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Engineering Electromagnetics

Time – Varying Fields & Maxwell’s Equations

Contents

Time – Varying Fields & Maxwell's Equations

1. Faraday's Law
2. Displacement Current
3. Maxwell's Equations in Point Form
4. Maxwell's Equations in Integral Form
5. The Retarded Potentials

Faraday's Law (1)

$$\text{emf} = -\frac{d\Phi}{dt} \quad \text{V}$$

http://micro.magnet.fsu.edu/electromag/electricity/inductance.html

Emf is nonzero if one of any:

- A time-changing flux linking a stationary closed path
- Relative motion between a steady flux and a closed path
- A combination of the two

Minus sign ?

Lenz's law

http://www.engineering-timelines.com/how/electricity/transformer.asp

Faraday's Law (2)

$$\left.\begin{aligned} \text{emf} &= -\frac{d\Phi}{dt} \\ \text{emf} &= \oint \mathbf{E}.d\mathbf{L} \\ \Phi &= \int_S \mathbf{B}.d\mathbf{S} \end{aligned}\right\} \rightarrow \text{emf} = \oint \mathbf{E}.d\mathbf{L} = -\frac{d}{dt}\int_S \mathbf{B}.d\mathbf{S}$$

$$\left.\begin{aligned} &\rightarrow \text{emf} = \oint \mathbf{E}.d\mathbf{L} = -\frac{d}{dt}\int_S \mathbf{B}.d\mathbf{S} \\ &\mathbf{B} = \mathbf{B}(t) \end{aligned}\right\} \rightarrow \text{emf} = \oint \mathbf{E}.d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S}$$

$$\left.\begin{aligned} &\rightarrow \text{emf} = \oint \mathbf{E}.d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S} \\ &\text{Stokes' theorem: } \oint \mathbf{E}.d\mathbf{L} = \int_S (\nabla\times\mathbf{E}).d\mathbf{S} \end{aligned}\right\}$$

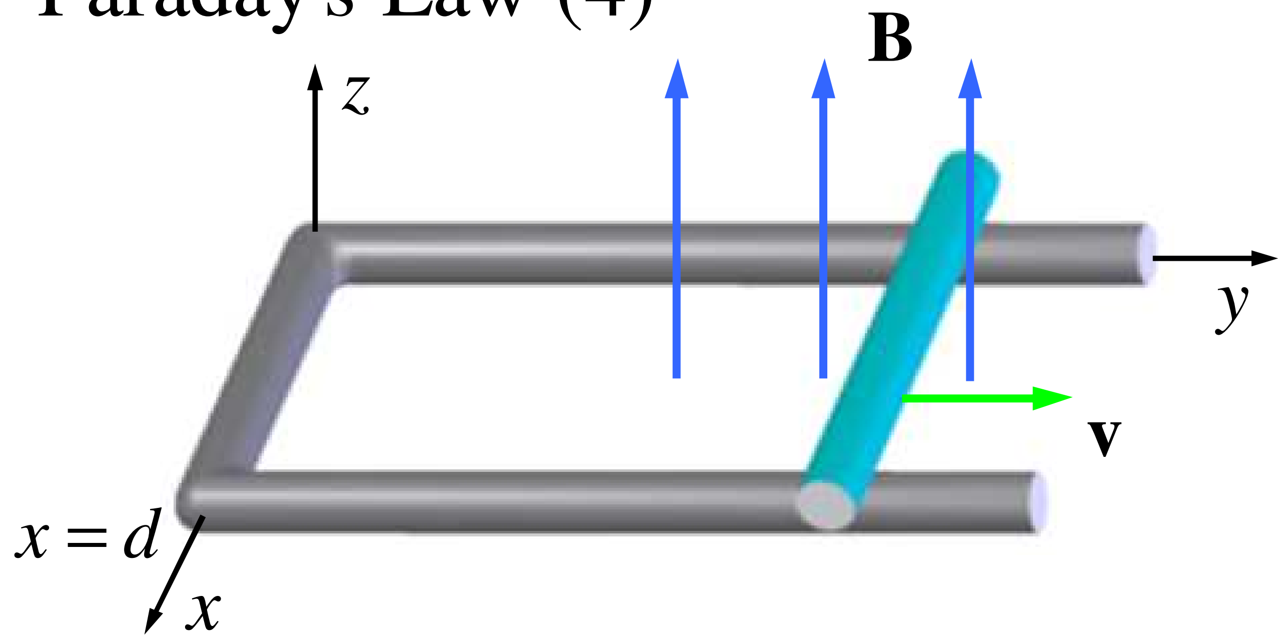
$$\rightarrow \int_S (\nabla\times\mathbf{E}).d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S} \quad \rightarrow (\nabla\times\mathbf{E}).d\mathbf{S} = -\frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S}$$

$$\rightarrow \boxed{\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

Faraday's Law (3)

$$\left.\begin{aligned} &\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ &\text{emf} = \oint \mathbf{E}.d\mathbf{L} = -\int_S \frac{\partial\mathbf{B}}{\partial t}.d\mathbf{S} \\ &\frac{\partial\mathbf{B}}{\partial t} = 0 \quad (\text{steady}) \end{aligned}\right\} \longrightarrow \begin{cases} \oint \mathbf{E}.d\mathbf{L} = 0 \\ \nabla\times\mathbf{E} = 0 \end{cases}$$

Faraday's Law (4)



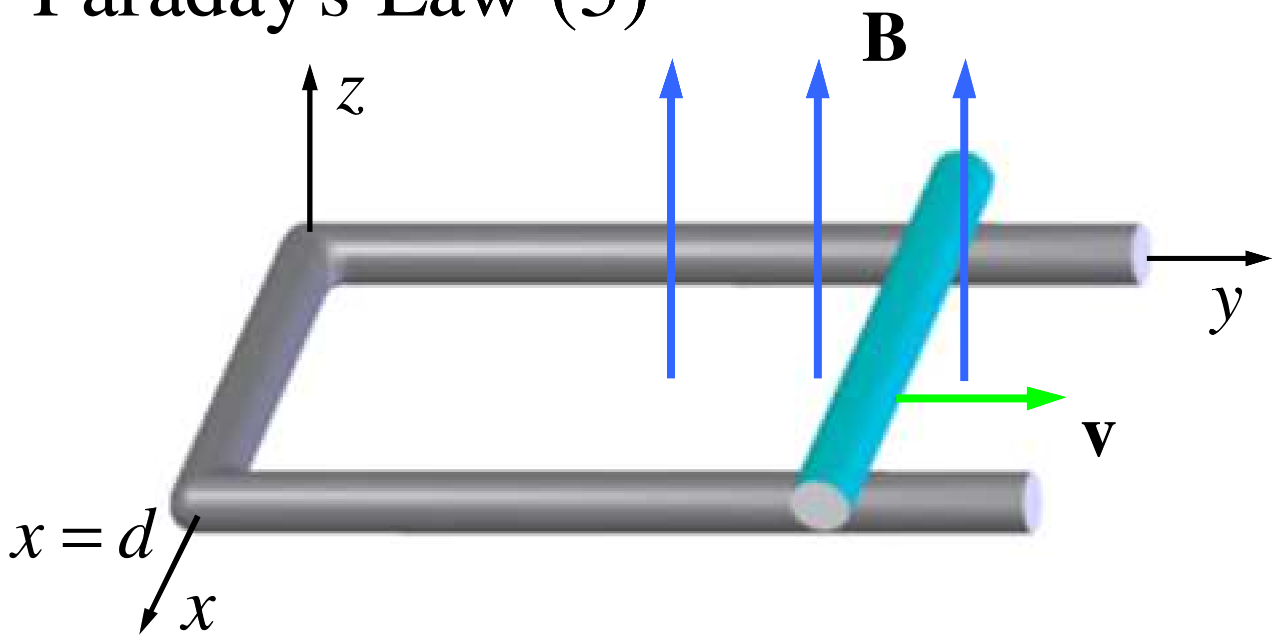
$$\left.\begin{aligned} \Phi &= \int_S \mathbf{B}.d\mathbf{S} = Byd \\ \text{emf} &= -\frac{d\Phi}{dt} \end{aligned}\right\} \rightarrow \text{emf} = -B\frac{dy}{dt}d = -Bvd$$

