Linear Algebra

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1. Linear Algebra

1.1. Scalars

- Scalars are single numbers
- Calculate Body Mass Index (BMI)
- Weight = 70 kg, height = 1.75 m

- $$\text{BMI} = \frac{\text{Weight}}{\text{height}^2} = \frac{70}{1.75^2} = 22.86$$

1.1. Vectors

- Vectors are scalar arrays
- Total Calories Burned
- Durations = [0.5, 1.0, 0.75], calories = [600, 400, 500]

- Total calories = $0.5 \times 600 + 1.0 \times 400 + 0.75 \times 500$
  $= 1075$

1.1. Matrices

- Matrices are 2D arrays
- Sales Data Analysis
- Sales = [[100, 120, 110], [80, 90, 85], [150, 130, 140]]

  Row 1: 100 + 120 + 110 = 330.

- Row 2: 80 + 90 + 85 = 255.

  Row 3: 150 + 130 + 140 = 420.

- Result: [330, 255, 420]

1.2. Reduction

- Reduction operations aggregate elements of a tensor along one or more axes, reducing its dimensionality

- Reduction is used to summarize data, extract key statistics, or simplify tensors for further processing in applications like data analysis, machine learning

1.2. Reduction

	Math	Physical	English	AVG
A	10	9	8	9
B	9	9	9	9

1.2. Reduction

- Strengths:
  - Provides a concise summary of performance.
  - Reduces data dimensionality for easier processing.
  - Applicable in grading systems and data analysis.

- Weaknesses:
  - Loses individual score details.
  - May mask variability (e.g., outliers like a low score).
  - Axis choice (e.g., per student vs. per subject) affects interpretation.

1.2. Reduction

Aspect	Reduction	Dimensional Reduction
Definition	Aggregates tensor elements (e.g., sum, mean, max) along one or more axes, reducing dimensionality.	Reduces the number of features or dimensions in a dataset while preserving key information.
Purpose	Summarizes data into simpler metrics (e.g., averages, totals) for analysis or processing.	Simplifies high-dimensional data for visualization, computation, or model efficiency.
Techniques	Sum, mean, max, min, applied along specific axes (e.g., row-wise mean of a matrix).	PCA, t-SNE, LDA, autoencoders, feature selection, or embedding methods.
Applications	Grading systems, image intensity averaging, statistical summaries.	Machine learning, data visualization, feature engineering, image compression.

1.3. Norm

- Norms measure the magnitude or size of a vector or matrix, quantifying properties like distance, error, or energy. Common norms include the L1 norm, L2 norm and Frobenius norm.

- Norms are used to evaluate errors in predictions, regularize models to prevent overfitting, or measure signal strength in applications like machine learning, signal processing, and optimization.

1.3. Norm

Given:

- Predicted values: $[3,\ 1,\ 5]$
- Actual values: $[4,\ 2,\ 7]$

Compute the error

$$\text{Error} = [3-4,\ 1-2,\ 5-7] = [-1,\ -1,\ -2]$$

Compute the ℓ_1 norm

$$\ell_1 = |-1| + |-1| + |-2| = 1 + 1 + 2 = 4$$

Compute the ℓ_2 norm

$$\ell_2 = \sqrt{(-1)^2 + (-1)^2 + (-2)^2} = \sqrt{1+1+4} = \sqrt{6} \approx 2.45$$

1.3. Norm

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 9 & 6 \\ 3 & 4 & 8 \end{bmatrix}$$

$$\|\mathbf{X}\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}^2}.$$

Square each element

$$\begin{bmatrix} 2^2 & 3^2 & 5^2 \\ 1^2 & 9^2 & 6^2 \\ 3^2 & 4^2 & 8^2 \end{bmatrix} = \begin{bmatrix} 4 & 9 & 25 \\ 1 & 81 & 36 \\ 9 & 16 & 64 \end{bmatrix}$$

Sum all squared elements

$$4 + 9 + 25 + 1 + 81 + 36 + 9 + 16 + 64 = 245$$

Take the square root

$$\|A\|_F = \sqrt{245} \approx 15.65$$

2. Geometry & Linear Algebraic Ops

2.1. Hyperplane

2.2. Linear Dependence.

2.1. Hyperplane

In 2D space, a **hyperplane** is simply a **line**. The general equation of a hyperplane in 2D is:

$$ax + by + c = 0$$

Where:

- a, b, c are real numbers, with a and/or $b \neq 0$.
- (x, y) are the coordinates of points on the line.

