

QUY NHON UNIVERSITY - EVORA UNIVERSITY

Algebraic Varieties - Robin Harshorne

Chapter I

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Dear professor Pedro M. Marques, thank you very much for help me studying this book. This is a free course from Evora University, Portugal. I will remember forever your help in my heart. Hope all the best for you!.

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Projective Varieties

Exercise 2.1 Prove the "homogeneous Nullstellensatz", which say if $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if f is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in \mathbb{P}^n , then $f^q \in \mathfrak{a}$ for some $q > 0$.

Solution:

Assume that $\mathfrak{a} = S$, then we can get $q = 1$.

Assume that $\mathfrak{a} \neq S$ and

$$\mathfrak{a} = \langle f_i | f_i \text{ is a homogeneous polynomial, } i = 1, \dots, n \rangle,$$

then $\deg f_i > 0$. We will show that

$$\mathbb{A}^{n+1} \supseteq Z(\mathfrak{a}) = \{P \in \mathbb{A}^{n+1} | P = 0 \text{ or } \bar{P} \in Z(\mathfrak{a}) \subseteq \mathbb{P}^n\},$$

where

$$\mathbb{A}^{n+1} \supseteq Z(\mathfrak{a}) = \{P \in \mathbb{A}^{n+1} | f_i(P) = 0, i = 1, \dots, n\} = U,$$

$$\mathbb{P}^n \supseteq Z(\mathfrak{a}) = \{P \in \mathbb{P}^n | f_i(P) = 0, i = 1, \dots, n\} = V.$$

If $P = 0$ then $f_i(0) = 0$ for all $i = 1, \dots, n$. Hence $P = 0 \in U$.

If $P \neq 0$, suppose that

$$P = (x_0, \dots, x_n) \in \mathbb{A}^{n+1},$$

then we have a point \bar{P} in \mathbb{P}^n corresponding to the point P ,

$$\bar{P} = (x_0 : \dots : x_n) \in \mathbb{P}^n.$$

Therefore, if $P \in U$, since f_i are the homogeneous polynomials,

$$f_i(\alpha x_0, \dots, \alpha x_n) = \alpha^{d_i} f_i(P) \text{ where } \alpha \neq 0 \text{ and } d_i = \deg f_i,$$

then $P_\alpha = (\alpha x_0, \dots, \alpha x_n) \in U$ for all $\alpha \neq 0$, thus $\bar{P} \in V$.

Conversely, if $\bar{P} \in V$ then $P_\alpha = (\alpha x_0, \dots, \alpha x_n) \in U$ for all $\alpha \neq 0$.

Thus we have

$$\mathbb{A}^{n+1} \supseteq Z(\mathfrak{a}) = \{P \in \mathbb{A}^{n+1} | P = 0 \text{ or } \bar{P} \in Z(\mathfrak{a}) \subseteq \mathbb{P}^n\}.$$

Now, we come back to the homogeneous polynomial f . Let

$$P = (x_0, \dots, x_n) \in Z(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}.$$

If $P = 0$ then $f(P) = 0$. If $P \neq 0$ then

$$\bar{P} = (x_0 : \dots : x_n) \in Z(\mathfrak{a}) \subseteq \mathbb{P}^n.$$

We have $f(\bar{P}) = 0$, hence $f(P) = 0$, therefore, $f \in I(Z(\mathfrak{a}))$ where $Z(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}$, then $f^q \in \mathfrak{a}$ for some $q > 0$.

Exercise 2.2 For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

- i) $Z(\mathfrak{a}) = \emptyset$ in \mathbb{P}^n ;
- ii) $\sqrt{\mathfrak{a}}$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;
- iii) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.

Solution:

i) \Rightarrow ii)

By Exercise 2.1, we have that if $Z(\mathfrak{a}) = \emptyset$ in \mathbb{P}^n , then in \mathbb{A}^{n+1} , either $Z(\mathfrak{a}) = \emptyset$ or $Z(\mathfrak{a}) = \{0\}$.

If $Z(\mathfrak{a}) = \emptyset$ then $\sqrt{\mathfrak{a}} = I(Z(\mathfrak{a})) = S = K[x_0, \dots, x_n]$.

If $Z(\mathfrak{a}) = \{0\}$ then $\sqrt{\mathfrak{a}} = I(Z(\mathfrak{a})) = \langle x_0, \dots, x_n \rangle$ (Example 1.1.4), hence

$$\sqrt{\mathfrak{a}} = S \setminus K = \bigoplus_{d>0} S_d.$$

ii) \Rightarrow iii)

If $\sqrt{\mathfrak{a}} = S = K[x_0, \dots, x_n]$ then $1 \in \sqrt{\mathfrak{a}}$, then $1 \in \mathfrak{a}$, hence $\mathfrak{a} = S$.

If $\sqrt{\mathfrak{a}} = \bigoplus_{d>0} S_d$, then for all $d \in \mathbb{N} \setminus \{0\}$, $x_i^d \in \sqrt{\mathfrak{a}}$ for all $i = 0, \dots, n$.

Assume that if there exist $i \in \{0, \dots, n\}$ such that $x_i^d \notin \mathfrak{a}$ for all $d \in \mathbb{N} \setminus \{0\}$, then $x_i \notin \sqrt{\mathfrak{a}}$ (contradiction). Therefore, there are $d_i \in \mathbb{N} \setminus \{0\}$ such that $x_i^{d_i} \in \mathfrak{a}$ for all $i = 0, \dots, n$. We get $t = \max\{d_i | i = 0, \dots, n\}$, then $x_i^t \in \mathfrak{a}$ for all $i = 0, \dots, n$.

We consider $S_{t(n+1)}$, for each $x_0^{\alpha_0} \dots x_n^{\alpha_n}$ is in $S_{t(n+1)}$, then

$$\alpha_0 + \alpha_1 + \dots + \alpha_n = t(n+1),$$

where $\alpha_i \in \mathbb{N}$ for all $i = 0, \dots, n$.

If $\alpha_i < t$ for all $i = 0, \dots, n$, then

$$\alpha_0 + \alpha_1 + \dots + \alpha_n < t(n+1) \text{ (contradiction).}$$

Therefore, there exist i such that $x_0^{\alpha_0} \dots x_n^{\alpha_n} = x_i^t \cdot f$ for some $f \in S$, then $x_0^{\alpha_0} \dots x_n^{\alpha_n} \in \mathfrak{a}$, hence $S_{t(n+1)} \subseteq \mathfrak{a}$.

iii) \Rightarrow i)

Suppose that $Z(\mathfrak{a}) \neq \emptyset$, let $P = (p_0 : \dots : p_n) \in Z(\mathfrak{a}) \subseteq \mathbb{P}^n$, then $P' = (p_0, \dots, p_n) \in Z(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}$ and $P' \neq 0$.

One has $\mathfrak{a} \supseteq S_d$ for some $d > 0$, then $f(P') = 0$ for all $f \in S_d$. Since $P' \neq 0$, there is $p_i \neq 0$ for some i . Consider the polynomial x_i^d is in S_d , then

$$x_i^d(P') = p_i^d \neq 0 \text{ (contradiction).}$$

Therefore, $Z(\mathfrak{a}) = \emptyset$.

Exercise 2.3

- a) If $T_1 \subseteq T_2$ are subsets of S^h , then $Z(T_1) \supseteq Z(T_2)$.
- b) If $Y_1 \subseteq Y_2$ are subsets of \mathbb{P}^n , then $I(Y_1) \supseteq I(Y_2)$.
- c) For any two subsets Y_1, Y_2 of \mathbb{P}^n , $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
- d) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
- e) For any subset $Y \subseteq \mathbb{P}^n$, $Z(I(Y)) = \overline{Y}$.

Solution:

a) Let $P \in Z(T_2)$, so we have $f(P) = 0$ for all $f \in T_2$. Hence $f(P) = 0$ for all $f \in T_1 \subseteq T_2$, therefore, $P \in Z(T_1)$. Thus $Z(T_1) \supseteq Z(T_2)$.

b) Let $f \in I(Y_2)$, then $f(P) = 0$ for all $P \in Y_2$, so $f(P) = 0$ for all $P \in Y_1$, so $f \in I(Y_1)$. Thus $I(Y_1) \supseteq I(Y_2)$.

c) Let $f \in I(Y_1 \cup Y_2)$, thus $f(P) = 0$ for all $P \in Y_1 \cup Y_2$, then $f \in I(Y_1)$ and $f \in I(Y_2)$.

Conversely, let $f \in I(Y_1) \cap I(Y_2)$, then f vanishes on Y_1 and then f vanishes on Y_2 , so then f vanishes on $Y_1 \cup Y_2$, therefore $f \in I(Y_1 \cup Y_2)$.

Thus $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

d) $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$.

Let $f \in \sqrt{\mathfrak{a}}$, there is $n > 0$ such that $f^n \in \mathfrak{a}$, then $f^n(P) = 0$ for all $P \in Z(\mathfrak{a})$, so $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ hence $f \in I(Z(\mathfrak{a}))$.

Conversely, if f is a homogeneous polynomial and $\deg f > 0$, such that $f \in I(Z(\mathfrak{a}))$, then Exercise 2.1 shows that $f \in \sqrt{\mathfrak{a}}$. If $f = 0$ then $f \in I(Z(\mathfrak{a}))$ and $f \in \sqrt{\mathfrak{a}}$. If $f = \alpha \in I(Z(\mathfrak{a}))$ for some $\alpha \in K \setminus \{0\}$ then $Z(\mathfrak{a}) = \emptyset$ (contradiction).

Thus $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

e) Let Y be a subset of \mathbb{P}^n , prove that $Z(I(Y)) = \overline{Y}$.

\overline{Y} is the closure of Y in \mathbb{P}^n , this is an algebraic set smallest containing Y . Since $Z(I(Y))$ is an algebraic set containing Y , one has $\overline{Y} \subseteq Z(I(Y))$.

On the other hand, suppose that $\overline{P} = Z(T)$ for some $T \subseteq S^h$. So $Y \subseteq Z(T)$, so by b) we have $I(Y) \supseteq I(Z(T))$, and by a) we have $Z(I(Y)) \subseteq Z(T)$.

Thus $Z(I(Y)) = \overline{Y}$.

Exercise 2.4

- a) *There is a 1-1 inclusion reversing correspondence between algebraic set in \mathbb{P}^n , and homogeneous radical ideal of S not equal to S_+ , given by $Y \rightarrow I(Y)$ and $\mathfrak{a} \rightarrow Z(\mathfrak{a})$.*
- b) *An algebraic set $Y \subseteq \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.*
- c) *Show that \mathbb{P}^n itself is irreducible.*

Solution:

a) Let A be the set of algebraic set in \mathbb{P}^n , and let B be the set of homogeneous radical ideal of S not equal to S_+ . Let ϕ and φ be the maps

$$\begin{aligned}\phi : A &\rightarrow B \\ Y &\mapsto I(Y),\end{aligned}$$

$$\begin{aligned}\varphi : B &\rightarrow A \\ \mathfrak{a} &\mapsto Z(\mathfrak{a}).\end{aligned}$$

For any $Y \in A$, then $\overline{Y} = Y$, we have $(\varphi \circ \phi)(Y) = Z(I(Y)) = \overline{Y} = Y$, thus $\varphi \circ \phi = Id_A$.

For any $\mathfrak{a} \in B$, so $\sqrt{\mathfrak{a}} = \mathfrak{a}$, if $\mathfrak{a} \neq S$ then $Z(\mathfrak{a}) \neq \emptyset$, so $(\phi \circ \varphi)(\mathfrak{a}) = I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$, and if $\mathfrak{a} = S$ then $Z(\mathfrak{a}) = \emptyset$, we have $(\phi \circ \varphi)(S) = \phi(Z(S)) = \phi(\emptyset) = S$, thus $\phi \circ \varphi = Id_B$.

Thus ϕ is a surjective and $\phi^{-1} = \varphi$. So we obtain the result.

b) Let $Y \subseteq \mathbb{P}^n$ be an algebraic set.

