QUY NHON UNIVERSITY - EVORA UNIVERSITY

Algebraic Varieties - Robin Hartshorne Chapter I

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June 7, 2018

Dear professor Pedro M. Marques, thank you very much for help me studying this book. This is a free course from Evora University, Portugal. I will remember forever your help in my heart. Hope all the best for you!.

Quy Nhon, 07/06/2018.

Projective Varieties

Exercise 2.1 *Prove the "homogeneous Nullstellensatz", which say if $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if f is a homogeneous polynomial with $degf > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbb{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$.*

Solution:

Assume that $\mathfrak{a} = S$, then we can get $q = 1$.

Assume that $\mathfrak{a} \neq S$ and

$$\mathfrak{a} = \langle f_i | f_i \; is \; a \; homogeneous \; polynomial, i = 1, ..., n\rangle,$$

then $degf_i > 0$. We will show that

$$\mathbb{A}^{n+1} \supseteq Z(\mathfrak{a}) = \{P \in \mathbb{A}^{n+1} | P = 0 \, or \, \overline{P} \in Z(\mathfrak{a}) \subseteq \mathbb{P}^n\},$$

where

$$\mathbb{A}^{n+1} \supseteq Z(\mathfrak{a}) = \{P \in \mathbb{A}^{n+1} | f_i(P), i = 1, ..., n\} = U,$$

$$\mathbb{P}^n \supseteq Z(\mathfrak{a}) = \{P \in \mathbb{P}^n | f_i(P), i = 1, ..., n\} = V.$$

If $P = 0$ then $f_i(0) = 0$ for all $i = 1, ..., n$. Hence $P = 0 \in U$.

If $P \neq 0$, suppose that

$$P = (x_0, ..., x_n) \in \mathbb{A}^{n+1},$$

then we have a point $\overline{P}$ in $\mathbb{P}^n$ corresponding to the point P,

$$\overline{P} = (x_0 : ... : x_n) \in \mathbb{P}^n.$$

Therefore, if $P \in U$, since f_i are the homogeneous polynomials,

$$f_i(\alpha x_0, ..., \alpha x_n) = \alpha^{d_i} f_i(P) \text{ where } \alpha \neq 0 \text{ and } d_i = degf_i,$$

then $P_\alpha = (\alpha x_0, ..., \alpha x_n) \in U$ for all $\alpha \neq 0$, thus $\overline{P} \in V$.

Conversely, if $\overline{P} \in V$ then $P_\alpha = (\alpha x_0, ..., \alpha x_n) \in U$ for all $\alpha \neq 0$.

Thus we have

$$\mathbb{A}^{n+1} \supseteq Z(\mathfrak{a}) = \{P \in \mathbb{A}^{n+1} | P = 0 \, or \, \overline{P} \in Z(\mathfrak{a}) \subseteq \mathbb{P}^n\}.$$

Now, we comeback to the homogeneous polynomial f. Let

$$P = (x_0, ..., x_n) \in Z(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}.$$

If $P = 0$ then $f(P) = 0$. If $P \neq 0$ then

$$\overline{P} = (x_0 : ... : x_n) \in Z(\mathfrak{a}) \subseteq \mathbb{P}^n.$$

We have $f(\overline{P}) = 0$, hence $f(P) = 0$, therefore, $f \in I(Z(\mathfrak{a}))$ where $Z(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}$, then $f^q \in \mathfrak{a}$ for some $q > 0$.

Exercise 2.2 *For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:*

i) $Z(\mathfrak{a}) = \emptyset$ *in* $\mathbb{P}^n$*;*

ii) $\sqrt{\mathfrak{a}}$ *either* S *or the ideal* $S_+ = \bigoplus_{d>0} S_d$*;*

iii) $\mathfrak{a} \supseteq S_d$ *for some* $d > 0$.

Solution:

i) $\Rightarrow$ *ii)*

By Exercise 2.1, we have that if $Z(\mathfrak{a}) = \emptyset$ in $\mathbb{P}^n$, then in $\mathbb{A}^{n+1}$, either $Z(\mathfrak{a}) = \emptyset$ or $Z(\mathfrak{a}) = \{0\}$.

If $Z(\mathfrak{a}) = \emptyset$ then $\sqrt{\mathfrak{a}} = I(Z(\mathfrak{a})) = S = K[x_0, ..., x_n]$.

If $Z(\mathfrak{a}) = \{0\}$ then $\sqrt{\mathfrak{a}} = I(Z(\mathfrak{a})) = \langle x_0, ..., x_n \rangle$ (Example 1.1.4), hence

$$\sqrt{\mathfrak{a}} = S \diagdown K = \bigoplus_{d>0} S_d.$$

ii) $\Rightarrow$ *iii)*

If $\sqrt{\mathfrak{a}} = S = K[x_0, ..., x_n]$ then $1 \in \sqrt{\mathfrak{a}}$, then $1 \in \mathfrak{a}$, hence $\mathfrak{a} = S$.

If $\sqrt{\mathfrak{a}} = \bigoplus_{d>0} S_d$, then for all $d \in \mathbb{N} \setminus \{0\}$, $x_i^d \in \sqrt{\mathfrak{a}}$ for all $i = 0, .., n$.

Assume that if there exist $i \in \{0, ..., n\}$ such that $x_i^d \notin \mathfrak{a}$ for all $d \in \mathbb{N} \setminus \{0\}$, then $x_i \notin \sqrt{\mathfrak{a}}$ (contradiction). Therefore, there are $d_i \in \mathbb{N} \setminus \{0\}$ such that $x_i^{d_i} \in \mathfrak{a}$ for all $i = 0, ..., n$. We get $t = max\{d_i | i = 0, ..., n\}$, then $x_i^t \in \mathfrak{a}$ for all $i = 0, ..., n$.

We consider $S_{t(n+1)}$, for each $x_0^{\alpha_0} ... x_n^{\alpha_n}$ is in $S_{t(n+1)}$, then

$$\alpha_0 + \alpha_1 + ... + \alpha_n = t(n+1),$$

where $\alpha_i \in \mathbb{N}$ for all $i = 0, ..., n$.

If $\alpha_i < t$ for all $i = 0, ..., n$, then

$$\alpha_0 + \alpha_1 + ... + \alpha_n < t(n+1) \text{ (contradiction)}.$$

Therefore, there exist i such that $x_0^{\alpha_0} ... x_n^{\alpha_n} = x_i^t . f$ for some $f \in S$, then $x_0^{\alpha_0} ... x_n^{\alpha_n} \in \mathfrak{a}$, hence $S_{t(n+1)} \in \mathfrak{a}$.

iii) $\Rightarrow$ *i)*

Suppose that $Z(\mathfrak{a}) \neq \emptyset$, let $P = (p_0 : ... : p_n) \in Z(\mathfrak{a}) \subseteq \mathbb{P}^n$, then $P' = (p_0, ..., p_n) \in Z(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}$ and $P' \neq 0$.

One has $\mathfrak{a} \supseteq S_d$ for some $d > 0$, then $f(P') = 0$ for all $f \in S_d$. Since $P' \neq 0$, there is $p_i \neq 0$ for some i. Consider the polynomial x_i^d is in S_d, then

$$x_i^d(P') = p_i^d \neq 0 \text{ (contradiction)}.$$

Therefore, $Z(\mathfrak{a}) = \emptyset$.

Exercise 2.3

a) If $T_1 \subseteq T_2$ are subsets of S^h, then $Z(T_1) \supseteq Z(T_2)$.

b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.

c) For any two subsets Y_1, Y_2 of $\mathbb{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

d) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

e) For any subset $Y \subseteq \mathbb{P}^n$, $Z(I(Y)) = \overline{Y}$.

Solution:

a) Let $P \in Z(T_2)$, so we have $f(P) = 0$ for all $f \in T_2$. Hence $f(P) = 0$ for all $f \in T_1 \subseteq T_2$, therefore, $P \in Z(T_1)$. Thus $Z(T_1) \supseteq Z(T_2)$

b) Let $f \in I(Y_2)$, then $f(P) = 0$ for all $P \in Y_2$, so $f(P) = 0$ for all $P \in Y_1$, so $f \in I(Y_1)$. Thus $I(Y_1) \supseteq I(Y_2)$.

c) Let $f \in I(Y_1 \cup Y_2)$, thus $f(P) = 0$ for all $P \in Y_1 \cup Y_2$, then $f \in I(Y_1)$ and $f \in I(Y_2)$.

Conversely, let $f \in I(Y_1) \cap I(Y_2)$, then f vanishes on Y_1 and then f vanishes on Y_2 , so then f vanishes on $Y_1 \cup Y_2$, therefore $f \in I(Y_1 \cup Y_2)$.

Thus $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

d) $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$.

Let $f \in \sqrt{\mathfrak{a}}$, there is $n > 0$ such that $f^n \in \mathfrak{a}$, then $f^n(P) = 0$ for all $P \in Z(\mathfrak{a})$, so $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ hence $f \in I(Z(\mathfrak{a}))$.

Conversely, if f is a homogeneous polynomial and $\deg f > 0$, such that $f \in I(Z(\mathfrak{a}))$, then Exercise 2.1 shows that $f \in \sqrt{\mathfrak{a}}$. If $f = 0$ then $f \in I(Z(\mathfrak{a}))$ and $f \in \sqrt{\mathfrak{a}}$. If $f = \alpha \in I(Z(\mathfrak{a}))$ for some $\alpha \in K \setminus \{0\}$ then $Z(\mathfrak{a}) = \emptyset$ (contradiction.

Thus $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

e) Let Y be a subset of $\mathbb{P}^n$, prove that $Z(I(Y)) = \overline{Y}$.

$\overline{Y}$ is the closure of Y in $\mathbb{P}^n$, this is an algebraic set smallest containing Y. Since $Z(I(Y))$ is an algebraic set containing Y , one has $\overline{Y} \subseteq Z(I(Y))$.

On the other hand, suppose that $\overline{P} = Z(T)$ for some $T \subseteq S^h$. So $Y \subseteq Z(T)$, so by b) we have $I(Y) \supseteq I(Z(T))$, and by a) we have $Z(I(Y)) \subseteq Z(T)$.

Thus $Z(I(Y)) = \overline{Y}$.

Exercise 2.4

a) There is a 1-1 inclusion reversing correspondence between algebraic set in $\mathbb{P}^n$, and homogeneous radical ideal of S not equal to S_+, given by $Y \to I(Y)$ and $\mathfrak{a} \to Z(\mathfrak{a})$.

b) An algebraic set $Y \subseteq \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

c) Show that $\mathbb{P}^n$ itself is irreducible.

Solution:

a) Let A be the set of algebraic set in $\mathbb{P}^n$, and let B be the set of homogeneous radical ideal of S not equal to S_+. Let ϕ and φ be the maps

$$\begin{aligned} \phi : A &\to B \\ Y &\mapsto I(Y), \end{aligned}$$

$$\begin{aligned} \varphi : B &\to A \\ \mathfrak{a} &\mapsto Z(\mathfrak{a}). \end{aligned}$$

For any $Y \in A$, then $\overline{Y} = Y$, we have $(\varphi \circ \phi)(Y) = Z(I(Y)) = \overline{Y} = Y$, thus $\varphi \circ \phi = Id_A$.

For any $\mathfrak{a} \in B$, so $\sqrt{\mathfrak{a}} = \mathfrak{a}$, if $\mathfrak{a} \neq S$ then $Z(\mathfrak{a}) \neq \emptyset$, so $(\phi \circ \varphi)(\mathfrak{a}) = I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$, and if $\mathfrak{a} = S$ then $Z(\mathfrak{a}) = \emptyset$, we have $(\phi \circ \varphi)(S) = \phi(Z(S)) = \phi(\emptyset) = S$, thus $\phi \circ \varphi = Id_B$.