Faraday's Law (5)

$$\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$$

$$\rightarrow \frac{\mathbf{F}}{Q} = \mathbf{v} \times \mathbf{B}$$

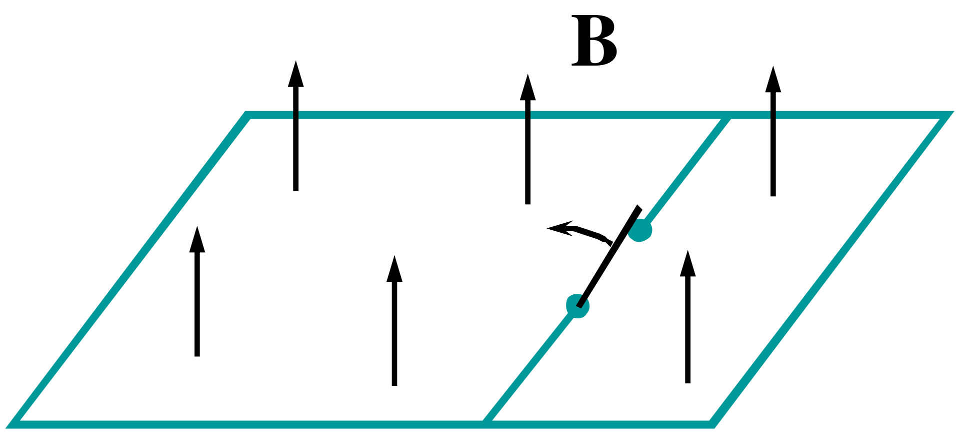
$$\mathbf{E}_m = \mathbf{v} \times \mathbf{B}$$



$$\text{emf} = \oint \mathbf{E}_m . d\mathbf{L} = \oint (\mathbf{v} \times \mathbf{B}) . d\mathbf{L} = \int_d^0 vBdx = -Bvd$$

Faraday's Law (6)

$$\text{emf} = \oint \mathbf{E}.d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S} + \oint (\mathbf{v} \times \mathbf{B}).d\mathbf{L}$$



Ex. 1

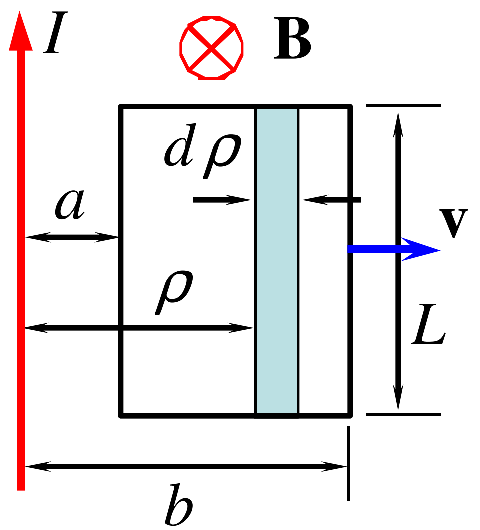
Faraday's Law (7)

A single turn loop is situated in air, with a uniform magnetic field normal to its plane. The area of the loop is 10 m^2. If the rate of change of flux density is 5 Wb/m^2/s, what is the emf appearing at the terminals of the loop?

$$\left.\begin{aligned} \text{emf} &= -N\frac{d\Phi}{dt} \\ \Phi &= BS \end{aligned}\right\} \rightarrow \text{emf} = -\frac{dB}{dt}S = 5\times 10 = 50\ \text{V}$$

Ex. 2

Faraday's Law (8)