Assume that Y is irreducible. Let f, g be in S^h such that $fg \in I(Y)$, so $Z(fg) \supseteq Y$, then Y is a subset of $Z(f) \cup Z(g)$. We can decompose Y as

$$Y = (Z(f) \cap Y) \cup (Z(g) \cap Y).$$

We have $Z(f) \cap Y$ and $Z(g) \cap Y$ are closed subsets of \mathbb{P}^n , then $Y = Z(f) \cap Y$ or $Y = Z(g) \cap Y$. If $Y = Z(f) \cap Y$ then $Y \subseteq Z(f)$, thus $f \in I(Y)$. If $Y = Z(g) \cap Y$ then $g \in I(Y)$. Hence $I(Y)$ is a prime ideal.

Conversely, suppose that $I(Y)$ is a prime ideal and $Y = Y_1 \cup Y_2$ with Y_1, Y_2 are closed subsets in \mathbb{P}^n . We have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2) = I(Y)$, then $I(Y_1) \subseteq I(Y)$ or $I(Y_2) \subseteq I(Y)$, thus $Y_1 \supseteq Y$ or $Y_2 \supseteq Y$, so $Y_1 = Y$ or $Y_2 = Y$, hence $Y \subseteq \mathbb{P}^n$ is irreducible.

c) We have $\mathbb{P}^n = Z(0)$, since 0 is the prime ideal, thus \mathbb{P}^n is irreducible.

Exercise 2.5

- a) \mathbb{P}^n is a noetherian topological space.
- b) Every algebraic set $Y \in \mathbb{P}^n$ can be written uniquely as a union of irreducible algebraic set, no one containing another.

Solution:

a) \mathbb{P}^n is a topological space, a closed set in \mathbb{P}^n is a algebraic set. Let $\{Y_i\}_{i \in \mathbb{N}}$ be a descending chain of closed subsets,

$$Y_0 \supseteq Y_1 \supseteq \dots,$$

then we have a ascending chain $\{I(Y_i)\}_{i \in \mathbb{N}}$ of ideals in S . Since S is noetherian ring, then there exist m such that $I(Y_m) = I(Y_{m+k})$ for all $k \in \mathbb{N}$, therefore, $Y_m = Y_{m+k}$ for all $k \in \mathbb{N}$, thus the descending chain $\{Y_i\}_{i \in \mathbb{N}}$ is stationary. Hence \mathbb{P}^n is a noetherian topological space.

b) See proposition 1.5

Exercise 2.6 If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$.

Exercise 2.7

- a) $\dim \mathbb{P}^n = n$.
- b) If $Y \subseteq \mathbb{P}^n$ is a quasi-projective variety, then $\dim \bar{Y} = \dim Y$.

Solution:

a) Since $I(\mathbb{P}^n) = 0$, then $S(\mathbb{P}^n) = S$, by Exercise 2.6, we have

$$n + 1 = \dim S = \dim S(\mathbb{P}^n) = \dim \mathbb{P}^n + 1,$$

thus $\dim \mathbb{P}^n = n$.

b)

Exercise 2.8 A projective variety $Y \subseteq \mathbb{P}^n$ has dimension $n - 1$ if and only if it is the zero set of single irreducible homogeneous polynomial f of positive degree. Y is called a **hypersurface** in \mathbb{P}^n .

Solution:

Let f be a single irreducible homogeneous polynomial f of positive degree. Then (f) is a prime ideal and $\text{height}(f) = 1$. Get $Y = Z(f)$, we have the relation as follows

$$\dim S(Y) + \text{height}(f) = \dim S = n + 1,$$

hence $\dim S(Y) = n$. We have,

$$\dim S(Y) = \dim Y + 1,$$

then $\dim Y = n - 1$.

Conversely, assume that projective variety $Y \subseteq \mathbb{P}^n$ has dimension $n - 1$.

Exercise 2.10 Let $Y \subseteq \mathbb{P}^n$ be a nonempty algebraic set, and let $\theta : \mathbb{A}^{n+1} - \{0\} \rightarrow \mathbb{P}^n$ be the map which sends the point with affine coordinate (a_0, \dots, a_n) to the point with homogeneous coordinates (a_0, \dots, a_n) . We define affine cone over Y to be

$$C(Y) = \theta^{-1}(Y) \cup (0, \dots, 0)$$

- a) Show that $C(Y)$ is an algebraic set in \mathbb{A}^{n+1} , whose ideal is equal to $I(Y)$.
- b) $C(Y)$ is irreducible if and only if Y is irreducible.
- c) $\dim C(Y) = \dim Y + 1$.

Solution:

a) Prove that $C(Y) = Z(I(Y))$. We have $I(Y)$ is a homogeneous radical ideal of S . Then there are f_1, \dots, f_k the homogeneous polynomials such that

$$I(Y) = \langle f_1, \dots, f_k \rangle.$$

If $P = 0$ then $f(0) = 0$ for all $f \in C(Y)$. If $P \neq 0$ and $P \in C(Y)$, then $P \in Y$, so we have $f(P) = 0$ for all f_1, \dots, f_k . Hence we have $C(Y) \subseteq Z(I(Y))$.

Conversely, let $P \neq 0$ and $P \in Z(I(Y))$, then $f_1(P) = \dots = f_k(P) = 0$, then $P \in Y$, and $P \in \theta^{-1}(Y)$, thus $P \in C(Y)$.

Therefore $C(Y) = Z(I(Y))$.

b) $C(Y)$ is irreducible if and only if $I(Y)$ is a prime ideal, if and only if Y is irreducible.

Morphism

Exercise 3.13 *The local ring of a subvariety.*

Let X be a variety of \mathbb{A}^n , and let $Y \subseteq X$ be a subvariety. Let $\mathcal{O}_{Y,X}$ be the set of equivalence class $\langle U, f \rangle$ such that $U \subseteq X$ is open, $U \cap Y \neq \emptyset$, and f is a regular function on U . We say that $\langle U, f \rangle$ is equivalent to $\langle V, g \rangle$ if $f = g$ on $U \cap V$.

Consider ideal

$$\mathfrak{m} = \{ \langle U, f \rangle \in \mathcal{O}_{Y,X} : f(P) = 0, \forall P \in U \cap Y \}.$$

Let $\langle U, f \rangle \in \mathcal{O}_{Y,X} \setminus \mathfrak{m}$, then $f(P) \neq 0$ for some $P \in U \cap Y$, so $1/f$ is regular in some open neighborhood V_P of P , therefore,

$$\langle V_P, 1/f \rangle = \langle V_P, f \rangle^{-1} = \langle U, f \rangle^{-1},$$

thus \mathfrak{m} is a maximal ideal and $\mathcal{O}_{Y,X}$ is a local ring.

If Y is a point P , we get \mathcal{O}_P . If $Y = X$ we get $K(X)$ the rational function on X .

We have

$$\begin{aligned} \mathcal{O}_{Y,X}/\mathfrak{m} &= \{ \langle U, f \rangle + \mathfrak{m} : \langle U, f \rangle \in \mathcal{O}_{Y,X} \} \text{ and} \\ K(Y) &= \{ \langle U, f \rangle : f \text{ is regular on open set } U \subseteq Y \}. \end{aligned}$$

Since U is open in X , $U \cap Y$ is open in Y . Consider the map

$$\begin{aligned} \pi : \mathcal{O}_{Y,X} &\rightarrow K(Y) \\ \langle U, f \rangle &\mapsto \langle U \cap Y, f \rangle, \end{aligned}$$

π is a ring homomorphism.

Hence $\pi(\mathfrak{m}) = 0$, and if $\pi(\langle U, f \rangle) = \langle U \cap Y, f \rangle = 0$, then $f(U \cap Y) = 0$, so $\langle U, f \rangle \in \text{Ker}(\pi)$. Thus $\text{Ker}(\pi) = \mathfrak{m}$.

For any $\langle V, f \rangle \in K(Y)$, let U be a open set of X such that $U \subseteq V$, so $U \cap V = U$ is a open subset of V . We have $\langle U, f \rangle = \langle V, f \rangle$, then π is surjective. so we have a ring isomorphism

$$\mathcal{O}_{Y,X}/\mathfrak{m} \cong K(Y)$$

Consider ideal

$$\mathfrak{a} = \{ f \in A(X) : f(P) = 0, \forall P \in Y \}.$$

Lemma: *The local ring $A(X)_{\mathfrak{a}}$ is isomorphic to $\mathcal{O}_{Y,X}$.*

The ideals of $A(X)_{\mathfrak{a}}$ correspond to the ideals of $A(X)$ contained in \mathfrak{a} , then $\dim(A(X)_{\mathfrak{a}}) = \text{height}(\mathfrak{a})$, so $\dim(\mathcal{O}_{Y,X}) = \text{height}(\mathfrak{a})$.

By 1.8A, we have

$$\text{height}(\mathfrak{a}) + \dim(A(X)/\mathfrak{a}) = \dim(A(X)).$$

We also have $A(X)/\mathfrak{a} \cong A(Y)$ and $\dim(A(X)) = \dim X, \dim(A(Y)) = \dim Y$ so we have a result

$$\dim(\mathcal{O}_{Y,X}) = \text{height}(\mathfrak{a}) = \dim(A(X)) - \dim(A(Y)).$$

Exercise 3.14 *Projection from a Point.*

Let

$$\mathbb{P}^n = \{(x_0 : \cdots : x_{n+1}) \in \mathbb{P}^{n+1} : x_{n+1} = 0\},$$

and let $P \in \mathbb{P}^{n+1} - \mathbb{P}^n$, this mean that

$$P = (p_0 : \cdots : p_n : 1).$$

Let $Q = (q_0 : \cdots : q_{n+1}) \in \mathbb{P}^{n+1} - P$, for $q_{n+1} = 0$ or $q_{n+1} = 1$, the intersection I of the line containing P and Q with \mathbb{P}^n is

$$I = (q_0 - p_0 : \cdots : q_n - p_n : 0) \text{ if } q_{n+1} = 1 \text{ and } I = Q \text{ if } q_{n+1} = 0,$$

therefore, the mapping φ can be written by

$$\varphi : \mathbb{P}^{n+1} - \{P\} \rightarrow \mathbb{P}^n$$

$$Q = (q_0 : \cdots : q_{n+1}) \mapsto Q \text{ if } Q \in \mathbb{P}^n$$

$$Q = (q_0 : \cdots : q_{n+1}) \mapsto (q_0 - p_0 q_{n+1} : \cdots : q_n - p_n q_{n+1} : 0) \text{ if } Q \notin \mathbb{P}^n$$

1. φ is a continuous map;
2. for every open subset $V \subseteq Y$, and for every regular function $f : V \rightarrow k$, the function

$$f \circ \varphi : \varphi^{-1}(V) \rightarrow k$$

$$Q \mapsto (f \circ \varphi)(Q) = f(\varphi(Q)),$$

we need to show that $\varphi^{-1}(V) \neq \emptyset$ if $V \neq \emptyset$. For every $I = (x_0 : \cdots : x_n : 0) \in \mathbb{P}^n$, we can choose $Q = I$, so $\varphi(Q) = I$, then φ is surjective, thus $\varphi^{-1}(V) \neq \emptyset$ if $V \neq \emptyset$.

If $S \in \varphi^{-1}(V) \setminus \mathbb{P}^n$, then $\varphi(S) \in V$ and let $V_{\varphi(S)}$ be an open neighborhood of $\varphi(S)$ in V such that $f = g/h$ on $V_{\varphi(S)}$, where homogeneous polynomials $h, g \in K[x_0, \dots, x_n]$ the same degree, $h(x) \neq 0$ for every $x \in V_{\varphi(S)}$. Then

$$f \circ \varphi = g \circ \varphi / h \circ \varphi$$

on an open subset $U_S = \varphi^{-1}(V_{\varphi(S)}) \cap \{x_{n+1} \neq 0\}$. Since

$$(g \circ \varphi)(S) = g(s_0 - p_0 s_{n+1}, \dots, s_n - p_n s_{n+1}) \in K[x_0, \dots, x_n, x_{n+1}],$$

then we have $g \circ \varphi, h \circ \varphi$ on U_S is the homogeneous polynomials the same degree, so $f \circ \varphi$ is regular at S .

