AGENDA

Digital Signal Processing

BH001:

1. Target/Purpose of this training course
2. Introduction of Digital Signal Processing
3. Analog vs. Digital Processing Methods
4. Introduction of Digital Filter

BH002:

1. Audio Signal Processing & Audio Codec
2. Video Signal Processing & Video Codec
3. SoC Architecture (SoC: System on Chip)

Day 1 / AM

Day 1 / PM
Day 2 / PM

System Solution

Digital Signal Processing

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AGENDA

BH001:

1. Target/Purpose of this training course

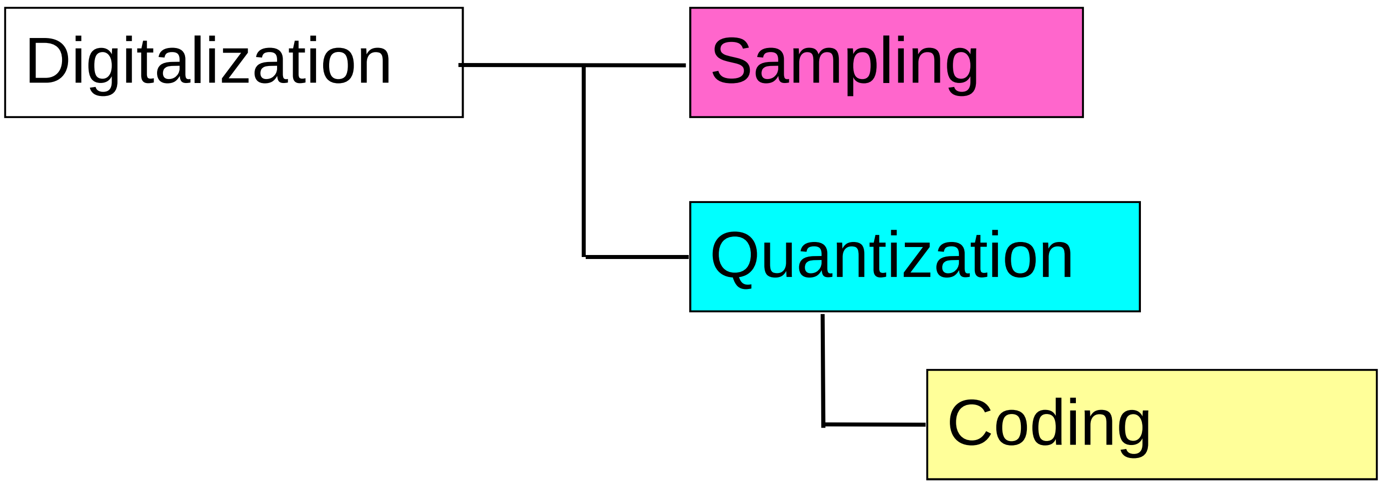
2. Introduction of Digital Signal Processing

3. Summary of Analog vs. Digital Processing

4. Introduction of Digital Filter

2. Introduction of Digital Signal Processing

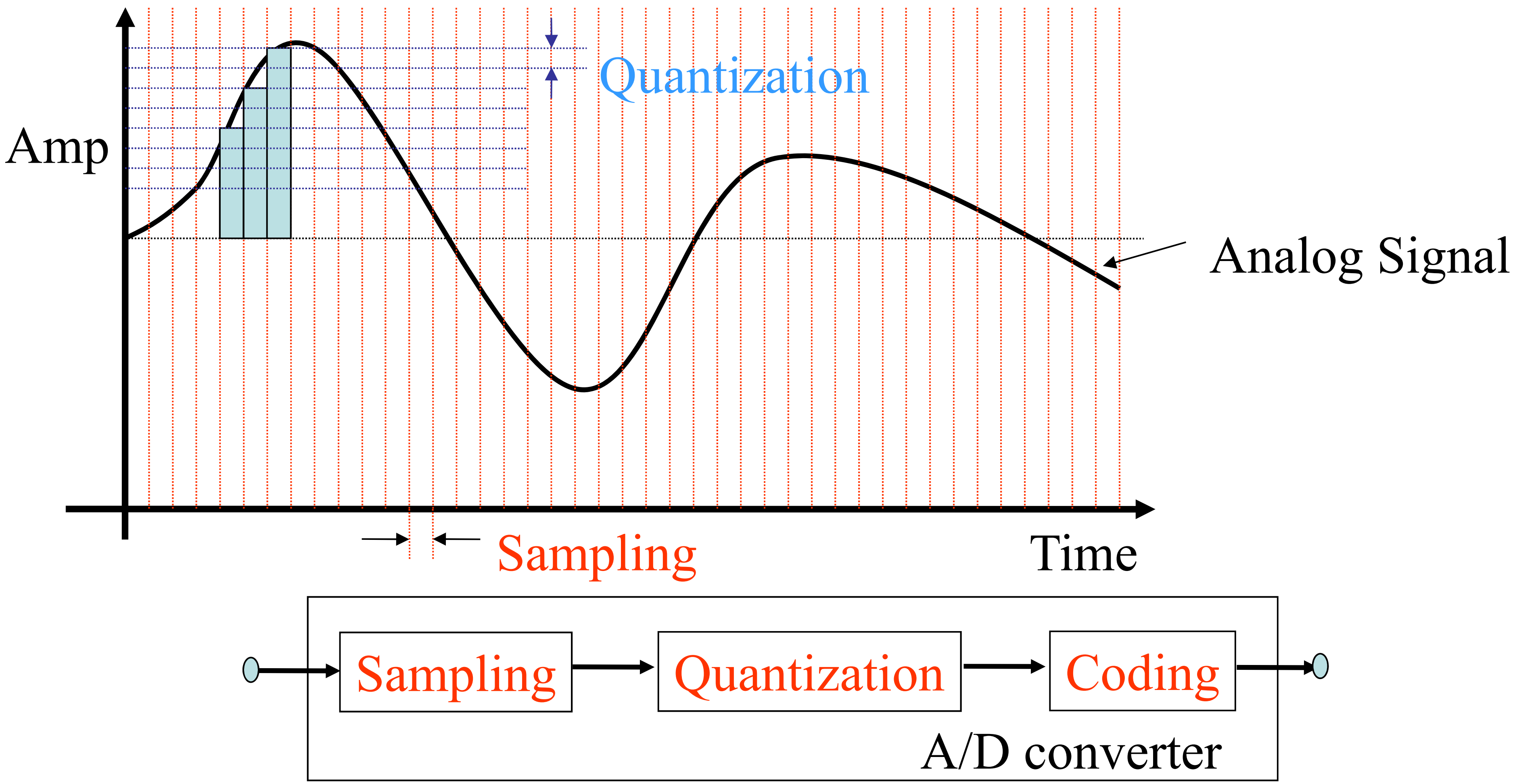
- Digitalization is composed of 2 phase operation



- Sampling : Discrete on time domain
- Quantization : Make steps of Amplitude
- Coding : Representing in digit number

2. Introduction of Digital Signal Processing

Digital Signal Processing



2-1-1. Sampling

- Sampling theory

The signal x(t), whose frequency band is limited to fm, is uniquely defined by the sampled signal, which is sampled from x(t) by sampling frequency **fs >= 2 * fm**.

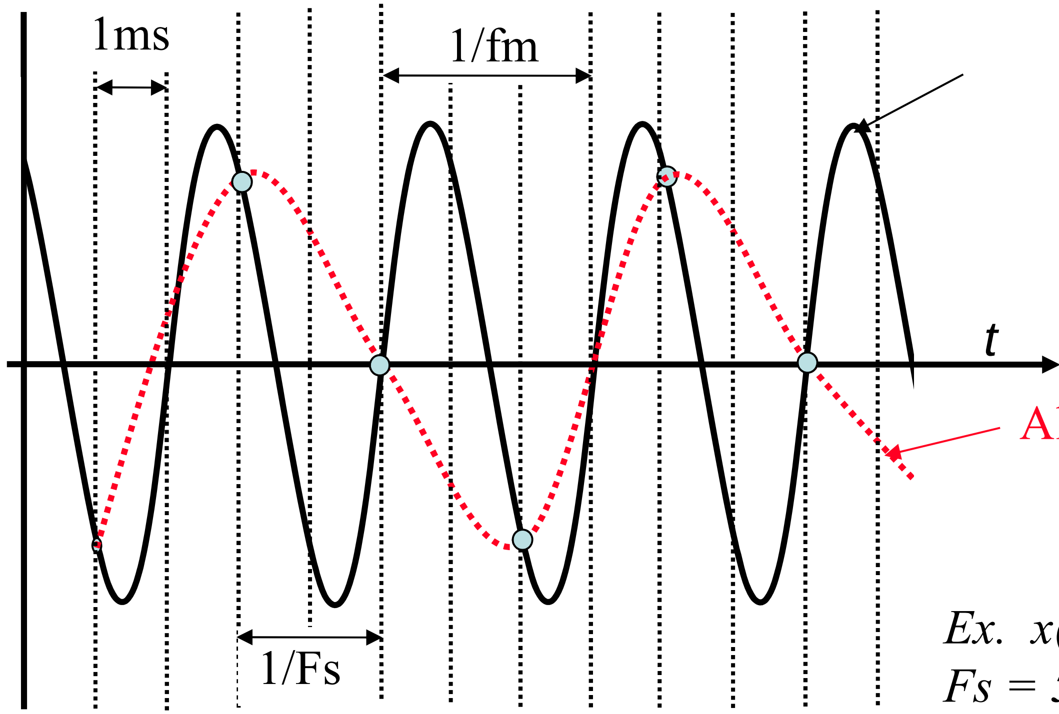
- Sampling theory ;

  1) The sampling frequency should be 2 times higher than the signal band width.   (Nyquist frequency)
  2) If 1) is realized, the signal can be reproduced from sampled data.
  3) If the sampling frequency is not higher than Nyquist frequency, **aliasing** error occurs.

Sampling frequency: How many sample / second (Hz)
Sampling period: The time distance between 2 samples (second)

Sampling frequency = 1 / Sampling period

2-1-2. Aliasing



If sampling frequency Fs < 2 fm ,
Aliasing signal appears.

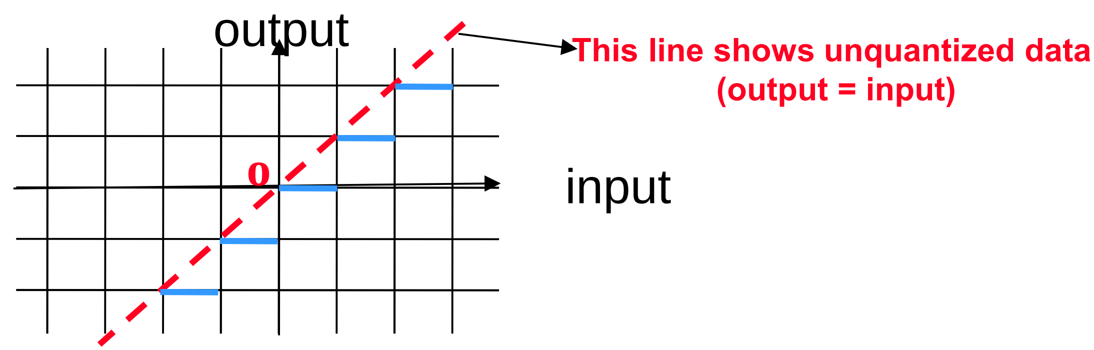
Ex. $x(t) = \sin(2\pi t*333)$ $fm = 333Hz$
$Fs = 500Hz$

Aliasing signal has fa = 167 Hz

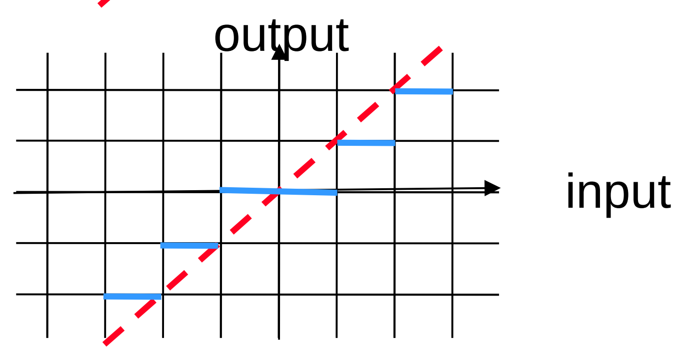
2-2-1. Quantization

- Quantization : mandatory to be operated by digital system
  The amplitude of the signal is quantized to some proper adjacent grid

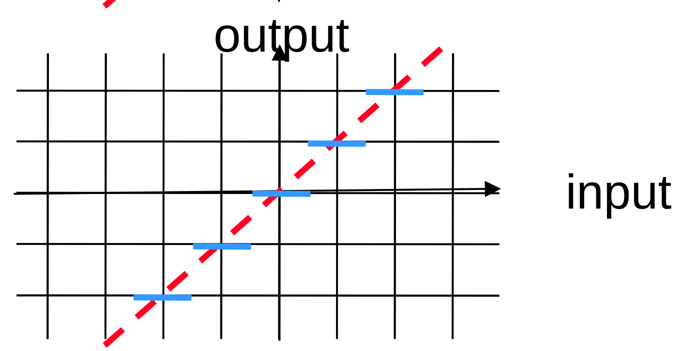
- Truncation (round off)



- Signed truncation

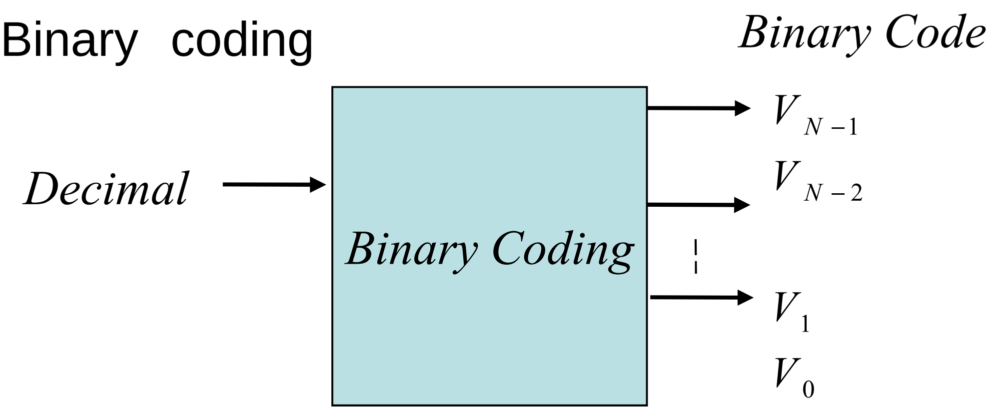


- Rounding



2-2-2. Coding

■Binary coding



Decimal　　　Binary Code

$$V_q = 2^{N-1}V_{N-1} + 2^{N-2}V_{N-2} + \cdots + 2\,V_1 + V_0$$

Example(3bit)

Decimal (Unsigned)	Binary (Unsigned)	Binary (Signed)	Decimal (Signed)
7	111	011	3
6	110	010	2
5	101	001	1
4	100	000	0
3	011	111	-1
2	010	110	-2
1	001	101	-3
0	000	100	-4

Sign bit

■Dynamic Range

Dynamic Range is the ratio between the largest and smallest possible values

$$DR = 20\log(7 - 0) = 20\log(3 - (-4)) = 16.9\text{dB}$$

$$DR = 20 \log FS \qquad FS: \ (\text{Full Scale} = \text{Max} - \text{Min})$$

2-2-3. Quantization & Coding

- Quantization causes "error".
  There must be quality control to maintain the final accuracy of the output according to the system (product) requirement.

- Generally, the accuracy is controlled by bit width.
  (How many bits per sample)
  Ex: For Audio, there are 16 bits / sample

- High bit width -> High accuracy -> Poor Storage

-> How to get balance depends on the application (Audio processing, Video processing, Image processing etc.). It is defined in **Standard** of each field.

