AIX – MARSEILLE UNIVERSITY

MASTER THESIS

INTRODUCTION TO TORIC VARIETIES

Speciality : Fundamental Mathematics

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Introduction and acknowledgements

The main goal of this work is to study the basic theories of the toric varieties.

The procedure of the construction of the toric varieties associates to a cone σ in space $\mathbb{R}^n$ successively: the dual of cone σ, a monoid S_σ, a finitely generated $\mathbb{C}-$algebra R_σ and finally an algebraic variety X_σ. We will describe the steps of the procedure:

$$\sigma \to \overset{\vee}{\sigma} \to S_\sigma \to R_\sigma \to X_\sigma.$$

Consider the group action of the torus $((\mathbb{C}^*)^n)$ on the toric varieties, the relations between combinatorial geometry and algebraic geometry, the compactness and the smoothness of the toric varieties.

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1 From combinatorial geometry to toric varieties

1.1 Cones

Definition 1.1.1 Let $A = \{v_1, ..., v_r\}$ be a finite set of vectors in $\mathbb{R}^n$, the set

$$\sigma = \{x \in \mathbb{R}^n | x = \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_r v_r, \lambda_i \geq 0, \lambda_i \in \mathbb{R}\}$$

is called a **convex polyhedral cone**. The vectors $v_1, ..., v_r$ are called **generators** of the cone σ.

Convention: If $A = \emptyset$ then $\sigma = \{0\}$.

In $\mathbb{R}^n$, let e_i be the $i - th$ column of the identity matrix, so we have $e_1 = (1, 0, ..., 0), ..., e_n = (0, 0, ..., 1)$. Then $(e_1, ..., e_n)$ is a basis of $\mathbb{R}^n$.

Example 1.1.2 In $\mathbb{R}^2$ with a basis (e_1, e_2),

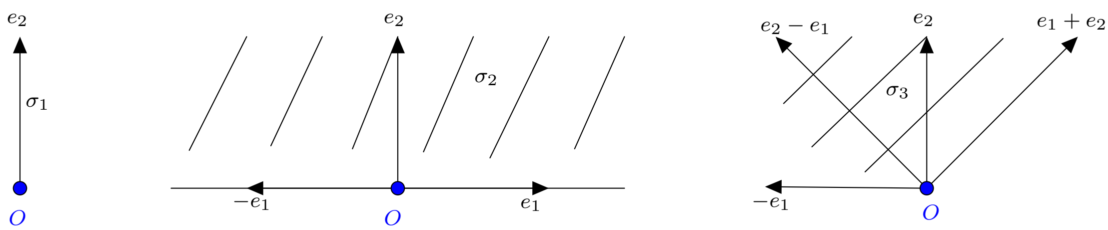

Fig. 1 Examples of cones

- σ_1 is generated by vector e_2.
- σ_2 is generated by vectors $(e_1, -e_1, e_2)$.
- σ_3 is generated by vectors $(-e_1, e_2 - e_1, e_2, e_1 + e_2)$, the vectors $(-e_1, e_1 + e_2)$ are also generators of σ_3.

Definition 1.1.3 The **dimension** of a cone σ, denoted by $dim\sigma$, is the dimension of the smallest linear space containing σ.

Example 1.1.4 In Example 1.1.2, one has $dim\sigma_1 = 1, dim\sigma_2 = dim\sigma_3 = 2$.

In the following, N will denote by a fixed lattice $N \cong \mathbb{Z}^n \subset \mathbb{R}^n$.

Definition 1.1.5 A cone σ is a **lattice** (or **rational**) cone if it admits generators $v_1, ..., v_r$ such that v_i belongs to N for all i.

A cone is **strongly convex** if it does not contain any straight line going through the origin (i.e $\sigma \bigcap (-\sigma) = \{0\}$).

Example 1.1.6 In Example 1.1.2, the cones σ_1 and σ_3 are the strongly convex lattice cones. The cone σ_2 is not strongly convex.

Definition 1.1.7 Let $(\mathbb{R}^n)^*$ be the dual space of $\mathbb{R}^n$ and $\langle , \rangle$ the dual pairing.

The **dual** of cone σ is the set

$$\overset{\vee}{\sigma} = \{u \in (\mathbb{R}^n)^* | \langle u, v \rangle \geq 0, \forall v \in \sigma\}.$$

The set of vectors $(e_1^*, ..., e_n^*)$ is a dual basis of $(\mathbb{R}^n)^*$, where e_i^* is the $i-th$ column of the identity matrix.

Given a lattice N in $\mathbb{R}^n$, we define the dual lattice by $M = Hom_Z(N, \mathbb{Z}) \cong \mathbb{Z}^n$ in $(\mathbb{R}^n)^*$.

Lemma 1.1.8 *Let σ be a convex polyhedral cone generated by the vectors $(v_1, v_2, ..., v_r)$, then $\overset{\vee}{\sigma} = \bigcap_{i=1}^{r} \overset{\vee}{\tau_i}$ where τ_i is the ray, generated by the vector v_i.*

PROOF:

For every v_i, we have a cone $\tau_i = \mathbb{R}_{\geq 0} v_i$ and

$$\overset{\vee}{\tau_i} = \{u \in (\mathbb{R}^n)^* | \langle u, v_i \rangle \geq 0\}.$$

Then

$$\overset{\vee}{\sigma} = \{u \in (\mathbb{R}^n)^* | \langle u, v_i \rangle \geq 0, i = 1, ..., r\} = \bigcap_{i=1}^{r} \{u \in \mathbb{R}^n | \langle u, v_i \rangle \geq 0\}.$$

So we have $\overset{\vee}{\sigma} = \bigcap_{i=1}^{r} \overset{\vee}{\tau_i}$

Lemma 1.1.9 *Let $v \in N$ and set $\tau = \mathbb{R}_{\geq 0} v$. Then $\overset{\vee}{\tau}$ is a lattice cone.*

PROOF:

We set

$$v^{\perp} = \{u = (u_1, ..., u_n) \in (\mathbb{R}^n)^* | \langle u, v \rangle = 0\}.$$

If $v = 0$ then $\overset{\vee}{\tau} = (\mathbb{R}^n)^*$.

If $v \neq 0$, suppose that $v = \sum_{i=1}^{n} \lambda_i e_i$, then $v^* = \sum_{i=1}^{n} \lambda_i e_i^*$, we have

$$\langle v^*, v \rangle = \sum_{i=1}^{n} \lambda_i^2 > 0.$$

One has

$$\overset{\vee}{\tau} = \{u \in (\mathbb{R}^n)^* | \langle u, v \rangle \geq 0\}.$$

For every $u \in \overset{\vee}{\tau}$, set $t = \dfrac{\langle u, v \rangle}{\langle v^*, v \rangle}$. Hence, one has $\langle u - tv^*, v \rangle = 0$. This means that $u - tv^* \in v^{\perp}$, then u belongs to $\mathbb{R}_{\geq 0} v^* + v^{\perp}$.

So we obtain

$$\overset{\vee}{\tau} = \mathbb{R}_{\geq 0} v^* + v^{\perp}.$$

Since $v \in N$, $v^* \in M$. We will show that $v^{\perp}$ is a lattice cone.

Suppose that $v = (v_1, ..., v_n) \in \mathbb{R}^n$, set the matrix

$$A = \begin{bmatrix} v_1 & ... & v_n \end{bmatrix} \in \mathbb{R}^{1 \times n},$$

then

$$v^{\perp} = \{u = (u_1, ..., u_n) \in (\mathbb{R}^n)^* | \langle u, v \rangle = u_1 v_1 + ... + u_n v_n = 0\} = KerA.$$

Since $v_1, ..., v_n \in \mathbb{Z}$, we can get $n-1$ vectors $a_1, ..., a_{n-1}$in $\mathbb{Z}^n$ such that

$$v^{\perp} = \sum_{i=1}^{n-1} \mathbb{R} a_i = \sum_{i=1}^{n-1} \mathbb{R}_{\geq 0} a_i + \sum_{i=1}^{n-1} \mathbb{R}_{\geq 0}(-a_i).$$

Therefore, $v^{\perp}$ is a lattice cone. Hence, $\overset{\vee}{\tau}$ is a lattice cone.

Example 1.1.10 Let us denote by (e_1^*, e_2^*) the dual basis of $(\mathbb{R}^2)^*$.

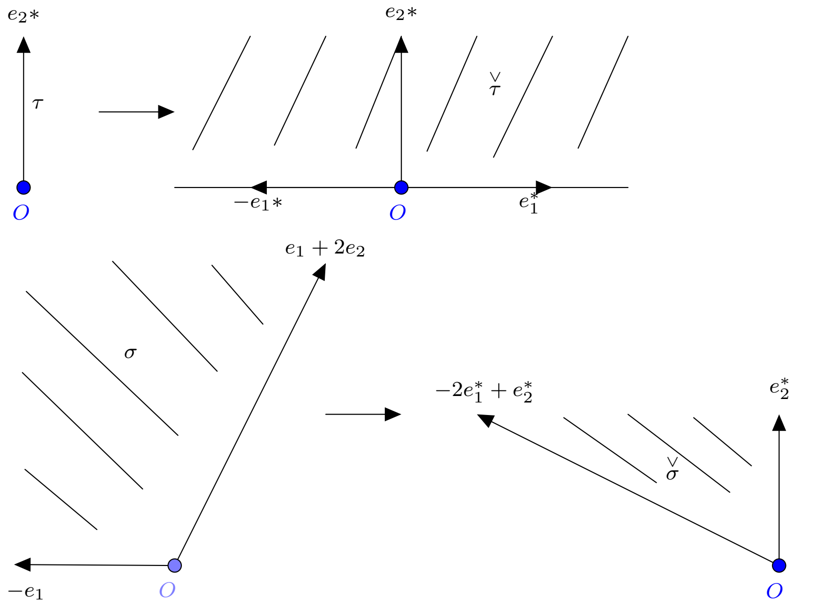

Fig. 2 Dual of cones τ and σ.

Proposition 1.1.11

i) If σ is a convex polyhedral cone, then $\overset{\vee}{\sigma}$ is also a convex polyhedral cone.

ii) If σ is a lattice cone then $\overset{\vee}{\sigma}$ is a lattice cone (relative to the lattice M).

PROOF:
i) Suppose that σ is generated by $(v_1, ..., v_r)$, then

$$\sigma = \{\sum_{i=1}^{r} \lambda_i v_i | \lambda_i \geq 0\}$$

By **Weyl-Minkowski's Theorem** for cones, there are vectors $a_1, ..., a_k \in \mathbb{R}^n$ such that

$$\sigma = \{x \in \mathbb{R}^n | a_i^t x \geq 0, \forall i = 1, ..., k\}.$$

Then

$$\overset{\vee}{\sigma} = \{\sum_{i=k}^{r} \lambda_i a_i^* | \lambda_i \geq 0\},$$

where $a_i^* := a_i \in (\mathbb{R}^n)^*$. Hence $\overset{\vee}{\sigma}$ is a convex polyhedral cone.

ii) See [7], page 7, Proposition 1.3.

Proposition 1.1.12 *If σ is a convex polyhedral cone, then $(\overset{\vee}{\sigma})^\vee = \sigma$.*

PROOF:

Suppose that σ is generated by $(v_1, ..., v_r)$. We have

$$(\overset{\vee}{\sigma})^\vee = \{v \in \mathbb{R}^n | \langle u, v\rangle \geq 0, \forall u \in \overset{\vee}{\sigma}\}.$$

Let $v \in \sigma$, then $\langle u, v\rangle \geq 0$ for all $u \in \overset{\vee}{\sigma}$. Hence, we have $\sigma \subset (\overset{\vee}{\sigma})^\vee$.

Conversely, suppose that $v \in (\overset{\vee}{\sigma})^\vee, v \notin \sigma$.

Since $v \notin \sigma$, there do not exist $\lambda_i \geq 0$ such that $\sum_{i=1}^{r} \lambda_i v_i = v$. **By Farkas' Lemma**, there is $u \in \overset{\vee}{\sigma}$ such that $\langle u, v\rangle < 0$. Contradiction with $v \in (\overset{\vee}{\sigma})^\vee$.

Therefore $(\overset{\vee}{\sigma})^\vee \subset \sigma$ and we obtain the result.

Definition 1.1.13 In fact, a convex polyhedral cone σ can also be defined as intersection of half-spaces. For each (co)vector $u \in (\mathbb{R}^n)^*$, we define a half-space by

$$H_u = \{v \in \mathbb{R}^n | \langle u, v\rangle \geq 0\}.$$

If the cone $\overset{\vee}{\sigma}$ is generated by the vectors $(u_1, u_2, ..., u_t)$, then

$$\sigma = \{v \in \mathbb{R}^n | \langle u_1, v\rangle \geq 0, .., \langle u_t, v\rangle \geq 0\} = \bigcap_{i=1}^{t} \{v \in \mathbb{R}^n | \langle u_i, v\rangle \geq 0\}.$$

So one has

$$\sigma = \bigcap_{i=1}^{t} H_{u_i}.$$

Notice that if σ is a strongly convex cone, then $\overset{\vee}{\sigma}$ is not necessarily a strongly convex cone (cone τ in Example 1.1.10 is an example).

Proposition 1.1.14 *The sum of two convex polyhedral cones is also a convex polyhedral cone. The intersection of two convex polyhedral cones is also a convex polyhedral cone.*

PROOF:

Suppose σ_1 and σ_2 are two convex polyhedral cones. σ_1 is generated by the vectors $(a_1, .., a_r)$, and σ_2 is generated by the vectors $(b_1, ..b_k)$.

$$\sigma_1 + \sigma_2 = \{\sum_{i=1}^{r} \lambda_i a_i + \sum_{i=1}^{k} \beta_i b_i | \lambda_i, \beta_i \in \mathbb{R}_{\geq 0}\}.$$

So $\sigma_1+\sigma_2$ is a convex polyhedral cone generated by the vectors $(a_1, ..a_r, b_1, ..b_k)$. Suppose $\sigma = \sigma_1 + \sigma_2$,

$$\overset{\vee}{\sigma} = \{u \in (\mathbb{R}^n)^* | \langle u, v\rangle \geq 0, \forall v \in \sigma\} = \bigcap_{i=1}^{2} \{u \in (\mathbb{R}^n)^* | \langle u, v\rangle \geq 0, \forall v \in \sigma_i\}.$$

Therefore, we have

$$\overset{\vee}{\sigma} = \overset{\vee}{\sigma}_1 \cap \overset{\vee}{\sigma}_2.$$

Finally,

$$\sigma_1 \cap \sigma_2 = (\overset{\vee}{\sigma}_1)^\vee \cap (\overset{\vee}{\sigma}_2)^\vee = (\overset{\vee}{\sigma}_1 + \overset{\vee}{\sigma}_2)^\vee$$

is a convex polyhedral cone, so we obtain the result.

1.2 Faces

Definition 1.2.1 Let σ be a cone and $\lambda \in \overset{\vee}{\sigma} \bigcap M$, then

$$\tau = \sigma \bigcap \lambda^{\perp} = \{v \in \sigma | \langle \lambda, v\rangle = 0\}$$

is called a face of σ. We will denote by $\tau < \sigma$.

A cone is a **face** of itself, other faces are called proper faces.

A one-dimensional face is called an edge.

Example 1.2.2 In Example 1.1.2, the cone σ_1 has 4 faces σ_1, $\tau_1 = \mathbb{R}_{\geq 0} e_1, \tau_2 = \mathbb{R}_{\geq 0} e_2$ and $\{0\}$.

Property 1.2.3 *Let σ be a cone generated by the vectors $(v_1, ..., v_r)$, then*

i) Every face τ of σ is a convex polyhedral cone, σ has a finite number of faces.

ii) Every intersection of faces of σ is a face of σ.

iii) Every face of a face is a face.

PROOF:
i) There is $\lambda \in \overset{\vee}{\sigma} \bigcap M$ such that

$$\tau = \sigma \bigcap \lambda^{\perp} = \{v \in \sigma | \langle \lambda, v \rangle = 0\}.$$

Let $\{a_1, ..., a_t\}$ be the set of all elements v_i belonging to $A = \{v_1, ..., v_r\}$ such that $\langle \lambda, v_i \rangle = 0$.

