eigendecompositions.

1. find eigenvalues and eigenvector of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

- eigenvalues: $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 4\lambda + 3 = 0$$

$$\Rightarrow \lambda_1 = 3 \; ; \; \lambda_2 = 1.$$

- eigenvectors

for $\lambda_1 = 3$

$$A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

solve $(A - 3I)x = 0$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0$

$\Rightarrow x = x_2$

eigenvector $\begin{matrix} x_1 \\ x_1 \end{matrix}$

$\Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda_2 = 1$

$$A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

solve $(A - I)x = 0$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

eigenvector

$$\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2. To find eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

- eigenvalues : solve $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(2-\lambda) - (1)(0)$$

$$= (2-\lambda)^2 = 0$$

$\lambda = 2$ (repeated eigenvalue)

- eigenvector for $\lambda = 2$

$$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

solve $(A - 2I)x = 0$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix} = 0$$

$\Rightarrow x_2 = 0$

eigenvector $\begin{bmatrix} x_1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- strange aspect : repeated eigenvalue with only one independent eigenvector (defective matrix)

3. determine smallest eigenvalue of $A = \begin{matrix} 3.0 & 0.1 & 0.3 & 1.0 \\ 0.1 & 1.0 & 0.1 & 0.2 \\ 0.3 & 0.1 & 5.0 & 0.0 \\ 1.0 & 0.2 & 0.0 & 1.8 \end{matrix}$ less than 0.5

(3)

- gershgorin disk

r_1 : center $= 3.0$, radius $= |0.1| + |0.3| + |1.0| = 1.4$, disc : $[3.0 - 1.4, 3.0 + 1.4] = [1.6, 4.4]$

r_2 : center $= 1.0$, radius $= |0.1| + |0.1| + |0.2| = 0.4$, disc : $[1.0 - 0.4, 1.0 + 0.4] = [0.6, 1.4]$

r_3 : center $= 5.0$, radius $= |0.3| + |0.1| + |0.0| = 0.4$, disc : $[5.0 - 0.4, 5.0 + 0.4] = [4.6, 5.4]$

r_4 : center $= 1.8$, radius $= |1.0| + |0.2| + |0.0| = 1.2$, disc : $[1.8 - 1.2, 1.8 + 1.2] = [0.6, 3.0]$

- $[1.6, 4.4] \cup [0.6, 3.0] \cup [1.6, 4.4] \cup [4.6, 5.4] = [0.6, 5.4]$

the smallest possible eigenvalue is 0.6

→ the smallest eigenvalue can not be less than 0.5.