2-3 Exercise

Sampled by 44.1kHz
Linear PCM by 16-bit
Stereo

- Calculate the maximum frequency CD can re-produce.
- Calculate the bit rate of the Digitalized data of CD in [Bit/sec].
- Calculate data size of 56[min] stereo music in CD.
- Calculate Dynamic Range of CD data.
- Suppose an analog signal which has 40kHz component, what is happed to the digitalized signal sampled by 44.1kHz?

2-3 Exercise

Sampled by 44.1kHz
Linear PCM by 16-bit
Stereo

- Calculate the maximum frequency CD can re-produce.
  fs >= 2 * fm -> fm <= fs / 2
- Calculate the bit rate of the Digitalized data of CD in [Bit/sec].
  bitrate = (44.1 * 1000) * 16 * 2
- Calculate data size of 56[min] stereo music in CD.
  datasize = bitrate * (56 * 60)
- Calculate Dynamic Range of CD data.
  DR = $20\log(2^{16} - 1)$
- Suppose an analog signal which has 40kHz component, what is happed to the digitalized signal sampled by 44.1kHz?
  Alias 4.1kHz

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BH002:

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2. Video Signal Processing & Video Codec

3. SoC Architecture (SoC: System on Chip)

3. Keywords of Digital Signal Processing

Merits and Demerits of Digital Processing

VS

Analog

Digital

Source: http://wikipedia.org

3. Keywords of Digital Signal Processing

Merits and Demerits of Digital Processing

- Digital processing has various demerits, but digital signal processing technology has overcome them and semiconductor technology realized it with reasonable cost.

	Analog	Digital
Complexity	Simple	Complex
Cost	Reasonable	Expensive
Quality	Good : for original signal Poor : for repeating copy & signal transfer	Good : for original signal Good : for repeating copy & signal transfer
Stability	Poor : for time variant, etc	Good : for time variant, etc
Portability	Difficult	Easy

3-1 Analog signal processing vs. Digital signal processing

Analog		Digital
Signal Analysis Fourier Transform	**Digitalize** (ADC) →	**Signal Analysis** DFT: Discrete Fourier Transform FFT: Fast Fourier transform DCT: Discrete Cosine transform
Signal Processing Laplace Transform	← **I-Digitalize** (DAC+LP filter)	**Signal Processing** Z Transform

3-2. Fourier Transform

Analog

Signal Analysis

Fourier Transform

Signal Processing

Laplace Transform

Digitalize

(ADC)

I-Digitalize

(DAC+LP filter)

Digital

Signal Analysis

DFT: Discrete Fourier Transform

FFT: Fast Fourier transform

DCT: Discrete Cosine transform

Signal Processing

Z Transform

3-2-1. Fourier Series Expansion (Trigonometric form)

- Cyclic signal can be expressed by series of trigonometric functions (sine/cosine)
- Fourier Series for Cyclic Signals

$$x(t) = a_0 + \sum_{K=1}^{\infty} a_k \cos k\omega_0 t + \sum_{K=1}^{\infty} b_k \sin k\omega_0 t$$

T: basic cycle period,

$\omega_0 = 2\pi / T$ *: basic angular frequency,*

$F = 1 / T$ *: basic frequency*

Fourier Series coefficients

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,dt$$

3-2-1. Fourier Series Expansion (Trigonometric form)

- Cyclic signal can be expressed by series of trigonometric functions (sine/cosine)
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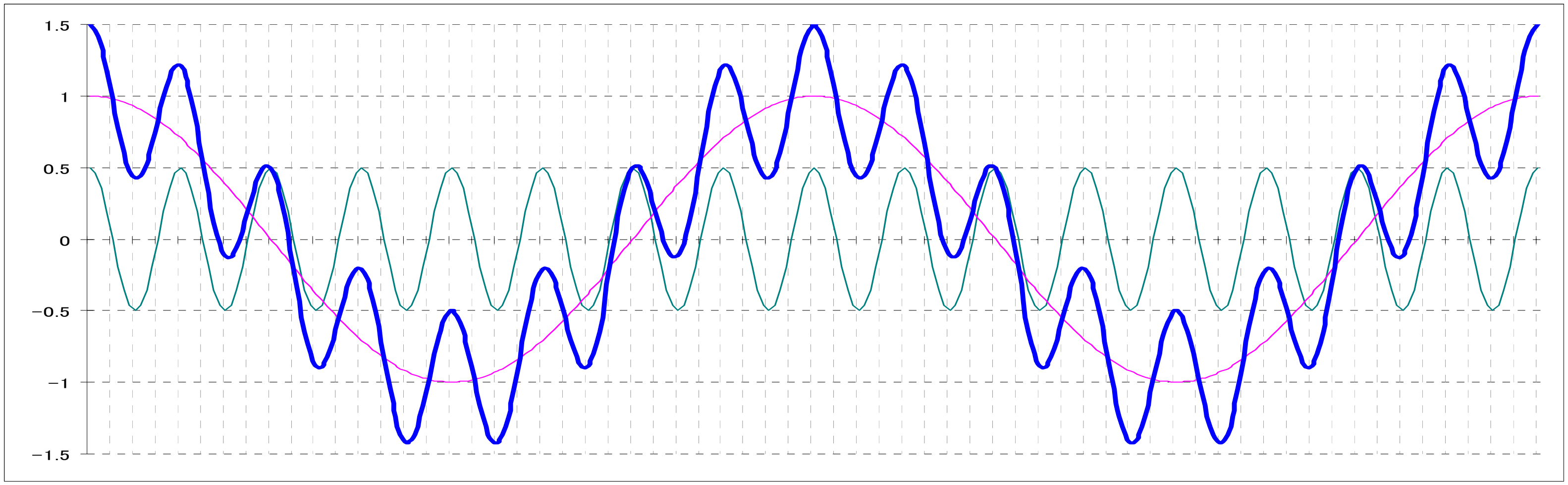


3-2-2. Fourier Series Expansion (Complex form)

- Euler's formula:
  $e^{jx} = \cos x + j \sin x\ ,\ \ e^{-jx} = \cos x - j \sin x$
- $a_0 = c_0$
  $a_k = c_k + c_{-k}$
  $b_k = j(c_k - c_{-k})$

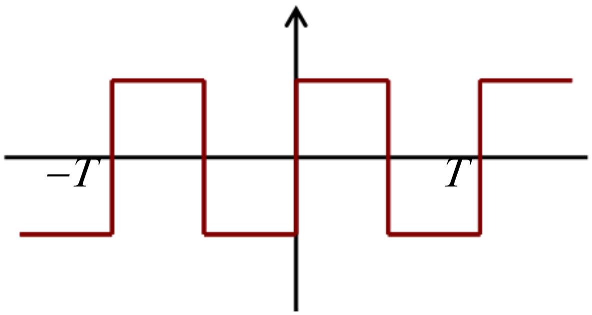
$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + \sum_{k=1}^{\infty} b_k \sin k\omega_0 t$$

$$\Downarrow$$

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

3-2-3. Fourier Series – rectangular wave

- Example 1: Rectangular wave

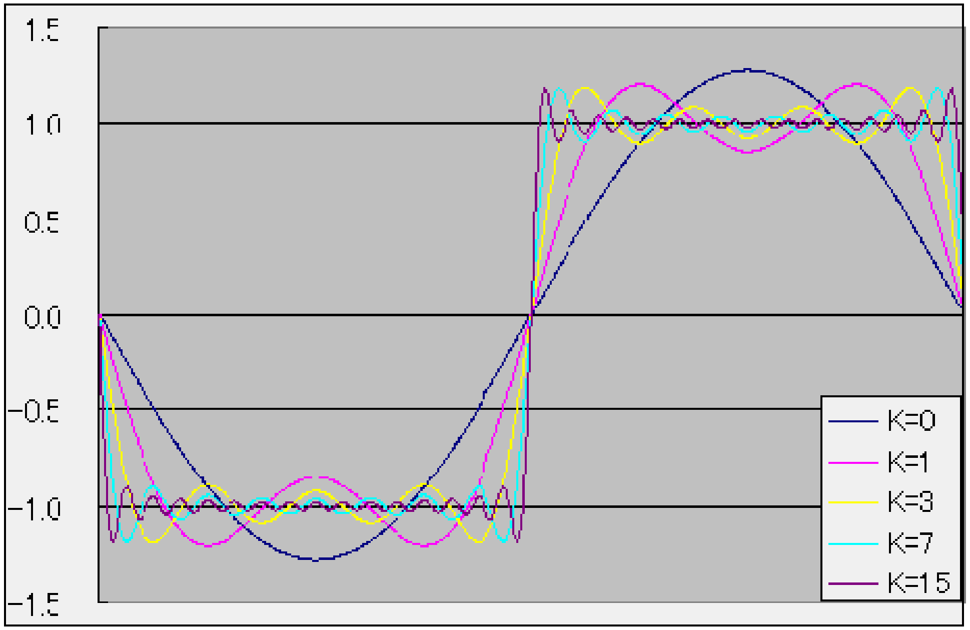


$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt \quad = 0$$

$$a_n = \frac{2}{T}\int_0^T x(t) \cos n\omega_0 t\, dt = 0$$

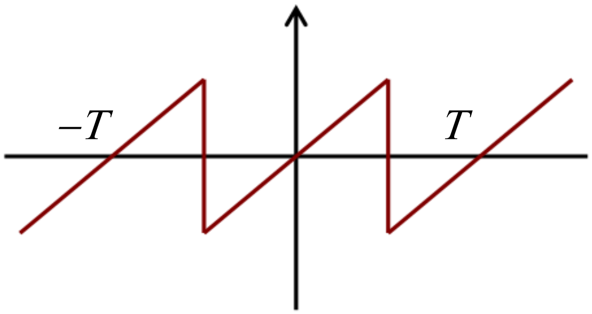
$$b_n = \frac{2}{T}\int_0^T x(t) \sin n\omega_0 t\, dt$$

$$= \begin{cases} \dfrac{4}{\pi n} & \text{when } n \text{ is odd} \\ 0 & \text{when } n \text{ is even} \end{cases}$$



3-2-4. Fourier Series – saw wave

- Example 2: Saw wave



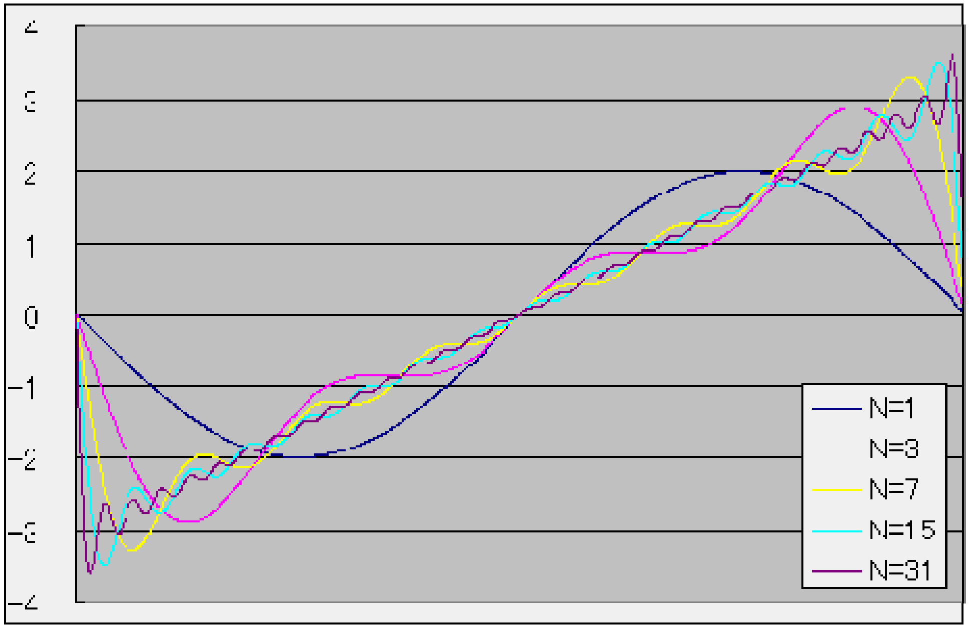
$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt = 0$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\,dt = 0$$

$$b_n = \frac{2}{T}\int_0^T t\sin n\omega_0 t\,dt$$

$$= \frac{2}{T}\left[-x\frac{\cos(nx)}{\pi n}\right]_{-T/2}^{T/2} + \frac{2}{T}\int\frac{\cos(nx)}{\pi n}dt$$

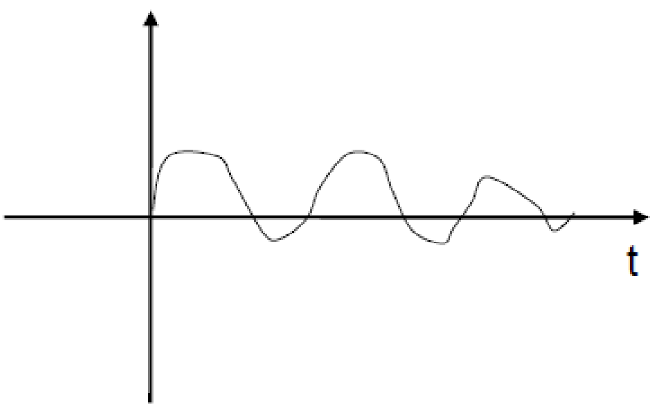
$$= (-1)^{n+1}\frac{2}{\pi}$$



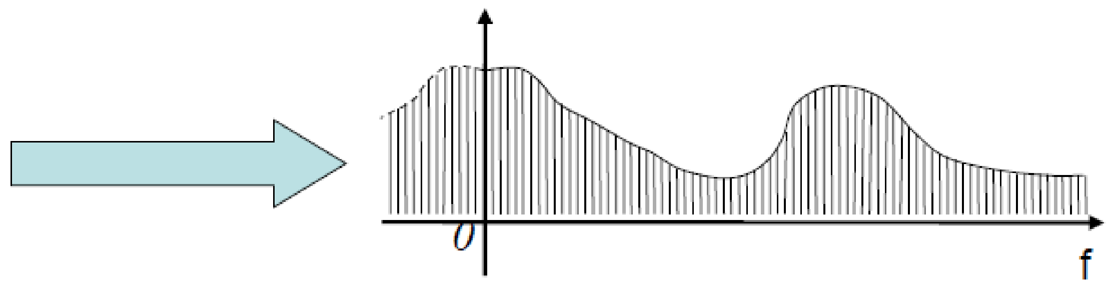
3-2-5. Fourier Transform

- Fourier Series is applied only for **cyclic signal**.
- Fourier Tranform is an extension of Fourier Series that can be applied for **non-cyclic** signal (cyclic signal whose T = ∞).