Let $v \in \tau$, suppose that $v = \sum_{i=1}^{r} \alpha_i v_i$. If $\langle \lambda, v_j \rangle > 0$ for some j, then $\alpha_j = 0$ (because $\langle \lambda, v \rangle = 0$).

Hence, we have

$$\tau = \sum_{i=1}^{t} \mathbb{R}_{\geq 0} a_i.$$

This means that τ is a convex polyhedral cone.

The set A has a finite number of subsets; therefore, σ has a finite number of faces.

ii) Suppose that τ_1, τ_2 are two faces of σ, so there are $\lambda_1, \lambda_2 \in \overset{\vee}{\sigma} \bigcap M$ such that $\tau_1 = \sigma \bigcap \lambda_1^{\perp}, \tau_2 = \sigma \bigcap \lambda_2^{\perp}$. Show that $\tau_1 \cap \tau_2$ is a face of σ.

Firstly, we will prove that

$$\sigma \cap (\lambda_1^{\perp} \cap \lambda_2^{\perp}) = \sigma \cap (\lambda_1 + \lambda_2)^{\perp}.$$

If $v \in \sigma \cap (\lambda_1^{\perp} \cap \lambda_2^{\perp})$, then $\langle \lambda_1, v \rangle = \langle \lambda_2, v \rangle = 0$, so $\langle \lambda_1 + \lambda_2, v \rangle = 0$, hence $v \in \sigma \cap (\lambda_1 + \lambda_2)^{\perp}$.

Conversely, if $v \in \sigma \cap (\lambda_1 + \lambda_2)^{\perp}$ then $\langle \lambda_1 + \lambda_2, v \rangle = \langle \lambda_1, v \rangle + \langle \lambda_2, v \rangle = 0$. Since $\lambda_1, \lambda_2 \in \overset{\vee}{\sigma} \bigcap M$, we have $\langle \lambda_1, v \rangle \geq 0$ and $\langle \lambda_2, v \rangle \geq 0$. Hence $\langle \lambda_1, v \rangle = \langle \lambda_2, v \rangle = 0$ and this implies that $v \in \lambda_1^{\perp} \cap \lambda_2^{\perp}$.

Finally, we have

$$\tau_1 \cap \tau_2 = (\sigma \cap \lambda_1^{\perp}) \cap (\sigma \cap \lambda_2^{\perp}) = \sigma \cap (\lambda_1^{\perp} \cap \lambda_2^{\perp}) = \sigma \cap (\lambda_1 + \lambda_2)^{\perp}.$$

We have $\lambda_1 + \lambda_2 \in \overset{\vee}{\sigma} \bigcap M$; therefore, $\tau_1 \cap \tau_2$ is a face of σ.

iii) Suppose that $\gamma < \tau < \sigma$, $\tau = \sigma \cap \lambda^{\perp}, \gamma = \tau \cap \alpha^{\perp}$ where $\lambda \in \overset{\vee}{\sigma} \bigcap M, \alpha \in \overset{\vee}{\tau} \bigcap M$.

We will prove that $\gamma < \sigma$.

Firstly, we will prove that if $v_i \in \sigma \setminus \tau$, then there is $k \in \mathbb{Z}_{\geq 0}$ such that $\langle \alpha + k\lambda, v_i \rangle > 0$.

For every $v_i \in \sigma \setminus \tau$, we have $\langle \lambda, v_i \rangle > 0$, $\langle \alpha, v_i \rangle \in \mathbb{R}$, so there is $k_i \in \mathbb{Z}_{\geq 0}$ such that $\langle \alpha + k_i\lambda, v_i \rangle > 0$. Choose $k = max\{k_i\}$, then $\langle \alpha + k\lambda, v_i \rangle > 0$ for all $v_i \in \sigma \setminus \tau$.

If $v_i \in \tau$ then $\langle \alpha + k\lambda, v_i \rangle \geq 0$.

Therefore, for every $v_i \in \sigma$, we have $\langle \alpha + k\lambda, v_i \rangle \geq 0$, then $\alpha + k\lambda \in \sigma$.

Finally, we prove that

$$\gamma = \sigma \cap (\alpha + k\lambda)^{\perp}.$$

If $v \in \gamma$ then $v \in \tau$ and $v \in \tau$, so $\langle \lambda, v \rangle = 0$, $\langle \alpha, v \rangle = 0$, hence $\langle \alpha + k\lambda, v \rangle = 0$, this means that $v \in \sigma \cap (\alpha + k\lambda)^{\perp}$.

Suppose that $v = \sum_{i=1}^{r} \beta_i v_i \in \sigma \cap (\alpha + k\lambda)^{\perp}$.

If $v_i \notin \tau$ then $\langle \alpha + k\lambda, v_i \rangle > 0$, then $\beta_i = 0$ (because $\langle \alpha + k\lambda, v \rangle = 0$). Hence, if $\beta_i \neq 0$ then $v_i \in \tau$, so $\langle \lambda, v \rangle = 0$. Since $\langle \alpha + k\lambda, v \rangle = 0$, one has $\langle \alpha, v \rangle = 0$, then $v \in \gamma$.

Remark 1.2.4 *Indeed, if $\tau < \sigma$ then $\overset{\vee}{\sigma} \subset \overset{\vee}{\tau}$.*

Remark 1.2.5 *Let τ be a face of σ. If x and y belong to σ and $x + y$ belongs to τ, then x and y belong to τ.*

Proposition 1.2.6 *Let $\tau = \sigma \bigcap \lambda^{\perp}$ be a face of σ ($\lambda \in \overset{\vee}{\sigma} \cap M$), then*

$$\overset{\vee}{\tau} = \overset{\vee}{\sigma} + \mathbb{R}_{\geq 0}(-\lambda).$$

PROOF:

- We show that $\sigma \cap \overset{\vee}{(-\lambda)} = \sigma \cap \lambda^{\perp}$.

If $v \in \sigma \cap \overset{\vee}{(-\lambda)}$ then $\langle -\lambda, v \rangle \geq 0$ and $\langle \lambda, v \rangle \geq 0$, so we have $\langle \lambda, v \rangle = 0$, hence $v \in \sigma \cap \lambda^{\perp}$.

If $v \in \sigma \cap \lambda^{\perp}$ then $\langle \lambda, v \rangle = 0$, so $v \in \sigma \cap \overset{\vee}{(-\lambda)}$.

- Consider the following relation

$$\tau = \sigma \bigcap \lambda^{\perp} = \sigma \cap \overset{\vee}{(-\lambda)} = (\overset{\vee}{\sigma} + \mathbb{R}_{\geq 0}(-\lambda))^{\vee}.$$

So we have the result

$$\overset{\vee}{\tau} = \overset{\vee}{\sigma} + \mathbb{R}_{\geq 0}(-\lambda).$$

Example 1.2.7 Let us consider the following example.

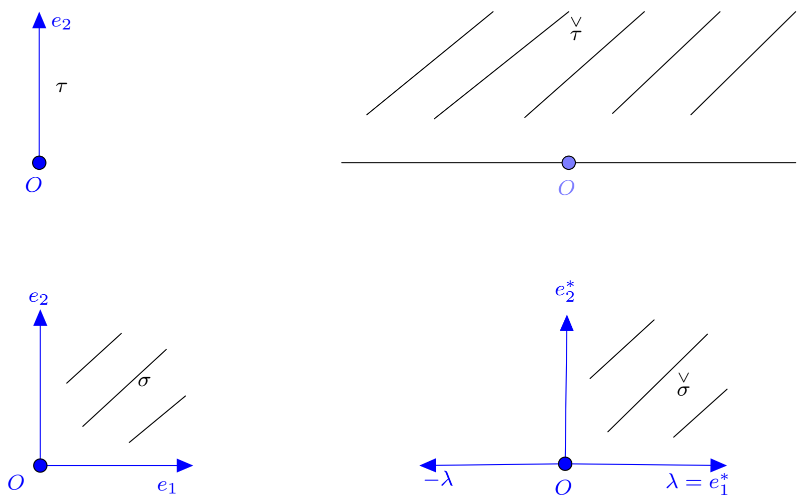

Fig. 3

Firstly, consider τ is a face of σ. One has $\tau = \sigma \bigcap \lambda^{\perp}$, and

$$\overset{\vee}{\tau} = \overset{\vee}{\sigma} + \mathbb{R}_{\geq 0}(-\lambda).$$

Secondly, consider the face $\{0\}$, we have $\overset{\vee}{\{0\}} = (\mathbb{R}^n)^*$.
Set $\eta = e_1^* + e_2^*$, we have $\eta \in \overset{\vee}{\sigma} \bigcap M$, and then

$$(\mathbb{R}^n)^* = \overset{\vee}{\sigma} + \mathbb{R}_{\geq 0}(-\eta).$$

Definition 1.2.8 The **relative interior** of a cone σ is the interior of σ as a subset of the space $\mathbb{R}\sigma$ generated by σ. A point of the relative interior is obtained by taking a strictly positive linear combination of $dim\sigma$ linearly independent vectors among the generators of σ.

For any vector v in σ, there is a face $\tau < \sigma$ such that v is in the relative interior of τ.

Property 1.2.9 *Let σ be a convex polyhedral cone σ and $v \in \sigma$. If v is in the relative interior of σ then $\overset{\vee}{\sigma} \cap v^{\perp} = \sigma^{\perp}$.*

PROOF:

Suppose that σ is generated by the vectors $(v_1, ..., v_r)$ and $dim\sigma = d$.

Let the set $A = \{v_1, ..., v_r\}$.

Assume that v is in the relative interior of σ, then there are vectors $a_1, ..., a_d$ in A such that the set of vectors $B = \{a_1, ..., a_d\}$ is linearly independent and

$$v = \sum_{i=1}^{d} \lambda_i a_i \quad (\lambda_i > 0, \forall i = 1, ..., d).$$

Since $dim(\sigma) = d$ and B is linearly independent. For every vector $x \in \sigma$, we have

$$x \in \sum_{i=1}^{d} \mathbb{R} a_i.$$

We have

$$v^{\perp} = \{u \in (\mathbb{R}^n)^* | \langle u, v \rangle = 0\}.$$

If $u \in \overset{\vee}{\sigma} \cap v^{\perp}$, then $\langle u, v_i \rangle \geq 0, \forall i = 1, ..., r$ and $\langle u, v \rangle = 0$. Hence $\langle u, a_i \rangle = 0, \forall i = 1, ..., d$. Then $\langle u, x \rangle = 0$ for all $x \in \sigma$, then $u \in \sigma^{\perp}$.

If $u \in \sigma^{\perp}$ then $\langle u, v_i \rangle = 0$ for $i = 1, ..., r$, hence $u \in \overset{\vee}{\sigma} \cap v^{\perp}$.

Finally, we have $\overset{\vee}{\sigma} \cap v^{\perp} = \sigma^{\perp}$.

Proposition 1.2.10 *Let τ be a face of σ, then $\overset{\vee}{\sigma} \cap \tau^{\perp}$ is a face of $\overset{\vee}{\sigma}$ with $dim(\tau) + dim(\overset{\vee}{\sigma} \cap \tau^{\perp}) = n$. This provides a one-to-one correspondence between faces of σ and faces of $\overset{\vee}{\sigma}$.*

PROOF:

• If $\tau < \sigma$ then $\overset{\vee}{\sigma} \cap \tau^{\perp}$ is a face of $\overset{\vee}{\sigma}$.

Faces of $\overset{\vee}{\sigma}$ are the cones $\overset{\vee}{\sigma} \cap v^{\perp}$ with $v \in (\overset{\vee}{\sigma})^{\vee} \cap N = \sigma \cap N$.

Let v be in the relative interior of τ. By Property 1.2.9, we have

$$\overset{\vee}{\sigma} \cap \tau^{\perp} = \overset{\vee}{\sigma} \cap (\overset{\vee}{\tau} \cap v^{\perp}) = (\overset{\vee}{\sigma} \cap \overset{\vee}{\tau}) \cap v^{\perp} = \overset{\vee}{\sigma} \cap v^{\perp}$$

is a face of $\overset{\vee}{\sigma}$.

• $dim(\tau) + dim(\overset{\vee}{\sigma} \cap \tau^{\perp}) = n$.

There is $\lambda \in \overset{\vee}{\sigma} \cap M$ such that $\tau = \sigma \cap \lambda^{\perp}$, and we have

$$\overset{\vee}{\tau} = \overset{\vee}{\sigma} + \mathbb{R}_{\geq 0}(-\lambda).$$

Since $v \in \tau$, we have $\langle \lambda, v \rangle = 0$. Then one has $\mathbb{R}_{\geq 0}(-\lambda) \subset v^{\perp}$, hence

$$\tau^{\perp} = \overset{\vee}{\tau} \cap v^{\perp} = (\overset{\vee}{\sigma} + \mathbb{R}_{\geq 0}(-\lambda)) \cap v^{\perp} = \overset{\vee}{\sigma} \cap v^{\perp} + \mathbb{R}_{\geq 0}(-\lambda).$$

Since $\lambda \in \overset{\vee}{\sigma} \cap M$, one has

$$dim(\overset{\vee}{\sigma} \cap v^{\perp}) = dim(\overset{\vee}{\sigma} \cap v^{\perp} + \mathbb{R}_{\geq 0}(-\lambda)) = dim(\tau^{\perp}).$$

Therefore,

$$dim(\tau) + dim(\overset{\vee}{\sigma} \cap \tau^{\perp}) = dim(\tau) + dim(\overset{\vee}{\sigma} \cap v^{\perp}) = dim(\tau) + dim(\tau^{\perp}) = n.$$

- Provide a one-to-one correspondence between faces of σ and faces of $\overset{\vee}{\sigma}$.

Consider the finite sets of cones

$$A = \{\tau | \tau < \sigma\},$$
$$B = \{\tau | \tau < \overset{\vee}{\sigma}\}.$$

Consider the mapping

$$\Theta : A \to B$$
$$\tau \mapsto \overset{\vee}{\sigma} \cap \tau^{\perp}$$

and

$$\Theta' : B \to A$$
$$\tau \mapsto \sigma \cap \tau^{\perp}$$

Let γ be a face of $\overset{\vee}{\sigma}$. This means that $\gamma = \overset{\vee}{\sigma} \cap v^{\perp}$ for some $v \in \sigma \cap N$. There is a face τ of σ such that v is in the relative interior of τ. Then $\Theta(\tau) = \gamma$, hence Θ is surjective. Similarly, we have Θ' is surjective.

We denote by the number of all elements of A by $|A|$. Since Θ, Θ' are surjective, then we have $|A| \geq |B|, |A| \leq |B|$, so $|A| = |B|$. Then Θ is bijective. Therefore, we have a one-to-one correspondence between faces of σ and faces of $\overset{\vee}{\sigma}$.

Remark 1.2.11 *Let σ be a convex polyhedral cone. For every vector v in σ, there exists a unique face $\tau < \sigma$ such that v is in the relative interior of τ.*

In fact, let τ_1 and τ_2 be the faces of σ.

Suppose that v is in the relative interior of the faces τ_1 and τ_2, then we have

$$\overset{\vee}{\sigma} \cap \tau_1^{\perp} = \overset{\vee}{\sigma} \cap \tau_2^{\perp} = \overset{\vee}{\sigma} \cap v^{\perp}.$$

This means that two faces $\overset{\vee}{\sigma} \cap \tau_1^{\perp}$ and $\overset{\vee}{\sigma} \cap \tau_2^{\perp}$ are the same, so we have $\tau_1 = \tau_2$. (By applying the one-to-one correspondence between faces of σ and faces of $\overset{\vee}{\sigma}$).

Remark 1.2.12 *Let σ be a strongly convex cone in $\mathbb{R}^n$, then $dim\overset{\vee}{\sigma} = n$.*

1.3 Monoids

Definition 1.3.1 A semi-group (i.e. a non empty set S with an associative operation $+ : S \times S \to S$) is called a **monoid** if it is commutative, has a zero element ($0 + s = s, \forall s \in S$) and satisfies the simplification law, i.e:

$$s + t = s' + t \Rightarrow s = s' \text{ for } s, s', t \in S.$$

Lemma 1.3.2 *If σ is a cone then $\sigma \cap M$ is a monoid.*

PROOF:

- Let x and y be in $\sigma \cap M$, then $x + y = y + x$ is in $\sigma \cap M$.
- The zero element $0 \in \sigma \cap M$, and $v + 0 = v$ for all $v \in \sigma \cap M$.
- Since $\sigma \cap M \subset M$, the operation satisfies the simplification law.

