QUY NHON UNIVERSITY - EVORA UNIVERSITY

Algebraic Varieties - Robin Hartshorne Chapter II

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Dear professor Pedro M. Marques, thank you very much for help me studying this book. This is a free course from Evora University, Portugal. I will remember forever your help in my heart. Hope all the best for you!.

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Sheaves

Exercise 1.1 (Constant presheaf). Let $\mathcal{F}$ be the presheaf $U \mapsto A$ for all $U \neq \emptyset$. Then we have $\mathcal{F}(U) = A$ for all $U \neq \emptyset$. Let $\mathcal{A}$ be the constant sheaf. To show that $\mathcal{A} \cong \mathcal{F}^+$, according to Proposition 1.1, we have to prove that $\mathcal{F}_P^+ \cong \mathcal{A}_p$ for every $P \in X$. Note that for any P, $\mathcal{F}_P = \mathcal{F}_P^+$, this shows that we have to prove $\mathcal{F}_P \cong \mathcal{A}_P$ for every $P \in X$.

Let any $P \in X$, we have $\mathcal{F}_P = \lim\limits_{\longrightarrow U \ni P} \mathcal{F}(U) = A$. With an open set U contains P, there is a connected component $V \subseteq U$ containing P , which is open set, we have $\mathcal{A}(V) = A$. Hence $\mathcal{A}_P = \lim\limits_{\longrightarrow U \ni P} \mathcal{A}(U) = \lim\limits_{\longrightarrow U \supseteq V \ni P} \mathcal{A}(V) \cong A$. Therefore $\mathcal{F}_P \cong \mathcal{A}_P$ for every $P \in X$.

Exercise 1.2

1. $\bullet (ker\varphi)_P = ker(\varphi_P)$

   We have

$$\begin{aligned} \varphi(U) : \mathcal{F}(U) &\longrightarrow \mathcal{G}(U) \\ t &\mapsto \varphi(U)(t). \end{aligned}$$

$$\begin{aligned} \varphi_P : \mathcal{F}_P &\longrightarrow \mathcal{G}_P \\ \langle U, t\rangle &\mapsto \langle U, \varphi(U)(t)\rangle. \end{aligned}$$

   Let $\langle U, t\rangle \in (ker\varphi)_P$, for U is an open neighborhood of P, $t \in ker(\varphi(U))$, calculate $\varphi_P(\langle U, t\rangle)$ we have

$$\varphi_P(\langle U, t\rangle) = \langle U, \varphi(U)(t)\rangle = \langle U, 0\rangle.$$

   This shows that $\langle U, t\rangle \in ker\varphi_P$. Then

$$(ker\varphi)_P \subseteq ker(\varphi_P).$$

   Conversely, let $\langle U, t\rangle \in ker(\varphi_P)$, for U is an open neighborhood of P, $t \in \mathcal{F}(U)$, since $\varphi_P(\langle U, t\rangle) = \langle U, 0\rangle = \langle U, \varphi(U)(t)\rangle$, then there is an open set V containing P such that $\varphi(V)(t) = 0$ on V, then $\langle V, t\rangle \in (ker\varphi)_P$. Since $U \cap V \neq \emptyset$ (containing P) then $\langle U, t\rangle = \langle V, t\rangle$, this shows that

$$(ker\varphi)_P \supseteq ker(\varphi_P).$$

$\bullet (im\varphi)_P = im(\varphi_P)$

Let $\langle U, t\rangle \in (im\varphi)_P$, for U is is an open neighborhood of P, $t \in im\varphi(U)$, then there is $s \in \mathcal{F}(U)$ such that $\varphi(U)(s) = t$. We have $\varphi_P(\langle U, s\rangle) = \langle U, t\rangle$, this shows that $\langle U, t\rangle \in im(\varphi_P)$.

Conversely, let $\langle U, t\rangle \in im\varphi_P$, there is $\langle V, s\rangle \in \mathcal{F}_P$ such that $\varphi_P(\langle V, s\rangle) = \langle U, t\rangle = \langle V, \varphi(V)(s)\rangle$. Then there is an open set W containing P such that $\varphi(W)(s) = t$ on W. Then we have $t \in im(\varphi(W))$, and $\langle W, t\rangle \in (im\varphi)_P$. Since U, W contain P, then $U \cap W \neq \emptyset$, then $\langle U, t\rangle = \langle W, t\rangle \in (im\varphi)_P$.

Thus we have

$$(im\varphi)_P = im(\varphi_P).$$

2. (a) If φ is injective, then ker $\varphi = 0$, thus ker $\varphi(U) = 0$ for all open set U. For any $P \in X$, let $\langle U, t\rangle \in (\text{ker } \varphi)_P = 0$, for $P \in U$ and $t \in$ ker $\varphi(U)$, then we have $t|_U = 0$, this shows that $(\text{ker } \varphi)_P = 0$ for all $P \in X$. By 1., we have ker $\varphi_P = 0$ for every P.

   Converse, if ker $\varphi_P = 0$ for every P, by 1., we have $(\text{ker } \varphi)_P = 0$ for all P. For any open set U of X, we have to show that ker $\varphi(U) = 0$. Let $t \in$ker $\varphi(U)$, and a point $Q \in U$, so we have $\langle U, t\rangle \in (\text{ker } \varphi)_Q$, hence we have that $\langle U, t\rangle = 0 = \langle U, 0\rangle$, then $t|_U = 0|_U = 0$. This shows that $t = 0$. Thus φ is injective.

   (b) If φ is surjective, then im $\varphi = \mathcal{G}$, hence $\mathcal{G}_P = (\text{im } \varphi)_P$ for all $P \in X$, by 1., we have $\mathcal{G}_P =$ im φ_P. This shows that φ_P is surjective.

   Converse, note that φ surjective need not $\varphi(U)$ surjective for all open U. If φ_P is surjective for all $P \in X$, then we have im $\varphi_P = \mathcal{G}_P$, according to Proposition 1.1, we have im $\varphi \cong \mathcal{G}$. This shows that φ is surjective. (So we can use this proposition to prove (a) again: $(\text{ker } \varphi)_P = 0$ for all $P \Longrightarrow$ ker $\varphi = 0$.)

3. We consider two sequences

$$\ldots \mathcal{F}^{i-1} \overset{\varphi^{i-1}}{\to} \mathcal{F}^{i} \overset{\varphi^{i}}{\to} \mathcal{F}^{i+1} \ldots \tag{1}$$

$$\ldots \mathcal{F}_P^{i-1} \overset{\varphi_P^{i-1}}{\to} \mathcal{F}_P^{i} \overset{\varphi_P^{i}}{\to} \mathcal{F}_P^{i+1} \ldots \tag{2}$$

Suppose that (1) is exact, then we have that φ^{i-1} is injective, φ^i is surjective and im $\varphi^{i-1} =$ ker φ^i, by 1 and 2, we have that φ_P^{i-1} is injective, φ_P^i is surjective and im $\varphi_P^{i-1} =$ ker φ_P^i, for all $P \in X$, then (2) is exact for all P. The converse of these clause is true, then we have that (2) is exact for all $P \in X \Rightarrow$ (1) is exact.