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad \omega = 2\pi f$$



Time domain



Frequency domain

3-3. Discrete Fourier Transform

Analog

Analysis

Fourier Transform

Processing

Laplace Transform

Digitalize

(ADC)

I-Digitalize

(DAC+LP filter)

Digital

Analysis

DFT: Discrete Fourier Transform

FFT: Fast Fourier transform

DCT: Discrete Cosine transform

Processing

Z Transform

3-3-1. Discrete-time Fourier Transform

- Fourier Transform for Discrete-time signal:

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$$

Discrete Fourier Transform (DFT) of a cyclic signal with period N:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi kn}{N}}$$

$$e^{jx} = \cos x + j\sin x$$
$$e^{-jx} = \cos x - j\sin x$$

3-3-2. Discrete-time Fourier Transform

Key Features of Discrete-time Fourier Transform

	Time domain	Frequency Domain
Linearity	$a\,x_1(n) + b\,x_2(n)$	$a\,X_1(e^{j\omega}) + b\,X_2(e^{j\omega})$
Time shift	$x(n-k)$	$X(e^{j\omega})\,e^{-j\omega k}$
Convolution	$\sum x_1(k)x_2(n-k)$	$X_1(e^{j\omega})\,X_2(e^{j\omega})$
Frequency shift	$X(n)e^{jw0n}$	$X(e^{j(\omega-\omega 0)})$
Symmetry of spectrum	$All\ x(n)\ are\ real$	$X(e^{j\omega}) = \overline{X}(e^{-j\omega})$

3-4. Fast Fourier Transform

Analog

Analysis

Fourier Transform

Processing

Laplace Transform

Digitalize

(ADC)

I-Digitalize

(DAC+LP filter)

Digital

Analysis

DFT: Discrete Fourier Transform

FFT: Fast Fourier transform

DCT: Discrete Cosine transform

Processing

Z Transform

3-4-1. FFT (Fast Fourier Transform)

Digital Signal Processing

- FFT is Fast calculation algorithm of DFT, for $N = 2^m$ cases
- DFT definition;

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

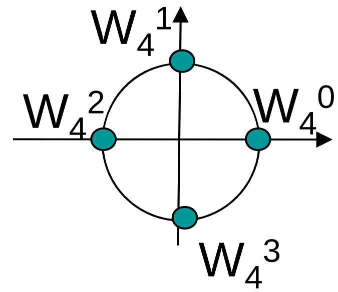
$$W_N^{kn} = e^{-\frac{j2\pi kn}{N}}$$

- When N = 4, X(k) is expressed as followings:

$X(0) = W_4^{0}\, x(0) + W_4^{0}\, x(1) + W_4^{0}\, x(2) + W_4^{0}\, x(3)$
$X(1) = W_4^{0}\, x(0) + W_4^{1}\, x(1) + W_4^{2}\, x(2) + W_4^{3}\, x(3)$
$X(2) = W_4^{0}\, x(0) + W_4^{2}\, x(1) + W_4^{4}\, x(2) + W_4^{6}\, x(3)$
$X(3) = W_4^{0}\, x(0) + W_4^{3}\, x(1) + W_4^{6}\, x(2) + W_4^{9}\, x(3)$

16 multiplication
12 addition

3-4-2. FFT (Fast Fourier Transform)



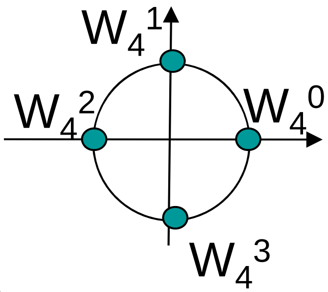
- $W_N^0 = 1$
- $W_N^{nk} = W_N^{(nk) \bmod N}$
- $W_N^x * W_N^y = W_N^{x+y}$

$$
\begin{aligned}
X(0) &= W_4^0\, x(0) + W_4^0\, x(1) + W_4^0\, x(2) + W_4^0\, x(3)\\
X(1) &= W_4^0\, x(0) + W_4^1\, x(1) + W_4^2\, x(2) + W_4^3\, x(3)\\
X(2) &= W_4^0\, x(0) + W_4^2\, x(1) + W_4^4\, x(2) + W_4^6\, x(3)\\
X(3) &= W_4^0\, x(0) + W_4^3\, x(1) + W_4^6\, x(2) + W_4^9\, x(3)
\end{aligned}
$$

=>

$$
\begin{aligned}
X(0) &= x(0) + x(2) + x(1) + x(3)\\
X(1) &= x(0) + W_4^2\, x(2) + [x(1) + W_4^2\, x(3)]\, W_4^1\\
X(2) &= x(0) + x(2) + [x(1) + x(3)]\, W_4^2 \quad (W_4^4 = W_4^0\ ,\ W_4^6 = W_4^2)\\
X(3) &= x(0) + W_4^2\, x(2) + [x(1) + W_4^2\, x(3)]\, W_4^1\, W_4^2 \quad (W_4^9 = W_4^5)
\end{aligned}
$$

3-4-3. FFT (Fast Fourier Transform)



- $W_4^{2} = -1$

$$
\begin{aligned}
X(0) &= x(0) + x(2) + x(1) + x(3) \\
X(1) &= x(0) + W_4^{2}\,x(2) + [x(1) + W_4^{2}\,x(3)]\,W_4^{1} \\
X(2) &= x(0) + x(2) + [x(1) + x(3)]\,W_4^{2} \\
X(3) &= x(0) + W_4^{2}\,x(2) + [x(1) + W_4^{2}\,x(3)]\,W_4^{1}\,W_4^{2}
\end{aligned}
$$

=>

$$
\begin{aligned}
X(0) &= x(0) + x(2) + x(1) + x(3) \\
X(1) &= x(0) - x(2) + [x(1) - x(3)]\,W_4^{1} \\
X(2) &= x(0) + x(2) - [x(1) + x(3)] \\
X(3) &= x(0) - x(2) - [x(1) - x(3)]\,W_4^{1}
\end{aligned}
$$

FFT
2 multiplication
8 addition

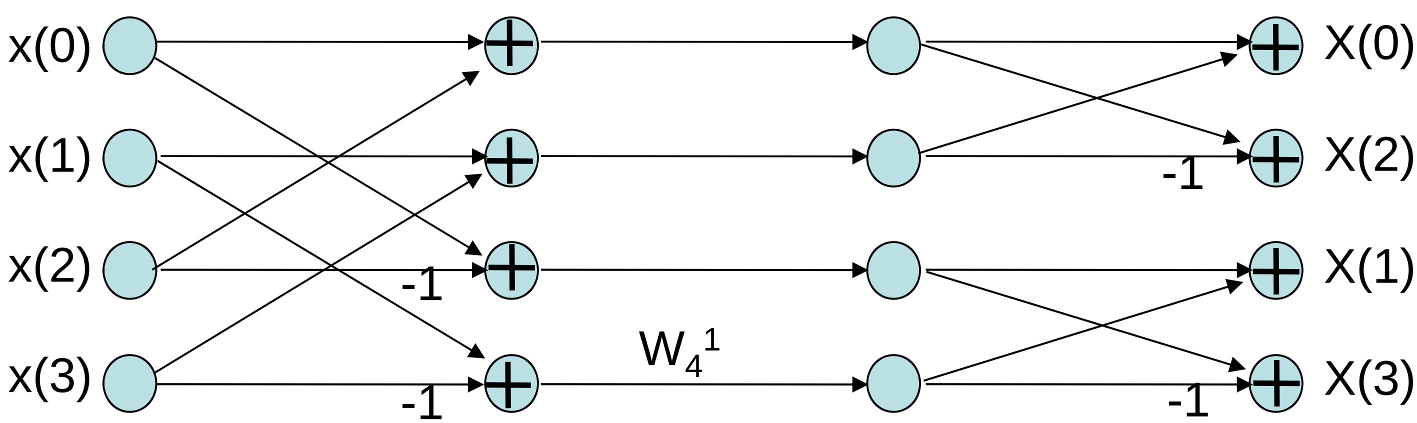
VS

DFT
16 multiplication
12 addition

3-4-4. FFT (Fast Fourier Transform)

- 4 points FFT flow is shown below

$$X(0) = x(0) + x(2) + x(1) + x(3)$$
$$X(1) = x(0) - x(2) + [x(1) - x(3)]\, W_4^1$$
$$X(2) = x(0) + x(2) - [x(1) + x(3)]$$
$$X(3) = x(0) - x(2) - [x(1) - x(3)]\, W_4^1$$



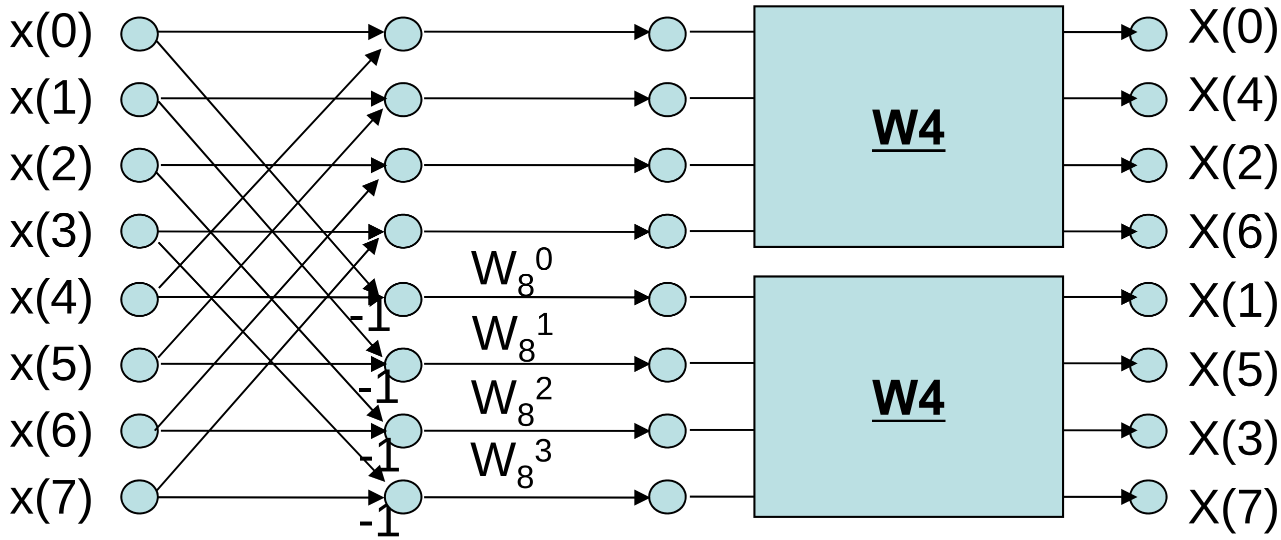
FFT
2 multiplication
8 addition

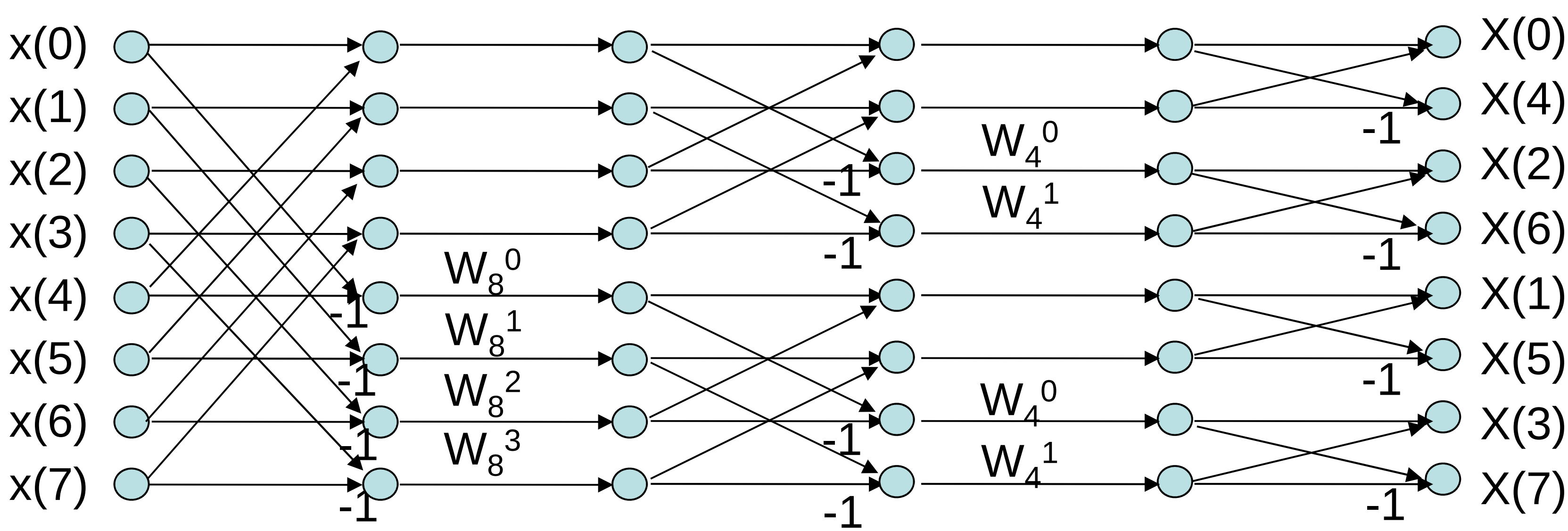
VS

DFT
16 multiplication
12 addition

3-4-5. FFT (Fast Fourier Transform)

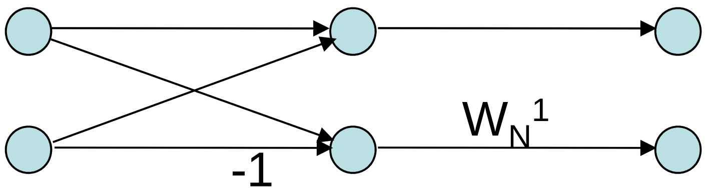
- 8 points FFT flow is shown below;





3-4-6. FFT (Fast Fourier Transform)

- Butterfly operation;
  The FFT is composed by the combination of following operation



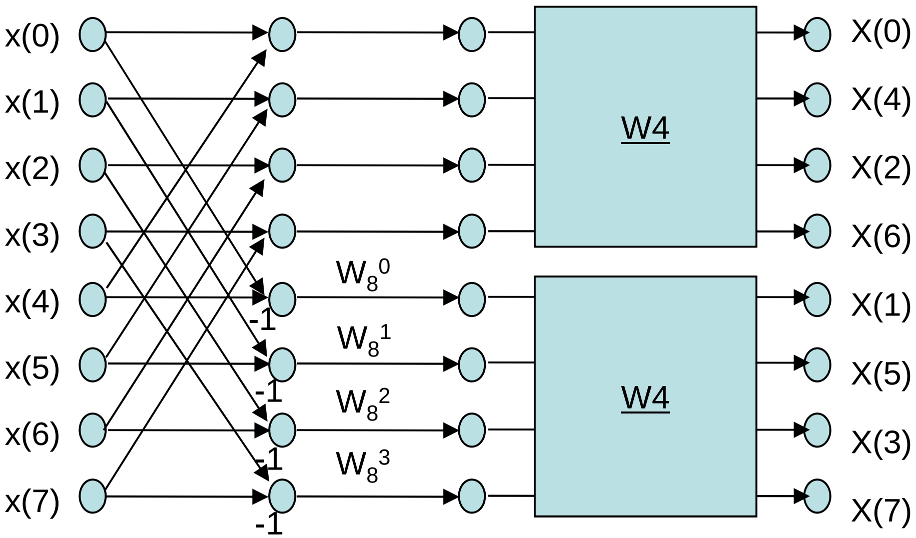
Calculation numbers

	Original (DFT)	FFT
Multiplex (complex)	N^2	$\frac{N(\log_2 N - 1)}{2}$
ADD (complex)	N(N-1)	$N \log_2 N$

3-4-7. FFT (Fast Fourier Transform)

Digital Signal Processing

Bit reverse operation: useful operation re-ordering



Bit reverse operation

normal order		*reverse order*	
decimal	*binary*	*binary*	*decimal*
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

3-5. Descrete Cosine Transform

Analog

Analysis

Fourier Transform

Processing

Laplace Transform

Digitalize

(ADC)

I-Digitalize

(DAC+LP filter)

Digital

Analysis

DFT: Discrete Fourier Transform

FFT: Fast Fourier transform

DCT: Discrete Cosine transform

Processing

Z Transform

3-5-1. Discrete Cosine Transform

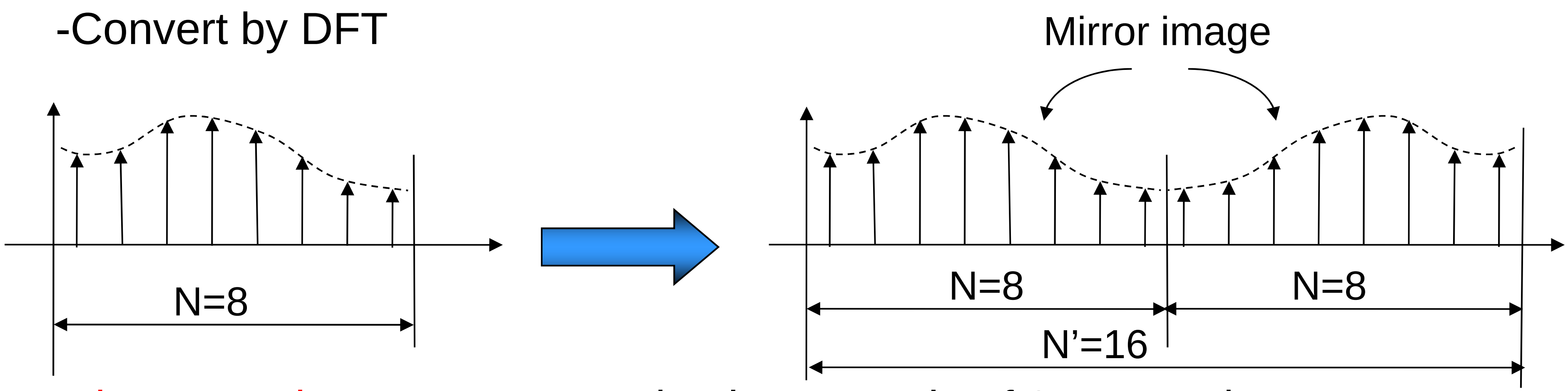
$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$W_N^{kn} = e^{-\frac{j2\pi kn}{N}}$$

Digital Signal Processing

- DCT is modified from DFT. DFT is very useful tool, but the coefficients of DFT is complex. So the signal of N real samples is converted to DFT of N complex samples (include real part and image part), so N coef. become 2N coef. This is not suitable for data compression use.

