# CS 70, Spring 2015 — Homework 1

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## Problem 1

1a. (a) TRUE: $\forall x \in \mathbb{Z}, x+1 \in \mathbb{Z}$
  (b) FALSE: $\forall x \in \mathbb{Z}$, if $\frac{x}{2} \in \mathbb{Z}$ then $\frac{x+1}{2} \in \mathbb{Z}$
  (c) TRUE: $\forall x \in \mathbb{Z}$, if $\frac{x}{2} \notin \mathbb{Z}$ then $\frac{x+1}{2} \in \mathbb{Z}$
  (d) FALSE: $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ s.t. $xy = 1$
  (e) TRUE: $\forall x \in \mathbb{Q}, \exists y \in \mathbb{Z}$ s.t. $xy = 1$

1b. The expression $\sum_{i=0}^{n} f(i)$ is equal to the following expressions:

  (a) $(\ \sum_{i=0}^{n-1} f(i)\ ) + f(n)$
  (b) $f(0) + f(1) + ... + f(n)$
  (c) $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} f(i) + \sum_{j=\lfloor \frac{n}{2} \rfloor + 1}^{n} f(j)$

1c. The expression $\prod_{i=0}^{n} f(i)$ is equal to the following expressions:

  (a) $f(0) * f(1) * f(2) * ... * f(n-1) * f(n)$
  (b) $f(n) \prod_{i=0}^{n-1} f(i)$
  (c) $f(n) \prod_{i=0}^{\frac{n}{2}-1} \prod_{j=2i}^{2i+1} f(j)$
  (d) $\prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} f(i) \prod_{j=\lfloor \frac{n}{2} \rfloor + 1}^{n} f(j)$

## Problem 2

2a. (a) For all integer $x$, $x$ is either less than $\sqrt{48}$ or greater than 7

(b) $(\exists x \in \mathbb{Z})\ ((x \geq \sqrt{48}) \wedge (x \leq 7))$

(c) TRUE, there exists such integer that matches this proposition

2b. (a) For all real numbers $x$, and for all real numbers $y$, there exists a real number $z$ such that $x$ is less than $y$ is less than $z$.

(b) $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(\forall z \in \mathbb{R})\ (x \geq z \geq y)$

(c) FALSE, there exists no such real number that matches this proposition

2c. (a) For all integers $x$, there exists an integer $y$, such that $y$ subtracted from $x$ is greater than 16.

(b) $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})\ (x - y \leq 16)$

(c) FALSE, there exists no such integer that matches this proposition

2d. (a) There exists a real number $x$, for all numbers $y$ such that $zy$ is greater than 1.

(b) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\ (xy \leq 1)$

(c) TRUE, there exists such real number that matches this proposition

2e. (a) For all real numbers $y$, there exists a real number $x$ such that $xy$ is greater than 16.

(b) $(\exists y \in \mathbb{R})\ (\forall x \in \mathbb{R})\ (xy \leq 1)$

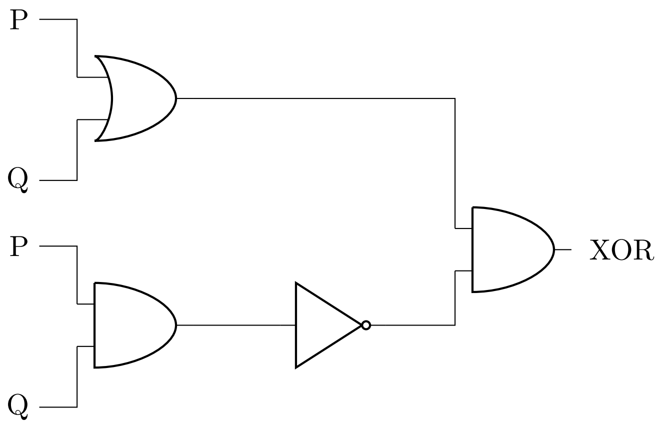
(c) FALSE, there exists no such real number that matches this proposition

## Number 3

3a. Show, using a truth table, that $P \oplus Q$ is equivalent to $(P \vee Q) \wedge \neg(P \wedge Q)$.

| $P$ | $Q$ | $P \vee Q$ | $\neg(P \wedge Q)$ | $(P \vee Q) \wedge \neg(P \wedge Q)$ |
|---|---|---|---|---|
| T | T | T | F | F |
| T | F | T | T | T |
| F | T | T | T | T |
| F | F | F | T | F |

3b. Drawing using logic gates NOT, AND, OR to create XOR



## Number 4

4a. Prove best casket the suitor should choose.
Realize that casket Lead is either true or false. Know that if the casket is made by Bellini, meaning the statement is true, it is the only casket made by Bellini and other caskets are made by Cellini so everything else will be false. However casket Lead is made by Cellini, meaning the statement is false, there will be more than one casket made by Bellini. Since there are only two left, both of them have to be true. Cases:

(a) Case 1, when Lead is made by Cellini (false): Gold is made by Bellini (true) and Silver is made by Bellini (true). This means that Gold has a 0% chance of not having a dagger, and Silver has a 100% chance of not having a dagger. Note that there is only one dagger, the dagger is already in the Gold casket. Therefore the Lead casket has 100% chance of not having a dagger.