If $S \in \varphi^{-1}(V) \cap \mathbb{P}^n$, let U_S be an open neighborhood in \mathbb{P}^n , then $\varphi(U_S) = U_S$, therefore $g \circ \varphi = g, h \circ \varphi = h$, so $f \circ \varphi$ is regular at S .

Thus $\varphi^{-1}(V)$ is regular. Hence φ is a morphism.

3.15 Products of Affine Varieties. Let $X \in \mathbb{A}^n$ and let $Y \in \mathbb{A}^m$ be affine varieties.

(a) Show that $X \times Y \subseteq \mathbb{A}^{m+n}$ with its induced topology is irreducible.

Firstly, we proof that $X \times Y$ is an affine algebra set in \mathbb{A}^{m+n} .

Suppose that

$$\begin{aligned} X &= Z(f_1, \dots, f_s) \\ Y &= Z(g_1, \dots, g_t), \end{aligned}$$

where $f_i \in K[x_1, \dots, x_n] \subset K[x_1, \dots, x_n, y_1, \dots, y_m]$ for all $i = 1, \dots, s$ and $g_i \in K[y_1, \dots, y_m] \subset K[x_1, \dots, x_n, y_1, \dots, y_m]$ for all $i = 1, \dots, t$.

Then

$$X \times Y = Z(f_1, \dots, f_s, g_1, \dots, g_t).$$

Indeed, Let $(a_1, \dots, a_n, b_1, \dots, b_m) \in X \times Y$, if and only if $(a_1, \dots, a_n) \in X$ and $(b_1, \dots, b_m) \in Y$, if and only if

$$\begin{aligned} f_i(a_1, \dots, a_n, b_1, \dots, b_m) &= 0, \forall i = 1, \dots, s, \\ g_i(a_1, \dots, a_n, b_1, \dots, b_m) &= 0, \forall i = 1, \dots, t, \end{aligned}$$

if and only if $(a_1, \dots, a_n, b_1, \dots, b_m) \in Z(f_1, \dots, f_s, g_1, \dots, g_t)$.

Secondly, show that $X \times Y$ is irreducible.

The second projective map

$$\begin{aligned} \pi_2 : X \times Y &\rightarrow Y \\ (x, y) &\mapsto y. \end{aligned}$$

Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$.

Fix $y \in Y$, then the map

$$\begin{aligned} \varphi : X &\rightarrow X \times Y \\ x &\mapsto (x, y) \end{aligned}$$

is continuous, the subset

$$X_y^i = \{x \in X : (x, y) \in Z_i\} = \varphi^{-1}(X \times \{y\} \cap Z_i)$$

is closed in X because $X \times \{y\} \cap Z_i$ is closed in $X \times Y$, for $i = 1, 2$. Let

$$X_i = \{x \in X : x \times Y \subseteq Z_i\}, \text{ for } i = 1, 2.$$

Now, we proof that $X_i = \bigcap_{y \in Y} X_y^i$, for $i = 1, 2$. Let $a \in X_i$, then $a \times Y \subseteq Z_i$, so $(a, y) \in Z_i$ for all $y \in Y$, hence for every $y \in Y$, $a \in X_y^i = \{x \in X : (x, y) \in Z_i\}$, thus $a \in \bigcap_{y \in Y} X_y^i$. Conversely, let $a \in \bigcap_{y \in Y} X_y^i$, then $a \in X_y^i$ for all $y \in Y$, so $(a, y) \in Z_i$ for all $y \in Y$, then $a \times Y \subseteq Z_i$, thus $a \in X_i$. Since closed sets are stable under arbitrary intersections, it shows that X_i is closed in X for $i = 1, 2$.

Now, we proof that $X = X_1 \cup X_2$. We can see that $X_1 \cup X_2 \subseteq X$ is easy. Conversely, let $x \in X$, let $Y_x = x \times Y$, we have $\pi_2(Y_x) = Y$, so Y_x is homeomorphic to Y via the map $\pi_2|_{Y_x}$. Since Y is irreducible, Y_x is closed and irreducible. We have

$$Y_x = (Z_1 \cap Y_x) \cup (Z_2 \cap Y_x),$$

therefore, $Z_1 \cap Y_x = Y_x$ or $Z_2 \cap Y_x = Y_x$, hence $Y_x \subseteq Z_1$ or $Y_x \subseteq Z_2$, it shows that $x \in X_1$ or $x \in X_2$, thus $x \in X_1 \cup X_2$.

With the results $X = X_1 \cup X_2$, X_i are close in X , we have $X = X_1$ or $X = X_2$. If $X_1 = X$, since $X \times Y = \bigcup_{x \in X} (x \times Y) = \bigcup_{x \in X_1} (x \times Y) \subseteq Z_1$, it shows that $Z_1 = X \times Y$. In a similar way, if $X = X_2$, then $Z_2 = X \times Y$. Thus $X \times Y$ is irreducible.

- (b) Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$

Consider the bilinear map

$$\begin{aligned} \varphi : A(X) \times A(Y) &\rightarrow A(X) \otimes_k A(Y) \\ (f, g) &\mapsto f \otimes g \end{aligned}$$

and the map

$$\begin{aligned} \theta : A(X) \times A(Y) &\rightarrow A(X \times Y) \\ (f, g) &\mapsto f \cdot g, \end{aligned}$$

for every $\alpha, \beta \in k, f_1, f_2 \in A(X), g \in A(Y)$, we have $\theta(\alpha f_1 + \beta f_2, g) = (\alpha f_1 + \beta f_2)g = \alpha f_1 \cdot g + \beta f_2 \cdot g = \alpha \theta(f_1, g) + \beta \theta(f_2, g)$, then θ is a bilinear

map, so by Universal property of the tensor products, we have the linear map ϕ such that $\theta = \phi \circ \varphi$, and

$$\begin{aligned}\phi : A(X) \otimes_k A(Y) &\rightarrow A(X \times Y) \\ f \otimes g &\rightarrow f.g.\end{aligned}$$

We have $\theta(\overline{x_i}, \overline{y_j}) = \overline{x_i}.\overline{y_j}$ for all $\overline{x_i} \in A(X)$ and $\overline{y_j} \in A(Y)$, hence, for $\in A(X \times Y)$, so $\prod_i \overline{x_i}^{t_i} \prod_j \overline{y_j}^{t_j} = \theta(\prod_i \overline{x_i}^{t_i}, \prod_j \overline{y_j}^{t_j})$, hence θ is surjective, then ϕ is surjective.

ϕ a a ring homomorphism.

Let $\sum_{i=1}^t f_i \otimes g_i \in \ker \phi, t \geq 1$, suppose that $g_t \neq 0$ so there is $b \in Y$ such that $g(b) \neq 0$, we have $\sum_{i=1}^t f_i g_i = 0$, for every $x \in X$ and if $t \geq 2$, we have

$$\sum_{i=1}^t (f_i g_i)(x, b) = \sum_{i=1}^t f_i(x) g_i(b) = \sum_{i=1}^{t-1} f_i(x) g_i(b) + f_t(x) g_t(b) = 0,$$

this shows that

$$f_t(x) = -g_t(b)^{-1} \sum_{i=1}^{t-1} f_i(x) g_i(b) = - \sum_{i=1}^{t-1} g_t(b)^{-1} g_i(b) f_i(x),$$

hence

$$\begin{aligned}\sum_{i=1}^t f_i \otimes g_i &= \sum_{i=1}^{t-1} f_i \otimes g_i + f_t \otimes g_t = \sum_{i=1}^{t-1} f_i \otimes g_i + \left(- \sum_{i=1}^{t-1} g_t(b)^{-1} g_i(b) f_i\right) \otimes g_t \\ &= \sum_{i=1}^{t-1} f_i \otimes (g_i - g_t(b)^{-1} g_i(b) g_t)\end{aligned}$$

we replace $g'_i = g_i - g_t(b)^{-1} g_i(b) g_t$, so

$$\sum_{i=1}^t f_i \otimes g_i = \sum_{i=1}^{t-1} f_i \otimes g'_i,$$

we calculate similar with $\sum_{i=1}^{t-1} f_i \otimes g'_i$, finally, we will have $f \in A(X), g \in$

$A(Y)$ such that $\sum_{i=1}^t f_i \otimes g_i = f \otimes g$, if $g \neq 0$, there is $b \in Y$ such that $g(b) \neq 0$, for every $x \in X$, we have $f g(x, b) = f(x).g(b) = 0$, this shows that $f = 0$, hence $f \otimes g = 0 \otimes g = 0$, thus ϕ is injective.

Consequence, ϕ is a ring isomorphism and $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

3.20 Group varieties. A group variety consists of a variety Y together with a morphism $\mu : Y \times Y \rightarrow Y$ such that the set of the point Y with the operation given by μ is a group, and such that the inverse map $y \rightarrow y^{-1}$ is also a morphism of $Y \rightarrow Y$.

- (a) The *additive group* \mathbb{G}_a is given by the varieties \mathbb{A}^1 and the morphism $\mu : \mathbb{A}^2 \rightarrow \mathbb{A}^1$ defined by $\mu(a, b) = a + b$. Show it is a group variety.

We need to verify the maps

$$\begin{aligned}\mu : \mathbb{A}^1 \times \mathbb{A}^1 &\rightarrow \mathbb{A}^1 \\ (a, b) &\mapsto a + b,\end{aligned}$$

$$\begin{aligned}i : \mathbb{A}^1 &\rightarrow \mathbb{A}^1 \\ x &\mapsto -x\end{aligned}$$

are morphism. We can see that μ and i are the morphisms because they are defined by the polynomials.

We have $i(x) = x$ if and only if $x = 0$, so the identity element is 0, addition on \mathbb{A}^1 is the addition on field k , then \mathbb{G}_a is a group variety.

- (b) The *multiplicative group* \mathbb{G}_m is given by variety $\mathbb{A}^1 - \{0\}$ and the morphism $\mu(a, b) = a.b$, show it is a group variety.

* $\mathbb{A}^1 - \{0\} \cong \{(x, y) \in \mathbb{A}^2 : xy - 1 = 0\}$. This show that $\mathbb{A}^1 - \{0\}$ is a variety of \mathbb{A}^2 .

* $(\mathbb{A}^1 - \{0\}, .)$ is a group, the identity element is 1, the inverse of x is $\frac{1}{x}$.

* The maps μ and

$$\begin{aligned}i : \mathbb{A}^1 - \{0\} &\rightarrow \mathbb{A}^1 - \{0\} \\ x &\mapsto \frac{1}{x}\end{aligned}$$

are the morphisms.

Thus we have \mathbb{G}_m is a group variety.

- (c) If G is a group variety, and X is any variety, show that the set $Hom(X, G)$ has a natural group structure.

Suppose that G is a group variety defined by the operation μ , then μ

is a morphism. For $f, g \in \text{Hom}(X, G)$, consider the morphism $\nu \in \text{Hom}(X, G)$ constructs by

$$\begin{aligned}\nu : X &\rightarrow X \times X \rightarrow G \times G \rightarrow G \\ x &\mapsto (x, x) \mapsto (f(x), g(x)) \mapsto \mu(f(x), g(x)),\end{aligned}$$

suppose that e is the identity element of G , then the map $f(x) = e$ for all x is the identity element of $\text{Hom}(X, G)$. So $\text{Hom}(X, G)$ is a group is defined by the operation

$$\begin{aligned}\text{Hom}(X, G) \times \text{Hom}(X, G) &\rightarrow \text{Hom}(X, G) \\ (f, g) &\mapsto \mu(f, g), \mu(f, g)(x) = \mu(f(x), g(x)).\end{aligned}$$

Rational maps

Exercise 4.1. Let f and g be the regular functions on U and V , $f = g$ on $U \cap V$, the function h is defined by

$$h(P) = \begin{cases} f(P) & \text{if } P \in U \\ g(P) & \text{if } P \in V, \end{cases}$$

for every points $P \in U \cup V$, then $P \in U$ or $P \in V$, if $P \in U$, then $f(P) = h(P)$, since f is regular at P , so we have that h is regular at P . Similarly, if $P \in V$, then h is regular at P . Thus h is regular on $U \cup V$.