Definition 1.3.3 A monoid S is **finitely generated** if there are elements $a_1, a_2, ..a_k$ in S such that

$$\forall s \in S, s = \lambda_1 a_1 + ... + \lambda_k a_k \text{ where } \lambda_i \in \mathbb{Z}_{\geq 0}.$$

Such elements $a_1, a_2, ..a_k$ are called generators of the monoid.

Lemma 1.3.4 ***(Gordon's Lemma)***. *If σ is a convex polyhedral lattice cone, then $\sigma \cap N$ is a finitely generated monoid.*

PROOF:

Let σ be a convex polyhedral lattice cone, generated by the vectors $(v_1, ..., v_r)$ such that $v_i \in \sigma \cap N$ for all $i = 1, ...r$.

Consider the map

$$f : \mathbb{R}^r \to \mathbb{R}$$
$$t = (t_1, ..., t_r) \mapsto \sum_{i=1}^{r} t_i v_i$$

Set $K = f([0;1]^r)$, so K is compact.

Let $x \in \mathbb{R}^r$ and set

$$B(x,1) = \{y \in \mathbb{R}^r | d(y,x) < 1\}.$$

One has $K \subset \bigcup_{x \in K} B(x,1)$, so there exists n_0 in $\mathbb{N}$ such that $K \subset \bigcup_{i=1}^{n_0} B(x_i,1)$.

Since $B(x_i,1) \cap N$ is a finite set, one has $K \cap N$ is a finite set.

Let us show that $K \cap N$ and $v_1, ..., v_r$ generates $\sigma \cap N$.

Let $v = \sum_{i=1}^{r} \lambda_i v_i$ be in $\sigma \cap N$, then

$$v = \sum_{i=1}^{r} (n_i + r_i) v_i = \sum_{i=1}^{r} n_i v_i + \sum_{i=1}^{r} r_i v_i \in N$$

where $n_i \in \mathbb{Z}_{\geq 0}, 0 \leq r_i \leq 1$.

Therefore, $u = \sum_{i=1}^{r} r_i v_i \in K \cap N$.

So $v = n_1 v_1 + ... + n_r v_r + u$, and we obtain the result.

Proposition 1.3.5 *Let σ be a convex polyhedral lattice cone, then $\overset{\vee}{\sigma} \cap M$ is a finitely generated monoid. We will denote $\overset{\vee}{\sigma} \cap M$ by S_σ.*

PROOF:

By Proposition 1.1.1, $\overset{\vee}{\sigma}$ is also a convex polyhedral lattice cone.

By Lemma 1.3.4, S_σ is a finitely generated monoid.

Example 1.3.6 In $\mathbb{R}^n$, consider the 0-dimensional cone $\sigma = \{0\}$.

We have $\overset{\vee}{\sigma} = (\mathbb{R}^n)^*$, then the monoid $S_\sigma = \overset{\vee}{\sigma} \cap M = M$. Hence, S_σ is generated by the vectors $(e_1^*, ...e_n^*, -e_1^*, ..., -e_n^*)$.

Example 1.3.7 In $\mathbb{R}^2$, let σ be the cone generated by $(-2e_1 + e_2, e_2)$.

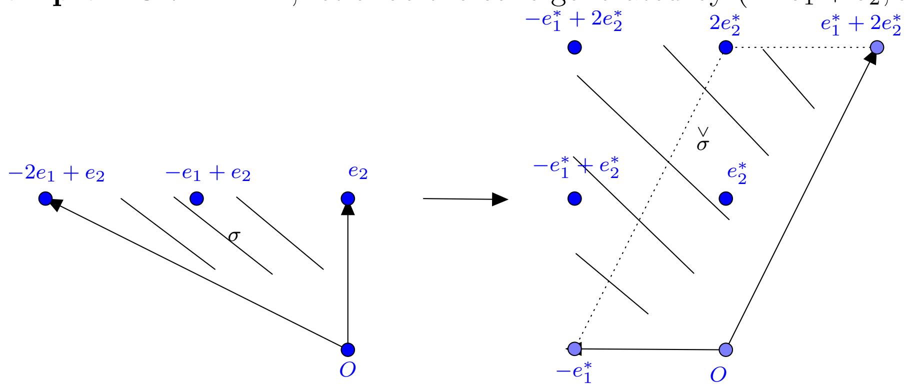

Fig. 4

- $\sigma \cap N$ is generated by the vectors $(-2e_1 + e_2, e_2, -e_1 + 2e_2)$.
- $S_\sigma = \overset{\vee}{\sigma} \cap M$ is generated by the vectors $(-e_1^*, e_2^*, e_1^* + 2e_2^*)$.

Proposition 1.3.8 *Let σ be a rational convex polyhedral cone and $\tau = \sigma \cap \lambda^\perp$ is a face of σ, with $\lambda \in S_\sigma \cap M$, then*

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}.(-\lambda).$$

PROOF:
According to Proposition 1.2.6, we have

$$\overset{\vee}{\tau} = \overset{\vee}{\sigma} + \mathbb{R}_{\geq 0}(-\lambda).$$

By taking the intersection of both sides by M, give us

$$\overset{\vee}{\tau} \cap M = (\overset{\vee}{\sigma} + \mathbb{R}_{\geq 0}(-\lambda)) \cap M.$$

Since $\mathbb{R}_{\geq 0}(-\lambda) \subset M$, one gets

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}.(-\lambda).$$

Example 1.3.9 In Example 1.2.7

$$\overset{\vee}{\tau} = \overset{\vee}{\sigma} + \mathbb{R}_{\geq 0}(-e_1^*).$$

S_τ is generated by the vectors $(e_1^*, , e_2^*, -e_1^*)$.
S_σ is generated by the vectors (e_1^*, e_2^*).
Therefore, we have

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}.(-e_1^*).$$

2 Affine toric varieties

2.1 Laurent polynomials

Let us denote by $\mathbb{C}[z, z^{-1}] = \mathbb{C}[z_1, ..., z_n, z_1^{-1}, ..., z_n^{-1}]$ the ring of **Laurent polynomials**. A Laurent monomial is written by $\lambda.z^a = \lambda z_1^{\alpha_1}...z_n^{\alpha_n}$, where $\lambda \in \mathbb{C}^*$ and $a = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n$.

One of the important facts in the definition of toric varieties, and the key of the second step, that is the mapping

$$\begin{aligned} \theta : \mathbb{Z}^n &\to \mathbb{C}[z, z^{-1}] \\ a = (\alpha_1, ..., \alpha_n) &\mapsto z^a = z_1^{\alpha_1}...z_n^{\alpha_n}, \end{aligned}$$

which is an isomorphism between the additive group $\mathbb{Z}^n$ and the multiplicative group of monic Laurent monomials. Monic means that the coefficient of the monomal is 1.

Proposition 2.1.1 *The mapping*

$$\begin{aligned} \theta : \mathbb{Z}^n &\to \mathbb{C}[z, z^{-1}] \\ a = (\alpha_1, ..., \alpha_n) &\mapsto z^a = z_1^{\alpha_1}...z_n^{\alpha_n} \end{aligned}$$

is an isomorphism between the additive group $\mathbb{Z}^n$ and the multiplicative group of monic Laurent monomials.

PROOF:
• θ is an homomorphism:
Let $a = (\alpha_1, ..., \alpha_n), b = (\beta_1, ..., \beta_n) \in \mathbb{Z}^n$, then

$$\theta(a+b) = \theta(\alpha_1+\beta_1, ..., \alpha_n+\beta_n) = z_1^{\alpha_1+\beta_1}...z_n^{\alpha_1+\beta_n} = z_1^{\alpha_1}...z_n^{\alpha_n}z_1^{\beta_1}...z_n^{\beta_n} = z^a.z^b$$

Therefore, $\theta(a + b) = \theta(a)\theta(b)$.
• θ is injective:
Let $a = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n$. If $\theta(a) = 1$ then $\alpha_1 = ... = \alpha_n = 0$, so $a = 0$, this means that θ is injective.
• θ is surjective:
Each monic $z^a = z_1^{\alpha_1}...z_n^{\alpha_1}$ in $\mathbb{C}[z, z^{-1}]$, choose $a = (\alpha_1, ..., \alpha_n)$ and we have $\theta(a) = z^a$, so θ is surjective.
Finally, we have θ is an isomorphism.

Definition 2.1.2 The **support** of a Laurent polynomial $f = \sum_{\text{finite}} \lambda_a z^a$ is defined by

$$supp(f) = \{a \in \mathbb{Z}^n : \lambda_a \neq 0\}.$$

Proposition 2.1.3 *Let σ be a lattice cone, the ring*

$$R_\sigma = \{f \in \mathbb{C}[z, z^{-1}] : supp(f) \subset S_\sigma\}$$

is a finitely generated monomial algebra (i.e. is a $\mathbb{C}$-algebra generated by Laurent monomials).

PROOF:

According to Proposition 1.3.5, S_σ is a finitely generated monoid. Suppose that S_σ is generated by the vectors $(a_1, ..., a_t)$ with $a_i \in \mathbb{Z}^n$ for all i.

We prove that R_σ is generated by the monomials $(z^{a_1}, ..., z^{a_t})$.

Take a monic Laurent monomial $z^a \in R_\sigma$, then $a \in S_\sigma$, so there are $\lambda_1, ..., \lambda_t \in \mathbb{Z}_{\geq 0}$ such that

$$a = \lambda_1 a_1 + ... + \lambda_t a_t.$$

Hence

$$z^a = (z^{a_1})^{\lambda_1} ... (z^{a_t})^{\lambda_t}.$$

So for every monic Laurent monomial in R_σ is generated by the monomials $(z^{a_1}, ..., z^{a_t})$. If $f \in R_\sigma$ then all its monic Laurent monomials belong to R_σ; therefore, we have the result.

2.2 Some basic results of algebraic geometry

Let $\mathbb{C}[\xi] = \mathbb{C}[\xi_1, ..., \xi_k]$ be the ring of polynomials in k variables over $\mathbb{C}$.

Definition 2.2.1 Let $E = \{f_1, ..., f_r\} \subset \mathbb{C}[\xi]$, then

$$V(E) = \{x \in \mathbb{C}^k : f_1(x) = ... = f_r(x) = 0\}$$

is called the **affine algebraic set** defined by E. Let I be the ideal generated by E, then $V(I) = V(E)$.

Definition 2.2.2 Let $X \subset \mathbb{C}^k$, then

$$I(X) = \{f \in \mathbb{C}[\xi] : f_{|X} = 0\}$$

is an ideal, called the **vanishing ideal** of X.

Proposition 2.2.3 *For $x = (x_1, ..., x_k)$ in $\mathbb{C}^k$, let us consider $E = \{\xi_1 - x_1, ..., \xi_k - x_k\}$. Then $V(E) = \{x\}$ and $I(\{x\}) = \mathbb{C}[\xi](\xi_1 - x_1) + ... + \mathbb{C}[\xi](\xi_k - x_k)$ is a maximal ideal. We will denote $I(\{x\})$ by $\mathcal{M}_x$.*

PROOF:

- $V(E) = \{x \in \mathbb{C}^k : \xi_1 - x_1 = ... = \xi_k - x_k = 0\} = \{x\}$.
- $\mathcal{M}_x = \{f \in \mathbb{C}[\xi] : f(x) = 0\}$ is the ideal generated by E, then we have

$$\mathcal{M}_x = \mathbb{C}[\xi](\xi_1 - x_1) + ... + \mathbb{C}[\xi](\xi_k - x_k).$$

- The mapping

$$\begin{aligned} \phi : \mathbb{C}[\xi] &\to \mathbb{C} \\ f &\mapsto f(x) \end{aligned}$$

is a surjective homomorphism.

We have $Ker\phi = I(\{x\})$, hence $\mathbb{C}[\xi]/I(\{x\}) \cong \mathbb{C}$. Since $\mathbb{C}$ is a field, then $\mathcal{M}_x$ is a maximal ideal.

Theorem 2.2.4 ***(Weak version of the Nullstellensatz)***: *Every maximal ideal in $\mathbb{C}[\xi]$ can be written by $\mathcal{M}_x$ for a point x.*

Corollary 2.2.5 *The correspondence $x \longleftrightarrow \mathcal{M}_x$ is a one - to - one correspondence between points in $\mathbb{C}^k$ and maximal ideals $\mathcal{M}$ of $\mathbb{C}[\xi]$.*

$$\mathbb{C}^k \longleftrightarrow \{\mathcal{M} \subset \mathbb{C}[\xi] : \mathcal{M} \text{ is a maximal ideal}\} =: Spec(\mathbb{C}[\xi]).$$

Lemma 2.2.6 *Let I be an ideal of $\mathbb{C}[\xi]$, then*

$$V(I) = \{x \in \mathbb{C}^k | I \subset \mathcal{M}_x\}.$$

PROOF:

Let $y \in V(I)$. For every $f \in I$, we have $f(y) = 0$, hence $f \in \mathcal{M}_y$. And then $I \subset \mathcal{M}_y$. Therefore, $y \in \{x \in \mathbb{C}^k | I \subset \mathcal{M}_x\}$

Conversely, let $x \in \mathbb{C}^k$ such that $I \subset \mathcal{M}_x$, this means that $f(x) = 0$ for all $f \in I$, then $x \in V(I)$.

Definition 2.2.7 Let us denote the vanishing ideal of $V(I)$ by $I_V = I(V(I))$, then $R_V = \mathbb{C}[\xi]/I_V$ is the coordinate ring of the affine algebraic set $V(I)$. It is generated as a $\mathbb{C}$ - algebra by the classes $\overline{\xi_j}$ of the coordinate function ξ_j.

The generators $\overline{\xi_j} = \xi_j + I_V$ of R_V are the restrictions of coordinate functions to the affine algebraic set V.

If $I = \{0\}$, then $V(I) = \mathbb{C}^k$ and we have $R_V = \mathbb{C}^k$.

Corollary 2.2.8 *There is a one - to - one correspondence*

$$V \longleftrightarrow \{\mathcal{M} \subset R_V | \mathcal{M} \text{ is a maximal ideal}\} =: Spec(R_V).$$

By considering the Zariski topology on each side, we obtain an homeomorphism

$$V \cong Spec(R_V).$$

2.3 Affine toric varieties

Definition 2.3.1 The **affine toric variety** corresponding to a rational, polyhedral, strongly convex cone σ is $X_\sigma := Spec(R_\sigma)$.

Example 2.3.2 In Example 1.3.7, let $a_1 = -e_1^*, a_2 = e_2^*$ and $a_3 = e_1^* + 2e_2^*$ be a system of generators of S_σ.

The isomorphism θ is given by

$$\begin{aligned} \theta : \mathbb{Z}^2 &\to \mathbb{C}[z_1, z_2, z_1^{-1}, z_2^{-1}] \\ a_1 &\mapsto z_1^{-1} && = u_1 \\ a_2 &\mapsto z_2 && = u_2 \\ a_3 &\mapsto z_1.z_2^2 && = u_3 \end{aligned}$$

By Proposition 2.1.3,

$$R_\sigma = \mathbb{C}[u_1, u_2, u_3].$$

The relation between a_1, a_2 and a_3 is $2a_2 = a_1 + a_3$, this provides the relation $u_2^2 = u_1.u_3$ between u_1, u_2, u_3 .

Consider the mapping

$$\begin{aligned} i : \mathbb{C}[\xi_1, \xi_2, \xi_3] &\to \mathbb{C}[u_1, u_2, u_3] \\ \xi_1 &\mapsto u_1 \\ \xi_2 &\mapsto u_2 \\ \xi_3 &\mapsto u_3 \end{aligned}$$

For every $f \in \mathbb{C}[\xi_1, \xi_2, \xi_3]$, if $i(f) = 0$ then $f \in Ker(i) = \mathbb{C}[\xi](\xi_2^2 - \xi_1.\xi_3)$. Indeed, i is surjective, hence, we have

$$R_\sigma = \mathbb{C}[u_1, u_2, u_3] \cong \mathbb{C}[\xi_1, \xi_2, \xi_3]/Ker(i).$$

Therefore,

$$X_\sigma = V(Ker(i)) = \{\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3 | \xi_2^2 = \xi_1.\xi_3\}.$$

Exercise 2.3.3 In $\mathbb{R}^n$, let σ be a rational, polyhedral, strictly convex cone. How to find the affine toric variety X_σ?