- DCT is developed as follows;
  -Make 2 N sample signal by using N sample data to make mirror image.
  -Convert by DFT



Imaginary part becomes zero and only N sample of 2N are unique.

3-5-2. Discrete Cosine Transform (DCT)

- For the expanded 2 N sample data, DFT coefficients are;

$$X(k) = \sum_{n=0}^{2N-1} x(n)\, e^{\frac{-j2\pi nk}{N'}}, \qquad n, k=0, 1, , ,2N-1 \quad N'=2N$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{2N}} + \sum_{n=N}^{2N-1} x(n)\, e^{\frac{-j2\pi nk}{2N}}$$

- Shift frequency axis by 0.5 base frequency (n=0.5),and set m=2N-1-n (n=2N-1-m) and replace n.

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j\pi k(n+0.5)}{2N}} + \sum_{m=0}^{N-1} x(2N-1-m)\, e^{\frac{-j\pi k(2N-1-m+0.5)}{2N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j\pi k(n+0.5)}{2N}} + \sum_{m=0}^{N-1} x(2N-1-m)\, e^{-j\left(2\pi k - \frac{\pi k(m+0.5)}{2N}\right)}$$

- As x(n) (n=N,N+1,,2N-1) are mirror image of x(n) (n=0,1,N-1), x(2N-1-m)=x(n),

$$x(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j\pi k(2n+1)}{2N}} + \sum_{n=0}^{N-1} x(n)\, e^{\frac{j\pi k(2n+1)}{2N}} = 2\sum_{n=0}^{N-1} x(n) \cos\frac{\pi k(2n+1)}{2N}$$

3-6. Laplace Transform

Analog

Analysis

Fourier Transform

Signal Processing

Laplace Transform

Digitalize

(ADC)

I-Digitalize

(DAC+LP filter)

Digital

Analysis

DFT: Discrete Fourier Transform

FFT: Fast Fourier transform

DCT: Discrete Cosine transform

Signal Processing

Z Transform

3-6-1. Laplace Transform - definition

- Laplace Transform definition:

$$F(s) = \int_{0}^{\infty} f(t)\, e^{-st}\, dt$$

- Laplace Transform is very useful to solve the differential equations.
- Laplace Transform is used to analyze and process analog signal.

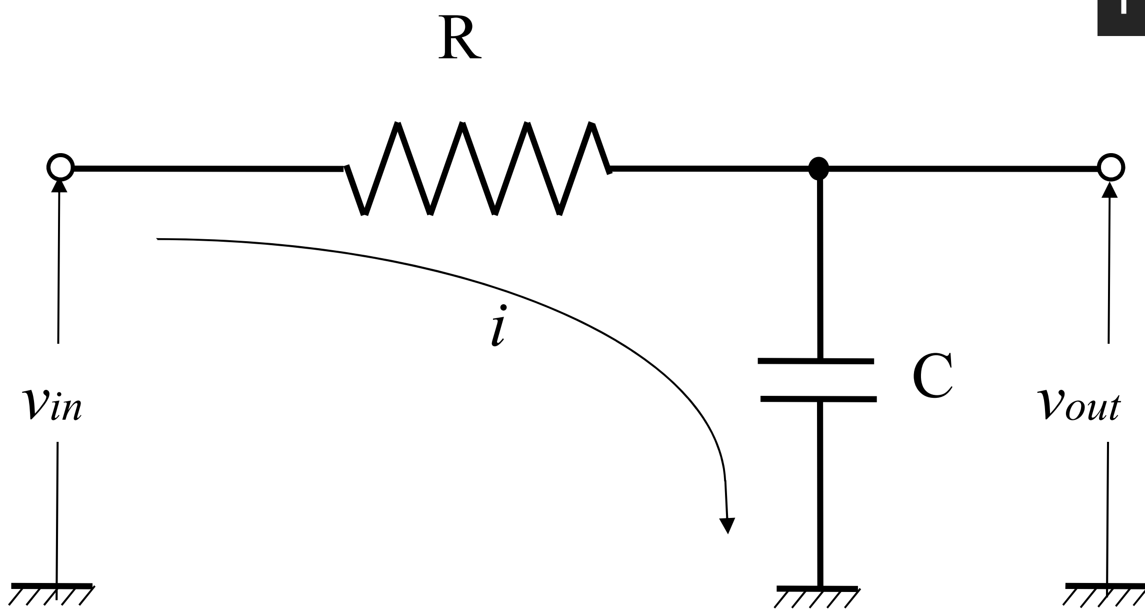
3-6-2. Laplace Transform - functions

- Laplace transforms of elementary functions

f (t)	F (s)
$\delta(t)$	1
$u(t)$	$1/s$
$t^n / n!$	$1 / s^{n+1}$
e^{-at} (a : real or complex)	$1/(s+a)$
$t^n e^{-at} / n!$ (a : real or complex)	$1/(s+a)^{n+1}$
$\cos \beta t$	$s/(s^2 + \beta^2)$
$\sin \beta t$	$\beta/(s^2 + \beta^2)$

3-6-3. Laplace Transform

- Example:



According to Physical theory and Kirchhoff's laws

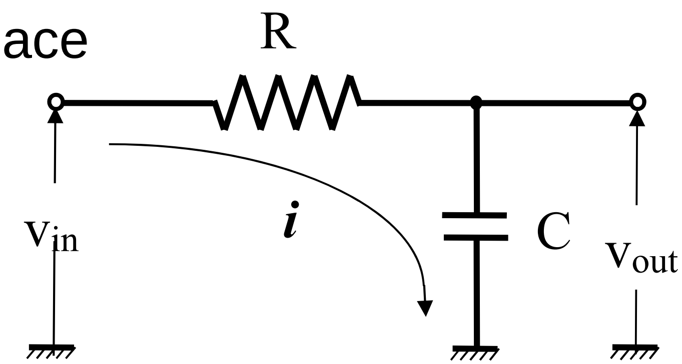
$$v_{in} = Ri + \frac{1}{C}\int i\,dt$$

$$v_{out} = \frac{1}{C}\int i\,dt$$

How to calculate v_{out} from v_{in}?

3-6-4. Laplace Transform

- Example;
  Calculate Transfer characteristics using Laplace



$$v_{in} = Ri + \frac{1}{C}\int idt$$

$$v_{out} = \frac{1}{C}\int idt$$

Using Laplace transform;

$$V_{in}(s) = R\,I(s) + \frac{1}{Cs} I(s)$$

$$V_{out}(s) = \frac{1}{Cs} I(s)$$

then, $I(s) = Cs\,V_{out}(s)$, and replace I(s)

$$V_{in}(s) = (R + \frac{1}{Cs})\,Cs\,V_{out}(s)$$

$$V_{out}(s) = \frac{1}{RCs + 1} V_{in}(s)$$

3-7. Z Transform

Analog

Analysis

Fourier Transform

Processing

Laplace Transform

Digitalize

(ADC)

I-Digitalize

(DAC+LP filter)

Digital

Analysis

DFT: Discrete Fourier Transform

FFT: Fast Fourier transform

DCT: Discrete Cosine transform

Processing

Z Transform

3-7-1. Z Transform

- Z transform definition:

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$$

Compare with Fourier Transform: $z = e^{j\omega}$

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$$

3-7-1. Z Transform

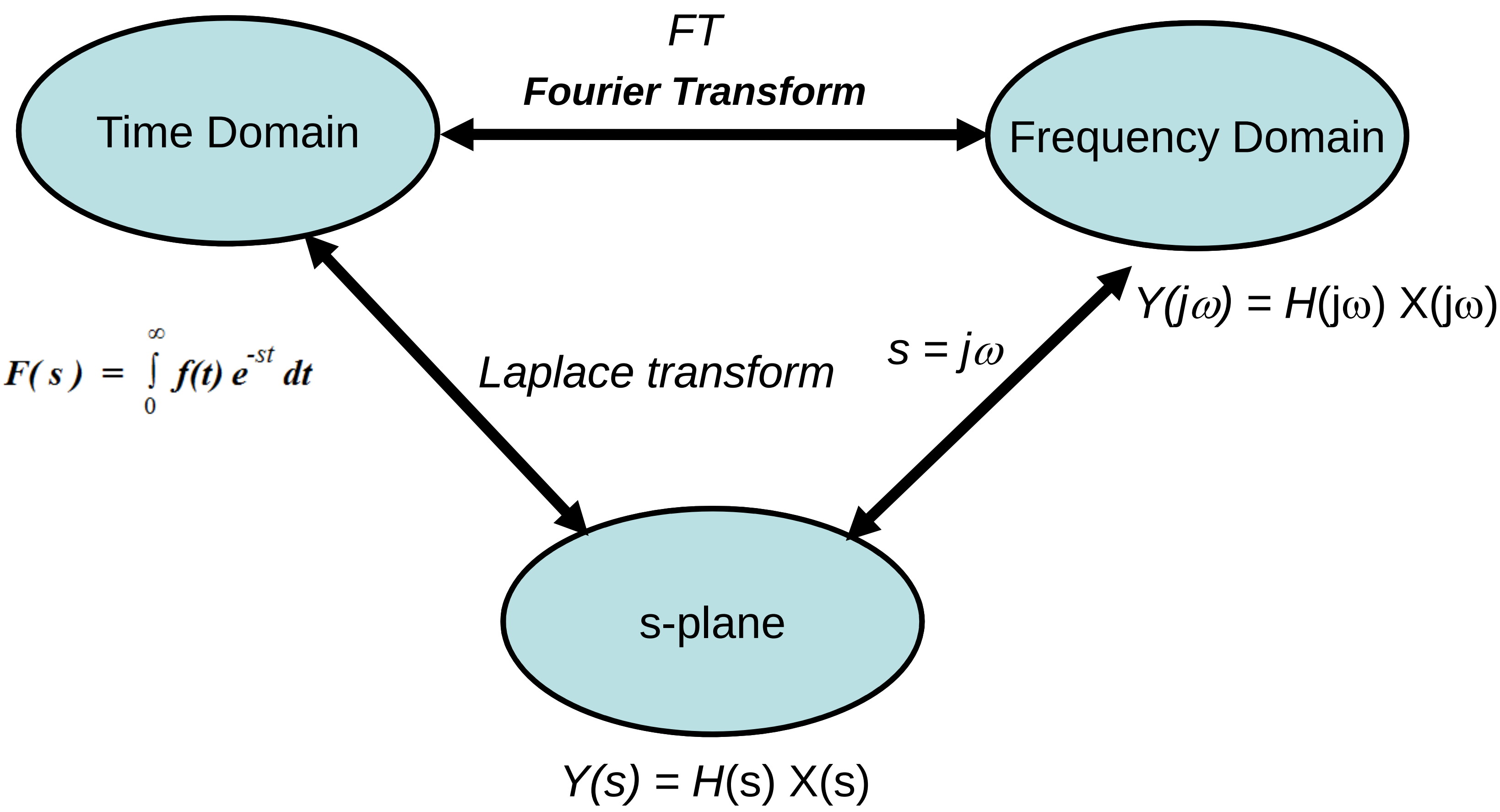
- Z transform is used to analyze and process digital signal as well as design digital filter.

$$\sum x_1(k)x_2(n-k) = X_1(z).X_2(z)$$

The convolution in Time domain become multiplication in z domain.

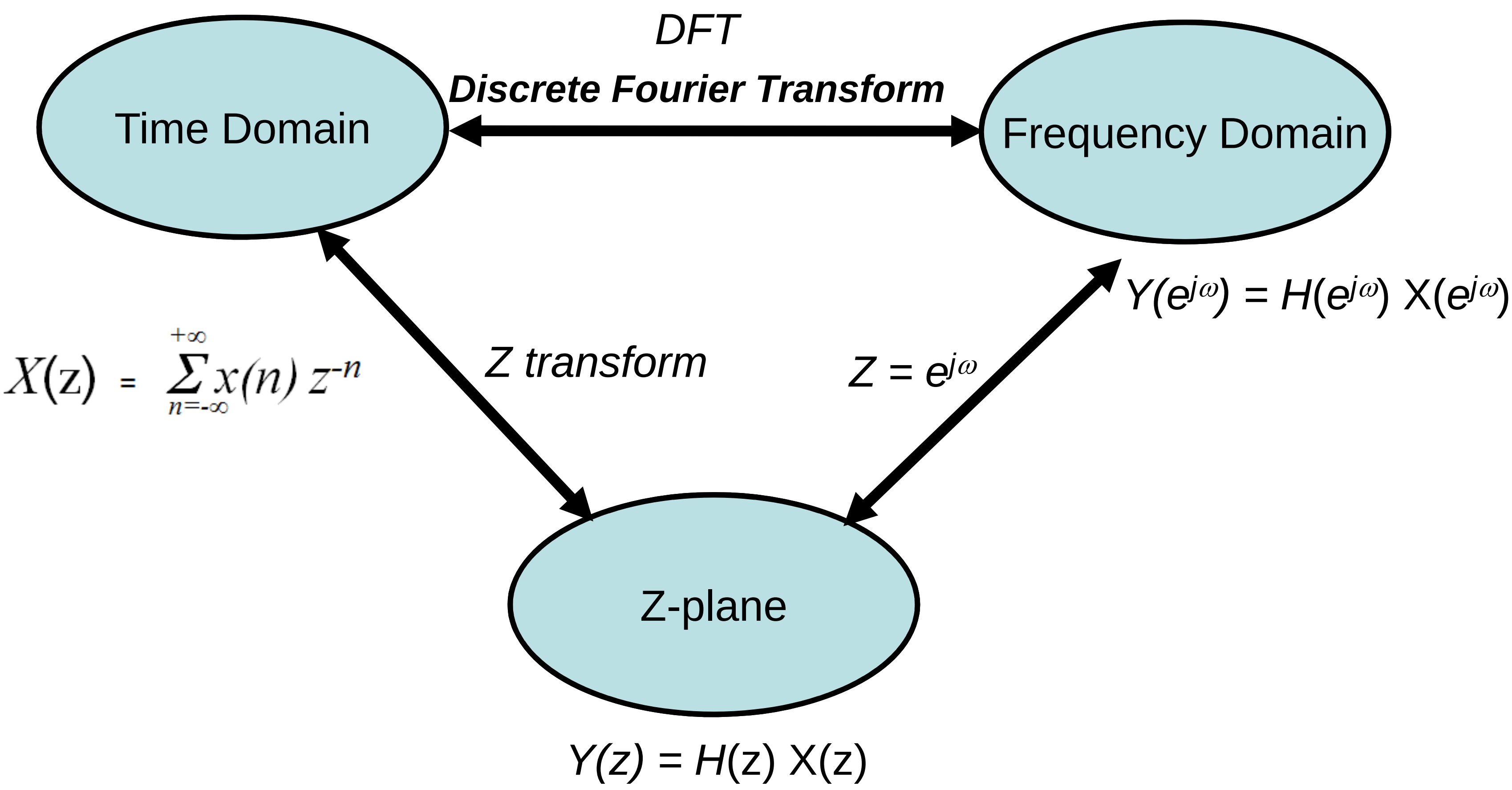
3-8-1. Summary of Analog signal processing