Exercise 1.3

1. If φ is sujective, then we have φ_P is surjective for all P. Let an open set U of X. For every $s \in \mathcal{G}(U)$, let a point $P_i \in U$, we consider $\mathcal{G}_{P_i}$. We have $\langle U, s\rangle$ is an element of $\mathcal{G}_{P_i}$. Since φ_{P_i} is surjective, then there is $\langle V_i, t_i\rangle \in \mathcal{F}_{P_i}$ such that such that $\varphi_{P_i}(\langle V_i, t_i\rangle) = \langle U, s\rangle$, then $\langle V_i, \varphi(t_i)\rangle) = \langle U, s\rangle$. Set $U_i = V_i \cap U \neq \emptyset$, since $t_i \in \mathcal{F}(V_i)$, then we can see $t_i \in \mathcal{F}(U_i)$ and $\varphi(t_i) = s|_{U_i}$. For each $P_i \in U$, we have a neighborhood U_i is of P_i and $t_i \in \mathcal{F}(U_i)$, $\varphi(t_i) = s|_{U_i}$. Let P_i run all U to get $U \subset \bigcup\limits_i U_i$, which is a covering of U.

   Converse, for any $P \in X$, for every $\langle U, s\rangle \in \mathcal{G}_P$ for $U \ni P$ and $s \in \mathcal{G}(U)$. There is a covering $\{U_i\}$ of U, there are $t_i \in \mathcal{F}(U_i)$ such that $\varphi(t_i) = s|_{U_i}$. Since $P \in U$, then there is $U_j \in \{U_i\}$ such that $P \in U_j$. We consider the element $\langle U_j, t_j\rangle \in \mathcal{F}_P$. We have

   $$\varphi_P(\langle U_j, t_j\rangle) = \langle U_j, \varphi(t_j)\rangle = \langle U_j, s\rangle.$$

   Note that

   $$\langle U, s\rangle = \langle U_j, s\rangle \in \mathcal{G}_P.$$

   This shows that φ_P is surjective, this is true for all $P \in X$, then φ is surjective.

2. (Give an example). We call $\mathcal{G}$ the sheaf of holomorphic function on $\mathbb{C}$, we consider the map $\varphi : \mathcal{G} \to \mathcal{G}^*$ defined by

   $$\begin{aligned}\varphi(U) : \mathcal{G}(U) &\to \mathcal{G}(U)^* \\ f &\mapsto exp(f).\end{aligned}$$

   For any $P \in \mathbb{C}$, there is an open set U containing P of $\mathbb{C}$, which is simply connected, then we can define the logarithm of all non-zero function $f \in \mathcal{G}(U)^*$ on U, this shows that φ_P is surjective for all $P \in \mathbb{C}$. Therefore φ is surjective.

   Now, let $U = \mathcal{C}^*$, since U is not simply connected, then we can not define the the logarithm of all non-zero function $f \in \mathcal{G}(U)^*$ on U, this shows that $\varphi(U)$ is not surjective.

Exercise 1.4

(a) For any $P \in X$, we have $\mathcal{F}_P = \mathcal{F}_P^+, \mathcal{G}_P = \mathcal{G}_P^+$, then $\varphi_P^+ = \varphi_P$. If φ is injective then $\varphi_P^+ = \varphi_P$ is injective, this implies that φ^+ is injective.

(b) Call $\varphi(\mathcal{F})$ the presheaf image of φ, then im $\varphi = \varphi(\mathcal{F})^+$. We have a natural injective $i : \varphi(\mathcal{F}) \to \mathcal{G}$, by (a), we have the induced map $i^+ :$ im $\varphi \to \mathcal{G}$, which is injective. Then im φ is a subsheaf of $\mathcal{G}$.

Exercise 1.6

1. Since $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$, by definition of the subsheaf, for any open subset U of X, we have $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$, then there is a natural map $\mathcal{F}'(U) \to \mathcal{F}(U)$, which is injective. Then there is a natural map $\mathcal{F}' \to \mathcal{F}$, which is injective.

   The sheaf $\mathcal{F}/\mathcal{F}'$ associated to the presheaf $U \to \mathcal{F}(U)/\mathcal{F}'(U)$. For any point P, we have

   $$(\mathcal{F}/\mathcal{F}')_P = \mathcal{F}_P/\mathcal{F}'_P.$$

   The natural map p of $\mathcal{F}$ to the presheaf $U \to \mathcal{F}(U)/\mathcal{F}'(U)$ is defined by

   $$\begin{aligned} p(U) : \mathcal{F}(U) &\to \mathcal{F}(U)/\mathcal{F}'(U) \\ t &\mapsto \bar{t} \end{aligned}$$

   Then p^+ is the natural map of $\mathcal{F}$ to the sheaf $\mathcal{F}/\mathcal{F}'$, and $p_P = p_P^+ : \mathcal{F}_P \to (\mathcal{F}/\mathcal{F}')_P$.

   For an element $\langle U, s\rangle \in (\mathcal{F}/\mathcal{F}')_P$, with $U \ni P$ and $s \in \mathcal{F}(U)/\mathcal{F}'(U)$. Since $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$, then $p(U)$ is surjective. This implies that there exits $t \in \mathcal{F}(U)$ such that $p(t) = s$. We have $\langle U, t\rangle \in \mathcal{F}_P$ and

   $$p_P(\langle U, t\rangle) = \langle U, p(t)\rangle = \langle U, s\rangle.$$

   This shows that p_P is surjective, then p_P^+ is surjective, thus p^+ is surjective.

   We have ker $p(U) = \mathcal{F}'(U)$ for all open set U. then ker $p = \mathcal{F}'$, which is a sheaf. Then ker $p^+ = (\text{ker}p)^+ = \mathcal{F}'$. Thus we have an exact sequence

   $$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0.$$

2. Suppose that we have an exact sequence

   $$0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \to 0.$$

   Therefore, α is injective, β is surjective and im α = ker β.

   $\mathcal{F}'$ is isomorphic to the presheaf given by $U \mapsto$ im $\alpha(U)$. Hence $\mathcal{F}'$ is isomorphic to the sheaf im α, which is a subsheaf of $\mathcal{F}$.

   Since β is surjective, Exercise 1.7 shows that $\mathcal{F}/\text{ker}\beta$ is isomorphic to im $\beta = \mathcal{F}''$. Therefore, the quotient sheaf $(\mathcal{F}/\text{im } \alpha)^+$ is isomorphic to the sheaf $\mathcal{F}''$.

Exercise 1.8 If

$$0 \to \mathcal{F}' \overset{\alpha}{\to} \mathcal{F} \overset{\beta}{\to} \mathcal{F}''$$

is an exact sequence of the sheaves, then α is injective and im α = ker β. Since α is injective , we have $\alpha(U) : \mathcal{F}'(U) \to \mathcal{F}(U)$ is injective for all open subset U of X. We only need to prove that im $\alpha(U)$ = ker $\beta(U)$ for all open subset U. Let any point P of X, by Exercise 1.1, we have

$$\text{im } \alpha_P = (\text{im } \alpha)_P = (\text{ker } \beta)_P = \text{ker } \beta_P,$$

and α_P is injective, then the sequence

$$0 \to \mathcal{F}'_P \overset{\alpha_P}{\to} \mathcal{F}_P \overset{\beta_P}{\to} \mathcal{F}''_P$$

is exact.

