Nguyễn Công Phương

Engineering Electromagnetics

Energy and Potential

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Energy & Potential

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Moving a Point Charge in an Electric Field (1)

- Moving a charge Q a distance $d\mathbf{L}$ in an $\mathbf{E}$, the force on Q arising from the electric field:

$$\mathbf{F}_E = Q\mathbf{E}$$

- The component in the direction $d\mathbf{L}$:

$$F_{EL} = \mathbf{F}.\mathbf{a}_L = Q\mathbf{E}.\mathbf{a}_L$$

- $\mathbf{a}_L$: a unit vector in the direction of $d\mathbf{L}$
- $\rightarrow$ the force must be applied:

$$F_{eff} = -\,Q\mathbf{E}.\mathbf{a}_L$$

- The expenditure of energy:

$$dW = -\,Q\mathbf{E}.\mathbf{a}_L dL = -\,Q\mathbf{E}.d\mathbf{L}$$

Moving a Point Charge in an Electric Field (2)

- The expenditure of energy required to move Q in $\mathbf{E}$:

$$dW = -Q\mathbf{E}.d\mathbf{L}$$

- $dW = 0$ if:
  - $Q = 0$, $\mathbf{E} = 0$, $d\mathbf{L} = 0$, or
  - $\mathbf{E}$ is perpendicular to $d\mathbf{L}$
- The work needed to move the charge a finite distance:

$$\boxed{W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E}.d\mathbf{L}}$$

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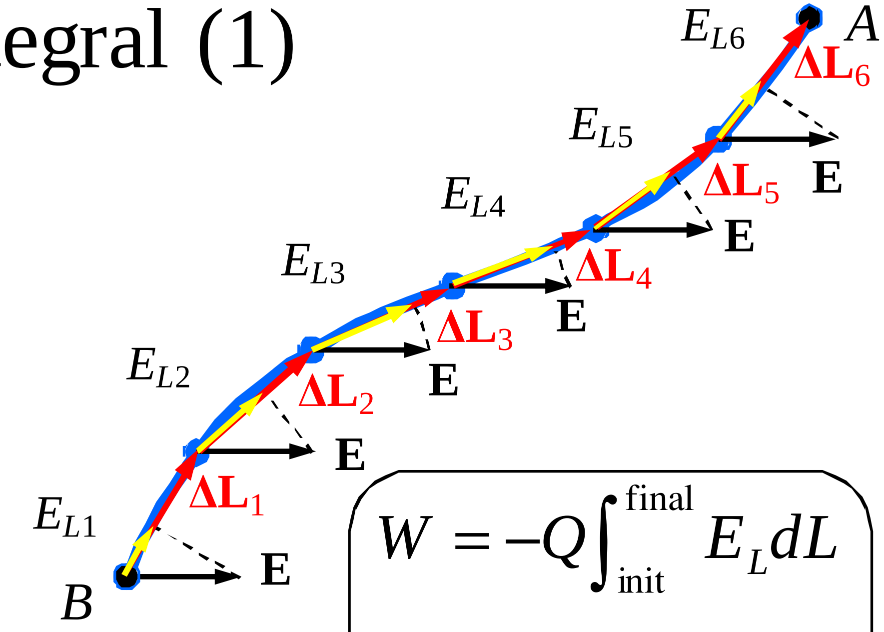
The Line Integral (1)



$$W = -Q\int_{\text{init}}^{\text{final}} E_L dL = -Q\mathbf{E}.\mathbf{L}_{BA} \quad \text{(uniform } \mathbf{E}\text{)}$$