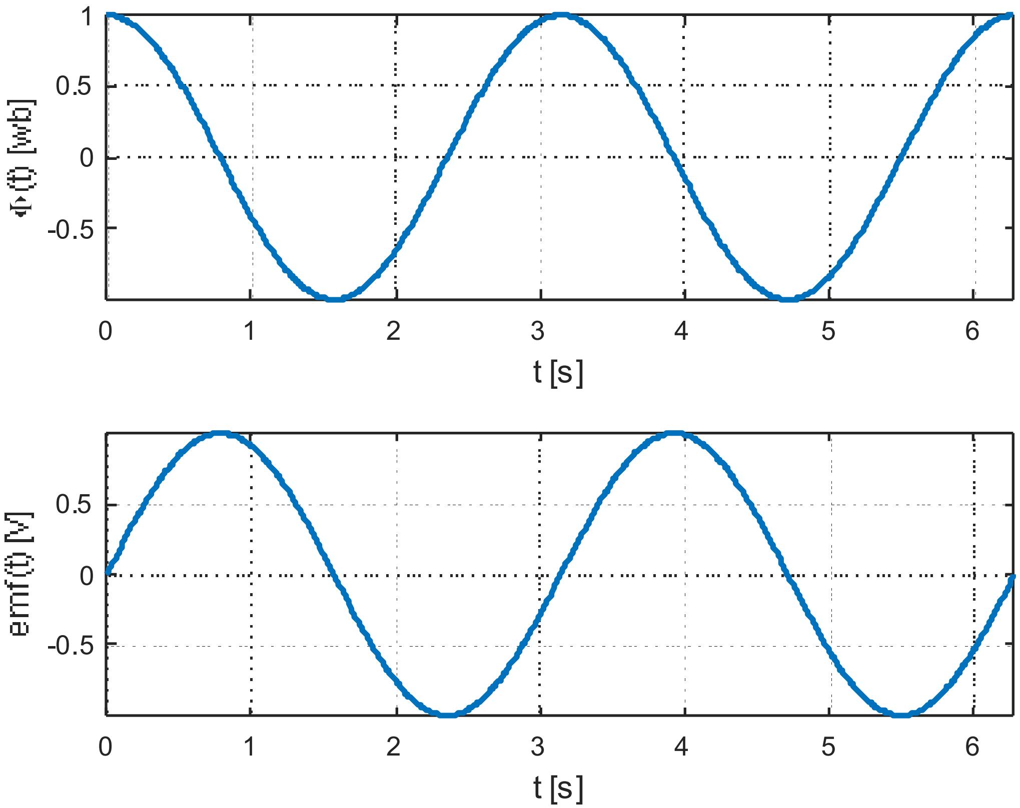


$$\Phi(t) = \frac{\mu_0 IL}{2\pi} \ln \frac{b_0 + vt}{a_0 + vt}$$

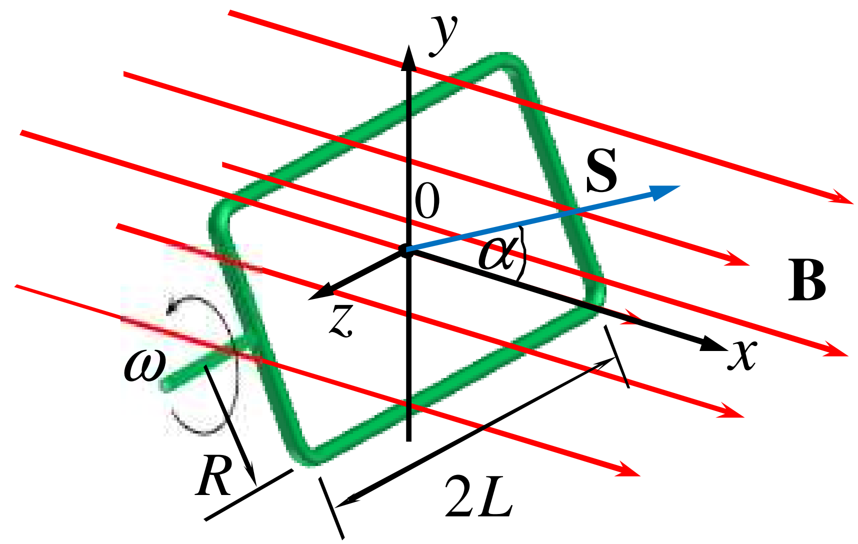
$$\to \text{emf} = -\frac{d\Phi(t)}{dt} = \frac{\mu_0 IL}{2\pi} \cdot \frac{(b_0 - a_0)v}{(a_0 + vt)^2} \cdot \frac{a_0 + vt}{b_0 + vt} = \frac{\mu_0 IL}{2\pi} \cdot \frac{(b_0 - a_0)v}{(a_0 + vt)b_0 + vt}$$

Ex. 3

Faraday's Law (9)



Method 1

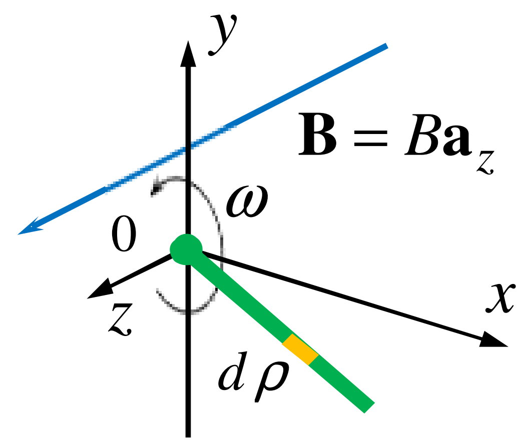


$$\Phi = BS\cos\omega t$$

$$\rightarrow \text{emf} = -\frac{d\Phi(t)}{dt} = BS\omega\sin\omega t$$

Ex. 4

Faraday's Law (10)



A conductive strip of length L pivoted at one end is rotating freely in the xy-plane with an angular frequency ω in a uniform magnetic flux $\mathbf{B}$. Find the induced emf between the two ends of the strip?

$$\text{emf} = \int (\mathbf{v} \times \mathbf{B}).d\mathbf{L}$$

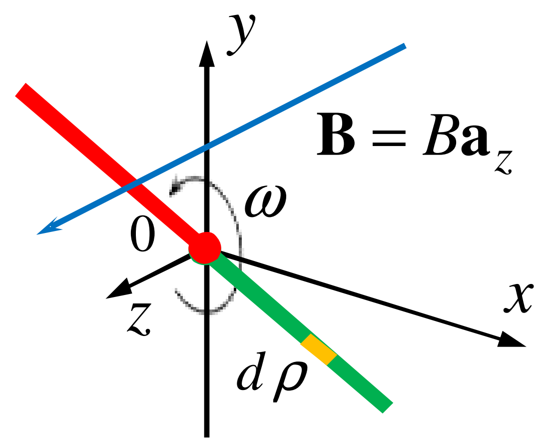
$$\mathbf{v} = \rho\omega\mathbf{a}_\varphi$$

$$\mathbf{v} \times \mathbf{B} = (\rho\omega\mathbf{a}_\varphi) \times (B\mathbf{a}_z) = \rho\omega B\mathbf{a}_\rho$$

$$\text{emf} = \int_0^L (\rho\omega B\mathbf{a}_\rho).d\rho\mathbf{a}_\rho = \int_0^L \rho\omega B d\rho = \omega B \int_0^L \rho d\rho = \frac{B\omega L^2}{2}$$

Ex. 5

Faraday's Law (11)



A conductive strip of length $2L$ pivoted at the midpoint is rotating freely in the xy-plane with an angular frequency ω in a uniform magnetic flux $\mathbf{B}$. Find the induced emf between the two ends of the strip?

$$\text{emf}_g = \frac{B\omega L^2}{2}$$

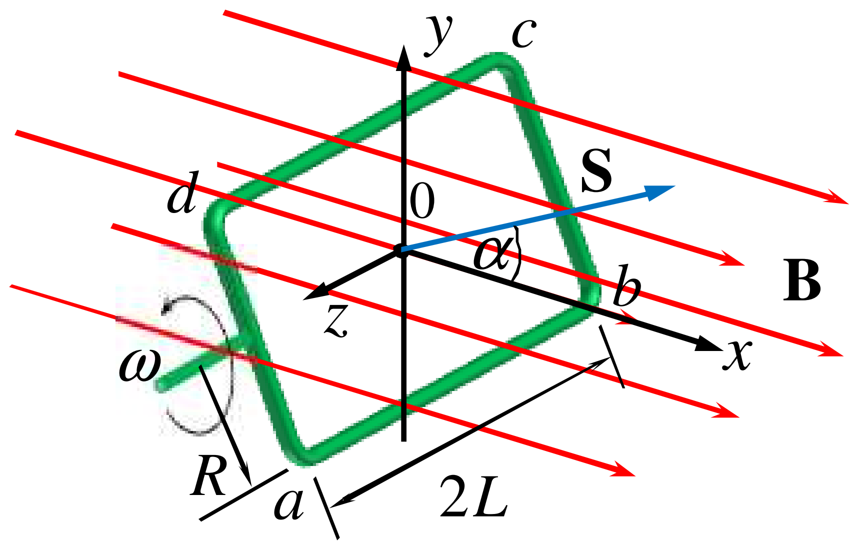
$$\text{emf}_r = \frac{B\omega L^2}{2}$$

$$\text{emf}_t = \text{emf}_g - \text{emf}_r = 0$$

Ex. 3

Faraday's Law (12)

Method 2



$$\text{emf} = \oint \mathbf{E}_m . d\mathbf{L} = \oint (\mathbf{v} \times \mathbf{B}) . d\mathbf{L}$$

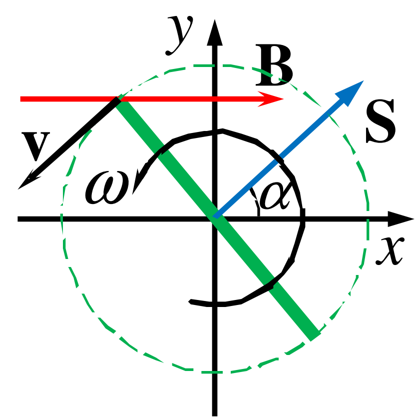
$$= \int_a^b + \int_b^c + \int_c^d + \int_d^a$$

$$\int_b^c = \int_d^a = 0$$

$$\text{emf}_{ab} = \int_a^b = \int_L^{-L} [(R\omega \mathbf{a}_\varphi) \times (B\mathbf{a}_x)] . (dz\mathbf{a}_z)$$