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For any open set U, we rewrite that

$$\begin{aligned} \alpha(U) : \mathcal{F}'(U) &\to \mathcal{F}(U) \\ t &\mapsto \alpha(t), \\ \beta(U) : \mathcal{F}(U) &\to \mathcal{F}''(U) \\ s &\mapsto \beta(s). \end{aligned}$$

Let $s \in$ im $\alpha(U)$, with a point $P \in U$, then $\langle U, s\rangle \in (\text{im } \alpha)_P$, thus we have $\langle U, s\rangle \in (\text{ker } \beta)_P$, this implies that $s \in$ ker $\beta(U)$. Conversely, let $t \in$ ker $\beta(U)$, with a point $P \in U$, we have $\langle U, t\rangle \in (\text{ker } \beta)_P = (\text{im } \alpha)_P$, this shows that $t \in$ im $\alpha(U)$. Then we have the sequence

$$0 \to \Gamma(U, \mathcal{F}') \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}'')$$

is exact. Since if β is surjective, we have $\beta(U)$ need not sujective for some U, thus the functor $\Gamma(U, .)$ need not exact.

$*$ Remark: if the sequence

$$0 \to \Gamma(U, \mathcal{F}') \overset{\alpha(U)}{\to} \Gamma(U, \mathcal{F}) \overset{\beta(U)}{\to} \Gamma(U, \mathcal{F}'')$$

is exact for all open U, then im $\alpha(U)$ = ker $\beta(U)$ for all open subset U, since ker β is actually a sheaf, this implies that the presheaf $U \mapsto$ im $\alpha(U)$ is a sheaf?.

Exercise 1.14 (support) Let $\mathcal{F}$ be a sheaf on X, let $s \in \mathcal{F}(U)$ be a section over an open subset U. We define the support of s by

$$\text{Supp s} = \{P \in U : s_P \neq 0\}.$$

We consider the subset $T = U - \text{Supp s}$ of U. Let any point $P \in T$, then $s_P = 0$, we have $s_P = \langle U, s\rangle = \langle U, 0\rangle$, then there is a open set $W \subseteq U = U \cap U$ containing P such that $s|_W = 0|_W$. For every point $Q \in W$, we have $s_Q = \langle W, s\rangle = 0$, therefore W is a subset of T, then $W = T \cap W$, this implies that W is an open subset of T. So, for any point $P \in T$, there is an open subset W containing P of T , thus T is an open set. Supp s is the complement of T in U, then Supp s is closed.

Exercise 1.15 (Sheaf $\mathcal{H}$om).

1. Set $\text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ has a natural structure of abelian group.

   Let $\varphi, \phi \in$ Hom $(\mathcal{F}|_U, \mathcal{G}|_U)$, for any $s \in \mathcal{F}(U)$, we have $(\varphi + \phi)(s) = \varphi(s) + \phi(s)$. Since $\mathcal{G}(U)$ is a abelian group, then $\varphi(s) + \phi(s) = \phi(s) + \varphi(s) = (\phi + \varphi)(s)$, this implies that $\varphi + \phi = \phi + \varphi$ on $\mathcal{F}(U)$. Hence, $\text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ has a natural structure of abelian group.

2. Presheaf $U \mapsto \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf.

   (i) If U is an open set, $\{V_i\}$ is an open covering of U, and if $\varphi \in$ Hom $(\mathcal{F}|_U, \mathcal{G}|_U)$ such that $\varphi|_{V_i} = 0$ for all i. For any $s \in \mathcal{F}(U)$, we need to prove that $\varphi(s) = 0$. We have $\varphi(s_i) = 0$ for all $s_i = s|_{V_i} \in \mathcal{F}(V_i)$. By the diagram on page 62, we have

   $$\varphi(s_i) = \varphi(s|_{V_i}) = \varphi(s)|_{V_i}, \text{ for all } i.$$

   Therefore, we have $\varphi(s)|_{V_i} = 0,$ for all i, since $\mathcal{G}$ is a sheaf, then we have $\varphi(s) = 0$, thus $\varphi = 0$.

   (ii) If U is an open set, $\{V_i\}$ is an open covering of U, and if we have the element $\varphi_i \in$ Hom $(\mathcal{F}|_{V_i}, \mathcal{G}|_{V_i})$ for all i, with the property that for each i, j

   $$\varphi_i|_{V_i \cap V_j} = \varphi_j|_{V_i \cap V_j}.$$

   We need to prove that there is $\varphi \in$ Hom $(\mathcal{F}|_U, \mathcal{G}|_U)$ such that $\varphi|_{V_i} = \varphi_i$.
   For any $s \in \mathcal{F}(U)$, set $s_i = s|_{V_i}$, we have

   $$\varphi_i(s_i)|_{V_i \cap V_j} = \varphi_j(s_j)|_{V_i \cap V_j},$$

Since $\mathcal{G}$ is a sheaf, there is $u \in \mathcal{G}(U)$ such that $u|_{V_i} = \varphi(s_i)$, and u is unique. Thus by this way, we have a morphim $\varphi : \mathcal{F}(U) \to \mathcal{G}(U), s \mapsto u$. Thus φ is the map, which we need to find.

Schemes

Exercise 2.1 For $f \in A$, we have the open subset

$$D(f) = \{\mathfrak{p} \in \text{spec } (A) : f \notin \mathfrak{p}\},$$

and the closed subset

$$V((f)) = \{\mathfrak{p} \in \text{spec } (A) : f \in \mathfrak{p}\}.$$

of spec A.
Let $S = \{f^n\}_{n \geq 0}$, the localization ring at f is

$$A_f = \{\frac{u}{f^i} : u \in A, i \in \mathbb{N}\},$$

and we have the correspondence one-to-one between prime ideals

$$\text{spec } A_f \xleftrightarrow{1-1} \{\mathfrak{p} \in \text{spec } A : \mathfrak{p} \cap S = \emptyset\} = D(f),$$
$$\mathfrak{p}^e = \{\frac{u}{f^i} : u \in \mathfrak{p}, i \in \mathbb{N}\} \longleftrightarrow \mathfrak{p}.$$

The map

$$\omega : D(f) \longrightarrow \text{spec } A_f$$
$$\mathfrak{p} \longmapsto \mathfrak{p}^e,$$

which is surjective, for any ideal I in spec A_f, there is $\mathfrak{p} \in D_f$ such that $I = \mathfrak{p}^e$ and $\omega^{-1}(I) = \mathfrak{p}$. We have that if $\mathfrak{p}^e \subset \mathfrak{m}^e$ then $\mathfrak{p} \subset \mathfrak{m}$, and the converse is also true. Let $V(\mathfrak{a}^e)$ be a closed subset of spec A_f, for some $\mathfrak{a}^e \in$ spec A_f, then

$$V(\mathfrak{a}^e) = \{\mathfrak{p}^e \in \text{spec } A_f : \mathfrak{p}^e \subseteq \mathfrak{a}^e\}.$$

The preimage of $V(\mathfrak{a}^e)$ is

$$\omega(V(\mathfrak{a}^e))^{-1} = \{\mathfrak{p} \in D(f) : \mathfrak{p} \subseteq \mathfrak{a}\},$$

which is closed in $D(f)$, this implies that ω is continuous. In the same way, we also have that ω^{-1} is continue. Thus ω is a homeomophism.