Thus ϕ is a surjective and $\phi^{-1} = \varphi$. So we obtain the result.

b) Let $Y \subseteq \mathbb{P}^n$ be an algebraic set.

Assume that Y is irreducible. Let f, g be in S^h such that $fg \in I(Y)$, so $Z(fg) \supseteq Y$, then Y is a subset of $Z(f) \cup Z(g)$. We can decompose Y as

$$Y = (Z(f) \cap Y) \cup (Z(g) \cap Y).$$

We have $Z(f) \cap Y$ and $Z(g) \cap Y$ are closed subsets of $\mathbb{P}^n$, then $Y = Z(f) \cap Y$ or $Y = Z(g) \cap Y$. If $Y = Z(f) \cap Y$ then $Y \subseteq Z(f)$, thus $f \in I(Y)$. If If $Y = Z(g) \cap Y$ then $g \in I(Y)$. Hence $I(Y)$ is a prime ideal.

Conversely, suppose that $I(Y)$ is a prime ideal and $Y = Y_1 \cup Y_2$ with Y_1, Y_2 are closed subsets in $\mathbb{P}^n$. We have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2) = I(Y)$, then $I(Y_1) \subseteq I(Y)$ or $I(Y_2) \subseteq I(Y)$, thus $Y_1 \supseteq Y$ or $Y_2 \supseteq Y$, so $Y_1 = Y$ or $Y_2 = Y$, hence $Y \subseteq \mathbb{P}^n$ is irreducible.

c) We have $\mathbb{P}^n = Z(0)$, since 0 is the prime ideal, thus $\mathbb{P}^n$ is irreducible.

Exercise 2.5

a) $\mathbb{P}^n$ is a noetherian topological space.

b) Every algebraic set $Y \in \mathbb{P}^n$ can be written uniquely as a union of irreducible algebraic set, no one containing another.

Solution:

a) $\mathbb{P}^n$ is a topological space, a closed set in $\mathbb{P}^n$ is a algebraic set. Let $\{Y_i\}_{i\in\mathbb{N}}$ be a descending chain of closed subsets,

$$Y_0 \supseteq Y_1 \supseteq \ldots,$$

then we have a ascending chain $\{I(Y_i)\}_{i\in\mathbb{N}}$ of ideals in S. Since S is noetherian ring, then there exist m such that $I(Y_m) = I(Y_{m+k})$ for all $k \in \mathbb{N}$, therefore, $Y_m = Y_{m+k}$ for all $k \in \mathbb{N}$, thus the descending chain $\{Y_i\}_{i\in\mathbb{N}}$ is stationary. Hence $\mathbb{P}^n$ is a noetherian topological space.

b) See proposition 1.5

Exercise 2.6 *If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $dim S(Y) = dim Y + 1$.*

Exercise 2.7

a) $dim\ \mathbb{P}^n = n$.

b) If $Y \subseteq \mathbb{P}^n$ is a quasi-projective variety, then $dim\overline{Y} = dim Y$.

Solution:
a) Since $I(\mathbb{P}^n) = 0$, then $S(\mathbb{P}^n) = S$, by Exercise 2.6, we have

$$n + 1 = dim S = dim S(\mathbb{P}^n) = dim \mathbb{P}^n + 1,$$

thus $dim\mathbb{P}^n = n$.

b)

Exercise 2.8 *A projective variety $Y \subseteq \mathbb{P}^n$ has dimension $n-1$ if and only if it is the zero set of single irreducible homogeneous polynomial f of positive degree. Y is called a* ***hypersurface*** *in $\mathbb{P}^n$.*

Solution:

Let f be a single irreducible homogeneous polynomial f of positive degree. Then (f) ia a prime ideal and $height(f) = 1$. Get $Y = Z(f)$, we have the relation as follows

$$dimS(Y) + hight(f) = dimS = n + 1,$$

hence $dimS(Y) = n$. We have,

$$dimS(Y) = dimY + 1,$$

then $dimY = n - 1$.

Conversely, assume that projective variety $Y \subseteq \mathbb{P}^n$ has dimension $n - 1$.

Exercise 2.10 *Let $Y \subseteq \mathbb{P}^n$ be a nonempty algebraic set, and let $\theta : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ be the map which sends the point with affine coordinate $(a_0, ..., a_n)$ to the point with homogeneous coordinates $(a_0, \dots, a_n)$. We define affine cone over Y to be*

$$C(Y) = \theta^{-1}(Y) \cup (0, \dots, 0)$$

a) Show that $C(Y)$ is a algebraic set in $\mathbb{A}^{n+1}$, whose ideal is equal to $I(Y)$.

b) $C(Y)$ is irreducible if and only if Y is irreducible.

c) $dimC(Y) = dimY + 1$.

Solution:

a) Prove that $C(Y) = Z(I(Y))$. We have $I(Y)$ is a homogeneous radical ideal of S. Then there are $f_1, \dots, f_k$ the homogeneous polynomials such that

$$I(Y) = \langle f_1, \dots, f_k \rangle.$$

If $P = 0$ then $f(0) = 0$ for all $f \in C(Y)$. If $P \neq 0$ and $P \in C(Y)$, then $P \in Y$, so we have $f(P) = 0$ for all $f_1, \dots, f_k$. Hence we have $C(Y) \subseteq Z(I(Y))$.

Conversely, let $P \neq 0$ and $P \in Z(I(Y))$, then $f_1(P) = \cdots = f_k(P) = 0$, then $P \in Y$, and $P \in \theta^{-1}(Y)$, thus $P \in C(Y)$.

Therefore $C(Y) = Z(I(Y))$.

b) $C(Y)$ is irreducible if and only if $I(Y)$ is a prime ideal, if and only if Y irreducible.

Morphism

Exercise 3.13 *The local ring of a subvariety.*

Let X be a variety of $\mathbb{A}^n$, and let $Y \subseteq X$ be a subvariety. Let $\mathcal{O}_{Y,X}$ be the set of equivalence class $< U, f >$ such that $U \subseteq X$ is open, $U \cap Y \neq \emptyset$, and f is a regular function on U. We say that $< U, f >$ is equivalent to $< V, g >$ if $f = g$ on $U \cap V$.

Consider ideal

$$\mathfrak{m} = \{< U, f > \in \mathcal{O}_{Y,X} : f(P) = 0, \forall P \in U \cap Y\}.$$

Let $< U, f > \in \mathcal{O}_{Y,X} \setminus \mathfrak{m}$, then $f(P) \neq 0$ for some $P \in U \cap Y$, so $1/f$ is regular in some open neighborhood V_P of P, therefore,

$$< V_P, 1/f > = < V_P, f >^{-1} = < U, f >^{-1},$$

thus $\mathfrak{m}$ is a maximal ideal and $\mathcal{O}_{Y,X}$ is a local ring.

If Y is a point P, we get $\mathcal{O}_P$. If $Y = X$ we get $K(X)$ the rational function on X.

We have

$$\mathcal{O}_{Y,X}/\mathfrak{m} = \{< U, f > + \mathfrak{m} : < U, f > \in \mathcal{O}_{Y,X}\} \text{ and}$$
$$K(Y) = \{< U, f >: f \text{ is regular on open set} U \subseteq Y\}.$$

Since U is open in X, $U \cap Y$ is open in Y. Consider the map

$$\pi : \mathcal{O}_{Y,X} \to K(Y)$$
$$< U, f > \mapsto < U \cap Y, f >,$$

π is a ring homomorphism.
Hence $\pi(\mathfrak{m}) = 0$, and if $\pi(< U, f >) = < U \cap Y, f > = 0$, then $f(U \cap Y) = 0$, so $< U, f > \in Ker(\pi)$. Thus $Ker(\pi) = \mathfrak{m}$.
For any $< V, f > \in K(Y)$, let U be a open set of X such that $U \subseteq V$, so $U \cap V = U$ is a open subset of V. We have $< U, f > = < V, f >$, then π is surjective. so we have a ring isomorphim

$$\mathcal{O}_{Y,X}/\mathfrak{m} \cong K(Y)$$

Consider ideal

$$\mathfrak{a} = \{f \in A(X) : f(P) = 0, \forall P \in Y\}.$$

Lemma: *The local ring $A(X)_{\mathfrak{a}}$ is isomorphic to $\mathcal{O}_{Y,X}$.*

The ideals of $A(X)_{\mathfrak{a}}$ correspond to the ideals of $A(X)$ contained in $\mathfrak{a}$, then $dim(A(X)_{\mathfrak{a}}) = height(\mathfrak{a})$, so $dim(\mathcal{O}_{Y,X}) = height(\mathfrak{a})$.
By 1.8A, we have

$$height(\mathfrak{a}) + dim(A(X)/\mathfrak{a}) = dim(A(X)).$$

We also have $A(X)/\mathfrak{a} \cong A(Y)$ and $dim(A(X)) = dimX, dim(A(Y)) = dimY$ so we have a result

$$dim(\mathcal{O}_{Y,X}) = height(\mathfrak{a}) = dim(A(X)) - dim(A(Y).$$

Exercise 3.14 *Projection from a Point.*

Let

$$\mathbb{P}^n = \{(x_0 : \cdots : x_{n+1}) \in \mathbb{P}^{n+1} : x_{n+1} = 0\},$$

and let $P \in \mathbb{P}^{n+1} - \mathbb{P}^n$, this mean that

$$P = (p_0 : \cdots : p_n : 1).$$

Let $Q = (q_0 : \cdots : q_{n+1}) \in \mathbb{P}^{n+1} - P$, for $q_{n+1} = 0$ or $q_{n+1} = 1$,the intersection I of the line containing P and Q with $\mathbb{P}^n$ is

$$I = (q_0 - p_0 : \cdots : q_n - p_n : 0) \text{ if } q_{n+1} = 1 \text{ and } I = Q \text{ if } q_{n+1} = 0,$$

therefore, the mapping φ can be written by

$$\begin{aligned} &\varphi : \mathbb{P}^{n+1} - \{P\} \to \mathbb{P}^n \\ Q = (q_0 : \cdots : q_{n+1}) &\mapsto Q \;\text{ if } Q \in \mathbb{P}^n \\ Q = (q_0 : \cdots : q_{n+1}) &\mapsto (q_0 - p_0 q_{n+1} : \cdots : q_n - p_n q_{n+1} : 0) \;\text{ if } Q \notin \mathbb{P}^n \end{aligned}$$

1. φ is a continuous map;

2. for every open subset $V \subseteq Y$, and for every regular function $f : V \to k$, the function

$$\begin{aligned} f \circ \varphi : \varphi^{-1}(V) &\to k \\ Q &\mapsto (f \circ \varphi)(Q) = f(\varphi(Q)), \end{aligned}$$

we need to show that $\varphi^{-1}(V) \neq \emptyset$ if $V \neq \emptyset$. For every $I = (x_0 : \cdots : x_n : 0) \in \mathbb{P}^n$, we can choose $Q = I$, so $\varphi(Q) = I$, then φ is surjective, thus $\varphi^{-1}(V) \neq \emptyset$ if $V \neq \emptyset$.

If $S \in \varphi^{-1}(V) \setminus \mathbb{P}^n$, then $\varphi(S) \in V$ and let $V_{\varphi(S)}$ be an open neighborhood of $\varphi(S)$ in V such that $f = g/h$ on $V_{\varphi(S)}$, where homogeneous polynomials $h, g \in K[x_0, \ldots, x_n]$ the same degree, $h(x) \neq 0$ for every $x \in V_{\varphi(S)}$. Then

$$f \circ \varphi = g \circ \varphi / h \circ \varphi$$

on an open subset $U_S = \varphi^{-1}(V_{\varphi(S)}) \cap \{x_{n+1} \neq 0\}$. Since

$$(g \circ \varphi)(S) = g(s_0 - p_0 s_{n+1}, \ldots, s_n - p_n s_{n+1}) \in K[x_0, \ldots, x_n, x_{n+1}],$$

then we have $g \circ \varphi, h \circ \varphi$ on U_S is the homogeneous polynomials the same degree, so $f \circ \varphi$ is regular at S.