2.1. Hyperplane

Let:

- $a = 2$
- $b = 3$
- $c = 6$

The equation becomes:

$$2x + 3y + 6 = 0$$

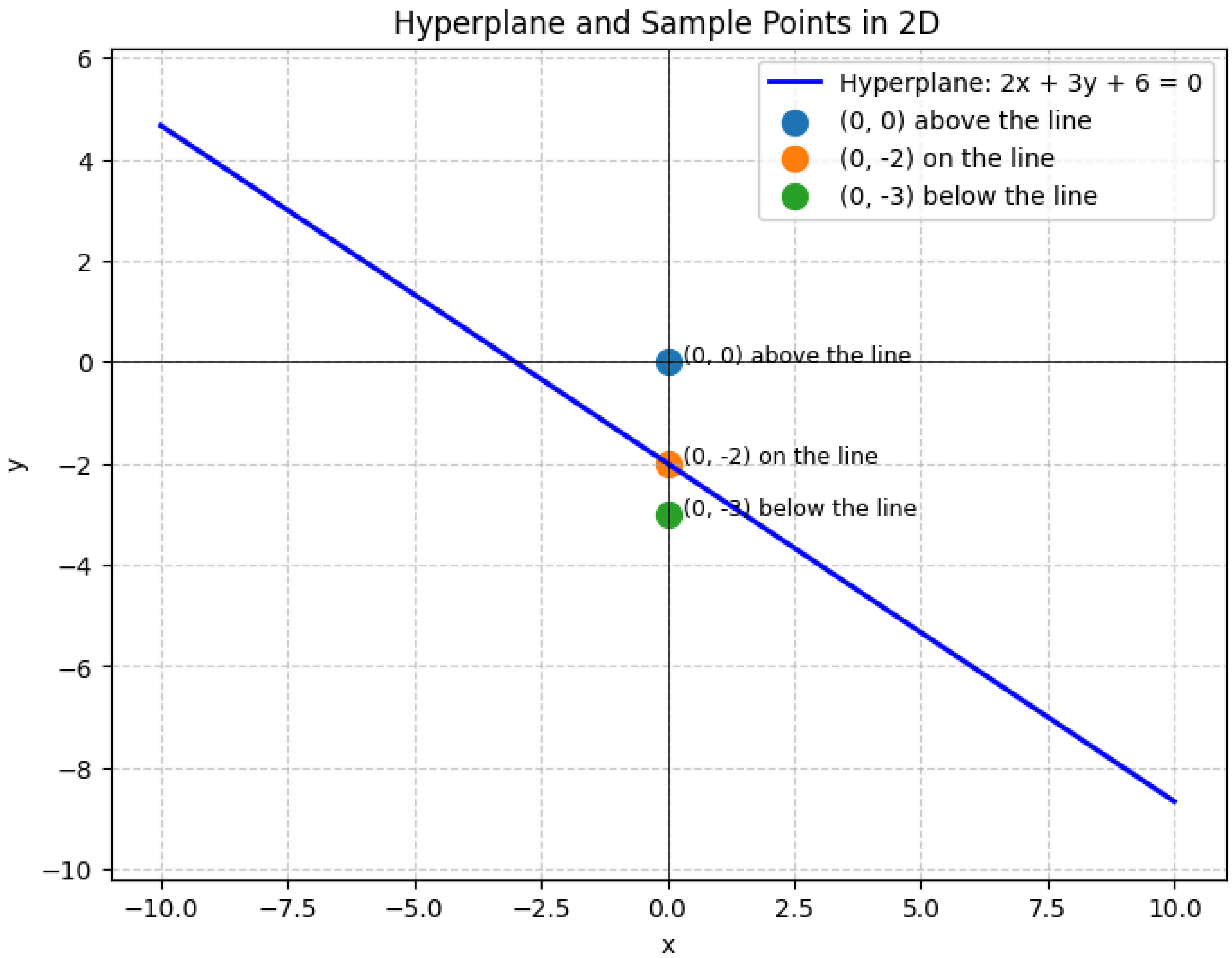
$$3y = -2x - 6$$

$$y = -\frac{2}{3}x - 2$$

2.1. Hyperplane

Point	Expression $2x + 3y + 6$	Value	Position relative to line
(0, 0)	$0 + 0 + 6$	6	**Above the line** (positive)
(0, -2)	$0 - 6 + 6$	0	**On the line**
(0, -3)	$0 - 9 + 6$	-3	**Below the line** (negative)

2.1. Hyperplane



2.2. Linear Dependence

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$a \cdot \mathbf{v}_1 + b \cdot \mathbf{v}_2 + c \cdot \mathbf{v}_3 = \mathbf{0}$$

$$a \begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} + b \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + c \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2.2. Linear Dependence

$$a \cdot 1 + b \cdot 2 + c \cdot 3 = 0 \rightarrow a + 2b + 3c = 0$$
$$a \cdot 9 + b \cdot 2 + c \cdot 6 = 0 \rightarrow 9a + 2b + 6c = 0$$
$$a \cdot 6 + b \cdot 2 + c \cdot 9 = 0 \rightarrow 6a + 2b + 9c = 0$$

$$a = b = c = 0$$

Conclusion: The three vectors are linearly independent.

