as follows (see Figure 2).

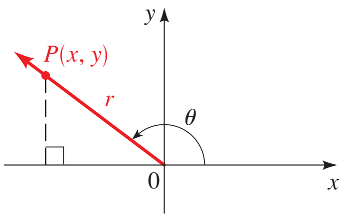


FIGURE 2

DEFINITION OF THE TRIGONOMETRIC FUNCTIONS

Let θ be an angle in standard position, and let $P(x, y)$ be a point on the terminal side. If $r = \sqrt{x^2 + y^2}$ is the distance from the origin to the point $P(x, y)$, then

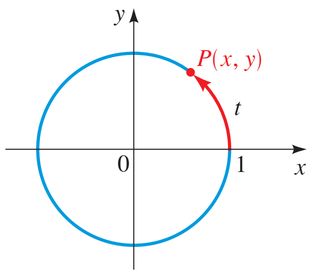
$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

$$\csc\theta = \frac{r}{y} \quad (y \neq 0) \qquad \sec\theta = \frac{r}{x} \quad (x \neq 0) \qquad \cot\theta = \frac{x}{y} \quad (y \neq 0)$$

Since division by 0 is an undefined operation, certain trigonometric functions are not defined for certain angles. For example, $\tan 90° = y/x$ is undefined because $x = 0$. The angles for which the trigonometric functions may be undefined are the angles for which

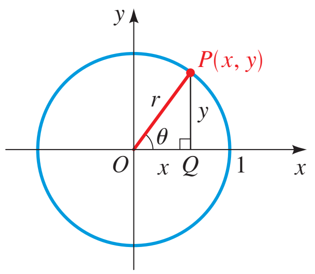
Relationship to the Trigonometric Functions of Real Numbers

You may have already studied the trigonometric functions defined by using the unit circle (Chapter 6). To see how they relate to the trigonometric functions of an *angle*, let's start with the unit circle in the coordinate plane.



$P(x, y)$ is the terminal point determined by t.

Let $P(x, y)$ be the terminal point determined by an arc of length t on the unit circle. Then t subtends an angle θ at the center of the circle. If we drop a perpendicular from P onto the point Q on the x-axis, then triangle $\triangle OPQ$ is a right triangle with legs of length x and y, as shown in the figure.



Triangle OPQ is a right triangle.

Now, by the definition of the trigonometric functions of the *real number* t we have

$$\sin t = y$$

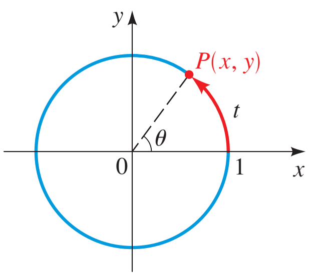
$$\cos t = x$$

By the definition of the trigonometric functions of the *angle* θ we have

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x$$

If θ is measured in radians, then $\theta = t$. (See the figure below.) Comparing the two ways of defining the trigonometric functions, we see that they are identical. In other words, as functions they assign identical values to a given real number. (The real number is the radian measure of θ in one case or the length t of an arc in the other.)



The radian measure of angle θ is t.

Why then do we study trigonometry in two different ways? Because different applications require that we view the trigonometric functions differently. (See *Focus on Modeling*, pages 499, 568, and 617, and Sections 5.2, 5.5, and 5.6.)

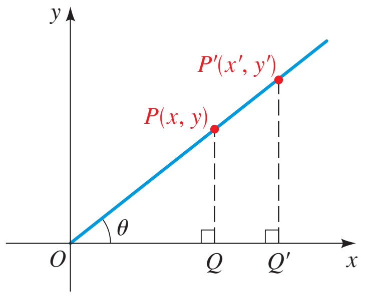
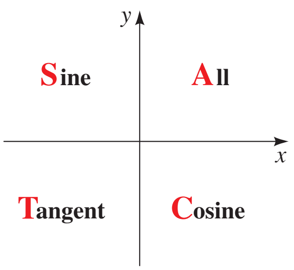


FIGURE 3

either the x- or y-coordinate of a point on the terminal side of the angle is 0. These are **quadrantal angles**—angles that are coterminal with the coordinate axes.

It is a crucial fact that the values of the trigonometric functions do *not* depend on the choice of the point $P(x, y)$. This is because if $P'(x', y')$ is any other point on the terminal side, as in Figure 3, then triangles POQ and $P'OQ'$ are similar.

The following mnemonic device can be used to remember which trigonometric functions are positive in each quadrant: **A**ll of them, **S**ine, **T**angent, or **C**osine.



You can remember this as "All Students Take Calculus."

Evaluating Trigonometric Functions at Any Angle

From the definition we see that the values of the trigonometric functions are all positive if the angle θ has its terminal side in Quadrant I. This is because x and y are positive in this quadrant. [Of course, r is always positive, since it is simply the distance from the origin to the point $P(x, y)$.] If the terminal side of θ is in Quadrant II, however, then x is negative and y is positive. Thus in Quadrant II the functions $\sin\theta$ and $\csc\theta$ are positive, and all the other trigonometric functions have negative values. You can check the other entries in the following table.

SIGNS OF THE TRIGONOMETRIC FUNCTIONS

Quadrant	Positive Functions	Negative Functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

We now turn our attention to finding the values of the trigonometric functions for angles that are not acute.

EXAMPLE 1 ■ Finding Trigonometric Functions of Angles

Find **(a)** $\cos 135°$ and **(b)** $\tan 390°$.

SOLUTION

(a) From Figure 4 we see that $\cos 135° = -x/r$. But $\cos 45° = x/r$, and since $\cos 45° = \sqrt{2}/2$, we have

$$\cos 135° = -\frac{\sqrt{2}}{2}$$

(b) The angles 390° and 30° are coterminal. From Figure 5 it's clear that $\tan 390° = \tan 30°$, and since $\tan 30° = \sqrt{3}/3$, we have

$$\tan 390° = \frac{\sqrt{3}}{3}$$

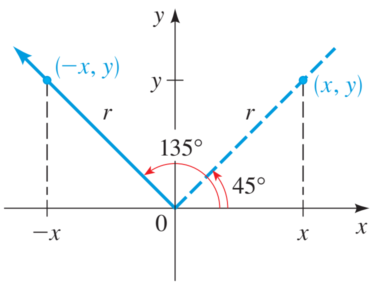


FIGURE 4

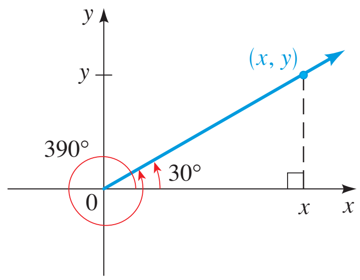


FIGURE 5

Now Try Exercises 13 and 15 ■

From Example 1 we see that the trigonometric functions for angles that aren't acute have the same value, except possibly for sign, as the corresponding trigonometric functions of an acute angle. That acute angle will be called the *reference angle*.

REFERENCE ANGLE

Let θ be an angle in standard position. The **reference angle** $\bar{\theta}$ associated with θ is the acute angle formed by the terminal side of θ and the x-axis.

Figure 6 shows that to find a reference angle $\bar{\theta}$, it's useful to know the quadrant in which the terminal side of the angle θ lies.

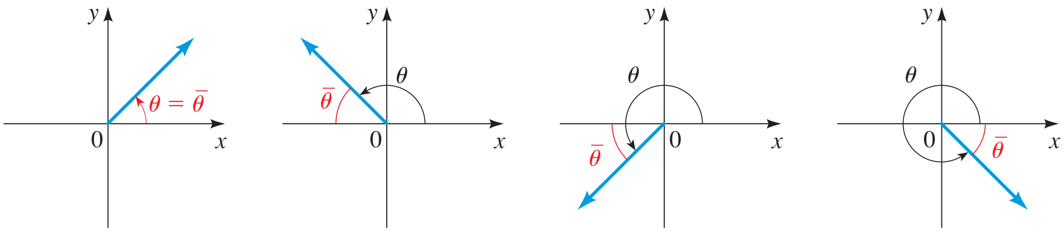


FIGURE 6 The reference angle $\bar{\theta}$ for an angle θ

EXAMPLE 2 ■ Finding Reference Angles

Find the reference angle for **(a)** $\theta = \dfrac{5\pi}{3}$ and **(b)** $\theta = 870°$.

SOLUTION

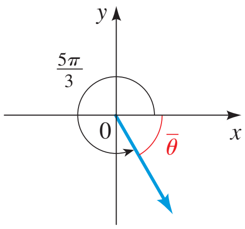


FIGURE 7

(a) The reference angle is the acute angle formed by the terminal side of the angle $5\pi/3$ and the x-axis (see Figure 7). Since the terminal side of this angle is in Quadrant IV, the reference angle is

$$\bar{\theta} = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$$

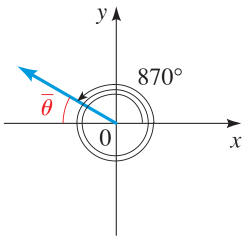


FIGURE 8

(b) The angles 870° and 150° are coterminal [because $870 - 2(360) = 150$]. Thus the terminal side of this angle is in Quadrant II (see Figure 8). So the reference angle is

$$\bar{\theta} = 180° - 150° = 30°$$

Now Try Exercises 5 and 9 ■

EVALUATING TRIGONOMETRIC FUNCTIONS FOR ANY ANGLE

To find the values of the trigonometric functions for any angle θ, we carry out the following steps.

1. Find the reference angle $\bar{\theta}$ associated with the angle θ.
2. Determine the sign of the trigonometric function of θ by noting the quadrant in which θ lies.
3. The value of the trigonometric function of θ is the same, except possibly for sign, as the value of the trigonometric function of $\bar{\theta}$.

EXAMPLE 3 ■ Using the Reference Angle to Evaluate Trigonometric Functions

Find **(a)** sin 240° and **(b)** cot 495°.

SOLUTION

(a) This angle has its terminal side in Quadrant III, as shown in Figure 9. The reference angle is therefore $240° - 180° = 60°$, and the value of sin 240° is negative. Thus

$$\sin 240° = \underbrace{-}_{\text{Sign}}\underbrace{\sin 60°}_{\text{Reference angle}} = -\frac{\sqrt{3}}{2}$$

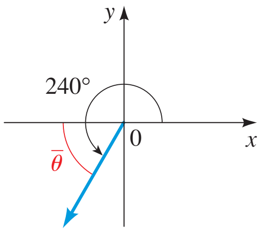


FIGURE 9

S | A / T | C — sin 240° is *negative*.

(b) The angle 495° is coterminal with the angle 135°, and the terminal side of this angle is in Quadrant II, as shown in Figure 10. So the reference angle is $180° - 135° = 45°$, and the value of cot 495° is negative. We have

$$\underbrace{\cot 495° = \cot 135°}_{\text{Coterminal angles}} = \underbrace{-}_{\text{Sign}}\underbrace{\cot 45°}_{\text{Reference angle}} = -1$$

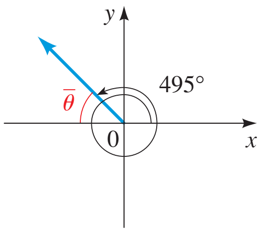


FIGURE 10

S | A / T | C — tan 495° is *negative*, so cot 495° is *negative*.

Now Try Exercises 19 and 21

EXAMPLE 4 ■ Using the Reference Angle to Evaluate Trigonometric Functions

Find **(a)** $\sin \frac{16\pi}{3}$ and **(b)** $\sec\left(-\frac{\pi}{4}\right)$.

SOLUTION

(a) The angle $16\pi/3$ is coterminal with $4\pi/3$, and these angles are in Quadrant III (see Figure 11). Thus the reference angle is $(4\pi/3) - \pi = \pi/3$. Since the value of sine is negative in Quadrant III, we have

$$\underbrace{\sin\frac{16\pi}{3} = \sin\frac{4\pi}{3}}_{\text{Coterminal angles}} = \underbrace{-}_{\text{Sign}}\underbrace{\sin\frac{\pi}{3}}_{\text{Reference angle}} = -\frac{\sqrt{3}}{2}$$

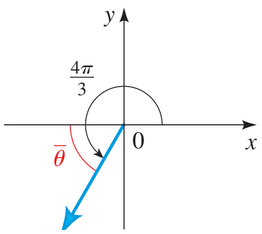


FIGURE 11

S | A / T | C — $\sin\frac{16\pi}{3}$ is *negative*.

(b) The angle $-\pi/4$ is in Quadrant IV, and its reference angle is $\pi/4$ (see Figure 12). Since secant is positive in this quadrant, we get

$$\sec\left(-\frac{\pi}{4}\right) = \underbrace{+}_{\text{Sign}}\underbrace{\sec\frac{\pi}{4}}_{\text{Reference angle}} = \sqrt{2}$$

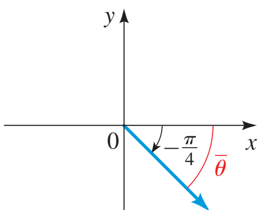


FIGURE 12

S | A / T | C — $\cos(-\frac{\pi}{4})$ is *positive*, so $\sec(-\frac{\pi}{4})$ is *positive*.

Now Try Exercises 25 and 27

■ Trigonometric Identities

The trigonometric functions of angles are related to each other through several important equations called **trigonometric identities**. We've already encountered the reciprocal identities. These identities continue to hold for any angle θ, provided that both sides of the equation are defined. The Pythagorean identities are a consequence of the Pythagorean Theorem.*

FUNDAMENTAL IDENTITIES

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

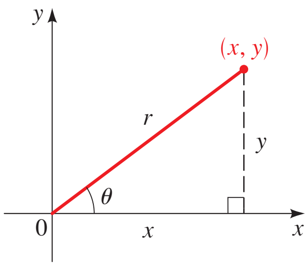


FIGURE 13

Proof Let's prove the first Pythagorean identity. Using $x^2 + y^2 = r^2$ (the Pythagorean Theorem) in Figure 13, we have

$$\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

Thus $\sin^2\theta + \cos^2\theta = 1$. (Although the figure indicates an acute angle, you should check that the proof holds for all angles θ.) ■

See Exercise 76 for the proofs of the other two Pythagorean identities.

EXAMPLE 5 ■ Expressing One Trigonometric Function in Terms of Another

(a) Express $\sin\theta$ in terms of $\cos\theta$.

(b) Express $\tan\theta$ in terms of $\sin\theta$, where θ is in Quadrant II.

SOLUTION

(a) From the first Pythagorean identity we get

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

where the sign depends on the quadrant. If θ is in Quadrant I or II, then $\sin\theta$ is positive, so

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$

whereas if θ is in Quadrant III or IV, $\sin\theta$ is negative, so

$$\sin\theta = -\sqrt{1 - \cos^2\theta}$$

(b) Since $\tan\theta = \sin\theta/\cos\theta$, we need to write $\cos\theta$ in terms of $\sin\theta$. By part (a)

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

and since $\cos\theta$ is negative in Quadrant II, the negative sign applies here. Thus

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{-\sqrt{1 - \sin^2\theta}}$$

Now Try Exercise 41 ■

*We follow the usual convention of writing $\sin^2\theta$ for $(\sin\theta)^2$. In general, we write $\sin^n\theta$ for $(\sin\theta)^n$ for all integers n except $n = -1$. The superscript $n = -1$ will be assigned another meaning in Section 5.4. Of course, the same convention applies to the other five trigonometric functions.

EXAMPLE 6 ■ Evaluating a Trigonometric Function

If $\tan\theta = \frac{2}{3}$ and θ is in Quadrant III, find $\cos\theta$.

SOLUTION 1 We need to write $\cos\theta$ in terms of $\tan\theta$. From the identity $\tan^2\theta + 1 = \sec^2\theta$ we get $\sec\theta = \pm\sqrt{\tan^2\theta + 1}$. In Quadrant III, $\sec\theta$ is negative, so

$$\sec\theta = -\sqrt{\tan^2\theta + 1}$$

Thus

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{-\sqrt{\tan^2\theta + 1}}$$

$$= \frac{1}{-\sqrt{\left(\frac{2}{3}\right)^2 + 1}} = \frac{1}{-\sqrt{\frac{13}{9}}} = -\frac{3}{\sqrt{13}}$$

If you wish to rationalize the denominator, you can express $\cos\theta$ as

$$-\frac{3}{\sqrt{13}}\cdot\frac{\sqrt{13}}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}$$

SOLUTION 2 This problem can be solved more easily by using the method of Example 2 of Section 5.2. Recall that, except for sign, the values of the trigonometric functions of any angle are the same as those of an acute angle (the reference angle). So, ignoring the sign for the moment, let's sketch a right triangle with an acute angle $\bar{\theta}$ satisfying $\tan\bar{\theta} = \frac{2}{3}$ (see Figure 14). By the Pythagorean Theorem the hypotenuse of this triangle has length $\sqrt{13}$. From the triangle in Figure 14 we immediately see that $\cos\bar{\theta} = 3/\sqrt{13}$. Since θ is in Quadrant III, $\cos\theta$ is negative, so

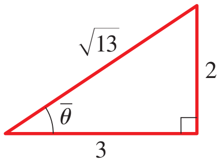


FIGURE 14

$$\cos\theta = -\frac{3}{\sqrt{13}}$$

Now Try Exercise 47 ■

EXAMPLE 7 ■ Evaluating Trigonometric Functions

If $\sec\theta = 2$ and θ is in Quadrant IV, find the other five trigonometric functions of θ.

SOLUTION We sketch a triangle as in Figure 15 so that $\sec\bar{\theta} = 2$. Taking into account the fact that θ is in Quadrant IV, we get

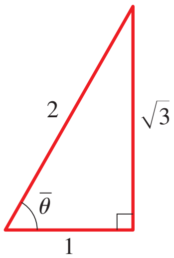


FIGURE 15

$$\sin\theta = -\frac{\sqrt{3}}{2} \qquad \cos\theta = \frac{1}{2} \qquad \tan\theta = -\sqrt{3}$$

$$\csc\theta = -\frac{2}{\sqrt{3}} \qquad \sec\theta = 2 \qquad \cot\theta = -\frac{1}{\sqrt{3}}$$

Now Try Exercise 49 ■

■ Areas of Triangles

We conclude this section with an application of the trigonometric functions that involves angles that are not necessarily acute. More extensive applications appear in Sections 5.5 and 5.6.

The area of a triangle is $\mathcal{A} = \frac{1}{2} \times \text{base} \times \text{height}$. If we know two sides and the included angle of a triangle, then we can find the height using the trigonometric functions, and from this we can find the area.

If θ is an acute angle, then the height of the triangle in Figure 16(a) is given by $h = b\sin\theta$. Thus the area is

$$\mathcal{A} = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2}ab\sin\theta$$

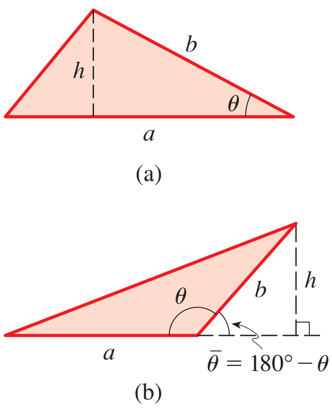


FIGURE 16

If the angle θ is not acute, then from Figure 16(b) we see that the height of the triangle is

$$h = b\sin(180° - \theta) = b\sin\theta$$

This is so because the reference angle of θ is the angle $180° - \theta$. Thus in this case also the area of the triangle is

$$\mathscr{A} = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2}ab \sin\theta$$

AREA OF A TRIANGLE

The area $\mathscr{A}$ of a triangle with sides of lengths a and b and with included angle θ is

$$\mathscr{A} = \tfrac{1}{2}ab \sin\theta$$

EXAMPLE 8 ■ Finding the Area of a Triangle

Find the area of triangle ABC shown in Figure 17.

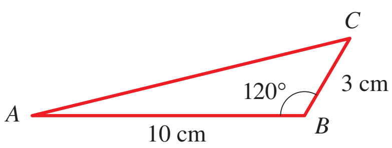


FIGURE 17

SOLUTION The triangle has sides of length 10 cm and 3 cm, with included angle 120°. Therefore

$$\begin{aligned}
\mathscr{A} &= \tfrac{1}{2}ab \sin\theta \\
&= \tfrac{1}{2}(10)(3)\sin 120° \\
&= 15 \sin 60° && \text{Reference angle} \\
&= 15\frac{\sqrt{3}}{2} \approx 13 \text{ cm}^2
\end{aligned}$$

Now Try Exercise 57

5.3 EXERCISES

CONCEPTS

1. If the angle θ is in standard position and $P(x, y)$ is a point on the terminal side of θ, and r is the distance from the origin to P, then

$$\sin\theta = \frac{\square}{\square} \qquad \cos\theta = \frac{\square}{\square} \qquad \tan\theta = \frac{\square}{\square}$$

2. The sign of a trigonometric function of θ depends on the ________ in which the terminal side of the angle θ lies.

In Quadrant II, $\sin\theta$ is ________ (positive / negative).