If $S \in \varphi^{-1}(V) \cap \mathbb{P}^n$, let U_S be an open neighborhood in $\mathbb{P}^n$, then $\varphi(U_S) = U_S$, therefore $g \circ \varphi = g, h \circ \varphi = h$, so $f \circ \varphi$ is regular at S. Thus $\varphi^{-1}(V)$ is regular. Hence φ is a morphism.

3.15 *Products of Affine Varieties.* Let $X \in \mathbb{A}^n$ and let $Y \in \mathbb{A}^m$ be affine varieties.

(a) Show that $X \times Y \subseteq \mathbb{A}^{m+n}$ with its induced topology is irreducible.

Firstly, we proof that $X \times Y$ is an affine algebra set in $\mathbb{A}^{m+n}$.

Suppose that

$$\begin{aligned} X &= Z(f_1, \ldots, f_s) \\ Y &= Z(g_1, \ldots, g_t), \end{aligned}$$

where $f_i \in K[x_1, \ldots, x_n] \subset K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ for all $i = 1, \ldots, s$ and $g_i \in K[y_1, \ldots, y_m] \subset K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ for all $i = 1, \ldots, t$.

Then

$$X \times Y = Z(f_1, \ldots, f_s, g_1, \ldots, g_t).$$

Indeed, Let $(a_1, \ldots, a_n, b_1 \ldots, b_m) \in X \times Y$, if and only if $(a_1, ..., a_n) \in X$ and $(b_1, ..., b_m) \in Y$, if and only if

$$\begin{aligned} f_i(a_1, \ldots, a_n, b_1 \ldots, b_m) &= 0, \forall i = 1, \ldots, s, \\ g_i(a_1, \ldots, a_n, b_1 \ldots, b_m) &= 0, \forall i = 1, \ldots, t, \end{aligned}$$

if and only if $(a_1, \ldots, a_n, b_1, \ldots, b_m) \in Z(f_1, \ldots, f_s, g_1, \ldots, g_t)$.

Secondly, show that $X \times Y$ is irreducible.
The second projective map

$$\begin{aligned} \pi_2 : X \times Y &\to Y \\ (x, y) &\mapsto y. \end{aligned}$$

Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$.
Fix $y \in Y$, then the map

$$\begin{aligned} \varphi : X &\to X \times Y \\ x &\mapsto (x, y) \end{aligned}$$

is continuous, the subset

$$X_y^i = \{x \in X : (x, y) \in Z_i\} = \varphi^{-1}(X \times \{y\} \cap Z_i)$$

is closed in X because $X \times \{y\} \cap Z_i$ is closed in $X \times Y$, for $i = 1, 2$. Let

$$X_i = \{x \in X : x \times Y \subseteq Z_i\}, \text{ for } i = 1, 2.$$

Now, we proof that $X_i = \bigcap\limits_{y \in Y} X_y^i$, for $i = 1, 2$. Let $a \in X_i$, then $a \times Y \subseteq Z_i$, so $(a, y) \in Z_i$ for all $y \in Y$, hence for every $y \in Y, a \in X_y^i = \{x \in X : (x, y) \in Z_i\}$, thus $a \in \bigcap\limits_{y \in Y} X_y^i$. Conversely, let $a \in \bigcap\limits_{y \in Y} X_y^i$, then $a \in X_y^i$ for all $y \in Y$, so $(a, y) \in Z_i$ for all $y \in Y$, then $a \times Y \subseteq Z_i$, thus $a \in X$. Since closed sets are stable under arbitrary intersections, it shows that X_i is closed in X for $i = 1, 2$.
Now, we proof that $X = X_1 \cup X_2$. We can see that $X_1 \cup X_2 \subseteq X$ is easy. Conversely, let $x \in X$, let $Y_x = x \times Y$, we have $\pi_2(Y_x) = Y$, so Y_x is homeomophic to Y via the map $\pi_2|_{Y_x}$. Since Y is irreducible, Y_x is closed and irreducible. We have

$$Y_x = (Z_1 \cap Y_x) \cup (Z_2 \cap Y_x),$$

therefore, $Z_1 \cap Y_x = Y_x$ or $Z_2 \cap Y_x = Y_x$, hence $Y_x \subseteq Z_1$ or $Y_x \subseteq Z_2$, it shows that $x \in X_1$ or $x \in X_2$, thus $x \in X_1 \cup X_2$.
With the results $X = X_1 \cup X_2$, X_i are close in X, we have $X = X_1$ or $X = X_2$. If $X_1 = X$, since $X \times Y = \bigcup\limits_{x \in X} (x \times Y) = \bigcup\limits_{x \in X_1} (x \times Y) \subseteq Z_1$, it shows that $Z_1 = X \times Y$. In a similar way, if $X = X_2$, then $Z_2 = X \times Y$. Thus $X \times Y$ is irreducible.

(b) Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$

Consider the bilinear map

$$\begin{aligned} \varphi : A(X) \times A(Y) &\to A(X) \otimes_k A(Y) \\ (f, g) &\mapsto f \otimes g \end{aligned}$$

and the map

$$\begin{aligned} \theta : A(X) \times A(Y) &\to A(X \times Y) \\ (f, g) &\mapsto f.g, \end{aligned}$$

for every $\alpha, \beta \in k, f_1, f_2 \in A(X), g \in A(Y)$, we have $\theta(\alpha f_1 + \beta f_2, g) = (\alpha f_1 + \beta f_2)g = \alpha f_1.g + \beta f_2.g = \alpha\theta(f_1, g) + \beta\theta(f_2, g)$, then θ is a bilinear map, so by Universal property of the tensor products, we have the linear map ϕ such that $\theta = \phi \circ \varphi$, and

$$\begin{aligned} \phi : A(X) \otimes_k A(Y) &\to A(X \times Y) \\ f \otimes g &\to f.g. \end{aligned}$$

We have $\theta(\overline{x_i}, \overline{y_j}) = \overline{x_i}.\overline{y_j}$ for all $\overline{x_i} \in A(X)$ and $\overline{y_j} \in A(Y)$, hence, for $\in A(X \times Y)$, so $\prod\limits_i \overline{x_i}^{t_i} \prod\limits_j \overline{y_j}^{t_j} = \theta(\prod\limits_i \overline{x_i}^{t_i}, \prod\limits_j \overline{y_j}^{t_j})$, hence θ is surjective, then ϕ is surjective.

ϕ a a ring homomorphism.

Let $\sum\limits_{i=1}^{t} f_i \otimes g_i \in ker\phi, t \geq 1$, suppose that $g_t \neq 0$ so there is $b \in Y$ such that $g(b) \neq 0$, we have $\sum\limits_{i=1}^{t} f_i g_i = 0$, for every $x \in X$ and if $t \geq 2$, we have

$$\sum_{i=1}^{t}(f_i g_i)(x, b) = \sum_{i=1}^{t} f_i(x)g_i(b) = \sum_{i=1}^{t-1} f_i(x)g_i(b) + f_t(x)g_t(b) = 0,$$

this shows that

$$f_t(x) = -g_t(b)^{-1} \sum_{i=1}^{t-1} f_i(x)g_i(b) = -\sum_{i=1}^{t-1} g_t(b)^{-1}g_i(b)f_i(x),$$

hence

$$\begin{aligned} \sum_{i=1}^{t} f_i \otimes g_i = \sum_{i=1}^{t-1} f_i \otimes g_i + f_t \otimes g_t &= \sum_{i=1}^{t-1} f_i \otimes g_i + (-\sum_{i=1}^{t-1} g_t(b)^{-1}g_i(b)f_i) \otimes g_t \\ &= \sum_{i=1}^{t-1} f_i \otimes (g_i - g_t(b)^{-1}g_i(b)g_t) \end{aligned}$$

we replace $g'_i = g_i - g_t(b)^{-1}g_i(b)g_t$, so

$$\sum_{i=1}^{t} f_i \otimes g_i = \sum_{i=1}^{t-1} f_i \otimes g'_i,$$

we calculate similar with $\sum\limits_{i=1}^{t-1} f_i \otimes g'_i$, finaly, we will have $f \in A(X), g \in A(Y)$ such that $\sum\limits_{i=1}^{t} f_i \otimes g_i = f \otimes g$, if $g \neq 0$, there is $b \in Y$ such that $g(b) \neq 0$, for every $x \in X$, we have $fg(x, b) = f(x).g(b) = 0$, this shows that $f = 0$, hence $f \otimes g = 0 \otimes g = 0$, thus ϕ is injective.

Consequence, ϕ is a ring isomorphism and $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

3.20 *Group varieties.* A group variety consists of a variety Y together with a morphism $\mu : Y \times Y \to Y$ such that the set of the point Y with the operation given by μ is a group, and such that the inverse map $y \to y^{-1}$ is also a morphism of $Y \to Y$.

(a) The *additive group* $\mathbb{G}_a$ is given by the varieties $\mathbb{A}^1$ and the morphism $\mu : \mathbb{A}^2 \to \mathbb{A}^1$ defined by $\mu(a, b) = a + b$. Show it is a group variety.

We need to verify the maps

$$\begin{aligned} \mu : \mathbb{A}^1 \times \mathbb{A}^1 &\to \mathbb{A}^1 \\ (a, b) &\mapsto a + b, \end{aligned}$$

$$\begin{aligned} i : \mathbb{A}^1 &\to \mathbb{A}^1 \\ x &\mapsto -x \end{aligned}$$

are morphism. We can see that μ and i are the morphisms because they are defined by the polynomials.
We have $i(x) = x$ if and only if $x = 0$, so the the identity element is 0, addition on $\mathbb{A}^1$ is the addition on field k, then $\mathbb{G}_a$ is a group variety.

(b) The *multiplicative group* $\mathbb{G}_m$ is given by variety $\mathbb{A}^1 - \{0\}$ and the morphism $\mu(a, b) = a.b$, show it is a group variety.

- $\mathbb{A}^1 - \{0\} \cong \{(x, y) \in \mathbb{A}^2 : xy - 1 = 0\}$. This show that $\mathbb{A}^1 - \{0\}$ is a variety of $\mathbb{A}^2$.
- $(\mathbb{A}^1 - \{0\}, .)$ is a group, the identity element is 1, the inverse of x is $\frac{1}{x}$.
- The maps μ and

$$\begin{aligned} i : \mathbb{A}^1 - \{0\} &\to \mathbb{A}^1 - \{0\} \\ x &\mapsto \frac{1}{x} \end{aligned}$$

are the morphisms.

Thus we have $\mathbb{G}_m$ is a group variety.

(c) If G is a group variety, and X is any variety, show that the set $Hom(X, G)$ has a natural group structure.
Suppose that G is a group variety defined by the operation μ, then μ

is a morphism. For $f, g \in Hom(X, G)$, consider the morphism $\nu \in Hom(X, G)$ constructs by

$$\begin{aligned} \nu : X \to X \times X \to \ G \times G \ \to \ G \\ x \mapsto (x, x) \mapsto (f(x), g(x)) \mapsto \mu(f(x), g(x)), \end{aligned}$$

suppose that e is the identity element of G, then the map $f(x) = e$ for all x is the identity element of $Hom(X, G)$. So $Hom(X, G)$ is a group is defined by the operation

$$\begin{aligned} Hom(X, G) \times Hom(X, G) \to Hom(X, G) \\ (f, g) \mapsto \mu(f, g), \mu(f, g)(x) = \mu(f(x), g(x)). \end{aligned}$$

Rational maps

Exercise 4.1. Let f and g be the regular functions on U and V, $f = g$ on $U \cap V$, the function h is defined by

$$h(P) = \begin{cases} f(P) \text{ if } P \in U \\ g(P) \text{ if } P \in V, \end{cases}$$

for every points $P \in U \cup V$, then $P \in U$ or $P \in V$, if $P \in U$, then $f(P) = h(P)$, since f is regular at P, so we have that h is regular at P. Similarly, if $P \in V$, then h is regular at P. Thus h is regular on $U \cup V$.