$$W = dW_1 + dW_2 + \ldots + dW_6$$

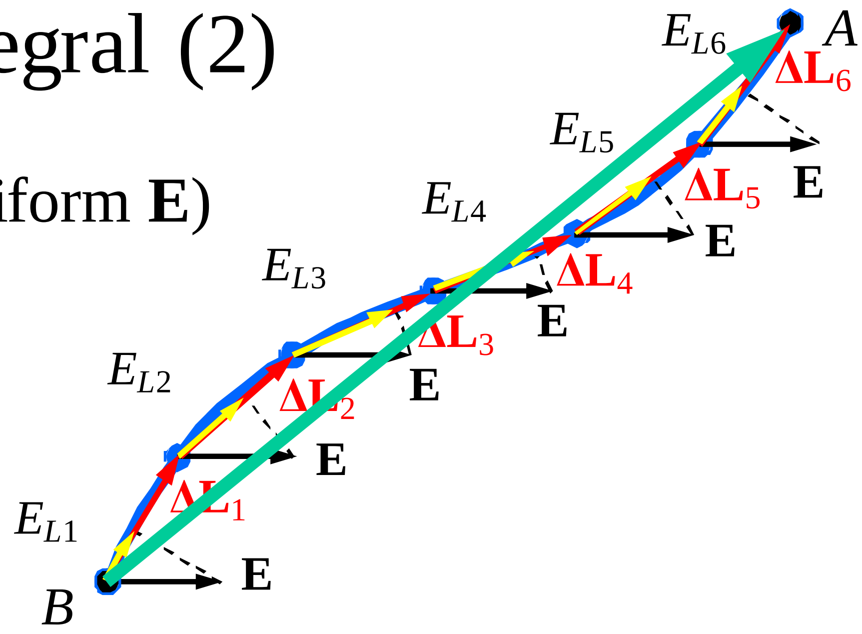
$$= -QE_{L1}.\Delta L_1 - QE_{L2}.\Delta L_2 - \ldots - QE_{L6}.\Delta L_6$$

$$\left.\begin{aligned} &= -Q\mathbf{E}_1.\Delta\mathbf{L}_1 - Q\mathbf{E}_2.\Delta\mathbf{L}_2 - \ldots - Q\mathbf{E}_6.\Delta\mathbf{L}_6 \\ &\mathbf{E}_1 = \mathbf{E}_2 = \ldots = \mathbf{E}_6 = \mathbf{E} \end{aligned}\right\}$$

$$\left.\begin{aligned} &\rightarrow W = -Q\mathbf{E}.(\Delta\mathbf{L}_1 + \Delta\mathbf{L}_2 + \ldots + \Delta\mathbf{L}_6) \\ &\Delta\mathbf{L}_1 + \Delta\mathbf{L}_2 + \ldots + \Delta\mathbf{L}_6 = \mathbf{L}_{BA} \end{aligned}\right\} \rightarrow W = -Q\mathbf{E}.\mathbf{L}_{BA} \quad \text{(uniform } \mathbf{E}\text{)}$$

The Line Integral (2)

$$W = -Q\int_{\text{init}}^{\text{final}} E_L dL = -Q\mathbf{E}.\mathbf{L}_{BA} \qquad \text{(uniform } \mathbf{E}\text{)}$$



$$\left.\begin{array}{l} W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E}.d\mathbf{L} \\ \text{Uniform } \mathbf{E} \end{array}\right\}$$

$$\rightarrow W = -Q\mathbf{E}.\int_B^A d\mathbf{L} = -Q\mathbf{E}.\mathbf{L}_{BA}$$

The Line Integral (3)

Ex. 1

Given $\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y + z\mathbf{a}_z$ V/m. Find the work needed in carrying 2 C from $B(1; 0; 1)$ to $A(0.8; 0.6; 1)$ along:
a) the shorter arc of the circle $x^2 + y^2 = 1$, $z = 1$; b) the straight-line path from B to A

$$\left.\begin{aligned} W &= -Q\int_B^A \mathbf{E}.d\mathbf{L} \\ d\mathbf{L} &= dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z \end{aligned}\right\}$$

$$\to W = -2\int_B^A (y\mathbf{a}_x + x\mathbf{a}_y + z\mathbf{a}_z).(dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z)$$

$$= -2\int_{x=1}^{x=0.8} y\,dx - 2\int_{y=0}^{y=0.6} x\,dy - 2\int_1^1 z\,dz$$

$$= -2\int_{x=1}^{x=0.8} \sqrt{1-x^2}\,dx - 2\int_{y=0}^{y=0.6} \sqrt{1-y^2}\,dy - 0$$

$$= -\left[x\sqrt{1-x^2} + \sin^{-1}x\right]_1^{0.8} - \left[y\sqrt{1-y^2} + \sin^{-1}y\right]_0^{0.6} \boxed{= -0.96 \text{ J}}$$

