CS 70, Spring 2015 — Homework 2

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Problem 1

1a. $\forall x P(x) \equiv \neg\exists x \neg P(x)$

TRUE because if we follow De Morgan's Law, this happens:

$\neg\exists x \neg P(x) \equiv \forall x \neg\neg P(x)$

$\forall x \neg\neg P(x) \equiv \forall x P(x)$

Therefore, after simplifying: $\forall x P(x) \equiv \forall x P(x)$

1b. $\forall x \exists y P(x,y) \equiv \forall y \exists x P(x,y)$

FALSE, using the counterexample: $P(x,y)$ will be true if $y > 0$, and for all x, the other statement cannot be true since we already decided that $P(x,y)$ can ONLY be try for when $y > 0$.

1c. $P \Rightarrow Q \equiv \neg P \Rightarrow Q$

FALSE, using counterexample: assume P is False and Q is False.

Then $P \Rightarrow Q \equiv TRUE$

However $\neg P \equiv TRUE$

Therefore $\neg P \Rightarrow Q \equiv FALSE$

Hence $TRUE \not\equiv FALSE$

1d. $(P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q) \equiv P \Leftrightarrow Q$

TRUE, from the following proof:

$A :=$ left side of the statement

$B :=$ right side of the statement

We assume $A \equiv TRUE$.

A can only be true if $P \equiv TRUE, Q \equiv TRUE$ or $P \equiv FALSE, Q \equiv FALSE$

Then B is true proven by the following truth table:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
T	T	T	T	T
F	F	T	T	T

Similarly, $B \equiv TRUE$ when $P \equiv TRUE, Q \equiv TRUE$ or $P \equiv FALSE, Q \equiv FALSE$

Problem 2

2a. $\forall n P(x)$ is true.

This can hold but not always due to what we have proven: $\forall k \in \mathbb{N}$, if $P(k)$ is true, $P(k+2)$ is true. We skipped the consecutive number right after k, which is denoted as $k+1$. Therefore what we have proven is that for every other number after k is true not every number after k is true.

2b. If $P(0)$ is true then $\forall n P(n+2)$ is true.

This can hold but not always because in this statement, we set our $k = 0$, therefore only the following would be true under what we've proven: $k = 0 + 2 = 2, k = 2 + 2 = 4, k = 4 + 2$.... As we can see, all of these numbers are even. But when we say $\forall n$, this means any $n + 2$ is suppose to be true. A counterexample is when $n = 2n+1$. As a result $2n+1+2 = 2(n+1)+1$ which is odd. Therefore it does not work when n is odd.

2c. If $P(0)$ is true then $\forall n P(2n)$ is true.

This always holds. The statement set $k = 0$. As exampled above, all the numbers that we have proven is true are even numbers. $k = 2n$ is an even number. Therefore this statement is true because $\forall n, 2n$ is even by definition.

2d. $\forall n P(n)$ is false.

This can hold but not always. The statement is similar to 2a in the way that we have proven that we stated every other number is true. Therefore the numbers in between is false so half of it is true and half of it is true.

2e. If $P(0)$ and $P(1)$ are true then $\forall n P(n)$ is true.

This always holds. Because now we assume k can be two things $k = 1$ and $k = 0$. We already stated in what numbers $k = 0$ proves to be true, and it is all even numbers when $k = 0$ in 2b. Now we notice that any $k + 2$ when starting at 1 will be odd because of the definition of an odd number. Notice the pattern: $k = 1, k = 1 + 2 = 3, k = 3 + 2 = 5$....

2f. ($\forall n \leq 10$ $P(n)$ is true) and ($\forall n > 20$ $P(n)$ is false)

This never holds because allowing n be 1 or 2 and making the statement $P(1)$ and $P(2)$ is enough to set the foundation such that all of the natural numbers following will be true because 1 covers all the odd numbers and 2 covers all the even numbers.

2g. $[\forall n \; prime \Rightarrow P(n)] \Rightarrow [\forall n \geq 11 \; P(n)]$

This always holds because as explained in 2e, number 2 will cover all of the even numbers. Similarly, the consecutive number, 3, will cover all of the odd numbers. Therefore any number after 3 will be true.

Problem 3

3a. Prove that $3 + 11 + 19 + ... + (8n - 5) = 4n^2 - n$ for all integers $n \geq 1$.

We will proceed to prove this by induction.

Base Case: $n = 1$, so $(8(1) - 5) = 3$. $4(1^2) - 1 = 3$

Hypothesis: Assume for $n = k \geq 1$, $3 + 11 + 19 + ... + (8k - 5) = 4k^2 - k$

Step: we want: $3 + 11 + 19 + ... + (8k - 5) + (8(k + 1) - 5) = 4(k + 1)^2 - (k + 1)$

i. Add $(8(k + 1) - 5)$ to both sides:
$3 + 11 + 19 + ... + (8k - 5) + (8(k + 1) - 5) = 4k^2 - k + 8(k + 1) - 5$

ii. Manipulate the right hand side:
$LHS = 4k^2 - k + 8k + 8 - 5$
$LHS = 4k^2 - k - 1 + 8k + 8 - 4$
$LHS = 4k^2 + 8k + 4 - (k + 1)$
$LHS = 4(k^2 + 2k + 1) - (k + 1)$
$LHS = 4(k + 1)^2 - (k + 1)$

iii. Hence, what we originally wanted to prove. QED

3b. Prove that $1^3 + 2^3 + 3^3 + ... + n^3 = (1 + 2 + 3 + ... + n)^2$ for all integers $n \geq 1$

We will proceed to prove this by induction.