In Quadrant III, $\cos\theta$ is ________ (positive / negative).

In Quadrant IV, $\sin\theta$ is ________ (positive / negative).

3. (a) If θ is in standard position, then the reference angle $\bar{\theta}$ is the acute angle formed by the terminal side of θ and the ________. So the reference angle for $\theta = 100°$ is $\bar{\theta} =$ ________, and that for $\theta = 190°$ is $\bar{\theta} =$ ________.

(b) If θ is any angle, the value of a trigonometric function of θ is the same, except possibly for sign, as the value of the trigonometric function of $\bar{\theta}$. So $\sin 100° = \sin$ ____, and $\sin 190° = -\sin$ ____.

4. The area $\mathscr{A}$ of a triangle with sides of lengths a and b and with included angle θ is given by the formula $\mathscr{A} =$ ________. So the area of the triangle with sides 4 and 7 and included angle $\theta = 30°$ is ________.

SKILLS

5–12 ■ Reference Angle Find the reference angle for the given angle.

5. (a) $120°$	**(b)** $200°$	**(c)** $285°$
6. (a) $175°$	**(b)** $310°$	**(c)** $730°$
7. (a) $225°$	**(b)** $810°$	**(c)** $-105°$
8. (a) $99°$	**(b)** $-199°$	**(c)** $359°$
9. (a) $\dfrac{7\pi}{10}$	**(b)** $\dfrac{9\pi}{8}$	**(c)** $\dfrac{10\pi}{3}$
10. (a) $\dfrac{5\pi}{6}$	**(b)** $\dfrac{10\pi}{9}$	**(c)** $\dfrac{23\pi}{7}$
11. (a) $\dfrac{5\pi}{7}$	**(b)** -1.4π	**(c)** 1.4
12. (a) 2.3π	**(b)** 2.3	**(c)** -10π

13–36 ■ Values of Trigonometric Functions Find the exact value of the trigonometric function.

13. $\cos 150°$ **14.** $\sin 240°$ **15.** $\tan 330°$
16. $\sin(-30°)$ **17.** $\cot(-120°)$ **18.** $\csc 300°$
19. $\csc(-630°)$ **20.** $\cot 210°$ **21.** $\cos 570°$
22. $\sec 120°$ **23.** $\tan 750°$ **24.** $\cos 660°$
25. $\sin \dfrac{3\pi}{2}$ **26.** $\cos \dfrac{4\pi}{3}$ **27.** $\tan\left(-\dfrac{4\pi}{3}\right)$
28. $\cos\left(-\dfrac{11\pi}{6}\right)$ **29.** $\csc\left(-\dfrac{5\pi}{6}\right)$ **30.** $\sec \dfrac{7\pi}{6}$
31. $\sec \dfrac{17\pi}{3}$ **32.** $\csc \dfrac{5\pi}{4}$ **33.** $\cot\left(-\dfrac{\pi}{4}\right)$
34. $\cos \dfrac{7\pi}{4}$ **35.** $\tan \dfrac{5\pi}{2}$ **36.** $\sin \dfrac{11\pi}{6}$

37–40 ■ Quadrant in Which an Angle Lies Find the quadrant in which θ lies from the information given.

37. $\sin\theta < 0$ and $\cos\theta < 0$
38. $\tan\theta < 0$ and $\sin\theta < 0$
39. $\sec\theta > 0$ and $\tan\theta < 0$
40. $\csc\theta > 0$ and $\cos\theta < 0$

41–46 ■ Expressing One Trigonometric Function in Terms of Another Write the first trigonometric function in terms of the second for θ in the given quadrant.

41. $\tan\theta$, $\cos\theta$; θ in Quadrant III
42. $\cot\theta$, $\sin\theta$; θ in Quadrant II
43. $\cos\theta$, $\sin\theta$; θ in Quadrant IV
44. $\sec\theta$, $\sin\theta$; θ in Quadrant I
45. $\sec\theta$, $\tan\theta$; θ in Quadrant II
46. $\csc\theta$, $\cot\theta$; θ in Quadrant III

47–54 ■ Values of Trigonometric Functions Find the values of the trigonometric functions of θ from the information given.

47. $\sin\theta = -\frac{4}{5}$, θ in Quadrant IV
48. $\tan\theta = \frac{4}{3}$, θ in Quadrant III
49. $\cos\theta = \frac{7}{12}$, $\sin\theta < 0$
50. $\cot\theta = -\frac{8}{9}$, $\cos\theta > 0$
51. $\csc\theta = 2$, θ in Quadrant I
52. $\cot\theta = \frac{1}{4}$, $\sin\theta < 0$
53. $\cos\theta = -\frac{2}{7}$, $\tan\theta < 0$
54. $\tan\theta = -4$, $\sin\theta > 0$

55–56 ■ Values of an Expression If $\theta = \pi/3$, find the value of each expression.

55. $\sin 2\theta$, $2\sin\theta$ **56.** $\sin^2\theta$, $\sin(\theta^2)$

57–60 ■ Area of a Triangle Find the area of the triangle with the given description.

57. A triangle with sides of length 7 and 9 and included angle 72°
58. A triangle with sides of length 10 and 22 and included angle 10°
59. An equilateral triangle with side of length 10
60. An equilateral triangle with side of length 13

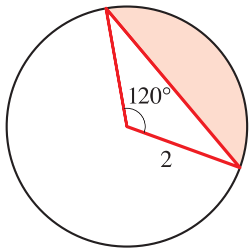
61. Finding an Angle of a Triangle A triangle has an area of 16 in^2, and two of the sides have lengths 5 in. and 7 in. Find the sine of the angle included by these two sides.

62. Finding a Side of a Triangle An isosceles triangle has an area of 24 cm^2, and the angle between the two equal sides is $5\pi/6$. Find the length of the two equal sides.

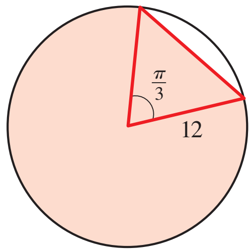
SKILLS Plus

63–64 ■ Area of a Region Find the area of the shaded region in the figure.

63.



64.



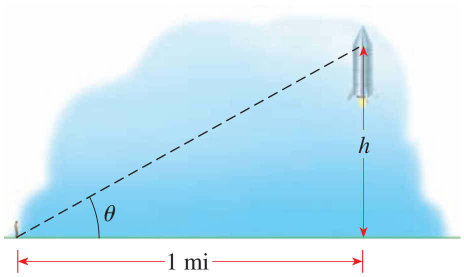
APPLICATIONS

65. Height of a Rocket A rocket fired straight up is tracked by an observer on the ground 1 mi away.

(a) Show that when the angle of elevation is θ, the height of the rocket (in ft) is $h = 5280\tan\theta$.

(b) Complete the table to find the height of the rocket at the given angles of elevation.

θ	20°	60°	80°	85°
h				



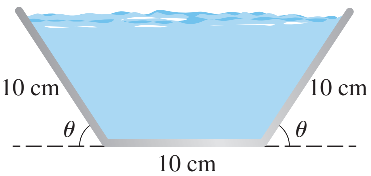
66. Rain Gutter A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle θ. (See the figure on the next page.)

(a) Show that the cross-sectional area of the gutter is modeled by the function

$$A(\theta) = 100\sin\theta + 100\sin\theta\cos\theta$$

(b) Graph the function A for $0 \le \theta \le \pi/2$.

(c) For what angle θ is the largest cross-sectional area achieved?

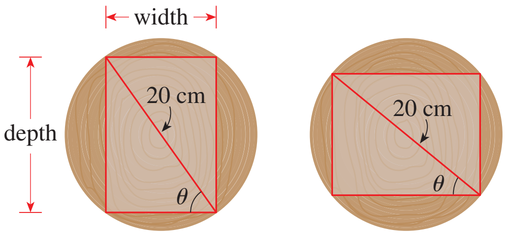


67. Wooden Beam A rectangular beam is to be cut from a cylindrical log of diameter 20 cm. The figures show different ways this can be done.

(a) Express the cross-sectional area of the beam as a function of the angle θ in the figures.

(b) Graph the function you found in part (a).

(c) Find the dimensions of the beam with largest cross-sectional area.



68. Strength of a Beam The strength of a beam is proportional to the width and the square of the depth. A beam is cut from a log as in Exercise 67. Express the strength of the beam as a function of the angle θ in the figures.

69. Throwing a Shot Put The range R and height H of a shot put thrown with an initial velocity of v_0 ft/s at an angle θ are given by

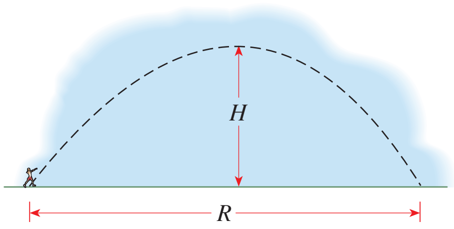
$$R = \frac{v_0^2 \sin(2\theta)}{g}$$

$$H = \frac{v_0^2 \sin^2\theta}{2g}$$

On the earth $g = 32$ ft/s^2, and on the moon $g = 5.2$ ft/s^2. Find the range and height of a shot put thrown under the given conditions.

(a) On the earth with $v_0 = 12$ ft/s and $\theta = \pi/6$

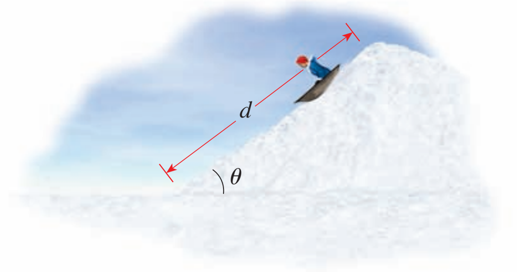
(b) On the moon with $v_0 = 12$ ft/s and $\theta = \pi/6$



70. Sledding The time in seconds that it takes for a sled to slide down a hillside inclined at an angle θ is

$$t = \sqrt{\frac{d}{16 \sin \theta}}$$

where d is the length of the slope in feet. Find the time it takes to slide down a 2000-ft slope inclined at 30°.



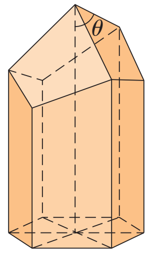
71. Beehives In a beehive each cell is a regular hexagonal prism, as shown in the figure. The amount of wax W in the cell depends on the apex angle θ and is given by

$$W = 3.02 - 0.38 \cot \theta + 0.65 \csc \theta$$

Bees instinctively choose θ so as to use the least amount of wax possible.

(a) Use a graphing device to graph W as a function of θ for $0 < \theta < \pi$.

(b) For what value of θ does W have its minimum value? [*Note:* Biologists have discovered that bees rarely deviate from this value by more than a degree or two.]



72. Turning a Corner A steel pipe is being carried down a hallway that is 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide.

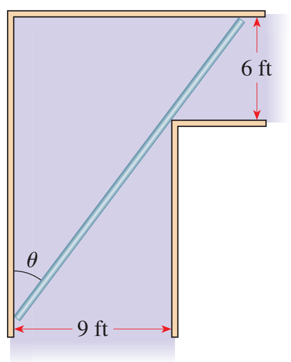
(a) Show that the length of the pipe in the figure is modeled by the function

$$L(\theta) = 9 \csc \theta + 6 \sec \theta$$

(b) Graph the function L for $0 < \theta < \pi/2$.

(c) Find the minimum value of the function L.

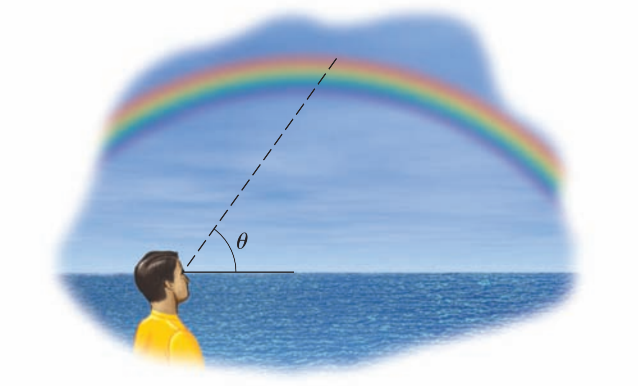
(d) Explain why the value of L you found in part (c) is the length of the longest pipe that can be carried around the corner.



73. Rainbows Rainbows are created when sunlight of different wavelengths (colors) is refracted and reflected in raindrops. The angle of elevation θ of a rainbow is always the same. It can be shown that $\theta = 4\beta - 2\alpha$, where

$$\sin \alpha = k \sin \beta$$

and $\alpha = 59.4°$ and $k = 1.33$ is the index of refraction of water. Use the given information to find the angle of elevation θ of a rainbow. [*Hint:* Find $\sin \beta$, then use the $\boxed{\text{SIN}^{-1}}$ key on your calculator to find β.] (For a mathematical explanation of rainbows see *Calculus Early Transcendentals,* 7th Edition, by James Stewart, page 282.)



DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

74. DISCUSS: Using a Calculator To solve a certain problem, you need to find the sine of 4 rad. Your study partner uses his calculator and tells you that

$$\sin 4 = 0.0697564737$$

On your calculator you get

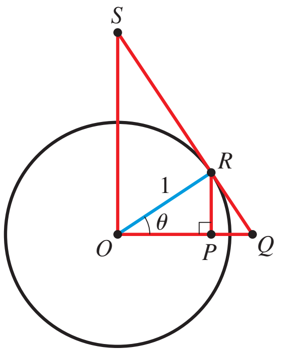
$$\sin 4 = -0.7568024953$$

What is wrong? What mistake did your partner make?

75. DISCUSS ■ DISCOVER: Viète's Trigonometric Diagram In the 16th century the French mathematician François Viète (see page 119) published the following remarkable diagram. Each of the six trigonometric functions of θ is equal to the length of a line segment in the figure. For instance, $\sin \theta = |PR|$, since from $\triangle OPR$ we see that

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{|PR|}{|OR|} = \frac{|PR|}{1} = |PR|$$

For each of the five other trigonometric functions, find a line segment in the figure whose length equals the value of the function at θ. [*Note:* The radius of the circle is 1, the center is O, segment QS is tangent to the circle at R, and $\angle SOQ$ is a right angle.]



76. PROVE: Pythagorean Identities To prove the following Pythagorean identities, start with the first Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$, which was proved in the text, and then divide both sides by an appropriate trigonometric function of θ.

(a) $\tan^2\theta + 1 = \sec^2\theta$ **(b)** $1 + \cot^2\theta = \csc^2\theta$

77. DISCUSS ■ DISCOVER: Degrees and Radians What is the smallest positive real number x with the property that the sine of x degrees is equal to the sine of x radians?

5.4 INVERSE TRIGONOMETRIC FUNCTIONS AND RIGHT TRIANGLES

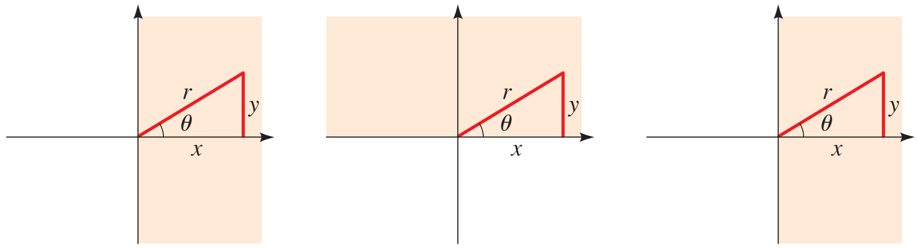
■ The Inverse Sine, Inverse Cosine, and Inverse Tangent Functions ■ Solving for Angles in Right Triangles ■ Evaluating Expressions Involving Inverse Trigonometric Functions

The graphs of the inverse trigonometric functions are studied in Section 6.5.

Recall that for a function to have an inverse, it must be one-to-one. Since the trigonometric functions are not one-to-one, they do not have inverses. So we restrict the domain of each of the trigonometric functions to intervals on which they attain all their values and on which they are one-to-one. The resulting functions have the same range as the original functions but are one-to-one.

■ The Inverse Sine, Inverse Cosine, and Inverse Tangent Functions

Let's first consider the sine function. We restrict the domain of the sine function to angles θ with $-\pi/2 \le \theta \le \pi/2$. From Figure 1 we see that on this domain the sine function attains each of the values in the interval $[-1, 1]$ exactly once and so is one-to-one. Similarly, we restrict the domains of cosine and tangent as shown in Figure 1.



$\sin\theta = \frac{y}{r}$ $\qquad$ $\cos\theta = \frac{x}{r}$ $\qquad$ $\tan\theta = \frac{y}{x}$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\qquad$ $0 \le \theta \le \pi$ $\qquad$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

FIGURE 1 Restricted domains of the sine, cosine, and tangent functions

On these restricted domains we can define an inverse for each of these functions. By the definition of inverse function we have

$$\sin^{-1}x = y \quad \Leftrightarrow \quad \sin y = x$$
$$\cos^{-1}x = y \quad \Leftrightarrow \quad \cos y = x$$
$$\tan^{-1}x = y \quad \Leftrightarrow \quad \tan y = x$$

We summarize the domains and ranges of the inverse trigonometric functions in the following box.

THE INVERSE SINE, INVERSE COSINE, AND INVERSE TANGENT FUNCTIONS

The sine, cosine, and tangent functions on the restricted domains $[-\pi/2, \pi/2]$, $[0, \pi]$, and $(-\pi/2, \pi/2)$, respectively, are one-to one and so have inverses. The inverse functions have domain and range as follows.

Function	Domain	Range
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}$	$\mathbb{R}$	$(-\pi/2, \pi/2)$

The functions $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ are sometimes called **arcsine**, **arccosine**, and **arctangent**, respectively.

Since these are inverse functions, they reverse the rule of the original function. For example, since $\sin \pi/6 = \frac{1}{2}$, it follows that $\sin^{-1}\frac{1}{2} = \pi/6$. The following example gives further illustrations.

EXAMPLE 1 ■ Evaluating Inverse Trigonometric Functions

Find the exact value.

(a) $\sin^{-1}\dfrac{\sqrt{3}}{2}$ $\qquad$ **(b)** $\cos^{-1}\left(-\frac{1}{2}\right)$ $\qquad$ **(c)** $\tan^{-1} 1$

SOLUTION

(a) The angle in the interval $[-\pi/2, \pi/2]$ whose sine is $\sqrt{3}/2$ is $\pi/3$. Thus $\sin^{-1}(\sqrt{3}/2) = \pi/3$.

(b) The angle in the interval $[0, \pi]$ whose cosine is $-\frac{1}{2}$ is $2\pi/3$. Thus $\cos^{-1}(-\frac{1}{2}) = 2\pi/3$.

(c) The angle in the interval $(-\pi/2, \pi/2)$ whose tangent is 1 is $\pi/4$. Thus $\tan^{-1}1 = \pi/4$.

Now Try Exercise 5 ■

EXAMPLE 2 ■ Evaluating Inverse Trigonometric Functions

Find approximate values for the given expression.

(a) $\sin^{-1}(0.71)$ **(b)** $\tan^{-1}2$ **(c)** $\cos^{-1}2$

SOLUTION We use a calculator to approximate these values.

(a) Using the INV SIN, or SIN⁻¹, or ARC SIN key(s) on the calculator (with the calculator in radian mode), we get

$$\sin^{-1}(0.71) \approx 0.78950$$

(b) Using the INV TAN, or TAN⁻¹, or ARC TAN key(s) on the calculator (with the calculator in radian mode), we get

$$\tan^{-1}2 \approx 1.10715$$

(c) Since $2 > 1$, it is not in the domain of $\cos^{-1}$, so $\cos^{-1}2$ is not defined.

Now Try Exercises 9, 13, and 15 ■

Solving for Angles in Right Triangles

In Section 5.2 we solved triangles by using the trigonometric functions to find the unknown sides. We now use the inverse trigonometric functions to solve for *angles* in a right triangle.

EXAMPLE 3 ■ Finding an Angle in a Right Triangle

Find the angle θ in the triangle shown in Figure 2.

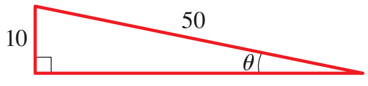


FIGURE 2

SOLUTION Since θ is the angle opposite the side of length 10 and the hypotenuse has length 50, we have

$$\sin\theta = \frac{10}{50} = \frac{1}{5} \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}}$$

Now we can use $\sin^{-1}$ to find θ.

$$\theta = \sin^{-1}\tfrac{1}{5} \qquad \text{Definition of } \sin^{-1}$$

$$\theta \approx 11.5° \qquad \text{Calculator (in degree mode)}$$

Now Try Exercise 17 ■

EXAMPLE 4 ■ Solving for an Angle in a Right Triangle

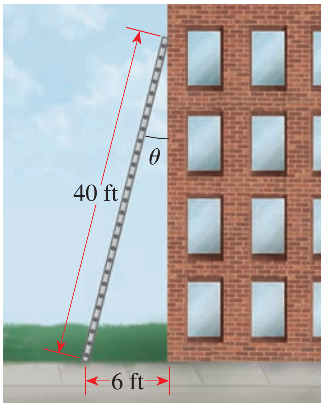


FIGURE 3

A 40-ft ladder leans against a building. If the base of the ladder is 6 ft from the base of the building, what is the angle formed by the ladder and the building?