The Line Integral (4)

Ex. 1

Given $\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y + z\mathbf{a}_z$ V/m. Find the work needed in carrying 2 C from $B(1; 0; 1)$ to $A(0.8; 0.6; 1)$ along:
a)the shorter arc of the circle $x^2 + y^2 = 1$, $z = 1$; b)the straight-line path from B to A

$$\left.\begin{aligned} W &= -Q\int_B^A \mathbf{E}.d\mathbf{L} \\ d\mathbf{L} &= dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z \end{aligned}\right\}$$

$$\to W = -2\int_B^A (y\mathbf{a}_x + x\mathbf{a}_y + z\mathbf{a}_z).(dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z)$$

$$\left.\begin{aligned} &= -2\int_{x=1}^{x=0.8} y\,dx - 2\int_{y=0}^{y=0.6} x\,dy - 2\int_1^1 z\,dz \\ &y - y_B = \frac{y_A - y_B}{x_A - x_B}(x - x_B) \to y = -3(x-1) \end{aligned}\right\}$$

$$\to W = 6\int_{x=1}^{x=0.8} (x-1)dx - 2\int_{y=0}^{y=0.6}\left(1 - \frac{y}{3}\right)dy - 0 \; \boxed{= -0.96 \text{ J}}$$

The Line Integral (5)

$$d\mathbf{L} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z \quad \text{(Descartes)}$$

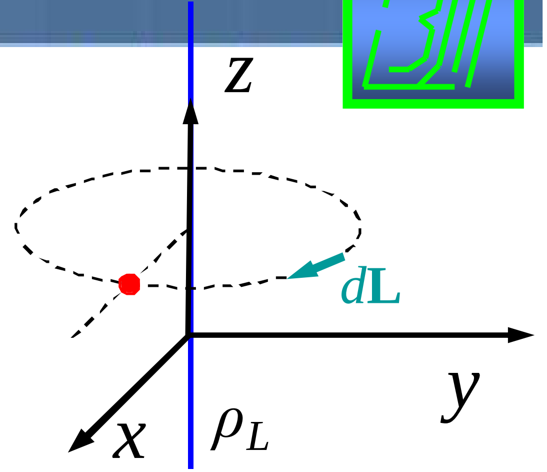
$$d\mathbf{L} = d\rho\mathbf{a}_\rho + \rho d\varphi\mathbf{a}_\varphi + dz\mathbf{a}_z \quad \text{(Cylindrical)}$$

$$d\mathbf{L} = dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta d\varphi\mathbf{a}_\varphi \quad \text{(Spherical)}$$

The Line Integral (6)

Ex. 2

Find the work needed in carrying the charge Q about a circular path centered at the line charged.



$$\left.\begin{array}{l} W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E}.d\mathbf{L} \\ \mathbf{E} = \dfrac{\rho_L}{2\pi\varepsilon_0\rho}\mathbf{a}_\rho \\ d\mathbf{L} = d\rho\mathbf{a}_\rho + \rho d\varphi\mathbf{a}_\varphi + dz\mathbf{a}_z \\ d\rho = 0 \\ dz = 0 \end{array}\right\} \rightarrow W = -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\varepsilon_0\rho}\mathbf{a}_\rho.\rho d\varphi\mathbf{a}_\varphi$$

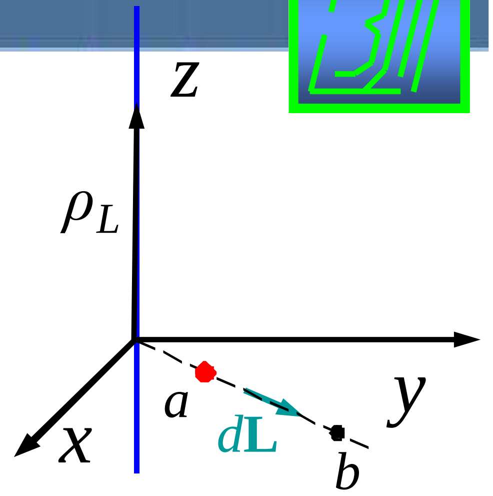
$$\left.\begin{array}{l} = -Q\displaystyle\int_0^{2\pi} \frac{\rho_L}{2\pi\varepsilon_0} d\varphi\mathbf{a}_\rho.\mathbf{a}_\varphi \\ \mathbf{a}_\rho.\mathbf{a}_\varphi = 1\times 1\times\cos 90^{\circ} \end{array}\right\}$$

$$\rightarrow W = -Q\frac{\rho_L}{2\pi\varepsilon_0}\int_0^{2\pi}\cos 90^{\circ}\, d\varphi \boxed{= 0}$$