Base Case: $n = 1$, $1^3 = 1 = 1^2 = 1$

Hypothesis: Assume for $n = k \geq 1$, $1^3 + 2^3 + 3^3 + ... + k^3 = (1 + 2 + 3 + ... + k)^2$

Step:

i. Note that $(1 + 2 + 3 + ... + k)^2 = (\sum_{i=1}^{k} i)^2 = (\frac{k(k+1)}{2})^2$

ii. Therefore what we want to prove is $1^3 + 2^3 + 3^3 + ... + k^3 + (k + 1)^3 = (\frac{(k+1)(k+2)}{2})^2$

iii. Using original equation, we add $(k + 1)^3$ to both sides:
$1^3 + 2^3 + 3^3 + ... + k^3 + (k + 1)^3 = (\frac{k(k+1)}{2})^2 + (k + 1)^3$

iv. Now simplify the right hand side:
$LHS = (\frac{(k(k+1))^2}{4}) + \frac{4(k+1)^3}{4}$
$LHS = (\frac{k^2(k+1)^2+4(k+1)^3}{4})$
$LHS = (\frac{(k+1)^2(k^2+4(k+1)}{4})$
$LHS = (\frac{(k+1)^2(k^2+4k+4}{4})$
$LHS = (\frac{(k+1)^2(k+2)^2}{4}))$
$LHS = (\frac{(k+1)(k+2)}{2})^2$

v. Hence we've proven what we originally wanted to prove. QED

Problem 4

Let $x \in \mathbb{R}$ be such that $a_1 = x + \frac{1}{x} \in \mathbb{Q}$. Using strong induction, show that for each integer $n \geq 1$, $a_n = x^n + \frac{1}{x^n} \in \mathbb{Q}$.

We will prove this by strong induction.

Base Cases:
$n = 1$, $a_1 = x + \frac{1}{x} = \frac{x^2+1}{x}$
$n = 2$, $a_1 = x^2 + \frac{1}{x^2} = \frac{x^4+1}{x^2}$

(a) Notice how a_1 and a_2 looks really similar. We will attempt to represent it in a way that uses the previous term(s) because this is *strong* induction.

(b) We will now attempt to manipulate a_1 such that it will look like a_2:

Notice the x^4 at the top, and x^2, so we might want to square a_1
$a_1^2 = \frac{(x^2+1)^2}{x^2} = \frac{x^4+2x^2+1}{x^2} = \frac{x^4+1}{x^2} + \frac{2x^2}{x^2} = \frac{x^4+1}{x^2} + 2 = a_2 + 2$
Rearranging for $a_2 = a_1^2 - 2$
We can now say that $a_2 \in \mathbb{Q}$ because $\mathbb{Q}$ is closed under multiplication and addition.

$n = 3$, $a_3 = x^3 + \frac{1}{x^3} = \frac{x^6+1}{x^3}$

(a) Again, notice how a_3 looks very similar to both a_2 and a_1. Using this, we will attempt to represent it in a way that uses the previous term(s).

(b) We will attempt to manipulate a_1 to look like a_3:

Notice x^6 at the top and x^3 at the bottom. We can get this by $a_1 * a_2$
$a_1 * a_2 = \frac{x^2+1}{x} * \frac{x^4+1}{x^2} = \frac{(x^2+1)(x^4+1)}{x^3} = \frac{x^6+x^2+x^4+1}{x^3} = \frac{x^6+1}{x^3} + \frac{x^2+x^4}{x^3} = \frac{x^6+1}{x^3} + \frac{1+x^2}{x} = a_3 + a_1$
Rearranging for $a_3 = a_1 * a_2 - a_1$
We can now say that $a_3 \in \mathbb{Q}$ because $\mathbb{Q}$ is closed under multiplication and addition.

From this we can realize that we need to use some form of strong induction/proof stating that it can use the previous terms, which we have proven above.

Hypothesis: assume $n = k$, $a_k = x^k + \frac{1}{x^k} = \frac{x^{2k}+1}{x^k}$ and that for all $n = j$ such that for all $1 \leq j \leq k$, $a_j = x^j + \frac{1}{x^j} = \frac{x^{2j}+1}{x^j} \in \mathbb{Q}$

Step:

(a) We want: $a_{k+1} = x^{k+1} + \frac{1}{x^{k+1}} = \frac{x^{2(k+1)}+1}{x^{k+1}}$