Now, we prove that

$$\omega_*(\mathcal{O}_X|_{D(f)}) \cong \mathcal{O}_{\text{spec } (A_f)}.$$

This is equivalent to

$$\mathcal{O}_{\text{spec } A_f, \omega(\mathfrak{p})} \cong (\mathcal{O}_X|_{D(f)})_{\mathfrak{p}}, \text{ for every } \mathfrak{p} \in D_f. \tag{3}$$

By definition of the restriction schemes (page 65), we have $(\mathcal{O}_X|_{D(f)})_{\mathfrak{p}} = (\mathcal{O}_X)_{\mathfrak{p}}$. By Proposition 2.2a, we have $(\mathcal{O}_X)_{\mathfrak{p}} \cong A_{\mathfrak{p}}$. The same way, we have

$$\mathcal{O}_{\text{spec } A_f, \omega(\mathfrak{p})} = (\mathcal{O}_{\text{spec } A_f})_{\omega(\mathfrak{p})} = (A_f)_{\omega(\mathfrak{p})}.$$

Then (1) is equivalent to

$$(A_f)_{\omega(\mathfrak{p})} \cong A_{\mathfrak{p}}, \text{ for very } \mathfrak{p} \in D_f. \tag{4}$$

Let any $\mathfrak{p} \in D_f$, we have $A_{\mathfrak{p}} = \{\frac{a}{s} : a \in A, s \in A - \mathfrak{p}\}$. Since $\omega(\mathfrak{p}) = \mathfrak{p}^e = \{\frac{u}{f^i} : u \in \mathfrak{p}, i \in \mathbb{N}\}$, then $(A_f)_{\omega(\mathfrak{p})} = \{\frac{a}{s} : a \in A_f, s \in A_f - \omega(\mathfrak{p})\}$. Note that if $a \in A_f$, then we can write $a = \frac{u_1}{f^i}$, for some $u_1 \in A, i \in \mathbb{N}$. Since $f^i \notin \mathfrak{p}$, then we have $a \in A_{\mathfrak{p}}$. And if $s \in A_f - \omega(\mathfrak{p})$, then we can write $s = \frac{u_2}{f^j}$, for some $u_2 \in A - \mathfrak{p}, j \in \mathbb{N}$. So s is an unit in $A_{\mathfrak{p}}$, $s^{-1} = \frac{f^j}{u_2}$. Thus we can define the map

$$\begin{aligned} \varphi : (A_f)_{\omega(\mathfrak{p})} &\to A_{\mathfrak{p}} \\ \frac{a}{s} &\mapsto a.s^{-1}. \end{aligned}$$

The map φ is a ring homomorphism. Indeed,

$$\begin{aligned} \varphi(\frac{a_1}{s_1} + \frac{a_2}{s_2}) &= \varphi(\frac{a_1 s_2 + a_2 s_1}{s_1 s_2}) = (a_1 s_2 + a_2 s_1)(s_1 s_2)^{-1} \\ &= a_1 s_1^{-1} + a_2 s_2^{-2} = \varphi(\frac{a_1}{s_1}) + \varphi(\frac{a_2}{s_2}). \\ \varphi(\frac{a_1}{s_1}.\frac{a_2}{s_2}) &= \varphi(\frac{a_1 a_2}{s_1 s_2}) = (a_1 a_2)(s_1 s_2)^{-1} \\ &= a_1 s_1^{-1} a_2 s_2^{-1} = \varphi(\frac{a_1}{s_1})\varphi(\frac{a_2}{s_2}). \end{aligned}$$

Let any $\frac{u}{v} \in A_{\mathfrak{p}}$, for $u \in A, v \in A - \mathfrak{p}$, then $u' = \frac{u}{1} \in A_f, v' = \frac{v}{1} \in A_f - \omega(\mathfrak{p})$, hence we have

$$\frac{u}{v} = \frac{u}{1}.(\frac{v}{1})^{-1} = \varphi(\frac{u'}{v'}).$$

This shows that φ is surjective.
Suppose that $\varphi(\frac{a}{s}) = as^{-1} = 0 \in A_{\mathfrak{p}}$, for some $a \in A_f, v \in A_f - \omega(\mathfrak{p})$, then $a = 0$, so $\frac{a}{s} = 0$, this implies that φ is injective.
Therefore φ is a ring isomorphism, give us (2). Thus the conclusion of our proof

$$(D(f), \mathcal{O}_X|_{D(f)}) \cong (\text{spec } A_f, \mathcal{O}_{\text{spec } A_f}).$$

Exercise 2.2 For any open set U of X, let any $\mathfrak{p} \in U$, then $\mathfrak{p} \in X$, since X is a scheme, by definition, there is an open set V containing $\mathfrak{p}$ of X such that V is a affine scheme, this means that there is a ring A such that $V = \text{spec } A$. We have that $\{D(f)\}_{f \in A}$ is a base of V, since $U \cap V$ is an open set of V, then $U \cap V = \bigcup\limits_{i \in I} D(f_i)$. Since $\mathfrak{p} \in V \cap U$, then there $f_k \in A$, for $k \in I$ such that $\mathfrak{p} \in D(f_k)$. According to Exercise 2.1, we have $(D(f_k), \mathcal{O}_X|D(f_k)) = \text{spec } A_{f_k}$, then $(D(f_k), \mathcal{O}_X|D(f_k))$ is an affine scheme.

The set $D(f_k)$ is an open neighborhood of $\mathfrak{p}$ in U, we denote $\mathcal{O}_U = \mathcal{O}_X|_U$, by definition of restriction schemes, for any $x \in D(f_k)$, we have

$$(\mathcal{O}_U|_{D(f_k)})_x = (\mathcal{O}_X|_U)_x = (\mathcal{O}_X)_x = (\mathcal{O}_X|_{D(f_k)})_x,$$

this shows that $\mathcal{O}_U|_{D(f_k)} = \mathcal{O}_X|_{D(f_k)}$, then $(D(f_k), \mathcal{O}_U|_{D(f_k)}) = (D(f_k), \mathcal{O}_X|_{D(f_k)})$ is an affine scheme. Therefore $(U, \mathcal{O}_U)$ is a scheme. Let A be a commutative ring, denote

$$nu(A) = \{a \in A : a^n = 0 \textbf{ for some } n \in \mathbb{N}\},$$

then $nu(A)$ is a ideal of A, denote $A_{red} = A/nu(A)$, we have A_{red} is a reduced ring. We consider three claims in commutative algebra.

Claim 1. Let any prime ideal $\mathfrak{p} \neq 0$ of A, then $nu(A)$ is a subset of p.

Claim 2. Let I be an ideal of A, there is an inclusion-preserving correspondence one-to-one between the set of (prime) ideal containing I and the set of (prime) ideal of A/I.

We define the map

$$\varphi : specA \longrightarrow specA_{red}$$
$$\mathfrak{p} \mapsto \mathfrak{p}/nu(A).$$

By Claim 1 and Claim 2, we have φ is an inclusion-preserving sujective. And implies that φ is a homeomorphism on Zarisky topology.

Claim 3 For any $f \in A$, we have $(A_f)_{red} = (A_{red})_{\overline{f}}$, for $\overline{f}$ is the image of f in A_{red}.

Exercise 2.3 (Reduced schemes)

(a) $(\Rightarrow)$ Suppose that $(X, \mathcal{O}_X)$ is a reduced scheme. For any $p \in X$, let any $t_p \in \mathcal{O}_{X,p}$ such that $t_p^n = 0$ for some $n \in \mathbb{N}$, there is an open set U of

X, and $s \in \mathcal{O}(U)$ such that $t_p = \langle s, U\rangle$. We have $t_p^n = \langle s^n, U\rangle = 0$, ($s^n \in \mathcal{O}(U)$ since $\mathcal{O}(U)$ is a ring,) then $s^n = 0$ in $\mathcal{O}(V)$, for some open set V of U. Since $(X, \mathcal{O}_X)$ is a reduced scheme, then $\mathcal{O}(V)$ is a reduced ring, this shows that $s = 0$, then $t_p = 0$, thus $\mathcal{O}_{X,p}$ is a reduced ring.