If f is a rational function on X , there is an open set U of X such that f is regular on U . Consider the set

$$\mathcal{T} = \{U : U \text{ is open on } X, f \text{ is regular on } U\}.$$

The set $\bigcup_{U \in \mathcal{T}} U$ is open in X , this set is the largest open set of X on which f is regular. We say that f is defined at the points of this set.

Exercise 4.3. Let f be the rational function on \mathbb{P}^2 given by $f = x_1/x_0$.

a) f is regular on $U_0 = \{(x_0 : x_1 : x_2) : x_0 \neq 0\}$, f is defined on the points of U_0 . Suppose that we have an open set U such that f is regular on U and $U \supsetneq U_0$, then there is $P = (p_0 : p_1 : p_2) \in U \setminus U_0$, so $p_0 = 0$, this shows that f is not regular at P (contradiction), hence U_0 is the largest open set on which f is regular. The corresponding regular function on U_0 is x_1/x_0 , we can replace x_0 by 1.

b) Consider the embedding

$$\begin{aligned} \tau : \mathbb{A}^1 &\hookrightarrow \mathbb{P}^1 \\ x &\mapsto (x : 1). \end{aligned}$$

See f as a rational map from \mathbb{P}^2 to \mathbb{A}^1 , we have the resulting map

$$\begin{aligned} \varphi : \mathbb{P}^2 &\rightarrow \mathbb{P}^1 \\ (x_0 : x_1 : x_2) &\mapsto (x_1/x_0 : 1). \end{aligned}$$

therefore, $\varphi = \tau \circ f$. Since $f|_U$ is a morphism, then $\varphi|_{U_0} = \tau \circ f|_{U_0}$ is a morphism. Describe $\varphi|_{U_0}$,

$$\begin{aligned} \varphi|_{U_0} : U_0 &\rightarrow \mathbb{P}^1 \\ (x_0 : x_1 : x_2) &\mapsto (x_1/x_0 : 1) = (x_1 : x_0). \end{aligned}$$

Exercise 4.4. *Rational varieties.* A variety Y is rational if it is birationally equivalent to \mathbb{P}^n for some n (or if $K(Y)$ is a pure transcendental extension of k)

a) Any *conic* in \mathbb{P}^2 is a rational curve.

Exercise 3.1-(a) shows that any conic in \mathbb{P}^2 is isomorphic to \mathbb{P}^1 . Since if f is a morphism then f is a rational map. Then we have that the conics in \mathbb{P}^2 and \mathbb{P}^1 are birationally equivalent, thus the conics in \mathbb{P}^2 are rational.

b) The *cuspidal cubic* $y^2 - x^3$ is a rational curve.

Let $V = \{(x, y) \in \mathbb{A}^2 : x^3 - y^2 = 0\} = \{(t^2, t^3) \in \mathbb{A}^2 : t \in \mathbb{A}^1\}$. Consider the map

$$\begin{aligned}\varphi : \mathbb{A}^1 &\rightarrow V \\ t &\mapsto (t^2, t^3),\end{aligned}$$

then the inverse of φ is

$$\begin{aligned}\varphi^{-1} : V &\rightarrow \mathbb{A}^1 \\ (x, y) &\mapsto \begin{cases} 0 & \text{if } x = 0 \\ y/x & \text{if } x \neq 0. \end{cases}\end{aligned}$$

In deed, if $x \neq 0, (x, y) \in V$, we have

$$\varphi \circ \varphi^{-1}(x, y) = \varphi(y/x) = (y^2/x^2, y^3/x^3) = (x^3/x^2, y^3/y^2) = (x, y),$$

and for every $t \in \mathbb{A}^1$, $\varphi^{-1} \circ \varphi(t) = t$, then φ is a birational map. Thus the cuspidal cubic is rational.

c) Projection φ from the point $P = (0 : 0 : 1)$ to the line $z = 0$.

For every $Q \in \mathbb{P}^2$, we can write $Q = (q_0 : q_1 : 0)$ or $Q = (q_0 : q_1 : 1)$, the projection can be written as follows

$$\begin{aligned}\varphi : \mathbb{P}^2 - P &\rightarrow \mathbb{P}^1 \\ (q_0 : q_1 : 0) &\mapsto (q_0 : q_1), \\ (q_0 : q_1 : 1) &\mapsto (q_0 : q_1).\end{aligned}$$

The curve $(C) : y^2z = x^2(x + z) \subseteq \mathbb{P}^2$, suppose that $Q = (q_0 : q_1 : q_2) \in (C)$, if $q_2 \neq 0$, then we can write again $Q = (q_0 : q_1 : 1)$. If $q_2 = 0$, then $q_1 = 0, q_0 = 1$. Thus

$$\begin{aligned}\theta = \varphi|_{(C)} : (C) &\rightarrow \mathbb{P}^1 \\ (q_0 : q_1 : 1) &\mapsto (q_0 : q_1), \\ (0 : 1 : 0) &\mapsto (0 : 1).\end{aligned}$$

The invert of θ is

$$\begin{aligned}\theta^{-1} : \mathbb{P}^1 &\rightarrow (C) \\ (q_0 : q_1) &\mapsto\end{aligned}$$

if $z = 1$ then $y^2 = x^3 + x^2$, so $x^3 = y^2 - x^2$

Exercise 4.5 The quadratic surface $Q : xy = zw$ in \mathbb{P}^3 is birational to \mathbb{P}^2 , but not isomorphic to \mathbb{P}^2 .

Solution: We consider the map

$$\begin{aligned}\varphi : Q &\rightarrow \mathbb{P}^2 \\ (x : y : z : t) &\mapsto \begin{cases} (x : y : z) & \text{if } z \neq 0 \\ (x : y : t) & \text{if } z = 0. \end{cases}\end{aligned}$$

Let $U = \{(x : y : z : t) \in Q : z \neq 0\}$ be an open subset of Q , since $\varphi|_U$ is a morphism of U to \mathbb{P}^2 , then φ is a rational map.

We consider the map

$$\begin{aligned}\phi : \mathbb{P}^2 &\rightarrow Q \\ (x : y : z) &\mapsto \begin{cases} (x : y : z : \frac{xy}{z}) & \text{if } z \neq 0 \\ (x : 0 : y : 0) & \text{if } z = 0. \end{cases}\end{aligned}$$

Let $U' = \{(x : y : z) \in \mathbb{P}^2 : z \neq 0\}$ be an open subset of \mathbb{P}^2 , since $\phi|_{U'}$ is a morphism of U' to Q , then ϕ is a rational map.

Let $P = (x : y : z : t) \in U \subset Q$, since $z \neq 0$, it shows that $t = \frac{xy}{z}$, we have

$$(\phi \circ \varphi)(P) = \phi(x : y : z) = (x : y : z : \frac{xy}{z}) = (x : y : z : t) = P.$$

Hence $\phi \circ \varphi = id_Q$.

Let $P' = (x : y : z) \in U' \subset \mathbb{P}^2$, we have

$$(\varphi \circ \phi)(P) = \varphi(x : y : z : \frac{xy}{z}) = (x : y : z) = P'.$$

Hence $\varphi \circ \phi = id_{\mathbb{P}^2}$.

Therefore φ is a birational map, consequence that the quadratic surface $Q : xy = zw$ in \mathbb{P}^3 is birational to \mathbb{P}^2 . But it is not isomorphic to \mathbb{P}^2 , we can prove this sentence as follows.

The linear varieties

$$L_1 : \begin{cases} x = z \\ y = w \end{cases}, L_2 : \begin{cases} x = 2z \\ 2y = w \end{cases}$$

are the lines of Q , we have $L_1 \cap L_2 = \emptyset$ in \mathbb{P}^3 . But Exercise 2.11.c shows that the intersection of two lines in \mathbb{P}^2 is not empty. This shows that Q is not isomorphic to \mathbb{P}^2 .

Exercise 4.6. *Plane Cremona Transformations.* A birational map of \mathbb{P}^2 into itself is called a Plane Cremona Transformations. We give an example, called a quadratic transformation. It is the rational map $\varphi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ given by $(a_0 : a_1 : a_2) \rightarrow (a_1a_2 : a_0a_2 : a_0a_1)$ when no two of a_0, a_1, a_2 are 0.

a) Show that φ is birational map, and is its own inverse?

We consider the open set $U = \{(a_0 : a_1 : a_2) \in \mathbb{P}^2 : a_0 \neq 0, a_1 \neq 0, a_2 \neq 0\}$ of \mathbb{P}^2 . We have

$$\begin{aligned} \varphi^2(a_0 : a_1 : a_2) &= \varphi(a_1a_2 : a_0a_2 : a_0a_1) = (a_0^2a_1a_2 : a_0a_1^2a_2 : a_0a_1a_2^2) \\ &= (a_0 : a_1 : a_2) \end{aligned}$$

for any $(a_0 : a_1 : a_2) \in U$, then $\varphi^2 = id_{\mathbb{P}^2}$, thus φ is birational map, and it is its own inverse.

b) Find open set $U, V \subseteq \mathbb{P}^2$ such that $\varphi : U \rightarrow V$ is an isomorphism.

We choose

$$U = V = \{(a_0 : a_1 : a_2) \in \mathbb{P}^2 : a_0 \neq 0, a_1 \neq 0, a_2 \neq 0\}.$$

We have that φ is a morphism of U to U , the inverse is itself.

c) Find the open set where φ is defined, describe the corresponding morphisms.

φ is defined by the open set $\mathbb{P}^2 - \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}$.

Exercise 4.10. Let Y be the cuspidal curve $y^2 = x^3$ in A^2 . Blow up Y at the point $O = (0, 0)$.

Let X be the closed subset of $A^2 \times \mathbb{P}^1$ defined by $\{((x, y), (u : t)) \in A^2 \times \mathbb{P}^1 : xt = uy\}$. The map $\varphi : X \rightarrow A^2$ is the blowing-up of the point O . The exceptional curve E is $\varphi^{-1}(0) \cong \mathbb{P}^1$. We find the inverse image of Y in X by considering the equations $y^2 = x^3$ and $xt = uy$. Since \mathbb{P}^1 is covered by $u \neq 0$ and $t \neq 0$, then we consider separately.

If $u \neq 0$, then we can set $u = 1$, and use t as a parameter. Then we have the equations

$$\begin{cases} y = xt \\ y^2 = x^3 \end{cases} \Rightarrow x^2 t^2 - x^3 = 0 \Rightarrow x^2(t^2 - x) = 0.$$

We solve this equations and we have

$$\begin{cases} x = y = 0 \\ t \in \mathbb{A}^1 \end{cases} \text{ or } \begin{cases} y = xt \\ t^2 - x = 0. \end{cases}$$

Hence

$$\tilde{Y} = \{((x, y), (1 : t)) \in \mathbb{A}^2 \times \mathbb{P}^1 : xt - y, t^2 - x\}$$

We have that E meets \tilde{Y} at one point $t = 0$.

Therefore, we have the morphism $\varphi : \tilde{Y} - ((0, 0), (1 : 0)) \longrightarrow Y$, and the rational map $\varphi : \tilde{Y} \longrightarrow Y$.