The Line Integral (7)

Ex. 3

Find the work done in carrying a charge Q from $\rho = a$ to $\rho = b$.



$$W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E}.d\mathbf{L}$$

$$\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\mathbf{a}_\rho$$

$$d\mathbf{L} = d\rho\mathbf{a}_\rho + \rho d\varphi\mathbf{a}_\varphi + dz\mathbf{a}_z$$

$$d\varphi = 0$$

$$dz = 0$$

$$\to W = -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\varepsilon_0\rho}\mathbf{a}_\rho.d\rho\mathbf{a}_\rho$$

$$= -Q\int_a^b \frac{\rho_L}{2\pi\varepsilon_0}\frac{d\rho}{\rho}$$

$$= -\frac{Q\rho_L}{2\pi\varepsilon_0}\ln\frac{b}{a}$$

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Potential Difference & Potential (1)

$$W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E}.d\mathbf{L}$$

- *Potential difference V*: work done in moving a unit positive charge from one point to another in an electric field:

$$\boxed{\text{Potential difference} = V = -\int_{\text{init}}^{\text{final}} \mathbf{E}.d\mathbf{L}}$$

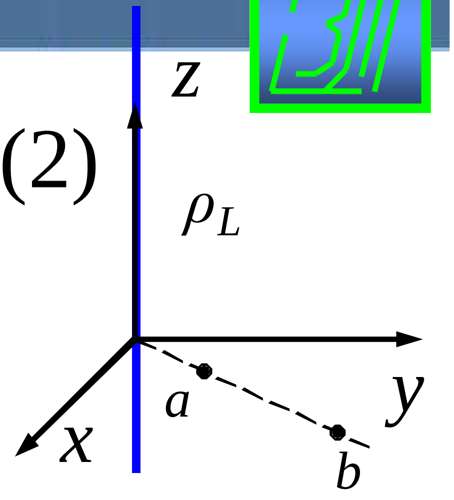
- Potential difference between points *A* & *B*:

$$V_{AB} = -\int_{B}^{A} \mathbf{E}.d\mathbf{L}$$

- Unit: volt (V, J/C)

Potential Difference & Potential (2)

Ex.



Find the potential difference between $\rho = a$ & $\rho = b$.

Work done in carrying Q from a to b:

$$W = -\frac{Q\rho_L}{2\pi\varepsilon_0}\ln\frac{b}{a}$$

→ work done in carrying Q from b to a:

$$\left.\begin{aligned} W &= \frac{Q\rho_L}{2\pi\varepsilon_0}\ln\frac{b}{a} \\ V_{ab} &= \frac{W}{Q} \end{aligned}\right\} \rightarrow V_{ab} = \frac{\rho_L}{2\pi\varepsilon_0}\ln\frac{b}{a}$$

Potential Difference & Potential (3)

- Potential difference between points A & B: $V_{AB} = -\int_B^A \mathbf{E}.d\mathbf{L}$
- No B?
- → potential (absolute potential) at A
- → still need a reference point:
  - "ground"
  - Infinity
- If the potential at A is V_A & that at B is V_B, then:

$$V_{AB} = V_A - V_B$$

- (provided V_A & V_B have the same zero reference point)