SOLUTION:

Step 1: *Find a system of generators of S_σ.*

By Proposition 1.3.5, suppose that S_σ is generated by $(a_1, ..., a_k)$, where $a_i = (\alpha_i^1, ..., \alpha_i^n)$.

Step 2: *Find the relations between coordinates.*

By the isomorphism θ, we obtain monic Laurent monomials $u_i = z^{\alpha_i^1} ... z^{\alpha_i^n} \in \mathbb{C}[z, z^{-1}]$ for $i = 1, ..., k$. The $\mathbb{C}$ -algebra $R_\sigma = \mathbb{C}[u_1, ..., u_k]$ can be represented by

$$R_\sigma = \mathbb{C}[\xi_1, ..., \xi_k]/I_\sigma$$

for some ideal I_σ that we must determine.

Find all linear relations between generators of S_σ. (The number of linear relations between them is finite, see Exercise 2.3.4). Each linear relation can be written as

$$\sum_{j=1}^{k} \nu_j a_j = \sum_{j=1}^{k} \mu_j a_j \qquad \nu_j, \mu_j \in \mathbb{Z}_{\geq 0}.$$

We obtain the relation between coordinates

$$u_1^{\nu_1} ... u_k^{\nu_k} = u_1^{\mu_1} ... u_k^{\mu_k},$$

and finally we have the binomial relation

$$\xi_1^{\nu_1} ... \xi_k^{\nu_k} = \xi_1^{\mu_1} ... \xi_k^{\mu_k}.$$

Step 3: *I_σ is generated by all binomial relations and $X_\sigma = V(I_\sigma)$.*

Exercise 2.3.4 Let $v_1, ..., v_k \in \mathbb{Z}^n$, find all $\alpha_1, ..., \alpha_k \in \mathbb{R}$ such that

$$\alpha_1 v_1 + ... + \alpha_k v_k = 0.$$

SOLUTION:

Suppose that $(e_1, ..., e_n)$ is the basis of $\mathbb{R}^n$, where $e_1 = (1, 0, .., 0), ..., e_n = (0, ..., 0, 1) \in \mathbb{Z}^n$.

There are $a_{ji} \in \mathbb{Z}$ (for $j = 1, ..., n$ and $i = 1, ..., k$) such that

$$v_i = a_{1i} e_1 + ... + a_{ni} e_n \text{ for } i = 1, ..., k.$$

By

$$\alpha_1 v_1 + ... + \alpha_k v_k = 0,$$

we have

$$e_1 \sum_{i=1}^{k} a_{1i}\alpha_i + ... + e_n \sum_{i=1}^{k} a_{ni}\alpha_i = 0,$$

if and only if

$$\sum_{i=1}^{k} a_{1i}\alpha_i = ... = \sum_{i=1}^{k} a_{ni}\alpha_i = 0,$$

if and only if $(\alpha_1, ..., \alpha_k) \in KerA$, where A is the matrix

$$A = \begin{bmatrix} a_{11} & ... & a_{1k} \\ . & & . \\ . & & . \\ . & & . \\ a_{n1} & ... & a_{nk} \end{bmatrix}.$$

$KerA$ is a linear space, $dim(KerA) = n - rank(A) = d$. We can get a basis $(u_1, ..., u_d)$ of $KerA$, where $u_i \in \mathbb{Z}^n$ for all $i = 1, ..., d$.

Theorem 2.3.5 *Let σ be a lattice cone in $\mathbb{R}^n$ and $A = (a_1, ..., a_k)$ a system of generators of S_σ, the corresponding toric variety X_σ is represented by the affine toric variety $V(I_\sigma) \subset \mathbb{C}^k$ where I_σ is an ideal of $\mathbb{C}[\xi_1, ..., \xi_k]$ generated by finitely many binomials corresponding to the relations between elements of A.*

PROOF:

By Exercise 2.3.4, the number of binomial relations corresponding to relations between elements of A is finite. For each binomial relation, we have one binomial.

In the rest of the proof, we show that every element of I_σ is a sum of binomials of the previous type. (See [4], Theorem VI.2.7.)

Example 2.3.6 Let us consider the cone $\sigma = \{0\}$, the dual cone is $\overset{\vee}{\sigma} = (\mathbb{R}^n)^*$. We can choose different systems of generators of S_σ, for example

$$A_1 = (e_1^*, ..., e_n^*, -e_1^*, ..., -e_n^*),$$
$$A_2 = (e_1^*, ..., e_n^*, -(e_1^* + ... + e_n^*)).$$

Let us take the first system of generators A_1. The corresponding monomial $\mathbb{C}$ -algebra is

$$R_\sigma = \mathbb{C}[z_1, ..., z_n, z_1^{-1}, ..., z_n^{-1}] = \mathbb{C}[\xi_1, ..., \xi_{2n}]/I_\sigma,$$

where

$$I_\sigma = \mathbb{C}[\xi](\xi_1.\xi_{n+1} - 1) + ... + \mathbb{C}[\xi](\xi_n.\xi_{2n} - 1).$$

Therefore

$$X_\sigma = V(\xi_1.\xi_{n+1} - 1, ..., \xi_n.\xi_{2n} - 1).$$

We set

$$\mathbb{T} = \{(x_1, ..., x_n) \in \mathbb{C}^n | x_i \neq 0, i = 1, ..., n\} = (\mathbb{C}^*)^n.$$

Consider the projection

$$\pi : \mathbb{C}^{2n} \to \mathbb{C}^n$$
$$(x_1, ..., x_{2n}) \mapsto (x_1, ..., x_n).$$

So X_σ is homeomorphic to $\mathbb{T}$.

For the second system of generators A_2, we have

$$R_\sigma = \mathbb{C}[z_1, ..., z_n, z_1^{-1}...z_n^{-n}] = \mathbb{C}[\xi_1, ..., \xi_{n+1}]/I_\sigma,$$

where

$$I_\sigma = \mathbb{C}[\xi](\xi_1...\xi_{n+1} - 1).$$

In this case,

$$X_\sigma = V(\xi_1...\xi_{n+1} - 1) \subset \mathbb{C}^{n+1}$$

Consider the projection

$$\pi : \mathbb{C}^{n+1} \to \mathbb{C}^n$$
$$(x_1, ..., x_{n+1}) \mapsto (x_1, ..., x_n).$$

Hence X_σ is also homeomorphic to $\mathbb{T}$.

Definition 2.3.7 The set $\mathbb{T} = (\mathbb{C}^*)^n$ is called the **complex algebraic n-torus**.

We have dim $\mathbb{T}$ = n.

Remark 2.3.8 *$\mathbb{T}$ is a closed subset of $\mathbb{C}^{2n}$, but as a subspace of $\mathbb{C}^n$, it is not closed.*

Proposition 2.3.9 *Let σ be a lattice cone in $\mathbb{R}^n$, then affine toric variety X_σ contains the torus $\mathbb{T}$ as a Zariski open dense subset.*

PROOF:

Let $(a_1, ..., a_k)$ be a system of generators for the monoid S_σ and let $V(I_\sigma) \subset \mathbb{C}^k$ be a representation of X_σ. Each a_i is written by $a_i = (\alpha_i^1, ..., \alpha_i^n)$ with $\alpha_i^j \in \mathbb{Z}$, and $t = (t_1, ..., t_n) \in \mathbb{T} = (\mathbb{C}^*)^n$ with $t_i \neq 0$ for $i = 1, ..., n$.

Consider the relation

$$\sum_{j=1}^{k} \nu_j a_j = \sum_{j=1}^{k} \mu_j a_j \qquad \nu_j, \mu_j \in \mathbb{Z}_{\geq 0}.$$

For every $t \in \mathbb{T}$, let $t^{a_i} = t_1^{\alpha_i^1} ... t_n^{\alpha_i^n} \in \mathbb{C}^*$, we have

$$(t^{a_1})^{\nu_1} ... (t^{a_k})^{\nu_k} = t^{(\nu_1 a_1 + ... + \nu_k a_k)} = t^{(\mu_1 a_1 + ... + \mu_k a_k)} = (t^{a_1})^{\mu_1} ... (t^{a_k})^{\mu_k}.$$

This means that $(t^{a_1}, ..., t^{a_k})$ satisfies the binomial relation

$$\xi_1^{\nu_1} ... \xi_k^{\nu_k} = \xi_1^{\mu_1} ... \xi_k^{\mu_k}.$$

Consider the embedding

$$\begin{gathered} h : \mathbb{T} \to X_\sigma \\ t = (t_1, ..., t_n) \mapsto (t^{a_1}, ..., t^{a_k}) \end{gathered}$$

We prove that h is bijective from $\mathbb{T}$ to $X_\sigma \cap (\mathbb{C}^*)^k$.

Firstly, we need to prove that there is $b \in S_\sigma$ such that all points $b + e_i^*$ are in S_σ.

If $\sigma = \{0\}$, then it is obvious.

Assume that $\sigma \neq \{0\}$ is generated by $(v_1, ..., v_r)$.

Since σ is strongly convex cone, there is $b_1 \in S_\sigma$ such that $\langle b_1, v_j \rangle > 0$ for all $j = 1, ..., r$.

Let us fix $i \in \{1, ..., n\}$.

If $b_i + e_i^* \in S_\sigma$ then $b_{i+1} = b_i$. We have $\langle b_{i+1}, v_j \rangle > 0$ for all $j = 1, ..., r$.

If $b_i + e_i^* \notin S_\sigma$, for every v_j such that $\langle b_i + e_i^*, v_j \rangle < 0$, then $\langle b_i, v_j \rangle > 0$ and $\langle e_i^*, v_j \rangle < 0$, then there exists $k_j \in \mathbb{Z}_{>0}$ such that $\langle k_j.b_i + e_i^*, v_j \rangle > 0$. Let $n_i = max\{k_j\}$, then $n_i.b_i + e_i^* \in S_\sigma$, and $\langle n_i.b_i + e_i^*, v_j \rangle > 0$ for all $j = 1, ..., r$. In this case we let $b_{i+1} = n_i.b_i + e_i^*$.

Choosing $b = b_n$, we have the result.

The Laurent monomials $z^b = f_0(u), z^{b+e_i^*} = f_i(u)$ are in $R_\sigma = \mathbb{C}[u] \subset \mathbb{C}[z, z^{-1}]$.

Let $h(t) = x$ be a point in $X_\sigma \cap (\mathbb{C}^*)^k$, then $f_i(h(t)) = t_i f_0(h(t))$, we have $t_i = f_i(h(t))/f_0(h(t))$. Therefore h is injective .

For every $x \in X_\sigma \cap (\mathbb{C}^*)^k$, we have

$$x = h((f_1(x)/f_0(x), ..., f_n(x)/f_0(x)).$$

Hence h is surjective from $\mathbb{T}$ to $X_\sigma \cap (\mathbb{C}^*)^k$.

Finally, since $X_\sigma \cap (\mathbb{C}^*)^k$ is dense in X_σ, one gets $h(\mathbb{T})$ is dense in X_σ. Then affine toric variety X_σ contains the torus $\mathbb{T}$ as a Zariski open dense subset.

Remark 2.3.10 *If σ is a rational, polyhedral, strictly convex cone in $\mathbb{R}^n$ then $dim_{\mathbb{C}} X_\sigma = n$.*

By Proposition 2.3.9 and $dim_{\mathbb{C}} X_\sigma$ is finite, then $dim_{\mathbb{C}} X_\sigma = dim \mathbb{T} = n$.

Example 2.3.11 In the case of Example 1.3.7, let $a_1 = -e_1^*, a_2 = e_2^*$ and $a_3 = e_1^* + 2e_2^*$, then

$$\begin{gathered} h : \mathbb{T} \to X_\sigma \\ t = (t_1, t_2) \mapsto (t_1^{-1}, t_2, t_1 t_2^2) \end{gathered}$$

Example 2.3.12 Let σ be the cone in $\mathbb{R}^n$ generated by $(e_1, ..., e_p)$ with $p < n$. Then S_σ is generated by $(e_1^*, ..., e_n^*, -(e_{p+1}^* + ... + e_n^*))$.

$$R_\sigma = \mathbb{C}[z_1, ..., z_n, z_{p+1}^{-1} ... z_n^{-1}] = \mathbb{C}[\xi_1, ..., \xi_{n+1}]/I_\sigma,$$

where

$$I_\sigma = \mathbb{C}[\xi](\xi_{p+1} ... \xi_{n+1} - 1).$$

Therefore

$$X_\sigma = V(I_\sigma) = \mathbb{C}^p \times (\mathbb{C}^*)^{n-p}.$$

If $p = n$ then $X_\sigma = \mathbb{C}^n$.

X_σ is smooth for all $p \leq n$.

Remark 2.3.13 *Let us denote $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. A lattice homomorphism $\varphi : N' \to N$ defines a homomorphism of real vector spaces $\varphi_{\mathbb{R}} : N'_{\mathbb{R}} \to N_{\mathbb{R}}$. Assume that $\varphi_{\mathbb{R}}$ maps a (polyhedral, rational, strictly convex) cone σ' of N' to a (polyhedral, rational, strictly convex) cone σ of N. Then the dual map $\overset{\vee}{\varphi} : M \to M'$ provides a map $S_\sigma \to S_{\sigma'}$. It defines a map $R_\sigma \to R_{\sigma'}$ and a map $X_{\sigma'} \to X_\sigma$.*

We apply this remark to an example.

Example 2.3.14 This is the example of a 2-dimensional affine toric variety.

Let us consider in $\mathbb{R}^2$ the cone generated by e_2 and $pe_1 - qe_2$, for integers $p, q \in \mathbb{Z}_{>0}$ such that $0 < q < p$ and $(p, q) = 1$.

Then $R_\sigma = \oplus\mathbb{C}(z_1^i z_2^j)$, the sum over (i, j) with $j \leq \frac{p}{q} i$. Let N' be the sublattice of N generated by $pe_1 - qe_2$ and e_2, i.e. by pe_1 and e_2. Let us call σ' the cone σ considered in N' instead of N. Then σ' is generated by two generators of the lattice N', so $X_{\sigma'}$ is $\mathbb{C}^2$.