We can write again \tilde{Y} as follows

$$\tilde{Y} = \{((t^2, t^3), (1 : t)) \in \mathbb{A}^2 \times \mathbb{P}^1 : t \in \mathbb{A}^1\}.$$

Hence, we consider the map

$$\begin{aligned} \tau : \tilde{Y} &\rightarrow \mathbb{A}^1 \\ ((t^2, t^3), (1 : t)) &\mapsto t. \end{aligned}$$

τ is a morphism, τ is surjective, the inverse of τ is

$$\begin{aligned} \tau^{-1} : \mathbb{A}^1 &\rightarrow \tilde{Y} \\ t &\mapsto ((t^2, t^3), (1 : t)). \end{aligned}$$

This shows that $\tilde{Y} \cong \mathbb{A}^1$.

If $t \neq 0$, we have the equations

$$\begin{cases} x = yu \\ y^2 = x^3 \end{cases} \Rightarrow y^2 - y^3 u^3 = 0 \Rightarrow y^2(1 - yu^3) = 0.$$

We solve this equations and we have

$$\begin{cases} x = y = 0 \\ u \in \mathbb{A}^1 \end{cases} \text{ or } \begin{cases} x = yu \\ yu^3 - 1 = 0. \end{cases}$$

Therefore we have

$$\tilde{Y}_u = \{((x, y), (u : 1)) \in \mathbb{A}^2 \times \mathbb{P}^1 : x - uy, yu^3 - 1\}$$

We have E doesn't meet \tilde{Y}_u .

Nonsingular Varieties

Exercise 5.1. Locate the singular points and sketch the following curve in \mathbb{A}^2 . Assume $\text{char } k \neq 2$.

- (a) $x^2 = x^4 + y^4$;
- (b) $xy = x^6 + y^6$;
- (c) $x^3 = x^2 + x^4 + y^4$;
- (d) $x^2y + xy^2 = x^4 + y^4$;

Solution:

- (a) $f(x, y) = x^2 - x^4 - y^4$, then we have

$$\begin{aligned}\frac{\partial f(x, y)}{\partial x} &= 2x - 4x^3, \\ \frac{\partial f(x, y)}{\partial y} &= -4y^3.\end{aligned}$$

So, the Jacobi matrix at (x_0, y_0) is the matrix

$$J = \begin{vmatrix} 2x_0 - 4x_0^3 & -4y_0^3 \end{vmatrix}.$$

Suppose $Y = V(f(x, y))$, then $\dim Y = 2 - 1 = 1$. So Y is nonsingular at the point (x_0, y_0) if $\text{rank } J = 1$. This shows that either $2x_0 - 4x_0^3 \neq 0$ or $-4y_0^3 \neq 0$, then either $(x \neq 0, x \neq \sqrt{\frac{1}{2}} \text{ and } x \neq -\sqrt{\frac{1}{2}})$ or $y \neq 0$.

Hence, the points can be the singular points $(0, 0), (\sqrt{\frac{1}{2}}, 0), (-\sqrt{\frac{1}{2}}, 0)$.

But $(\sqrt{\frac{1}{2}}, 0), (-\sqrt{\frac{1}{2}}, 0)$ is not belong to Y . It shows that Y has only singular point $(0, 0)$.

We fix $y_0 \in k$, then $x^2 - x^4 - y_0^4$ is an even function, then its graph remains unchanged after reflection about the y -axis. So we see Figure 4., Tacnode satisfies this condition.

- (b) $xy = x^6 + y^6$;

Let (x_0, y_0) be the singular point of $V(xy - x^6 - y^6)$, then x_0, y_0 is a solution of the equations

$$\begin{cases} xy - x^6 - y^6 = 0 \\ y - 6x^5 = 0 \\ x - 6y^5 = 0 \end{cases} \Leftrightarrow \begin{cases} xy - xy/6 - yx/6 = 0 \\ x^5 = y/6 \\ y^5 = x/6, \end{cases}$$

it shows that $xy = 0$, then we have $x = 0$ or $y = 0$, if $x = 0$ then $y = 0$, if $y = 0$ then $x = 0$, thus $x = y = 0$.

From the equation $xy = x^6 + y^6$, we have $xy \geq 0$, so the picture of Node satisfies.

(c) $x^3 = x^2 + x^4 + y^4$;

Let (x_0, y_0) be the singular point of $V(x^3 - x^2 - x^4 - y^4)$, then (x_0, y_0) is a solution of the equations

$$\begin{cases} x^3 - x^2 - x^4 - y^4 = 0 \\ 3x^2 - 2x - 4x^3 = 0 \\ -4y^3 = 0 \end{cases} \Leftrightarrow \begin{cases} x^2(x - 1 + x^2) = 0 \\ x(3x - 2 - 4x^2) = 0 \\ y = 0 \end{cases}$$

if α is a solution of $x - 1 + x^2$, then $\alpha^2 = 1 - \alpha$, if α is a solution of $3x - 2 - 4x^2$, then $\alpha^2 = (3\alpha - 2)/4$, so we have $1 - \alpha = (3\alpha - 2)/4$, thus $\alpha = 6/7$, but $6/7$ is not a solution of $x - 1 + x^2$. Hence this equations have only one solution $(0, 0)$.

The equation $x^3 = x^2 + x^4 + y^4$ shows that $x \geq 0$, this is Cusp. (I am considering the picture on $\mathbb{A}^2 \cap \mathbb{R}^2$.)

(d) $x^2y + xy^2 = x^4 + y^4$;

The picture is Triple point.

Let (x_0, y_0) be the singular point of $V(x^2y + xy^2 - x^4 - y^4)$, then (x_0, y_0) is a solution of the equations

$$\begin{cases} x^2y + xy^2 - x^4 - y^4 = 0 \\ 2xy + y^2 - 4x^3 = 0 \\ x^2 + 2xy - 4y^3 = 0 \end{cases} \Leftrightarrow \begin{cases} x^2y + xy^2 - x^4 - y^4 = 0 \\ x^3 = \frac{2xy + y^2}{4} \\ y^3 = \frac{x^2 + 2xy}{4} \end{cases}$$

then we have

$$\begin{aligned} x^2y + xy^2 - x^4 - y^4 &= 0 \\ \Leftrightarrow x^2y + xy^2 - x \frac{2xy + y^2}{4} - y \frac{x^2 + 2xy}{4} &= 0 \\ \Leftrightarrow 4x^2y + 4xy^2 - 2x^2y - xy^2 - x^2y - 2xy^2 &= 0 \\ \Leftrightarrow xy(4x + 4y - 2x - y - x - 2y) &= 0 \\ \Leftrightarrow xy(x + y) &= 0 \\ \Leftrightarrow xy = 0 \text{ or } x = -y \end{aligned}$$

So, if $xy = 0$ then $x = 0$ or $y = 0$, if $x = 0$ then $y = 0$, if $y = 0$ then $x = 0$, so $(0, 0)$ is a solution.

If $x = -y$, then we have $x^2(-x) + xx^2 - x^4 - x^4 = 0$, then $x^4 = 0$, thus $x = 0$, hence $y = -x = 0$. Thus we have only one solution $(0, 0)$.

Exercise 5.2. Locate the singular points

(a) $xy^2 = z^2$.

Let $f(x, y, z) = xy^2 - z^2$, then we have

$$\begin{aligned}\frac{\partial f}{\partial x} &= y^2, \\ \frac{\partial f}{\partial y} &= 2xy, \\ \frac{\partial f}{\partial z} &= -2z.\end{aligned}$$

So, the Jacobi matrix at (x_0, y_0, z_0) is the matrix

$$J = \begin{vmatrix} y_0^2 & 2x_0y_0 & -2z_0 \end{vmatrix}.$$

Hence, (x_0, y_0, z_0) is a singular point if $y_0^2 = 2x_0y_0 = -2z_0 = 0$, it shows that $y_0 = z_0 = 0, x_0$ free. The singular points on the x -axis. The picture of Pinch point.

(b) $x^2 + y^2 = z^2$.

Let $g(x, y, z) = x^2 + y^2 - z^2$, then we have

$$\begin{aligned}\frac{\partial g}{\partial x} &= 2x, \\ \frac{\partial g}{\partial y} &= 2y, \\ \frac{\partial g}{\partial z} &= -2z.\end{aligned}$$

(x_0, y_0, z_0) is a singular point if $2x_0 = 2y_0 = -2z_0 = 0$, then we have $x_0 = y_0 = z_0 = 0$, and $(0, 0, 0) \in V(g)$, so $V(g)$ has only one singular point $(0, 0, 0)$. The picture is Conical double point.

Exercise 5.3. For $f(x, y) \in k[x, y]$, we can write f as a sum

$$f = f_0 + f_1 + \cdots + f_d,$$

where f_i is a homogeneous polynomial of degree i in x and y . The multiplicity of $P = (0, 0)$ on Y , denote $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. If $P = (0, 0) \in Y$ then $f_0 = 0$, so $\mu_P(Y) > 0$.

- (a) Show that $\mu_P(Y) = 1 \Leftrightarrow P = (0, 0)$ is a nonsingular point of Y .
(b) Find the multiplicity of each of the singular points in Ex. 5.1 above.

Solution:

- (a) $\mu_P(Y) = 1 \Leftrightarrow f_1 \neq 0 \Leftrightarrow$ there are a, b such that $f_1 = ax + by$, $a \neq 0$ or $b \neq 0 \Leftrightarrow \frac{\partial f_1(x, y)}{\partial x} = a \neq 0$ or $\frac{\partial f_1(x, y)}{\partial y} = b \neq 0 \Leftrightarrow \frac{\partial f(x, y)}{\partial x}(0, 0) = a \neq 0$ or $\frac{\partial f(x, y)}{\partial y}(0, 0) = b \neq 0 \Leftrightarrow P = (0, 0)$ is a nonsingular point of Y .

Remark: for any $P \in Y$, we can change the coordinates such that P be comes to the point $(0, 0)$, so we have $\mu_P(Y)$, which is found by the new polynomial after changing coordinates.

- (b) Find the multiplicity of each of the singular points in Ex. 5.1 above.

All of them have only one singular point $P = (0, 0)$.

For 5.1 a, $\mu_P(V(x^2 - x^4 - y^4)) = 2$

For 5.1 b, $\mu_P(V(xy - x^6 - y^6)) = 2$

For 5.1 c, $\mu_P(V(x^3 - y^2 - x^4 - y^4)) = 2$

For 5.1 d, $\mu_P(V(x^2y + xy^2 - x^4 - y^4)) = 3$.

Exercise 5.6.

- (a) • Let Y be the cusp, then $Y = Z(x^3 - y^2 - x^4 - y^4)$, Y has only singular point $O(0, 0)$. We will blow up Y at O . Let X be the blowing up at O of \mathbb{A}^2 , then X is the closed subset of $\mathbb{A}^2 \times \mathbb{P}^1$ defined by the equation $xu = yt$, with $((x, y), (t : u)) \in \mathbb{A}^2 \times \mathbb{P}^1$. We have the inverse of Y in X considering by the equations $x^3 - y^2 - x^4 - y^4 = 0$ and $xu = yt$.

On the chart $t \neq 0$, we can set $t = 1$, then $y = xu$ and $x^2(x - u^2 - x^2 - x^2u^4) = 0$. Then $x = y = 0, u$ free, this is E , or $x - u^2 - x^2 - x^2u^4 = 0, y = xu$, this is \tilde{Y}_t . We replace $y = xu$, we have \tilde{Y}_t is defined by

$x - u^2 - x^2 - y^2u^2 = 0, y - xu = 0$. Set $f(x, y, u) = x - u^2 - x^2 - y^2u^2$, then

$$\begin{aligned}\frac{\partial f}{\partial x} &= 1 - 2x \\ \frac{\partial f}{\partial y} &= -2yu^2 \\ \frac{\partial f}{\partial u} &= -2y^2u.\end{aligned}$$

We have $\tilde{Y}_t \cap E$ at $(0, 0, 0)$. The Jacobi matrix at $(0, 0, 0)$ is

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since $\tilde{Y} - \varphi^{-1}(O)$ is isomorphic to $Y - O$, then $\dim \tilde{Y} = \dim Y = 1, \text{rank} J = 2 = 3 - 1$. So \tilde{Y} is nonsingular point at $(0, 0, 0)$.