($\Leftarrow$) Let any open set U of X, let any $t \in \mathcal{O}(U)$ such that $t^n = 0$ for some n, let any point $p \in U$, then $t_p = \langle t, U\rangle \in \mathcal{O}_{X,p}$, we have $t_p^n = \langle t^n, U\rangle = 0$, since $\mathcal{O}_{X,p}$ is a reduce ring, then $t_p = 0$, this implies that $t = 0$ for some open set V_P containing P of U. Thus $t = 0$ on U. Hence $\mathcal{O}(U)$ is a reduced ring. Therefore $(X, \mathcal{O}_X)$ is a reduced scheme.

(b) Firstly, for every ring A, we prove that

$$(\varphi, \varphi^{\#}) : (specA, (\mathcal{O}_{specA})_{red}) \cong (specA_{red}, \mathcal{O}_{specA_{red}}). \tag{5}$$

Indeed, φ is a homeomorphism. Denote $V = specA, V_{red} = specA_{red}$, for every point $p \in V$, the local ring

$$((\mathcal{O}_V)_{red})_p = \lim_{\substack{\rightarrow \\ U \ni p}} (\mathcal{O}_V(U)_{red}) \tag{6}$$

$$= \lim_{\substack{\rightarrow \\ U \supset D(f) \ni p}} (\mathcal{O}_V(D(f))_{red}), f \in A. \tag{7}$$

$$\mathcal{O}_{V_{red},\varphi(p)} = \lim_{\substack{\rightarrow \\ U \supset D(\overline{f}) \ni \varphi(p)}} (\mathcal{O}_{V_{red}}(D(\overline{f}))), \overline{f} \in A_{red}. \tag{8}$$

By Proposition 2.2b, we have $\mathcal{O}_V(D(f)) = A_f$, by Claim 3, we have $\mathcal{O}_V(D(f))_{red} = (A_f)_{red} = (A_{red})_{\overline{f}} = \mathcal{O}_{V_{red}}(D(\overline{f}))$. Thus, by (3), (4), we have

$$((\mathcal{O}_V)_{red})_p = \mathcal{O}_{V_{red},\varphi(p)}$$

Then $\varphi_*(\mathcal{O}_V)_{red} = \mathcal{O}_{V_{red}}$, and $(V, (\mathcal{O}_V)_{red}) \cong (V_{red}, \mathcal{O}_{V_{red}})$.

Now, if $(X, \mathcal{O}_X)$ is an affine scheme, then there is a ring A such that $(X, \mathcal{O}_X) \cong (specA, \mathcal{O}_{specA})$, then by (1), $(X, (\mathcal{O}_X)_{red})$ is a scheme. For any $p \in X$, since $(X, \mathcal{O}_X)$ is a scheme, then there is an open set U of X such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. Then $(U, (\mathcal{O}_X|_U)_{red})$ is a scheme. Since $(\mathcal{O}_X)_{red}|_U = (\mathcal{O}_X|_U)_{red}$, then $(U, (\mathcal{O}_X)_{red}|_U)$ is an affine scheme. Thus $(X, (\mathcal{O}_X)_{red})$ is a scheme.

In the case X is an affine scheme, then there is a ring A such that $(X, \mathcal{O}_X) \cong (specA, \mathcal{O}_{specA})$, we consider the map $\pi : A \longrightarrow A_{red}, \pi(f) =$

$\overline{f} = f + nu(A)$, which is a ring homomorphism, by Proposition 2.3 b, then π induces a natural morphism of locally ringed spaces

$$(f, f^{\#}) : (spec(A_{red}), \mathcal{O}_{spec(A_{red})}) \longrightarrow (specA, \mathcal{O}_{specA}),$$

the map f is defined by (seeing the proof of Proposition 2.3b)

$$\begin{aligned} f : spec(A_{red}) &\longrightarrow specA \\ \mathfrak{p} &\longmapsto \pi^{-1}(\mathfrak{p}). \end{aligned}$$

Note that $spec(A_{red}) = \{\mathfrak{p}/(nu(A)) \in spec(A_{red}) : \mathfrak{p} \in specA\}$, and so $\pi^{-1}(\mathfrak{p}/(nu(A)) = \mathfrak{p}$, this shows us that $f = \varphi^{-1}$, the map φ is defined in Claim 2, which is a homeomorphism. Therefore f is a homeomorphism. By (1), we have

$$(X, (\mathcal{O}_X)_{red}) = (specA, (\mathcal{O}_{specA})_{red}) \cong (specA_{red}, \mathcal{O}_{specA_{red}}), \tag{9}$$

then there is a morphism of schemes $X_{red} \to X$, which is a homeomorphism.

In the general case, if $(X, \mathcal{O}_X)$ is a scheme, for any open set U of X, the natural map $\phi(U) : \mathcal{O}_X(U) \to \mathcal{O}_X(U)_{red}$, which is a ring morphism, this give us the presheaf morphism $\phi : \mathcal{O}_X \longrightarrow (U \mapsto \mathcal{O}_X(U)_{red})$, then we have the induced sheaf morphism $\phi^+ : \mathcal{O}_X \longrightarrow (O_X)_{red}$.
Considering the identity map $i_X : X \to X$, then i_X is a homeomorphism. The direct image sheaf $(i_X)_*(O_X)_{red} = (O_X)_{red}$, then we have the sheaf morphism $i_X^{\#} = \phi^+ : \mathcal{O}_X \longrightarrow (i_X)_*(O_X)_{red}$. Hence, we have the scheme morphism

$$(i_X, i_X^{\#}) : (X, (\mathcal{O}_X)_{red}) \longrightarrow (X, \mathcal{O}_X),$$

which is a homeomorphism.

(c) Lemma: A ring morphism $f : A \to B$ induces the morphism ring $g : A_{red} \to B_{red}$ with $g(\overline{x}) = \overline{f(x)}$.

Proof. We check that g is a map.
For any $\overline{x}, \overline{y} \in A_{red}$ such that $\overline{x} = \overline{y}$, then $x = y + nu(A)$, since f is a ring morphism, then $f(x) = f(y + nu(A)) = f(y) + nu(A) = \overline{f(y)}$. Then g is a map. Actually g is a morphism. □

We have the scheme morphism

$$(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y),$$

with $f^{\#} : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$, for any open set U of Y, we have the ring morphism $f^{\#}(U) : \mathcal{O}_Y(U) \to (f_*\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U))$, then it induces the ring morphism $f^{\#}(U)_r : \mathcal{O}_Y(U)_{red} \to \mathcal{O}_X(f^{-1}(U))_{red}$, since X is reduced, then $\mathcal{O}_X(f^{-1}(U))_{red} = \mathcal{O}_X(f^{-1}(U))$, then we have the ring morphism $f^{\#}(U)_r : \mathcal{O}_Y(U)_{red} \to \mathcal{O}_X(f^{-1}(U))$, this gives us the presheaf $f_r^{\#} : (U \mapsto \mathcal{O}_Y(U)_{red}) \longrightarrow f_*\mathcal{O}_X$, then we have the induced sheaf morphism $(f_r^{\#})^+ : (\mathcal{O}_Y)_{red} \longrightarrow f_*\mathcal{O}_X$. This gives us the scheme morphism

$$(g, g^{\#}) = (f, (f_r^{\#})^+) : (X, \mathcal{O}_X) \longrightarrow (Y, (\mathcal{O}_Y)_{red}).$$

Finally, we prove that the composing $(g, g^{\#})$ with the natural scheme morphism $(i_Y, i_Y^{\#}) : (Y, (\mathcal{O}_Y)_{red}) \longrightarrow (Y, \mathcal{O}_Y)$ equal to $(f, f^{\#})$, thus we need only prove that $f^{\#} = g^{\#} \circ i_Y^{\#}$.