SOLUTION First we sketch a diagram as in Figure 3. If θ is the angle between the ladder and the building, then

$$\sin\theta = \frac{6}{40} = 0.15 \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}}$$

Now we use $\sin^{-1}$ to find θ.

$$\theta = \sin^{-1}(0.15) \qquad \text{Definition of } \sin^{-1}$$
$$\theta \approx 8.6° \qquad \text{Calculator (in degree mode)}$$

Now Try Exercise 39 ■

EXAMPLE 5 ■ The Angle of a Beam of Light

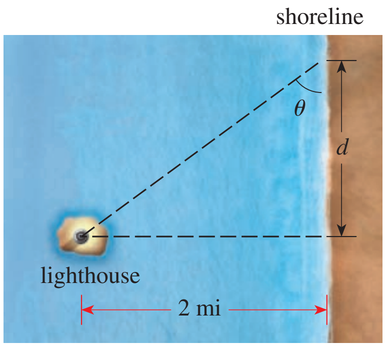


FIGURE 4

A lighthouse is located on an island that is 2 mi off a straight shoreline (see Figure 4). Express the angle formed by the beam of light and the shoreline in terms of the distance d in the figure.

SOLUTION From the figure we see that

$$\tan\theta = \frac{2}{d} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

Taking the inverse tangent of both sides, we get

$$\tan^{-1}(\tan\theta) = \tan^{-1}\left(\frac{2}{d}\right) \qquad \text{Take } \tan^{-1} \text{ of both sides}$$
$$\theta = \tan^{-1}\left(\frac{2}{d}\right) \qquad \text{Property of inverse functions: } \tan^{-1}(\tan\theta) = \theta$$

Now Try Exercise 41 ■

In Sections 5.5 and 5.6 we will learn how to solve any triangle (not necessarily a right triangle). The angles in a triangle are always in the interval $(0, \pi)$ (or between 0° and 180°). We'll see that to solve such triangles, we need to find all angles in the interval $(0, \pi)$ that have a specified sine or cosine. We do this in the next example.

EXAMPLE 6 ■ Solving a Basic Trigonometric Equation on an Interval

Find all angles θ between 0° and 180° satisfying the given equation.

(a) $\sin\theta = 0.4$ **(b)** $\cos\theta = 0.4$

SOLUTION

(a) We use $\sin^{-1}$ to find one solution in the interval $[-\pi/2, \pi/2]$.

$$\sin\theta = 0.4 \qquad \text{Equation}$$
$$\theta = \sin^{-1}(0.4) \qquad \text{Take } \sin^{-1} \text{ of each side}$$
$$\theta \approx 23.6° \qquad \text{Calculator (in degree mode)}$$

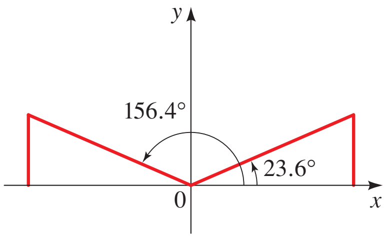


FIGURE 5

Another solution with θ between 0° and 180° is obtained by taking the supplement of the angle: $180° - 23.6° = 156.4°$ (see Figure 5). So the solutions of the equation with θ between 0° and 180° are

$$\theta \approx 23.6° \qquad \text{and} \qquad \theta \approx 156.4°$$

(b) The cosine function is one-to-one on the interval $[0, \pi]$, so there is only one solution of the equation with θ between 0° and 180°. We find that solution by taking $\cos^{-1}$ of each side.

$$\cos\theta = 0.4$$
$$\theta = \cos^{-1}(0.4) \qquad \text{Take } \cos^{-1} \text{ of each side}$$
$$\theta \approx 66.4° \qquad \text{Calculator (in degree mode)}$$

The solution is $\theta \approx 66.4°$

Now Try Exercises 25 and 27 ■

Evaluating Expressions Involving Inverse Trigonometric Functions

Expressions like $\cos(\sin^{-1} x)$ arise in calculus. We find exact values of such expressions using trigonometric identities or right triangles.

EXAMPLE 7 ■ Composing Trigonometric Functions and Their Inverses

Find $\cos\left(\sin^{-1} \frac{3}{5}\right)$.

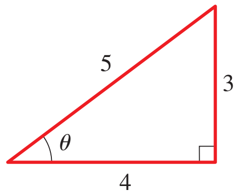


FIGURE 6

$\cos \theta = \dfrac{4}{5}$

SOLUTION 1 Let $\theta = \sin^{-1} \frac{3}{5}$. Then θ is the number in the interval $[-\pi/2, \pi/2]$ whose sine is $\frac{3}{5}$. Let's interpret θ as an angle and draw a right triangle with θ as one of its acute angles, with opposite side 3 and hypotenuse 5 (see Figure 6). The remaining leg of the triangle is found by the Pythagorean Theorem to be 4. From the figure we get

$$\begin{aligned} \cos\left(\sin^{-1} \tfrac{3}{5}\right) &= \cos \theta && \theta = \sin^{-1} \tfrac{3}{5} \\ &= \frac{4}{5} && \cos \theta = \frac{\text{adj}}{\text{hyp}} \end{aligned}$$

So $\cos\left(\sin^{-1} \frac{3}{5}\right) = \frac{4}{5}$.

SOLUTION 2 It's easy to find $\sin\left(\sin^{-1} \frac{3}{5}\right)$. In fact, by the cancellation properties of inverse functions, this value is exactly $\frac{3}{5}$. To find $\cos\left(\sin^{-1} \frac{3}{5}\right)$, we first write the cosine function in terms of the sine function. Let $u = \sin^{-1} \frac{3}{5}$. Since $-\pi/2 \le u \le \pi/2$, $\cos u$ is positive, and we can write the following:

$$\begin{aligned} \cos u &= +\sqrt{1 - \sin^2 u} && \cos^2 u + \sin^2 u = 1 \\ &= \sqrt{1 - \sin^2\left(\sin^{-1} \tfrac{3}{5}\right)} && u = \sin^{-1} \tfrac{3}{5} \\ &= \sqrt{1 - \left(\tfrac{3}{5}\right)^2} && \text{Property of inverse functions: } \sin\left(\sin^{-1} \tfrac{3}{5}\right) = \tfrac{3}{5} \\ &= \sqrt{1 - \tfrac{9}{25}} = \sqrt{\tfrac{16}{25}} = \tfrac{4}{5} && \text{Calculate} \end{aligned}$$

So $\cos\left(\sin^{-1} \frac{3}{5}\right) = \frac{4}{5}$.

Now Try Exercise 29 ■

EXAMPLE 8 ■ Composing Trigonometric Functions and Their Inverses

Write $\sin(\cos^{-1} x)$ and $\tan(\cos^{-1} x)$ as algebraic expressions in x for $-1 \le x \le 1$.

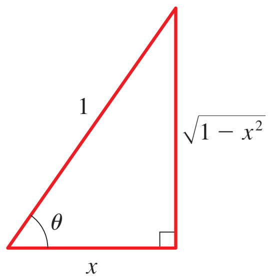


FIGURE 7

$\cos \theta = \dfrac{x}{1} = x$

SOLUTION 1 Let $\theta = \cos^{-1} x$; then $\cos \theta = x$. In Figure 7 we sketch a right triangle with an acute angle θ, adjacent side x, and hypotenuse 1. By the Pythagorean Theorem the remaining leg is $\sqrt{1 - x^2}$. From the figure we have

$$\sin(\cos^{-1} x) = \sin \theta = \sqrt{1 - x^2} \qquad \text{and} \qquad \tan(\cos^{-1} x) = \tan \theta = \frac{\sqrt{1 - x^2}}{x}$$

SOLUTION 2 Let $u = \cos^{-1} x$. We need to find $\sin u$ and $\tan u$ in terms of x. As in Example 7 the idea here is to write sine and tangent in terms of cosine. Note that $0 \le u \le \pi$ because $u = \cos^{-1} x$. We have

$$\sin u = \pm\sqrt{1 - \cos^2 u} \qquad \text{and} \qquad \tan u = \frac{\sin u}{\cos u} = \frac{\pm\sqrt{1 - \cos^2 u}}{\cos u}$$

To choose the proper signs, note that u lies in the interval $[0, \pi]$ because $u = \cos^{-1} x$. Since $\sin u$ is positive on this interval, the $+$ sign is the correct choice. Substituting $u = \cos^{-1} x$ in the displayed equations and using the cancellation property $\cos(\cos^{-1} x) = x$, we get

$$\sin(\cos^{-1} x) = \sqrt{1 - x^2} \qquad \text{and} \qquad \tan(\cos^{-1} x) = \frac{\sqrt{1 - x^2}}{x}$$

Now Try Exercises 35 and 37 ■

Note: In Solution 1 of Example 8 it might seem that because we are sketching a triangle, the angle $\theta = \cos^{-1} x$ must be acute. But it turns out that the triangle method works for any x. The domains and ranges of all six inverse trigonometric functions have been chosen in such a way that we can always use a triangle to find $S(T^{-1}(x))$, where S and T are any trigonometric functions.

5.4 EXERCISES

CONCEPTS

1. For a function to have an inverse, it must be ____________. To define the inverse sine function, we restrict the ____________ of the sine function to the interval ____________.

2. The inverse sine, inverse cosine, and inverse tangent functions have the following domains and ranges.

(a) The function $\sin^{-1}$ has domain ________ and range ________.

(b) The function $\cos^{-1}$ has domain ________ and range ________.

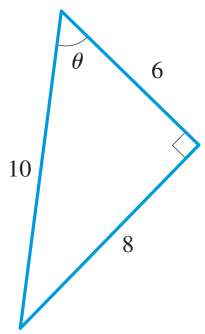
(c) The function $\tan^{-1}$ has domain ________ and range ________.

3. In the triangle shown we can find the angle θ as follows.

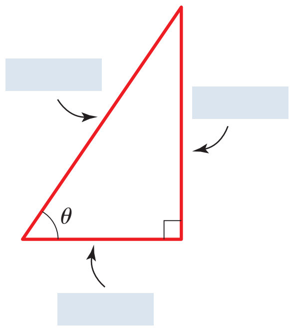
(a) $\theta = \sin^{-1} \dfrac{\square}{\square}$

(b) $\theta = \cos^{-1} \dfrac{\square}{\square}$

(c) $\theta = \tan^{-1} \dfrac{\square}{\square}$



4. To find $\sin\left(\cos^{-1} \frac{5}{13}\right)$, we let $\theta = \cos^{-1}\left(\frac{5}{13}\right)$ and complete the right triangle at the top of the next column. We find that $\sin\left(\cos^{-1} \frac{5}{13}\right) =$ ________.



SKILLS

5–8 ■ Evaluating Inverse Trigonometric Functions Find the exact value of each expression, if it is defined. Express your answer in radians.

5. **(a)** $\sin^{-1} 1$ **(b)** $\cos^{-1} 0$ **(c)** $\tan^{-1} \sqrt{3}$

6. **(a)** $\sin^{-1} 0$ **(b)** $\cos^{-1}(-1)$ **(c)** $\tan^{-1} 0$

7. **(a)** $\sin^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$ **(b)** $\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$ **(c)** $\tan^{-1}(-1)$

8. **(a)** $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$ **(b)** $\cos^{-1}\left(-\frac{1}{2}\right)$ **(c)** $\tan^{-1}(-\sqrt{3})$

9–16 ■ Evaluating Inverse Trigonometric Functions Use a calculator to find an approximate value (in radians) of each expression rounded to five decimal places, if it is defined.

9. $\sin^{-1}(0.30)$ **10.** $\cos^{-1}(-0.2)$

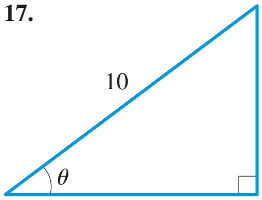
11. $\cos^{-1} \frac{1}{3}$ **12.** $\sin^{-1} \frac{5}{6}$

13. $\tan^{-1} 3$ **14.** $\tan^{-1}(-4)$

15. $\cos^{-1} 3$ **16.** $\sin^{-1}(-2)$

17–22 ■ Finding Angles in Right Triangles Find the angle θ in degrees, rounded to one decimal place.

17.



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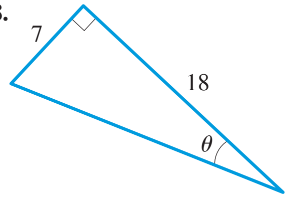

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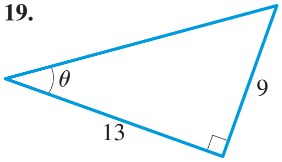

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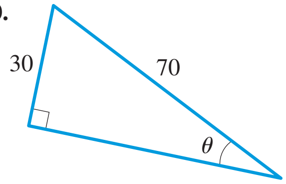

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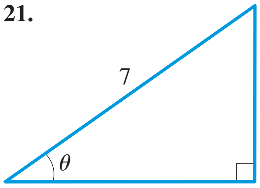

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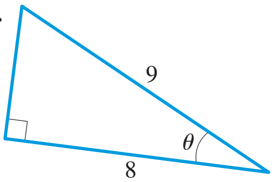


23–28 ■ Basic Trigonometric Equations Find all angles θ between 0° and 180° satisfying the given equation. Round your answer to one decimal place.

23. $\sin\theta = \frac{2}{3}$ **24.** $\cos\theta = \frac{3}{4}$

25. $\cos\theta = -\frac{2}{5}$ **26.** $\tan\theta = -20$

27. $\tan\theta = 5$ **28.** $\sin\theta = \frac{4}{5}$

29–34 ■ Value of an Expression Find the exact value of the expression.

29. $\cos\left(\sin^{-1}\frac{4}{5}\right)$ **30.** $\cos\left(\tan^{-1}\frac{4}{3}\right)$ **31.** $\sec\left(\sin^{-1}\frac{12}{13}\right)$

32. $\csc\left(\cos^{-1}\frac{7}{25}\right)$ **33.** $\tan\left(\sin^{-1}\frac{12}{13}\right)$ **34.** $\cot\left(\sin^{-1}\frac{2}{3}\right)$

35–38 ■ Algebraic Expressions Rewrite the expression as an algebraic expression in x.

35. $\cos(\sin^{-1}x)$ **36.** $\sin(\tan^{-1}x)$

37. $\tan(\sin^{-1}x)$ **38.** $\cos(\tan^{-1}x)$

APPLICATIONS

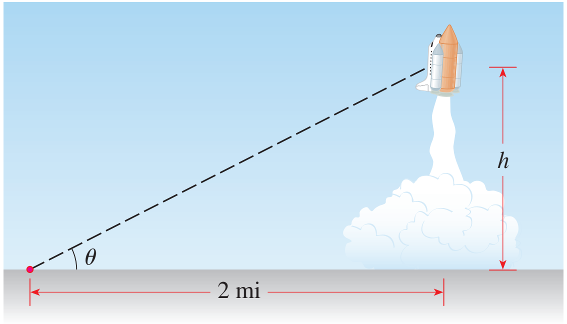
39. Leaning Ladder A 20-ft ladder is leaning against a building. If the base of the ladder is 6 ft from the base of the building, what is the angle of elevation of the ladder? How high does the ladder reach on the building?

40. Angle of the Sun A 96-ft tree casts a shadow that is 120 ft long. What is the angle of elevation of the sun?

41. Height of the Space Shuttle An observer views the space shuttle from a distance of 2 mi from the launch pad.

(a) Express the height of the space shuttle as a function of the angle of elevation θ.

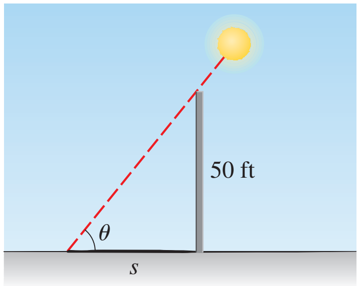
(b) Express the angle of elevation θ as a function of the height h of the space shuttle.



42. Height of a Pole A 50-ft pole casts a shadow as shown in the figure.

(a) Express the angle of elevation θ of the sun as a function of the length s of the shadow.

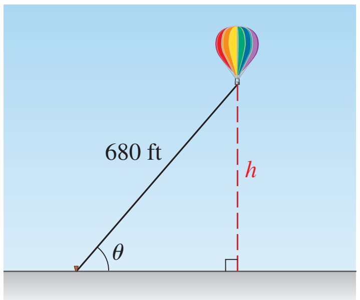
(b) Find the angle θ of elevation of the sun when the shadow is 20 ft long.



43. Height of a Balloon A 680-ft rope anchors a hot-air balloon as shown in the figure.

(a) Express the angle θ as a function of the height h of the balloon.

(b) Find the angle θ if the balloon is 500 ft high.



44. View from a Satellite The figures on the next page indicate that the higher the orbit of a satellite, the more of the earth the satellite can "see." Let θ, s, and h be as in the figure, and assume that the earth is a sphere of radius 3960 mi.

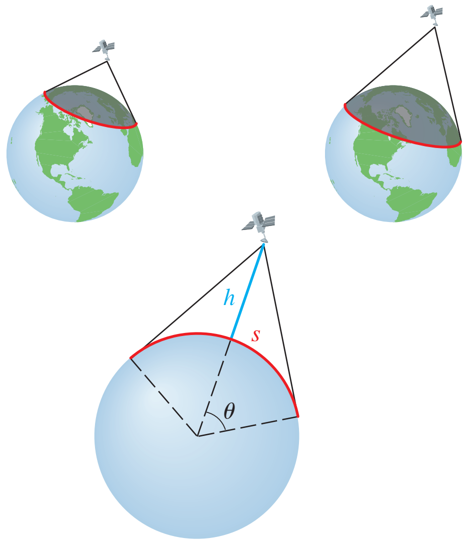
(a) Express the angle θ as a function of h.

(b) Express the distance s as a function of θ.

(c) Express the distance s as a function of h. [*Hint:* Find the composition of the functions in parts (a) and (b).]

(d) If the satellite is 100 mi above the earth, what is the distance s that it can see?

(e) How high does the satellite have to be to see both Los Angeles and New York, 2450 mi apart?



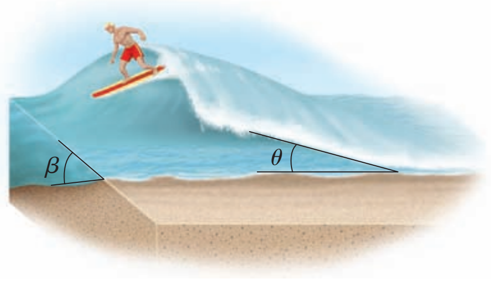
45. Surfing the Perfect Wave For a wave to be surfable, it can't break all at once. Robert Guza and Tony Bowen have shown that a wave has a surfable shoulder if it hits the shoreline at an angle θ given by

$$\theta = \sin^{-1}\left(\frac{1}{(2n + 1)\tan\beta}\right)$$

where β is the angle at which the beach slopes down and where $n = 0, 1, 2, \ldots$.

(a) For $\beta = 10°$, find θ when $n = 3$.

(b) For $\beta = 15°$, find θ when $n = 2$, 3, and 4. Explain why the formula does not give a value for θ when $n = 0$ or 1.



DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

46. PROVE: Inverse Trigonometric Functions on a Calculator Most calculators do not have keys for $\sec^{-1}$, $\csc^{-1}$, or $\cot^{-1}$. Prove the following identities, and then use these identities and a calculator to find $\sec^{-1}2$, $\csc^{-1}3$, and $\cot^{-1}4$.

$$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right) \qquad x \ge 1$$

$$\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right) \qquad x \ge 1$$

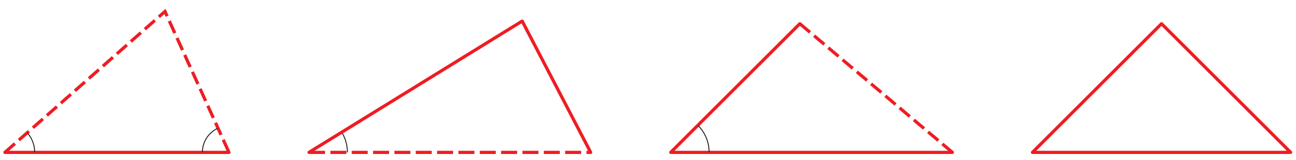
$$\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right) \qquad x > 0$$

5.5 THE LAW OF SINES

■ The Law of Sines ■ The Ambiguous Case

In Section 5.2 we used the trigonometric ratios to solve right triangles. The trigonometric functions can also be used to solve *oblique triangles*, that is, triangles with no right angles. To do this, we first study the Law of Sines here and then the Law of Cosines in the next section.