On the chart $u \neq 0$, set $u = 1$, we have $x = yt$, the considering the equations $y^3t^3 - y^2 - y^4t^4 - y^4 = 0$ and $x = yt$. So $y^2(yt^3 - 1 - y^2t^4 - y^2) = 0$, then \tilde{Y}_u is defined by the equations $yt^3 - 1 - y^2t^4 - y^2 = 0$ and $x = yt$. We have $\tilde{Y}_u \cap E = \emptyset$.

This shows that $\varphi^{-1}(O) \cap \tilde{Y} = \{((0, 0), (1 : 0))\}$. $\tilde{Y} - \varphi^{-1}(O)$ is isomorphic to $Y - O$, then $\tilde{Y} - \varphi^{-1}(O)$ is nonsingular point, and \tilde{Y} is nonsingular point at $(0, 0, 0)$, thus \tilde{Y} is nonsingular point.

• Let C be the Node, $C = Z(xy - x^6 - y^6)$ has only singular point at $O = (0, 0)$.

On the chart $t \neq 0$, consider the equations $y = xu$ and $xy - x^6 - y^6 = 0$, then we have $x^2u - x^6 - x^6u^6 = 0$, so $x^2(u - x^4 - x^4u^6)$, thus we have \tilde{C}_t defined by the equations $u - x^4 - x^4u^6 = 0$ and $y - xu = 0$. $\tilde{C}_t \cap E$ at $(0, 0, 0) := ((0, 0), (1 : 0))$. The Jacobi matrix at $(0, 0, 0)$ is

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since $\text{rank} J = 2 = \text{codim} \tilde{C}$, then \tilde{C} is nonsingular at $((0, 0), (1 : 0))$.

On the chart $u \neq 0$, set $u = 1$, consider the equations $x = yt$ and $xy - x^6 - y^6 = 0$, then we have $y^2t - y^6 - y^6t^6 = 0$, so $y^2(t - y^4 - y^4t^6)$, thus

we have \tilde{C}_u is defined by the equations $t - y^4 - y^4 t^6 = 0$ and $x - yt = 0$. $\tilde{C}_u \cap E$ at $(0, 0, 0) := ((0, 0), (0 : 1))$. We do similar way on the chart $t \neq 0$. Consequence, \tilde{C} is nonsingular point.

- (b) $Y = Z(f)$, and let P be a node. Make a linear change of coordinates so that P be come to the point $(0, 0)$, then $f = xy + g(x, y)$, where $g(x, y)$ has only term of degree greater then 2, $a^2 + b^2 + c^2 \neq 0$.

On the chart $t \neq 0$, we can set $t = 1$, consider the equations $f(x, y) = 0$ and $y = xu$, we have

$$xy + g(x, y) = ux^2 + g(x, xu) = 0,$$

hence,

$$x^2(u + x^{-2}g(x, xu)) = 0.$$

One has $x^{-2}g(x, xu) = h(x, u)$ has only term of degree greater than 0, so we have $h(0, 0) = 0$. By this equation, we get two irreducible components, the Exceptional curve E on the chart $t \neq 0$ defined by $x = 0, y = 0, u$ is free, \tilde{Y} is defined by the equations $u + h(x, u) = 0, y - xu = 0$. Since $h(0, 0) = 0$, then $u = 0$, thus $E \cap \tilde{Y} = (0, 0, 0) =: ((0, 0), (1 : 0))$. The Jacobi matrix at $(0, 0, 0)$ is

$$J_1 = \begin{bmatrix} \frac{\partial h}{\partial x}(0, 0, 0) & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We have $\dim Y = \dim \tilde{Y} = 1$, then

$$\text{rank } J_1 = \text{codim } \tilde{Y},$$

it shows that \tilde{Y} is nonsingular at $((0, 0), (1 : 0))$.

On the chart $u \neq 0$, we can set $u = 1$, consider the equations $f(x, y) = 0$ and $x = yt$, we have

$$xy + g(x, y) = ty^2 + g(y, yt) = 0,$$

hence,

$$y^2(t + x^{-2}g'(y, yt)) = 0.$$

One has $y^{-2}g(y, yt) = h'(y, t)$ has only term of degree greater than 0, so we have $h'(0, 0) = 0$. By this equation, we get two irreducible components, the Exceptional curve E on the chart $u \neq 0$ defined by $x = 0, y = 0, t$ is free, \tilde{Y} is defined by the equations $t + h'(y, t) = 0, x - yt = 0$. Since $h'(0, 0) = 0$, then $t = 0$, thus $E \cap \tilde{Y} = (0, 0, 0) =: ((0, 0), (0 : 1))$. The Jacobi matrix at $(0, 0, 0)$ is

$$J_2 = \begin{bmatrix} \frac{\partial h'}{\partial x}(0, 0, 0) & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We have $\dim Y = \dim \tilde{Y} = 1$, then

$$\text{rank } J_1 = \text{codim } \tilde{Y},$$

it shows that \tilde{Y} is nonsingular at $((0, 0), (0 : 1))$. Thus, $E \cap \tilde{Y} = \{((0, 0), (0 : 1)), ((0, 0), (0 : 1))\}$, and they are nonsingular points.

- (c) Let $P \in Y = Z(x^2 - x^4 - y^4)$ be the tacnode, and $\varphi : \tilde{Y} \rightarrow Y$ is the blowing up at P . Show that $\varphi^{-1}(P)$ is a node.

On the chart $t \neq 0$, we can set $t = 1$, consider the equations $x^2 - x^4 - y^4 = 0$ and $y = xu$, we have

$$x^2 - x^4 - y^4 = x^2 - x^4 - x^4 u^4 = x^2(1 - x^2 - x^2 u^4) = 0,$$

By this equation, we get two irreducible components, the Exceptional curve E on the chart $t \neq 0$ defined by $x = 0, y = 0, u$ is free, \tilde{Y} is defined by the equations $1 - x^2 - x^2 u^4 = 0, y - xu = 0$. Then we have $E \cap \tilde{Y} = \emptyset$. On the chart $u \neq 0$, we can set $u = 1$, consider the equations $x^2 - x^4 - y^4 = 0$ and $x = yt$, we have

$$x^2 - x^4 - y^4 = t^2 y^2 - y^4 t^4 - y^4 = y^2(t^2 - y^2 t^4 - y^2) = 0,$$

By this equation, we get two irreducible components, the Exceptional curve E on the chart $u \neq 0$ defined by $x = 0, y = 0, t$ is free, \tilde{Y} is defined by the equations $t^2 - y^2 t^4 - y^2, x - yt = 0$. Thus $E \cap \tilde{Y} = (0, 0, 0) =: ((0, 0), (0 : 1)) = \varphi^{-1}(P)$.

Since $t^2 - y^2 t^4 - y^2 = t^2 - y^2 - y^2 t^4 = (t - y)(t + y) - y^2 t^4$, with change of $s = t - y, v = t + y$, $\varphi^{-1}(P)$ is a node. Using b), then tacnode can be resolved by two successive blowings-up.

- (d) $Y = Z(y^3 - x^5)$. Since the lowest term equal to 3, then we have a triple point.

On the chart $t \neq 0$, we can set $t = 1$, consider the equations $y^3 - x^5 = 0$ and $y = xu$. Then we have $x^3u^3 - x^5 = x^3(u^3 - x^2) = 0$. By this equation, we get two irreducible components, the Exceptional curve E on the chart $t \neq 0$ defined by $x = 0, y = 0, u$ is free, \tilde{Y} is defined by the equations $1 - x^2 - x^2u^4 = 0, y - xu = 0$. Then we have $E \cap \tilde{Y} = ((0, 0), (1, 0))$. The point $(0, 0)$ of $u^3 - x^2 = 0$ is a tacnode point. By c), this point can be resolved.

On the chart $u \neq 0$, we can set $u = 1$, consider the equations $y^3 - x^5 = 0$ and $x = yt$. Then we have $y^3 - y^5t^5 = y^3(1 - y^2t^5) = 0$. By this equation, we get two irreducible components, the Exceptional curve E on the chart $t \neq 0$ defined by $x = 0, y = 0, t$ is free, \tilde{Y} is defined by the equations $1 - y^2t^5, x - yt = 0$. Then we have $E \cap \tilde{Y} = \emptyset$.

Exercise 5.7. f is a homogeneous polynomial of degree greater than 1. $Y = Z(f) \subseteq \mathbb{P}^2$ is nonsingular point, $X = Z(f) \subseteq \mathbb{A}^3$.

- (a) Show that X has one singular point, namely P .

If $Q \neq P = (0, 0, 0)$ then Q is nonsingular point (since Y is nonsingular point). With the point $P = (0, 0, 0)$, the degree of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are greater than 0, then $\frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$. Thus $P \in X$ is a singular point.

- (b) \tilde{X} is nonsingular point.

$\tilde{X} \subseteq \mathbb{A}^3 \times \mathbb{P}^2$, suppose $((x, y, z), (t : u : v)) \in \tilde{X}$, then x, y, z, u, v, t satisfies the equations $f(x, y, z) = 0, xu = yt, xv = zt, yv = zu$.

On the chart $t \neq 0$, we can set $t = 1$, then we consider the equations $f(x, y, z) = 0, xu = y, xv = z, yv = zu$, then $f(x, xu, xv) = 0$, since f is a homogeneous polynomial, set $d = \text{degree}(f)$, we have $x^d f(1, u, v) = 0$. By this equation, we have two components: the exceptional curve E is defined by $x = y = z = 0, u, v$ is free; \tilde{X} is defined by the equations $f(1, u, v) = 0, xu = y, xv = z, yv = zu$. Thus

$$\tilde{X} \cap E = \{((0, 0, 0), (1 : u : v)) | f(1, u, v) = 0\} \cong Y \cap U_0$$

with the chart $U_0 = \{t \neq 0\}$.

Similar, on the chart $u \neq 0$, set $u = 1$, we have

$$\tilde{X} \cap E = \{((0, 0, 0), (t : 1 : v)) | f(t, 1, v) = 0\} \cong Y \cap U_1$$

with the chart $U_1 = \{u \neq 0\}$.

On the chart $v \neq 0$, set $v = 1$, we have

$$\tilde{X} \cap E = \{((0, 0, 0), (t : u : 1)) | f(t, u, 1) = 0\} \cong Y \cap U_2$$

with the chart $U_2 = \{v \neq 0\}$.

Therefore,

$$\tilde{X} \cap E \cong Y. \text{ (the answer of question c)}$$

Since Y is nonsingular point, then $\tilde{X} \cap E$ is nonsingular point. This show that \tilde{X} is nonsingular point. (the answer of question b)).

Exercise 5.12. Quadric Hypersurfaces.

(a)

(b)

(c) Suppose that $f = x_0^2 + x_1^2 + \cdots + x_r^2$, $Q = Z(f) \subseteq \mathbb{P}^n$, then we have the Jacobi matrix at (x_0, x_1, \dots, x_n) is

$$J = \begin{bmatrix} 2x_0 & 2x_1 & \cdots & 2x_r & 0 & \cdots & 0 \end{bmatrix}.$$

Then

$$Z = \text{Sing}Q = \{(0 : \cdots : 0 : x_{r+1} : \cdots : x_n) \in \mathbb{P}^n | x_i \in k, \forall i = r+1, \dots, n\}.$$

Z is a linear variety, $\dim Z = n - (r+1) = n - r - 1$, thus $Z = \emptyset$ if $n = r$, then Q is nonsingular point.

(d) Suppose that $f = x_0^2 + x_1^2 + \cdots + x_r^2$, $Q = Z(f) \subseteq \mathbb{P}^n$, consider the embed

$$\begin{aligned} \mathbb{P}^r &\hookrightarrow \mathbb{P}^n \\ (x_0 : \cdots : x_r) &\mapsto (x_0 : \cdots : x_r : 0 : \cdots : 0) \end{aligned}$$

Let $Q' = Z(x_0^2 + x_1^2 + \cdots + x_r^2) \in \mathbb{P}^r$, Using c), we have that Q' is nonsingular point.