If f is a rational function on X, there is an open set U of X such that f is regular on U. Consider the set

$$\mathcal{T} = \{U : U \text{ is open on X}, f \text{ is regular on} \, U\}.$$

The set $\bigcup\limits_{U \in \mathcal{T}} U$ is open in X, this set is the largest open set of X on which f is regular. We say that f is defined at the points of this set.

Exercise 4.3. Let f be the rational function on $\mathbb{P}^2$ given by $f = x_1/x_0$.

a) f is regular on $U_0 = \{(x_0 : x_1 : x_2) : x_0 \neq 0\}$, f is defined on the points of U_0. Suppose that we have an open set U such that f is regular on U and $U \supsetneq U_0$, then there is $P = (p_0 : p_1 : p_2) \in U \setminus U_0$, so $p_0 = 0$, this shows that f is not regular at P (contradiction), hence U_0 is the largest open set on which f is regular. The corresponding regular function on U_0 is x_1/x_0, we can replace x_0 by 1.

b) Consider the embedding

$$\begin{aligned} \tau : \mathbb{A}^1 &\hookrightarrow \mathbb{P}^1 \\ x &\mapsto (x : 1). \end{aligned}$$

See f as a rational map from $\mathbb{P}^2$ to $\mathbb{A}^1$, we have the resulting map

$$\begin{aligned} \varphi : \mathbb{P}^2 &\to \mathbb{P}^1 \\ (x_0 : x_1 : x_2) &\mapsto (x_1/x_0 : 1). \end{aligned}$$

therefore, $\varphi = \tau \circ f$. Since $f \mid_U$ is a morphism, then $\varphi \mid_{U_0} = \tau \circ f \mid_{U_0}$ is a morphism. Describe $\varphi \mid_{U_0}$,

$$\begin{aligned} \varphi \mid_{U_0} : U_0 &\to \mathbb{P}^1 \\ (x_0 : x_1 : x_2) &\mapsto (x_1/x_0 : 1) = (x_1 : x_0). \end{aligned}$$

Exercise 4.4. *Rational varieties.* A variety Y is rational if it is birationally equivalent to $\mathbb{P}^n$ for some n (or if $K(Y)$ ia a pure transcendental extension of k)

a) Any *conic* in $\mathbb{P}^2$ is a rational curve.

Exercise 3.1-(a) shows that any conic in $\mathbb{P}^2$ isomorphism to $\mathbb{P}^1$. Since if f is a morphism then f is a rational map. Then we have that the conics in $\mathbb{P}^2$ and $\mathbb{P}^1$ are birationally equivalent, thus the conics in $\mathbb{P}^2$ are rational.

b) The *cuspidal cubic* $y^2 - x^3$ is a rational curve.

Let $V = \{(x, y) \in \mathbb{A}^2 : x^3 - y^2 = 0\} = \{(t^2, t^3) \in \mathbb{A}^2 : t \in \mathbb{A}^1\}$. Consider the map

$$\begin{aligned} \varphi : \mathbb{A}^1 &\to V \\ t &\mapsto (t^2, t^3), \end{aligned}$$

then the invert of φ is

$$\begin{aligned} \varphi^{-1} : V &\to \mathbb{A}^1 \\ (x, y) &\mapsto \begin{cases} 0 \text{ if } x = 0 \\ y/x \text{ if } x \neq 0. \end{cases} \end{aligned}$$

In deed, if $x \neq 0, (x, y) \in V$, we have

$$\varphi \circ \varphi^{-1}(x, y) = \varphi(y/x) = (y^2/x^2, y^3/x^3) = (x^3/x^2, y^3/y^2) = (x, y),$$

and for every $t \in \mathbb{A}^1$, $\varphi^{-1} \circ \varphi(t) = t$, then φ is a birational map. Thus the cuspidal cubic is rational.

c) Projection φ from the point $P = (0 : 0 : 1)$ to the line $z = 0$.

For every $Q \in \mathbb{P}^2$, we can write $Q = (q_0 : q_1 : 0)$ or $Q = (q_0 : q_1 : 1)$, the projection can be written as follows

$$\begin{aligned} \varphi : \mathbb{P}^2 - P &\to \mathbb{P}^1 \\ (q_0 : q_1 : 0) &\mapsto (q_0 : q_1), \\ (q_0 : q_1 : 1) &\mapsto (q_0 : q_1). \end{aligned}$$

The curve $(C) : y^2 z = x^2(x + z) \subseteq \mathbb{P}^2$, suppose that $Q = (q_0 : q_1 : q_2) \in (C)$, if $q_2 \neq 0$, then we can write again $Q = (q_0 : q_1 : 1)$. If $q_2 = 0$, then $q_1 = 0, q_2 = 1$. Thus

$$\begin{aligned} \theta = \varphi\,|_{(C)} : (C) &\to \mathbb{P}^1 \\ (q_0 : q_1 : 1) &\mapsto (q_0 : q_1), \\ (0 : 1 : 0) &\mapsto (0 : 1). \end{aligned}$$

The invert of θ is

$$\begin{aligned}\theta^{-1} : \mathbb{P}^1 &\to (C)\\ (q_0 : q_1) &\mapsto\end{aligned}$$

if $z = 1$ then $y^2 = x^3 + x^2$, so $x^3 = y^2 - x^2$

Exercise 4.5 The quadratic surface $Q : xy = zw$ in $\mathbb{P}^3$ is birational to $\mathbb{P}^2$, but not isomorphic to $\mathbb{P}^2$.

Solution: We consider the map

$$\begin{aligned}\varphi : Q &\to \mathbb{P}^2\\ (x : y : z : t) &\mapsto \begin{cases} (x : y : z) \text{ if } z \neq 0\\ (x : y : t) \text{ if } z = 0.\end{cases}\end{aligned}$$

Let $U = \{(x : y : z : t) \in Q : z \neq 0\}$ be an open subset of Q, since $\varphi_{|U}$ is a morphism of U to $\mathbb{P}^2$, then φ is a rational map.

We consider the map

$$\begin{aligned}\phi : \mathbb{P}^2 &\to Q\\ (x : y : z) &\mapsto \begin{cases} (x : y : z : \frac{xy}{z}) \text{ if } z \neq 0\\ (x : 0 : y : 0) \text{ if } z = 0.\end{cases}\end{aligned}$$

Let $U' = \{(x : y : z) \in Q : z \neq 0\}$ be an open subset of $\mathbb{P}^2$, since $\phi_{|U'}$ is a morphism of U' to Q, then ϕ is a rational map.

Let $P = (x : y : z : t) \in U \subset Q$, since $z \neq 0$, it shows that $t = \frac{xy}{z}$, we have

$$(\phi \circ \varphi)(P) = \phi(x : y : z) = (x : y : z : \frac{xy}{z}) = (x : y : z : t) = P.$$

Hence $\phi \circ \varphi = id_Q$.

Let $P' = (x : y : z) \in U' \subset \mathbb{P}^2$, we have

$$(\varphi \circ \phi)(P) = \varphi(x : y : z : \frac{xy}{z}) = (x : y : z) = P'.$$

Hence $\varphi \circ \phi = id_{\mathbb{P}^2}$.

Therefore φ is a birational map, consequence that the quadratic surface $Q : xy = zw$ in $\mathbb{P}^3$ is birational to $\mathbb{P}^2$. But it is not isomorphic to $\mathbb{P}^2$, we can prove this sentence as follows.

The linear varieties

$$L_1 : \begin{cases} x = z \\ y = w \end{cases}, L_2 : \begin{cases} x = 2z \\ 2y = w \end{cases}$$

are the lines of Q, we have $L_1 \cap L_2 = \emptyset$ in $\mathbb{P}^3$. But Exercise 2.11.c shows that the intersection of two lines in $\mathbb{P}^2$ is not empty. This shows that Q is not isomorphic to $\mathbb{P}^2$.

Exercise 4.6. *Plane Cremona Transformations.* A birational map of $\mathbb{P}^2$ into itself is called a Plane Cremona Transformations. We give an example, called a quadratic transformation. It is the rational map $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ given by $(a_0 : a_1 : a_2) \to (a_1a_2 : a_0a_2 : a_0a_1)$ when no two of a_0, a_1, a_2 are 0.

a) Show that φ is birational map, and is it own inverse?

   We consider the open set $U = \{(a_0 : a_1 : a_2) \in \mathbb{P}^2 : a_0 \neq 0, a_1 \neq 0, a_2 \neq 0\}$ of $\mathbb{P}^2$. We have

$$\begin{aligned} \varphi^2(a_0 : a_1 : a_2) &= \varphi(a_1a_2 : a_0a_2 : a_0a_1) = (a_0^2a_1a_2 : a_0a_1^2a_2 : a_0a_1a_2^2) \\ &= (a_0 : a_1 : a_2) \end{aligned}$$

   for any $(a_0 : a_1 : a_2) \in U$, then $\varphi^2 = id_{\mathbb{P}^2}$, thus φ is birational map, and it is own inverse.

b) Find open set $U, V \subseteq \mathbb{P}^2$ such that $\varphi : U \to V$ is an isomorphism.

   We choose

$$U = V = \{(a_0 : a_1 : a_2) \in \mathbb{P}^2 : a_0 \neq 0, a_1 \neq 0, a_2 \neq 0\}.$$

   We have that φ is a morphism of U to U, the inverse is itself.

c) Find the open set where φ is defined, describe the corresponding morphisms.

   φ is defined by the openset $\mathbb{P}^2 - \{(1 : 0 : 0), (0 : 1 : 0, (0 : 0 : 1)\}$.

Exercise 4.10. Let Y be the cuspidal curve $y^2 = x^3$ *in* A^2. Blow up Y at the point $O = (0, 0)$.

Let X be the closed subset of $\mathbb{A}^2 \times \mathbb{P}^1$ defined by $\{((x, y), (u : t)) \in \mathbb{A}^2 \times \mathbb{P}^1 : xt = uy\}$. The map $\varphi : X \to \mathbb{A}^2$ is the blowing-up of the point O. The exceptional curve E is $\varphi^{-1}(0) \cong \mathbb{P}^1$. We find the inverse image of Y in X by considering the equations $y^2 = x^3$ and $xt = uy$. Since $\mathbb{P}^1$ is covered by $u \neq 0$ and $t \neq 0$, then we consider separately.

If $u \neq 0$, then we can set $u = 1$, and use t as a parameter. Then we have the equations

$$\begin{cases} y = xt \\ y^2 = x^3 \end{cases} \Rightarrow x^2t^2 - x^3 = 0 \Rightarrow x^2(t^2 - x) = 0.$$

We solve this equations and we have

$$\begin{cases} x = y = 0 \\ t \in \mathbb{A}^1 \end{cases} \text{or} \begin{cases} y = xt \\ t^2 - x = 0. \end{cases}$$

Hence

$$\widetilde{Y} = \{((x, y), (1 : t)) \in \mathbb{A}^2 \times \mathbb{P}^1 : xt - y, t^2 - x\}$$

.

We have that E meets $\widetilde{Y}$ at one point $t = 0$.

Therefore, we have the morphism $\varphi : \widetilde{Y} - ((0, 0), (1 : 0)) \longrightarrow Y$, and the rational map $\varphi : \widetilde{Y} \longrightarrow Y$.