- The time domain, Frequency domain and s-plane have following relations;



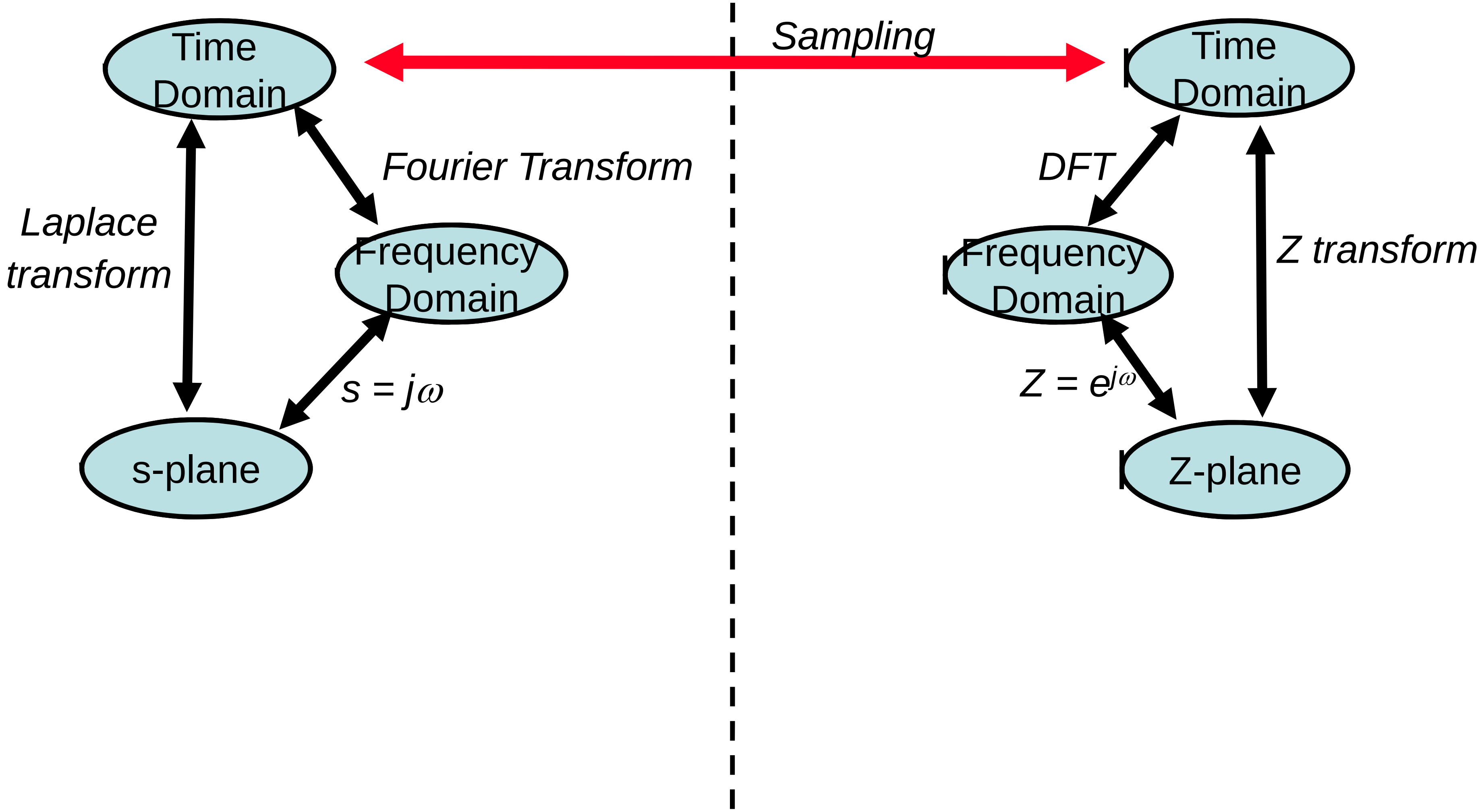
3-8-2. Summary of Digital signal processing

- The time domain, Frequency domain and Z-plane have following relations;



3-8-2. Summary of Analog-Digital signal processing

- The analog-digital signal processing have following relations;



AGENDA

BH001:

1. Target/Purpose of this training course

2. Introduction of Digital Signal Processing

3. Analog vs. Digital Processing Method

4. Introduction of Digital Filter

BH002:

1. Audio Signal Processing & Audio Codec

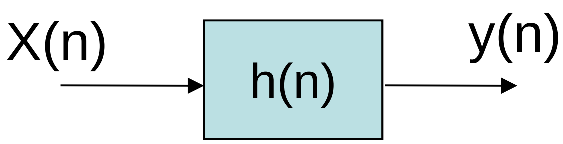
2. Video Signal Processing & Video Codec

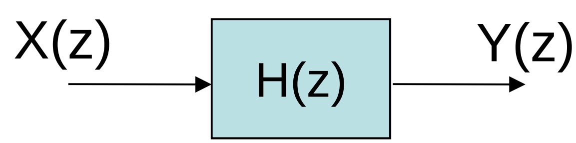
3. SoC Architecture (SoC: System on Chip)

4. Introduction of Digital Filter

- Filter is a process that remove some unwanted component or feature from a signal.

- Filter is represented by transfer function.





$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \quad \text{(convolution)}$$

$$Y(z) = H(z)\, X(z)$$

4. Introduction of Digital Filter

- Filter Example:

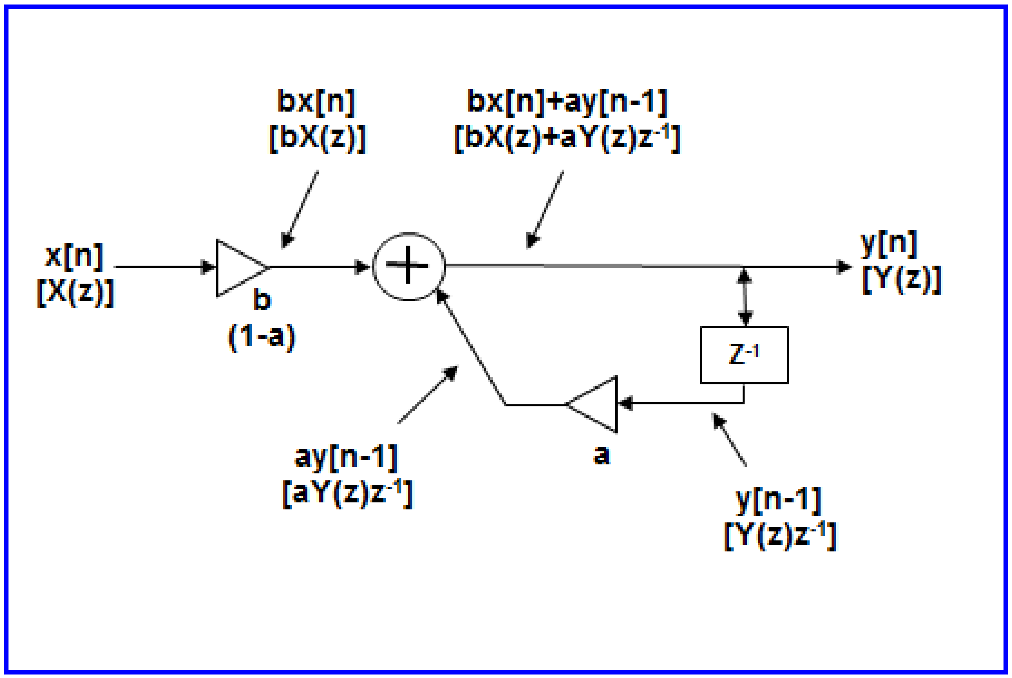


Figure : Block Diagram of difference equation of
y[n] = ay[n-1]+bx[n]

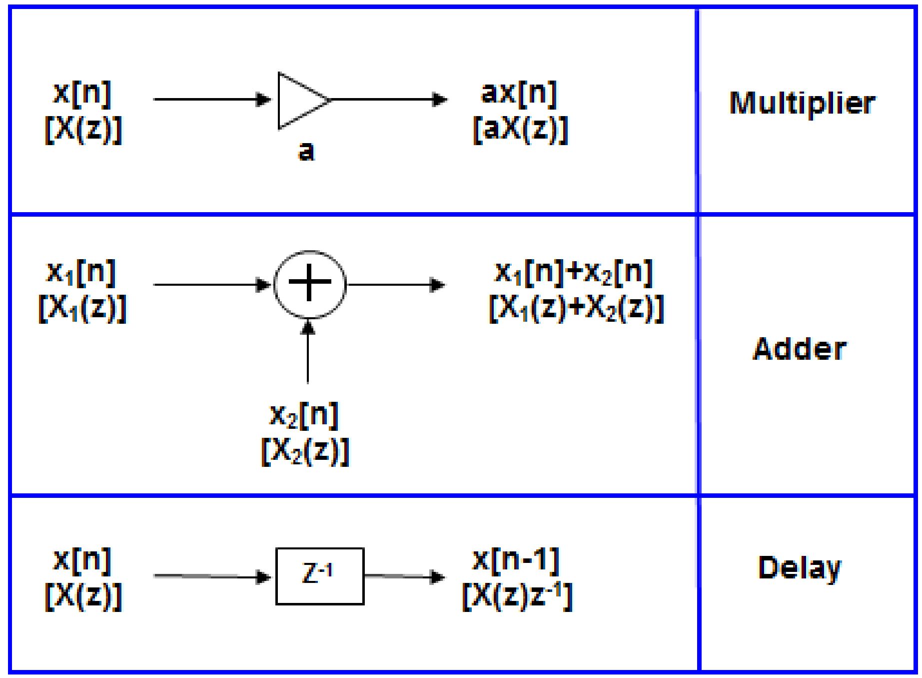
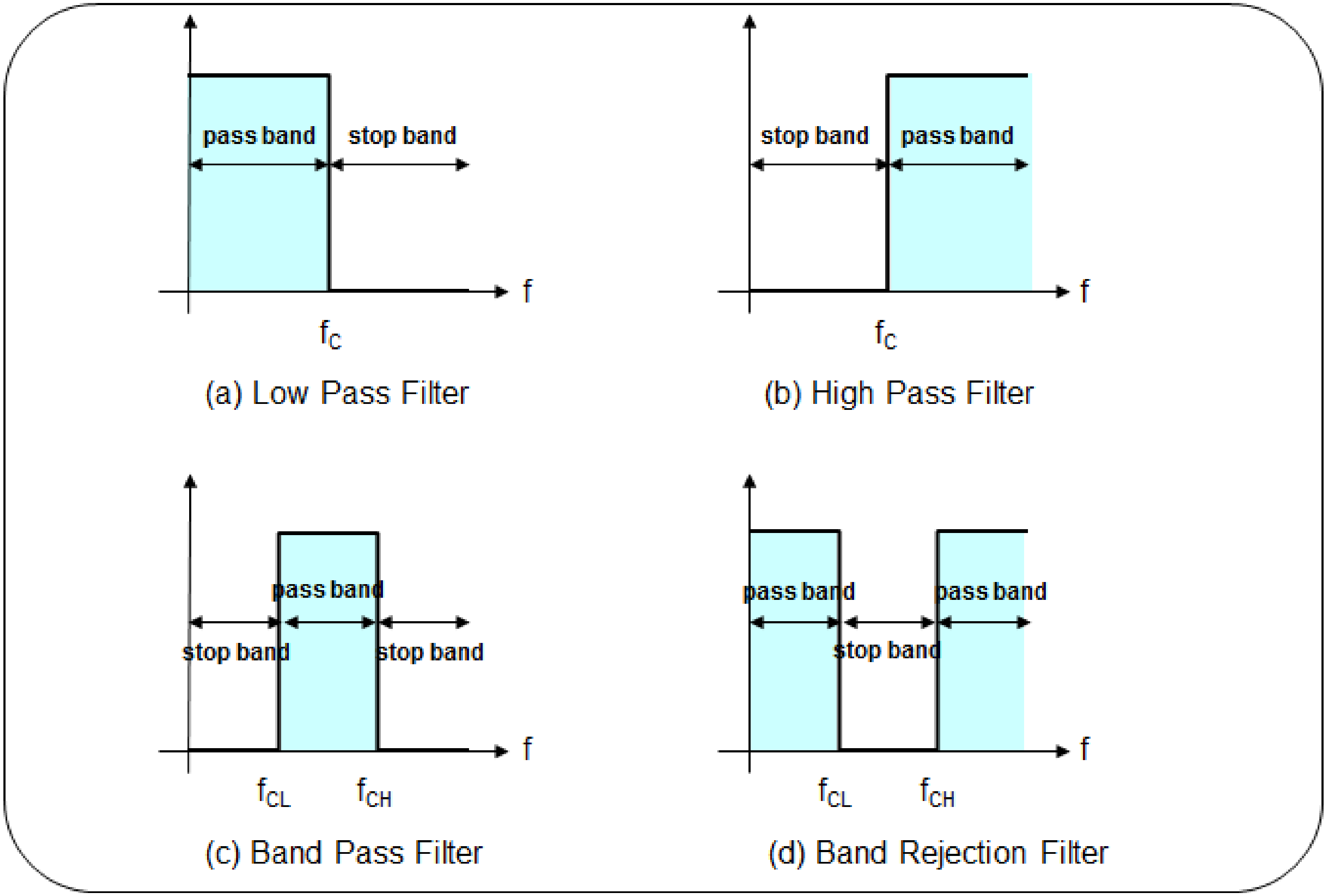


Figure : Elements of Block Diagram

4. Introduction of Digital Filter

Digital Signal Processing

- When processing on frequency domain, there are 4 kinds of filters



Thank You for Attendance!