Recall

$$Z = \text{Sing}Q = \{(0 : \cdots : 0 : x_{r+1} : \cdots : x_n) \in \mathbb{P}^n | x_i \in k, \forall i = r+1, \dots, n\}.$$

By define of the embed, we have $Q' \cap Z = \emptyset$, and $\dim Z = n - r - 1$, thus Q is a cone with axis Z over a nonsingular point quadric hypersurface Q' .

Exercise 5.10. For the point P on a varieties X , let \mathfrak{m} be the local ring \mathcal{O}_P . We define the Zariski tangent space $T_P(X)$ of X at P to be the dual k -vector space of $\mathfrak{m}/\mathfrak{m}^2$

- (a) By Proposition 5.2A, we have $\dim \mathfrak{m}/\mathfrak{m}^2 \geq \dim \mathcal{O}_P$, since $T_P(X)$ of X at P to be the dual k -vector space of $\mathfrak{m}/\mathfrak{m}^2$, then $\dim T_P(X) = \dim \mathfrak{m}/\mathfrak{m}^2$, thus $\dim T_P(X) \geq \dim \mathcal{O}_P$.

By Theorem 3.2, $\dim \mathcal{O}_P = \dim X$, then we have the result

$$\dim T_P(X) \geq \dim X.$$

We have $\dim T_P(X) = \dim X \iff \dim \mathfrak{m}/\mathfrak{m}^2 = \dim \mathcal{O}_P$, then \mathcal{O}_P is a regular local ring, then X is nonsingular at P .

- (b) For any morphism $\varphi : X \rightarrow Y$, there is a natural map induced k -linear map $T_P(\varphi) : T_P(X) \rightarrow T_{\varphi(P)}(Y)$.

We recall

$$\mathfrak{m}_P = \{f \in \mathcal{O}_P : f(P) = 0\}.$$

Let $f \in \mathfrak{m}_{\varphi(P)}$, then $f(\varphi(P)) = 0$, it shows that $f \circ \varphi \in \mathfrak{m}_P$, hence, we define the map φ^* as follows

$$\begin{aligned} \varphi^* : \mathfrak{m}_{\varphi(P)} &\rightarrow \mathfrak{m}_P \\ f &\mapsto f \circ \varphi. \end{aligned}$$

Let $fg \in \mathfrak{m}_{\varphi(P)}^2$, with $f, g \in \mathfrak{m}_{\varphi(P)}$, then $f \circ \varphi, g \circ \varphi \in \mathfrak{m}_P$, so $(fg) \circ \varphi \in \mathfrak{m}_P^2$, thus we define the k -linear map

$$\begin{aligned} \tau : \mathfrak{m}_{\varphi(P)}/\mathfrak{m}_{\varphi(P)}^2 &\rightarrow \mathfrak{m}_P/\mathfrak{m}_P^2 \\ f + \mathfrak{m}_{\varphi(P)}^2 &\mapsto f \circ \varphi + \mathfrak{m}_P^2 = \varphi^*(f) + \mathfrak{m}_P^2. \end{aligned}$$

We take the dual of τ giving us the k -linear map

$$\begin{aligned} T_P(\varphi) : T_P(X) &\rightarrow T_{\varphi(P)}(Y) \\ f &\mapsto f \circ \tau. \end{aligned}$$

- (c) If φ is the vertical projection of the parabola $x = y^2$ onto the x -axis, show that induced map $T_0(\varphi)$ of tangent space at the origin is the zero map.

We set $X = Z(x - y^2)$.

The map φ is defined by

$$\begin{aligned}\varphi : X &\rightarrow \mathbb{A}^1 \\ (x, y) &\mapsto x.\end{aligned}$$

Hence, $\varphi(0, 0) = 0$, the map φ^* is define by

$$\begin{aligned}\varphi^* : \mathfrak{m}_0 &\rightarrow \mathfrak{m}_{(0,0)} \\ f &\mapsto f \circ \varphi.\end{aligned}$$

We have $\mathfrak{m}_0 = (x)$, then let $f \in \mathfrak{m}_0$, there is $g \in k[x]$ such that $f = x.g$, so $\varphi^*(f) = (x.g) \circ \varphi$. For every $(x, y) \in X$, we have

$$(x.g) \circ \varphi(x, y) = (xg)(x) = (xg)(y^2) = y^2g(y^2).$$

We see that $y \in \mathfrak{m}_{(0,0)}$, then $y^2 \in \mathfrak{m}_{(0,0)}^2$, it shows that $\varphi^*(f) \in \mathfrak{m}_{(0,0)}^2$ for any $f \in \mathfrak{m}_0$.

The k -linear map

$$\begin{aligned}\tau : \mathfrak{m}_0/\mathfrak{m}_0^2 &\rightarrow \mathfrak{m}_{(0,0)}/\mathfrak{m}_{(0,0)}^2 \\ f + \mathfrak{m}_0^2 &\mapsto \varphi^*(f) + \mathfrak{m}_{(0,0)}^2 = 0.\end{aligned}$$

This shows that $\tau = 0$, then the dual of τ is the zero map.

Exercise 5.11. The Elliptic Quartic Curve in \mathbb{P}^3 . Let Y be the algebraic set in \mathbb{P}^3 defined by the equations $x^2 - xz - yw = 0$ and $yz - xw - zw = 0$. Let P be the point $(x, y, z, w) = (0, 0, 0, 1)$, and let φ denote the projection from P to the plane $w = 0$. Show that φ induces an isomorphism of $Y - P$ with the plane cubic curve $y^2z - x^3 + xz^2 = 0$ minus the point $(1, 0, -1)$. Then show that Y is an irreducible nonsingular curve. It is called *the elliptic quadratic curve* in \mathbb{P}^3 . Since it is defined by two equations it is another example of a complete intersection.

Solution:

We have $Y = Z(x^2 - xz - yw, yz - xw - zw)$. The projection from the point P is given by

$$\begin{aligned}\varphi : \mathbb{P}^3 - P &\rightarrow \mathbb{P}^2 \\ [x : y : z : w] &\mapsto [x : y : z]\end{aligned}$$

Set $T = Z(y^2z - x^3 + xz^2)$

We will show that $\varphi(Y - P) \subseteq T$.

Let $[x : y : z : w] \in Y - P$, since $[x : y : z : w] \in \mathbb{P}^3$, then we can consider two cases: $w = 0; w = 1$.

If $w = 0$, then $(x, y, z) \neq (0, 0, 0)$ and

$$\begin{cases} x^2 - xz &= 0 \\ yz &= 0 \end{cases}$$

$$\Rightarrow \begin{cases} x(x - z)(x + z) &= 0 \\ y^2z &= 0 \end{cases},$$

therefore $y^2z - x(x - z)(x + z) = 0$, so $y^2z - x^3 + xz^2 = 0$. It shows that $\varphi([x : y : z : w]) = [x : y : z] \in T$.

If $w = 1$, then

$$\begin{cases} x^2 - xz - y &= 0 \\ yz - x - z &= 0 \end{cases}$$

$$\Rightarrow \begin{cases} y &= x^2 - xz \\ yz &= x + z \end{cases},$$

hence, $y^2z = (x^2 - xz)(x + z) = x(x - z)(x + z) = x(x^2 - z^2) = x^3 - xz^2$, thus $y^2z - x^3 + xz^2 = 0$. It shows that $\varphi([x : y : z : w]) = [x : y : z] \in T$.

Thus we have $\varphi(Y - P) \subseteq T$.

Let $[x : y : z] \in T$, we have $y^2z - x^3 + xz^2 = 0$, we consider two cases:
 $y = 0; y \neq 0$.

If $y = 0$ then $x^3 - xz^2 = 0$, so $x = 0$ or $x + z = 0$ or $x - z = 0$. If $x = 0$ then we have the point $[0 : 0 : 1] \in T$, if $x = -z$ then we have the point $[-1 : 0 : 1] \in T$, if $x = z$ then we have the point $[1 : 0 : 1] \in T$. We get $[0 : 0 : 1 : 0], [1 : 0 : 1 : 0] \in Y$ and $\varphi([0 : 0 : 1 : 0]) = [0 : 0 : 1], \varphi([1 : 0 : 1 : 0]) = [1 : 0 : 1]$, but for any $w, [-1 : 0 : 1 : w]$ does not satisfy the equations $x^2 - xz - yw = 0$ and $yz - xw - zw = 0$. This shows that $\varphi(Y - P) \subseteq T - \{[-1 : 0 : 1]\}$.

If $y \neq 0$, then we get $[x : y : z : \frac{x^2 - xz}{y}]$, this point satisfies the equations $x^2 - xz - yw = 0$ and $yz - xw - zw = 0$. Thus $[x : y : z : \frac{x^2 - xz}{y}] \in Y$ and $\varphi([x : y : z : \frac{x^2 - xz}{y}]) = [x : y : z] \in T$.

Hence

$$\varphi(Y - P) = T - \{[-1 : 0 : 1]\}.$$

We consider the morphism defined by

$$\begin{aligned} \phi : T' = T - \{[-1 : 0 : 1]\} &\rightarrow Y - P \\ [0 : 0 : 1] &\mapsto [0 : 0 : 1 : 0] \\ [1 : 0 : 1] &\mapsto [1 : 0 : 1 : 0] \\ [x : y : z] &\mapsto [x : y : z : \frac{x^2 - xz}{y}] \text{ for } y \neq 0. \end{aligned}$$

We have $\varphi|_{Y-P} \circ \phi = id_{Y-P}, \phi \circ \varphi|_{Y-P} = id_{T'}$, this shows that $\varphi|_{Y-P}$ is an isomorphism.

$T = Z(y^2z - x^3 + xz^2)$ is irreducible?, then $T - \{[-1 : 0 : 1]\}$ is irreducible, then $Y - P$ is irreducible, then Y is irreducible.

Nonsingular Curves

Exercise 6.1. Recall that a curve is rational if it is birationally equivalent to \mathbb{P}^1 . Let Y be a nonsingular rational curve which is not isomorphic to \mathbb{P}^1 .

1. Show that Y is isomorphic to an open subset of \mathbb{A}^1 .

By Proposition 6.7, Y is isomorphic to an abstract nonsingular curve X , by Corollary 6.10, X is isomorphic to an open subset U of a nonsingular projective curve T .

Since Y be a nonsingular rational curve which is not isomorphic to \mathbb{P}^1 , then U is infinite, thus T is isomorphic to \mathbb{P}^1 . The space \mathbb{P}^1 has two chart U_0 and U_1 , which are isomorphic to \mathbb{A}^1 . Since $U \subsetneq \mathbb{P}^1$ then $U \subseteq U_0$ or $U \subseteq U_1$, so Y is isomorphic to an open subset U of \mathbb{A}^1 .

2. Show that Y is an affine.

Assume that Y is isomorphic to $\mathbb{P}^1 - \{a_1, \dots, a_r\} = U$.

Let $Z = Z((x - a_1)(x - a_2) \dots (x - a_r)y - 1)$ be a variety in \mathbb{A}^2 , then U is isomorphic to Z . Indeed, we consider the maps

$$\begin{aligned} \varphi : U &\rightarrow Z \\ x &\mapsto (x, \frac{1}{(x - a_1)(x - a_2) \dots (x - a_r)}), \end{aligned}$$

$$\begin{aligned} \varphi^{-1} : Z &\rightarrow U \\ (x, y) &\mapsto x. \end{aligned}$$

Then φ is an isomorphism.

3. Show that $A(Y)$ is a unique factorization domain.

We have

$$A(Y) \cong \frac{k(x, y)}{(x - a_1)(x - a_2) \dots (x - a_r)y - 1},$$

$$A(Y) \cong \mathcal{O}(Y) = k[t, \frac{1}{t-a_1}, \dots, \frac{1}{t-a_r}],$$

$k[t]$ is a unique factorization domain, the localization of a UFD is UFD, it shows that $A(Y)$ is UFD.