In this situation, the inclusion $N' \subset N$ provides a map $X_{\sigma'} \to X_\sigma$ (Remark 2.3.13). The map can be made explicitly in the following way:

Let us denote by x and y the monomials corresponding to the generators e_1^* and e_2^* of the dual lattice M. The dual lattice $M' \supset M$ corresponding to N' is generated by $\frac{1}{p} e_1^*$ and e_2^*. The monomials corresponding to these generators are u and y such that $u^p = x$. The monoid $S_{\sigma'}$ is generated by $\frac{1}{p} e_1^*$ and $\frac{q}{p} e_1^* + e_2^*$, then

$$R_{\sigma'} = \mathbb{C}[u, u^q y] = \mathbb{C}[u, v] \text{ with } v = u^q y.$$

On the other hand,

$$R_\sigma = \oplus\mathbb{C}[x^i y^j] = \oplus\mathbb{C}[u^{pi-qj} v^j] \text{ , the sum over } (i, j) \text{ with } j \leq \tfrac{p}{q} i.$$

Consider the group of $p - th$ roots of unity

$$\Gamma_p = \{z \in \mathbb{C} | z^p - 1 = 0\} \cong \mathbb{Z}/p\mathbb{Z}$$

Consider the group action φ of Γ_p on $R_\sigma = \mathbb{C}[u, v]$ given by

$$\begin{aligned} \varphi : \Gamma_p \times \mathbb{C}[u, v] &\to \mathbb{C}[u, v] \\ (\zeta, f) &\mapsto \zeta.f = f(\zeta u, \zeta^q v) \end{aligned}$$

Set

$$\mathbb{C}[u, v]^{\Gamma_p} = \{f \in \mathbb{C}[u, v] | \zeta.f = f, \forall \zeta \in \Gamma_p\}.$$

We have

$$(u^{pi-qj} v^j)(\zeta u, \zeta^q v) = \zeta^{pi} u^{pi-qj} v^j = u^{pi-qj} v^j \text{ for all } j \leq \tfrac{p}{q} i.$$

This shows that

$$R_\sigma \subset \mathbb{C}[u, v]^{\Gamma_p}.$$

For any $u^t v^k \in \mathbb{C}[u, v]$, $t, k \in \mathbb{Z}_{\geq 0}$, we have

$$u^t v^k (\zeta u, \zeta^q v) = \zeta^{t+kq} u^t v^k.$$

Since $(p,q)=1$, one has

$$t+kq \equiv 0 \ mod(p) \text{ if and only if } t = pi-kq \text{ for some } i.$$

Since $t \geq 0$, we have $k \leq \frac{p}{q}i$, hence $u^t v^k = u^{pi-qk}v^k$ in R_σ.
This implies that

$$R_\sigma = \mathbb{C}[u,v]^{\Gamma_p}$$

Consider the group action φ of Γ_p on $X_{\sigma'}$.

$$\begin{aligned} \varphi : \Gamma_p \times X_{\sigma'} &\to X_{\sigma'} \\ (\zeta,(u,v)) &\mapsto (\zeta u, \zeta^q v) \end{aligned}$$

Then $X_{\sigma'}/\Gamma_p$ is the quotient of the action, that is the set of all orbits of $X_{\sigma'}$ under the action of Γ_p. The orbit of (u,v) is

$$\Gamma_p.(u,v) = \{(\zeta u, \zeta^q v) | \zeta \in \Gamma_p\}.$$

We will show that $X_\sigma = X_{\sigma'}/\Gamma_p$.
The inclusion $R_\sigma \subset R_{\sigma'}$ induces a map

$$\begin{aligned} \theta : Spec(R_{\sigma'}) &\to Spec(R_\sigma) \\ \mathcal{M} &\mapsto \mathcal{M} \cap R_\sigma \end{aligned}$$

which is surjective.

For every $(u_0,v_0),(u_1,v_1) \in \mathbb{C}^2$, we have the maximal ideals $\mathcal{M}_0, \mathcal{M}_1 \in Spec(R_{\sigma'})$ corresponding to $(u_0,v_0),(u_1,v_1)$. One has

$$\mathcal{M}_0 \cap R_\sigma = \{f \in R_\sigma | f(u_0,v_0) = 0\}$$

Let $f = u^p - u_0^p \in \mathcal{M}_0 \cap R_\sigma$, $g = u^{p-q}v - u_0^{p-q}v_0 \in \mathcal{M}_0 \cap R_\sigma$.

Assume that $\mathcal{M}_0 \cap R_\sigma = \mathcal{M}_1 \cap R_\sigma$, this means that $f(u_1,v_1)=0$ and $g(u_1,v_1)=0$, hence

$$u_1^p = u_0^p \text{ and } u_1^{p-q}v_1 - u_0^{p-q}v_0 = 0.$$

Therefore there is $\zeta \in \Gamma_p$ such that $u_1 = \zeta u_0$, and

$$\zeta^{p-q}u_0^{p-q}v_1 - u_0^{p-q}v_0 = 0.$$

If $u_0 \neq 0$ then

$$\zeta^{p-q}v_1 = v_0.$$

This shows that $v_1 = \zeta^q v_0$.

If $u_0 = 0$, let $h = v^p - v_0^p \in \mathcal{M}_0 \cap R_\sigma$, one has $v_1^p = v_0^p$, then there is $\zeta_1 \in \Gamma_p$ such that $v_1 = \zeta_1^p v_0$.

In two cases ($u_0 = 0$ or $u_0 \neq 0$), we have $(u_1, v_1) = (\zeta u_0, \zeta^q v_0)$ for some $\zeta \in \Gamma_p$.

This implies that (u_1, v_1) is in the orbit of (u_0, v_0).

Conversely, suppose that (u, v) and $(\zeta u, \zeta^q v)$ correspond to the maximal ideals $\mathcal{M}, \zeta\mathcal{M} \in Spec(R_{\sigma'})$. Since $R_\sigma = \mathbb{C}[u, v]^{\Gamma_p}$, we have

$$\mathcal{M} \cap R_\sigma = \{f \in R_\sigma | f(u, v) = 0\} = \{f \in R_\sigma | f(\zeta u, \zeta^q v) = 0\} = \zeta\mathcal{M} \cap R_\sigma.$$

Finally we have the result $X_\sigma \cong X_{\sigma'}/\Gamma_p = \mathbb{C}^2/\Gamma_p$.

3 Toric varieties

3.1 Fans

Definition 3.1.1 A **fan** Δ in the space $\mathbb{R}^n$ is a finite union of cones such that

i) Every cone of Δ is a strongly convex, polyhedral, rational cone.

ii) Every face of a cone of Δ is a cone of Δ.

iii) If σ and σ' are the cones of Δ, then $\sigma \cap \sigma'$ is a common face of σ and σ'.

In the following, unless specified, all cones we will consider will be polyhedral, rational cones.

Example 3.1.2 Example of fans

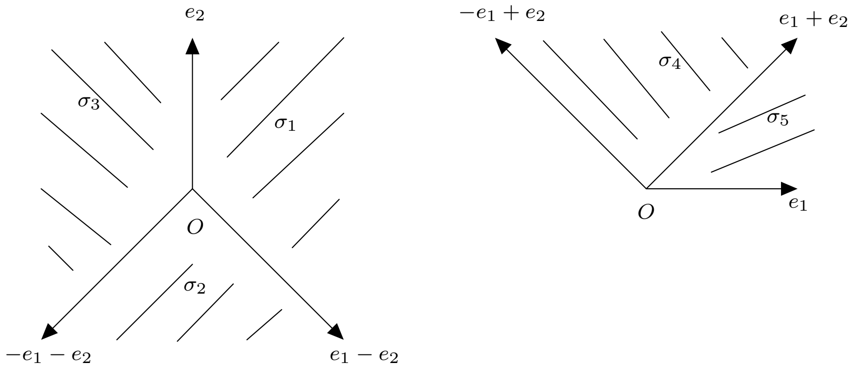

Fig. 5

Example 3.1.3 Let us denote by (t_0, t_1, t_2) the homogeneous coordinates of the space $\mathbb{P}^2$. Let $z_1 = t_1/t_0, z_2 = t_2/t_0$.

$\mathbb{P}^2$ has three coordinate charts

$U_0 = \{(t_0 : t_1 : t_2) \in \mathbb{P}^2 | t_0 \neq 0\} \cong \mathbb{C}^2_{(z_1,z_2)}$, corresponding to the algebra $\mathbb{C}[z_1, z_2]$,

$U_1 = \{(t_0 : t_1 : t_2) \in \mathbb{P}^2 | t_1 \neq 0\} \cong \mathbb{C}^2_{(z_1^{-1}, z_1^{-1}z_2)}$, corresponding to the algebra $\mathbb{C}[z_1^{-1}, z_1^{-1}z_2]$,

$U_2 = \{(t_0 : t_1 : t_2) \in \mathbb{P}^2 | t_2 \neq 0\} \cong \mathbb{C}^2_{(z_2^{-1}, z_1z_2^{-1})}$, corresponding to the algebra $\mathbb{C}[z_2^{-1}, z_1z_2^{-1}]$.

Let us consider in $\mathbb{R}^2$ the following fan:

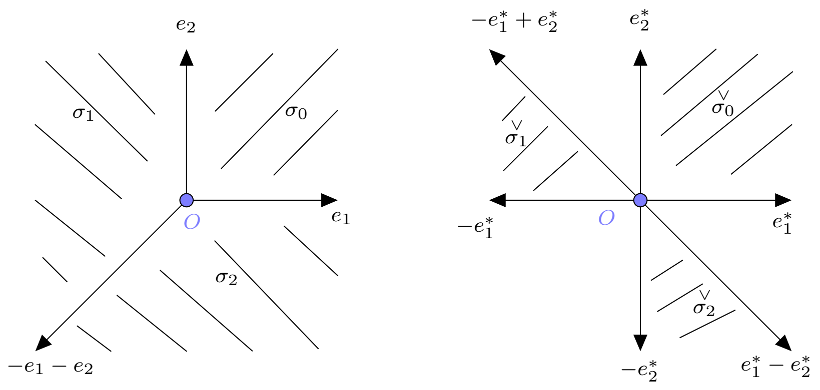

Fig. 6

i) S_{σ_0} is generated by (e_1^*, e_2^*), hence $R_{\sigma_0} = \mathbb{C}[z_1, z_2]$, then $X_{\sigma_0} = \mathbb{C}^2_{(z_1,z_2)} = U_0$.

ii) S_{σ_1} is generated by $(-e_1^*, -e_1^* + e_2^*)$, hence $R_{\sigma_0} = \mathbb{C}[z_1^{-1}, z_1^{-1}z_2]$, then $X_{\sigma_1} = \mathbb{C}^2_{(z_1^{-1}, z_1^{-1}z_2)} = U_1$.

iii) S_{σ_2} is generated by $(-e_2^*, e_1^* - e_2^*)$, hence $R_{\sigma_2} = \mathbb{C}[z_2^{-1}, z_1 z_2^{-1}]$, then $X_{\sigma_2} = \mathbb{C}^2_{(z_2^{-1}, z_1 z_2^{-1})} = U_2$.

We have $\sigma_0 \cap \sigma_1 = \tau$, which is the cone generated by e_2.
Let us glue X_{σ_0} and X_{σ_1} along X_τ.
We have

$$S_\tau = S_{\sigma_0} + \mathbb{Z}_{\geq 0}(-e_1^*) = S_{\sigma_1} + \mathbb{Z}_{\geq 0}(e_1^*).$$

Then $X_\tau = \mathbb{C}^*_{z_1} \times \mathbb{C}_{z_2}$ in X_{σ_0} and $X_\tau = \mathbb{C}^*_{z_1^{-1}} \times \mathbb{C}_{z_1^{-1}z_2}$ in X_{σ_1}. So we have

$$X_{\sigma_0} \diagdown (z_1 = 0) \cong X_\tau \cong X_{\sigma_1} \diagdown (z_1^{-1} = 0).$$

One has

$$X_\tau = \{(t_0 : t_1 : t_2) \in \mathbb{P}^2 | t_0 \neq 0, t_0 \neq 0\}$$

which is a subset of U_0 and U_1. Since $U_0 \cap U_1 = X_\tau$, the gluing X_{σ_0} and X_{σ_1} along X_τ is $U_0 \cup U_1 = \mathbb{P}^2 \setminus \{(0 : 0 : 1)\}$.

3.2 Toric varieties

In a general way, let τ be a face of a cone σ, then $\overset{\vee}{\tau} = \overset{\vee}{\sigma} + \mathbb{R}_{\geq 0}(-\lambda)$ where $\lambda \in \overset{\vee}{\sigma} \cap M$ and $\tau = \sigma \bigcap \lambda^{\perp}$ (Proposition 1.2.6).

The monoid S_τ is thus obtained from S_σ by adding one generator $-\lambda$. As λ can be chosen as an element of a system of generators $(a_1, ..., a_k)$ for S_σ. We may assume that $\lambda = a_k$ is the last vector in the system of generators of S_σ and we denote $a_{k+1} = -\lambda$. In order to obtain the relationships between the generators of S_τ, one has to consider previous relationships between the generators $(a_1, ..., a_k)$ of S_σ and the supplementary relationship $a_k + a_{k+1} = 0$.

This relationship corresponds to the multiplicative one $u_k u_{k+1} = 1$ and that is the only supplementary relationship we need in order to obtain R_τ from R_σ. As the generators u_i are precisely the coordinate function on the toric varieties X_σ and X_τ, this mean that the projection

$$\begin{aligned} \mathbb{C}^{k+1} &\to \mathbb{C}^k \\ (x_1, ..., x_k, x_{k+1}) &\mapsto (x_1, ..., x_k) \end{aligned}$$

identifies X_τ with the open subset of X_σ defined by $x_k \neq 0$. That can be written as follows.

Lemma 3.2.1 *There is a natural identification $X_\tau \cong X_\sigma \setminus (u_k = 0)$.*

Remark 3.2.2 *Let us suppose that τ is the common face of two cones σ and σ'. Lemma 3.2.1 allows us to glue together X_σ and $X_{\sigma'}$ along X_τ. This is performed in the following way:*

Let us write $(v_1, ..., v_l)$ the coordinates on $X_{\sigma'}$. By Lemma 3.2.1, there is a homeomorphism $X_\tau \cong X_{\sigma'} \setminus (u_l = 0)$, we obtain the gluing map

$$\psi_{\sigma,\sigma'} : X_\sigma \setminus (u_k = 0) \overset{\cong}{\to} X_\tau \overset{\cong}{\to} X_{\sigma'} \setminus (v_l = 0).$$

Definition 3.2.3 (Toric varieties) Let Δ be a fan in $\mathbb{R}^n$. Consider the disjoint union $\coprod_{\sigma \in \Delta} X_\sigma$ where two points $x \in X_\sigma$ and $x' \in X_{\sigma'}$ are identified if $\psi_{\sigma,\sigma'}(x) = x'$. The resulting space X_Δ is called a toric variety. It is topological space endowed with an open covering by the affine toric varieties X_σ for $\sigma \in \Delta$. It is an algebraic variety whose charts are defined by binomial relations.

In fact, we have shown that , for a face τ of a cone σ, one has inclusions:

$$\tau \hookrightarrow \sigma$$
$$\overset{\vee}{\tau} \hookleftarrow \overset{\vee}{\sigma}$$
$$R_\tau \hookleftarrow R_\sigma$$
$$X_\tau \hookrightarrow X_\sigma.$$

Proposition 3.2.4 *Let Δ be a fan in $\mathbb{R}^n$. Consider the disjoint union $\coprod_{\sigma \in \Delta} X_\sigma$. We write $x \sim x'$ if $\psi_{\sigma,\sigma'}(x) = x'$ for some $\sigma, \sigma' \in \Delta$. Then the relation $\sim$ is an equivalence relation.*

PROOF:
• Reflexivity:

$$\psi_{\sigma,\sigma} : X_\sigma \overset{\cong}{\to} X_\sigma \overset{\cong}{\to} X_\sigma$$

So we have $x \sim x$.
• Symmetry: If $x \sim x'$,then there is the map

$$\psi_{\sigma,\sigma'} : X_\sigma \setminus (u_k = 0) \overset{\cong}{\to} X_\tau \overset{\cong}{\to} X_{\sigma'} \setminus (v_l = 0)$$

such that $\psi_{\sigma,\sigma'}(x) = x'$. Therefore we have $\psi_{\sigma',\sigma}(x') = x$, then $x' \sim x$.
• Transitivity: If $x \sim x'$ and $x' \sim x''$, there are the maps

$$\psi_{\sigma,\sigma'} : X_\sigma \setminus (u_k = 0) \overset{\cong}{\to} X_\tau \overset{\cong}{\to} X_{\sigma'} \setminus (v_l = 0)$$
$$\psi_{\sigma',\sigma''} : X_{\sigma'} \setminus (v_m = 0) \overset{\cong}{\to} X_{\tau'} \overset{\cong}{\to} X_{\sigma''} \setminus (s_r = 0)$$

hence

$$\psi_{\tau,\tau'} : X_\tau \setminus (v_m = 0) \overset{\cong}{\to} X_{\sigma'} \setminus (v_m = 0, v_l = 0) \overset{\cong}{\to} X_{\tau'} \setminus (v_l = 0)$$

and $\psi_{\tau,\tau'}(x) = x''$.

Remark 3.2.5 $X_\Delta = \coprod_{\sigma \in \Delta} X_\sigma / \sim$.

Proposition 3.2.6 *Every $n-$dimensional toric variety contains the torus $\mathbb{T} = (\mathbb{C}^*)^n$ as an Zariski open dense subset.*
PROOF:

The torus $\mathbb{T}$ corresponds to the zero cone, which is a face of every $\sigma \in \Delta$. (i.e. $\mathbb{T} = X_{\{0\}}$). The embedding of the torus into every affine toric variety X_σ has been shown in Proposition 2.3.9. By the previous identifications, all the tori corresponding to affine toric varieties X_σ in X_Δ are identified as an open dense subset in X_Δ.