$$= \int_L^{-L} [BR\omega \sin(\omega t)(-\mathbf{a}_z)] . (dz\mathbf{a}_z)$$

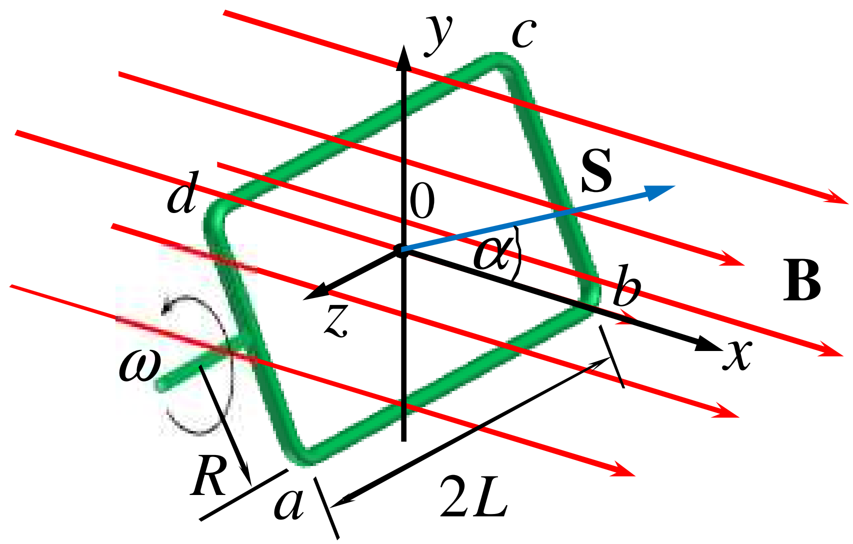
$$= -BR\omega \sin \omega t \int_L^{-L} dz = 2LBR\omega \sin \omega t$$



Ex. 3

Faraday's Law (13)

Method 2



$$\text{emf} = \oint \mathbf{E}_m.d\mathbf{L} = \oint (\mathbf{v}\times\mathbf{B}).d\mathbf{L}$$

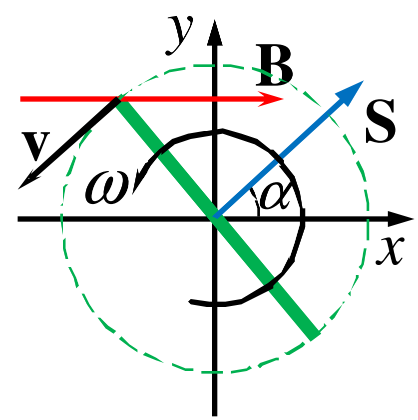
$$\text{emf}_{bc} = \text{emf}_{da} = 0$$

$$\text{emf}_{ab} = 2LBR\omega\sin\omega t$$

$$\text{emf}_{cd} = 2LBR\omega\sin\omega t$$

$$\text{emf} = \text{emf}_{ab} + \text{emf}_{cd}$$

$$= 4LBR\omega\sin\omega t = \boxed{BS\omega\sin\omega t}$$

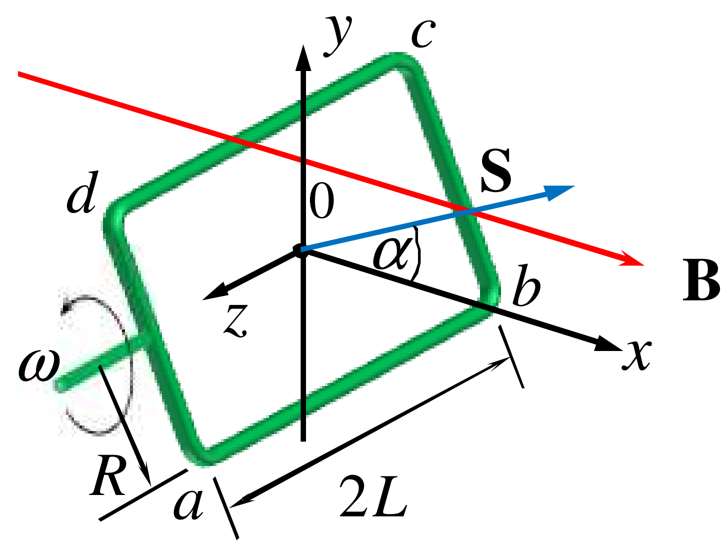


Ex. 6

Faraday's Law (14)

Method 1

$\boxed{\mathbf{B} = B_m \sin\omega t \mathbf{a}_\rho.}$



$$\Phi = \int_S \mathbf{B}.d\mathbf{S} = \int_S (B_m \sin\omega t \mathbf{a}_\rho).(dS\mathbf{a}_S)$$

$$= \int_S B_m \sin\omega t \cos\omega t dS$$

$$= B_m \sin\omega t \cos\omega t \int_S dS = B_m S \sin\omega t \cos\omega t = \frac{1}{2} B_m S \sin 2\omega t$$

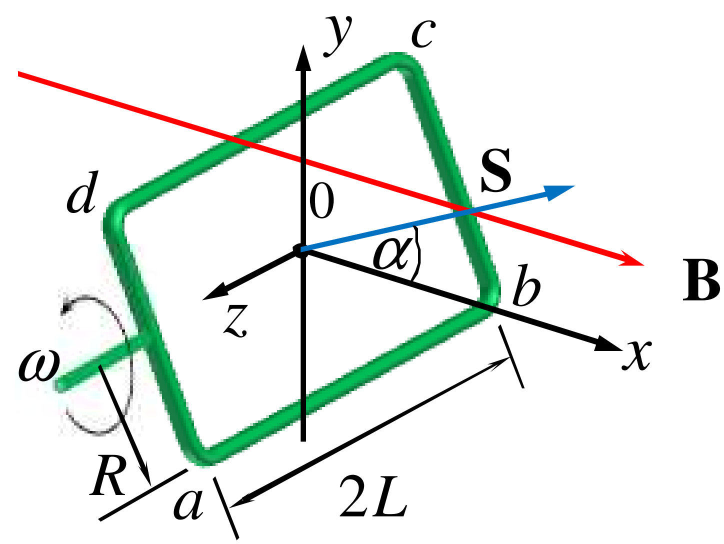
$$\text{emf} = -\frac{d\Phi}{dt} = -B_m S\omega \cos 2\omega t$$

Ex. 6

Faraday's Law (15)

$\boxed{\mathbf{B} = B_m \sin\omega t \mathbf{a}_\rho.}$

Method 2



$$\text{emf} = -\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S} + \oint (\mathbf{v}\times\mathbf{B}).d\mathbf{L}$$

$$-\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S} = -\int_S \left(\frac{\partial B_m \sin\omega t}{\partial t}\mathbf{a}_\rho\right).(dS\mathbf{a}_S)$$

$$= -\int_S \left(B_m \omega \cos\omega t \mathbf{a}_\rho\right).(dS\mathbf{a}_S)$$

$$= -B_m \omega \cos\omega t \int_S \mathbf{a}_\rho.dS\mathbf{a}_S = -B_m \omega \cos\omega t \int_S \cos\omega t dS$$

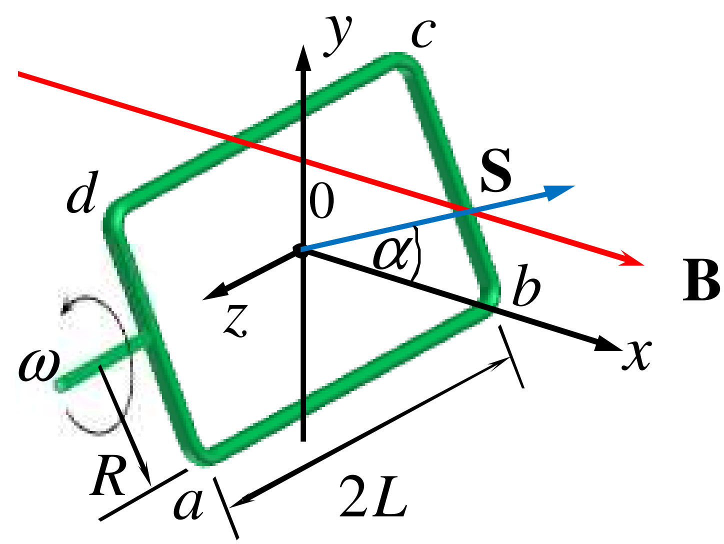
$$= -B_m \omega \cos^2\omega t \int_S dS = -B_m S \omega \cos^2\omega t$$