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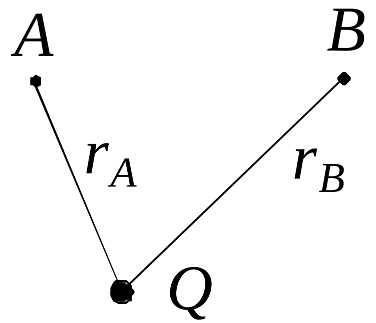
The Potential Field of a Point Charge (1)



$$\left.\begin{aligned} V_{AB} &= -\int_B^A \mathbf{E}.d\mathbf{L} \\ \mathbf{E} &= \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r \\ d\mathbf{L} &= dr\mathbf{a}_r \end{aligned}\right\} \rightarrow V_{AB} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2} dr$$

$$\left.\begin{aligned} &= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) \\ &r_B \rightarrow \infty \end{aligned}\right\} \rightarrow V_A = \frac{Q}{4\pi\varepsilon_0 r_A}$$

$$\boxed{V = \frac{Q}{4\pi\varepsilon_0 r}}$$

(Potential field of a point charge)

The Potential Field of a Point Charge (2)

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

- The potential at any point distant r from a point charge Q
- The zero reference is the potential at infinite radius
- $Q/4\pi\varepsilon_0 r$ (J) must be done in carrying a 1-C charge from infinity to any point r meters from the charge Q
- If $\dfrac{Q}{4\pi\varepsilon_0 r_B} = C_1 \rightarrow V = \dfrac{Q}{4\pi\varepsilon_0 r} + C_1$
- The potential difference does not depend on C_1

The Potential Field of a Point Charge (3)

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

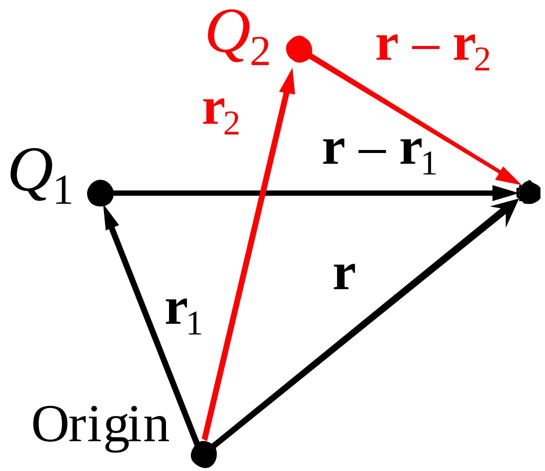
- The potential field of a point charge
- A scalar field, & no unit vector
- *Equipotential surface*: a surface composed of all those points having the same value of potential
- No work is required in moving a charge around *on* an equipotential surface
- The equipotential surfaces in the potential field of a point charge are spheres centered at the point charge

Energy & Potential

The Potential Field of a System of Charges (1)

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}_1|}$$

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}_1|} + \frac{Q_2}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}_2|}$$



$$\left.\begin{aligned} V(\mathbf{r}) &= \frac{Q_1}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}_1|} + \frac{Q_2}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}_2|} + \ldots + \frac{Q_n}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}_n|} = \sum_{m=1}^{n} \frac{Q_m}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}_m|} \\ Q_m &= \rho_v \Delta v_m \end{aligned}\right\}$$

$$\rightarrow V(\mathbf{r}) = \frac{\rho_v(\mathbf{r}_1)\Delta v_1}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}_1|} + \frac{\rho_v(\mathbf{r}_2)\Delta v_2}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}_2|} + \ldots + \frac{\rho_v(\mathbf{r}_n)\Delta v_n}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}_n|}$$

The Potential Field of a System of Charges (2)

$$\left.\begin{array}{r} V(\mathbf{r}) = \dfrac{\rho_v(\mathbf{r}_1)\Delta v_1}{4\pi\varepsilon_0 \left|\mathbf{r}-\mathbf{r}_1\right|} + \dfrac{\rho_v(\mathbf{r}_2)\Delta v_2}{4\pi\varepsilon_0 \left|\mathbf{r}-\mathbf{r}_2\right|} + \ldots + \dfrac{\rho_v(\mathbf{r}_n)\Delta v_n}{4\pi\varepsilon_0 \left|\mathbf{r}-\mathbf{r}_n\right|} \\ n \to \infty \end{array}\right\}$$

$$\to V(\mathbf{r}) = \int_V \frac{\rho_v(\mathbf{r}')dv'}{4\pi\varepsilon_0 \left|\mathbf{r}-\mathbf{r}'\right|}$$