In general, to solve a triangle, we need to know certain information about its sides and angles. To decide whether we have enough information, it's often helpful to make a sketch. For instance, if we are given two angles and the included side, then it's clear that one and only one triangle can be formed (see Figure 1(a)). Similarly, if two sides and the included angle are known, then a unique triangle is determined (Figure 1(c)). But if we know all three angles and no sides, we cannot uniquely determine the triangle because many triangles can have the same three angles. (All these triangles would be similar, of course.) So we won't consider this last case.



(a) ASA or SAA (b) SSA (c) SAS (d) SSS

FIGURE 1

In general, a triangle is determined by three of its six parts (angles and sides) as long as at least one of these three parts is a side. So the possibilities, illustrated in Figure 1, are as follows.

Case 1 One side and two angles (ASA or SAA)
Case 2 Two sides and the angle opposite one of those sides (SSA)
Case 3 Two sides and the included angle (SAS)
Case 4 Three sides (SSS)

Cases 1 and 2 are solved by using the Law of Sines; Cases 3 and 4 require the Law of Cosines.

The Law of Sines

The **Law of Sines** says that in any triangle the lengths of the sides are proportional to the sines of the corresponding opposite angles. To state this law (or formula) more easily, we follow the convention of labeling the angles of a triangle as A, B, and C and the lengths of the corresponding opposite sides as a, b, and c, as in Figure 2.

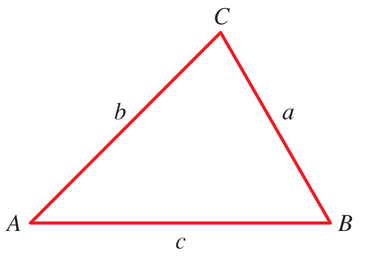


FIGURE 2

THE LAW OF SINES

In triangle ABC we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

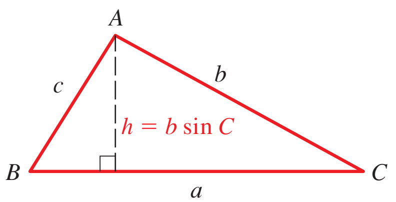


FIGURE 3

Proof To see why the Law of Sines is true, refer to Figure 3. By the formula in Section 5.3 the area of triangle ABC is $\frac{1}{2}ab \sin C$. By the same formula the area of this triangle is also $\frac{1}{2}ac \sin B$ and $\frac{1}{2}bc \sin A$. Thus

$$\tfrac{1}{2}bc \sin A = \tfrac{1}{2}ac \sin B = \tfrac{1}{2}ab \sin C$$

Multiplying by $2/(abc)$ gives the Law of Sines. ■

EXAMPLE 1 ■ Tracking a Satellite (ASA)

A satellite orbiting the earth passes directly overhead at observation stations in Phoenix and Los Angeles, 340 mi apart. At an instant when the satellite is between these two stations, its angle of elevation is simultaneously observed to be 60° at Phoenix and 75° at Los Angeles. How far is the satellite from Los Angeles?

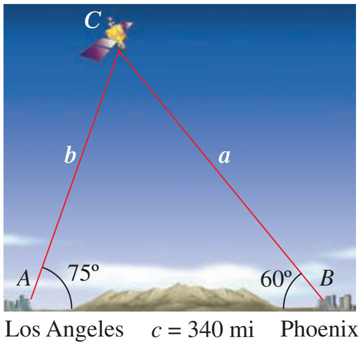


FIGURE 4

SOLUTION We need to find the distance b in Figure 4. Since the sum of the angles in any triangle is 180°, we see that $\angle C = 180° - (75° + 60°) = 45°$ (see Figure 4), so we have

$$\frac{\sin B}{b} = \frac{\sin C}{c} \qquad \text{Law of Sines}$$

$$\frac{\sin 60°}{b} = \frac{\sin 45°}{340} \qquad \text{Substitute}$$

$$b = \frac{340 \sin 60°}{\sin 45°} \approx 416 \qquad \text{Solve for } b$$

The distance of the satellite from Los Angeles is approximately 416 mi.

Now Try Exercises 5 and 31 ■

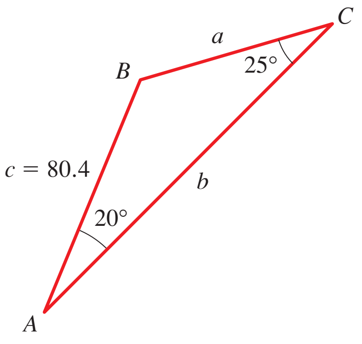


FIGURE 5

EXAMPLE 2 ■ Solving a Triangle (SAA)

Solve the triangle in Figure 5.

SOLUTION First, $\angle B = 180^\circ - (20^\circ + 25^\circ) = 135^\circ$. Since side c is known, to find side a, we use the relation

$$\frac{\sin A}{a} = \frac{\sin C}{c} \qquad \text{Law of Sines}$$

$$a = \frac{c \sin A}{\sin C} = \frac{80.4 \sin 20^\circ}{\sin 25^\circ} \approx 65.1 \qquad \text{Solve for } a$$

Similarly, to find b, we use

$$\frac{\sin B}{b} = \frac{\sin C}{c} \qquad \text{Law of Sines}$$

$$b = \frac{c \sin B}{\sin C} = \frac{80.4 \sin 135^\circ}{\sin 25^\circ} \approx 134.5 \qquad \text{Solve for } b$$

Now Try Exercise 13 ■

■ The Ambiguous Case

In Examples 1 and 2 a unique triangle was determined by the information given. This is always true of Case 1 (ASA or SAA). But in Case 2 (SSA) there may be two triangles, one triangle, or no triangle with the given properties. For this reason, Case 2 is sometimes called the **ambiguous case**. To see why this is so, we show in Figure 6 the possibilities when angle A and sides a and b are given. In part (a) no solution is possible, since side a is too short to complete the triangle. In part (b) the solution is a right triangle. In part (c) two solutions are possible, and in part (d) there is a unique triangle with the given properties. We illustrate the possibilities of Case 2 in the following examples.

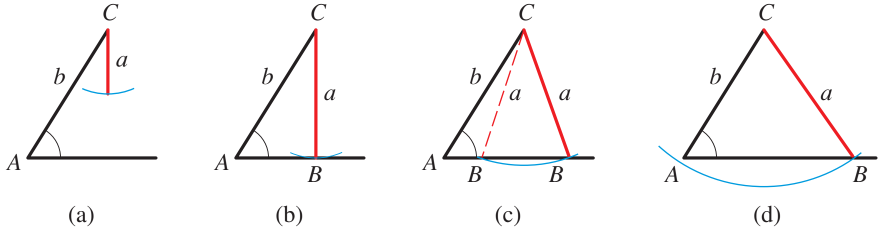


FIGURE 6 The ambiguous case

EXAMPLE 3 ■ SSA, the One-Solution Case

Solve triangle ABC, where $\angle A = 45^\circ$, $a = 7\sqrt{2}$, and $b = 7$.

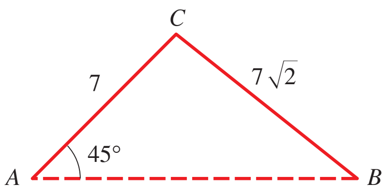


FIGURE 7

SOLUTION We first sketch the triangle with the information we have (see Figure 7). Our sketch is necessarily tentative, since we don't yet know the other angles. Nevertheless, we can now see the possibilities.

We first find $\angle B$.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \text{Law of Sines}$$

$$\sin B = \frac{b \sin A}{a} = \frac{7}{7\sqrt{2}} \sin 45^\circ = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2} \qquad \text{Solve for } \sin B$$

We consider only angles smaller than 180°, since no triangle can contain an angle of 180° or larger.

Which angles B have $\sin B = \frac{1}{2}$? From the preceding section we know that there are two such angles smaller than 180° (they are 30° and 150°). Which of these angles is compatible with what we know about triangle ABC? Since $\angle A = 45^\circ$, we cannot have $\angle B = 150°$, because $45° + 150° > 180°$. So $\angle B = 30°$, and the remaining angle is $\angle C = 180° - (30° + 45°) = 105°$.

Now we can find side c.

$$\frac{\sin B}{b} = \frac{\sin C}{c} \qquad \text{Law of Sines}$$

$$c = \frac{b \sin C}{\sin B} = \frac{7 \sin 105°}{\sin 30°} = \frac{7 \sin 105°}{\frac{1}{2}} \approx 13.5 \qquad \text{Solve for } c$$

Now Try Exercise 19 ■

In Example 3 there were two possibilities for angle B, and one of these was not compatible with the rest of the information. In general, if $\sin A < 1$, we must check the angle and its supplement as possibilities, because any angle smaller than 180° can be in the triangle. To decide whether either possibility works, we check to see whether the resulting sum of the angles exceeds 180°. It can happen, as in Figure 6(c), that both possibilities are compatible with the given information. In that case, two different triangles are solutions to the problem.

The *supplement* of an angle θ (where $0 \le \theta \le 180°$) is the angle $180° - \theta$.

EXAMPLE 4 ■ SSA, the Two-Solution Case

Solve triangle ABC if $\angle A = 43.1°$, $a = 186.2$, and $b = 248.6$.

SOLUTION From the given information we sketch the triangle shown in Figure 8. Note that side a may be drawn in two possible positions to complete the triangle. From the Law of Sines

$$\sin B = \frac{b \sin A}{a} = \frac{248.6 \sin 43.1°}{186.2} \approx 0.91225$$

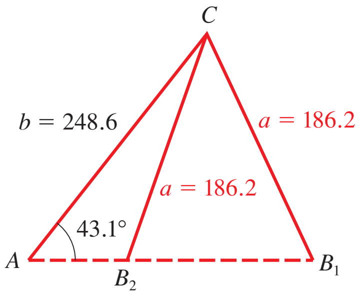


FIGURE 8

There are two possible angles B between 0° and 180° such that $\sin B = 0.91225$. Using a calculator, we find that one of the angles is

$$\sin^{-1}(0.91225) \approx 65.8°$$

The other angle is approximately $180° - 65.8° = 114.2°$. We denote these two angles by B_1 and B_2 so that

$$\angle B_1 \approx 65.8° \qquad \text{and} \qquad \angle B_2 \approx 114.2°$$

Thus two triangles satisfy the given conditions: triangle $A_1B_1C_1$ and triangle $A_2B_2C_2$.

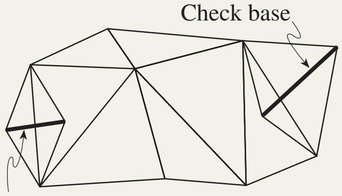
Solve triangle $A_1B_1C_1$:

$$\angle C_1 \approx 180° - (43.1° + 65.8°) = 71.1° \qquad \text{Find } \angle C_1$$

Thus $$c_1 = \frac{a_1 \sin C_1}{\sin A_1} \approx \frac{186.2 \sin 71.1°}{\sin 43.1°} \approx 257.8 \qquad \text{Law of Sines}$$

ALAN ODDIE/Photo Edit

Surveying is a method of land measurement used for mapmaking. Surveyors use a process called *triangulation* in which a network of thousands of interlocking triangles is created on the area to be mapped. The process is started by measuring the length of a *baseline* between two surveying stations. Then, with the use of an instrument called a *theodolite*, the angles between these two stations and a third station are measured. The Law of Sines is then used to calculate the two other sides of the triangle formed by the three stations. The calculated sides are used as baselines, and the process is repeated over and over to create a network of triangles. In this method the only distance measured is the initial baseline; all other distances are calculated from the Law of Sines. This method is practical because it is much easier to measure angles than distances.



One of the most ambitious mapmaking efforts of all time was the Great Trigonometric Survey of India (see Problem 8, page 502) which required several expeditions and took over a century to complete. The famous expedition of 1823, led by **Sir George Everest**, lasted 20 years. Ranging over treacherous terrain and encountering the dreaded malaria-carrying mosquitoes, this expedition reached the foothills of the Himalayas. A later expedition, using triangulation, calculated the height of the highest peak of the Himalayas to be 29,002 ft. The peak was named in honor of Sir George Everest.

Today, with the use of satellites, the height of Mt. Everest is estimated to be 29,028 ft. The very close agreement of these two estimates shows the great accuracy of the trigonometric method.

Solve triangle $A_2B_2C_2$:

$$\angle C_2 \approx 180° - (43.1° + 114.2°) = 22.7° \qquad \text{Find } \angle C_2$$

Thus $$c_2 = \frac{a_2 \sin C_2}{\sin A_2} \approx \frac{186.2 \sin 22.7°}{\sin 43.1°} \approx 105.2 \qquad \text{Law of Sines}$$

Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are shown in Figure 9.

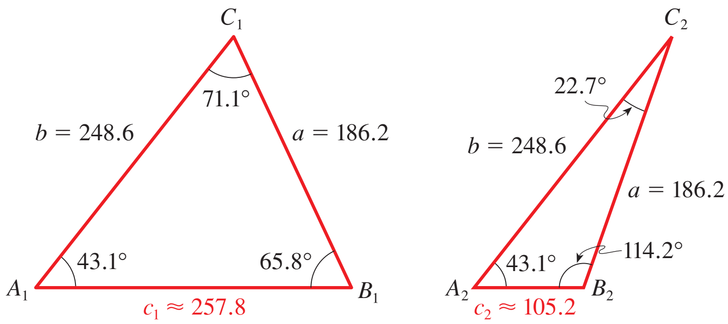


FIGURE 9

Now Try Exercise 23 ■

The next example presents a situation for which no triangle is compatible with the given data.

EXAMPLE 5 ■ SSA, the No-Solution Case

Solve triangle ABC, where $\angle A = 42°$, $a = 70$, and $b = 122$.

SOLUTION To organize the given information, we sketch the diagram in Figure 10. Let's try to find $\angle B$. We have

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \text{Law of Sines}$$

$$\sin B = \frac{b \sin A}{a} = \frac{122 \sin 42°}{70} \approx 1.17 \qquad \text{Solve for } \sin B$$

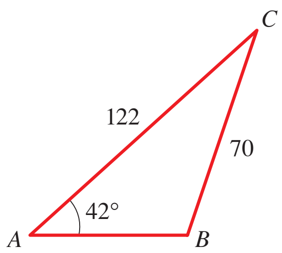
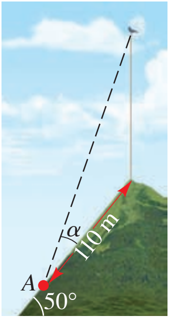


FIGURE 10

Since the sine of an angle is never greater than 1, we conclude that no triangle satisfies the conditions given in this problem.

Now Try Exercise 21 ■

EXAMPLE 6 ■ Calculating a Distance



A bird is perched on top of a pole on a steep hill, and an observer is located at point A on the side of the hill, 110 m downhill from the base of the pole, as shown in the figure. The angle of inclination of the hill is 50°, and the angle α in the figure is 9°. Find the distance from the observer to the bird.

SOLUTION We first sketch a diagram as shown in Figure 11. We want to find the distance b in the figure. Triangle ADB is a right triangle, so $\angle DBA = 90° - 50° = 40°$. It follows that $\angle ABC = 180° - 40° = 140°$.

Now in triangle ABC we have $\angle A = 9°$ and $\angle B = 140°$, so $\angle C = 180° - 149° = 31°$. By the Law of Sines we have

$$\frac{\sin B}{b} = \frac{\sin C}{c} \qquad \text{Law of Sines}$$

Substituting $\angle B = 140°$, $\angle C = 31°$, and $c = 110$, we get

$$\frac{\sin 140°}{b} = \frac{\sin 31°}{110}$$

$$b = \frac{110 \sin 140°}{\sin 31°} \qquad \text{Solve for } b$$

$$\approx 137.3 \qquad \text{Calculator}$$

So the distance from the observer to the bird is about 137 m.

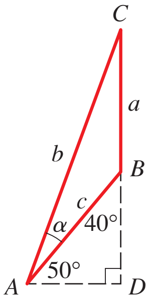


FIGURE 11

Now Try Exercise 37

5.5 EXERCISES

CONCEPTS

1. In triangle ABC with sides a, b, and c the Law of Sines states that

$$\frac{\square}{\square} = \frac{\square}{\square} = \frac{\square}{\square}$$

2. The four cases in which we can solve a triangle are

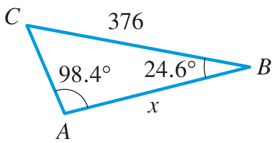
ASA SSA SAS SSS

(a) In which of these cases can we use the Law of Sines to solve the triangle?

(b) Which of the cases listed can lead to more than one solution (the ambiguous case)?

SKILLS

3–8 ■ Finding an Angle or Side Use the Law of Sines to find the indicated side x or angle θ.

3.


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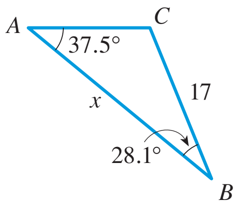

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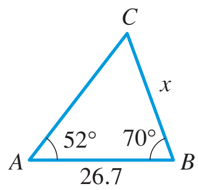

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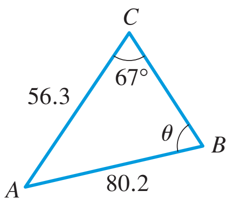

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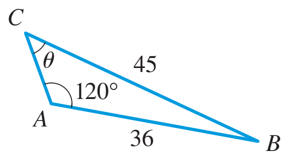

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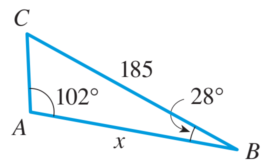
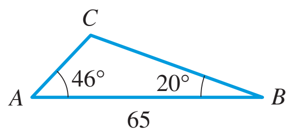


9–12 ■ Solving a Triangle Solve the triangle using the Law of Sines.

9.


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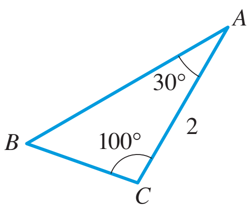
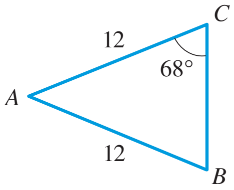
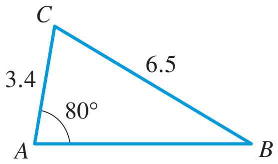


11.



12.



13–18 ■ Solving a Triangle Sketch each triangle, and then solve the triangle using the Law of Sines.

13. $\angle A = 50°, \quad \angle B = 68°, \quad c = 230$

14. $\angle A = 23°, \quad \angle B = 110°, \quad c = 50$

15. $\angle A = 30°, \quad \angle C = 65°, \quad b = 10$

16. $\angle A = 22°, \quad \angle B = 95°, \quad a = 420$

17. $\angle B = 29°, \quad \angle C = 51°, \quad b = 44$

18. $\angle B = 10°, \quad \angle C = 100°, \quad c = 115$

19–28 ■ Solving a Triangle Use the Law of Sines to solve for all possible triangles that satisfy the given conditions.

19. $a = 28, \quad b = 15, \quad \angle A = 110°$

20. $a = 30, \quad c = 40, \quad \angle A = 37°$

21. $a = 20, \quad c = 45, \quad \angle A = 125°$

22. $b = 45, \quad c = 42, \quad \angle C = 38°$

23. $b = 25, \quad c = 30, \quad \angle B = 25°$

24. $a = 75, \quad b = 100, \quad \angle A = 30°$

25. $a = 50, \quad b = 100, \quad \angle A = 50°$

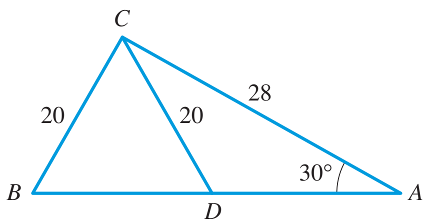
26. $a = 100, \quad b = 80, \quad \angle A = 135°$

27. $a = 26, \quad c = 15, \quad \angle C = 29°$

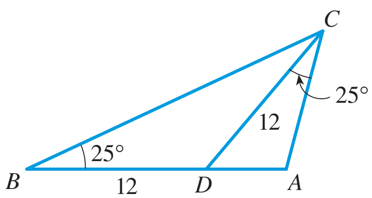
28. $b = 73, \quad c = 82, \quad \angle B = 58°$

SKILLS Plus

29. Finding Angles For the triangle shown, find **(a)** $\angle BCD$ and **(b)** $\angle DCA$.



30. Finding a Side For the triangle shown, find the length AD.

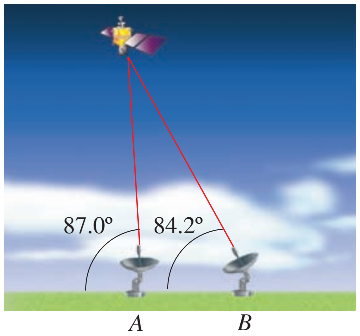


APPLICATIONS

31. Tracking a Satellite The path of a satellite orbiting the earth causes the satellite to pass directly over two tracking stations A and B, which are 50 mi apart. When the satellite is on one side of the two stations, the angles of elevation at A and B are measured to be 87.0° and 84.2°, respectively.