Ex. 6

Faraday's Law (16)

$$\boxed{\mathbf{B} = B_m \sin\omega t\,\mathbf{a}_\rho.}$$

Method 2



$$\text{emf} = -\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S} + \oint (\mathbf{v}\times\mathbf{B}).d\mathbf{L}$$

$$\oint (\mathbf{v}\times\mathbf{B}).d\mathbf{L} = 2\int_a^b (\mathbf{v}\times\mathbf{B}).d\mathbf{L}$$

$$= 2\int_L^{-L} [(R\omega\mathbf{a}_\varphi)\times(B_m \sin\omega t\,\mathbf{a}_\rho)].(dz\mathbf{a}_z)$$

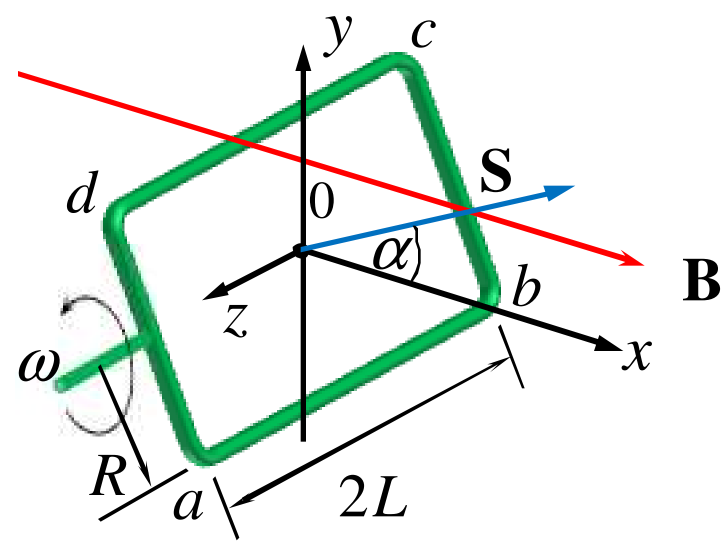
$$= 2\int_L^{-L} [(R\omega B_m \sin^2(-\mathbf{a}_z)].(dz\mathbf{a}_z) = B_m S\omega \sin^2\omega t$$

Ex. 6

Faraday's Law (17)

Method 2

$\boxed{\mathbf{B} = B_m \sin\omega t\, \mathbf{a}_\rho.}$



$$\text{emf} = -\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S} + \oint (\mathbf{v}\times\mathbf{B}).d\mathbf{L}$$

$$-\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S} = -B_m S\omega \cos^2 \omega t$$

$$\oint (\mathbf{v}\times\mathbf{B}).d\mathbf{L} = B_m S\omega \sin^2 \omega t$$

$$\text{emf} = -B_m S\omega\cos^2 \omega t + B_m S\omega \sin^2 \omega t = \boxed{-B_m S\omega \cos 2\omega t}$$

Ex. 7

Faraday's Law (18)

A circular conducting loop of radius R lies in the xy plane. Find the emf of the loop if $\mathbf{B} = 0.5\sin 500t\mathbf{a}_x + 0.3\sin 400t\mathbf{a}_y + 0.9\cos 314t\mathbf{a}_z$ T?

$$\Phi = \int_S \mathbf{B}.d\mathbf{S}$$

$$= \int_S (0.5\sin 500t\mathbf{a}_x + 0.3\sin 400t\mathbf{a}_y + 0.9\cos 314t\mathbf{a}_z).(dS\mathbf{a}_z)$$

$$= \int_S (0.9\cos 314t)dS$$

$$= 0.9\cos 314t\int_S dS$$

$$= 0.9\cos 314t(\pi R^2)$$

$$\text{emf} = -\frac{d\Phi}{dt} = -\frac{d}{dt}(0.9\pi R^2\cos 314t) = 888R^2\sin 314t \text{ V}$$

Time – Varying Fields & Maxwell's Equations

1. Faraday's Law
2. **Displacement Current**
3. Maxwell's Equations in Point Form
4. Maxwell's Equations in Integral Form
5. The Retarded Potentials

Displacement Current (1)

$$\left.\begin{array}{r}\nabla\times\mathbf{H}=\mathbf{J}\rightarrow\nabla.\nabla\times\mathbf{H}=\nabla.\mathbf{J}\\ \nabla.\nabla\times\mathbf{H}=0\\ \nabla.\mathbf{J}=-\dfrac{\partial\rho_v}{\partial t}\end{array}\right\}\rightarrow\frac{\partial\rho_v}{\partial t}=0\,(\text{unreasonable})$$

$$\left.\begin{array}{r}\left.\begin{array}{r}\nabla\times\mathbf{H}=\mathbf{J}+\mathbf{G}\rightarrow 0=\nabla.\mathbf{J}+\nabla.\mathbf{G}\\ \nabla.\mathbf{J}=-\dfrac{\partial\rho_v}{\partial t}\end{array}\right\}\rightarrow\nabla.\mathbf{G}=\dfrac{\partial\rho_v}{\partial t}\\ \nabla.\mathbf{D}=\rho_v\end{array}\right\}$$

$$\left.\begin{array}{r}\rightarrow\nabla.\mathbf{G}=\dfrac{\partial}{\partial t}(\nabla.\mathbf{D})=\nabla.\dfrac{\partial\mathbf{D}}{\partial t}\rightarrow\mathbf{G}=\dfrac{\partial\mathbf{D}}{\partial t}\\ \nabla\times\mathbf{H}=\mathbf{J}+\mathbf{G}\end{array}\right\}\rightarrow\boxed{\nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}}$$

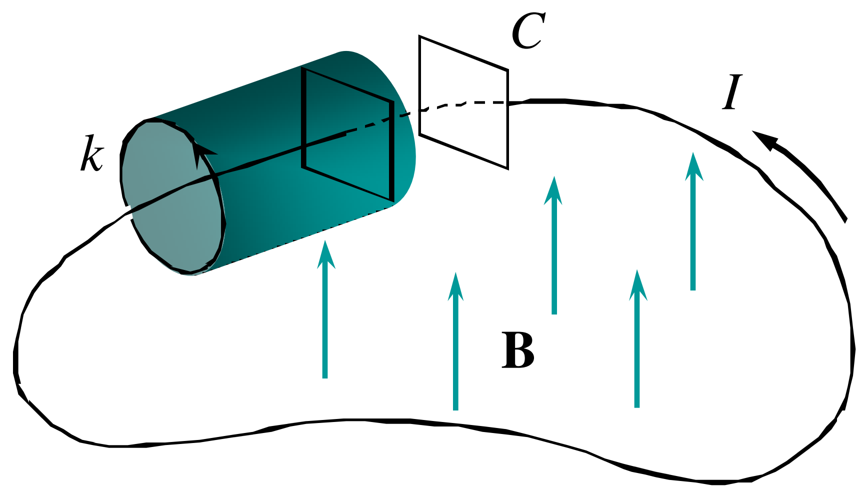
Displacement Current (2)

$$\left.\begin{array}{l}\nabla\times\mathbf{H}=\mathbf{J}+\dfrac{\partial\mathbf{D}}{\partial t}\\[2ex] \text{Define } \mathbf{J}_d=\dfrac{\partial\mathbf{D}}{\partial t}\end{array}\right\}\rightarrow\nabla\times\mathbf{H}=\mathbf{J}+\mathbf{J}_d$$