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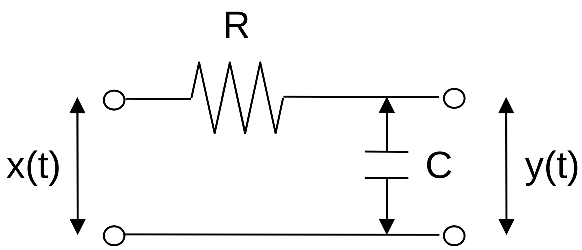
4. Digital Filter Design

4-1. Analog Logic and Discrete-time system

The filter that made by analog logics are one of analog signal processing system.
The relation in the time-domain for input and output signal is shown by differential equation.
The other side, discrete-time system that using discrete signal shows by difference equation.



$$\frac{dy(t)}{dt} + \frac{1}{CR} y(t) = \frac{1}{CR} x(t)$$

Figure: Differential equation by analog logic

Differential equation

$$\frac{dy(t)}{dt} + \frac{1}{CR} y(t) = \frac{1}{CR} x(t)$$

Convert to “Difference” from “Differential” equation

$$\frac{dy(t)}{dt} \Rightarrow \frac{y(nT) - y((n-1)T)}{T}$$

T is the sampling interval (in seconds)

$$\frac{CR}{CR + T} = a$$

Difference equation

$$y(nT) = ay((n-1)T) + (1-a)x(nT)$$

Final difference equation
(T is constant and can omit it)

$$y(n) = ay[n-1] + (1-a)x[n]$$

4-2. Basic Knowledge for Design Digital Filter

- A digital signal is a discrete-time signal that takes on only discrete set of values
- Difference equation shows the relations in the time-domain for discrete-time system
- When design the discrete-time system of the digital filter, design it based on a difference equation

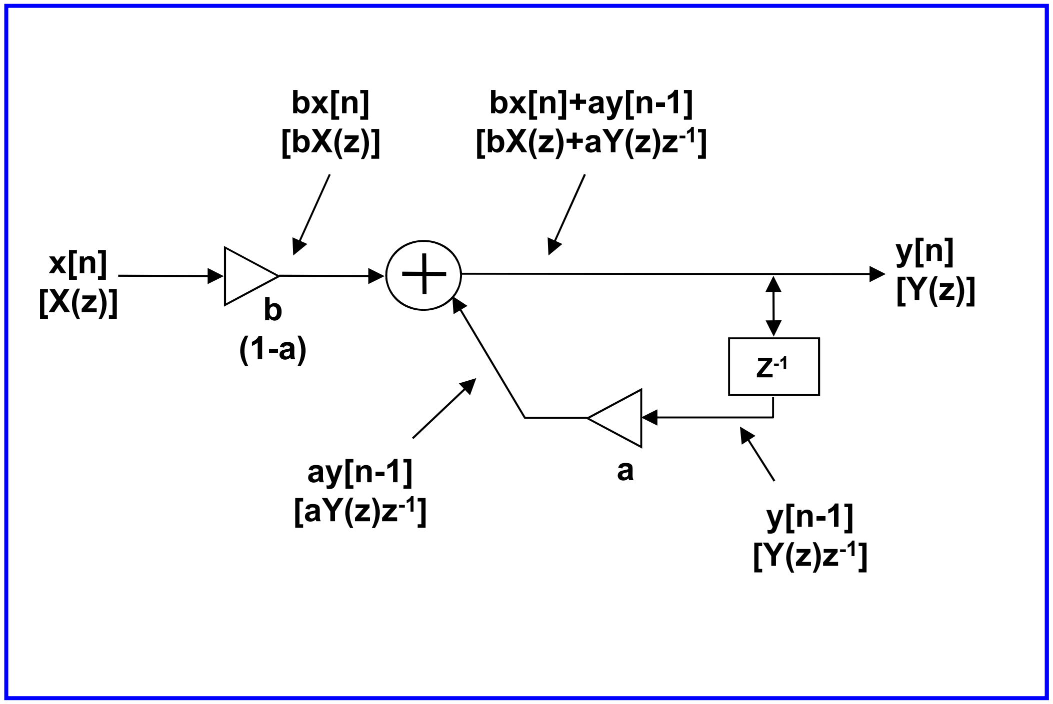


Figure : Block Diagram of difference equation of
$y[n] = ay[n-1]+bx[n]$

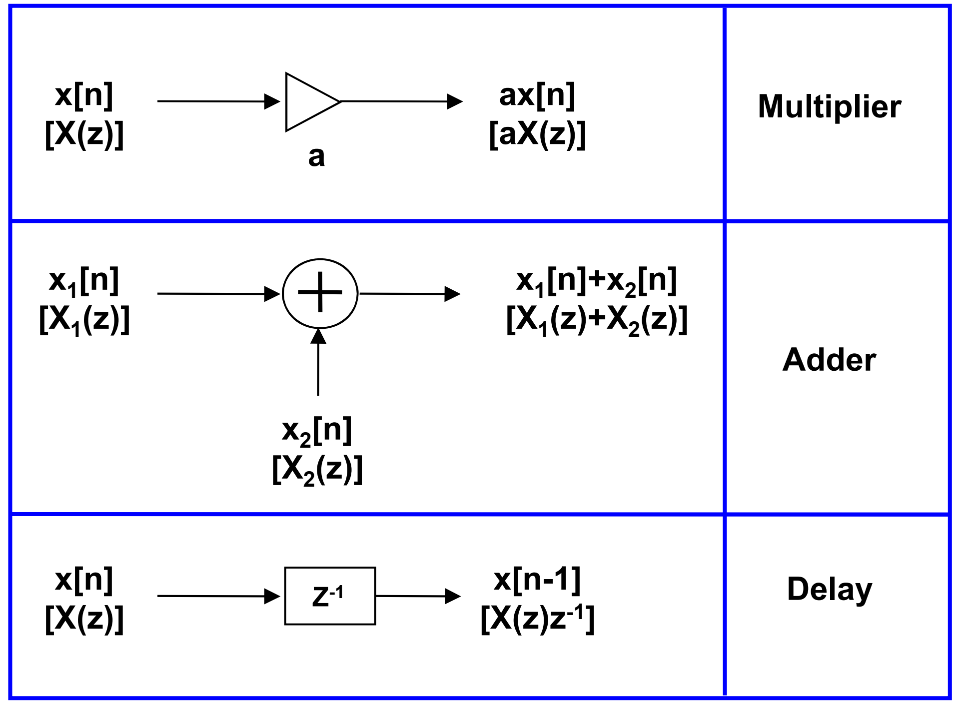


Figure : Elements of Block Diagram

4-3. Transfer Function

- Discrete-time system also can describe the discrete-time system by transfer function

In the case of input signal x[n] and output signal y[n],
the transfer function H(z) can be described by z transform of input X(z) and output Y(z).

$$H(z) = \frac{Y(z)}{X(z)}$$

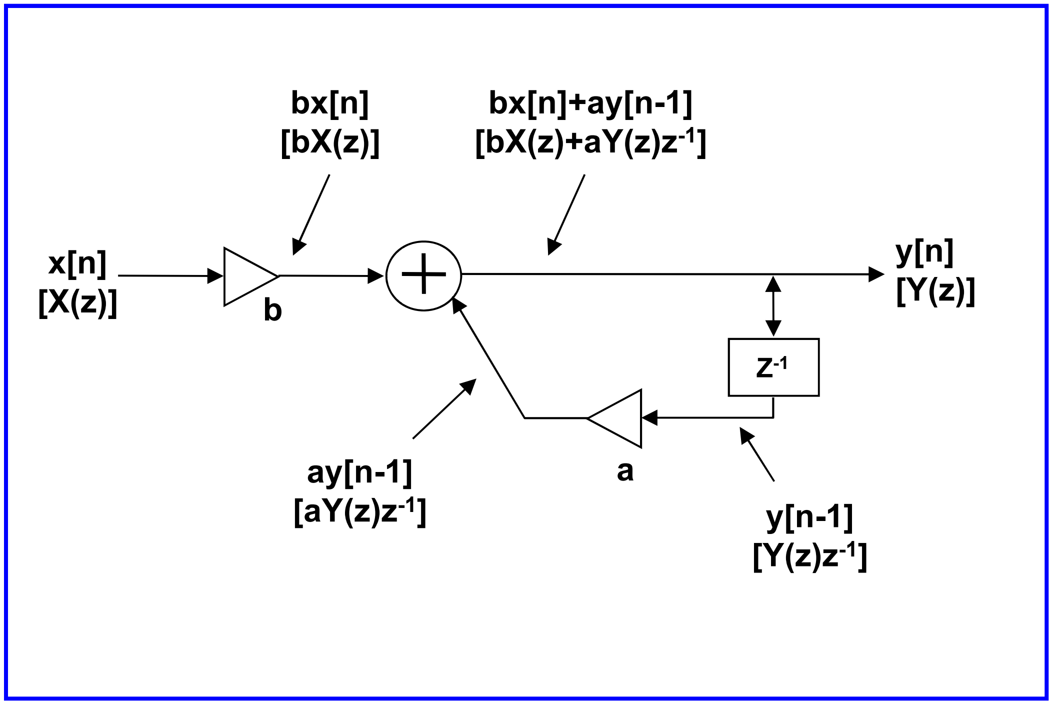


Figure : Block Diagram of difference equation of
y[n] = ay[n-1]+bx[n]

Difference equation

$$y[n] = ay[n-1] + bx[n]$$

Convert by z transform

$$Y(z) = aY(z)z^{-1} + bX(z)$$

Transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{1-az^{-1}}$$

4-4. Frequency Response

Digital Signal Processing

- Transfer function can describe by frequency response

Frequency response is the measure of any system's output spectrum in response to an input signal.
Transfer function is not the physical quantity that we can observe.
So we convert like this;

$$z = \exp(j\omega T)$$

And the transfer function

$$H(z) = \frac{b}{1 - az^{-1}}$$

convert to

$$H(j\omega t) = \frac{b}{1 - a\exp(-j\omega t)}$$

This formula shows the frequency characteristic.

Frequency characteristic is typically characterized by magnitude of the system's response, measured in decibels (dB), and the phase, measured in radians, versus frequency.

$$H(j\omega t) = A(\omega)\exp(j\theta(\omega))$$

Magnitude Characteristic:

$$A(\omega) = |H(\omega)| = \frac{|b|}{\sqrt{1 + a^2 - 2a\cos\omega t}}$$

Phase Characteristic:

$$\theta(\omega) = \arctan\frac{\mathrm{Im}[H(\exp(j\omega t))]}{\mathrm{Re}[H(\exp(j\omega t))]} = -\tan^{-1}\frac{a\sin\omega T}{1 - a\cos\omega T}$$

These formula are calculated by using the Euler's formula.

Euler's formula:

$$\exp(jx) = \cos x + j\sin x$$

$$\exp(-jx) = \cos x - j\sin x$$

4-5. Exercise1

Show that the formula of magnitude characteristic (B) based on formula (A) using by Euler’s formula

(A) $$H(j\omega t) = \frac{b}{1 - a\exp(-j\omega t)}$$

(B) $$A(\omega) = \frac{|b|}{\sqrt{1 + a^2 - 2a\cos\omega t}}$$

Euler’s formula:

$$\exp(jx) = \cos x + j\sin x$$

$$\exp(-jx) = \cos x - j\sin x$$

4-6. Explanation of Exercise1

Point 1: Calculate absolute value of complex number

Point 2: Use Euler’s formula

In the case of “z=a+bi”, conjugate be “z=a-bi”

$$|z| = \sqrt{z\bar{z}} = \sqrt{(a+bi)(a-bi)}$$

$$H(j\omega t) = \frac{b}{1 - a\exp(-j\omega t)}$$

$$|H(j\omega t)| = \frac{|b|}{|1 - a\exp(-j\omega t)|} = \frac{|b|}{\sqrt{(1 - a\exp(-j\omega t))(1 - a\exp(jwt))}}$$

$$= \frac{|b|}{\sqrt{1 - a\exp(-j\omega t) - a\exp(j\omega t) + a^2\exp(-j\omega t)\exp(j\omega t)}}$$

$$= \frac{|b|}{\sqrt{1 - 2a(\frac{\exp(-j\omega t) + \exp(j\omega t)}{2}) + a^2}} = \frac{|b|}{\sqrt{1 - 2a\cos(\omega t) + a^2}}$$

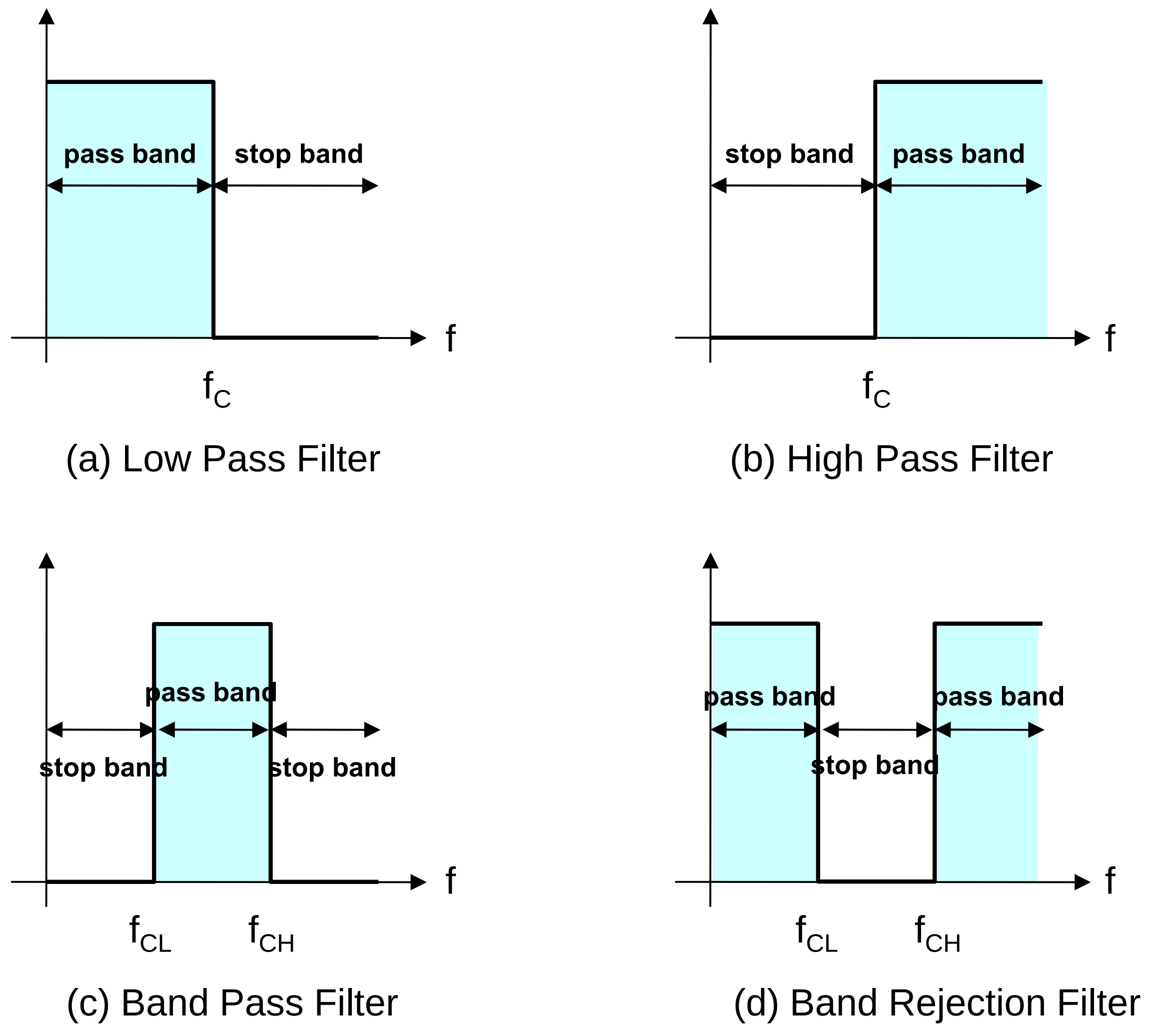
$$A(\omega) = \frac{|b|}{\sqrt{1 + a^2 - 2a\cos\omega t}}$$

From Euler’s formula:

$$\cos x = \frac{\exp(-jx) + \exp(jx)}{2}$$

4-7. Classification of Filter

Filter passes the signal of a certain frequency band, and have a operation to stop in the band except it.
Generally, there are 4 kind of filter.



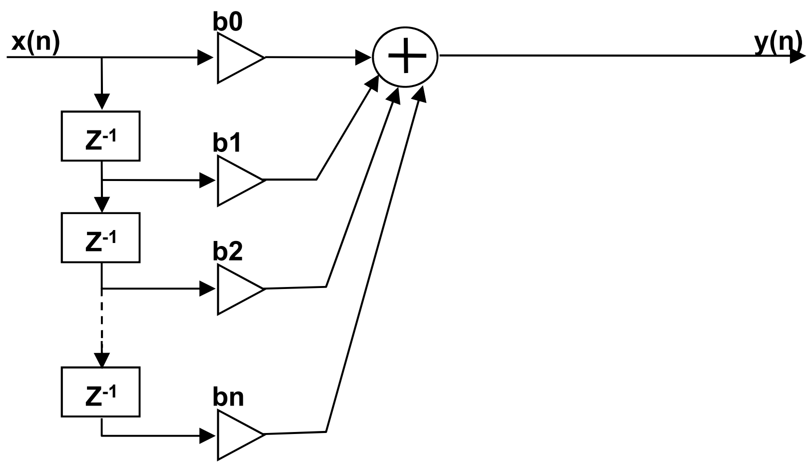
4-8. Digital Filter (FIR & IIR)

Table shows FIR & IIR filter to use by signal processing.