(a) How far is the satellite from station A?

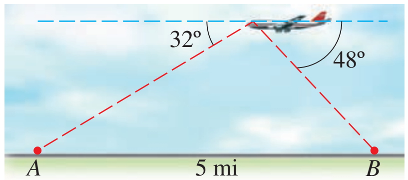
(b) How high is the satellite above the ground?



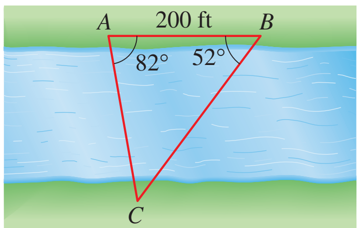
32. Flight of a Plane A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 5 mi apart, to be 32° and 48°, as shown in the figure.

(a) Find the distance of the plane from point A.

(b) Find the elevation of the plane.



33. Distance Across a River To find the distance across a river, a surveyor chooses points A and B, which are 200 ft apart on one side of the river (see the figure). She then chooses a reference point C on the opposite side of the river and finds that $\angle BAC \approx 82°$ and $\angle ABC \approx 52°$. Approximate the distance from A to C.

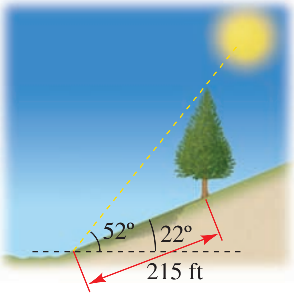


34. Distance Across a Lake Points A and B are separated by a lake. To find the distance between them, a surveyor locates a point C on land such that $\angle CAB = 48.6°$. He also measures CA as 312 ft and CB as 527 ft. Find the distance between A and B.

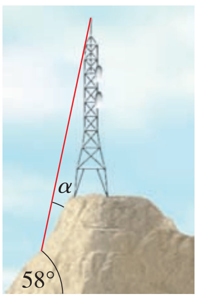
35. The Leaning Tower of Pisa The bell tower of the cathedral in Pisa, Italy, leans 5.6° from the vertical. A tourist stands 105 m from its base, with the tower leaning directly toward her. She measures the angle of elevation to the top of the tower to be 29.2°. Find the length of the tower to the nearest meter.

36. Radio Antenna A short-wave radio antenna is supported by two guy wires, 165 ft and 180 ft long. Each wire is attached to the top of the antenna and anchored to the ground at two anchor points on opposite sides of the antenna. The shorter wire makes an angle of 67° with the ground. How far apart are the anchor points?

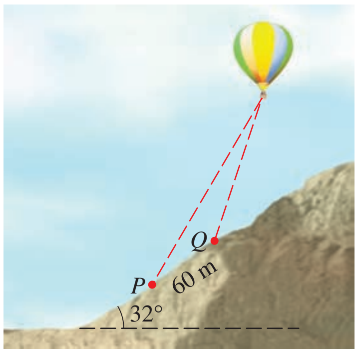
37. Height of a Tree A tree on a hillside casts a shadow 215 ft down the hill. If the angle of inclination of the hillside is 22° to the horizontal and the angle of elevation of the sun is 52°, find the height of the tree.



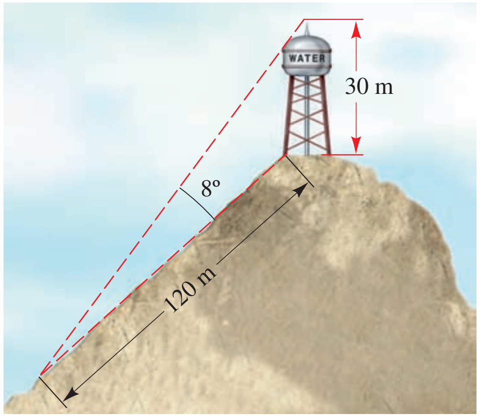
38. Length of a Guy Wire A communications tower is located at the top of a steep hill, as shown. The angle of inclination of the hill is 58°. A guy wire is to be attached to the top of the tower and to the ground, 100 m downhill from the base of the tower. The angle α in the figure is determined to be 12°. Find the length of cable required for the guy wire.



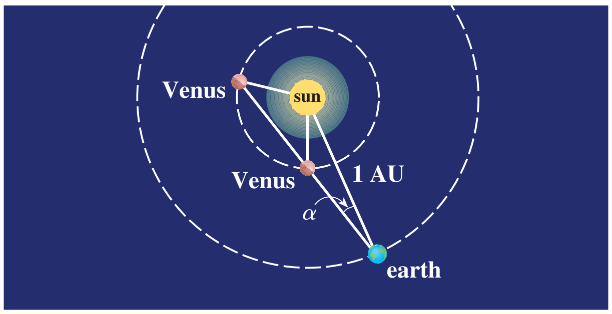
39. Calculating a Distance Observers at P and Q are located on the side of a hill that is inclined 32° to the horizontal, as shown. The observer at P determines the angle of elevation to a hot-air balloon to be 62°. At the same instant the observer at Q measures the angle of elevation to the balloon to be 71°. If P is 60 m down the hill from Q, find the distance from Q to the balloon.



40. Calculating an Angle A water tower 30 m tall is located at the top of a hill. From a distance of 120 m down the hill it is observed that the angle formed between the top and base of the tower is 8°. Find the angle of inclination of the hill.



41. Distances to Venus The *elongation* α of a planet is the angle formed by the planet, earth, and sun (see the figure). It is known that the distance from the sun to Venus is 0.723 AU (see Exercise 71 in Section 5.2). At a certain time the elongation of Venus is found to be 39.4°. Find the possible distances from the earth to Venus at that time in astronomical units (AU).



42. Soap Bubbles When two bubbles cling together in midair, their common surface is part of a sphere whose center D lies on the line passing through the centers of the bubbles (see the figure). Also, $\angle ACB$ and $\angle ACD$ each have measure 60°.

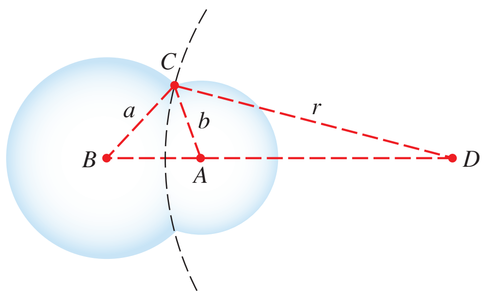
(a) Show that the radius r of the common face is given by

$$r = \frac{ab}{a - b}$$

[*Hint:* Use the Law of Sines together with the fact that an angle θ and its supplement $180° - \theta$ have the same sine.]

(b) Find the radius of the common face if the radii of the bubbles are 4 cm and 3 cm.

(c) What shape does the common face take if the two bubbles have equal radii?



DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

43. PROVE: Area of a Triangle Show that, given the three angles A, B, and C of a triangle and one side, say, a, the area of the triangle is

$$\text{area} = \frac{a^2 \sin B \sin C}{2 \sin A}$$

44. PROVE: Areas and the Ambiguous Case Suppose we solve a triangle in the ambiguous case. We are given $\angle A$ and sides a and b, and we find the two solutions $\triangle ABC$ and $\triangle A'B'C'$. Prove that

$$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle A'B'C'} = \frac{\sin C}{\sin C'}$$

45. DISCOVER: Number of Solutions in the Ambiguous Case We have seen that when the Law of Sines is used to solve a triangle in the SSA case, there may be two, one, or no solution(s). Sketch triangles like those in Figure 6 to verify the criteria in the table for the number of solutions if you are given $\angle A$ and sides a and b.

Criterion	Number of solutions
$a \geq b$	1
$b > a > b \sin A$	2
$a = b \sin A$	1
$a < b \sin A$	0

If $\angle A = 30°$ and $b = 100$, use these criteria to find the range of values of a for which the triangle ABC has two solutions, one solution, or no solution.

5.6 THE LAW OF COSINES

■ The Law of Cosines ■ Navigation: Heading and Bearing ■ The Area of a Triangle

■ The Law of Cosines

The Law of Sines cannot be used directly to solve triangles if we know two sides and the angle between them or if we know all three sides (these are Cases 3 and 4 of the preceding section). In these two cases the **Law of Cosines** applies.

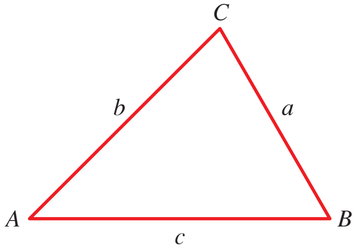


FIGURE 1

THE LAW OF COSINES

In any triangle ABC (see Figure 1) we have

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Proof To prove the Law of Cosines, place triangle ABC so that $\angle A$ is at the origin, as shown in Figure 2. The coordinates of vertices B and C are $(c, 0)$ and $(b \cos A, b \sin A)$, respectively. (You should check that the coordinates of these points will be the same if we draw angle A as an acute angle.) Using the Distance Formula, we get

$$\begin{aligned} a^2 &= (b \cos A - c)^2 + (b \sin A - 0)^2 \\ &= b^2 \cos^2 A - 2bc \cos A + c^2 + b^2 \sin^2 A \\ &= b^2(\cos^2 A + \sin^2 A) - 2bc \cos A + c^2 \\ &= b^2 + c^2 - 2bc \cos A \qquad \text{Because } \sin^2 A + \cos^2 A = 1 \end{aligned}$$

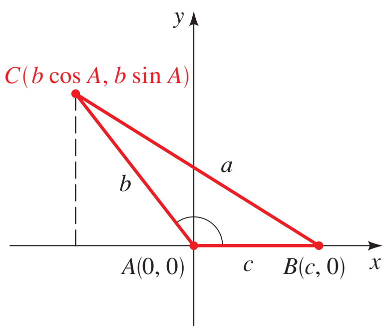


FIGURE 2

This proves the first formula. The other two formulas are obtained in the same way by placing each of the other vertices of the triangle at the origin and repeating the preceding argument. ■

In words, the Law of Cosines says that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of those two sides times the cosine of the included angle.

If one of the angles of a triangle, say, $\angle C$, is a right angle, then $\cos C = 0$, and the Law of Cosines reduces to the Pythagorean Theorem, $c^2 = a^2 + b^2$. Thus the Pythagorean Theorem is a special case of the Law of Cosines.

EXAMPLE 1 ■ Length of a Tunnel

A tunnel is to be built through a mountain. To estimate the length of the tunnel, a surveyor makes the measurements shown in Figure 3. Use the surveyor's data to approximate the length of the tunnel.

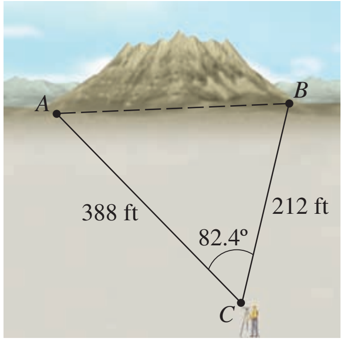


FIGURE 3

SOLUTION To approximate the length c of the tunnel, we use the Law of Cosines.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C && \text{Law of Cosines} \\ &= 212^2 + 388^2 - 2(212)(388)\cos 82.4^\circ && \text{Substitute} \\ &\approx 173730.2367 && \text{Use a calculator} \\ c &\approx \sqrt{173730.2367} \approx 416.8 && \text{Take square roots} \end{aligned}$$

Thus the tunnel will be approximately 417 ft long.

Now Try Exercises 3 and 39

EXAMPLE 2 ■ SSS, the Law of Cosines

The sides of a triangle are $a = 5$, $b = 8$, and $c = 12$ (see Figure 4). Find the angles of the triangle.

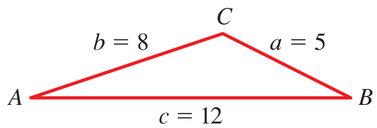


FIGURE 4

SOLUTION We first find $\angle A$. From the Law of Cosines, $a^2 = b^2 + c^2 - 2bc\cos A$. Solving for $\cos A$, we get

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{8^2 + 12^2 - 5^2}{2(8)(12)} = \frac{183}{192} = 0.953125$$

COS⁻¹
or
INV COS
or
ARC COS

Using a calculator, we find that $\angle A = \cos^{-1}(0.953125) \approx 18^\circ$. In the same way we get

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{5^2 + 12^2 - 8^2}{2(5)(12)} = 0.875$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{5^2 + 8^2 - 12^2}{2(5)(8)} = -0.6875$$

Using a calculator, we find that

$$\angle B = \cos^{-1}(0.875) \approx 29^\circ \qquad \text{and} \qquad \angle C = \cos^{-1}(-0.6875) \approx 133^\circ$$

Of course, once two angles have been calculated, the third can more easily be found from the fact that the sum of the angles of a triangle is 180°. However, it's a good idea to calculate all three angles using the Law of Cosines and add the three angles as a check on your computations.

Now Try Exercise 7

EXAMPLE 3 ■ SAS, the Law of Cosines

Solve triangle ABC, where $\angle A = 46.5°$, $b = 10.5$, and $c = 18.0$.

SOLUTION We can find a using the Law of Cosines.

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$= (10.5)^2 + (18.0)^2 - 2(10.5)(18.0)(\cos 46.5°) \approx 174.05$$

Thus $a \approx \sqrt{174.05} \approx 13.2$. We also use the Law of Cosines to find $\angle B$ and $\angle C$, as in Example 2.

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{13.2^2 + 18.0^2 - 10.5^2}{2(13.2)(18.0)} \approx 0.816477$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{13.2^2 + 10.5^2 - 18.0^2}{2(13.2)(10.5)} \approx -0.142532$$

Using a calculator, we find that

$$\angle B = \cos^{-1}(0.816477) \approx 35.3° \qquad \text{and} \qquad \angle C = \cos^{-1}(-0.142532) \approx 98.2°$$

To summarize: $\angle B \approx 35.3°$, $\angle C \approx 98.2°$, and $a \approx 13.2$. (See Figure 5.)

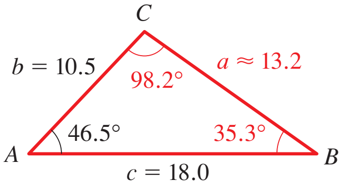


FIGURE 5

Now Try Exercise 13 ■

We could have used the Law of Sines to find $\angle B$ and $\angle C$ in Example 3, since we knew all three sides and an angle in the triangle. But knowing the sine of an angle does not uniquely specify the angle, since an angle θ and its supplement $180° - \theta$ both have the same sine. Thus we would need to decide which of the two angles is the correct choice. This ambiguity does not arise when we use the Law of Cosines, because every angle between 0° and 180° has a unique cosine. So using only the Law of Cosines is preferable in problems like Example 3.

■ Navigation: Heading and Bearing

In navigation a direction is often given as a **bearing**, that is, as an acute angle measured from due north or due south. The bearing N 30° E, for example, indicates a direction that points 30° to the east of due north (see Figure 6).

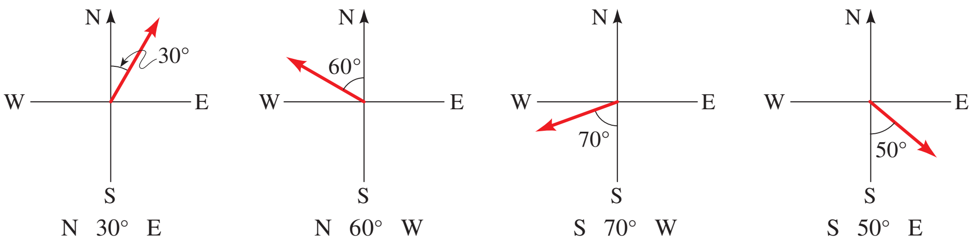


FIGURE 6

EXAMPLE 4 ■ Navigation

A pilot sets out from an airport and heads in the direction N 20° E, flying at 200 mi/h. After 1 h, he makes a course correction and heads in the direction N 40° E. Half an hour after that, engine trouble forces him to make an emergency landing.

(a) Find the distance between the airport and his final landing point.

(b) Find the bearing from the airport to his final landing point.

SOLUTION

(a) In 1 h the plane travels 200 mi, and in half an hour it travels 100 mi, so we can plot the pilot's course as in Figure 7. When he makes his course

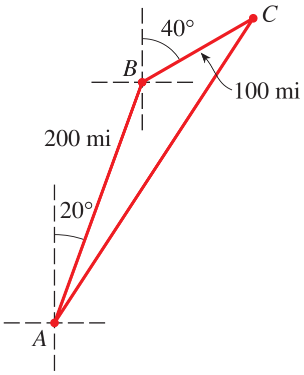


FIGURE 7

correction, he turns 20° to the right, so the angle between the two legs of his trip is $180° - 20° = 160°$. So by the Law of Cosines we have

$$b^2 = 200^2 + 100^2 - 2 \cdot 200 \cdot 100 \cos 160°$$
$$\approx 87{,}587.70$$

Thus $b \approx 295.95$. The pilot lands about 296 mi from his starting point.

(b) We first use the Law of Sines to find $\angle A$.

$$\frac{\sin A}{100} = \frac{\sin 160°}{295.95}$$

$$\sin A = 100 \cdot \frac{\sin 160°}{295.95}$$

$$\approx 0.11557$$

Using the $\boxed{\text{SIN}^{-1}}$ key on a calculator, we find that $\angle A \approx 6.636°$. From Figure 7 we see that the line from the airport to the final landing site points in the direction $20° + 6.636° = 26.636°$ east of due north. Thus the bearing is about N 26.6° E.

Another angle with sine 0.11557 is $180° - 6.636° = 173.364°$. But this is clearly too large to be $\angle A$ in $\angle ABC$.

Now Try Exercise 45

The Area of a Triangle

An interesting application of the Law of Cosines involves a formula for finding the area of a triangle from the lengths of its three sides (see Figure 8).

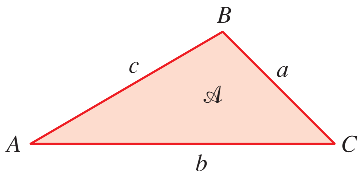


FIGURE 8

HERON'S FORMULA

The area $\mathscr{A}$ of triangle ABC is given by

$$\mathscr{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$ is the **semiperimeter** of the triangle; that is, s is half the perimeter.

Proof We start with the formula $\mathscr{A} = \frac{1}{2}ab \sin C$ from Section 5.3. Thus

$$\mathscr{A}^2 = \tfrac{1}{4}a^2b^2 \sin^2 C$$

$$= \tfrac{1}{4}a^2b^2(1 - \cos^2 C) \qquad \text{Pythagorean identity}$$

$$= \tfrac{1}{4}a^2b^2(1 - \cos C)(1 + \cos C) \qquad \text{Factor}$$

Next, we write the expressions $1 - \cos C$ and $1 + \cos C$ in terms of a, b, and c. By the Law of Cosines we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \qquad \text{Law of Cosines}$$

$$1 + \cos C = 1 + \frac{a^2 + b^2 - c^2}{2ab} \qquad \text{Add 1}$$

$$= \frac{2ab + a^2 + b^2 - c^2}{2ab} \qquad \text{Common denominator}$$

$$= \frac{(a + b)^2 - c^2}{2ab} \qquad \text{Factor}$$

$$= \frac{(a + b + c)(a + b - c)}{2ab} \qquad \text{Difference of squares}$$

Similarly,

$$1 - \cos C = \frac{(c + a - b)(c - a + b)}{2ab}$$

Substituting these expressions in the formula we obtained for $\mathscr{A}^2$ gives

$$\begin{aligned}
\mathscr{A}^2 &= \tfrac{1}{4}a^2b^2 \frac{(a + b + c)(a + b - c)}{2ab} \frac{(c + a - b)(c - a + b)}{2ab} \\
&= \frac{(a + b + c)}{2} \frac{(a + b - c)}{2} \frac{(c + a - b)}{2} \frac{(c - a + b)}{2} \\
&= s(s - c)(s - b)(s - a)
\end{aligned}$$

To see that the factors in the last two products are equal, note for example that

$$\begin{aligned}
\frac{a + b - c}{2} &= \frac{a + b + c}{2} - c \\
&= s - c
\end{aligned}$$

Heron's Formula now follows from taking the square root of each side. ■

EXAMPLE 5 ■ Area of a Lot

A businessman wishes to buy a triangular lot in a busy downtown location (see Figure 9). The lot frontages on the three adjacent streets are 125, 280, and 315 ft. Find the area of the lot.