Exercise 2.8

Lemma 1: If A is a local ring, the maximal ideal is $\mathfrak{p}$, then the local ring $A_{\mathfrak{p}} \cong A$.

Lemma 2: Let $f : A \longrightarrow B$ be the ring morphism. Let I be the ideal of A, J be the ideal of B such that $f(I) \subseteq J$, then induce the map

$$\begin{aligned} g : A/I &\longrightarrow B/J \\ \overline{x} &\longmapsto \overline{f(x)}. \end{aligned}$$

Lemma 3: $k[\varepsilon]/(\varepsilon^2)$ is a local ring, the maximal ideal is (ε).

Suppose that we have a morphism

$$(f, f^{\#}) : (Spec(k[\varepsilon]/(\varepsilon^2)), \mathcal{O}_{Spec(k[\varepsilon]/(\varepsilon^2))}) \longrightarrow (X, \mathcal{O}_X),$$

with

$$\begin{aligned} f : Spec(k[\varepsilon]/(\varepsilon^2)) &\longrightarrow X \\ (\varepsilon) &\longmapsto x. \\ f^{\#} : \mathcal{O}_X &\longrightarrow f_*(\mathcal{O}_{Spec(k[\varepsilon]/(\varepsilon^2))}). \end{aligned}$$

Suppose that $f((\varepsilon)) = x$, the local homomorphism of local rings at (ε) is the map

$$f_{(\varepsilon)}^{\#} : (\mathcal{O}_X)_{f((\varepsilon))} \longrightarrow (\mathcal{O}_{Spec(k[\varepsilon]/(\varepsilon^2))})_{(\varepsilon)}.$$

We have $(\mathcal{O}_X)_{f((\varepsilon))} = \mathcal{O}_x, (\mathcal{O}_{Spec(k[\varepsilon]/(\varepsilon^2))})_{(\varepsilon)} = (k[\varepsilon]/(\varepsilon^2))_{(\varepsilon)} = k[\varepsilon]/(\varepsilon^2)$. Hence, we write again the map

$$f_{(\varepsilon)}^{\#} : \mathcal{O}_x \longrightarrow k[\varepsilon]/(\varepsilon^2).$$

We have that $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_x$, then $f^{\#}_{(\varepsilon)}(\mathfrak{m}_x)$ is an ideal of $k[\varepsilon]/(\varepsilon^2)$, hence $f^{\#}_{(\varepsilon)}(\mathfrak{m}_x) \subseteq (\varepsilon)$, we use Lemma 2, this map induces a morphism

$$\varphi : k(x) = \mathcal{O}_x/(\mathfrak{m}_x) \longrightarrow (k[\varepsilon]/(\varepsilon^2))/(\varepsilon).$$

Note that $(k[\varepsilon]/(\varepsilon^2))/(\varepsilon) \cong k$, the proof is considered the morphism of $k[\varepsilon]/(\varepsilon^2)$ to k, this map is surjective and kernel of the map is (ε). Thus we have a morphism of $k(x)$ to k, since $k(x), k$ are the fields, then φ is injective, we also have $k \subseteq k(x)$, thus $k(x) = k$. Define a morphism $T_x \longrightarrow k$. Firstly, since $f^{\#}_{(\varepsilon)}(\mathfrak{m}_x) \subseteq (\varepsilon)$, we can defined the map

$$\begin{aligned} \lambda : \mathfrak{m}_x &\longrightarrow k \\ a &\longmapsto \frac{f^{\#}_{(\varepsilon)}(a)}{\varepsilon}, \end{aligned}$$

the map λ is a ring morphism, let any $a \in \mathfrak{m}_x^2$, since $\varepsilon^2 = 0$, then $\lambda(a) = 0$, this shows that $\lambda(\mathfrak{m}_x^2) = 0$, use Lemma 2, λ induces a morphism of $\mathfrak{m}_x/\mathfrak{m}_x^2$ to k, which is in T_x.

Conversely, we fix $x \in X$ for $k(x) = k$, an element $\varphi \in T_x, \varphi : \mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow k$. We have $\mathcal{O}_x/\mathfrak{m}_x = k(x) = k$, then $\mathcal{O}_x = k \bigoplus \mathfrak{m}_x$.

Definition (Algebras) Let $f : A \longrightarrow B$ be a homomorphism ring, we define a product

$$a.b = f(a).b.$$

Then B is an A-module. We call that B is an A-algebra. So A-algebra B is defined by the homomorphism $f : A \longrightarrow B$. B is a finitely-generated A-algebra if B is a finitely-generated A-module, this mean that there are $b_1, \dots, b_r$ in B such that

$$B = Ab_1 + \cdots + Ab_r.$$

We can see f as a module homormorphism $A-$module.

Let S be a multiplicatively closed subset of A, then $f(S)$ is also a multiplicatively closed subset of B, we have that f induces a ring homormorphism

$$\begin{aligned} f_S : S^{-1}A &\longrightarrow (f(S))^{-1}B \\ \frac{a}{s} &\longmapsto \frac{f(a)}{f(s)}. \end{aligned}$$

This give us a $S^{-1}A$-algebra $(f(S))^{-1}B$. And if B is a finitely-generated $A-$algebra, then $B = Ab_1 + \cdots + Ab_r$ for some b_i in B, this implies that

$$(f(S))^{-1}B = S^{-1}A.b_1/1 + \cdots + S^{-1}A.b_r/1$$

as a finitely-generated module, then $(f(S))^{-1}B$ is a finitely generated $S^{-1}A$-algebra.

First Properties of Schemes

Exercise 3.1
($\Rightarrow$) Suppose that f is a locally of finite type. Then there is a covering of Y by open affine subset $V_i = SpecB_i$, such that for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = SpecA_{ij}$, where each A_{ij} is a finitely generated B_i-algebra. For every open affine subset $V = SpecB$ of Y, for each i, $V \cap V_i$ is an open subset of V_i, then $V \cap V_i = \bigcup_{i_k} Spec(B_i)_{f_{i_k}}$, for $f_{i_k} \in B_i$. Since A_{ij} is a finitely generated B_i-algebra, then there is a ring homomorphism $\varphi : B_i \longrightarrow A_{ij}$, which defines the finitely generated B_i-algebra A_{ij}. For every f_{i_k} in B_i, φ induces a ring homomorphism $\varphi_{f_{i_k}} : (B_i)_{f_{i_k}} \longrightarrow (A_{ij})_{\varphi(f_{i_k})}$, this gives us a finitely generated $(A_{ij})_{\varphi(f_{i_k})}-$algebra $(B_i)_{f_{i_k}}$, and a scheme morphism of $Spec((A_{ij})_{\varphi(f_{i_k})})$ to $Spec((B_i)_{f_{i_k}})$. We have

$$V = SpecB = \bigcup_i (V \cap V_i) = \bigcup_i \bigcup_{i_k} Spec(B_i)_{f_{i_k}},$$

since $f^{-1}(V_i) = \bigcup_{i,ij} SpecA_{ij}$, then

$$f^{-1}(V) = \bigcup_{i,i_k} f^{-1}(Spec(B_i)_{f_{i_k}}) = \bigcup_{i,j,i_k} (Spec(A_{ij})_{\varphi(f_{i_k})}).$$

