geometry and linear algebraic operations

1. geometry of vectors

- vectors can be interpreted as either points or directions in space

- As points, vector difine locations relative to the origin

- As directions, vector indicate movement. Multiple vectors can represent the same direction if they parallel

- Vector addition is visualized as following one vector's direction and then another's, while subtraction show the direction from one point to another example :

consider two 2D vector : $v = [2, 3]$ and $w = [1, -1]$

- As points : v is at $(2, 3)$, and w is at $(1, -1)$

- As directions : v mean 2 right and 3 up

  w mean 1 right and 1 down

- vector addition : $v + w = [2+1, 3-1] = [3, 2]$

- vector subtraction : $v - w = [2-1, 3-(-1)] = [1, 4]$

2. dot products and angles

- the dot products of two vector u and v is

$u \cdot v = \sum_i u_i v_i$, which relate to the angle θ

between them via

$$u \cdot v = \|u\| \|v\| \cos(\theta)$$

- the angle is compute as

$$\theta = \arccos \frac{u \cdot v}{\|u\| \|v\|}$$

example :

compute the angle between $u = [1,0]$ and $v = [1,1]$

- dot product : $u.v = 1.1 + 0.1 = 1$

- norm : $\|u\| = \sqrt{1^2 + 0^2} = 1$; $\|v\| = \sqrt{1^2 + 1^2} = \sqrt{2}$

- cosine : $\cos(\theta) = \dfrac{1}{1.\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

- Angle : $\theta = \arccos \dfrac{1}{\sqrt{2}} = 45°$

2.1. cosine similarity

- cosine similarity measures how close 2 vector are in direction

$$\cos(\theta) = \frac{V \cdot W}{\|V\| \, \|W\|}$$

example: $v = [2, 1, 0]$ and $w = [4, 2, 0]$

$$\cos(\theta) = \frac{2.4 + 1.2 + 0.0}{\sqrt{4+1+0}\ \sqrt{16+4+0}}$$

$$= \frac{10}{\sqrt{5} . 2\sqrt{5}} = 1$$

3. Hyper plane

- A hyperplane in d dimensional is a (d-1) -dimensional subspace that divide the space into two half

- define by $w \cdot v = b$ wher w is a normal vector
  b is a scalar

Example:

consider a 2D hyperplane defined by

$w = [2, 1]$ and $w \cdot v = 1$

- equation : $2x + y = 1$
- point satisfying this form a line
- point where $2x + y > 1$ lie on one side,

and $2x + y < 1$ on the other.

4. geometry of linear tranformations

- A matrix represent a linear transformation that maps vectors via Av.
- geometryly, it can scale, rotate, or skew the space but cannot distort parts differently.
- the transformation is determined by where A sends basis vectors.

example:

use matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ to transform $v = [2, -1]$

- compute $Av = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1.2 + 2.(-1) \\ -1.2 + 3.(-1) \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ -5 \end{bmatrix}$$

- geometrically, the grid is skewed, and mapped to

(0, -5)

5. Linear Idependence

- vector are linear dependent if one can be written as a combination of others

Σ ai vi = 0 with not all ai = 0

- linear dependence indicate the vector span a lower-dimensional space.

- for a matrix, linearly dependent columns mean it compresses space.

$u = (1, 2, 3)$

$v = (2, 4, 6)$ $\qquad \{u, v, w\}$ linear independent?

$w = (1, 1, 1)$

- find $a, b, c \neq 0$

$$au + bv + cw = 0$$

$$\Leftrightarrow \quad a(1, 2, 3) + b(2, 4, 6) + c(1, 1, 1) = 0$$

$$a + 2b + c = 0$$

$$2a + 4b + c = 0$$

$$3a + 6b + c = 0$$

$a = -2$

$b = 1$

$c = 0$

conclușe $a, b, c \neq 0 \Rightarrow$ linear dependent.

6. Rank

- the rank of matrix is the number of linearly independent columns (or rows)

example: find rank of $B = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$

+ $b_1 = [2, -1]$
+ $b_2 = [4, -2]$
+ $b_2 = 2 b_1$

$\Rightarrow$ they are dependent

+ rank $= 1$, as only one column is independent

7. Invertibility

- A square matrix A is invertible if there exist A^{-1} such that $A^{-1} A = I$

- Invertibility requires full rank and non-zero determina
- The inverse undoes the transformation of A.

example: $A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$ is invertible?

- $$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Determinant : $ad - bc = 1.4 - 2.1 = 2 \neq 0$

- Inverse of A : $A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -0.5 & 0.5 \end{bmatrix}$

8. Determinant

- the determinant measures how a matrix scale areas / volumes

- for 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det A = ad - bc$

- $\det = 0$ means the matrix is singular

example : compute the determinant of $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

- $\det = 1 \times 3 - 2 \times (-1) = 5.$