	FIR (Finite Impulse Response) Filter	IIR (Infinite Impulse Response) Filter
Length of Time of Impulse Response	Finite	Infinite
Difference Equation	$y[n]=\sum_{i=0}^{N} b_i x[n-i]$ N : the filter order (it also calls Taps) b_i : the filter coefficients $x[n]$: the input signal	$y[n]=\sum_{i=0}^{P} b_i x[n-i]-\sum_{j=1}^{Q} a_j y[n-j]$ P : the feedforward filter order b_i : the feedforward filter coefficients Q : the feedback filter order a_i : the feedback filter coefficents $x[n]$: the input signal $y[n]$: the output signal
Transfer Function	$H(z)=Z\{h[n]\}$ $=\sum_{n=-\infty}^{\infty} h[n]z^{-n}$ $=\sum_{n=0}^{N} b_n z^{-n}$	$H(z)=\frac{Y(z)}{X(z)}$ $=\frac{\sum_{i=0}^{P} b_i z^{-i}}{\sum_{j=0}^{Q} a_j z^{-j}}$
Stability of Structure	Non-feed back Always stable	Feed back
Realization of Completely Accurate Liner Phase Characteristic	Possible	Approximation is possible
Error of Calculation	Do not appear so big	There is a case to appear big
Realization of Sharper Cutoff characteristic	High order filter	Low order filter

4-9. FIR Filter

Finite impulse response (FIR) filter is a type of a digital filter. The impulse response is ‘finite’ because it settles to zero in a finite number of sample intervals.



$$H(z) = Z\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=0}^{N} b_n z^{-n}$$

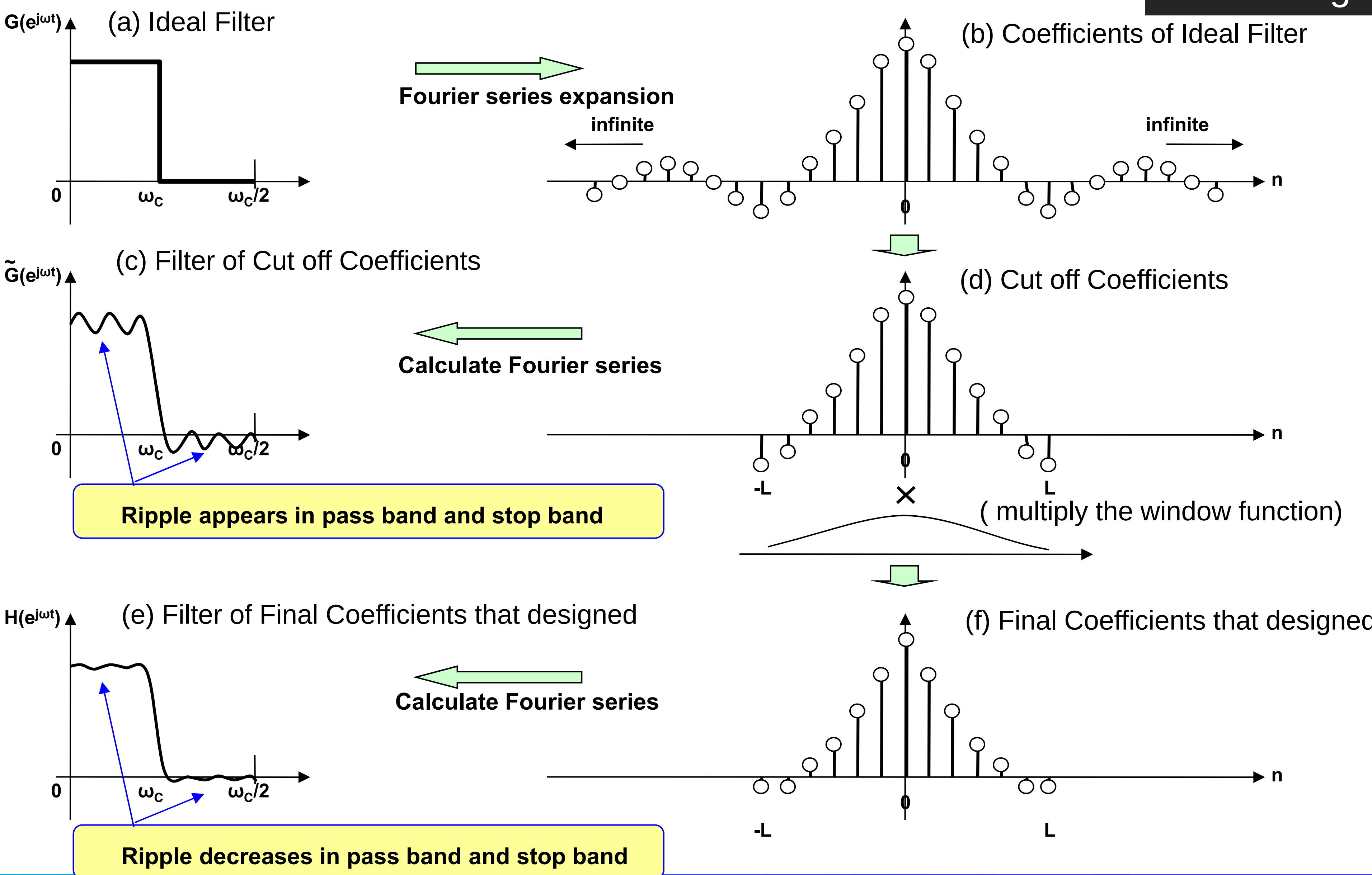
FIR filter’s useful properties which prefer to an IIR filter

Stable : This is due to the fact that all the poles are located at the origin and thus are located within unit circle.

Require no feedback : This means that any rounding errors are not compounded by summed iterations.
The same relative error occurs in each calculation.
This is also makes implementation simpler.

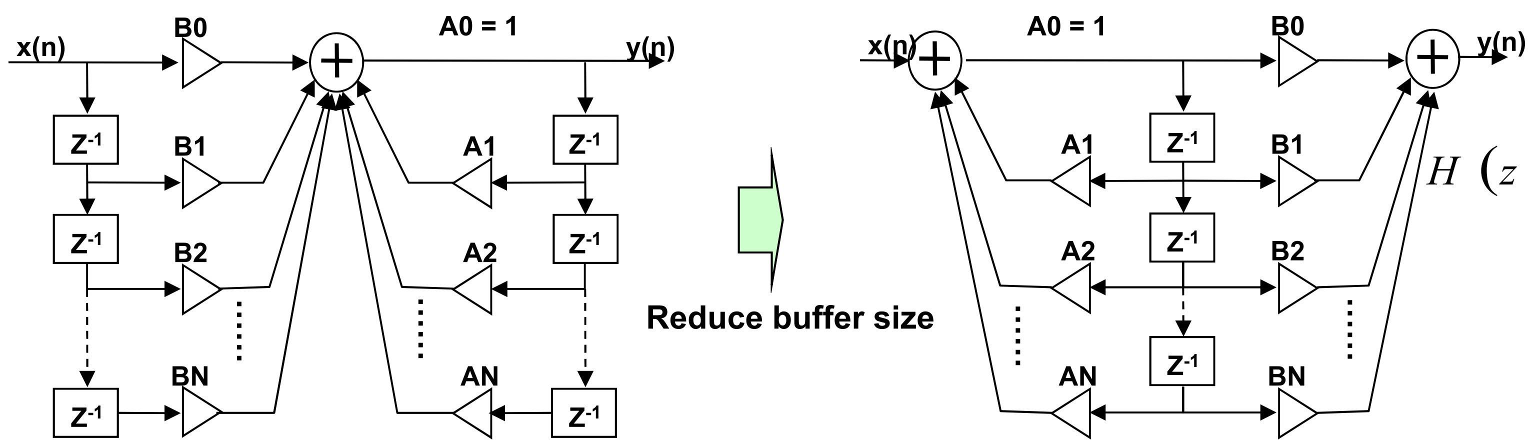
Designed to be linear phase : This means the phase change is proportional to the frequency.
This is usually desired for phase-sensitive applications.

4-10. Design method of FIR Filter (Window Design Method)



4-11. IIR Filter

Infinite impulse response (IIR) filter is type of a digital filter. IIR systems have an impulse response function That is non-zero over an infinite length of time.



$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{P} b_i z^{-i}}{\sum_{j=0}^{Q} a_j z^{-j}}$$

IIR filter's useful properties which prefer to an FIR filter

Fast and cheap : This means it can realize the filter by low order .

Sharp : This means an IIR filter can achieve a much sharper transition region roll-off than FIR filter of the same order.

***Note that unlike with FIR filters, in designing IIR filters it is necessary to carefully consider "time zero" case in which the outputs of the filter have no yet been clearly defined.**

4-12. Design Method of IIR Filter (s-z Transfer Method) 1/6

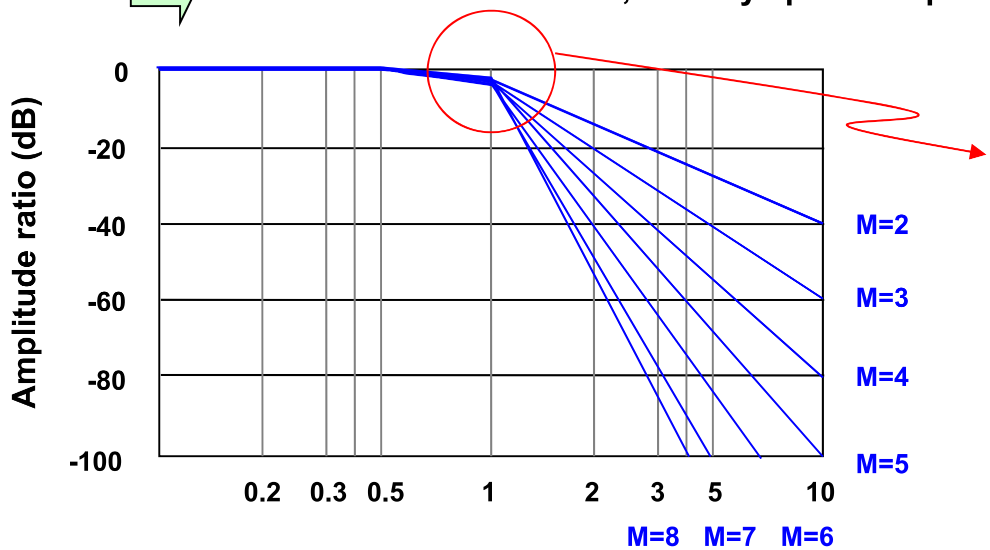
Design method of IIR filter use by s-z transform need to design the analog filter first. Butterworth characteristic is one of famous filter to calculate analog filter.

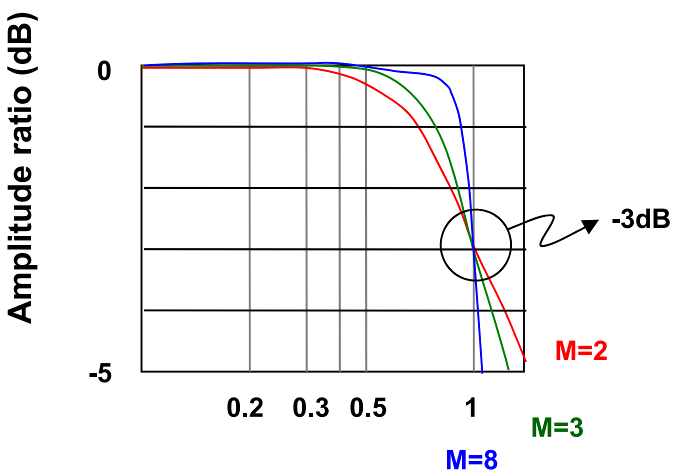
Butterworth Characteristic

Amplitude characteristic of M order low pass filter of Butterworth show like this:

$$\left|H_M(\omega)\right| = \frac{1}{\sqrt{1+\omega^{2M}}}$$

In the case of "ω=1", it always pass the point of -3dB





From amplitude characteristic, transfer function describe like following formula.

M = even

$$H_M(s) = \prod_{m=1}^{M/2} \frac{1}{s^2 + (2\cos\theta_m)s + 1}$$

$$\theta_m = (2m-1)\pi/(2M), \qquad m = 1,2,\cdots,M/2$$

M = odd

$$H_M(s) = \frac{1}{s+1} \prod_{m=1}^{(M-1)/2} \frac{1}{s^2 + (2\cos\theta_m)s + 1}$$

$$\theta_m = m\pi/M, \qquad m = 1,2,\cdots,(M-1)/2$$

4-13. Design Method of IIR Filter (s-z Transfer Method) 2/6

From the result of Butterworth characteristic, use the following table and need to do the frequency transform.

Low Band Pass Filter	$s \to s/\omega_0$
High Band Pass Filter	$s \to \omega_0/s$
Band Pass Filter	$s \to (s^2+\omega_0^2)/\{s(\omega_2-\omega_1)\}$
Band Rejection Filter	$s \to s(\omega_2+\omega_1)/(s^2+\omega_0^2)$

ω_1: Cutoff angular frequency of low band side

ω_2: Cutoff angular frequency of high band side $\omega_0 = \sqrt{\omega_1 \cdot \omega_2}$

For example, transfer function of high band pass filter based on Butterworth characteristic that have M order (M : even) and ω0 cutoff angular frequency to be following formula.

$$H_M(s) = \prod_{m=1}^{M/2} \frac{1}{s^2 + (2\cos\theta_m)s + 1}$$

$$s \to \omega_0/s$$

$$\Rightarrow \quad H_M(s) = \prod_{m=1}^{M/2} \frac{s^2}{s^2 + (2\omega_0\cos\theta_m)s + \omega_0^2}$$

4-14. Design Method of IIR Filter (s-z Transfer Method) 3/6

Use s-z transform to get transfer function of digital filter from transfer function of analog filter.
In s-z transform, there are 3 kind of transform.

- **Standard z-transform**
- **Bilinear z-transform**
- **Matched z-transform**

Explain about these basic transform.

Standard z-transform

By using Standard z-transform, transfer function of digital filter to have an impulse response same as sampling of basic analog filter's impulse response. So it also call impulse invariant.

The point is expand the transfer function of M order analog filter "G(s)" in partial fraction.

$$G(s) = \sum_{i=1}^{M} \frac{A_i}{s + a_i}$$

And after that convert to z transfer.

$$\frac{A_i}{s + a_i} \to \frac{A_i}{1 - \exp(-a_i T) \cdot z^{-1}}$$

Bilinear z-transform

By using Bilinear z-transform, it is difference from standard z-transform and it can design many kind of filter such as low pass, band pass, high pass, notch filter and so on.