In nonconducting medium $\mathbf{J}=0\rightarrow\nabla\times\mathbf{H}=\dfrac{\partial\mathbf{D}}{\partial t}$

$$\left.\begin{array}{l}I_d=\int_S\mathbf{J}_d.d\mathbf{S}=\int_S\dfrac{\partial\mathbf{D}}{\partial t}.d\mathbf{S}\\[2ex] \int_S(\nabla\times\mathbf{H}).d\mathbf{S}=\int_S\mathbf{J}.d\mathbf{S}+\int_S\dfrac{\partial\mathbf{D}}{\partial t}.d\mathbf{S}\\[2ex] \oint\mathbf{H}.d\mathbf{L}=\int_S(\nabla\times\mathbf{H}).d\mathbf{S}\end{array}\right\}\rightarrow\boxed{\oint\mathbf{H}.d\mathbf{L}=I+I_d=I+\int_S\dfrac{\partial\mathbf{D}}{\partial t}.d\mathbf{S}}$$

Displacement Current (3)



$$\text{emf} = V_0 \cos \omega t$$

$$\rightarrow I = -\omega C V_0 \sin \omega t$$

$$= -\omega \frac{\varepsilon S}{d} V_0 \sin \omega t$$

$$\oint_k \mathbf{H}.d\mathbf{L} = I_k$$

$$\left.\begin{aligned} D &= \varepsilon E = \varepsilon\left(\frac{V_0}{d}\cos \omega t\right) \\ I_d &= \int_S \frac{\partial \mathbf{D}}{\partial t}.d\mathbf{S} = \frac{\partial D}{\partial t} S \end{aligned}\right\} \rightarrow I_d = -\omega \frac{\varepsilon S}{d} V_0 \sin \omega t$$

Ex. 1 Displacement Current (4)

Given a magnetic field in free space as $\mathbf{H} = H_0\sin(\omega t - \beta z)\mathbf{a}_y$ A/m. Determine the displacement current density.

$$\left.\begin{aligned} \mathbf{J}_d &= \frac{\partial \mathbf{D}}{\partial t} \\ \nabla\times\mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\ \mathbf{J} &= 0 \end{aligned}\right\}$$

$$\rightarrow \mathbf{J}_d = \nabla\times\mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

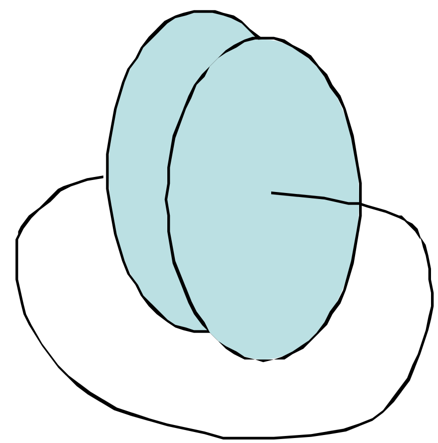
$$= H_0\beta\cos(\omega t - \beta z)\mathbf{a}_x$$

Ex. 2

Displacement Current (5)

A parallel – plate capacitor consists of two circular plates of radius R. Suppose that the capacitor is being charged at a uniform rate so that the EFI between the plates changes at a constant rate $dE/dt = 10^{12}$ V/m/s. Find the displacement current for the capacitor?

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$



$$I_d = \int_S \mathbf{J}_d d\mathbf{S} = \int_S \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} d\mathbf{S} = \int_S \varepsilon_0 \frac{\partial E}{\partial t} dS$$

$$= \varepsilon_0 \frac{\partial E}{\partial t} \int_S dS = \varepsilon_0 \frac{dE}{dt} \pi R^2$$

Time – Varying Fields & Maxwell's Equations

1. Faraday's Law
2. Displacement Current
3. **Maxwell's Equations in Point Form**
4. Maxwell's Equations in Integral Form
5. The Retarded Potentials

Maxwell's Equations in Point Form (1)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla . \mathbf{D} = \rho_v$$

$$\nabla . \mathbf{B} = 0$$

Ex. 1 Maxwell's Equations in Point Form (2)

Given an electric field $\mathbf{E} = A\cos\omega(t - z/c)\mathbf{a}_y$. Determine the time-dependent MFI $\mathbf{H}$ in free space?

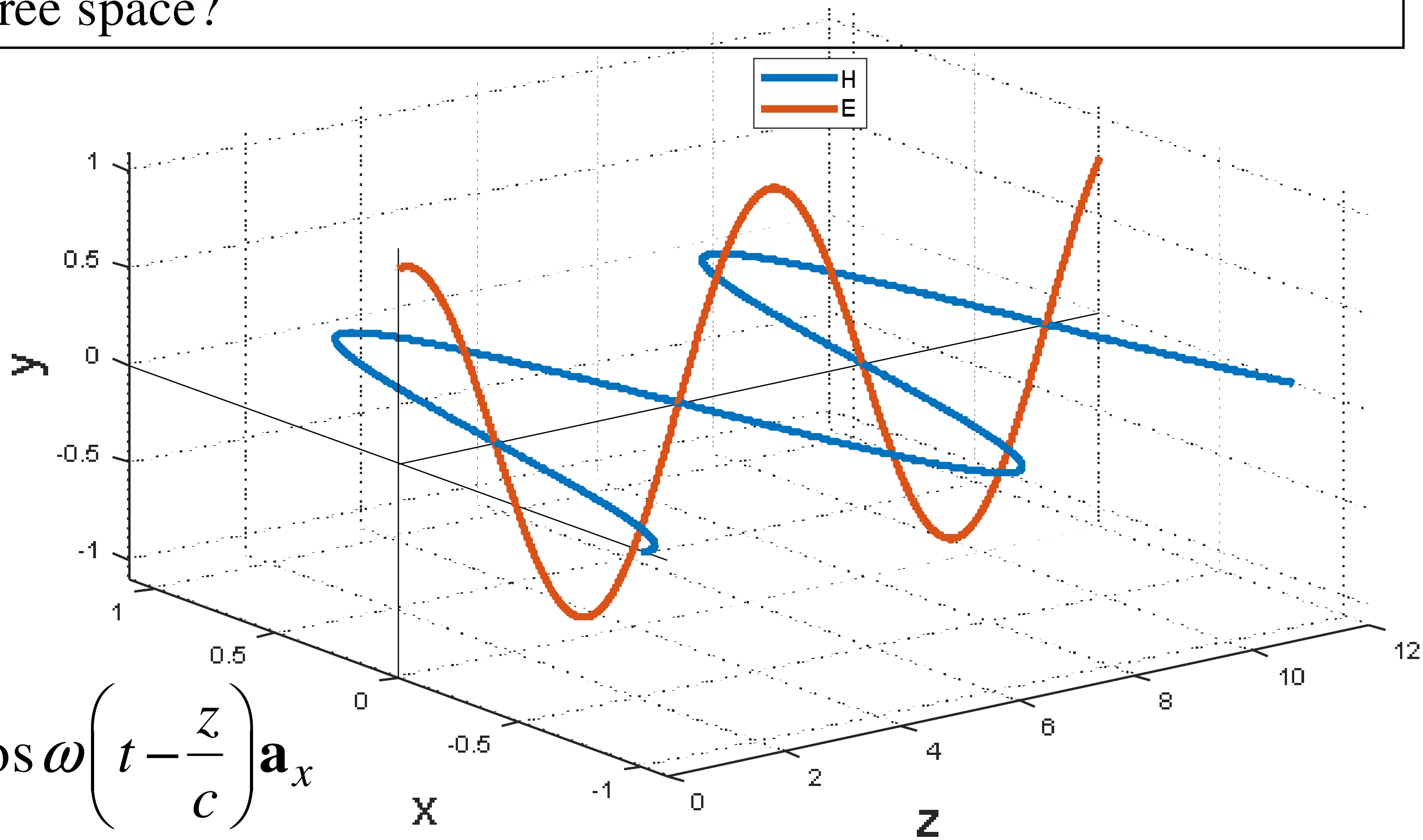
$$\left.\begin{aligned} \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} = -\mu_0\frac{\partial\mathbf{H}}{\partial t} \\ \nabla\times\mathbf{E} &= -\frac{\partial E_y}{\partial z}\mathbf{a}_x = -\frac{\omega}{c}A\sin\omega\left(t-\frac{z}{c}\right) \end{aligned}\right\}$$