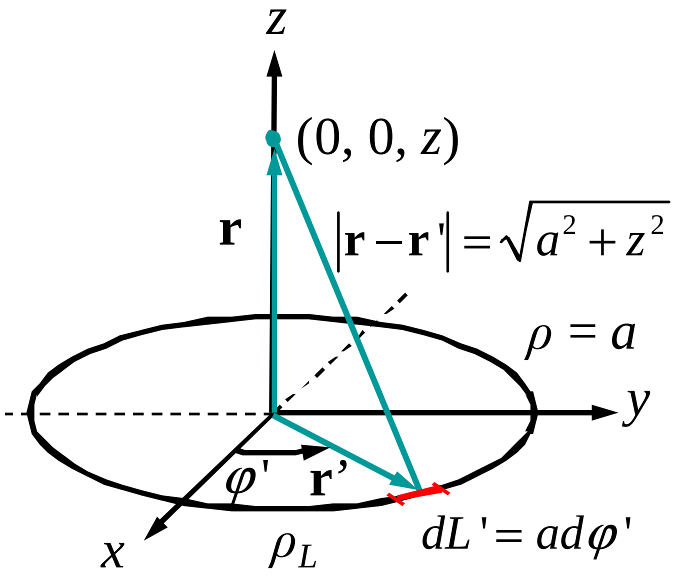
$$V(\mathbf{r}) = \int \frac{\rho_L(\mathbf{r}')dL'}{4\pi\varepsilon_0 \left|\mathbf{r}-\mathbf{r}'\right|}$$

$$V(\mathbf{r}) = \int_S \frac{\rho_S(\mathbf{r}')dS'}{4\pi\varepsilon_0 \left|\mathbf{r}-\mathbf{r}'\right|}$$

The Potential Field of a System of Charges (3)

Ex. 1

Find the potential on the z axis.



$$V(\mathbf{r}) = \int \frac{\rho_L(\mathbf{r}')dL'}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|}$$

$$dL' = ad\varphi'$$

$$\left.\begin{array}{l}\mathbf{r} = z\mathbf{a}_z \\ \mathbf{r}' = a\mathbf{a}_\rho\end{array}\right\} \rightarrow |\mathbf{r}-\mathbf{r}'| = \sqrt{a^2+z^2}$$

$$\rightarrow V(\mathbf{r}) = \int_0^{2\pi} \frac{\rho_L a d\varphi'}{4\pi\varepsilon_0\sqrt{a^2+z^2}} = \frac{\rho_L a}{2\varepsilon_0\sqrt{a^2+z^2}}$$

The Potential Field of a System of Charges (4)

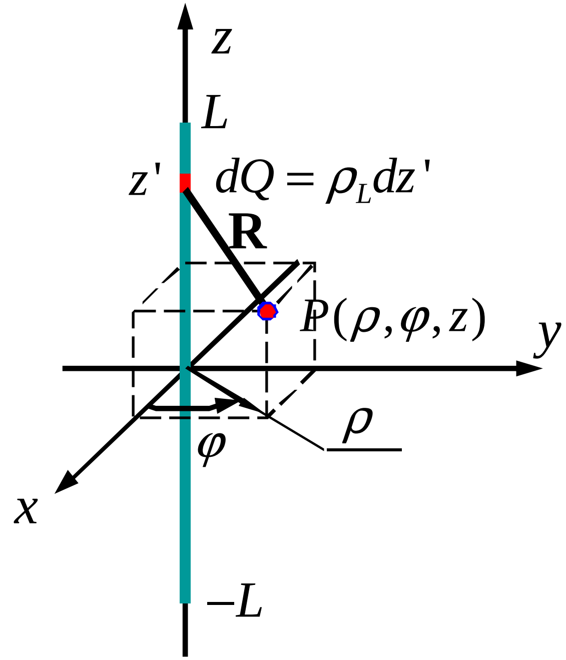
For a zero reference at infinity, then:

- The potential due to a single point charge: the work done in carrying a unit positive charge from infinity to the point at which we desire the potential, this work does not depend on the path chosen between these two points
- The potential field due to a number of point charges is the sum of the individual potential fields due to each charge
- The expression for potential: $V_A = -\int_{\infty}^{A} \mathbf{E}.d\mathbf{L}$
- The potential difference: $V_{AB} = V_A - V_B = -\int_{B}^{A} \mathbf{E}.d\mathbf{L}$
- For a static field: $\boxed{\oint \mathbf{E}.d\mathbf{L} = 0}$