The point is convert the transfer function of M order analog filter "G(s)" to z transfer.

$$s \to \frac{1 - z^{-1}}{1 + z^{-1}}$$

And there are relationship between analog angular frequency "ω_A" and digital angular frequency "ω_D" .

$$\omega_A = \tan\left(\frac{\pi \omega_D}{\omega_S}\right)$$

sampling angular frequency "ω_S"

4-15. Design Method of IIR Filter (s-z Transfer Method) 4/6

Example of Designing of Low Pass filter

Type of filter	2 order IIR
Sampling frequency	10k[Hz]
Type of characteristic	Butterworth characteristic
Cutoff frequency	1k[Hz]

$$H_M(s) = \prod_{m=1}^{M/2} \frac{\omega^2}{s^2 + (2\omega_0 \cos\theta_m)s + \omega_0^2}$$

$$M = 2, m = 1$$

$$\theta_m = (2m-1)\pi/2M = \pi/4 \Rightarrow \cos\theta_m = 1/\sqrt{2}$$

$$G(s) = \frac{\omega_0^2}{s^2 + \sqrt{2}\omega_0 s + \omega_0^2}$$

Use "standard z-transform" to calculate example case of low pass filter

Partial fraction expansion

$$G(s) = \frac{j\sqrt{2}\omega_0/2}{s + \sqrt{2}(1+j)\omega_0/2 + \omega_0^2} - \frac{j\sqrt{2}\omega_0/2}{s + \sqrt{2}(1+j)\omega_0/2 + \omega_0^2}$$

s-z transform

$$H(z) = \frac{j\sqrt{2}\omega_0/2}{1 - \exp(-\sqrt{2}(1+j)\omega_0 T/2)\cdot z^{-1}} - \frac{j\sqrt{2}\omega_0/2}{1 - \exp(-\sqrt{2}(1-j)\omega_0 T/2)\cdot z^{-1}}$$

$\omega_0 = 2\pi x 10^3$, $T = 10^{-4}$

$$H(z) = \frac{2449.203\, z^{-1}}{1 - 1.158046\, z^{-1} + 0.411241\, z^{-2}}$$

Revise the amplitude ratio :
In the case of low pass filter of basic analog filter, amplitude ratio be 1 at frequency 0.
(z=exp(j0T)=1, H(z)=1)

$$H(z) = \frac{0.253195\, z^{-1}}{1 - 1.158046\, z^{-1} + 0.411241\, z^{-2}}$$

4-16. Design Method of IIR Filter (s-z Transfer Method) 5/6

Use “bilinear z-transform” to calculate example case of low pass filter

Calculate $\omega_{A,0}$

$$\omega_{A,0} = \tan(\frac{\pi\omega_D}{\omega_s}) = \tan(\pi * \frac{1k}{10k}) = 0.324920$$

Input the value of $\omega_{A,0}$ to transfer function of G(s)

$$G(s) = \frac{\omega_0^2}{s^2 + \sqrt{2}\omega_0 s + \omega_0^2} = \frac{0.105573}{s^2 + 0.459506 \ s + 0.105573}$$

Convert to the z-transform

$$H(z) = \frac{0.105573}{(\frac{1-z^{-1}}{1+z^{-1}})^2 + 0.459506 \ \frac{1-z^{-1}}{1+z^{-1}} + 0.105573}$$

$$= \frac{0.105573 \ (1 + 2z^{-1} + z^{-2})}{1 - 2z^{-1} + z^{-2} + 0.459506 \ (1 - z^{-2}) + 0.105573 \ (1 + 2z^{-1} + z^{-2})}$$

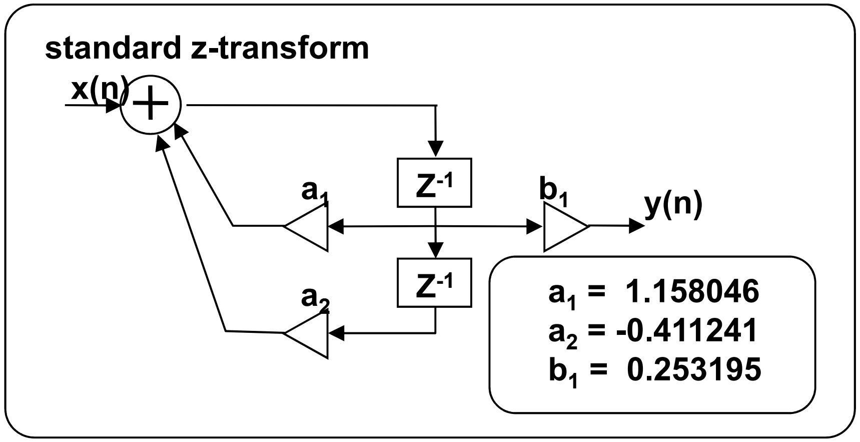
$$= \frac{0.105573 \ (1 + 2z^{-1} + z^{-2})}{1.565079 \ - 1.788854 \ z^{-1} + 0.646067 \ z^{-2}}$$

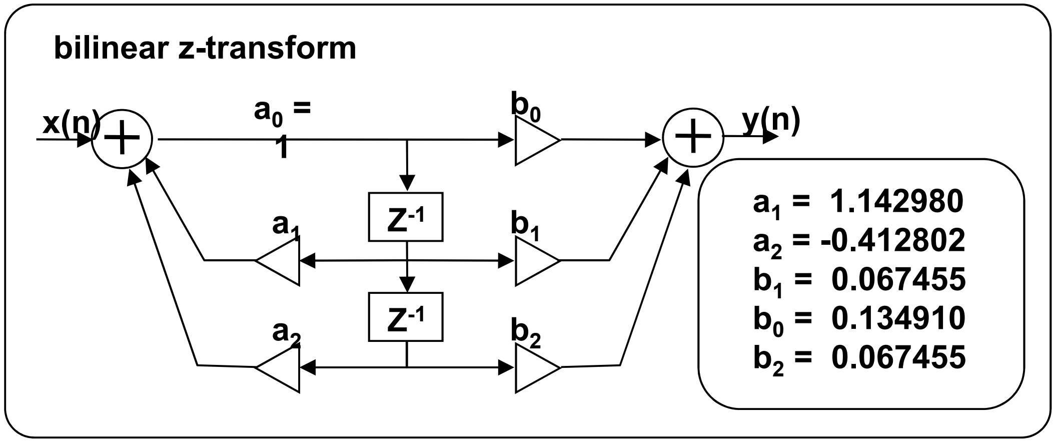
$$= \frac{0.067455 \ + 0.134910 \ z^{-1} + 0.067455 \ z^{-2}}{1 - 1.142980 \ z^{-1} + 0.412802 \ z^{-2}}$$

In this case, it doesn’t need to revise the value.
Because the amplitude ratio is under than 1 at frequency 0.

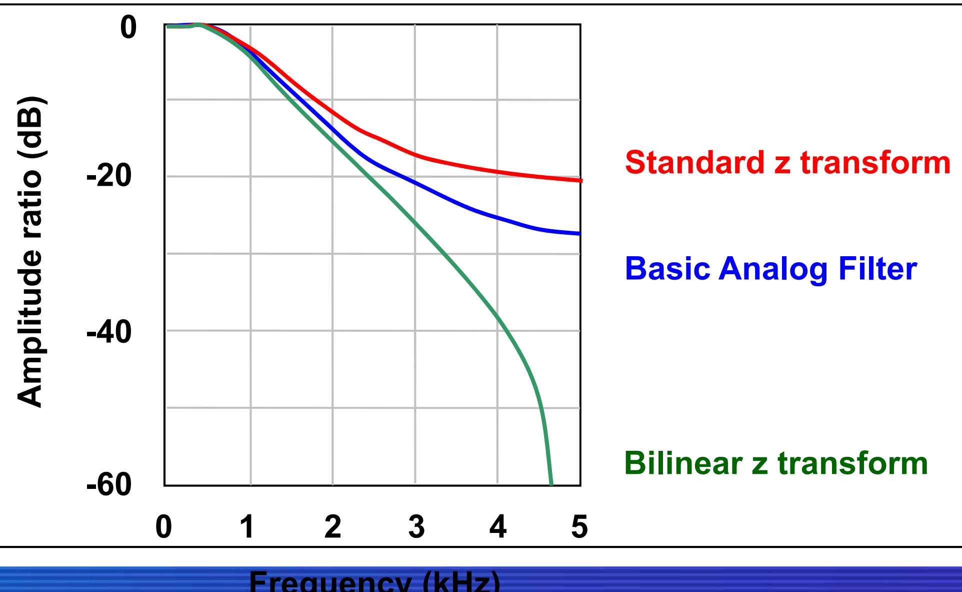
4-17. Design Method of IIR Filter (s-z Transfer Method) 6/6

These figure shows the block diagram that designed by "standard z-transform" and "bilinear z-transform".





This amplitude characteristic shows about the difference of low pass filter that designed by analog filter and "standard z-transform", "bilinear z-transform". The characteristic of "bilinear z-transform" realized to decline sharply around high frequency.



4-18. Exercise 2

Calculate that the transfer function of digital filter of (B) based on formula (A) using by Euler's formula and example formula

(A) $$H(z) = \frac{j\sqrt{2}\omega_0/2}{1-\exp(-\sqrt{2}(1+j)\omega_0 T/2)\cdot z^{-1}} - \frac{j\sqrt{2}\omega_0/2}{1-\exp(-\sqrt{2}(1-j)\omega_0 T/2)\cdot z^{-1}}$$

(B) $$H(z) = \frac{2449.203\, z^{-1}}{1-1.158046\, z^{-1} + 0.411241\, z^{-2}}$$

Euler's formula:

$$\exp(jx) = \cos x + j\sin x$$

$$\exp(-jx) = \cos x - j\sin x$$

Example formula:

$$\exp(-\sqrt{2}\omega_0 T) = \exp(-\sqrt{2} * 2 * \pi * 10^3 * 10^{-4}) = 1.158046$$

$$\exp(-\sqrt{2}\omega_0 T/2) = \exp(-\sqrt{2} * \pi * 10^3 * 10^{-4}) = 0.641281$$

$$2\cos(\sqrt{2}\omega_0 T/2) = 2\cos(\sqrt{2} * \pi * 10^3 * 10^{-4}) = 1.805834$$

$$\exp(-\sqrt{2}\omega_0 T/2) * 2\cos(\sqrt{2}\omega_0 T/2) = 0.411241$$

$$2\sin(\sqrt{2}\omega_0 T/2) = 2\sin(\sqrt{2} * \pi * 10^3 * 10^{-4}) = 0.859631$$

$$\sqrt{2}\omega_0/2 = \sqrt{2} * \pi * 10^3 = 4442.8829$$

$$(\sqrt{2}\omega_0/2) * \exp(-\sqrt{2}\omega_0 T/2) * 2\sin(\sqrt{2}\omega_0 T/2) = 2449.203$$

4-19. Explanation of Exercise 2

Digital Signal Processing

Point 1: Calculate absolute value of complex number

Point 2: Use Euler's formula

$$|z| = \sqrt{z\bar{z}} = \sqrt{(a+bi)(a-bi)}$$

$$H(z) = \frac{j\sqrt{2}\omega_0/2}{1-\exp(-\sqrt{2}(1+j)\omega_0 T/2)\cdot z^{-1}} - \frac{j\sqrt{2}\omega_0/2}{1-\exp(-\sqrt{2}(1-j)\omega_0 T/2)\cdot z^{-1}}$$

$$= \frac{(j\sqrt{2}\omega_0/2)(1-\exp(-\sqrt{2}(1-j)\omega_0 T/2)\cdot z^{-1}) - (j\sqrt{2}\omega_0/2)(1-\exp(-\sqrt{2}(1+j)\omega_0 T/2)\cdot z^{-1})}{(1-\exp(-\sqrt{2}(1+j)\omega_0 T/2)\cdot z^{-1})(1-\exp(-\sqrt{2}(1-j)\omega_0 T/2)\cdot z^{-1})}$$

$$= \frac{(j\sqrt{2}\omega_0/2)(\exp(-\sqrt{2}(1+j)\omega_0 T/2) - \exp(-\sqrt{2}(1-j)\omega_0 T/2))z^{-1}}{1+\exp(-\sqrt{2}(1+j)\omega_0 T/2 - \sqrt{2}(1-j)\omega_0 T/2)z^{-2} - \exp(-\sqrt{2}(1+j)\omega_0 T/2)z^{-1} - \exp(-\sqrt{2}(1-j)\omega_0 T/2)z^{-1}}$$

$$= \frac{(j\sqrt{2}\omega_0/2)\exp(-\sqrt{2}\omega_0 T/2)(\exp(-\sqrt{2}j\omega_0 T/2) - \exp(\sqrt{2}j\omega_0 T/2))z^{-1}}{1+\exp(-\sqrt{2}\omega_0 T)z^{-2} - \exp(-\sqrt{2}\omega_0 T/2)(\exp(-\sqrt{2}j\omega_0 T/2) + \exp(\sqrt{2}j\omega_0 T/2))z^{-1}}$$

$$= \frac{(j\sqrt{2}\omega_0/2)\exp(-\sqrt{2}\omega_0 T/2)(-2j\sin(\sqrt{2}j\omega_0 T/2))z^{-1}}{1+\exp(-\sqrt{2}\omega_0 T)z^{-2} - \exp(-\sqrt{2}\omega_0 T/2)(2\cos(\sqrt{2}j\omega_0 T/2))z^{-1}}$$

$$= \frac{(\sqrt{2}\omega_0/2)\exp(-\sqrt{2}\omega_0 T/2)(2\sin(\sqrt{2}\omega_0 T/2))z^{-1}}{1+\exp(-\sqrt{2}\omega_0 T)z^{-2} - \exp(-\sqrt{2}\omega_0 T/2)(2\cos(\sqrt{2}\omega_0 T/2))z^{-1}}$$

Input example formula

From Euler's formula:

$$H(z) = \frac{2449.203\, z^{-1}}{1-1.158046\, z^{-1} + 0.411241\, z^{-2}}$$

4-20. Exercise3

Design the transform function of digital filter using by “bilinear z-transform”.

Type of filter	**2 order IIR**
Sampling frequency	**10k[Hz]**
Type of characteristic	**Butterworth characteristic**
Cutoff frequency	**2k[Hz]**

$$H_M(s) = \prod_{m=1}^{M/2} \frac{\omega^2}{s^2 + (2\omega_0 \cos\theta_m)s + \omega_0^2}$$

$$M = 2, m = 1$$

$$\theta_m = (2m-1)\pi / 2M = \pi / 4 \Rightarrow \cos\theta_m = 1/\sqrt{2}$$

$$\tan(\pi * 0.01) = 0.031426$$
$$\tan(\pi * 0.02) = 0.062918$$
$$\tan(\pi * 0.05) = 0.158384$$
$$\tan(\pi * 0.1) = 0.324920$$
$$\tan(\pi * 0.2) = 0.726543$$

$$\cos(0) = 1$$
$$\cos(\pi) = -1$$
$$\cos(\pi / 4) = 1/\sqrt{2}$$

4-21. Explanation of Exercise 3

$$G(s) = \frac{s_0^2}{s^2 + \sqrt{2}\omega_0 s + \omega_0^2}$$

Calculate $\omega_{A,0}$

$$\omega_{A,0} = \tan(\frac{\pi\omega_D}{\omega_s}) = \tan(\pi * \frac{2k}{10k}) = 0.726542$$

Input the value of $\omega_{A,0}$ to transfer function of G(s)

$$G(s) = \frac{s_0^2}{s^2 + \sqrt{2}\omega_0 s + \omega_0^2} = \frac{s^2}{s^2 + 1.027486\ s + 0.527864}$$

Convert to the z-transform

$$H(z) = \frac{\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)^2}{(\frac{1 - z^{-1}}{1 + z^{-1}})^2 + 1.027486\ \frac{1 - z^{-1}}{1 + z^{-1}} + 0.527864}$$

$$= \frac{(1 - z^{-1})^2}{1 - 2z^{-1} + z^{-2} + 1.027486\ (1 - z^{-2}) + 0.527864\ (1 + 2z^{-1} + z^{-2})}$$

$$= \frac{1 - 2z^{-1} + z^{-2}}{2.55535\ - 0.944272\ z^{-1} + 0.500378\ z^{-2}}$$

$$= \frac{0.391336\ + 0.782672\ z^{-1} + 0.391336\ z^{-2}}{1 - 0.369627\ z^{-1} + 0.195816\ z^{-2}}$$