Example 3.2.7 Consider the following fan:

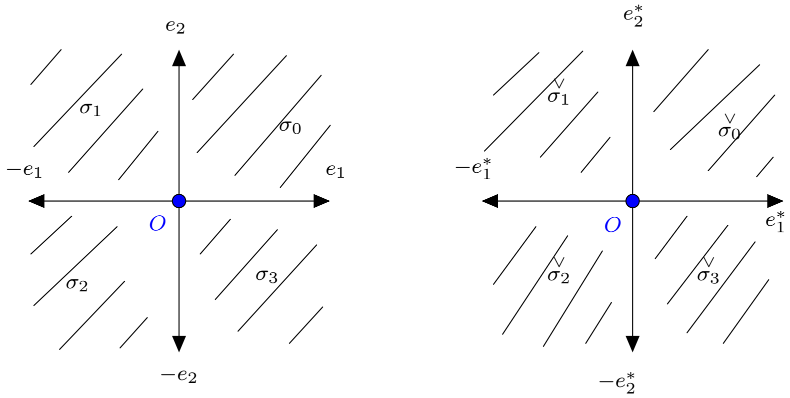

Fig. 7

We have that
S_{σ_0} is generated by (e_1^*, e_2^*), hence $R_{\sigma_0} = \mathbb{C}[z_1, z_2]$ and $X_{\sigma_0} = \mathbb{C}^2_{(z_1,z_2)}$,
S_{σ_1} is generated by $(-e_1^*, e_2^*)$, hence $R_{\sigma_1} = \mathbb{C}[z_1^{-1}, z_2]$ and $X_{\sigma_1} = \mathbb{C}^2_{(z_1^{-1},z_2)}$,
S_{σ_2} is generated by $(-e_1^*, -e_2^*)$, hence $R_{\sigma_2} = \mathbb{C}[z_1^{-1}, z_2^{-1}]$ and $X_{\sigma_2} = \mathbb{C}^2_{(z_1^{-1},z_2^{-1})}$,
S_{σ_3} is generated by $(e_1^*, -e_2^*)$, hence $R_{\sigma_3} = \mathbb{C}[z_1, z_2^{-1}]$ and $X_{\sigma_3} = \mathbb{C}^2_{(z_1,z_2^{-1})}$.

The face $\tau_1 = \sigma_0 \cap \sigma_1$, which is generated by e_2, then $X_{\tau_1} = \mathbb{C}^*_{z_1} \times \mathbb{C}_{z_2}$ in X_{σ_0} and $X_{\tau_1} = \mathbb{C}^*_{z_1^{-1}} \times \mathbb{C}_{z_2}$ in X_{σ_1}.

We have $X_{\sigma_0} = U_0 \times \mathbb{C}_{z_2}$ and $X_{\sigma_1} = U_1 \times \mathbb{C}_{z_2}$ where $U_0 = \{(t_0 : t_1) \in \mathbb{P}^1 | t_0 \neq 0\}$, and $U_1 = \{(t_0 : t_1) \in \mathbb{P}^1 | t_1 \neq 0\}$.

Let $U = \{(t_0 : t_1) \in \mathbb{P}^1 | t_0 \neq 0, t_1 \neq 0\} \cong \mathbb{C}^*$.

One has $X_{\tau_1} = U \times \mathbb{C}_{z_2} = X_{\sigma_0} \cap X_{\sigma_1}$ in $\mathbb{P}^1 \times \mathbb{C}_{z_2}$. Therefore the gluing X_{σ_0} and X_{σ_1} along X_{τ_1} is $\mathbb{P}^1 \times \mathbb{C}_{z_2}$.

The face $\tau_2 = \sigma_2 \cap \sigma_3$. Similarly, we have the gluing X_{σ_2} and X_{σ_3} along X_{τ_2} is $\mathbb{P}^1 \times \mathbb{C}_{z_2^{-1}}$.

Finally, we glue $\mathbb{P}^1 \times \mathbb{C}_{z_2}$ and $\mathbb{P}^1 \times \mathbb{C}_{z_2^{-1}}$ together along $\mathbb{P}^1 \times U$, we have $\mathbb{P}^1 \times \mathbb{P}^1$.

4 The torus action and the orbits

4.1 The torus action

Definition 4.1.1 The torus $\mathbb{T} = (\mathbb{C}^*)^n$ is a group operating on itself by multiplication. The **action of the torus** on each affine toric variety X_σ is described as follows:

Let $(a_1, ..., a_k)$ be a system of generators for the monoid S_σ. For the previous coordinates of $\mathbb{R}^n$, each a_i is written by $a_i = (\alpha_i^1, ..., \alpha_i^n)$ with $\alpha_i^1 \in \mathbb{Z}$, and $t \in \mathbb{T}$ is written by $t = (t_1, ..., t_n)$ where $t_j \in \mathbb{C}^*$. A point $x \in X_\sigma$ is written by $x = (x_1, ..., x_k) \in \mathbb{C}^k$. The action of $\mathbb{T}$ on X_σ is given by

$$\begin{aligned} \mathbb{T} \times X_\sigma &\to X_\sigma \\ (t, x) &\mapsto t.x = (t^{a_1}x_1, ..., t^{a_k}x_k) \end{aligned}$$

where $t^{a_i} = t_1^{\alpha_i^1} ... t_n^{\alpha_i^n} \in \mathbb{C}^*$.

Let $x = (x_1, ..., x_k)$ and $y = (y_1, ..., y_k)$ in $\mathbb{C}^k$ and satisfy the binomial relation

$$\xi_1^{\nu_1} ... \xi_k^{\nu_k} = \xi_1^{\mu_1} ... \xi_k^{\mu_k}.$$

Since

$$\begin{gathered} (x_1y_1)^{\nu_1} ... (x_ky_k)^{\nu_k} = (x_1)^{\nu_1} ... (x_k)^{\nu_k} (y_1)^{\nu_1} ... (y_k)^{\nu_k} = \\ (x_1)^{\mu_1} ... (x_k)^{\mu_k} (y_1)^{\mu_1} ... (y_k)^{\mu_k} = (x_1y_1)^{\mu_1} ... (x_ky_k)^{\mu_k}, \end{gathered}$$

then $xy = (x_1y_1, ..., x_ky_k)$ also satisfies this binomial relation.

Therefore, if x and y are in X_σ then xy is also in X_σ.

By Proposition 2.3.9, for every $t = (t_1, ..., t_n) \in \mathbb{T}$, one has $(t^{a_1}, ..., t^{a_k}) \in X_\sigma$. Then $t.x \in X_\sigma$ for all $x \in X_\sigma$.

Example 4.1.2 In the case of Example 1.3.7, let $a_1 = -e_1^*, a_2 = e_2^*$ and $a_3 = e_1^* + 2e_2^*$. The action of $\mathbb{T}$ on X_σ is the map

$$\begin{gathered} \mathbb{T} \times X_\sigma \to X_\sigma \\ (t, x) = ((t_1, t_2), (x_1, x_2, x_3)) \mapsto (t_1^{-1}x_1, t_2x_2, t_1t_2^2x_3). \end{gathered}$$

Remark 4.1.3 L*et τ be a face of σ. By Lemma 3.2.1, we can suppose that S_σ is generated by $(a_1, ..., a_k)$, S_τ is generated by $(a_1, ..., a_k, -a_k)$ and there is a natural identification*

$$\begin{gathered} p : X_\tau \overset{\cong}{\to} X_\sigma \setminus (x_k \neq 0) \\ (x_1, ..., x_k, x_{k+1}) \mapsto (x_1, ..., x_k). \end{gathered}$$

The action of $\mathbb{T}$ on X_σ is the map

$$\begin{aligned}\mathbb{T} \times X_\sigma &\to X_\sigma \\ (t,x) &\mapsto t \bullet_\sigma x = (t^{a_1}x_1, ..., t^{a_k}x_k).\end{aligned}$$

The action of $\mathbb{T}$ on X_τ is the map

$$\begin{aligned}\mathbb{T} \times X_\tau &\to X_\tau \\ (t,x) &\mapsto t \bullet_\tau x = (t^{a_1}x_1, ..., t^{a_k}x_k, t^{-a_k}x_{k+1}).\end{aligned}$$

Hence, for every $x = (x_1, ..., x_k) \in X_\sigma \setminus (x_k \neq 0)$, one has $p^{-1}(x) = (x_1, ..., x_k, x_k^{-1})$. Therefore

$$p^{-1}(t \bullet_\sigma x) = (t^{a_1}x_1, ..., t^{a_k}x_k, t^{-a_k}x_k^{-1}) = t \bullet_\tau p^{-1}(x).$$

Then $t \bullet_\sigma x = p(t \bullet_\tau p^{-1}(x))$.

For every $x = (x_1, ..., x_k, x_{k+1}) \in X_\tau$, we have

$$p(t \bullet_\tau x) = (t^{a_1}x_1, ..., t^{a_k}x_k) = t \bullet_\sigma p(x).$$

Theorem 4.1.4 *Let Δ be a fan in $\mathbb{R}^n$, the torus action on the affine toric varieties, for $\sigma \in \Delta$, provide a torus action on the toric variety X_Δ.*

PROOF:

Suppose that x and x' are identified, then there are σ and σ' in Δ such that $\psi_{\sigma,\sigma'}(x) = x'$, where $\psi_{\sigma,\sigma'}$ is the gluing map. We also have

$$\psi_{\sigma,\sigma'} : X_\sigma \setminus (u_k = 0) \overset{p_1^{-1}}{\to} X_\tau \overset{p_2}{\to} X_{\sigma'} \setminus (v_l = 0), \tau = \sigma \cap \sigma'.$$

Then $\psi_{\sigma,\sigma'}(x) = p_2(p_1^{-1}(x)) = x'$.

By Remark 4.1.3, we have

$t \bullet_{\sigma'} x' = t \bullet_{\sigma'} (p_2(p_1^{-1}(x))) = p_2(t \bullet_\tau p_1^{-1}(x)) = p_2^{-1}p_1(t \bullet_\sigma x) = \psi_{\sigma,\sigma'}(t \bullet_\sigma x)$.

This shows that $t \bullet_{\sigma'} x'$ and $t \bullet_\sigma x$ are identified in X_Δ.

4.2 Orbits

Let us consider the case $\Delta = \{0\}$, then $X_\Delta = (\mathbb{C}^*)^n$ is the torus. There is only one orbit which is the total space X_Δ and is the orbit of the point whose coordinates u_i are $(1, ..., 1)$ in $\mathbb{C}^n$.

In the general case, the apex $\sigma = \{0\}$ of Δ provides an open dense orbit which is the embedded torus $\mathbb{T} = (\mathbb{C}^*)^n$ (Proposition 3.2.6). Let us describe the other orbits.

There is a correspondence (see Corollary 2.2.5)

$$\mathbb{C}^k \longleftrightarrow \{\mathcal{M} \subset \mathbb{C}[\xi] : \mathcal{M} \textbf{ maximal ideal}\} \longleftrightarrow Hom_{\mathbb{C}-alg}(\mathbb{C}[\xi], \mathbb{C}).$$

With this correspondence, the point $x = (x_1, ..., x_k)$ corresponds to the ideal $\mathcal{M}_x = \mathbb{C}[\xi](\xi_1 - x_1) + ... + \mathbb{C}[\xi](\xi_k - x_k)$ and to the homomorphism $\varphi : \mathbb{C}[\xi] \to \mathbb{C}$ such that $Ker\varphi = \mathcal{M}_x$, (i.e. $\varphi(f) = f(x)$).

If I is an ideal in $\mathbb{C}[\xi]$, then $V = V(I) = \{x \in \mathbb{C}^k : I \subset \mathcal{M}_x\}$ and $I_V = I(V(I))$. The set V is an affine algebraic set whose coordinate ring is $R_V = \mathbb{C}[\xi]/I_V$ and we have the correspondence (see Corollary 2.2.8)

$$V \longleftrightarrow \{\mathcal{M} \subset R_V : \mathcal{M} \textbf{ maximal ideal}\} \longleftrightarrow Hom_{\mathbb{C}-alg}(R_V, \mathbb{C}).$$

As a semi- group, the dual lattice M is generated by $(e_1^*, ..., e_n^*, -e_1^*, ..., e_n^*)$ and the Laurent polynomial ring $\mathbb{C}[M]$ is generated by $(z_1, ..., z_n, z_1^{-1}, ..., z_n^{-n})$. We have identifications

$$\mathbb{T} = Spec(\mathbb{C}[M]) \cong Hom(M, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$$

where $N \cong Hom(M, \mathbb{C})$ and $Hom(M, \mathbb{C})$ are group homomorphisms.

All semi-groups $S_\sigma = \overset{\vee}{\sigma} \cap M$ are semi-groups of the lattice M and $\mathbb{C}[S_\sigma]$ is a sub - algebra of $\mathbb{C}[M]$. These sub-algebras are generated by monomials in variables u_i.

If S_σ is generated by $(a_1, ..., a_k)$, then elements $u_i = z^{a_i}, i = 1, ..., k$, are generators of the $\mathbb{C}$-sub-algebra $\mathbb{C}[S_\sigma]$, with multiplication $z^a z^{a'} = z^{a+a'}$ and $z^0 = 1$.

Remark 4.2.1 *Points of $Spec(\mathbb{C}[S_\sigma])$ correspond to homomorphisms of semi-groups of S_σ in $\mathbb{C}$ where $\mathbb{C} = \mathbb{C}^* \cup \{0\}$ is an abelian semi-group via multiplication:*

$$X_\sigma = Spec(\mathbb{C}[S_\sigma]) \cong Hom_{sg}(S_\sigma, \mathbb{C})$$

(semi-group homomorphisms). If $\varphi \in Hom_{sg}(S_\sigma, \mathbb{C})$, the point x corresponding to φ satisfies $\varphi(a) = z^a(x)$ (evaluation in x) for all $a \in S_\sigma$. This means that $\varphi(a_i)$ is the $i - th$ coordinate of x, i.e. $x = (\varphi(a_1), ..., \varphi(a_n)) \in \mathbb{C}^k$.

The action of $\mathbb{T}$ on X_σ can be interpreted in the following way:

$$\begin{aligned} t \in \mathbb{T} &\text{ is identified with the group homomorphism } M \xrightarrow{t} \mathbb{C}^*, \text{and} \\ x \in X_\sigma &\text{ is identified with the group homomorphism } S_\sigma \xrightarrow{x} \mathbb{C}, \text{then} \\ t.x \in X_\sigma &\text{ is identified with the group homomorphism } S_\sigma \xrightarrow{t.x} \mathbb{C}, u \mapsto t(u).x(u). \end{aligned}$$

Indeed, we have

$$t.x = (t^{a_1}x_1, ..., t^{a_k}x_k),$$

then $(t.x)(a_i) = t^{a_i}x_i = t(a_i).x(a_i)$, (by $t(a_i) = t^{a_i}$ and $x(a_i) = x_i$).

Hence, for every $u = \sum_{i=1}^{k} \lambda_i a_i \in S_\sigma$, where $\lambda_i \in \mathbb{Z}_{\geq 0}$, one has

$$(t.x)(u) = (t^{a_1}x_1)^{\lambda_1}...(t^{a_k}x_k)^{\lambda_k} = t^{\lambda_1 a_1 + ... + \lambda_k a_k}.x(\lambda_1 a_1)...x(\lambda_k a_k) =$$
$$t^u x(\lambda_1 a_1 + ... + \lambda_k a_k) = t(u).x(u).$$

Definition 4.2.2 Distinguished points. Let σ be a cone and X_σ the associated affine toric variety. We associate to each face τ of σ a distinguished point x_τ corresponding to the semi-group homomorphism defined on generators a of S_σ by

$$\varphi_\tau(a) = \begin{cases} 1 & \text{if } a \in \tau^\perp \\ 0 & \text{in other cases.} \end{cases}$$

Exercise 4.2.3 *Prove that $\varphi_\tau : S_\sigma \to \mathbb{C}$ is a semi-group homomorphism.*

SOLUTION:

Let $a, b \in S_\sigma$, consider two cases:

If $a, b \in \tau^\perp = \{u \in (\mathbb{R}^n)^* | \langle u, v\rangle = 0, \forall v \in \tau\}$, then $(a + b) \in \tau$, by the definition of φ_τ, we have

$$\varphi_\tau(a + b) = \varphi_\tau(a) = \varphi_\tau(b) = \varphi_\tau(a)\varphi_\tau(b) = 1.$$

If $a \notin \tau^\perp$, then there is $v \in \tau$ such that $\langle a, v\rangle > 0$, hence $\langle a + b, v\rangle > 0$ (since $\langle b, v\rangle \geq 0$) , this means that $a + b \notin \tau^\perp$. Then we have

$$\varphi_\tau(a + b) = \varphi_\tau(a) = \varphi_\tau(a)\varphi_\tau(b) = 0.$$

Therefore, we have the result.

