geometry linear algebraic ops (1)

1. $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$; $\vec{v_2} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

- dot product : $\vec{v_1} \cdot \vec{v_2} = 1 \times 3 + 0 \times 1 + (-1) \times 0 + 2 \times 1 = 5$

- magnitude $\vec{v_1}$: $|\vec{v_1}| = \sqrt{1^2 + 0^2 + (-1)^2 + 2^2} = \sqrt{6}$

- magnitude $\vec{v_2}$: $|\vec{v_2}| = \sqrt{3^2 + 1^2 + 0^2 + 1^2} = \sqrt{11}$

- $\cos \theta = \dfrac{5}{\sqrt{6} \cdot \sqrt{11}} = \dfrac{5}{\sqrt{66}}$

- $\theta = \arccos \dfrac{5}{\sqrt{66}}$

2. check $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ are inverses, check if their product is identify matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- compute the product $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix}$

$$\begin{bmatrix} 1\times 1 + 2\times 0 & 1\times(-2) + 2\times 1 \\ 0\times 1 + 1\times 0 & 0(-2) + 1\times 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- since the product is the identify matrix; the matrices are inverse

3.

- compute the determinant of $\begin{matrix} 2 & 3 \\ 1 & 2 \end{matrix}$

$\det = 2 \times 2 - 3 \times 1 = 1$

- compute the absolute value of determinant

$|\det| = |1| = 1$

- compute the new area

new area $= |\det| .$ original area $= 1 \times 100\, m^2 = 100\, m^2$

4. $\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\1\\-1\end{pmatrix}, \begin{pmatrix}3\\1\\1\end{pmatrix}$

- equation : $a\begin{pmatrix}1\\0\\-1\end{pmatrix} + b\begin{pmatrix}2\\1\\-1\end{pmatrix} + c\begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

- system

$$\begin{aligned} a + 2b + 3c &= 0 \quad (1)\\ b + c &= 0 \quad (2)\\ -a - b + c &= 0 \quad (3) \end{aligned}$$

$$a = b = c = 0$$

$\Rightarrow$ the set is linearly independent

4. $\begin{matrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{matrix}$

equation $a \begin{matrix} 3 \\ 1 \\ 1 \end{matrix} + b \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} + c \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$

system: $a + 2b + 3c = 0$ (1)

$b + c = 0$ (2)

$-a - b + c = 0$ (3)

$a = b = c = 0$

$\Rightarrow$ the set is linearly independent

4. $\begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{matrix}$

- equation : $a \begin{matrix} 1 \\ 1 \\ 0 \end{matrix} + b \begin{matrix} 0 \\ 1 \\ -1 \end{matrix} + c \begin{matrix} 1 \\ 0 \\ 1 \end{matrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$

- system: $a + c = 0$ (1)
  $a + b = 0$ (2)
  $-b + c = 0$ (3)

+ from (1), $a = -c$
+ substitute into (2) $\Rightarrow b = c$
+ substitute into (3) $\Rightarrow 0$
+ for any c, $a = -c$, $b = c$ works
+ for exaple $c = 1$, $a = -1$, $b = 1$ $\Rightarrow$ the set is linearly dependent

5. $A = \begin{matrix} c \\ d \end{matrix} \cdot a \; b$

- $A = \begin{matrix} c \\ d \end{matrix} \cdot a \; b = \begin{matrix} ca & cb \\ da & db \end{matrix}$

- determinant of A:

$\det(A) = (ca \times db) - (cb \times da)$

$= cabd - cbda$

$= 0$

since the determinant is always 0 for any a, b, c, d the statement is true

6. for vectors Ae_1 and Ae_2 to be orthogonal, their dot product must be zero: $(Ae_1) \cdot (Ae_2) = 0$.

- let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- compute Ae_1 : $Ae_1 = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$

- compute Ae_2 : $Ae_2 = A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$

- dot product condition: $(Ae_1)(Ae_2) = \begin{bmatrix} a \\ c \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix} = ab + cd = 0$

7. In Einstein notation

- Let $A = A^i_j$, where A^i_j are the components of the matrix
- The matrix A^4 is $(A^4)^i_j = A^i_k A^k_l A^l_m A^m_j$.
- the trace is the sum of diagonal elements:

$$tr(A^4) = (A^4)^i_i = A^i_k A^k_l A^l_m A^m_i$$