$$\rightarrow \mathbf{H} = \frac{\omega A}{c\mu_0}\int\sin\omega\left(t-\frac{z}{c}\right)\mathbf{a}_x$$

$$= -\frac{A}{c\mu_0}\cos\omega\left(t-\frac{z}{c}\right)\mathbf{a}_x$$

Ex. 1 Maxwell's Equations in Point Form (3)

Given an electric field $\mathbf{E} = A\cos\omega(t - z/c)\mathbf{a}_y$. Determine the time-dependent MFI $\mathbf{H}$ in free space?



$$\mathbf{H} = \frac{-A}{c\mu_0}\cos\omega\left(t - \frac{z}{c}\right)\mathbf{a}_x$$

Ex. 2 Maxwell's Equations in Point Form (4)

$$\boxed{\text{Find } \mathbf{E} \text{ if } \mathbf{B} = \begin{vmatrix} B_0 \cos(\omega t + \alpha)\mathbf{a}_z & (\rho \le a) \\ 0 & (\rho > a) \end{vmatrix}}$$

$$\left.\begin{array}{r} \oint \mathbf{E}.d\mathbf{L} = \int_S (\nabla \times \mathbf{E}).d\mathbf{S} \\ \mathbf{E} = E(\rho)\mathbf{a}_\varphi \end{array}\right\} \rightarrow \oint \mathbf{E}.d\mathbf{L} = 2\pi\rho E$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = \omega B \sin(\omega t + \alpha)\mathbf{a}_z$$

$$\rightarrow \int_S (\nabla \times \mathbf{E}).d\mathbf{S} = \omega B \sin(\omega t + \alpha) \int_S dS$$

$$\rho \le a \rightarrow \omega B \sin(\omega t + \alpha) \int_S dS = \omega B \sin(\omega t + \alpha)(\pi \rho^2)$$

$$\rho > a \rightarrow \omega B \sin(\omega t + \alpha) \int_S dS = \omega B \sin(\omega t + \alpha) \int_{S,\rho \le a} dS + 0 \int_{S,\rho > a} dS$$

$$= \omega B \sin(\omega t + \alpha)(\pi a^2)$$

Ex. 2 Maxwell's Equations in Point Form (5)

$$\text{Find } \mathbf{E} \text{ if } \mathbf{B} = \begin{vmatrix} B_0 \cos(\omega t + \alpha)\mathbf{a}_z & (\rho \le a) \\ 0 & (\rho > a) \end{vmatrix}$$

$$\left.\begin{aligned} &\oint \mathbf{E}.d\mathbf{L} = \int_S (\nabla \times \mathbf{E}).d\mathbf{S} \\ &\oint \mathbf{E}.d\mathbf{L} = 2\pi\rho E \\ &\int_S (\nabla \times \mathbf{E}).d\mathbf{S} = \begin{vmatrix} \omega B \sin(\omega t + \alpha)(\pi\rho^2) & (\rho \le a) \\ \omega B \sin(\omega t + \alpha)(\pi a^2) & (\rho > a) \end{vmatrix} \end{aligned}\right\}$$

$$\rightarrow 2\pi\rho E = \begin{vmatrix} \omega B \sin(\omega t + \alpha)(\pi\rho^2) & (\rho \le a) \\ \omega B \sin(\omega t + \alpha)(\pi a^2) & (\rho > a) \end{vmatrix}$$

$$\rightarrow \mathbf{E} = \begin{vmatrix} \frac{1}{2}\omega B \rho \sin(\omega t + \alpha)\mathbf{a}_\varphi & (\rho \le a) \\ \frac{1}{2}\omega B \frac{a^2}{\rho} \sin(\omega t + \alpha)\mathbf{a}_\varphi & (\rho > a) \end{vmatrix}$$

Ex. 3 Maxwell's Equations in Point Form (6)

Given a magnetic field in free space where there is neither current density nor charge, $\mathbf{B} = A\sin(\omega t - nx)\mathbf{a}_x + Ank\cos(\omega t - nx)\mathbf{a}_y$ (T) where A, n, & ω are constants. Use a Maxwell equation to derive the time-dependent part of $\mathbf{E}$?

$$\left.\begin{array}{r}\nabla\times\mathbf{H} = \mathbf{J} + \dfrac{\partial\mathbf{D}}{\partial t}\\ \mathbf{J} = 0\end{array}\right\} \rightarrow \nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} \rightarrow \frac{1}{\mu_0}\nabla\times\mathbf{B} = \varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \rightarrow \nabla\times\mathbf{B} = \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\nabla\times\mathbf{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\mathbf{a}_z$$

$$= An^2k\sin(\omega t - nx)\mathbf{a}_z$$

$$\rightarrow An^2k\sin(\omega t - nx)\mathbf{a}_z = \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\rightarrow \mathbf{E} = \frac{An^2k}{\mu_0\varepsilon_0}\mathbf{a}_z\int_0^t \sin(\omega t - nx)dt = \frac{An^2k}{\mu_0\varepsilon_0\omega}\cos(\omega t - nx)\mathbf{a}_z$$

Time – Varying Fields & Maxwell's Equations

1. Faraday's Law
2. Displacement Current
3. Maxwell's Equations in Point Form
4. **Maxwell's Equations in Integral Form**
5. The Retarded Potentials

Maxwell's Equations in Integral Form (1)

$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint \mathbf{E}.d\mathbf{L} = -\int_S \dfrac{\partial \mathbf{B}}{\partial t}.d\mathbf{S}$	$E_{t1} = E_{t2}$
$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint \mathbf{H}.d\mathbf{L} = I + \int_S \dfrac{\partial \mathbf{D}}{\partial t}.d\mathbf{S}$	$H_{t1} = H_{t2}$
$\nabla .\mathbf{D} = \rho_v$	$\oint_S \mathbf{D}.d\mathbf{S} = \int_V \rho_v dv$	$D_{N1} - D_{N2} = \rho_S$
$\nabla .\mathbf{B} = 0$	$\oint_S \mathbf{B}.d\mathbf{S} = 0$	$B_{N1} = B_{N2}$

Ex. Maxwell's Equations in Integral Form (2)

Find **E** given an magnetic field $\mathbf{B} = B_0 e^{bt}\mathbf{a}_z$?