SOLUTION The semiperimeter of the lot is

$$s = \frac{125 + 280 + 315}{2} = 360$$

By Heron's Formula the area is

$$\mathscr{A} = \sqrt{360(360 - 125)(360 - 280)(360 - 315)} \approx 17{,}451.6$$

Thus the area is approximately 17,452 ft^2.

FIGURE 9

Now Try Exercises 29 and 53 ■

5.6 EXERCISES

CONCEPTS

1. For triangle ABC with sides a, b, and c the Law of Cosines states

   $c^2 =$ ______________________

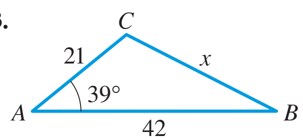
2. In which of the following cases must the Law of Cosines be used to solve a triangle?

   ASA SSS SAS SSA

SKILLS

3–10 ■ Finding an Angle or Side Use the Law of Cosines to determine the indicated side x or angle θ.

3.



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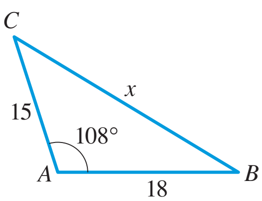

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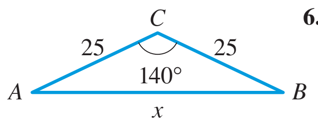

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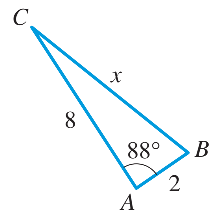

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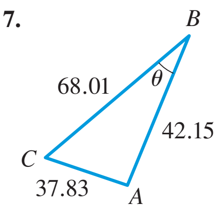

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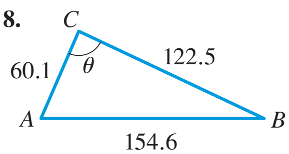

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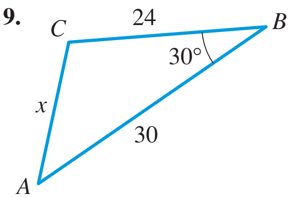

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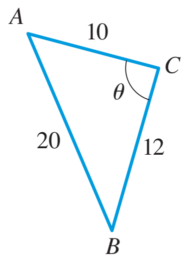
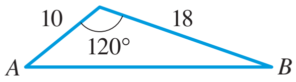


11–20 ■ Solving a Triangle Solve triangle ABC.

11.



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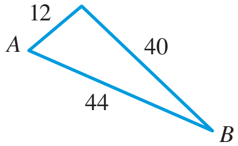


13. $a = 3.0, \quad b = 4.0, \quad \angle C = 53°$

14. $b = 60, \quad c = 30, \quad \angle A = 70°$

15. $a = 20, \quad b = 25, \quad c = 22$

16. $a = 10, \quad b = 12, \quad c = 16$

17. $b = 125, \quad c = 162, \quad \angle B = 40°$

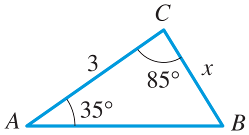
18. $a = 65, \quad c = 50, \quad \angle C = 52°$

19. $a = 50, \quad b = 65, \quad \angle A = 55°$

20. $a = 73.5, \quad \angle B = 61°, \quad \angle C = 83°$

21–28 ■ Law of Sines or Law of Cosines? Find the indicated side x or angle θ. (Use either the Law of Sines or the Law of Cosines, as appropriate.)

21.



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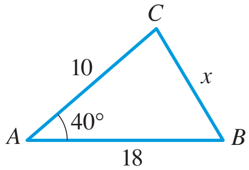

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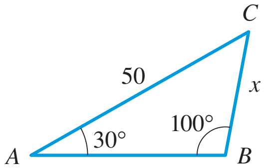
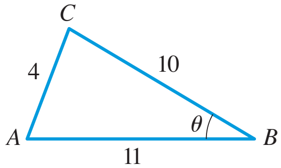


24.



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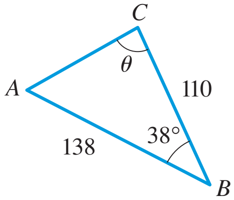

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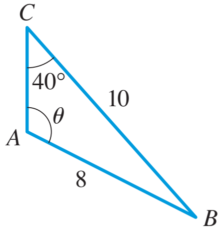

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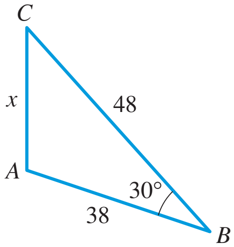

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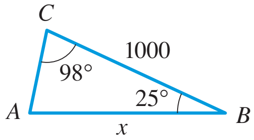


29–32 ■ Heron's Formula Find the area of the triangle whose sides have the given lengths.

29. $a = 9, \quad b = 12, \quad c = 15$

30. $a = 1, \quad b = 2, \quad c = 2$

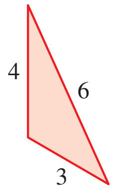
31. $a = 7, \quad b = 8, \quad c = 9$

32. $a = 11, \quad b = 100, \quad c = 101$

SKILLS Plus

33–36 ■ Heron's Formula Find the area of the shaded figure, rounded to two decimals.

33.



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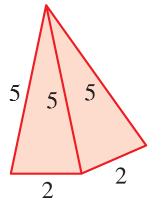

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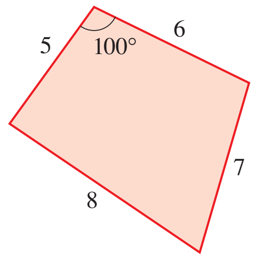

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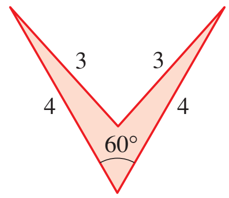
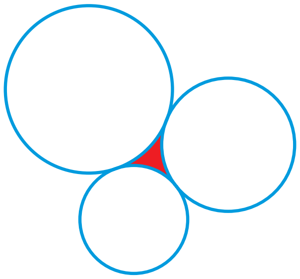
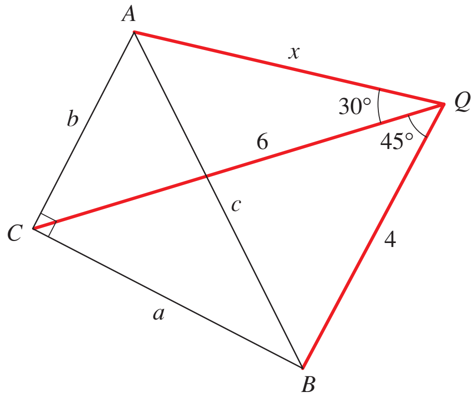


37. Area of a Region Three circles of radii 4, 5, and 6 cm are mutually tangent. Find the shaded area enclosed between the circles.

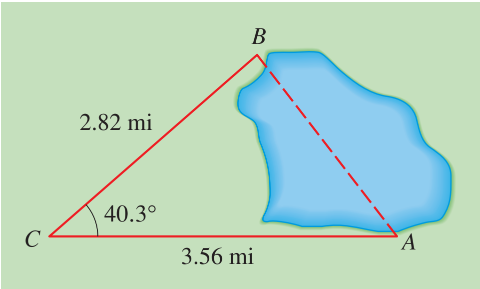


38. Finding a Length In the figure, triangle ABC is a right triangle, $CQ = 6$, and $BQ = 4$. Also, $\angle AQC = 30°$ and $\angle CQB = 45°$. Find the length of AQ. [*Hint:* First use the Law of Cosines to find expressions for a^2, b^2, and c^2.]



APPLICATIONS

39. Surveying To find the distance across a small lake, a surveyor has taken the measurements shown. Find the distance across the lake using this information.



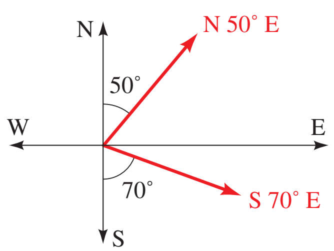
40. Geometry A parallelogram has sides of lengths 3 and 5, and one angle is 50°. Find the lengths of the diagonals.

41. Calculating Distance Two straight roads diverge at an angle of 65°. Two cars leave the intersection at 2:00 P.M., one traveling at 50 mi/h and the other at 30 mi/h. How far apart are the cars at 2:30 P.M.?

42. Calculating Distance A car travels along a straight road, heading east for 1 h, then traveling for 30 min on another road that leads northeast. If the car has maintained a constant speed of 40 mi/h, how far is it from its starting position?

43. Dead Reckoning A pilot flies in a straight path for 1 h 30 min. She then makes a course correction, heading 10° to the right of her original course, and flies 2 h in the new direction. If she maintains a constant speed of 625 mi/h, how far is she from her starting position?

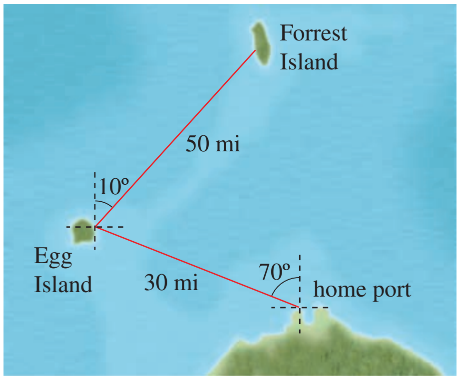
44. Navigation Two boats leave the same port at the same time. One travels at a speed of 30 mi/h in the direction N 50° E, and the other travels at a speed of 26 mi/h in a direction S 70° E (see the figure). How far apart are the two boats after 1 h?



45. Navigation A fisherman leaves his home port and heads in the direction N 70° W. He travels 30 mi and reaches Egg Island. The next day he sails N 10° E for 50 mi, reaching Forrest Island.

(a) Find the distance between the fisherman's home port and Forrest Island.

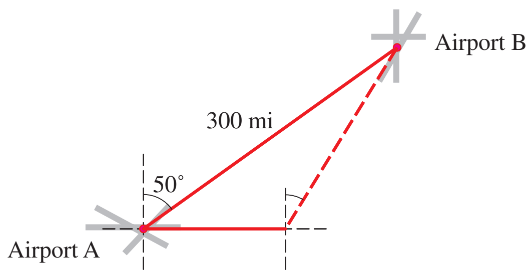
(b) Find the bearing from Forrest Island back to his home port.



46. Navigation Airport B is 300 mi from airport A at a bearing N 50° E (see the figure). A pilot wishing to fly from A to B mistakenly flies due east at 200 mi/h for 30 min, when he notices his error.

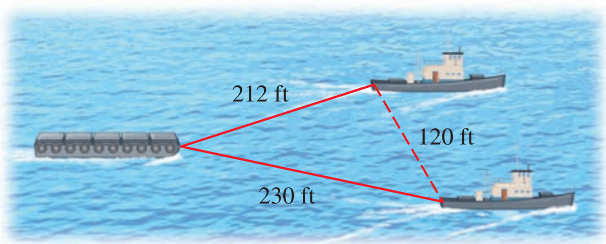
(a) How far is the pilot from his destination at the time he notices the error?

(b) What bearing should he head his plane to arrive at airport B?

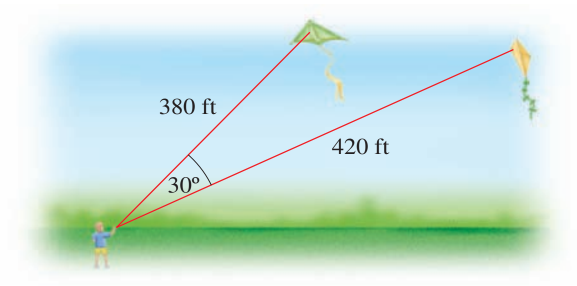


47. Triangular Field A triangular field has sides of lengths 22, 36, and 44 yd. Find the largest angle.

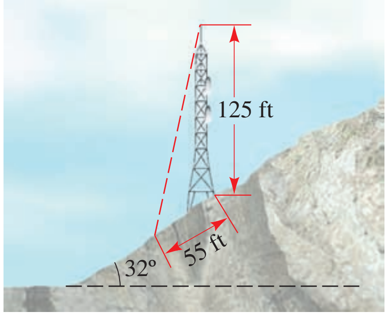
48. Towing a Barge Two tugboats that are 120 ft apart pull a barge, as shown. If the length of one cable is 212 ft and the length of the other is 230 ft, find the angle formed by the two cables.



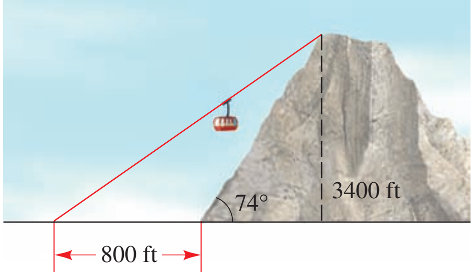
49. Flying Kites A boy is flying two kites at the same time. He has 380 ft of line out to one kite and 420 ft to the other. He estimates the angle between the two lines to be 30°. Approximate the distance between the kites.



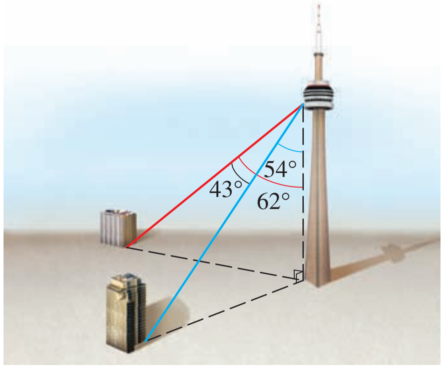
50. Securing a Tower A 125-ft tower is located on the side of a mountain that is inclined 32° to the horizontal. A guy wire is to be attached to the top of the tower and anchored at a point 55 ft downhill from the base of the tower. Find the shortest length of wire needed.



51. Cable Car A steep mountain is inclined 74° to the horizontal and rises 3400 ft above the surrounding plain. A cable car is to be installed from a point 800 ft from the base to the top of the mountain, as shown. Find the shortest length of cable needed.



52. CN Tower The CN Tower in Toronto, Canada, is the tallest free-standing structure in North America. A woman on the observation deck, 1150 ft above the ground, wants to determine the distance between two landmarks on the ground below. She observes that the angle formed by the lines of sight to these two landmarks is 43°. She also observes that the angle between the vertical and the line of sight to one of the landmarks is 62° and that to the other landmark is 54°. Find the distance between the two landmarks.



53. Land Value Land in downtown Columbia is valued at $20 a square foot. What is the value of a triangular lot with sides of lengths 112, 148, and 190 ft?

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

54. DISCUSS: Solving for the Angles in a Triangle The paragraph that follows the solution of Example 3 on page 484 explains an alternative method for finding $\angle B$ and $\angle C$, using the Law of Sines. Use this method to solve the triangle in the example, finding $\angle B$ first and then $\angle C$. Explain how you chose the appropriate value for the measure of $\angle B$. Which method do you prefer for solving an SAS triangle problem: the one explained in Example 3 or the one you used in this exercise?

55. PROVE: Projection Laws Prove that in triangle ABC

$$a = b\cos C + c\cos B$$

$$b = c\cos A + a\cos C$$

$$c = a\cos B + b\cos A$$

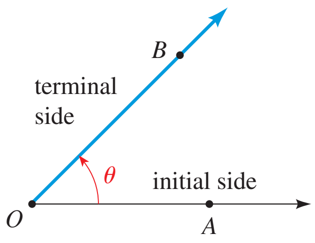
These are called the *Projection Laws*. [*Hint:* To get the first equation, add the second and third equations in the Law of Cosines and solve for a.]

CHAPTER 5 ■ REVIEW

PROPERTIES AND FORMULAS

Angles (p. 438)

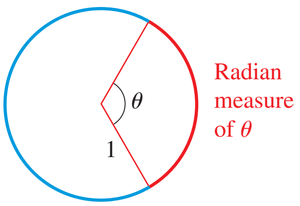
An **angle** consists of two rays with a common vertex. One of the rays is the **initial side**, and the other the **terminal side**. An angle can be viewed as a rotation of the initial side onto the terminal side. If the rotation is counterclockwise, the angle is **positive**; if the rotation is clockwise, the angle is **negative**.



Notation: The angle in the figure can be referred to as angle AOB, or simply as angle O, or as angle θ.

Angle Measure (p. 438)

The **radian measure** of an angle (abbreviated **rad**) is the length of the arc that the angle subtends in a circle of radius 1, as shown in the figure.



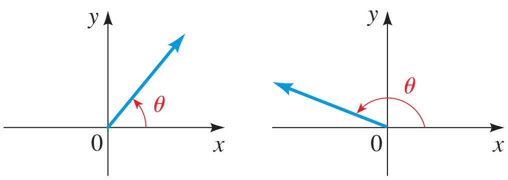
The **degree measure** of an angle is the number of degrees in the angle, where a degree is $\frac{1}{360}$ of a complete circle.

To convert degrees to radians, multiply by $\pi/180$.

To convert radians to degrees, multiply by $180/\pi$.

Angles in Standard Position (pp. 439, 460)

An angle is in **standard position** if it is drawn in the xy-plane with its vertex at the origin and its initial side on the positive x-axis.

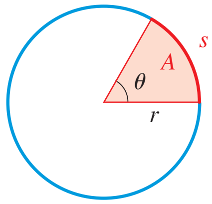


Two angles in standard position are **coterminal** if their sides coincide.

The **reference angle** $\overline{\theta}$ associated with an angle θ is the acute angle formed by the terminal side of θ and the x-axis.

Length of an Arc; Area of a Sector (pp. 441–442)

Consider a circle of radius r.



The **length s of an arc** that subtends a central angle of θ radians is $s = r\theta$.

The **area A of a sector** with central angle of θ radians is $A = \frac{1}{2}r^2\theta$.

Circular Motion (pp. 442–443)

Suppose a point moves along a circle of radius r and the ray from the center of the circle to the point traverses θ radians in time t. Let $s = r\theta$ be the distance the point travels in time t.

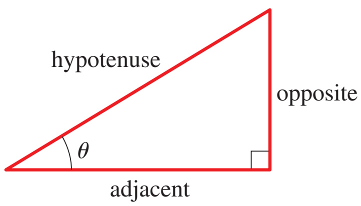
The **angular speed** of the point is $\omega = \theta/t$.

The **linear speed** of the point is $v = s/t$.

Linear speed v and angular speed ω are related by the formula $v = r\omega$.

Trigonometric Ratios (p. 448)

For a right triangle with an acute angle θ the trigonometric ratios are defined as follows.



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

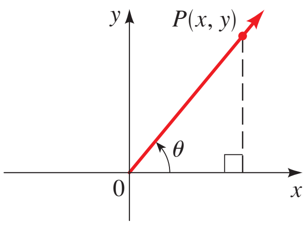
Special Trigonometric Ratios (p. 449)

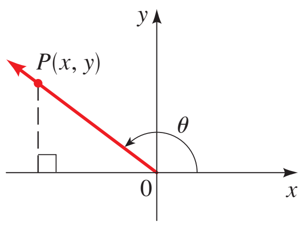
The trigonometric functions have the following values at the special values of θ.

θ	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Trigonometric Functions of Angles (p. 457)

Let θ be an angle in standard position, and let $P(x, y)$ be a point on the terminal side. Let $r = \sqrt{x^2 + y^2}$ be the distance from the origin to the point $P(x, y)$.





For nonzero values of the denominator the **trigonometric functions** are defined as follows.

$$\sin t = \frac{y}{r} \qquad \cos t = \frac{x}{r} \qquad \tan t = \frac{y}{x}$$

$$\csc t = \frac{r}{y} \qquad \sec t = \frac{r}{x} \qquad \cot t = \frac{x}{y}$$

Basic Trigonometric Identities (p. 462)

An identity is an equation that is true for all values of the variable. The basic trigonometric identities are as follows.

Reciprocal Identities:

$$\csc \theta = \frac{1}{\sin \theta} \qquad \sec \theta = \frac{1}{\cos \theta} \qquad \cot \theta = \frac{1}{\tan \theta}$$

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Area of a Triangle (p. 464)

The area $\mathcal{A}$ of a triangle with sides of lengths a and b and with included angle θ is

$$\mathcal{A} = \frac{1}{2}ab \sin \theta$$

Inverse Trigonometric Functions (p. 468)

Inverse functions of the trigonometric functions are defined by restricting the domains as follows.

Function	Domain	Range
$\sin^{-1}$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

The inverse trigonometric functions are defined as follows.