We can write again $\widetilde{Y}$ as follows

$$\widetilde{Y} = \{((t^2, t^3), (1 : t)) \in \mathbb{A}^2 \times \mathbb{P}^1 : t \in \mathbb{A}^1\}.$$

Hence, we consider the map

$$\begin{aligned} \tau : \widetilde{Y} &\to \mathbb{A}^1 \\ ((t^2, t^3), (1 : t)) &\mapsto t. \end{aligned}$$

τ is a morphism, τ is surjective, the inverse of τ is

$$\begin{aligned} \tau^{-1} : \mathbb{A}^1 &\to \widetilde{Y} \\ t &\mapsto ((t^2, t^3), (1 : t)). \end{aligned}$$

This shows that $\widetilde{Y} \cong \mathbb{A}^1$.

If $t \neq 0$, we have the equations

$$\begin{cases} x = yu \\ y^2 = x^3 \end{cases} \Rightarrow y^2 - y^3u^3 = 0 \Rightarrow y^2(1 - yu^3) = 0.$$

We solve this equations and we have

$$\begin{cases} x = y = 0 \\ u \in \mathbb{A}^1 \end{cases} \text{or} \begin{cases} x = yu \\ yu^3 - 1 = 0. \end{cases}$$

Therefore we have

$$\widetilde{Y_u} = \{((x, y), (u : 1)) \in \mathbb{A}^2 \times \mathbb{P}^1 : x - uy, yu^3 - 1\}$$

.

We have E doesn't meet $\widetilde{Y_u}$.

Nonsingular Varieties

Exercise 5.1. Locate the singular points and sketch the following curve in $\mathbb{A}^2$. Assume char $k \neq 2$.

(a) $x^2 = x^4 + y^4$;

(b) $xy = x^6 + y^6$;

(c) $x^3 = x^2 + x^4 + y^4$;

(d) $x^2y + xy^2 = x^4 + y^4$;

Solution:

(a) $f(x, y) = x^2 - x^4 - y^4$, then we have

$$\frac{\partial f(x,y)}{\partial x} = 2x - 4x^3,$$
$$\frac{\partial f(x,y)}{\partial y} = -4y^3.$$

So, the Jacobi matrix at (x_0, y_0) is the matrix

$$J = |2x_0 - 4x_0^3 \quad - 4y_0^3|.$$

Suppose $Y = V(f(x, y))$, then $dimY = 2 - 1 = 1$. So Y is nonsigular at the point (x_0, y_0) if $rankJ = 1$. This shows that either $2x_0 - 4x_0^3 \neq 0$ or $-4y_0^3 \neq 0$, then either ($x \neq 0$, $x \neq \sqrt{\frac{1}{2}}$ and $x \neq -\sqrt{\frac{1}{2}}$) or $y \neq 0$. Hence, the points can be the singular points $(0, 0), (\sqrt{\frac{1}{2}}, 0), (-\sqrt{\frac{1}{2}}, 0)$. But $(\sqrt{\frac{1}{2}}, 0), (-\sqrt{\frac{1}{2}}, 0)$ is not belong to Y. It shows that Y has only singular point $(0, 0)$.
We fix $y_0 \in k$, then $x^2 - x^4 - y_0^4$ is an even function, then its graph remains unchanged after reflection about the $y - axis$. So we see Figure 4., Tacnode satisfies this condition.

(b) $xy = x^6 + y^6$;

Let (x_0, y_0) be the singular point of $V(xy - x^6 - y^6)$, then $x_0, y_0)$ is a solution of the equations

$$\begin{cases} xy - x^6 - y^6 & = 0 \\ y - 6x^5 & = 0 \\ x - 6y^5 & = 0 \end{cases} \Leftrightarrow \begin{cases} xy - xy/6 - yx/6 & = 0 \\ x^5 & = y/6 \\ y^5 & = x/6, \end{cases}$$

it shows that $xy = 0$, then we have $x = 0$ or $y = 0$, if $x = 0$ then $y = 0$, if $y = 0$ then $x = 0$, thus $x = y = 0$.

From the equation $xy = x^6 + y^6$, we have $xy \geq 0$, so the picture of Node satisfies.

(c) $x^3 = x^2 + x^4 + y^4$;

Let (x_0, y_0) be the singular point of $V(x^3 - x^2 - x^4 - y^4)$, then (x_0, y_0) is a solution of the equations

$$\begin{cases} x^3 - x^2 - x^4 - y^4 & = 0 \\ 3x^2 - 2x - 4x^3 & = 0 \\ -4y^3 & = 0 \end{cases} \Leftrightarrow \begin{cases} x^2(x - 1 + x^2) & = 0 \\ x(3x - 2 - 4x^2) & = 0 \\ y & = 0, \end{cases}$$

if α is a solution of $x - 1 + x^2$, then $\alpha^2 = 1 - \alpha$, if α is a solution of $3x - 2 - 4x^2$, then $\alpha^2 = (3\alpha - 2)/4$, so we have $1 - \alpha = (3\alpha - 2)/4$, thus $\alpha = 6/7$, but $6/7$ is not a solution of $x - 1 + x^2$. Hence this equations have only one solution $(0, 0)$.

The equation $x^3 = x^2 + x^4 + y^4$ shows that $x \geq 0$, this is Cusp. (I am considering the picture on $\mathbb{A}^2 \cap \mathbb{R}^2$.)

(d) $x^2y + xy^2 = x^4 + y^4$;
The picture is Triple point.

Let (x_0, y_0) be the singular point of $V(x^2y + xy^2 - x^4 - y^4)$, then (x_0, y_0) is a solution of the equations

$$\begin{cases} x^2y + xy^2 - x^4 - y^4 & = 0 \\ 2xy + y^2 - 4x^3 & = 0 \\ x^2 + 2xy - 4y^3 & = 0 \end{cases} \Leftrightarrow \begin{cases} x^2y + xy^2 - x^4 - y^4 & = 0 \\ x^3 & = \frac{2xy+y^2}{4} \\ y^3 & = \frac{x^2+2xy}{4}, \end{cases}$$

then we have

$$\begin{aligned} & x^2y + xy^2 - x^4 - y^4 && = 0 \\ \Leftrightarrow & x^2y + xy^2 - x\frac{2xy + y^2}{4} - y\frac{x^2 + 2xy}{4} && = 0 \\ \Leftrightarrow & 4x^2y + 4xy^2 - 2x^2y - xy^2 - x^2y - 2xy^2 && = 0 \\ \Leftrightarrow & xy(4x + 4y - 2x - y - x - 2y) && = 0 \\ \Leftrightarrow & xy(x + y) && = 0 \\ \Leftrightarrow & xy = 0 \,\text{or}\, x = -y \end{aligned}$$

So, if $xy = 0$ then $x = 0$ or $y = 0$, if $x = 0$ then $y = 0$, if $y = 0$ then $x = 0$, so $(0, 0)$ is a solution.

If $x = -y$, then we have $x^2(-x) + xx^2 - x^4 - x^4 = 0$, then $x^4 = 0$, thus $x = 0$, hence $y = -x = 0$. Thus we have only one solution $(0, 0)$.

Exercise 5.2. Locate the singular points

(a) $xy^2 = z^2$.

Let $f(x, y, z) = xy^2 - z^2$, then we have

$$\frac{\partial f}{\partial x} = y^2,$$
$$\frac{\partial f}{\partial y} = 2xy,$$
$$\frac{\partial f}{\partial z} = -2z.$$

So, the Jacobi matrix at (x_0, y_0, z_0) is the matrix

$$J = |y_0^2 \quad 2x_0y_0 \quad -2z_0|.$$

Hence, (x_0, y_0, z_0) is a singular point if $y_0^2 = 2x_0y_0 = -2z_0 = 0$, it shows that $y_0 = z_0 = 0, x_0$ free. The singular points on the $x - axis$. The picture of Pinch point.

(b) $x^2 + y^2 = z^2$.

Let $g(x, y, z) = x^2 + y^2 - z^2$, then we have

$$\frac{\partial g}{\partial x} = 2x,$$
$$\frac{\partial g}{\partial y} = 2y,$$
$$\frac{\partial g}{\partial z} = -2z.$$

(x_0, y_0, z_0) is a singular point if $2x_0 = 2y_0 = -2z_0 = 0$, then we have $x_0 = y_0 = z_0 = 0$, and $(0, 0, 0) \in V(g)$, so $V(g)$ has only one singular point $(0, 0, 0)$. The picture is Conical double point.

Exercise 5.3. For $f(x,y) \in k[x,y]$, we can write f as a sum

$$f = f_0 + f_1 + \cdots + f_d,$$

where f_i is a homogeneous polynomial of degree i in x and y. The multiplicity of $P = (0,0)$ on Y, denote $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. If $P = (0,0) \in Y$ then $f_0 = 0$, so $\mu_P(Y) > 0$.

(a) Show that $\mu_P(Y) = 1 \Leftrightarrow P = (0,0)$ is a nonsingular point of Y.

(b) Find the multiplicity of each of the singular points in Ex. 5.1 above.

Solution:

(a) $\mu_P(Y) = 1 \Leftrightarrow f_1 \neq 0 \Leftrightarrow$ there are a, b such that $f_1 = ax + by$, $a \neq 0$ or $b \neq 0 \Leftrightarrow \frac{\partial f_1(x,y)}{\partial x} = a \neq 0$ or $\frac{\partial f(x,y)}{\partial y} = b \neq 0 \Leftrightarrow \frac{\partial f(x,y)}{\partial x}(0,0) = a \neq 0$ or $\frac{\partial f(x,y)}{\partial y}(0,0) = b \neq 0 \Leftrightarrow P = (0,0)$ is a nonsingular point of Y.

Remark: for any $P \in Y$, we can change the coordinates such that P be comes to the point $(0,0)$, so we have $\mu_P(Y)$, which is found by the new polynomial after changing coordinates.

(b) Find the multiplicity of each of the singular points in Ex. 5.1 above.

All of them have only one singular point $P = (0,0)$.

For 5.1 a, $\mu_P(V(x^2 - x^4 - y^4)) = 2$

For 5.1 b, $\mu_P(V(xy - x^6 - y^6)) = 2$

For 5.1 c, $\mu_P(V(x^3 - y^2 - x^4 - y^4)) = 2$

For 5.1 d, $\mu_P(V(x^2y + xy^2 - x^4 - y^4)) = 3$.

Exercise 5.6.

(a) • Let Y be the cusp, then $Y = Z(x^3 - y^2 - x^4 - y^4)$, Y has only singular point $O(0,0)$. We will blow up Y at O. Let X be the blowing up at O of $\mathbb{A}^2$, then X is the closed subset of $\mathbb{A}^2 \times \mathbb{P}^1$ defined by the equation $xu = yt$, with $((x,y),(t:u)) \in \mathbb{A}^2 \times \mathbb{P}^1$. We have the inverse of Y in X considering by the equations $x^3 - y^2 - x^4 - y^4 = 0$ and $xu = yt$.

On the chart $t \neq 0$, we can set $t = 1$, then $y = xu$ and $x^2(x - u^2 - x^2 - x^2u^4) = 0$. Then $x = y = 0, u$ free,this is E, or $x - u^2 - x^2 - x^2u^4 = 0, y = xu$, this is $\widetilde{Y_t}$. We replace $y = xu$, we have $\widetilde{Y_t}$ is defined by

$x - u^2 - x^2 - y^2u^2 = 0, y - xu = 0$. Set $f(x, y, u) = x - u^2 - x^2 - y^2u^2$, then

$$\frac{\partial f}{\partial x} = 1 - 2x$$
$$\frac{\partial f}{\partial y} = -2yu^2$$
$$\frac{\partial f}{\partial u} = -2y^2u.$$

We have $\widetilde{Y_t} \cap E$ at $(0, 0, 0)$. The Jacobi matrix at $(0, 0, 0)$ is

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since $\widetilde{Y} - \varphi^{-1}(O)$ is isomorphic to $Y - O$,, then $dim\widetilde{Y} = dimY = 1, rankJ = 2 = 3 - 1$. So $\widetilde{Y}$ is nonsingular point at $(0, 0, 0)$.