To prove easier, for each (i, i_k), we set $C_i = (B_i)_{f_{i_k}}, D_{ij} = (A_{ij})_{\varphi(f_{i_k})}$, with D_{ij} is a finitely generated C_i-algebra. So we can write V and $f^{-1}(V)$ again as follows

$$V = SpecB = \bigcup_i (SpecC_i),$$

$$f^{-1}(V) = \bigcup_{i,ij} (SpecD_{ij}).$$

Lemma 1: *Let $X = SpecA$ be an affine scheme, let $U = SpecB$ be an open subset of X, let $f \in A$ such that $D(f)$ open in U, let $\overline{f}$ be the image f in B, then $A_f \cong B_{\overline{f}}$. (This lemma is found by the page 83, Hartshorne'book.)*

For every $p \in V = SpecB$, then $p \in SpecC_i$ for some i, since $SpecC_i$ is an open subset of V, then

$$SpecC_i = \bigcup_{i_k} SpecB_{f_{i_k}}.$$

Thus, there is $f_p \in B$ such that $p \in SpecB_{f_p}$, then we have

$$D(f_p) = SpecB_{f_p} \subset SpecC_i \subset SpecB.$$

By Lemma 1, we have $B_{f_p} \cong (C_i)_{\overline{f_p}}$, this implies that $SpecB_{f_p} = Spec(C_i)_{\overline{f_p}}$. Note that $SpecB_{f_p}$ is an open neighborhood containing p, which is existing for every $p \in V$, then we can write as follows

$$V = SpecB = \bigcup_{p\in V} SpecB_{f_p} = \bigcup_{p\in V,i} Spec(C_i)_{\overline{f_p}}.$$

Thus

$$f^{-1}(V) = \bigcup_{p\in V,i,ij} Spec(D_{ij})_{\varphi(\overline{f_p})}.$$

For each i, $(D_{ij})_{\varphi(\overline{f_p})}$ is a finitely generated $(C_i)_{\overline{f_p}}-$algebra, hence, $(D_{ij})_{\varphi(\overline{f_p})}$ is a finitely generated $B_{f_p}-$algebra.

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Lemma 2: *Let A, B be the rings, let f be in A. If B is a finitely generated A_f-algebra, then B is a finitely generated $A-$algebra.*

Proof. Suppose that there are $b_1, \dots, b_r$ such that

$$B = A_f.b_1 + \cdots + A_f.b_r.$$

Hence, for every $b \in B$, there are $\frac{a_1}{f^{i_1}}, \dots, \dfrac{a_r}{f^{i_r}}$ in A_f such that

$$b = \frac{a_1}{f^{i_1}}.b_1 + \cdots + \frac{a_r}{f^{i_r}}.b_r.$$

We have a ring homomorphism $\lambda : A \longrightarrow A_f \longrightarrow B$, since $\frac{f}{1}$ is an unit in A_f, then $\overline{f} = \lambda(\frac{f}{1})$ is an unit in B. By the definition of the product of algebras, then

$$b = a_1.(b_1(\overline{f})^{-1}) + \cdots + a_r.(b_r(\overline{f})^{-1})).$$

Thus B is a A-algebra. □

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(Continuing our proof)

By Lemma 2, $(D_{ij})_{\varphi(\overline{f_p})}$ is a finitely generated $B-$algebra.

$(\Leftarrow)$. It is implied by definition.

Lemma 1. Let X be a affine schemes, then $\mathrm{sp}(X)$ is a quasi-compact.(Exercise 2.13-b, page 80.)

Proof. □

Lemma 2. A finite union of quasi-compact sets is a quasi-compact set.

Lemma 3. Let $f : \text{Spec } A \longrightarrow \text{Spec } B$ be a morphism of ringed space, then it induces a ring homomorphism $\varphi : B \longrightarrow A$ such that $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. (See Proposition 2.3.) For any $g \in B$, we define the open sets

$$D(g) = \{\mathfrak{p} \in \text{Spec } B : g \notin \mathfrak{p}\},$$
$$D(\varphi(g)) = \{\mathfrak{p} \in \text{Spec } A : \varphi(g) \notin \mathfrak{p}\}.$$

Then $f^{-1}(D(g)) = D(\varphi(g))$.

Proof. For any $\mathfrak{p} \notin D(\varphi(g))$, we have $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. Since $\mathfrak{p} \notin D(\varphi(g))$ then we have $\varphi(g) \in \mathfrak{p}$, this shows us $g \in \varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$, hence $f(\mathfrak{p}) \notin D(g)$. Thus if $f(\mathfrak{a}) \in D(g)$ then $\mathfrak{a} \in D(\varphi(g))$, therefore

$$f^{-1}(D(g)) = D(\varphi(g)).$$

□

Exercise 3.2 (Quasi-compact schemes).
($\Rightarrow$) Suppose that $f : X \longrightarrow Y$ of schemes is quasi-compact, then there is a cover Y by open affine V_i such that $f^{-1}(V_i)$ is quasi-compact for each i. For every open affine subset $V = \text{Spec } B$ of Y. Suppose that $V_i = \text{Spec } B_i$. For each i, $V \cap V_i$ is a open set of V_i. Then $V \cap V_i = \bigcup_{i_k} \text{Spec } (B_i)_{f_{i_k}}$, for some $f_{i_k} \in B_i$. Hence

$$V = \bigcup_{i,i_k} \text{Spec } (B_i)_{g_{i_k}}$$

Since V is a affine scheme, then we can take a finite subcover such that

$$V = \bigcup_{i=1}^{n} \bigcup_{k=1}^{u_i} \text{Spec } (B_i)_{g_{i_k}}, \text{for } u_i \in \mathbb{N}. \tag{10}$$

For each i, we have $f^{-1}(V_i) = \bigcup_{j=1}^{t_i} \text{Spec} A_{ij}$, for $t_i \in \mathbb{N}$, then we have the restriction map $f_{ij} : \text{Spec } A_{ij} \longrightarrow \text{Spec } B_i$ of f. By Lemma 3, we have

$$f_{ij}^{-1}(\text{Spec } (B_i)_{g_{i_k}}) = \text{Spec } (A_{ij})_{\varphi_{ij}(g_{i_k})},$$

for $\varphi_{ij} : B_i \longrightarrow A_{ij}$. This implies the inverse image

$$f^{-1}(\text{Spec } (B_i)_{g_{i_k}}) = \bigcup_{j=1}^{t_i} \text{Spec } (A_{ij})_{\varphi_{ij}(g_{i_k})}.$$

By (1), we have

$$\begin{aligned} f^{-1}(V) &= \bigcup_{i=1}^{n} \bigcup_{k=1}^{u_i} f^{-1}(\text{Spec } (B_i)_{g_{i_k}}) \\ &= \bigcup_{i=1}^{n} \bigcup_{k=1}^{u_i} \bigcup_{j=1}^{t_i} \text{Spec } (A_{ij})_{\varphi_{ij}(g_{i_k})}. \end{aligned}$$

Therefore, $f^{-1}(V)$ is a finite union of quasi-compact sets, then $f^{-1}(V)$ is a quasi-compact set.
($\Longleftarrow$) Obvious.