3. Eigendecompositions

3.1. Find Eigenvalues and Eigenvectors

3.2. Gershgorin Circle Theorem

3.1. Find Eigenvalues and Eigenvectors

Problem

Given:

$$A = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}$$

Find the **eigenvalues** and **eigenvectors** of A.

3.1. Find Eigenvalues and Eigenvectors

Step 1: Characteristic equation

We solve:

$$\det(A - \lambda I) = 0$$

Subtract λ from the diagonal entries:

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix}$$

The determinant is:

$$(5-\lambda)(4-\lambda) - (1)(2) = \lambda^2 - 9\lambda + 18$$

3.1. Find Eigenvalues and Eigenvectors

Step 2: Solve for eigenvalues

$$\lambda^2 - 9\lambda + 18 = 0$$

Factor:

$$(\lambda - 6)(\lambda - 3) = 0$$

Thus:

$$\boxed{\lambda_1 = 6, \quad \lambda_2 = 3}$$

3.1. Find Eigenvalues and Eigenvectors

Step 3: Eigenvector for $\lambda_1 = 6$

We solve:

$$(A - 6I)\mathbf{v} = \mathbf{0}$$

$$A - 6I = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

Let $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. The system is:

$$\begin{cases} -1x + 2y = 0 \\ 1x - 2y = 0 \end{cases}$$

Both equations are the same, so we take the first:

$$-x + 2y = 0 \quad \Rightarrow \quad x = 2y$$

Choose $y = 1 \Rightarrow x = 2$.

Eigenvector:

$$\boxed{\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

3.1. Find Eigenvalues and Eigenvectors

Step 4: Eigenvector for $\lambda_2 = 3$

We solve:

$$(A - 3I)\mathbf{v} = \mathbf{0}$$

$$A - 3I = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

The system:

$$\begin{cases} 2x + 2y = 0 \\ x + y = 0 \end{cases}$$

Again, the equations are dependent. From $x + y = 0$:

$$x = -y$$

Choose $y = -1 \Rightarrow x = 1$.

Eigenvector:

$$\boxed{\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

3.1. Find Eigenvalues and Eigenvectors

Step 5: Final result

- Eigenvalues:

$$\lambda_1 = 6, \quad \lambda_2 = 3$$

- Eigenvectors:

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

3.2. Gershgorin Circle Theorem

Problem

Given the matrix

$$A = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}$$

Estimate the eigenvalues using the **Gershgorin Circle Theorem**.

3.2. Gershgorin Circle Theorem

Step 1: Gershgorin Circle Theorem (Row-based version)

For each row i:

- **Center**: $c_i = a_{ii}$ (diagonal entry)
- **Radius**: $R_i = \sum_{j \neq i} |a_{ij}|$ (sum of absolute values of non-diagonal entries in that row)

All eigenvalues of A lie within at least one of the intervals:

$$[c_i - R_i,\ c_i + R_i]$$

3.2. Gershgorin Circle Theorem

Step 2: Calculate the Gershgorin intervals

Row 1:

- Center: $c_1 = 5$
- Radius: $R_1 = |2| = 2$
- Interval: $[5 - 2,\ 5 + 2] = [3, 7]$

Row 2:

- Center: $c_2 = 4$
- Radius: $R_2 = |1| = 1$
- Interval: $[4 - 1,\ 4 + 1] = [3, 5]$

3.2. Gershgorin Circle Theorem

Step 3: Conclusion

Union of the two intervals:

$$[3,7] \cup [3,5] = [3,7]$$

Therefore, **all eigenvalues of A lie in the range [3, 7]** (if they are real; if complex, their real part lies in these discs).

3.2. Gershgorin Circle Theorem

Step 4: Verification

Exact eigenvalues:

$$\det(A - \lambda I) = (5 - \lambda)(4 - \lambda) - 2 = \lambda^2 - 9\lambda + 18 = 0$$

$$\Rightarrow \lambda_1 = 6, \quad \lambda_2 = 3$$

Both eigenvalues are indeed in $[3, 7]$ ✔

THANK YOU