On the chart $u \neq 0$, set $u = 1$, we have $x = yt$, the considering the equations $y^3t^3 - y^2 - y^4t^4 - y^4 = 0$ and $x = yt$. So $y^2(yt^3 - 1 - y^2t^4 - y^2) = 0$, then $\widetilde{Y_u}$ is defined by the equations $yt^3 - 1 - y^2t^4 - y^2 = 0$ and $x = yt$. We have $\widetilde{Y_u} \cap E = \emptyset$.

This shows that $\varphi^{-1}(O) \cap \widetilde{Y} = \{((0, 0), (1 : 0))\}$. $\widetilde{Y} - \varphi^{-1}(O)$ is isomorphic to $Y - O$, then $\widetilde{Y} - \varphi^{-1}(O)$ is nonsingular point, and $\widetilde{Y}$ is nonsingular point at $(0, 0, 0)$, thus $\widetilde{Y}$ is nonsingular point.

• Let C be the Node, $C = Z(xy - x^6 - y^6)$ has only singular point at $O = (0, 0)$.

On the chart $t \neq 0$, consider the equations $y = xu$ and $xy - x^6 - y^6 = 0$, then we have $x^2u - x^6 - x^6u^6 = 0$, so $x^2(u - x^4 - x^4u^6)$,thus we have $\widetilde{C_t}$ defined by the equations $u - x^4 - x^4u^6 = 0$ and $y - xu = 0$. $\widetilde{C_t} \cap E$ at $(0, 0, 0) := ((0, 0), (1 : 0))$. The Jacobi matrix at $(0, 0, 0)$ is

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since $rankJ = 2 = codim\widetilde{C}$, then $\widetilde{C}$ is nonsingular at $((0, 0), (1 : 0))$.

On the chart $u \neq 0$, set $u = 1$, consider the equations $x = yt$ and $xy - x^6 - y^6 = 0$, then we have $y^2t - y^6 - y^6t^6 = 0$, so $y^2(t - y^4 - y^4t^6)$,thus we have $\widetilde{C}_u$ is defined by the equations $t - y^4 - y^4t^6 = 0$ and $x - yt = 0$. $\widetilde{C}_u \cap E$ at $(0,0,0) := ((0,0),(0:1))$. We do similar way on the chart $t \neq 0$. Consequence, $\widetilde{C}$ is nonsingular point.

(b) $Y = Z(f)$, and let P be a node. Make a linear change of coordinates so that P be come to the point $(0,0)$, then $f = xy + g(x,y)$, where $g(x,y)$ has only term of degree greater then 2, $a^2 + b^2 + c^2 \neq 0$.

On the chart $t \neq 0$, we can set $t = 1$, consider the equations $f(x,y) = 0$ and $y = xu$, we have

$$xy + g(x,y) = ux^2 + g(x,xu) = 0,$$

hence,

$$x^2(u + x^{-2}g(x,xu)) = 0.$$

One has $x^{-2}g(x,xu)) = h(x,u)$ has only term of degree greater than 0, so we have $h(0,0) = 0$. By this equation, we get two irreducible components, the Exceptional curve E on the chart $t \neq 0$ defined by $x = 0, y = 0, u$ is free, $\widetilde{Y}$ is defined by the equations $u + h(x,u) = 0, y - xu = 0$. Since $h(0,0) = 0$, then $u = 0$, thus $E \cap \widetilde{Y} = (0,0,0) =: ((0,0),(1:0))$. The Jacobi matrix at $(0,0,0)$ is

$$J_1 = \begin{bmatrix} \frac{\partial h}{\partial x}(0,0,0) & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We have $dimY = dim\widetilde{Y} = 1$, then

$$rankJ_1 = codim\widetilde{Y},$$

it shows that $\widetilde{Y}$ is nonsingular at $((0,0),(1:0))$.

On the chart $u \neq 0$, we can set $u = 1$, consider the equations $f(x,y) = 0$ and $x = yt$, we have

$$xy + g(x,y) = ty^2 + g(y,yt) = 0,$$

hence,

$$y^2(t + x^{-2}g'(y,yt)) = 0.$$

One has $y^{-2}g(y, yt)) = h'(y,t)$ has only term of degree greater than 0, so we have $h'(0,0) = 0$. By this equation, we get two irreducible components, the Exceptional curve E on the chart $u \neq 0$ defined by $x = 0, y = 0, t$ is free, $\widetilde{Y}$ is defined by the equations $t + h'(y,t) = 0, x - yt = 0$. Since $h'(0,0) = 0$, then $t = 0$, thus $E \cap \widetilde{Y} = (0,0,0) =: ((0,0),(0:1))$. The Jacobi matrix at $(0,0,0)$ is

$$J_2 = \begin{bmatrix} \frac{\partial h'}{\partial x}(0,0,0) & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We have $dimY = dim\widetilde{Y} = 1$, then

$$rankJ_1 = codim\widetilde{Y},$$

it shows that $\widetilde{Y}$ is nonsingular at $((0,0),(0:1))$. Thus, $E \cap \widetilde{Y} = \{((0,0),(0:1)),((0,0),(0:1))\}$, and they are nonsingular points.

(c) Let $P \in Y = Z(x^2 - x^4 - y^4)$ be the tacnode, and $\varphi : \widetilde{Y} \to Y$ is the blowing up at P. Show that $\varphi^{-1}(P)$ is a node.

On the chart $t \neq 0$, we can set $t = 1$, consider the equations $x^2 - x^4 - y^4 = 0$ and $y = xu$, we have

$$x^2 - x^4 - y^4 = x^2 - x^4 - x^4u^4 = x^2(1 - x^2 - x^2u^4) = 0,$$

By this equation, we get two irreducible components, the Exceptional curve E on the chart $t \neq 0$ defined by $x = 0, y = 0, u$ is free, $\widetilde{Y}$ is defined by the equations $1 - x^2 - x^2u^4 = 0, y - xu = 0$. Then we have $E \cap \widetilde{Y} = \emptyset$.

On the chart $u \neq 0$, we can set $u = 1$, consider the equations $x^2 - x^4 - y^4 = 0$ and $x = yt$, we have

$$x^2 - x^4 - y^4 = t^2y^2 - y^4t^4 - y^4 = y^2(t^2 - y^2t^4 - y^2) = 0,$$

By this equation, we get two irreducible components, the Exceptional curve E on the chart $u \neq 0$ defined by $x = 0, y = 0, t$ is free, $\widetilde{Y}$ is defined by the equations $t^2 - y^2t^4 - y^2, x - yt = 0$. Thus $E \cap \widetilde{Y} = (0,0,0) =: ((0,0),(0:1)) = \varphi^{-1}(P)$.

Since $t^2 - y^2t^4 - y^2 = t^2 - y^2 - y^2t^4 = (t-y)(t+y) - y^2t^4$, with change of $s = t - y, v = t + y$, $\varphi^{-1}(P)$ is a node. Using b), then tacnode cab br resolved by two successive blowings-up.

(d) $Y = Z(y^3 - x^5)$. Since the lowest term equal to 3, then we have a triple point.

On the chart $t \neq 0$, we can set $t = 1$, consider the equations $y^3 - x^5 = 0$ and $y = xu$. Then we have $x^3u^3 - x^5 = x^3(u^3 - x^2) = 0$. By this equation, we get two irreducible components, the Exceptional curve E on the chart $t \neq 0$ defined by $x = 0, y = 0, u$ is free, $\widetilde{Y}$ is defined by the equations $1 - x^2 - x^2u^4 = 0, y - xu = 0$. Then we have $E \cap \widetilde{Y} = ((0,0),(1,0))$. The point $(0,0)$ of $u^3 - x^2 = 0$ is a tacnode point. By c), this point can be resolved.

On the chart $u \neq 0$, we can set $u = 1$, consider the equations $y^3 - x^5 = 0$ and $x = yt$. Then we have $y^3 - y^5t^5 = y^3(1 - y^2t^5) = 0$. By this equation, we get two irreducible components, the Exceptional curve E on the chart $t \neq 0$ defined by $x = 0, y = 0, t$ is free, $\widetilde{Y}$ is defined by the equations $1 - y^2t^5, x - yt = 0$. Then we have $E \cap \widetilde{Y} = \emptyset$.

Exercise 5.7. f is a homogeneous polynomial of degree greater than 1. $Y = Z(f) \subseteq \mathbb{P}^2$ is nonsigular point , $X = Z(f) \subseteq \mathbb{A}^3$.

(a) Show that X has one singular point, namely P.

If $Q \neq P = (0,0,0)$ then Q is nonsingular point (since Y is nonsingular point). With the point $P = (0,0,0)$, the degree of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are greater then 0, then $\frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$. Thus $P \in X$ is a singular point.

(b) $\widetilde{X}$ is nonsingular point.

$\widetilde{X} \subseteq \mathbb{A}^3 \times \mathbb{P}^2$, suppose $((x,y,z),(t : u : v)) \in \widetilde{X}$, then x, y, z, u, v, t satisfies the equations $f(x,y,z) = 0, xu = yt, xv = zt, yv = zu$.

On the chart $t \neq 0$, we can set $t = 1$, then we consider the equations $f(x,y,z) = 0, xu = y, xv = z, yv = zu$, then $f(x, xu, xv) = 0$, since f is a homogeneous polynomial, set $d = degree(f)$, we have $x^d f(1,u,v) = 0$. By this equation, we have two components: the exceptional curve E is defined by $x = y = z = 0, u, v$ is free; $\widetilde{X}$ is defined by the equations $f(1,u,v) = 0, xu = y, xv = z, yv = zu$. Thus

$$\widetilde{X} \cap E = \{((0,0,0),(1 : u : v)) | f(1,u,v) = 0\} \cong Y \cap U_0$$

with the chart $U_0 = \{t \neq 0\}$.

Similar, on the chart $u \neq 0$, set $u = 1$, we have

$$\widetilde{X} \cap E = \{((0,0,0),(t:1:v)) | f(t,1,v) = 0\} \cong Y \cap U_1$$

with the chart $U_1 = \{u \neq 0\}$.

On the chart $v \neq 0$, set $v = 1$, we have

$$\widetilde{X} \cap E = \{((0,0,0),(t:u:1)) | f(t,u,1) = 0\} \cong Y \cap U_2$$

with the chart $U_2 = \{v \neq 0\}$.

Therefore,

$$\widetilde{X} \cap E \cong Y. \text{ (the answer of question c))}$$

Since Y is nonsinguar point, then $\widetilde{X} \cap E$ is nonsingular point. This show that $\widetilde{X}$ is nonsingular point.(the answer of question b)).

Exercise 5.12. Quadric Hypersurfaces.

(a)

(b)

(c) Suppose that $f = x_0^2 + x_1^2 + \cdots + x_r^2$, $Q = Z(f) \subseteq \mathbb{P}^n$, then we have the Jacobi matrix at $(x_0, x_1, \ldots, x_n)$ is

$$J = \begin{bmatrix} 2x_0 & 2x_1 & \ldots & 2x_r & 0 & \ldots & 0 \end{bmatrix}.$$

Then

$$Z = SingQ = \{(0 : \cdots : 0 : x_{r+1} : \cdots : x_n) \in \mathbb{P}^n | x_i \in k, \forall i = r+1, ..., n\}.$$

Z is a linear variety, $dimZ = n-(r+1) = n-r-1$, thus $Z = \emptyset$ if $n = r$, then Q is nonsingular point.

