eigendecompositions

1. finding eigenvalues

example: compute eigenvalues and eigenvector

$$\text{for } A = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$$

eigenvalue λ satisfy $\det(A - \lambda I) = 0$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

compute $A - \lambda I$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 0 \\ 0 & -2-\lambda \end{bmatrix}$$

$\det(A - \lambda I) = (4-\lambda) \cdot (-2-\lambda)$

set it to zero : $(4-\lambda) \cdot (-2-\lambda) = 0$

solve: $\lambda = 4$ or $\lambda = -2$.

eigenvalues are $\lambda_1 = 4$

$\lambda_2 = -2$.

find eigenvectors for $\lambda_1 = 4$

solve $(A - 4I)v = 0$

$$A - 4I = \begin{bmatrix} 4-4 & 0 \\ 0 & -2-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -6 \end{bmatrix}$$

the system is

$$\begin{bmatrix} 0 & 0 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-6y = 0 \Rightarrow y = 0$

$0x = 0 \Rightarrow x \in R$

choose $x = 1$, $y = 0$

eigenvector : $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

find eigenvectors for $\lambda_2 = -2$.

solve $(A - (-2) I) v = 0$

$$A + 2I = \begin{bmatrix} 4+2 & 0 \\ 0 & -2+2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}$$

the system is

$$\begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$6x = 0 \Rightarrow x = 0$

$0 \cdot y = 0 \Rightarrow y \in \mathbb{R}$

choose $x = 0$, $y = 1$

eigenvector $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

verification

for $\lambda_1 = 4$, $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$Av_1 = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for $\lambda = -2$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$Av_2 = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. Decomposing matrices

Decomposing $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$$

$$\det(A) = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = 0$$

$$\lambda_1 = 4 \; ; \; \lambda_2 = 2$$

for $\lambda_1 = 4$

$$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow -x + y = 0 \Rightarrow x = y$$

eigenvector :

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $\lambda_2 = 2$

$$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow x + y = 0 \Rightarrow x = -y$$

eigenvector : $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

form matrices for decompositions

- $W = [v_1 \ v_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

- $\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

compute W^{-1}:

determinant of W: $\det(W) = 1.(-1) - 1.1$

$= -2$

Inverse

$$W^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Reconstruct $A = W \Sigma W^{-1}$

- $W\Sigma = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix}$

- $W \Sigma W^{-1} = \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

3. operations on eigendecompositions

example Compute A^3 and A^{-1} for $A = \begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}$

- find eigenvalues and eigenvectors

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

$$(2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 4 - 1$$

$$= \lambda^2 - 4\lambda + 3 = 0$$

$\lambda_1 = 3$; $\lambda_2 = 1$.

for $\lambda_1 = 3$

$$A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow -x + y = 0$$

$$\Rightarrow x = y$$

eigenvector : $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda_2 = 1$

$$A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow x + y = 0$$

eigenvetor : $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

form decomposition

$$W = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} , \quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det(W) = -2 , \quad W^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

compute $A^3 = W \Sigma^3 W^{-1}$

$$\Sigma^3 = \begin{bmatrix} 3^3 & 0 \\ 0 & 1^3 \end{bmatrix} = \begin{bmatrix} 27 & 0 \\ 0 & 1 \end{bmatrix}$$

$$W \Sigma^3 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 27 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 27 & 1 \\ 27 & -1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 27 & 1 \\ 27 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$$

Compute $A^{-1} = W\Sigma^{-1}W^{-1}$

- $\Sigma^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{1} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$

- $W\Sigma^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{1}{3} & -1 \end{bmatrix}$

- $A^{-1} = \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{1}{3} & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

$$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{bmatrix}$$

4. eigen decompositions of symmetric matrices

$A = \begin{matrix} 4 & 1 \\ 1 & 4 \end{matrix}$

- find eigenvalues and eigenvectors

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix}$$

$$\det(A) = (4-\lambda)^2 - 1 = \lambda^2 - 8\lambda + 15 = 0$$

$$\lambda_1 = 5 \; ; \; \lambda_2 = 3$$

for $\lambda_1 = 5$

$$A - 5I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow -x + y = 0$$

$$\Rightarrow x = y$$

eigenvector : $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

normalize $\|v_1\| = \sqrt{2}$

so $v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

for $\lambda_2 = 3$

$$A - 3I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow x + y = 0$$

eigenvector : $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

normalize $\|v_2\| = \sqrt{2}$

so $v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

very fy orthogonality:

$$W^T W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

reconstruct $A = W \Sigma W^T$

$$W \Sigma = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{5}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{5}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & \frac{-3}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

5. Gershgorin circle Theorem:

$$A = \begin{bmatrix} 4 & 0.5 & 0.2 \\ 0.3 & 2 & 0.4 \\ 0.1 & 0.6 & 5 \end{bmatrix}$$

determine the center and radius of each Gershgorin disk

row 1 : [4, 0.5, 0.2]

- center = 4
- radius : $r_1 = |0.5| + |0.2| = 0.7$
- gershgorin disk - $4 - 0.7, 4 + 0.7 = [3.3, 4.7]$

row 2: [0.3 2 0.4]

- center = 2

- radius = |0.3| + |0.4| = 0.7

- gershgorin disk = [2 − 0.7 , 2 + 0.7] = [1.3 , 2.7]

row 3: [0.1 0.6 5]

- center = 5

- radius = |0.1| + |0.6| = 0.7

- gershgorin disk = [5 − 0.7 , 5 + 0.7] = [4.3 , 5.7]

- compute the actual eigenvalues (to verify)

+ $A - \lambda I = \begin{bmatrix} 4-\lambda & 0.5 & 0.2 \\ 0.3 & 2-\lambda & 0.4 \\ 0.1 & 0.6 & 5-\lambda \end{bmatrix}$

+ $\det(A - \lambda I) = 0$

$\Leftrightarrow \quad \lambda^3 - 11\lambda^2 + 37.93\lambda + 39.774 = 0$

$\lambda_1 = 1.68, \quad \lambda_2 = 4.47, \quad \lambda_3 = 4.85$

- verify

disk 1 $[3.3, 4.7] \rightarrow \lambda_2 = 4.47$ | disk 3 : $[4.3, 5.7]$

disk 2 $[1.3, 2.7] \rightarrow \lambda_1 = 1.68$ | $\rightarrow \lambda_3 = 4.85$