Example 4.2.4 In the case of Example 1.3.7, the generators of S_σ are $a_1 = -e_1^*, a_2 = e_2^*$ and $a_3 = e_1^* + 2e_2^*$.

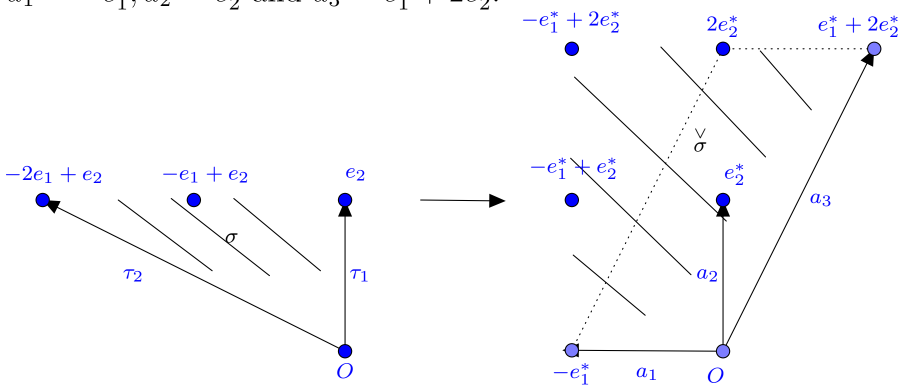

Fig. 8

The cone σ has four faces $\{0\}$, τ_1 is generated by e_2, τ_2 is generated by $-2e_1 + e_2$, and σ.

We have $\{0\}^\perp = (\mathbb{R}^2)^*, \tau_1^\perp = \mathbb{R}a_1$, $\tau_2^\perp = \mathbb{R}a_3, \sigma^\perp = \{0\}$.

The distinguished points:

$$\begin{aligned} x_{\{0\}} &= (\varphi_{\{0\}}(a_1), \varphi_{\{0\}}(a_2), \varphi_{\{0\}}(a_3)) = (1, 1, 1), \\ x_{\tau_1} &= (\varphi_{\tau_1}(a_1), \varphi_{\tau_1}(a_2), \varphi_{\tau_1}(a_3)) = (1, 0, 0), \\ x_{\tau_2} &= (\varphi_{\tau_2}(a_1), \varphi_{\tau_2}(a_2), \varphi_{\tau_2}(a_3)) = (0, 0, 1), \\ x_\sigma &= (\varphi_{\tau_2}(a_1), \varphi_{\tau_2}(a_2), \varphi_{\tau_2}(a_3)) = (0, 0, 0). \end{aligned}$$

Definition 4.2.5 Let σ be a cone in $\mathbb{R}^n$ and τ a face of σ. The **orbit** of $\mathbb{T}$ in X_σ corresponding to the face τ is the orbit of the distinguished point x_τ, we denote by O_τ.

$$O_\tau = \{t.x_\tau | t \in \mathbb{T} = (\mathbb{C}^*)^n\}.$$

Example 4.2.6 In Example 4.2.5, for each distinguished point, we have $O_{\{0\}} = \{t.x_{\{0\}} | t = (t_1, t_2) \in \mathbb{T} = (\mathbb{C}^*)^2\} = \{(t_1^{-1}, t_2, t_1 t_2^2) | (t_1, t_2) \in (\mathbb{C}^*)^2\}$, then

$$\begin{aligned} O_{\{0\}} &\cong (\mathbb{C}^*)^2. \\ O_\sigma &= \{(0, 0, 0\}, \\ O_{\tau_1} &= \mathbb{C}^*_{\xi_1} \times \{0\} \times \{0\}, \\ O_{\tau_2} &= \{0\} \times \{0\} \times \mathbb{C}^*_{\xi_3}. \end{aligned}$$

Consider the disjoint union $O_\sigma \coprod O_{\tau_1} \coprod O_{\tau_2} \coprod O_{\{0\}}$, we have

$$O_\sigma \coprod O_{\tau_1} \coprod O_{\tau_2} \coprod O_{\{0\}} = X_\sigma = \{(x_1, x_2, x_3) \in \mathbb{C}^3 | x_1 x_3 = x_2^2\}.$$

Indeed, suppose $x = (x_1, x_2, x_3) \in X_\sigma$.

If $x_2 \neq 0$ then $x \in O_{\{0\}}$.

If $x_2 = 0$ then $x_1 = 0$ or $x_3 = 0$, therefore $x \in O_\sigma \coprod O_{\tau_1} \coprod O_{\tau_2}$.

Proposition 4.2.7 *Let σ be a cone in $\mathbb{R}^n$ and O_σ be the orbit of the distinguished point x_σ. If $dim\sigma = k$ then*

$$O_\sigma \cong (\mathbb{C}^*)^{n-k}.$$

PROOF:

If $dim\sigma = n$ then $\sigma^\perp = \{0\}$. Hence $x_\sigma = (0, ..., 0) \in X_\sigma$, then $O_\sigma = \{0\}$.

If $\sigma = \{0\}$ then $x_\sigma = (1, ..., 1) \in X_\sigma$, then $O_\sigma = \mathbb{T} = \mathbb{T}_\mathbb{N}$.

If $dim\sigma < n$, let N_σ be the sublattice of N, which is generated by $\sigma \cap N$, then

$$N_\sigma = (\sigma \cap N) + (-\sigma \cap N).$$

Then we have a decomposition

$$N = N_\sigma \bigoplus N', \sigma = \sigma' \bigoplus \{0\},$$

where σ' is in a cone in $N_\sigma \subset \mathbb{R}^k$. Then $dimN'_\mathbb{R} = n - k$.

The dual decomposition $M = M' \bigoplus M''$, then $dimM''_\mathbb{R} = n - k$. One has

$$S_\sigma = ((\sigma')^\vee \cap M')) \bigoplus M'',$$
$$X_\sigma = X_{\sigma'} \times \mathbb{T}_{N'}.$$

Consider two toric actions on $X_{\sigma'}$ and $\mathbb{T}_{N'}$:

$$\mathbb{T}_{N_\sigma} \times X_{\sigma'} \to X_{\sigma'},$$
$$\mathbb{T}_{N'} \times \mathbb{T}_{N'} \to \mathbb{T}_{N'}.$$

By $\mathbb{T} = \mathbb{T}_{N_\sigma} \times \mathbb{T}_{N'}$, we have toric action on X_σ

$$\mathbb{T} \times (X_{\sigma'} \times \mathbb{T}_{N'}) \to X_{\sigma'} \times \mathbb{T}_{N'}.$$

Since $\sigma = \sigma' \bigoplus \{0\}$, then $x_\sigma = x_{\sigma'} \times x_{\{0\}} \in X_{\sigma'} \times \mathbb{T}_{N'}$. Hence

$$O_\sigma = O_{\sigma'} \times O_{\{0\}} \subset X_{\sigma'} \times \mathbb{T}_{N'}.$$

We have $dim(\sigma') = dim(N_\sigma)_\mathbb{R} = k$, then $O_{\sigma'} = \{0\} \subset X_{\sigma'}$. Then

$$O_\sigma \cong \mathbb{T}_{N'} = (\mathbb{C}^*)^{n-k}.$$

Remark 4.2.8 *In Proposition 4.2.7, we can suppose that the lattice N_σ is generated by $(e_1, ..., e_k)$, then N' is generated by $(e_{k+1}, ..., e_n)$. Hence the lattice M'' is generated by $(e^*_{k+1}, ..., e^*_n)$, then*

$$M'' = \tau^\perp \cap M.$$

Hence,

$$O_\sigma \cong \mathbb{T}_{N'} = Hom_{sg}(\tau^\perp \cap M, \mathbb{C}^*).$$

Theorem 4.2.9 *Let σ be a cone in $\mathbb{R}^n$, then*

$$X_\sigma = \coprod_{\tau<\sigma} O_\tau,$$

where O_τ is the orbit of the distinguished point x_τ.

PROOF:

We have $O_\tau \subset X_\sigma$ for all $\tau < \sigma$, then

$$X_\sigma \supset \coprod_{\tau<\sigma} O_\tau.$$

Conversely, for every $x \in X_\sigma$, x corresponds to a semi-homomorphism $\varphi : S_\sigma \to \mathbb{C}$ such that $\varphi(a) = z^a(x)$. Then $\varphi(0) = 1$, this shows that $\varphi^{-1}(\mathbb{C}^*) \neq \emptyset$.

Let $a, b \in S_\sigma$ such that $(a + b) \in \varphi^{-1}(\mathbb{C}^*)$, so $\varphi(a + b) \in \mathbb{C}^*$, then $\varphi(a)\varphi(b) \in \mathbb{C}^*$, hence a and $b \in \varphi^{-1}(\mathbb{C}^*)$. Then

$$\varphi^{-1}(\mathbb{C}^*) = \overset{\vee}{\sigma} \cap \tau^\perp \cap M \text{ for some } \tau < \sigma.$$

Hence $\varphi \in Hom_{sg}(\overset{\vee}{\sigma} \cap \tau^\perp \cap M, \mathbb{C}^*)$, this means that $x \in O_\tau$.

4.3 Compactness and smoothness

For each k in $\mathbb{Z}$, one has algebraic group homomorphism

$$\begin{aligned} \mathbb{C}^* &\to \mathbb{C}^* \\ z &\mapsto z^k \end{aligned}$$

providing the isomorphism $Hom_{alg.gr}(\mathbb{C}^*, \mathbb{C}^*) = \mathbb{Z}$.

Let N be a lattice, with dual lattice M, one has

$$(A) \qquad \mathbb{T} = \mathbb{T}_N = Hom(M, \mathbb{C}^*)$$

and with the choice of a basis for N, one has isomorphisms

$$(B) \qquad Hom(\mathbb{C}^*, \mathbb{T}) \cong Hom(\mathbb{Z}, N) \cong N.$$

Every one-parameter sub-group $\lambda : \mathbb{C}^* \to \mathbb{T}$ corresponds to an unique $v \in N$. Let us denote by λ_v the one-parameter sub-group corresponding to v. One has

$$v = (v_1, ..., v_n) \qquad \lambda_v(z) = (z^{v_1}, ..., z^{v_n}).$$

In a dual way, one has

$$Hom(\mathbb{T}, \mathbb{C}^*) \cong Hom(N, \mathbb{Z}) \cong M.$$

Every character corresponding to a unique $u \in M$. Let $\chi^u \in Hom(\mathbb{T}, \mathbb{C}^*)$ be the character corresponding to $u = (u_1, ..., u_n) \in M$. For $t = (t_1, ..., t_n) \in \mathbb{T}$, then $\chi^u(t) = t_1^{u_1}...t_n^{u_n}$. We will denote also by χ^u the corresponding function in the coordinate ring $\mathbb{C}[M]$.

Let us recall that a basis of the complex vectorial space $\mathbb{C}[M]$ is given by the elements χ^u with $u \in M$.The generators $u_i \in M$ correspond to the generators χ^{u_i} for the $\mathbb{C}-$algebra $\mathbb{C}[M]$. More precisely, if $(e_1, ..., e_n)$ is a basis for N, then $(e_1^*, ..., e_n^*)$ is a basis for N and $\chi^{e_i} = \chi_i$ a basis for the ring of Laurent polynomial with n variables over $\mathbb{C}[M]$.

If $z \in \mathbb{C}^*$, then $\lambda_v(z) \in \mathbb{T}$, and (by (A)), $\lambda_v(z)$ corresponds to a group homomorphism from M in $\mathbb{C}^*$. More explicitly

$$\lambda_v(z)(u) = \chi^u(\lambda_v(z)) = z^{\langle u,v \rangle},$$

where $\langle , \rangle$ is the dual pairing $M \bigotimes N \to \mathbb{Z}$, i.e.

$$\begin{array}{ccc} u \quad v & \mapsto & \langle u, v \rangle \\ M \times N & \to & \mathbb{Z} \\ Hom(\mathbb{T}, \mathbb{C}^*) \times Hom(\mathbb{C}^*, \mathbb{T}) & \to & Hom(\mathbb{C}^*, \mathbb{C}^*) \\ \chi \quad \lambda & \mapsto & z \mapsto z^{\langle u,v \rangle} = \chi^u(\lambda_v(z)) \end{array}$$

In fact, let $\lambda_v(z) = (t_1, ..., t_n) \in \mathbb{T}$ and $v = (v_1, ..., v_n) \in N$ then

$$\lambda_v(z)(u) = t_1^{u_1}...t_n^{u_n} = \chi^u(t) = \chi^u(\lambda_v(z)),$$

and

$$\lambda_v(z)(u) = (z^{v_1}, ..., z^{v_n})(u) = (z^{v_1})^{u_1}...(z^{v_n})^{u_n} = z^{u_1 v_1 + u_n v_n} = z^{\langle u,v \rangle}.$$

Example 4.3.1 Let σ be a cone generated by a part $(e_1, ..., e_p)$ of a basis of N, then $X_\sigma = \mathbb{C}^p \times \mathbb{C}^{n-p}$ (Example 2.3.12).

For $v = (v_1, ..., v_n) \in \mathbb{Z}^n$, then $\lambda_v(z) = (z^{v_1}, ..., z^{v_n})$. The limit $lim_{z\to 0}\lambda_v(z)$ exists and lies in X_σ if and only if all v_i are nonnegative and $v_i = 0$ for $i > p$. In other words, the limit exists in X_σ if and only if $v \in \sigma$. In that case, the limit is $(y_1, ..., y_n)$, where $y_i = 0$ if $v_i > 0$ and $y_i = 1$ if $v_i = 0$. The possible limits are the distinguished point x_τ for some face τ of σ.

We denote the union of the cones of Δ by $|\Delta|$.

Proposition 4.3.2 *Let v be in $|\Delta|$ and τ be a cone of Δ containing v in its relative interior. If $lim_{z\to 0}\lambda_v(z)$ exists then $lim_{z\to 0}\lambda_v(z) = x_\tau$.*

PROOF:

By Property 1.2.9, we have $\overset{\vee}{\tau} \cap v^\perp = \tau^\perp$.

For every $\sigma \in \Delta$ and containing τ, we work in X_σ. For $z \in \mathbb{C}^*$, we have $\lambda_v(z) = (z^{v_1}, ..., z^{v_n}) \in \mathbb{T} = Hom(M, \mathbb{C}^*)$, then one has the map

$$\begin{aligned} \lambda_v(z) : M &\longrightarrow \mathbb{C}^* \\ u &\longmapsto z^{\langle u,v\rangle} = \lambda_v(z)(u). \end{aligned}$$

Consider the restriction of $\lambda_v(z)$ on $S_\sigma = M \cap \overset{\vee}{\sigma}$, we have

$$\begin{aligned} \lambda_v(z) |_{S_\sigma} : S_\sigma &\longrightarrow \mathbb{C}^* \\ u &\longmapsto z^{\langle u,v\rangle} = \lambda_v(z)(u). \end{aligned}$$

Then $\lambda_v(z) |_{S_\sigma} \in Hom(S_\sigma, \mathbb{C}) = X_\sigma$.

For every $u \in S_\sigma$, we have $\langle u, v\rangle \geq 0$, and $\langle u, v\rangle = 0$ if and only if $u \in \overset{\vee}{\tau} \cap v^\perp = \tau^\perp$. Then

$$\begin{aligned} lim_{z\to 0}(\lambda_v(z) |_{S_\sigma}) : S_\sigma &\longrightarrow \mathbb{C} \\ u &\longmapsto \begin{cases} 1 & \text{if } u \in \tau^\perp \\ 0 & \text{in other cases.} \end{cases} \end{aligned}$$

This shows that $lim_{z\to 0}(\lambda_v(z) |_{S_\sigma}) = x_\tau$. Hence if $lim_{z\to 0}\lambda_v(z)$ exists then $lim_{z\to 0}\lambda_v(z) = x_\tau$.