(b) We need to manipulate a_k so that it looks like what we want. Notice the top is x gets added one to the exponent and the bottom one adds one exponent. So we attemp the following, using the previous term a_1:
$a_k * a_1 = \frac{x^{2k}+1}{x^k} * \frac{x^2+1}{x} = \frac{(x^{2k}+1)(x^2+1)}{x^{k+1}} = \frac{x^{2k+2}+x^2+x^{2k}+1}{x^{k+1}} = \frac{x^{2k+2}+1}{x^{k+1}} + \frac{x^2+x^{2k}}{x^{k+1}} = \frac{x^{2k+2}+1}{x^{k+1}} + \frac{x^{2k-2}+1}{x^{k-1}}$
Now we have $a_k * a_1 = a_{k+1} + \frac{x^{2k-2}+1}{x^{k-1}}$

(c) Notice how $\frac{x^{2k-2}+1}{x^{k-1}}$ follows a pattern... when we input $k = 3$, we get a_2, and when we input $k = 2$, we get a_1. Therefore this exists and it is known as j.

(d) We can now rewrite the formula as: $a_k * a_1 = a_{k+1} + a_{k-1}$

(e) And therefore $a_{k+1} = a_k * a_1 - a_{k-1}$ which $\in \mathbb{Q}$ because $\mathbb{Q}$ is closed under addition and multiplication.

(f) Hence we've proven that $a_n = x^n + \frac{1}{x^n} \in \mathbb{Q}, \forall n \geq 1$. QED

Problem 5

Let $a_0 = 1$ and $a_n = 2a_{n-1} + 7$. Prove that there is a constant $C > 0$, which does not depend on n, such that $a_n \leq C * 2^n$ for all $n \in \mathbb{N}$.

1. Base Case: $a_0 = 1$ and $a_1 = 2(a_0) + 7 = 9$
2. Hypothesis: (strengthen) there exists $n = k$ such that $a_k \leq C * 2^k - 7$, choosing 7 because it matched the equation.
3. Step:
   (a) We want $a_{k+1} \leq C * 2^{k+1} - 7$
   (b) Starting with $a_k \leq C * 2^k - 7$, we change the left side with a_{k+1} and try to manipulate it such that the right side looks like what we want.
   (c) $C * 2^k - 7$, notice that for a_{k+1}, the 2^{k+1}, we can get this if we multipled the whole RHS by 2:
   $2a_k \leq 2(C * 2^k - 7) = C * 2^{k+1} - 14$
   (d) Notice that we are 7 away from the equation we want, so we add 7 to both sides:
   $2a_k + 7 \leq C * 2^{k+1} - 7$
   (e) Now, we proved that a_{k+1} exists through the fact that we can manipulate a_k to become this.
   (f) The reason this strengthening works is because $C * 2^n - 7 < C * 2^n$. Therefore if it is less that $C * 2^n - 7$, then it is definitely less than $C * 2^n$. QED

Problem 6

6a. F because the base case is incorrect. We did not plug 1 into the equation to prove that the equation works in the first place.

6b. F because the person is using strong induction however the person did not include additional base cases to prove that it the base cases use each other.

Problem 7

Base Cases: $n = 0 \to x^0 = 1$
$n = 1 \to x^1 = x * x^0 = x^1$
$n = 2 \to x^2 = (x^{\frac{2}{2}})^2 = x^2$
$n = 10 \to x^{10} = (x^{\frac{10}{2}})^2 = (x^5)^2$
$n = 5 \to x^5 = x * x^4$
$n = 4 \to x^4 = (x^{\frac{4}{2}})^2 = (x^2)^2$

Hypothesis: there exists a k such that $x^k = Power(x, k)$ and there exists $j = 0, 1, 2, ..., k$ such that $x^j = Power(x, j)$.

Steps:

(a) Want $Power(x, k+1) = x^{k+1}$

(b) Case 1 (k is odd): $Power(x, k+1) = x * x^k$.
This works because of our hypothesis!

(c) Case 2 (k is even): $Power(x, k+1) = (x^{\frac{k}{2}})$.
This works because of our hypthosis!
Case 3 ($k + 1 = 0$): this is not possible because that would make $k = -1$ which is not possible.

(d) Therefore we covered all the cases and proven they work from the hypothesis. QED

Problem 8

Doing this problem out by hand for the first 4 plates, we see a pattern that will be represented in the base cases.

Base Cases: 0 plates = 0 moves
1 plate = 1 move
2 plates = 3 moves
3 plates = 7 moves
4 plates = 15 moves

Hypothesis: There exists a k plates such that it will take $2^k - 1$ moves.

Steps:

(a) We want to prove: for $k + 1$ plates, it will take $2^{k+1} - 1$ moves.

(b) So we start off with moving the first k plates on needle A off the biggest plate (number $k + 1$ plate). This takes $2^k - 1$ as we said above so that it will end up on all on one needle. For this instance, make that needle C.

(c) Now we move all the $k + 1$ plate to needle B. This takes an additional move.

(d) Then we have to remove the rest of the plates in the same manner previously such that it will stack upon the $k + 1$ plate in order. This takes $2^k - 1$ as we have determined earlier.

(e) Therefore it is: $2(2^k - 1) + 1 = 2^{k+1} - 1$ QED

Hence, plugging in 64, seeing that this is the pattern. We will get around 584 billion years because the end of the earth.