3. a) Assume $A \cup B$ is uncountable.

In order for this to be true, one of the two cases or both case must be true:

1) A is uncountable, so when A is combined with B, it will cause the new set $A \cup B$ to be uncountable.

2) Similarly, B is uncountable, so when combining and creating a new set $A \cup B$. This new set must be uncountable since we cannot count what the elements A are in the new set.

Or if A & B both cannot be counted, and as a result $A \cup B$ will be uncounted.

Therefore either A or B or both must be uncountable in order for $A \cup B$ to be uncountable.

Then the original statement is true, A & B are countably infinite sets, then $A \cup B$ is also countable infinite □

b) We know that $\mathbb{R}$ is uncountable. Using our knowledge of part A, we know that $\mathbb{R}$ is made of a union between irrational number and rational numbers — as in we can split any numbers into one of the two choices.

We know rational numbers are countable infinite from lecture.

Then from part A, since real numbers are uncountable, either its rational numbers, irrational numbers, or both must be uncountable for this to be true. So from this, we now know irrational numbers have to be uncountable in order for real numbers to be uncountable. □

HW 8

1. a) $\mathbb{Q}$ is countable infinite because it has the same cardinality as $\mathbb{N}$ & by definition, there is a bijection between $\mathbb{Q}$ & $\mathbb{N}$, so $\mathbb{Q}$ is countable infinite.

   b) $\mathbb{R}$ is uncountably by Cantor's Diagonalization – there is an infinite numbers of possibilities between just two numbers.

   c) $\mathbb{C}$ is uncountable b/c $\mathbb{R}$ is a subset of $\mathbb{C}$ & by transition property, due to the fact that $\mathbb{R}$ is uncountable, neither is $\mathbb{C}$.

   d) $\{0,1\}^*$ is countable infinite because we can rearrange into unique patterns to match to a subset of $\mathbb{N}$.

   e) $\{0,1,2\}^*$ is countable infinite because $\{0,1\}^*$ is a subset & there is only an additional number. Since we can map it to a $x \in \mathbb{N}$, it is countable.

   f) $\mathbb{N}^4$ is countable infinite because for all $\mathbb{N}$, there is $\mathbb{N}^4$ for it just because $\mathbb{N}^4$ exists in $\mathbb{N}$.

   g) $\mathbb{Z}^n$ is countable infinite since we can separate into such n^{th} dimension on a graph such that we will be able to map each one specifically to $\mathbb{N}$.

   h) $S = \{p(x) : p(x) = ax^2 + bx + c$, where $a,b,c \in \mathbb{Z}\}$ is countable infinite because we can rewrite this in ternary string & as stated we can state that it is countable finite.

   i) $T = \{P(x) : P(x) = a_n x^n + \ldots + a_1 x + a_0$, where $n \in \mathbb{N}$ & $a_1, a_0, \ldots, a_0 \in \mathbb{Z}\}$ is countable infinite because we can also create a ternary string out of this & use its property that we know of.

   j) H is countable infinite because we know $\mathbb{Q}$ is countable infinite & $P(x)$ & $Q(x)$ itselves are countable infinite.

   k) all primes are countable infinite because they exist in $\mathbb{N}$ so for all primes, we can link/map it to an $\mathbb{N}$.

2. a) Show one-to-one function $d : S \to T$
   Show one-to-one funtion $e : T \to S$

   We know that we can map everything from S to T because S have subsets of $\mathbb{N}$, and we see that for T, $\mathbb{N}$ maps to values, so we donot have to worry about this. Now we have a one-to one for $S \to T$. For the other way around, it's the values 0 and 1 which both exists in S because S is all the subset of $\mathbb{N}$ & therefore can map to it by default. Therefore this is a bijection.

   b) S seems very similar to $\mathbb{R}$ in the fact that it is possibly infinite and by that, there can be an infinite amount of ways to pair up the values in $\mathbb{N}$. As a result, we can use Cantor's diagonalization to see the infinite possibilities. And therefore, it would make S uncountable.

4. a) Not every increasing function is computable. The reason for this is that some functions themselves will use up the entire $\mathbb{N}$. Meaning there is no cap for the ending result for at least one function, and from this, it will make the specific function uncountable. If a function is uncountable then the statement given is not computable.

b) Note that because the functions are decreasing, we know that up to a certain $x_i \in \mathbb{N}$, it will go to 0. This means that the functions are countable since it will eventually reach 0. As a result, all decreasing functions is countable and therefore it is computable.

5. a) Yes, we can create a program to perform the task simply by halting/ending the program within the first line & ask the checking program whether not "Hello World" is outputed. This will check if program P actually ran the print statement (if at all) first.

b) This may not be possible because we cannot reduce the program to a certain point in order to figure out when the program P should have printed "Hello World". Essentially, meaning, we cannot find a halting condition to know when it is finalized that "Hello World" is not going to be printed for sure. If this program were to run on P, & P has no such print statement, our program will forever continue to search for the next line.

6. a) Because his hotel has a countable infinite sets of room, he can never really run out of room. So if an additional person arrive into the hotel, we can shift the people in the current room over by 1 to free up room 1. The new comer can now reside in room 1 while the previous person living/staying there moves to the next room, and the person in that room does the same until they reach the new room that is unoccupied.

b) Again, since the hotel has countable infinite sets of room, he can continue to expand to place people into rooms. Note we can consider that both sets of infinite people are "equal". Therefore if we shift all the current people in the rooms to ~~the~~ new rooms that are even corresponding to the room they are currently in, we make the old people staying there to stay in even number rooms. This leaves the odd number rooms for the new comers. Since both are infinitely countable, this works out.