(b) Case 2, when Lead is made by Bellini (true): Gold is made by Cellini (false) and Silver is made by Cellini (false). This means that Gold has a 100% of not having the dagger, and Silver has a 50% chance of not having the dagger. Silver has a 50% chance because if the casket is not empty, it means either it has a dagger or a portrait. Since we are looking for the best posibility to avoid the dagger, it is 50% since they are both equal chances. Because nothing was said about the Lead casket, we have to assume all posibilities (having a dagger, portrait or nothing at all) have equal chances. Therfore the Lead casket has a 66% chance of not having the dagger.

To figure out which casket has the best possibility, we add up all the percentages of not having a dagger from all caskets from case 1, and 2 by the total number of cases (2). For the Gold casket, there is a 50% chance of not having a dagger. For the Silver casket, there is a 75% chance of not having the dagger. For the Lead, there is a 83% chance of not having the dagger. Looking at these percentages, it is appears that the Lead casket is the best possible casket for the suite to choose to avoid the dagger.

4b. Select the casket with the portrait and determine the maker of each casket.
Proof:

(a) Assume Lead casket is made by Cellini (false)

(b) This means that there are less than two caskets made by Cellini.

(c) Less than two means only one is made by Cellini.

(d) Therefore only the Lead casket is made by Cellini.

(e) Therefore both of the other caskets, Gold and Silver, must have been made by Bellini

(f) Realize this is not possible because Bellini only writes the truth. And since there cannot be more than one portrait total, this instance is not possible.

(g) Therefore the only other possibility is if the Lead casket is made by Bellini (true)

(h) That means at least two of the caskets are made by Cellini.

(i) This means that the other two caskets are made by Cellini.

(j) Meaning both statements about Gold and Silver having the portrait is false.

(k) Realize that the portrait has to be in at least one of the caskets, we can safely assume the Lead casket has the portrait.

The suitor should tell Portia that the Lead casket is made by Bellini, and the other two caskets (Gold & Silver) is made by Cellini. As a result, the portrait is iniside the Lead casket. This was a proof by contrapositive.

## Number 5

This statement is false, proven by proof by contrapositive.
Prove: $(\forall x, y \in \mathbb{Z})\ 6 \nmid xy \Rightarrow (6 \nmid x \wedge 6 \nmid y)$

1. Assume $(\exists x, y \in \mathbb{Z})\ 6 \mid xy \Rightarrow (6 \mid x \vee 6 \mid y)$
2. By definition, $6 \mid x$ is equivalent to $x = 6A$
3. By definition, $6 \mid y$ is equivalent to $y = 6B$
4. By definition, $6 \mid xy$ is equivalent to $xy = 6C$
5. Note we can now write $xy = 6A(6B)$
6. We can rearrange into $xy = 6(6AB)$
7. Let $C = 6AB$
8. Then we have $xy = 6C$.
9. Then by definition, 6 does divide xy.
10. This makes our assumption true. And through proof by contrapositive, the original proposition is false.

## Number 6

(a) $\forall C(x)$, $C(x) \Rightarrow S(x)$

(b) $\exists E(x)$, $E(x) \nRightarrow S(x)$

(c) $\exists E(x)$, $E(x) \nRightarrow C(x)$

Prove: (c) follow from (a) and (b)

1. Assume $\forall E(x)$, $E(x) \Rightarrow C(x)$
2. Then $\exists E(x)$, $E(X) \nRightarrow S(x)$
3. However note, by our assumption, $\forall E(x)$, $E(x) \Rightarrow C(x)$
4. This makes the statement $\forall C(x)$, $C(x) \Rightarrow S(x)$ false
5. Since this contradicts our original statement to prove, our original statement is true through proof by contrapositive.

## Number 7

7a. (a) Statement for Amy having a winning strategy:

   i. Let x be moves made by Amy
   ii. Let y be moves made by Bob
   iii. Let A(x) be Amy winning using move x
   iv. Let B(y) be Bob winning using move y
   v. Then $\exists x$ *s.t.* $\forall y$, $A(x)$

(b) Statement for Bob having a winning strategy:

   i. Let x be moves made by Amy
   ii. Let y be moves made by Bob
   iii. Let A(x) be Amy winning using move x
   iv. Let B(y) be Bob winning using move y
   v. Then $\exists y$ *s.t.* $\forall x$, $B(y)$

(c) Proof that either Amy has a winning strategy or Bob does:

   i. Let Amy make her best possible move X.
   ii. Let Bob make his best possible move Y.
   iii. Case 1 - Amy Wins: moving to X was her winning move, regardless of what Bob's move was. Therefore this is, by definition, a winning strategy she has.
   iv. Case 2 - Bob Wins: moving to Y was his winning move, even if Amy's move was the best. Therefore this is, by definition, a winnign strategy he has.

Since these are the only two possible cases, both of them are proven to both have winning strategies.