$$\left.\begin{aligned} &\oint \mathbf{E}.d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S} \\ &\mathbf{E} = E(\rho)\mathbf{a}_\varphi \end{aligned}\right\}$$

$$\rightarrow \mathbf{E}.\oint d\mathbf{L} = -\frac{\partial \mathbf{B}}{\partial t}.\int_S d\mathbf{S}$$

$$\rightarrow E(2\pi\rho) = -bB_0 e^{bt}(\pi\rho^2)$$

$$\rightarrow E = -\frac{1}{2} bB_0 e^{bt}\pi\rho$$

$$\rightarrow \boxed{\mathbf{E} = -\frac{1}{2} bB_0 e^{bt}\pi\rho\mathbf{a}_\varphi}$$

Time – Varying Fields & Maxwell's Equations

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The Retarded Potentials (1)

$$\left.\begin{array}{r}\mathbf{E} = -\nabla V \rightarrow \nabla\times\mathbf{E} = -\nabla\times(\nabla V) \\ 0 = \nabla\times(\nabla V)\end{array}\right\} \rightarrow \left.\begin{array}{r}\nabla\times\mathbf{E} = 0 \\ \nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}\end{array}\right\} \rightarrow \underset{\text{(unreasonable)}}{\dfrac{\partial\mathbf{B}}{\partial t} = 0}$$

$$\left.\begin{array}{r}\mathbf{E} = -\nabla V + \mathbf{N} \rightarrow \nabla\times\mathbf{E} = -\nabla\times(\nabla V) + \nabla\times\mathbf{N} \\ \nabla\times(\nabla V) = 0 \\ \nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}\end{array}\right\} \rightarrow \left.\begin{array}{r}\nabla\times\mathbf{N} = -\dfrac{\partial\mathbf{B}}{\partial t} \\ \mathbf{B} = \nabla\times\mathbf{A}\end{array}\right\}$$

$$\rightarrow \nabla\times\mathbf{N} = -\frac{\partial}{\partial t}(\nabla\times\mathbf{A}) \rightarrow \nabla\times\mathbf{N} = -\nabla\times\frac{\partial\mathbf{A}}{\partial t} \rightarrow \mathbf{N} = -\frac{\partial\mathbf{A}}{\partial t}$$

$$\rightarrow \boxed{\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t}}$$

The Retarded Potentials (2)

$$\left.\begin{aligned}\mathbf{B} &= \nabla\times\mathbf{A}\\ \mathbf{E} &= -\nabla V - \frac{\partial\mathbf{A}}{\partial t}\\ \nabla\times\mathbf{H} &= \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\\ \nabla\mathbf{.D} &= \rho_v\end{aligned}\right\}$$

$$\rightarrow\begin{cases}\dfrac{1}{\mu}\nabla\times\nabla\times\mathbf{A} = \mathbf{J} + \varepsilon\left(-\nabla\dfrac{\partial V}{\partial t} - \dfrac{\partial^2\mathbf{A}}{\partial t^2}\right)\\ \varepsilon\left(-\nabla.\nabla V - \dfrac{\partial}{\partial t}\nabla\mathbf{.A}\right) = \rho_v\end{cases}$$

The Retarded Potentials (3)

$$\begin{cases} \dfrac{1}{\mu}\nabla\times\nabla\times\mathbf{A} = \mathbf{J} + \varepsilon\left(-\nabla\dfrac{\partial V}{\partial t} - \dfrac{\partial^2\mathbf{A}}{\partial t^2}\right) \\ \varepsilon\left(-\nabla.\nabla V - \dfrac{\partial}{\partial t}\nabla.\mathbf{A}\right) = \rho_v \end{cases}$$

$$\rightarrow \begin{cases} \nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A} = \mu\mathbf{J} - \mu\varepsilon\left(\nabla\dfrac{\partial V}{\partial t} + \dfrac{\partial^2\mathbf{A}}{\partial t^2}\right) \\ \nabla^2 V + \dfrac{\partial}{\partial t}(\nabla.\mathbf{A}) = -\dfrac{\rho_v}{\varepsilon} \end{cases}$$

The Retarded Potentials (4)

$$\left.\begin{cases} \nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A} = \mu\mathbf{J} - \mu\varepsilon\left(\nabla\dfrac{\partial V}{\partial t} + \dfrac{\partial^2\mathbf{A}}{\partial t^2}\right) \\ \nabla^2 V + \dfrac{\partial}{\partial t}(\nabla.\mathbf{A}) = -\dfrac{\rho_v}{\varepsilon} \end{cases} \\ \text{Define } \nabla.\mathbf{A} = -\mu\varepsilon\dfrac{\partial V}{\partial t} \right\}$$

$$\rightarrow \begin{cases} \nabla^2\mathbf{A} = -\mu\mathbf{J} + \mu\varepsilon\dfrac{\partial^2\mathbf{A}}{\partial t^2} \\ \nabla^2 V = -\dfrac{\rho_v}{\varepsilon} + \mu\varepsilon\dfrac{\partial^2 V}{\partial t^2} \end{cases}$$

The Retarded Potentials (5)

$$\left.\begin{aligned} V &= \int_V \frac{\rho_v dv}{4\pi\varepsilon R} \\ t' &= t - \frac{R}{v} \end{aligned}\right\} \to V = \int_V \frac{[\rho_v]dv}{4\pi\varepsilon R}$$

$$\text{Ex}: \quad \rho_v = e^{-r}\cos\omega t \to [\rho_v] = e^{-r}\cos\left[\omega\left(t - \frac{R}{v}\right)\right]$$

$$\mathbf{A} = \int_V \frac{\mu \mathbf{J}}{4\pi R} dv \to \boxed{\mathbf{A} = \int_V \frac{\mu[\mathbf{J}]}{4\pi R} dv}$$

$$Q \longrightarrow \mathbf{F} = \frac{Q_1 Q_2}{4\pi\varepsilon R^2}\mathbf{a}_R \longrightarrow \mathbf{E} = \frac{Q}{4\pi\varepsilon R^2}\mathbf{a}_R \longrightarrow \mathbf{D} = \varepsilon\mathbf{E}$$

$$\mathbf{F} \downarrow \quad W = -Q\int \mathbf{E}.d\mathbf{L} \longrightarrow V = -\int \mathbf{E}.d\mathbf{L} \longrightarrow C = \frac{Q}{V}$$

$$Q \downarrow \quad I = \frac{dQ}{dt} \longrightarrow R = \frac{V}{I}$$

$$\boxed{\begin{array}{l} \nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}; \ \nabla\times\mathbf{H} = \mathbf{J} + \dfrac{\partial\mathbf{D}}{\partial t} \\ \nabla.\mathbf{D} = \rho_v; \ \nabla.\mathbf{B} = 0 \end{array}}$$

$$I \downarrow \quad \mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\varphi \longrightarrow \mathbf{B} = \mu\mathbf{H} \longrightarrow \Phi = \int \mathbf{B}.d\mathbf{S} \longrightarrow L = \frac{\Phi}{I}$$

$$\mathbf{H} \downarrow \quad V_{m,ab} = -\int_b^a \mathbf{H}.d\mathbf{L}$$

$$\mathbf{B} = \mu\mathbf{H} \longrightarrow \mathbf{B} = \nabla\times\mathbf{A}$$

$$\Phi \downarrow \quad \text{emf} = -\frac{d\Phi}{dt} \qquad \Phi \longrightarrow M_{12} = \frac{N_2\Phi_{12}}{I_1}$$

$$\mathbf{B} \downarrow \quad \mathbf{F} = -I\oint \mathbf{B}\times d\mathbf{L} \longrightarrow \mathbf{T} = \mathbf{R}\times\mathbf{F}$$