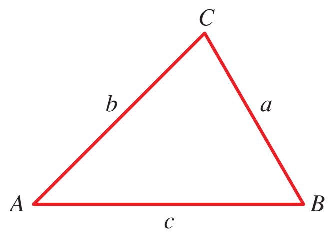
$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x$$

$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x$$

$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x$$

The Law of Sines and the Law of Cosines (pp. 475, 482)

We follow the convention of labeling the angles of a triangle as A, B, C and the lengths of the corresponding opposite sides as a, b, c, as in the figure.



For a triangle ABC we have the following laws.

The **Law of Sines** states that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The **Law of Cosines** states that

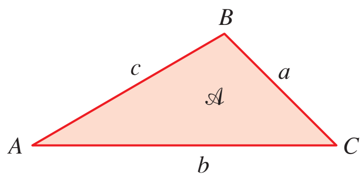
$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Heron's Formula (p. 485)

Let ABC be a triangle with sides a, b, and c.



Heron's Formula states that the area $\mathcal{A}$ of triangle ABC is

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$ is the semiperimeter of the triangle.

CONCEPT CHECK

1. **(a)** How is the degree measure of an angle defined?

(b) How is the radian measure of an angle defined?

(c) How do you convert from degrees to radians? Convert $45°$ to radians.

(d) How do you convert from radians to degrees? Convert 2 rad to degrees.

2. **(a)** When is an angle in standard position? Illustrate with a graph.

(b) When are two angles in standard position coterminal? Illustrate with a graph.

(c) Are the angles $25°$ and $745°$ coterminal?

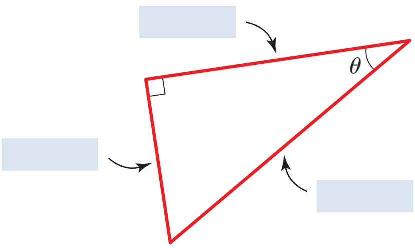
(d) How is the reference angle for an angle θ defined?

(e) Find the reference angle for $150°$.

3. **(a)** In a circle of radius r, what is the length s of an arc that subtends a central angle of θ radians?

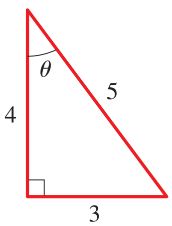
(b) In a circle of radius r, what is the area A of a sector with central angle θ radians?

4. **(a)** Let θ be an acute angle in a right triangle. Identify the opposite side, the adjacent side, and the hypotenuse in the figure.



(b) Define the six trigonometric ratios in terms of the adjacent and opposite sides and the hypotenuse.

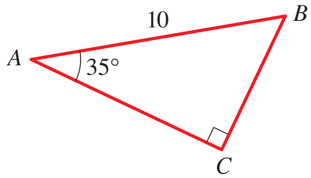
(c) Find the six trigonometric ratios for the angle θ shown in the figure.



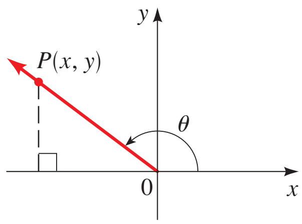
(d) List the special values of sine, cosine, and tangent.

5. **(a)** What does it mean to solve a triangle?

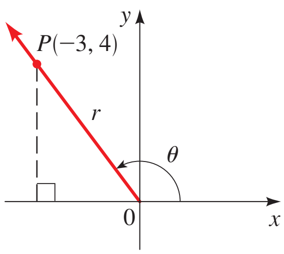
(b) Solve the triangle shown.



6. **(a)** Let θ be an angle in standard position, let $P(x, y)$ be a point on the terminal side, and let r be the distance from the origin to P, as shown in the figure. Write expressions for the six trigonometric functions of θ.



(b) Find the sine, cosine, and tangent for the angle θ shown in the figure.



7. In each of the four quadrants, identify the trigonometric functions that are positive.

8. **(a)** Describe the steps we use to find the value of a trigonometric function of an angle θ.

(b) Find $\sin 5\pi/6$.

9. **(a)** State the reciprocal identities.

(b) State the Pythagorean identities.

10. **(a)** What is the area of a triangle with sides of length a and b and with included angle θ?

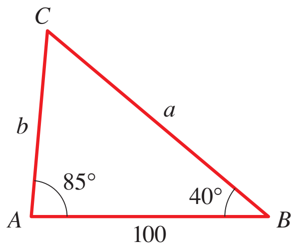
(b) What is the area of a triangle with sides of length a, b, and c?

11. **(a)** Define the inverse sine function, the inverse cosine function, and the inverse tangent function.

(b) Find $\sin^{-1}\frac{1}{2}$, $\cos^{-1}(\sqrt{2}/2)$, and $\tan^{-1}1$.

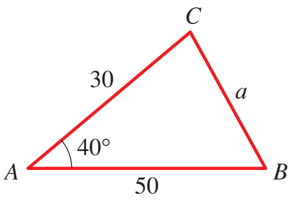
(c) For what values of x is the equation $\sin(\sin^{-1}x) = x$ true? For what values of x is the equation $\sin^{-1}(\sin x) = x$ true?

12. **(a)** State the Law of Sines.
(b) Find side a in the figure.



(c) Explain the ambiguous case in the Law of Sines.

13. **(a)** State the Law of Cosines.
(b) Find side a in the figure.



ANSWERS TO THE CONCEPT CHECK CAN BE FOUND AT THE BACK OF THE BOOK.

EXERCISES

1–2 ■ From Degrees to Radians Find the radian measure that corresponds to the given degree measure.

1. **(a)** 30° **(b)** 150° **(c)** −20° **(d)** −225°

2. **(a)** 105° **(b)** 72° **(c)** −405° **(d)** −315°

3–4 ■ From Radians to Degrees Find the degree measure that corresponds to the given radian measure.

3. **(a)** $\frac{5\pi}{6}$ **(b)** $-\frac{\pi}{9}$ **(c)** $-\frac{4\pi}{3}$ **(d)** 4

4. **(a)** $-\frac{5\pi}{3}$ **(b)** $\frac{10\pi}{9}$ **(c)** -5 **(d)** $\frac{11\pi}{3}$

5–10 ■ Length of a Circular Arc These exercises involve the formula for the length of a circular arc.

5. Find the length of an arc of a circle of radius 10 m if the arc subtends a central angle of $2\pi/5$ rad.

6. A central angle θ in a circle of radius 2.5 cm is subtended by an arc of length 7 cm. Find the measure of θ in degrees and radians.

7. A circular arc of length 25 ft subtends a central angle of 50°. Find the radius of the circle.

8. A circular arc of length 13π m subtends a central angle of 130°. Find the radius of the circle.

9. How many revolutions will a car wheel of diameter 28 in. make over a period of half an hour if the car is traveling at 60 mi/h?

10. New York and Los Angeles are 2450 mi apart. Find the angle that the arc between these two cities subtends at the center of the earth. (The radius of the earth is 3960 mi.)

11–14 ■ Area of a Circular Sector These exercises involve the formula for the area of a circular sector.

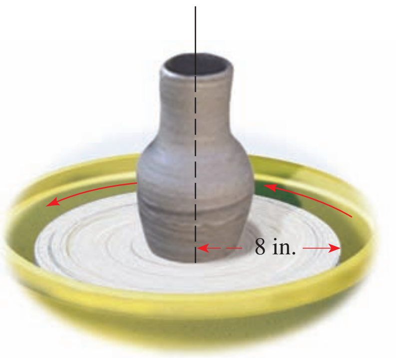
11. Find the area of a sector with central angle 2 rad in a circle of radius 5 m.

12. Find the area of a sector with central angle 52° in a circle of radius 200 ft.

13. A sector in a circle of radius 25 ft has an area of 125 ft^2. Find the central angle of the sector.

14. The area of a sector of a circle with a central angle of $11\pi/6$ radians is 50 m^2. Find the radius of the circle.

15. Angular Speed and Linear Speed A potter's wheel with radius 8 in. spins at 150 rpm. Find the angular and linear speeds of a point on the rim of the wheel.



16. Angular Speed and Linear Speed In an automobile transmission a *gear ratio* g is the ratio

$$g = \frac{\text{angular speed of engine}}{\text{angular speed of wheels}}$$

The angular speed of the engine is shown on the tachometer (in rpm).

A certain sports car has wheels with radius 11 in. Its gear ratios are shown in the following table. Suppose the car is in fourth gear and the tachometer reads 3500 rpm.

(a) Find the angular speed of the engine.

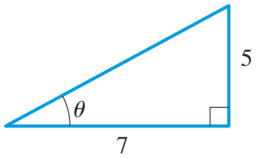
(b) Find the angular speed of the wheels.

(c) How fast (in mi/h) is the car traveling?

Gear	Ratio
1st	4.1
2nd	3.0
3rd	1.6
4th	0.9
5th	0.7

17–18 ■ Trigonometric Ratios Find the values of the six trigonometric ratios of θ.

17.



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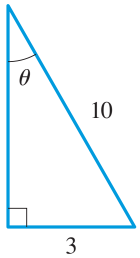
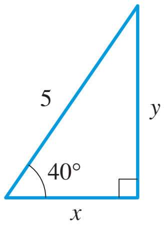


19–22 ■ Finding Sides in Right Triangles Find the sides labeled x and y, rounded to two decimal places.

19.



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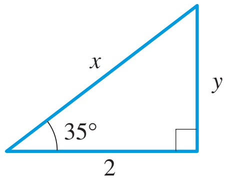

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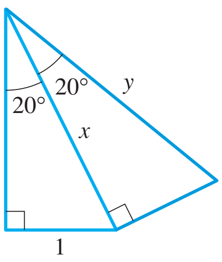

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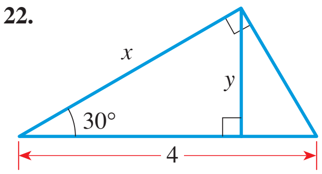
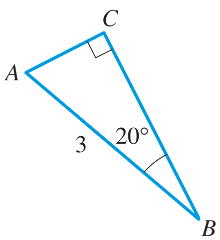


23–26 ■ Solving a Triangle Solve the triangle.

23.



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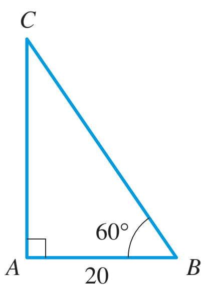

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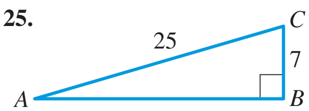

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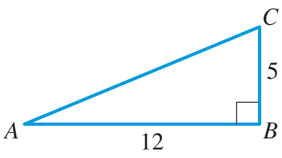
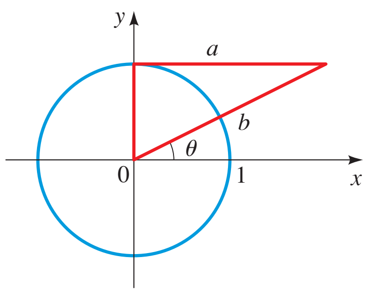


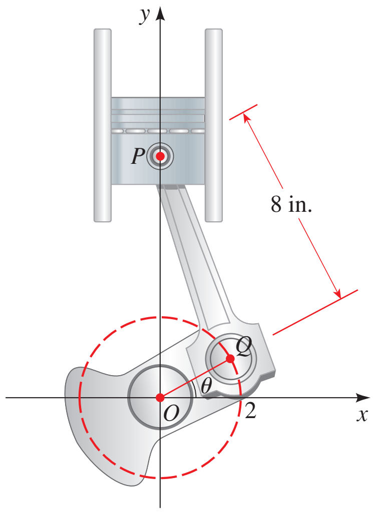
27. Trigonometric Ratios Express the lengths a and b in the figure in terms of the trigonometric ratios of θ.



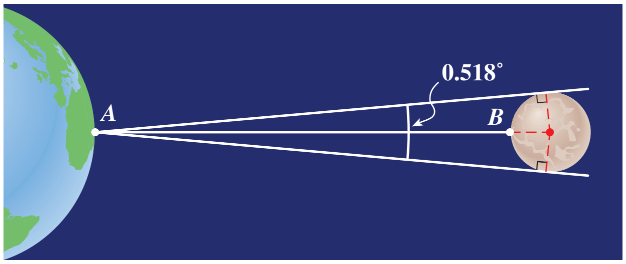
28. CN Tower The highest free-standing tower in North America is the CN Tower in Toronto, Canada. From a distance of 1 km from its base, the angle of elevation to the top of the tower is 28.81°. Find the height of the tower.

29. Perimeter of a Regular Hexagon Find the perimeter of a regular hexagon that is inscribed in a circle of radius 8 m.

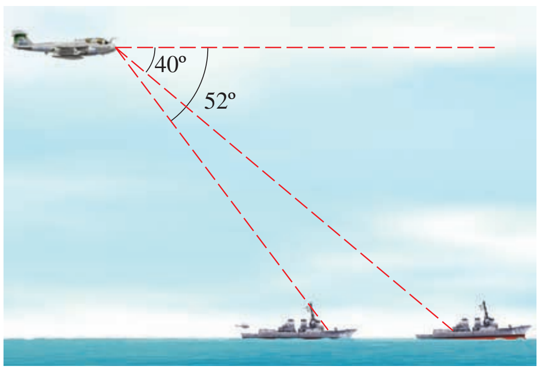
30. Pistons of an Engine The pistons in a car engine move up and down repeatedly to turn the crankshaft, as shown. Find the height of the point P above the center O of the crankshaft in terms of the angle θ.



31. Radius of the Moon As viewed from the earth, the angle subtended by the full moon is 0.518°. Use this information and the fact that the distance AB from the earth to the moon is 236,900 mi to find the radius of the moon.



32. Distance Between Two Ships A pilot measures the angles of depression to two ships to be 40° and 52° (see the figure). If the pilot is flying at an elevation of 35,000 ft, find the distance between the two ships.



33–44 ■ Values of Trigonometric Functions Find the exact value.

33. $\sin 315°$ **34.** $\csc \dfrac{9\pi}{4}$ **35.** $\tan(-135°)$

36. $\cos \dfrac{5\pi}{6}$ **37.** $\cot\left(-\dfrac{22\pi}{3}\right)$ **38.** $\sin 405°$

39. $\cos 585°$ **40.** $\sec \dfrac{22\pi}{3}$ **41.** $\csc \dfrac{8\pi}{3}$

42. $\sec \dfrac{13\pi}{6}$ **43.** $\cot(-390°)$ **44.** $\tan \dfrac{23\pi}{4}$

45. Values of Trigonometric Functions Find the values of the six trigonometric ratios of the angle θ in standard position if the point $(-5, 12)$ is on the terminal side of θ.

46. Values of Trigonometric Functions Find $\sin\theta$ if θ is in standard position and its terminal side intersects the circle of radius 1 centered at the origin at the point $(-\sqrt{3}/2, \frac{1}{2})$.

47. Angle Formed by a Line Find the acute angle that is formed by the line $y - \sqrt{3}x + 1 = 0$ and the x-axis.

48. Values of Trigonometric Functions Find the six trigonometric ratios of the angle θ in standard position if its terminal side is in Quadrant III and is parallel to the line $4y - 2x - 1 = 0$.

49–52 ■ Expressing One Trigonometric Function in Terms of Another Write the first expression in terms of the second, for θ in the given quadrant.

49. $\tan\theta$, $\cos\theta$; θ in Quadrant II

50. $\sec\theta$, $\sin\theta$; θ in Quadrant III

51. $\tan^2\theta$, $\sin\theta$; θ in any quadrant

52. $\csc^2\theta\cos^2\theta$, $\sin\theta$; θ in any quadrant

53–56 ■ Values of Trigonometric Functions Find the values of the six trigonometric functions of θ from the information given.

53. $\tan\theta = \sqrt{7}/3$, $\sec\theta = \frac{4}{3}$

54. $\sec\theta = \frac{41}{40}$, $\csc\theta = -\frac{41}{9}$

55. $\sin\theta = \frac{3}{5}$, $\cos\theta < 0$

56. $\sec\theta = -\frac{13}{5}$, $\tan\theta > 0$

57–60 ■ Value of an Expression Find the value of the given trigonometric expression.

57. If $\tan\theta = -\frac{1}{2}$ for θ in Quadrant II, find $\sin\theta + \cos\theta$.

58. If $\sin\theta = \frac{1}{2}$ for θ in Quadrant I, find $\tan\theta + \sec\theta$.

59. If $\tan\theta = -1$, find $\sin^2\theta + \cos^2\theta$.

60. If $\cos\theta = -\sqrt{3}/2$ and $\pi/2 < \theta < \pi$, find $\sin 2\theta$.

61–64 ■ Values of Inverse Trigonometric Functions Find the exact value of the expression.

61. $\sin^{-1}(\sqrt{3}/2)$ **62.** $\tan^{-1}(\sqrt{3}/3)$

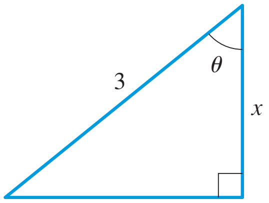
63. $\tan\left(\sin^{-1}\frac{2}{5}\right)$ **64.** $\sin\left(\cos^{-1}\frac{3}{8}\right)$

65–66 ■ Inverse Trigonometric Functions Rewrite the expression as an algebraic expression in x.

65. $\sin(\tan^{-1}x)$ **66.** $\sec(\sin^{-1}x)$

67–68 ■ Finding an Unknown Side Express θ in terms of x.

67.



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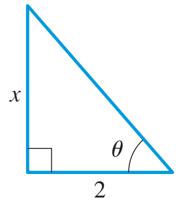
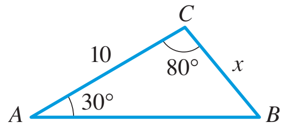


69–78 ■ Law of Sines and Law of Cosines Find the side labeled x or the angle labeled θ.

69.



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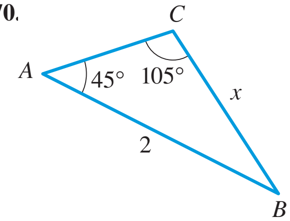

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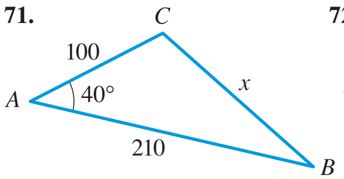

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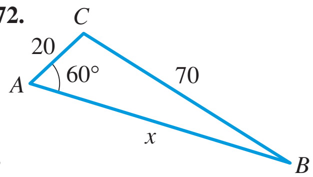

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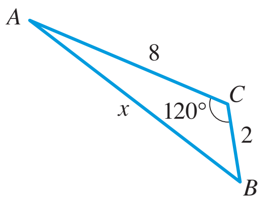

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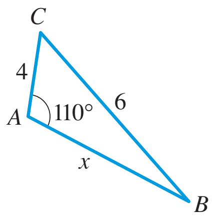

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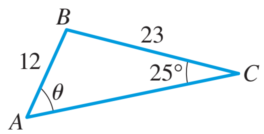

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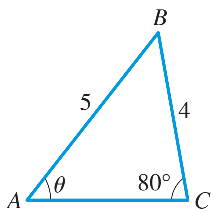

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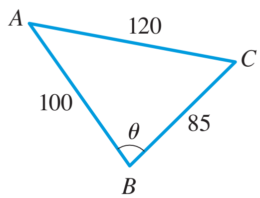

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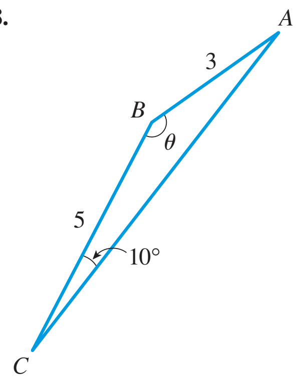
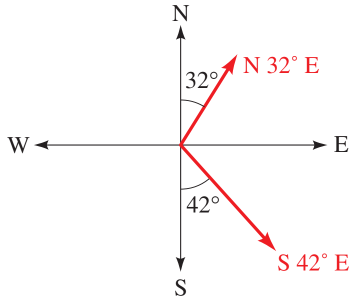
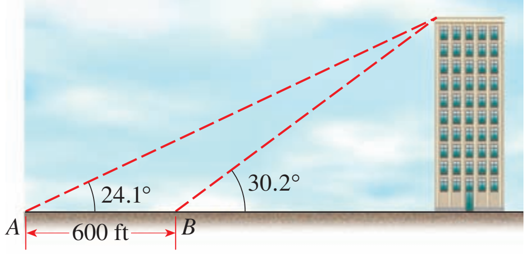


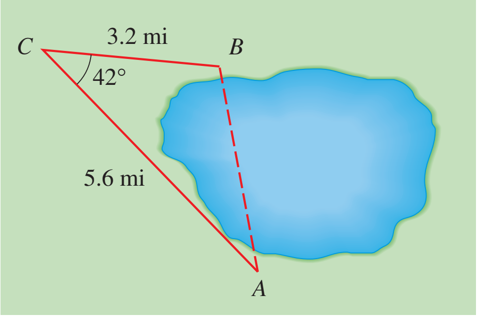
79. Distance Between Two Ships Two ships leave a port at the same time. One travels at 20 mi/h in a direction N 32° E, and the other travels at 28 mi/h in a direction S 42° E (see the figure). How far apart are the two ships after 2 h?