Exercise 3.3

(a) ($\Rightarrow$) If $f : X \longrightarrow Y$ is of finite type then there is a covering of Y by open affine subsets $V_i = \text{Spec } B_i$, for each i, $f^{-1}(V_i)$ can be covered by a finite number of $U_{ij} = \text{Spec } A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra. This define gives us that f is of local finite type, and f is quasi-compact.
($\Leftarrow$) Conversely, if $f : X \longrightarrow Y$ is of local finite type and quasi-compact. For every open affine $V = \text{Spec } B$ of Y, $f^{-1}(V)$ can be covered by open affine subsets $U_i = \text{Spec } A_i$, where each A_i is a finitely generated B-algebra. Since $f^{-1}(V)$ is quasi-compact, then $f^{-1}(V)$ can be covered by a finite number of U_i. This shows that f is of finite type.

(b) By 3.1, 3.2 and 3.3-a, we have this result.

(c) Suppose that $f : X \longrightarrow Y$ is finite of type. For every open affine subset $V = \text{Spec } B \subseteq Y$, and for every open affine subset $U = \text{Spec } A \subseteq f^{-1}(V)$, since $f^{-1}(V)$ is quasi-compact, then we have

$$f^{-1}(V) = \bigcup_{i=1}^{n} \text{Spec } A_i,$$

where each A_i is a finitely generated $B-$algebra. Since U is an open set of $f^{-1}(V)$, then there are $f_{i_k} \in A_i$ such that

$$U = \bigcup_{i=1}^{n} \bigcup_{k=1}^{u_i} \text{Spec } (A_i)_{f_{i_k}}.$$

For any $p \in U = \text{Spec } A$, there is f_{i_k} such that $p \in \text{Spec } (A_i)_{f_{i_k}}$, since $\text{Spec } (A_i)_{f_{i_k}}$ is an open set of Spec A, then there is $f_p \in A$ such that

$p \in \text{Spec } A_{f_p} \subseteq \text{Spec } (A_i)_{f_{i_k}}$. Thus we have

$$\text{Spec } A_{f_p} \subseteq \text{Spec } (A_i)_{f_{i_k}} \subseteq \text{Spec } A.$$

Therefore $A_{f_p} \cong (A_i)_{f_{i_k}}$, then Spec $A_{f_p} =$ Spec $(A_i)_{f_{i_k}}$, since Spec A_{f_p} is an open neighborhood of $p \in$ Spec A, then

$$U = \text{Spec } A = \bigcup_{p \in U} \text{Spec } A_{f_p}.$$

Since U is an affine scheme, then we can take a finite cover of open sets Spec A_{f_p}. Hence, there is a finite set $I \subset U$ such that

$$U = \text{Spec } A = \bigcup_{p \in I} \text{Spec } A_{f_p}.$$

For each i, since A_i is a finitely generated B-algebra, and $(A_i)_{f_{i_k}}$ is a finitely generated A_i-algebra, then $(A_i)_{f_{i_k}}$ is a finitely generated $B-$algebra. For any $p \in U$, since $A_{f_p} \cong (A_i)_{f_{i_k}}$ for some $i, f_{i_k} \in A_i$, then A_{f_p} is a finitely generated $B-$algebra.

Thus we can assume that

$$U = \text{Spec } A = \bigcup_{i=1}^{m} \text{Spec } A_{f_m} = \bigcup_{i=1}^{m} D(f_m). \tag{11}$$

where each A_{f_m} is a finitely generated $B-$algebra. that $\emptyset = \bigcap_{i=1}^{m} V(f_m)$.

By (2), we have that (1) implies that $\emptyset = \bigcap_{i=1}^{m} V(f_m)$. This implies that 1 belong to the ideal $(f_1, \ldots, f_m)$ of A. So for any $a \in A$, there are $a_1, \ldots, a_m \in A$ such that

$$a = a_1 f_1 + \cdots + a_m f_m.$$

Lemma 4. The followings conditions are equivalent for an integrally closed domain.

1. A is integrally closed;
2. A_p is integrally closed for every prime ideal p;
3. A_m is integrally closed for every maximal ideal m;

Proof. `https://en.wikipedia.org/wiki/Integrally_closed_domain` □

We suppose that A is integrally closed, we consider the affine scheme $X =$ Spec A, for every $p \in$ Spec A, since the local ring $\mathcal{O}_{X,p}$ equals to A_p, by Lemma 1, then $\mathcal{O}_{X,p}$ is integrally closed. Thus we have X is normal.

Definition 5. (Integral closure of a ring). Let A be a ring, for K is the quotient field, let any $b \in K$. Then b is said to be *integral* over A if there is a polynomial $f$in $A[x] - 0$, such that

$$f(b) = 0.$$

Let B be a ring in K, then we call that B is *integral* over A if every element of B is integral over A.

Exercise 3.8. Let $U = \text{Spec } A$, $V = \text{Spec } B$ be the open affine schemes of a scheme X. Firstly, we describe the open set $U \cap V$. Let any point $p \in U \cap V$, seeing this set as an open set of U, then there is $f \in A$ such that $p \in \text{Spec } A_f \subseteq U \cap V$. Seeing Spec A_f as an open set of V, then there is $g \in V$ such that

$$p \in \text{Spec } B_g \subset \text{Spec } A_f \subseteq U \cap V \subset \text{Spec } B.$$

This gives us $A_f \cong B_g$, and then Spec $B_g \cong$ Spec A_f, which is an open neighborhood of p, then we can write as follows

$$U \cap V = \bigcup_i \text{Spec } A_{f_i} = \bigcup_i \text{Spec } B_{g_i},$$

for every i, $A_{f_i} \cong B_{g_i}$.
We have the natural injective $i_1 : A \longrightarrow \widetilde{A}, i_2 : B \longrightarrow \widetilde{B}$. They induce the schemes morphism $\varphi : \text{Spec } \widetilde{A} \longrightarrow \text{Spec } A, \phi : \text{Spec } \widetilde{B} \longrightarrow \text{Spec } B$. By Lemma 3, for every i, we have

$$\varphi^{-1}(U \cap V) = \bigcup_i \varphi^{-1}(\text{Spec } A_{f_i}) = \bigcup_i \text{Spec } \widetilde{A}_{f_i}.$$

$$\phi^{-1}(U \cap V) = \bigcup_i \phi^{-1}(\text{Spec } B_{g_i}) = \bigcup_i \text{Spec } \widetilde{B}_{g_i}.$$

For every i, since $A_{f_i} \cong B_{g_i}$, then we have $\widetilde{A}_{f_i} \cong \widetilde{B}_{g_i}$, thus we have an isomorphism of $\varphi^{-1}(U \cap V)$ to $\phi^{-1}(U \cap V)$. Then we can glue Spec $\widetilde{A}$ and Spec $\widetilde{B}$ along $\varphi^{-1}(U \cap V)$ to obtain a normal scheme. Finally, we work on a covering of X, then one has a normal scheme $\widetilde{X}$, by our gluing, we have a morphim $\widetilde{X} \longrightarrow X$.