Proposition 4.3.3 *If v does not belong to any cone of Δ, then $lim_{z \to 0} \lambda_v(z)$ does not exist in X_Δ.*

PROOF:

Suppose that $v \notin \sigma$, then there is $u \in \overset{\vee}{\sigma}$ such that $\langle u, v \rangle < 0$. Therefore $lim_{z \to 0} z^{\langle u,v \rangle} = \infty$. This shows that $lim_{z \to 0} \lambda_v(z)$ does not exist in $Hom(M, \mathbb{C}^*)$. Then $lim_{z \to 0} \lambda_v(z)$ does not exist in X_Δ.

Definition 4.3.4 A cone σ defined by the set of vectors $(v_1, ..., v_r)$ is a **simplex** if all the vectors v_i are linearly independent. A fan Δ is simplicial if all cones of Δ are simplices.

Definition 4.3.5 A vector $v \in \mathbb{Z}^n$ is **primitive** if its coordinate are coprime. A cone is **regular** if the vectors $(v_1, ..., v_r)$ spanning the cone are primitive and there exist primitive vectors $(v_{r+1}, ..., v_n)$ such that $det(v_1, ..., v_n) = \pm 1$. In other words, the vectors $(v_1, ..., v_r)$ can be completed in a basis of the lattice N. A fan is regular if all its cones are regular cones.

Definition 4.3.6 A fan Δ is **complete** if its cones cover $\mathbb{R}^n$, i.e. $|\Delta| = \mathbb{R}^n$.

Theorem 4.3.7 *Let Δ be a fan in $\mathbb{R}^n$. If X_Δ is compact, then Δ is complete.*

PROOF:

If $|\Delta|$ is not all $\mathbb{R}^n$, then there is a vector v such that v does not belong to any cone (Δ is finite). In that case, $\lambda_v(z)$ does not have a limit in X_Δ when z goes to 0. That gives a contradiction with the compacity. (If X is compact, then every infinite subset of X has at least one limit point in X).

Remark 4.3.8 *Torus $\mathbb{T} = (\mathbb{C}^*)^n$ is smooth.*

Let $\sigma = \{0\}$ be a cone in $\mathbb{R}^n$, by Example 2.3.6, we have

$$\mathbb{T} \cong X_\sigma = V(\xi_1 ... \xi_{n+1} - 1) \subset \mathbb{C}^{n+1}.$$

Set $F(\xi_1, ..., \xi_{n+1}) = \xi_1 ... \xi_{n+1} - 1 \in \mathbb{C}[\xi_1, ..., \xi_{n+1}]$, then F is irreducible. The set of singular points of X_σ is

$$\begin{aligned} & V(\xi_1 ... \xi_{n+1} - 1) \cap V(\frac{\partial F}{\partial \xi_1}, ..., \frac{\partial F}{\partial \xi_{n+1}}) \\ = & V(\xi_1 ... \xi_{n+1} - 1) \cap V(\xi_2 ... \xi_{n+1}, ..., \xi_1 ... \xi_n) = \emptyset. \end{aligned}$$

Then $\mathbb{T} = (\mathbb{C}^*)^n$ is smooth.

Theorem 4.3.9 *Let Δ be a fan in $\mathbb{R}^n$. If Δ is regular, then X_Δ is smooth.*

PROOF:

Let $(e_1, ..., e_n)$ be a basis of N.

Suppose that σ_0 is generated by $(e_1, ..., e_p)$, then σ_0 is regular. By Example 2.3.12, we have $X_{\sigma_0} \cong \mathbb{C}^p \times (\mathbb{C}^*)^{n-p}$. Hence X_{σ_0} is smooth.

If σ is regular, then σ is generated by the vectors $(v_1, ..., v_r)$, and the set of the vectors $(v_1, ..., v_r, v_{r+1}, ..., v_n)$ is a basis of N, for some $r = 1, ..., n$.

Hence, there is a matrix $A \in M(n, \mathbb{Z})$ such that $Av_1 = e_1, ..., Av_n = e_n$. Consider the mapping

$$\begin{aligned} A : N &\to N \\ v &\mapsto Av. \end{aligned}$$

We have that A is an isomorphism (because $det(A) = \pm 1$).

One has that $A\sigma = \sigma'$ is generated by $(e_1, ..., e_r)$. Then $X_{\sigma'} \cong \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$ is smooth. By $\sigma \cong \sigma'$ then $X_\sigma \cong X_{\sigma'}$. Hence X_σ is smooth.

Theorem 4.3.10 *Let Δ be a fan in $\mathbb{R}^n$. If the toric variety X_Δ is smooth, then the fan Δ is regular.*

PROOF:

For every cone $\sigma \in \Delta$, we prove that σ is regular.

Since X_Δ is smooth, then X_σ is smooth for all $\sigma \in \Delta$.

Firstly, suppose that $dim\sigma = n$. Then we have $\sigma^\perp = \{0\}$, so the distinguished point $x_\sigma = (0, ..., 0) \in X_\sigma$.

Assume that $S_\sigma = \overset{\vee}{\sigma} \cap M$ is generated by the vectors $(a_1, ..., a_k)$, one has

$$R_\sigma = \mathbb{C}[S_\sigma] = \mathbb{C}[z^{a_1}, ..., z^{a_k}] = \mathbb{C}[\xi_1, ..., \xi_k]/I_\sigma,$$

and

$$X_\sigma = V(I_\sigma) \subset \mathbb{C}^k.$$

Let us denote by $\mathcal{M}$ the maximal ideal of R_σ corresponding to the point x_σ. Let $\varphi \in Hom_{sg}(S_\sigma, \mathbb{C})$ correspond to x_σ, then $\varphi(a) = z^a(x)$ for $a \in S_\sigma$.

One has

$$\mathcal{M} \cong Ker(\varphi).$$

Since $x_\sigma = (0, ..., 0) = (\varphi(a_1), ..., \varphi(a_k)) \in X_\sigma$, then $\varphi(a_1) = ... = \varphi(a_k) = 0$. This shows that $Ker\varphi = S_\sigma - \{0\}$ (because $\varphi(0) = 1$). Therefore, $\mathcal{M}$ is generated by all z^u such that $u \in S_\sigma - \{0\}$. So $\mathcal{M}^2$ is generated by z^u such that u is the sum of two elements of $S_\sigma - \{0\}$.

And $\mathcal{M}/\mathcal{M}^2$ is identified with the cotangent space at x_σ. Since X_σ is smooth, then R_σ is regular, so $dim R_\sigma = dim \mathcal{M}/\mathcal{M}^2$.

Since $dim R_\sigma = dim(X_\sigma) = dim \mathbb{T} = n$, then

$$dim \mathcal{M}/\mathcal{M}^2 = n.$$

In other words, $\mathcal{M}/\mathcal{M}^2$ has a basis the images of elements z^u for $u \in S_\sigma - \{0\}$, such that u is not the sum of two vectors in $S_\sigma - \{0\}$.

Let H be a set of vectors $u \in S_\sigma - \{0\}$ such that u is not the sum of two vectors in $S_\sigma - \{0\}$, then $H \subset \{a_1, ..., a_k\}$, and H generates S_σ. Hence H is finite. This shows that $\mathcal{M}/\mathcal{M}^2$ and H have n elements.

One has $dim \overset{\vee}{\sigma} = n$, then the elements of H are linearly independent, and $S_\sigma + (-S_\sigma) = M$. Then H is the basis of M. This implies that σ is generated by a basis of N and $X_\sigma = \mathbb{C}^n$.

Let us consider the general case, i.e. $dim\sigma = k \leq n$.

Consider the sub-lattice

$$N_\sigma = (\sigma \cap N) + (-\sigma \cap N).$$

Then we have a decomposition

$$N = N_\sigma \bigoplus N', \sigma = \sigma' \bigoplus \{0\},$$

where σ' is in a cone in $N_\sigma \subset \mathbb{R}^k$. Then $dim N'_\mathbb{R} = n - k$.

The dual decomposition $M = M' \bigoplus M''$, then $dim M''_\mathbb{R} = n - k$. One has

$$S_\sigma = ((\sigma')^\vee \cap M')) \bigoplus M'',$$
$$X_\sigma = X_{\sigma'} \times \mathbb{T}_{N'}.$$

Since X_σ is smooth, then $X_{\sigma'}$ is smooth.

The toric variety $X_{\sigma'}$ corresponds to the cone σ' in the lattice N_σ. Since $dim\sigma' = dim(N_\sigma)_\mathbb{R} = k$, then σ' is regular. This shows that σ is regular.

Example 4.3.11 Let us consider the following cone σ in $\mathbb{R}^2$.

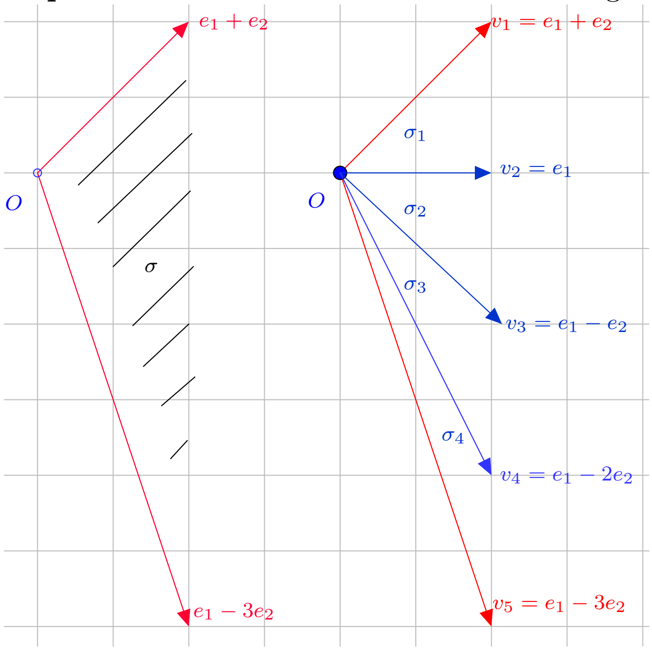

Fig. 9

We have that X_σ is not smooth, because σ is not regular. Now, we will decompose σ to get a regular fan Δ. So we will add the vectors between $e_1 + e_2$ and $e_1 - 3e_2$ to get the cones, which are regular.

Set $v_1 = e_1 + e_2, v_2 = e_1, v_3 = e_1 - e_2, v_4 = e_1 - 2e_2, v_5 = e_1 - 3e_3$.

Since $det(v_i, v_{i+1}) = \pm 1$, for $i = 1, ..., 4$. Then, for $i = 1, ..., 4$, the cone σ_i is generated by (v_i, v_{i+i}) is regular. By decomposition, we have a fan Δ, which is regular, then X_Δ is smooth.

We can see that $v_2 = v_1 + v_3, 2v_3 = v_2 + v_4$.

Proposition 4.3.12 *Let v_1, v_2, v_3 be the vectors in $\mathbb{R}^2$. If $det(v_1, v_2) = det(v_2, v_3) = 1$, then there is $\alpha \in \mathbb{Z}$ such that $\alpha v_2 = v_1 + v_3$.*

PROOF:

Suppose that $v_1 = (a_1, a_2), v_2 = (b_1, b_2), v_3 = (c_1, c_2)$. Then

$$det(v_1, v_2) = a_1 b_2 - a_2 b_1 = 1,$$

$$det(v_2, v_3) = b_1 c_2 - b_2 c_1 = 1.$$

Hence, (b_1, b_2) is a solution of the following system of equations

$$\begin{cases} a_2 x - a_1 y & = -1 \\ c_2 x - c_1 y & = 1. \end{cases}$$

Then,

$$b_1 = \frac{a_1 + c_1}{a_1c_2 - a_2c_1}, b_2 = \frac{a_2 + c_2}{a_1c_2 - a_2c_1}.$$

Therefore, set $\alpha = a_1c_2 - a_2c_1$, so we have $\alpha v_2 = v_1 + v_3$.

Proposition 4.3.13 *Let v_1, v_2 be the vectors in $\mathbb{R}^2$. If $det(v_1, v_2) = 1$ and $v_3 = \alpha v_2 - v_1$ for some $\alpha \in \mathbb{Z}$, then $det(v_2, v_3) = 1$.*

PROOF:

Assume that $v_1 = (a_1, a_2), v_2 = (b_1, b_2)$, then $v_3 = (\alpha b_1 - a_1, \alpha b_2 - a_2)$. Hence,

$$det(v_2, v_3) = det \begin{bmatrix} b_1 & \alpha b_1 - a_1 \\ b_2 & \alpha b_2 - a_2 \end{bmatrix} = det(a_1b_2 - a_2b_1) = det(v_1, v_2) = 1.$$

Example 4.3.14 Let us consider the cone σ generated by two vectors e_2 and $7e_1 - 3e_2$. We have that X_σ is not smooth. Then, let us decompose σ to get a regular fan.

• Step 1: Consider the **Hirzebruch-Jung** fraction of $\frac{7}{3}$:

$$\frac{7}{3} = \mathbf{3} - \frac{1}{\mathbf{2} - \frac{1}{\mathbf{2}}}$$

• Step 2: Set $v_0 = (0, 1) = e_2, v_1 = (1, 0)$. Calculate

$$\begin{aligned} v_2 &= \mathbf{3}v_1 - v_0 = (3, -1), \\ v_3 &= \mathbf{2}v_2 - v_1 = (5, -2), \\ v_4 &= \mathbf{2}v_3 - v_2 = (7, -3) = 7e_1 - 3e_2. \end{aligned}$$

• Step 3: Decompose σ by cones σ_i, generated by the vectors (v_i, v_{i+1}), for $i = 0, ..., 3$. Thus, we have a regular fan Δ, and then X_Δ is smooth. X_Δ is a resolution of singularities of X_σ.

Proposition 4.3.15 *Let $\sigma \subset N_\mathbb{R} \cong \mathbb{R}^2$ be a $2-$dimensional strongly convex lattice cone. Then there exists a basis (v_1, v_2) for N such that σ is generated by the vectors $(v_2, mv_1 - kv_2)$ where $(k, m) = 1$ and $0 \leq k < m$.*

PROOF:

Suppose that σ is generated by (u_1, u_2), where u_1, u_2 are primitive vectors.

Set $v_2 = u_2$, since u_2 is a primitive vector, then we can take it as a part of a basis of N. Hence we have a basis (v_1', v_2) for some $v_1' \in N$. And then, we have

$$u_1 = mv'_1 + lv_2 \text{ for some } m \neq 0.$$

Then we can assume that $m > 0$ (if $m < 0$, then we can get $v''_1 = -v'_1$). There are integers s, k such that $l = sm - k$, where $0 \leq k < m$. Let us take this integer s. Let $v_1 = v'_1 + sv_2$, then (v_1, v_2) is a basis of N and

$$u_2 = mv'_1 + lv_2 = m(v_1 - sv_2) + lv_2 = mv_1 + (l - ms)v_2 = mv_1 - kv_2.$$

Remark 4.3.16 *A $2-$dimensional strongly convex lattice cone σ in $\mathbb{R}^2$ is isomorphic to a cone σ', generated by the vectors $(e_2, me_1 - ke_2)$, where $(k, m) = 1$ and $0 \leq k < m$.*

Example 4.3.17 Let σ be a cone in $\mathbb{R}^2$, generated by the vectors $(v_1 = 4e_1 - 3e_2, v_2 = e_1 + e_2)$. We have that σ is not regular. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

one has $detA = 1$ and

$$Av_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix},$$
$$Av_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let σ' be a cone generated by the vectors $(u_1 = 7e_1 - 3e_2, u_2 = e_2)$. Then, $\sigma \overset{A}{\cong} \sigma'$, hence $X_\sigma \cong X_{\sigma'}$.

By Example 2.3.14, $X_{\sigma'} \cong \mathbb{C}^2/\Gamma_7$.

By Theorem 4.3.7, X_σ is not compact.

X_σ is not smooth. By Example 4.3.14, we can decompose σ' to get a regular fan Δ. Then X_Δ is smooth and is a resolution of singularities of X_σ.

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