Exercise 6.2. An Elliptic curve. $Y = Z(y^2 - x^3 + x)$, let $f(x, y) = y^2 - x^3 + x$.

1. Show that Y is nonsingular, and deduce $A = A(Y) = k(x, y)/(y^2 - x^3 + x)$ is integrally closed domain.

We have $f'_x(x, y) = -3x^2 + 1, f'_y(x, y) = 2y$, assume that $f'_x(x, y) = -3x^2 + 1 = 0, f'_y(x, y) = 2y = 0$, then we have two points $(\frac{1}{\sqrt{3}}, 0), (-\frac{1}{\sqrt{3}}, 0)$, the points are not in Y , then Y is nonsingular.

$A(Y)$ is integrally closed domain?

2. $k[x]$ is a polynomial ring. $A = \overline{k[x]}$?
3. The map

$$\begin{aligned}\sigma : A &\rightarrow A \\ f(x, y) &\mapsto f(x, -y)\end{aligned}$$

is automorphism. The norm $N(a) = a\sigma(a)$.

Let $f \in A$, since $y^2 = x^3 - x \in k[x]$, then we can write $f(x, y) = yg(x) + h(x)$. Then

$$N(f) = f\sigma(f) = (yg(x) + h(x)) \cdot (-yg(x) + h(x)) = h^2(x) - y^2g^2(x) \in k[x].$$

$$N(1) = 1 \cdot 1 = 1$$

$$N(fg) = fg\sigma(fg) = fgf(x, -y)g(x, -y) = f\sigma(f)g\sigma(g) = N(f)N(g).$$

Exercise 6.6. Automorphism of \mathbb{P}^1 . Think of \mathbb{P}^1 as $\mathbb{A}^1 \cup \{\infty\}$. Then we define a fractional linear transformation of \mathbb{P}^1 by sending $x \rightarrow (ax+b)/(cx+d)$, for $a, b, c, d \in k, ad - bc \neq 0$.

- (a) Show that Y a fraction linear transformation induces an automorphism of \mathbb{P}^1 . We denote the group of all these fraction linear transformations by **PLG(1)**.

For $a, b, c, d \in k, ad - bc \neq 0$, the fraction linear transformation is defined by the map

$$\begin{aligned}\varphi : \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \\ x &\mapsto \frac{ax+b}{cx+d} \text{ if } x \neq \infty, \frac{-c}{d}, \\ \infty &\mapsto \frac{a}{c}, \\ \frac{-d}{c} &\mapsto \infty.\end{aligned}$$

The inverse of φ is the map defined by

$$\begin{aligned}\phi : \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \\ x &\mapsto \frac{dx - b}{-cx + a} \text{ if } x \neq \infty, \frac{a}{c}, \\ \infty &\mapsto \frac{-d}{c}, \\ \frac{a}{c} &\mapsto \infty.\end{aligned}$$

We have

$$\begin{aligned}\phi(\varphi(x)) &= \frac{d \frac{ax+b}{cx+d} - b}{-c \frac{ax+b}{cx+d} + a} = \frac{adx - bcx}{ad - bc} = x, \\ \phi(\varphi(\infty)) &= \phi\left(\frac{a}{c}\right) = \infty, \\ \phi\left(\varphi\left(\frac{-d}{c}\right)\right) &= \phi(\infty) = \frac{-d}{c},\end{aligned}$$

it shows that $\phi \circ \varphi = id_{\mathbb{P}^1}$. And we also have $\varphi \circ \phi = id_{\mathbb{P}^1}$. This shows that φ is an isomorphism.

- (b) Let $Aut \mathbb{P}^1$ denote the group of all automorphism of \mathbb{P}^1 . Show that $Aut \mathbb{P}^1 \cong Aut k(x)$, the group of k -automorphisms of the field $k(x)$.

Let $\tau \in Aut \mathbb{P}^1$, then we have the map

$$\begin{aligned}k(x) &\longrightarrow k(x) \\ f &\mapsto f \circ \tau,\end{aligned}$$

which is in $Aut k(x)$.

For each $\chi \in Aut k(x)$, by Theorem 4.4, Corollary 4.5, there is a birational map $\nu : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. So there is an open set U of \mathbb{P}^1 such that $\nu : U \rightarrow \mathbb{P}^1$ is a morphism, by Proposition 6.8, we have a morphism $\bar{\nu} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, similarly way, we have a morphism $\bar{\nu}^{-1} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. I am hoping that $\bar{\nu}$ is a isomorphism??. Maybe we can see Lemma 4.1?, since $\bar{\nu} \circ \nu^{-1}$ agree with id on some open set, then $\bar{\nu} \circ \nu^{-1} = id$.

This shows that $Aut \mathbb{P}^1 \cong Aut k(x)$.

- (c) Now show that every automorphism of $k(x)$ is a fractional linear transformation, and deduce that every $\mathbf{PGL}(1) \rightarrow Aut \mathbb{P}^1$ is an isomorphism.

For each $\varphi \in \mathbf{PGL}(1)$, by a), we have $\varpi \in \text{Aut } \mathbb{P}^1$.

Let $\varphi \in \text{Aut}k(x) \cong \text{Aut}\mathbb{P}^1$, then there are $f, g \in k[x], g \neq 0$ such that $(f, g) = 1$ and $\varphi(x) = \frac{f(x)}{g(x)}$, if $\deg f$ or $\deg g$ are more than 1, φ is not in $\text{Aut}\mathbb{P}^1$, then we can assume that $f(x) = ax + b, g(x) = cx + d$, since $(f, g) = 1$, then $ad - bc \neq 0$. So $\varphi \in \mathbf{PGL}(1)$.

The Hilbert polynomial

Let X be a projective variety of \mathbb{P}^n . For $d \geq 1$, denote

$$I(X)_m = I(X) \cap k[x_0, \dots, x_n]_m.$$

Denote $S(X) = k[x_0, \dots, x_n]/I(X)$, since $I(X)$ is a homogeneous ideal, then $S(X)$ is a graded ring with decomposition

$$S(X) = \bigoplus_{m \geq 0} S(X)_m,$$

with $S(X)_m = k[x_0, \dots, x_n]_m/I(X)_m$.

The **Hilbert function** of X is given by

$$\varphi_X(l) = \dim_k(S(X)_l).$$

Note that $\dim(k[x_0, \dots, x_n]_m) = \binom{n+m}{n}$.

Intersections in Projective Space

Exercise 7.1

- (a) The d -uple embedding of \mathbb{P}^n in \mathbb{P}^N . Let M_0, \dots, M_N be all the monomials of degree d in the $n+1$ variables x_0, \dots, x_n , where $N = \binom{n+d}{n} - 1$.

We define the map

$$\begin{aligned} \rho_d : \mathbb{P}^n &\rightarrow \mathbb{P}^N \\ P = [a_0 : \dots : a_n] &\mapsto [M_0(P) : \dots : M_N(P)]. \end{aligned}$$

Since M_i are the polynomials, then ρ_d is a morphism, and ρ_d is called the d -uple embedding of \mathbb{P}^n in \mathbb{P}^N .

Let $\theta : k[x_0, \dots, x_n] \rightarrow k[x_0, \dots, x_n]$ be the homomorphism defined by sending y_i to M_i . Let $\mathfrak{a} = \ker \theta$, by Exercise 2.12, we have

$$k[x_0, \dots, x_n]/\mathfrak{a} \cong k[x_0, \dots, x_n]_d.$$

And $Z(\mathfrak{a}) = \rho_d(\mathbb{P}^n)$, then we have

$$S(Z(\mathfrak{a})) = k[x_0, \dots, x_n]_d = \bigoplus_{m \geq 0} (k[x_0, \dots, x_n]_d)_m,$$

with $(k[x_0, \dots, x_n]_d)_m = k[M_0, \dots, M_N]_m = k[x_0, \dots, x_n]_{md}$. This shows that

$$\varphi_{Z(\mathfrak{a})}(m) = \dim_k(S(Z(\mathfrak{a})_m)) = \binom{n+md}{n}.$$

Then the Hilbert polynomial of $Z(\mathfrak{a})$

$$\begin{aligned} P_{Z(\mathfrak{a})}(x) &= \binom{n+dx}{n} = \frac{(n+dx)!}{n!(dx)!} = \\ &= \frac{(n+dx) \dots (xd+1)}{n!} \\ &= \frac{(xd)^n}{n!} + \text{term of degree} < n. \end{aligned}$$

It shows that $\deg Z(\mathfrak{a}) = \frac{d^n}{n!} \dim(\mathbb{P}^n)! = d^n$.

(b) The Segre Embedding is defined by the map

$$\begin{aligned} \phi : \mathbb{P}^r \times \mathbb{P}^s &\rightarrow \mathbb{P}^N \\ (a_0, \dots, a_r) \times (b_0, \dots, b_s) &\mapsto (\dots, a_i b_j, \dots) \end{aligned}$$

where $N = rs + r + s$.

Note that ϕ is injective. Image of ϕ is a subvariety of \mathbb{P}^N . Indeed, we consider the map

$$\begin{aligned} \pi : k[\{z_{ij} : i = 0, \dots, r, j = 0, \dots, s\}] &\rightarrow k[x_0, \dots, x_r, y_0, \dots, y_s] \\ z_{ij} &\mapsto x_i y_j. \end{aligned}$$

Exercise 2.14 shows that $\text{im}(\phi) = Z(\mathfrak{m})$, where $\mathfrak{m} = \ker(\phi)$. Hence we have the coordinate ring of $Z(\mathfrak{m})$

$$k[\{z_{ij} : i = 0, \dots, r, j = 0, \dots, s\}]/\mathfrak{m} \cong \langle x_i y_j : i = 0, \dots, r, j = 0, \dots, s \rangle = M$$

We define

$$M_l = \langle ab : a \in k[x_0, \dots, x_r]_l, b \in k[y_0, \dots, y_s]_l \rangle \quad (1)$$

Then we have

$$M = \bigoplus_{l \geq 0} M_l.$$

Since (1), then we have

$$\dim(M_l) = \binom{r+l}{r} \binom{s+l}{s}.$$

It show us the Hilbert function

$$\begin{aligned} \varphi_{Z(\mathfrak{m})}(l) &= \binom{r+l}{r} \binom{s+l}{s}, l \in \mathbb{N}. \\ &= \frac{(r+l)!}{r!l!} \frac{(l+s)!}{l!s!} \\ &= \frac{(r+l) \dots (l+1)}{r!} \frac{(s+l) \dots (l+1)}{s!}. \end{aligned}$$

Hence, the Hilbert polynomial is

$$\begin{aligned} P_{Z(\mathfrak{m})}(x) &= \frac{(r+x) \dots (x+1)}{r!} \frac{(s+x) \dots (x+1)}{s!} \\ &= \frac{x^r}{r!} \frac{x^s}{s!} + \text{term of lower degree.} \end{aligned}$$

Since $\dim(\mathbb{P}^r \times \mathbb{P}^s) = r + s$, then we have

$$\deg Z(\mathfrak{m}) = \frac{(r+s)!}{r!s!} = \binom{r+s}{r}$$

Exercise 7.3 The dual Curve.

Let $Y = Z(F)$ be a projective variety in \mathbb{P}^n , let $P = (a_0, a_1, a_3)$ be a nonsingular point of Y . The tangent line to Y at P is defined by

$$T_P Y = F'_{x_0}(P)(x_0 - a_0) + F'_{x_1}(P)(x_1 - a_1) + F'_{x_2}(P)(x_2 - a_2).$$

Let A be any ring (commutative ring with identity). A multiplicatively closed subset S of A such that $1 \in S$ and S is closed under multiplication: in other words S is a sub-semigroup of the multiplicative semigroup of A . Define a relation \equiv on $A \times S$ as follows:

$$(a, s) \equiv (b, t) \Leftrightarrow (at - bs)u = 0.$$

Reference

[1] Robin Hartshorne, *Algebra Geometry, chapter I*, 1977.