(d) Suppose that $f = x_0^2 + x_1^2 + \cdots + x_r^2$, $Q = Z(f) \subseteq \mathbb{P}^n$, consider the embed

$$\mathbb{P}^r \hookrightarrow \mathbb{P}^n$$
$$(x_0 : \cdots : x_r) \mapsto (x_0 : \cdots : x_r : 0 \cdots : 0)$$

Let $Q' = Z(x_0^2 + x_1^2 + \cdots + x_r^2) \in \mathbb{P}^r$, Using c), we have that Q' is nonsingular point.

Recall

$$Z = SingQ = \{(0 : \cdots : 0 : x_{r+1} : \cdots : x_n) \in \mathbb{P}^n | x_i \in k, \forall i = r+1, ..., n\}.$$

By define of the embed, we have $Q' \cap Z = \emptyset$, and $dimZ = n-r-1$, thus Q is a cone with axis Z over a nonsingular point quadric hypersurface Q'.

Exercise 5.10. For the point P on a varieties X, let $\mathfrak{m}$ be the local ring $\mathcal{O}_P$. We define the Zariski tangent space $T_P(X)$ of X at P to be the dual $k-$vector space of $\mathfrak{m}/\mathfrak{m}^2$

(a) By Proposition 5.2A , we have $dim\mathfrak{m}/\mathfrak{m}^2 \geq dim\mathcal{O}_P$, since $T_P(X)$ of X at P to be the dual $k-$vector space of $\mathfrak{m}/\mathfrak{m}^2$, then $dimT_P(X) = dim\mathfrak{m}/\mathfrak{m}^2$, thus $dimT_P(X) \geq dim\mathcal{O}_P$.

By Theorem 3.2, $dim\mathcal{O}_P = dimX$, then we have the result

$$dimT_P(X) \geq dimX.$$

We have $dimT_P(X) = dimX \iff dim\mathfrak{m}/\mathfrak{m}^2 = dim\mathcal{O}_P$, then $\mathcal{O}_P$ is a regular local ring, then X is nonsingular at P.

(b) For any morphism $\varphi : X \to Y$, there is a natural map induced $k-$linear map $T_P(\varphi) : T_P(X) \to T_{\varphi(P)}(Y)$.

We recall

$$\mathfrak{m}_P = \{f \in \mathcal{O}_P : f(P) = 0\}.$$

Let $f \in \mathfrak{m}_{\varphi(P)}$, then $f(\varphi(P)) = 0$, it shows that $f \circ \varphi \in \mathfrak{m}_P$, hence, we define the map φ^* as follows

$$\begin{aligned} \varphi^* : \mathfrak{m}_{\varphi(P)} &\to \mathfrak{m}_P \\ f &\mapsto f \circ \varphi. \end{aligned}$$

Let $fg \in \mathfrak{m}^2_{\varphi(P)}$, with $f, g \in \mathfrak{m}_{\varphi(P)}$, then $f \circ \varphi, g \circ \varphi \in \mathfrak{m}_P$, so $(fg) \circ \varphi \in \mathfrak{m}^2_{\varphi(P)}$, thus we define the $k-$linear map

$$\begin{aligned} \tau : \mathfrak{m}_{\varphi(P)}/\mathfrak{m}^2_{\varphi(P)} &\to \mathfrak{m}_P/\mathfrak{m}^2_P \\ f + \mathfrak{m}^2_{\varphi(P)} &\mapsto f \circ \varphi + \mathfrak{m}^2_P = \varphi^*(f) + \mathfrak{m}^2_P. \end{aligned}$$

We take the dual of τ giving us the $k-$linear map

$$\begin{aligned} T_P(\varphi) : T_P(X) &\to T_{\varphi(P)}(Y) \\ f &\mapsto f \circ \tau. \end{aligned}$$

(c) If φ is the vertical projection of the parabola $x = y^2$ onto the $x-$axis, show that induced map $T_0(\varphi)$ of tangent space at the origin is the zero map.

We set $X = Z(x - y^2)$.

The map φ is defined by

$$\begin{aligned}\varphi : X &\to \mathbb{A}^1\\ (x, y) &\mapsto x.\end{aligned}$$

Hence, $\varphi(0, 0) = 0$, the map φ^* is define by

$$\begin{aligned}\varphi^* : \mathfrak{m}_0 &\to \mathfrak{m}_{(0,0)}\\ f &\mapsto f \circ \varphi.\end{aligned}$$

We have $\mathfrak{m}_0 = (x)$, then let $f \in \mathfrak{m}_0$, there is $g \in k[x]$ such that $f = x.g$, so $\varphi^*(f) = (x.g) \circ \varphi$. For every $(x, y) \in X$, we have

$$(x.g) \circ \varphi(x, y) = (xg)(x) = (xg)(y^2) = y^2 g(y^2).$$

We see that $y \in \mathfrak{m}_{(0,0)}$, then $y^2 \in \mathfrak{m}^2_{(0,0)}$, it shows that $\varphi^*(f) \in \mathfrak{m}^2_{(0,0)}$ for any $f \in \mathfrak{m}_0$.

The $k-$linear map

$$\begin{aligned}\tau : \mathfrak{m}_0/\mathfrak{m}_0^2 &\to \mathfrak{m}_{(0,0)}/\mathfrak{m}^2_{(0,0)}\\ f + \mathfrak{m}_0^2 &\mapsto \varphi^*(f) + \mathfrak{m}^2_{(0,0)} = 0.\end{aligned}$$

This shows that $\tau = 0$, then the dual of τ is the zero map.

Exercise 5.11. ***The Elliptic Quartic Curve in*** $\mathbb{P}^3$**.** Let Y be the algebraic set in $\mathbb{P}^3$ defined by the equations $x^2 - xz - yw = 0$ and $yz - xw - zw = 0$. Let P be the point $(x, y, z, w) = (0, 0, 0, 1)$, and let φ denote the projection from P to the plane $w = 0$. Show that φ induces an isomorphism of $Y - P$ with the plane cubic curve $y^2z - x^3 + xz^2 = 0$ minus the point $(1, 0, -1)$. Then show that Y is an irreducible nonsingular curve. It is called *the elliptic quadratic curve* in $\mathbb{P}^3$. Since it is defined by two equations it is another example of a complete intersection.

Solution:

We have $Y = Z(x^2 - xz - yw, yz - xw - zw)$. The projection from the point P is given by

$$\begin{aligned} \varphi : \mathbb{P}^3 - P &\to \mathbb{P}^2 \\ [x : y : z : w] &\mapsto [x : y : z] \end{aligned}$$

Set $T = Z(y^2z - x^3 + xz^2)$

We will show that $\varphi(Y - P) \subseteq T$.

Let $[x : y : z : w] \in Y - P$, since $[x : y : z : w] \in \mathbb{P}^3$, then we can consider two cases: $w = 0; w = 1$.

If $w = 0$, then $(x, y, z) \neq (0, 0, 0)$ and

$$\begin{cases} x^2 - xz &= 0 \\ yz &= 0 \end{cases}$$

$$\Rightarrow \begin{cases} x(x - z)(x + z) &= 0 \\ y^2z &= 0 \end{cases},$$

therefore $y^2z - x(x - z)(x + z) = 0$, so $y^2z - x^3 + xz^2 = 0$. It shows that $\varphi([x : y : z : w]) = [x : y : z] \in T$.

If $w = 1$, then

$$\begin{cases} x^2 - xz - y &= 0 \\ yz - x - z &= 0 \end{cases}$$

$$\Rightarrow \begin{cases} y &= x^2 - xz \\ yz &= x + z \end{cases},$$

hence, $y^2z = (x^2 - xz)(x + z) = x(x - z)(x + z) = x(x^2 - z^2) = x^3 - xz^2$, thus $y^2z - x^3 + xz^2 = 0$. It shows that $\varphi([x : y : z : w]) = [x : y : z] \in T$.

Thus we have $\varphi(Y-P) \subseteq T$.

Let $[x:y:z] \in T$, we have $y^2z - x^3 + xz^2 = 0$, we consider two cases: $y = 0; y \neq 0$.

If $y = 0$ then $x^3 - xz^2 = 0$, so $x = 0$ or $x + z = 0$ or $x - z = 0$. If $x = 0$ then we have the point $[0:0:1] \in T$, if $x = -z$ then we have the point $[-1:0:1] \in T$, if $x = z$ then we have the point $[1:0:1] \in T$. We get $[0:0:1:0], [1:0:1:0] \in Y$ and $\varphi([0:0:1:0]) = [0:0:1], \varphi([1:0:1:0]) = [1:0:1]$, but for any $w, [-1:0:1:w]$ does not satisfy the equations $x^2 - xz - yw = 0$ and $yz - xw - zw = 0$. This shows that $\varphi(Y-P) \subseteq T - \{[-1:0:1]\}$.

If $y \neq 0$, then we get $[x:y:z:\frac{x^2-xz}{y}]$, this point satisfies the equations $x^2 - xz - yw = 0$ and $yz - xw - zw = 0$. Thus $[x:y:z:\frac{x^2-xz}{y}] \in Y$ and $\varphi([x:y:z:\frac{x^2-xz}{y}]) = [x:y:z] \in T$.

Hence

$$\varphi(Y-P) = T - \{[-1:0:1]\}.$$

We consider the morphism defined by

$$\begin{aligned} \phi : T' = T - \{[-1:0:1]\} &\to Y - P \\ [0:0:1] &\mapsto [0:0:1:0] \\ [1:0:1] &\mapsto [1:0:1:0] \\ [x:y:z] &\mapsto [x:y:z:\frac{x^2-xz}{y}] \,\text{for}\, y \neq 0. \end{aligned}$$

We have $\varphi_{|_{Y-P}} \circ \phi = id_{Y-P}, \phi \circ \varphi_{|_{Y-P}} = id_{T'}$, this shows that $\varphi_{|_{Y-P}}$ is an isomorphism.

$T = Z(y^2z - x^3 + xz^2)$ is irreducible?, then $T - \{[-1:0:1]\}$ is irreducible, then $Y - P$ is irreducible, then Y is irreducible.

Nonsingular Curves

Exercise 6.1. Recall that a curve is rational if it is birationally equivalent to $\mathbb{P}^1$. Let Y be a nonsingular rational curve which is not isomorphic to $\mathbb{P}^1$.

1. Show that Y is isomorphic to an open subset of $\mathbb{A}^1$.

   By Proposition 6.7, Y is isomorphic to an abstract nonsingular curve X, by Corollary 6.10, X is isomorphic to an open subset U of a nonsingular projective curve T.

   Since Y be a nonsingular rational curve which is not isomorphic to $\mathbb{P}^1$, then U is infinite, thus T is isomorphic to $\mathbb{P}^1$. The space $\mathbb{P}^1$ has two chart U_0 and U_1, which are isomorphic to $\mathbb{A}^1$. Since $U \subsetneq \mathbb{P}^1$ then $U \subseteq U_0$ or $U \subseteq U_1$, so Y is isomorphic to an open subset U of $\mathbb{A}^1$.

2. Show that Y is an affine.

   Assume that Y is isomorphic to $\mathbb{P}^1 - \{a_1, \ldots, a_r\} = U$.

   Let $Z = Z((x-a_1)(x-a_2)\ldots(x-a_r)y - 1)$ be a variety in $\mathbb{A}^2$, then U is isomorphic to Z. Indeed, we consider the maps

$$\begin{aligned} \varphi : U &\to Z \\ x &\mapsto (x, \frac{1}{(x-a_1)(x-a_2)\ldots(x-a_r)}), \end{aligned}$$

$$\begin{aligned} \varphi^{-1} : Z &\to U \\ (x,y) &\mapsto x. \end{aligned}$$

   Then φ is an isomorphism.