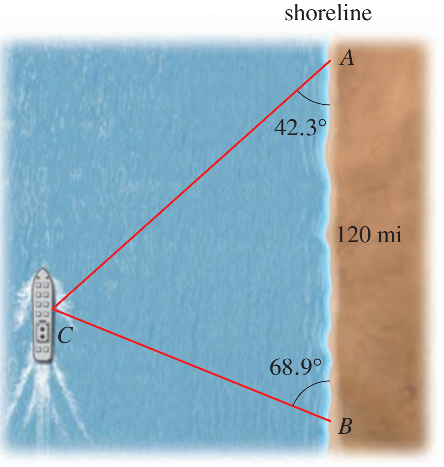
80. Height of a Building From a point A on the ground, the angle of elevation to the top of a tall building is 24.1°. From a point B, which is 600 ft closer to the building, the angle of elevation is measured to be 30.2°. Find the height of the building.



81. Distance Between Two Points Find the distance between points A and B on opposite sides of a lake from the information shown.



82. Distance Between a Boat and the Shore A boat is cruising the ocean off a straight shoreline. Points A and B are 120 mi apart on the shore, as shown. It is found that $\angle A = 42.3°$ and $\angle B = 68.9°$. Find the shortest distance from the boat to the shore.

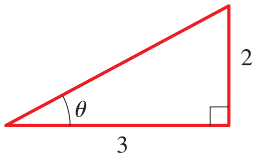


83. Area of a Triangle Find the area of a triangle with sides of length 8 and 14 and included angle 35°.

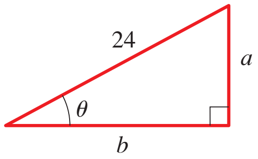
84. Heron's Formula Find the area of a triangle with sides of length 5, 6, and 8.

TEST

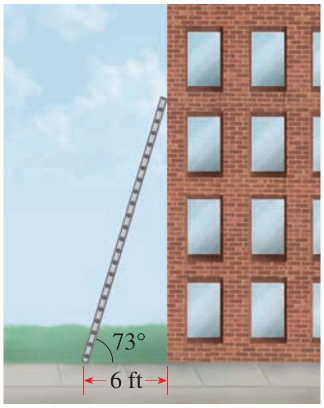
1. Find the radian measures that correspond to the degree measures 330° and $-135°$.
2. Find the degree measures that correspond to the radian measures $4\pi/3$ and -1.3.
3. The rotor blades of a helicopter are 16 ft long and are rotating at 120 rpm.
   (a) Find the angular speed of the rotor.
   (b) Find the linear speed of a point on the tip of a blade.
4. Find the exact value of each of the following.
   (a) $\sin 405°$ **(b)** $\tan(-150°)$ **(c)** $\sec \dfrac{5\pi}{3}$ **(d)** $\csc \dfrac{5\pi}{2}$
5. Find $\tan\theta + \sin\theta$ for the angle θ shown.



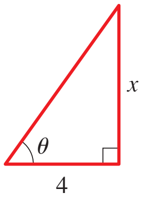
6. Express the lengths a and b shown in the figure in terms of θ.



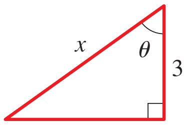
7. If $\cos\theta = -\frac{1}{3}$ and θ is in Quadrant III, find $\tan\theta \cot\theta + \csc\theta$.
8. If $\sin\theta = \frac{5}{13}$ and $\tan\theta = -\frac{5}{12}$, find $\sec\theta$.
9. Express $\tan\theta$ in terms of $\sec\theta$ for θ in Quadrant II.
10. The base of the ladder in the figure is 6 ft from the building, and the angle formed by the ladder and the ground is 73°. How high up the building does the ladder touch?



11. Express θ in each figure in terms of x.
    (a)



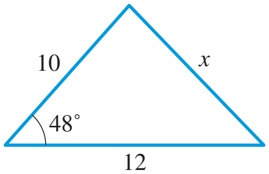
(b)



12. Find the exact value of $\cos\left(\tan^{-1} \frac{9}{40}\right)$.

13–18 ■ Find the side labeled x or the angle labeled θ.

13.



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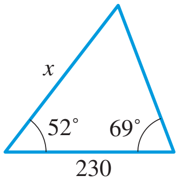

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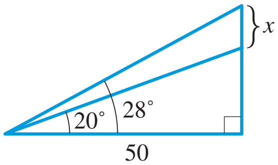

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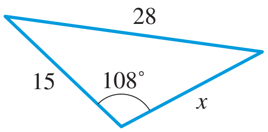

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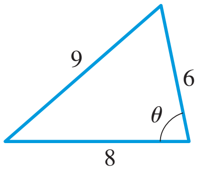

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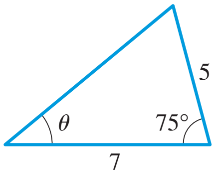
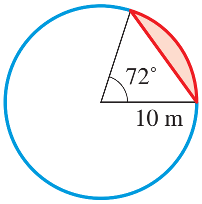


19. Refer to the figure below.

(a) Find the area of the shaded region.

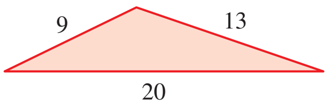
(b) Find the perimeter of the shaded region.



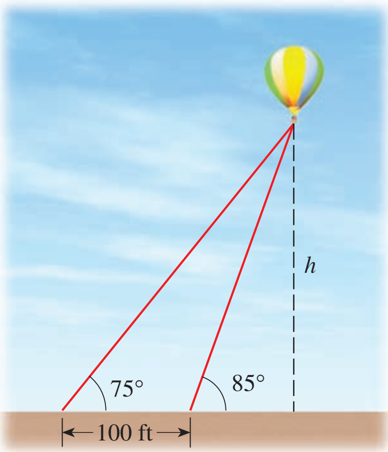
20. Refer to the figure below.

(a) Find the angle opposite the longest side.

(b) Find the area of the triangle.



21. Two wires tether a balloon to the ground, as shown. How high is the balloon above the ground?



How can we measure the height of a mountain or the distance across a lake? Obviously, it may be difficult, inconvenient, or impossible to measure these distances directly (that is, by using a tape measure or a yardstick). On the other hand, it is easy to measure *angles* involving distant objects. That's where trigonometry comes in: The trigonometric ratios relate angles to distances, so they can be used to *calculate* distances from the *measured* angles. In this *Focus* we examine how trigonometry is used to map a town. Modern mapmaking methods use satellites and the Global Positioning System, but mathematics remains at the core of the process.

Mapping a Town

A student wants to draw a map of his hometown. To construct an accurate map (or scale model), he needs to find distances between various landmarks in the town. The student makes the measurements shown in Figure 1. Note that only one distance is measured: that between City Hall and the first bridge. All other measurements are angles.

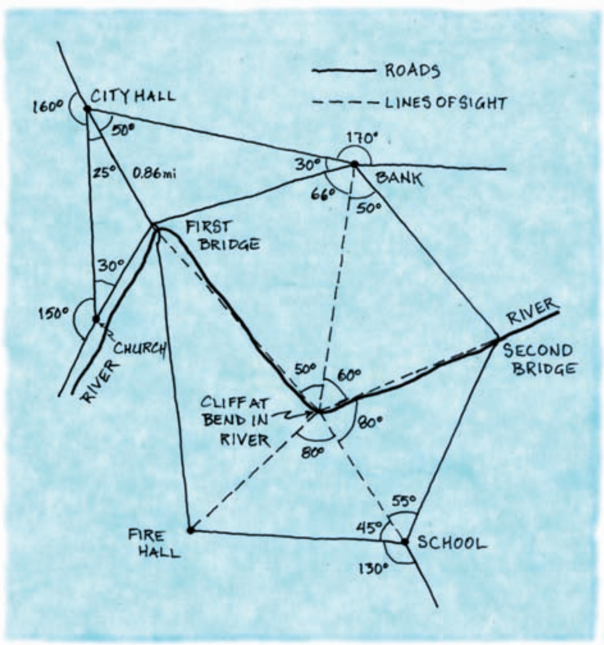


FIGURE 1

The distances between other landmarks can now be found by using the Law of Sines. For example, the distance x from the bank to the first bridge is calculated by applying the Law of Sines to the triangle with vertices at City Hall, the bank, and the first bridge.

$$\frac{x}{\sin 50^\circ} = \frac{0.86}{\sin 30^\circ} \qquad \text{Law of Sines}$$

$$x = \frac{0.86 \sin 50^\circ}{\sin 30^\circ} \qquad \text{Solve for } x$$

$$\approx 1.32 \text{ mi} \qquad \text{Calculator}$$

So the distance between the bank and the first bridge is 1.32 mi.

The distance we just found can now be used to find other distances. For instance, we find the distance y between the bank and the cliff as follows:

$$\frac{y}{\sin 64°} = \frac{1.32}{\sin 50°} \qquad \text{Law of Sines}$$

$$y = \frac{1.32 \sin 64°}{\sin 50°} \qquad \text{Solve for } y$$

$$\approx 1.55 \text{ mi} \qquad \text{Calculator}$$

Continuing in this fashion, we can calculate all the distances between the landmarks shown in the rough sketch in Figure 1. We can use this information to draw the map shown in Figure 2.

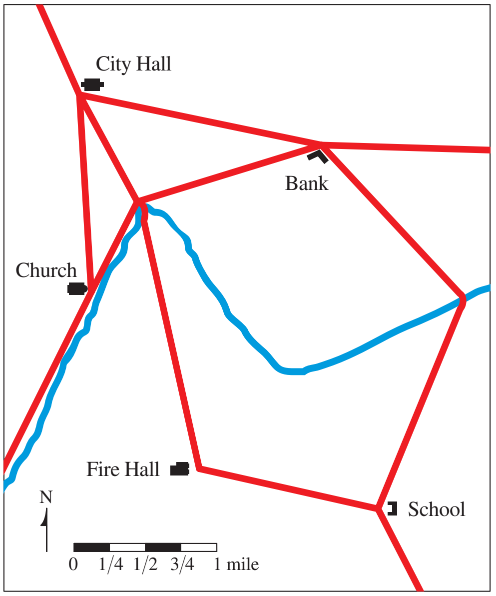


FIGURE 2

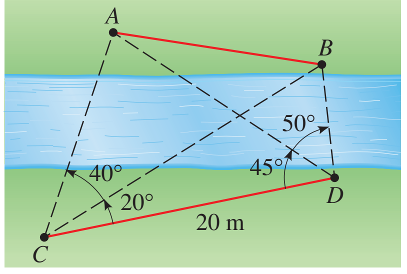
To make a topographic map, we need to measure elevation. This concept is explored in Problems 4–6.

PROBLEMS

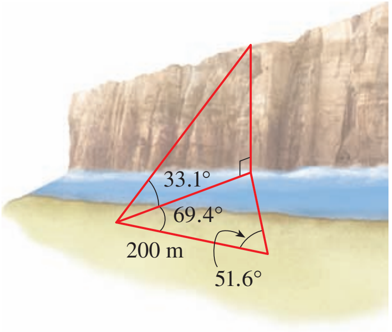
1. **Completing the Map** Find the distance between the church and City Hall.

2. **Completing the Map** Find the distance between the fire hall and the school. [*Hint:* You will need to find other distances first.]

3. **Determining a Distance** A surveyor on one side of a river wishes to find the distance between points A and B on the opposite side of the river. On her side she chooses points C and D, which are 20 m apart, and measures the angles shown in the figure below. Find the distance between A and B.



4. Height of a Cliff To measure the height of an inaccessible cliff on the opposite side of a river, a surveyor makes the measurements shown in the figure at the left. Find the height of the cliff.



5. Height of a Mountain To calculate the height h of a mountain, angles α and β and distance d are measured, as shown in the figure below.

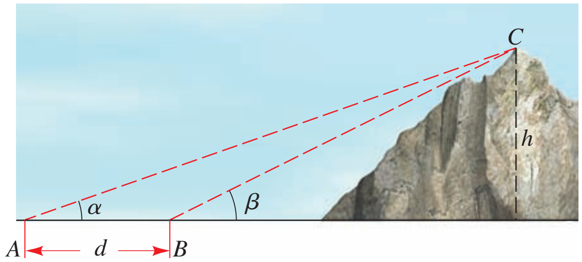
(a) Show that

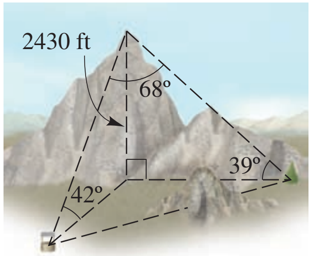
$$h = \frac{d}{\cot \alpha - \cot \beta}$$

(b) Show that

$$h = d\frac{\sin \alpha \sin \beta}{\sin(\beta - \alpha)}$$

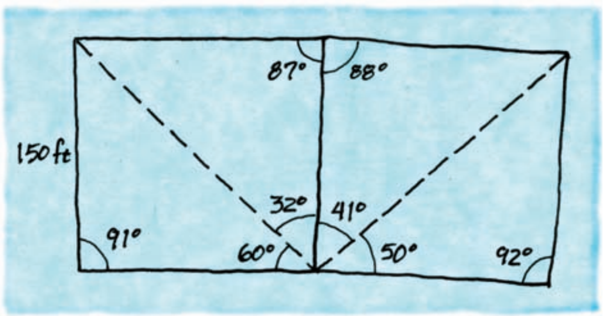
(c) Use the formulas from parts (a) and (b) to find the height of a mountain if $\alpha = 25°$, $\beta = 29°$, and $d = 800$ ft. Do you get the same answer from each formula?



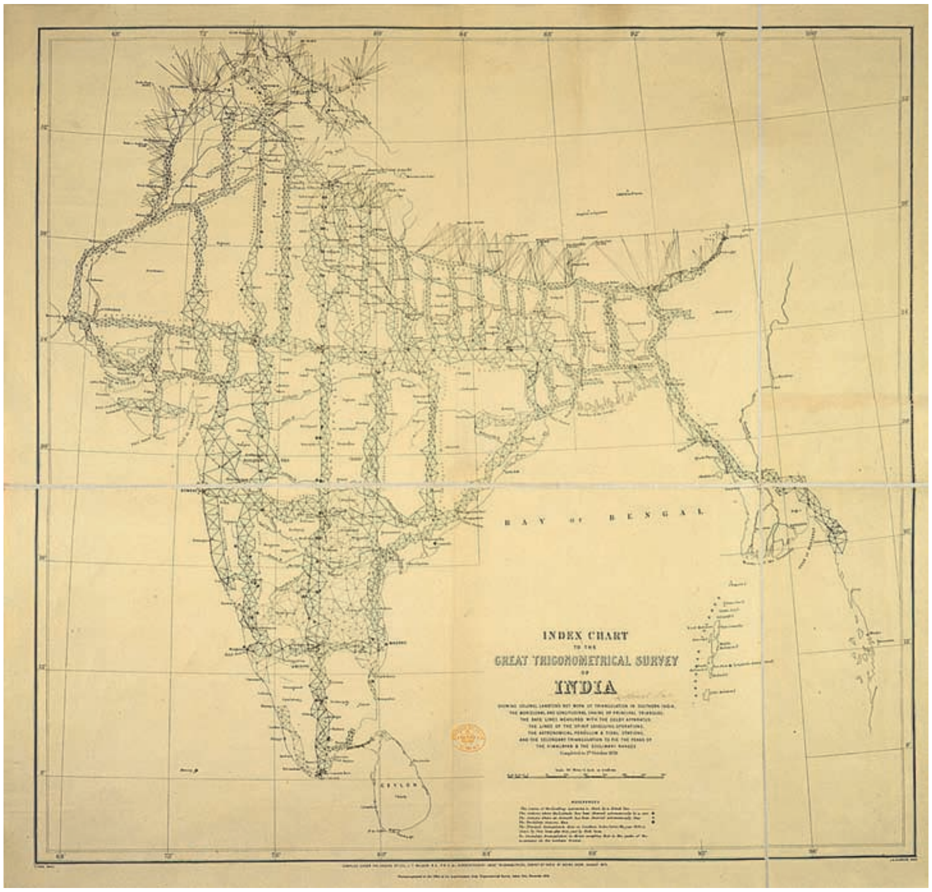


6. Determining a Distance A surveyor has determined that a mountain is 2430 ft high. From the top of the mountain he measures the angles of depression to two landmarks at the base of the mountain and finds them to be 42° and 39°. (Observe that these are the same as the angles of elevation from the landmarks as shown in the figure at the left.) The angle between the lines of sight to the landmarks is 68°. Calculate the distance between the two landmarks.

7. Surveying Building Lots A surveyor surveys two adjacent lots and makes the following rough sketch showing his measurements. Calculate all the distances shown in the figure, and use your result to draw an accurate map of the two lots.



8. Great Survey of India The Great Trigonometric Survey of India was one of the most massive mapping projects ever undertaken (see the margin note on page 478). Do some research at your library or on the Internet to learn more about the Survey, and write a report on your findings.



© The British Library Board. (Index Chart to the Great Trigonometrical Survey of India/Maps.144.e.24)

Doug Steakley/Lonely Planet Images/Getty Images

6 Trigonometric Functions: Unit Circle Approach

In this chapter we introduce two different but equivalent ways of viewing the trigonometric functions. One way is to view them as *functions of angles* (Chapter 5); the other is to view them as *functions of real numbers* (Chapter 6). The two approaches to trigonometry are independent of each other, so either Chapter 5 or Chapter 6 may be studied first. The applications of trigonometry are numerous, including signal processing, digital coding of music and videos, finding distances to stars, producing CAT scans for medical imaging, and many others. These applications are very diverse, and we need to study both approaches to trigonometry because the different approaches are required for different applications.

One of the main applications of trigonometry that we study in this chapter is periodic motion. If you've ever taken a Ferris wheel ride, then you know about periodic motion—that is, motion that repeats over and over. Periodic motion occurs often in nature, as in the daily rising and setting of the sun, the daily variation in tide levels, the vibrations of a leaf in the wind, and many more. We will see in this chapter how the trigonometric functions are used to model periodic motion.

6.1 THE UNIT CIRCLE

■ The Unit Circle ■ Terminal Points on the Unit Circle ■ The Reference Number

In this section we explore some properties of the circle of radius 1 centered at the origin. These properties are used in the next section to define the trigonometric functions.

■ The Unit Circle

The set of points at a distance 1 from the origin is a circle of radius 1 (see Figure 1). In Section 1.2 we learned that the equation of this circle is $x^2 + y^2 = 1$.

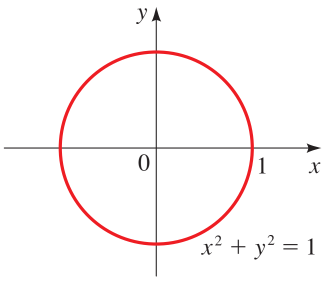


FIGURE 1 The unit circle

Circles are studied in Section 1.2, page 97.

THE UNIT CIRCLE

The **unit circle** is the circle of radius 1 centered at the origin in the *xy*-plane. Its equation is

$$x^2 + y^2 = 1$$

EXAMPLE 1 ■ A Point on the Unit Circle

Show that the point $P\left(\dfrac{\sqrt{3}}{3}, \dfrac{\sqrt{6}}{3}\right)$ is on the unit circle.

SOLUTION We need to show that this point satisfies the equation of the unit circle, that is, $x^2 + y^2 = 1$. Since

$$\left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{6}}{3}\right)^2 = \frac{3}{9} + \frac{6}{9} = 1$$

P is on the unit circle.

Now Try Exercise 3 ■

EXAMPLE 2 ■ Locating a Point on the Unit Circle

The point $P(\sqrt{3}/2, y)$ is on the unit circle in Quadrant IV. Find its y-coordinate.

SOLUTION Since the point is on the unit circle, we have

$$\left(\frac{\sqrt{3}}{2}\right)^2 + y^2 = 1$$

$$y^2 = 1 - \frac{3}{4} = \frac{1}{4}$$

$$y = \pm\frac{1}{2}$$

Since the point is in Quadrant IV, its y-coordinate must be negative, so $y = -\frac{1}{2}$.

Now Try Exercise 9 ■

■ Terminal Points on the Unit Circle

Suppose t is a real number. If $t \geq 0$, let's mark off a distance t along the unit circle, starting at the point $(1, 0)$ and moving in a counterclockwise direction. If $t < 0$, we mark off a distance $|t|$ in a clockwise direction (Figure 2). In this way we arrive at a