3. Show that $A(Y)$ is an unique factorization domain.

   We have

$$A(Y) \cong \frac{k(x,y)}{(x-a_1)(x-a_2)\ldots(x-a_r)y - 1},$$

$$A(Y) \cong \mathcal{O}(Y) = k[t, \tfrac{1}{t-a_1}, \ldots, \tfrac{1}{t-a_r}],$$

   k[t] is an unique factorization domain, the localization of an UFD is UFD, it shows that $A(Y)$ is UFD.

Exercise 6.2. An Elliptic curve. $Y = Z(y^2 - x^3 + x)$, let $f(x,y) = y^2 - x^3 + x$.

1. Show that Y is nonsingular, and deduce $A = A(Y) = k(x,y)/(y^2 - x^3 + x)$ is integrally closed domain.

   We have $f'_x(x,y) = -3x^2 + 1, f'_y(x,y) = 2y$, assume that $f'_x(x,y) = -3x^2+1 = 0, f'_y(x,y) = 2y = 0$, then we have two points $(\frac{1}{\sqrt{3}}, 0), (-\frac{1}{\sqrt{3}}, 0)$, the points are not in Y, then Y is nonsingular.

   $A(Y)$ is integrally closed domain?

2. $k[x]$ is a polynomial ring. $A = \overline{k[x]}$?

3. The map

$$\begin{aligned} \sigma : A &\to A \\ f(x,y) &\mapsto f(x,-y) \end{aligned}$$

   is automorphism. The norm $N(a) = a\sigma(a)$.

   Let $f \in A$, since $y^2 = x^3 - x \in k[x]$, then we can write $f(x,y) = yg(x) + h(x)$. Then

$$N(f) = f\sigma(f) = (yg(x) + h(x)).(-yg(x) + h(x)) = h^2(x) - y^2g^2(x) \in k[x].$$

   $N(1) = 1.1 = 1$
   $N(fg) = fg\sigma(fg) = fgf(x,-y)g(x,-y) = f\sigma(f)g\sigma(g) = N(f)N(g).$

Exercise 6.6. Automorphism of $\mathbb{P}^1$. Think of $\mathbb{P}^1$ as $\mathbb{A}^1 \cup \{\infty\}$. Then we define a fractional linear transformation of $\mathbb{P}^1$ by sending $x \to (ax+b)(cx+d)$, for $a, b, c, d \in k, ad - bc \neq 0$.

(a) Show that Y a fraction linear transformation induces an automorphism of $\mathbb{P}^1$. We denote the group of all these fraction linear transformations by **PlG(1).**

For $a, b, c, d \in k, ad - bc \neq 0$, the fraction linear transformation is defined by the map

$$\begin{aligned} \varphi : \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \\ x &\mapsto \frac{ax+b}{cx+d} \text{ if } x \neq \infty, \frac{-c}{d}, \\ \infty &\mapsto \frac{a}{c}, \\ \frac{-d}{c} &\mapsto \infty. \end{aligned}$$

The inverse of φ is the map defined by

$$\begin{aligned}\phi : \mathbb{P}^1 &\longrightarrow \mathbb{P}^1\\ x &\mapsto \frac{dx-b}{-cx+a} \text{ if } x \neq \infty, \frac{a}{c},\\ \infty &\mapsto \frac{-d}{c},\\ \frac{a}{c} &\mapsto \infty.\end{aligned}$$

We have

$$\begin{aligned}\phi(\varphi(x)) &= \frac{d\dfrac{ax+b}{cx+d}-b}{-c\dfrac{ax+b}{cx+d}+a} = \frac{adx-bcx}{ad-bc} = x,\\ \phi(\varphi(\infty)) &= \phi(\frac{a}{c}) = \infty,\\ \phi(\varphi(\frac{-d}{c})) &= \phi(\infty) = \frac{-d}{c},\end{aligned}$$

it shows that $\phi \circ \varphi = id_{\mathbb{P}^1}$. And we also have $\varphi \circ \phi = id_{\mathbb{P}^1}$. This shows that φ is an isomorphism.

(b) Let $Aut\mathbb{P}^1$ denote the group of all automorphism of $\mathbb{P}^1$. Show that Aut $\mathbb{P}^1 \cong$ Aut $k(x)$, the group of k-automorphisms of the field $k(x)$.

Let $\tau \in$ Aut $\mathbb{P}^1$, then we have the map

$$\begin{aligned}k(x) &\longrightarrow k(x)\\ f &\mapsto f \circ \tau,\end{aligned}$$

which is in Aut $k(x)$.

For each $\chi \in$ Aut $k(x)$, by Theorem 4.4, Corollary 4.5, there is a birational map $\nu : \mathbb{P}^1 \to \mathbb{P}^1$. So there is an open set U of $\mathbb{P}^1$ such that $\nu : U \to \mathbb{P}^1$ is a morphism, by Proposition 6.8, we have a morphism $\overline{\nu} : \mathbb{P}^1 \to \mathbb{P}^1$, similarly way, we have a morphism $\overline{\nu^{-1}} : \mathbb{P}^1 \to \mathbb{P}^1$. I am hoping that $\overline{\nu}$ is a isomorphism??. Maybe we can see Lemma 4.1?, since $\overline{\nu} \circ \overline{\nu^{-1}}$ agree with id on some open set, then $\overline{\nu} \circ \overline{\nu^{-1}} = id$.

This shows that Aut$\mathbb{P}^1 \cong$ Aut $k(x)$.

(c) Now show that every automorphism of $k(x)$ is a fractional linear transformation, and deduce that every $\mathbf{PGL(1)} \to$ Aut $\mathbb{P}^1$ is an isomorphism.

For each $\varphi \in \mathbf{PGL(1)}$, by a), we have $\varpi \in \text{Aut } \mathbb{P}^1$.

Let $\varphi \in Autk(x) \cong Aut\mathbb{P}^1$, then there are $f, g \in k[x], g \neq 0$ such that $(f, g) = 1$ and $\varphi(x) = \frac{f(x)}{g(x)}$, if deg$f$ or deg$g$ are more than 1, φ is not in $Aut\mathbb{P}^1$, then we can assume that $f(x) = ax + b, g(x) = cx + d$, since $(f, g) = 1$, then $ad - bc \neq 0$. So $\varphi \in \mathbf{PGL(1)}$.

The Hilbert polynomial

Let X be a projective variety of $\mathbb{P}^n$. For $d \geq 1$, denote

$$I(X)_m = I(X) \cap k[x_0, \dots, x_n]_m.$$

Denote $S(X) = k[x_0, \dots, x_n]/I(X)$, since $I(X)$ is a homogeneous ideal, then $S(X)$ is a graded ring with decomposition

$$S(X) = \bigoplus_{m \geq 0} S(X)_m,$$

with $S(X)_m = k[x_0, \dots, x_n]_m / I(X)_m$.

The ***Hilbert function*** of X is given by

$$\varphi_X(l) = dim_k(S(X)_m).$$

Note that $dim(k[x_0, \dots, x_n]_m) = \binom{n+m}{n}$.

Intersections in Projective Space

Exercise 7.1

(a) The $d-$uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$. Let $M_0, \dots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \dots, x_n$, where $N = \binom{n+d}{n} - 1$. We define the map

$$\begin{aligned} \rho_d : \mathbb{P}^n &\to \mathbb{P}^N \\ P = [a_0 : \cdots : a_n] &\mapsto [M_0(P) : \cdots : M_n(P)]. \end{aligned}$$

Since M_i are the polynomials, then ρ_d is a morphism, and ρ_d is called the $d-$uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$.

Let $\theta : k[x_0, \dots, x_N] \to k[x_0, \dots, x_n]$ be the homomorphism defined by sending y_i to M_i. Let $\mathfrak{a} = ker\theta$, by Exercise 2.12, we have

$$k[x_0, \dots, x_N]/\mathfrak{a} \cong k[x_0, \dots, x_n]_d.$$

And $Z(\mathfrak{a}) = \rho_d(\mathbb{P}^n)$, then we have

$$S(Z(\mathfrak{a})) = k[x_0, \dots, x_n]_d = \bigoplus_{m \geq 0} (k[x_0, \dots, x_n]_d)_m,$$

with $(k[x_0, \dots, x_n]_d)_m = k[M_0, \dots, M_N]_m = k[x_0, \dots, x_n]_{md}$. This shows that

$$\varphi_{Z(\mathfrak{a})}(m) = dim_k(S(Z(\mathfrak{a})_m) = \binom{n+md}{n}.$$

Then the Hilbert polynomial of $Z(\mathfrak{a})$

$$\begin{aligned} P_{Z(\mathfrak{a})}(x) = \binom{n+dx}{n} &= \frac{(n+xd)!}{n!(dx)!} = \\ &= \frac{(n+xd)\dots(xd+1)}{n!} \\ &= \frac{(xd)^n}{n!} + \text{ term of degree } < n. \end{aligned}$$

Is shows that deg $Z(\mathfrak{a}) = \dfrac{d^n}{n!} dim(\mathbb{P}^n)! = d^n$.

(b) The Segre Embedding is defined by the map

$$\begin{gathered} \phi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N \\ (a_0, \dots, a_r) \times (b_0, \dots, b_s) \mapsto (\dots, a_i b_j, \dots) \end{gathered}$$

where $N = rs + r + s$.

Note that ϕ is injective. Image of ϕ is a subvariety of $\mathbb{P}^N$. Indeed, we consider the map

$$\begin{aligned} \pi : k[\{z_{ij} : i = 0, \dots, r, j = 0, \dots, s\}] &\to k[x_0, \dots, x_r, y_0, \dots, y_s] \\ z_{ij} &\mapsto x_i y_j. \end{aligned}$$

Exercise 2.14 shows that $im(\phi) = Z(\mathfrak{m})$, where $\mathfrak{m} = ker(\phi)$. Hence we have the coordinate ring of $Z(\mathfrak{m})$

$$k[\{z_{ij} : i = 0, \dots, r, j = 0, \dots, s\}]/\mathfrak{m} \cong \langle x_i y_j : i = 0, \dots, r, j = 0, \dots, s \rangle = M$$

We define

$$M_l = \langle ab : a \in k[x_0, \dots, x_r]_l, b \in k[y_0, \dots, y_s]_l \rangle \tag{1}$$

Then we have

$$M = \bigoplus_{l \geq 0} M_l.$$

Since (1), then we have

$$dim(M_l) = \binom{r+l}{r}\binom{s+l}{s}.$$

It show us the Hilbert function

$$\begin{aligned}\varphi_{Z(\mathfrak{m})}(l) &= \binom{r+l}{r}\binom{s+l}{s}, l \in \mathbb{N}.\\ &= \frac{(r+l)!}{r!l!}\frac{(l+s)!}{l!s!}\\ &= \frac{(r+l)\dots(l+1)}{r!}\frac{(s+l)\dots(l+1)}{s!}.\end{aligned}$$

Hence, the Hilbert polynomial is

$$\begin{aligned}P_{Z(\mathfrak{m})}(x) &= \frac{(r+x)\dots(x+1)}{r!}\frac{(s+x)\dots(x+1)}{s!}\\ &= \frac{x^r}{r!}\frac{x^s}{s!} + \text{ term of lower degree.}\end{aligned}$$

Since $dim(\mathbb{P}^r \times \mathbb{P}^s) = r + s$, then we have

$$degZ(\mathfrak{m}) = \frac{(r+s)!}{r!s!} = \binom{r+s}{r}$$

Exercise 7.3 The dual Curve.
Let $Y = Z(F)$ be a projective variety in $\mathbb{P}^n$, let $P = (a_0, a_1, a_3)$ be a nonsingular point of Y. The tangent line to Y at P is defined by

$$T_P Y = F'_{x_0}(P)(x_0 - a_0) + F'_{x_1}(P)(x_1 - a_1) + F'_{x_2}(P)(x_2 - a_2).$$

Reference

[1] Robin Hartshorne, *Algebraic Geometry, chapter I*, 1977.