The Potential Field of a System of Charges (5)

Ex. 2

Investigate the uniform line charge density ρ_L of finite length $2L$ centered on the z axis.



$$V_{point\ charge} = \frac{Q}{4\pi\varepsilon_0 r}$$

$$\rightarrow dV = \frac{dQ}{4\pi\varepsilon_0 R} = \frac{\rho_L dz'}{4\pi\varepsilon_0 \sqrt{\rho^2 + (z - z')^2}}$$

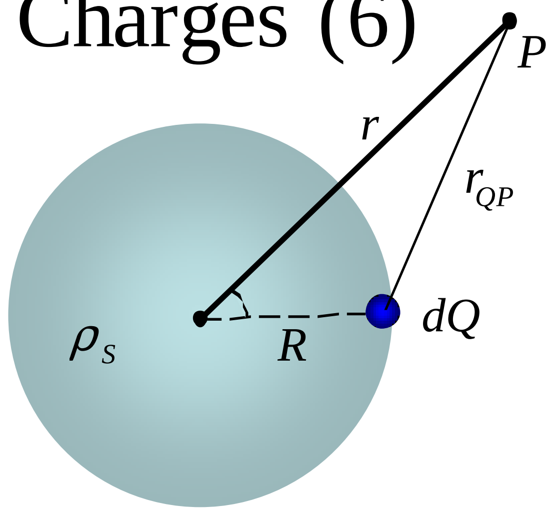
$$\rightarrow V = \int_{-L}^{L} \frac{\rho_L dz'}{4\pi\varepsilon_0 \sqrt{\rho^2 + (z - z')^2}}$$

$$= -\frac{\rho_L}{4\pi\varepsilon_0} \ln\left(\frac{z - L + \sqrt{\rho^2 + (z - L)^2}}{z + L + \sqrt{\rho^2 + (z - L)^2}}\right)$$

The Potential Field of a System of Charges (6)

Ex. 3

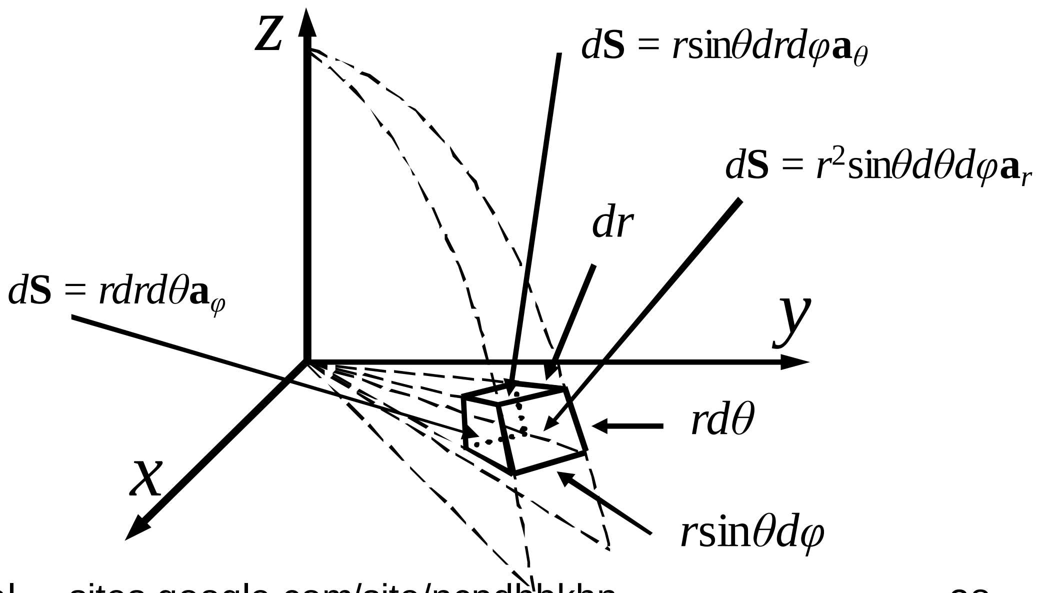
Investigate a sphere of radius *R* has a uniform surface charge density ρ_S.



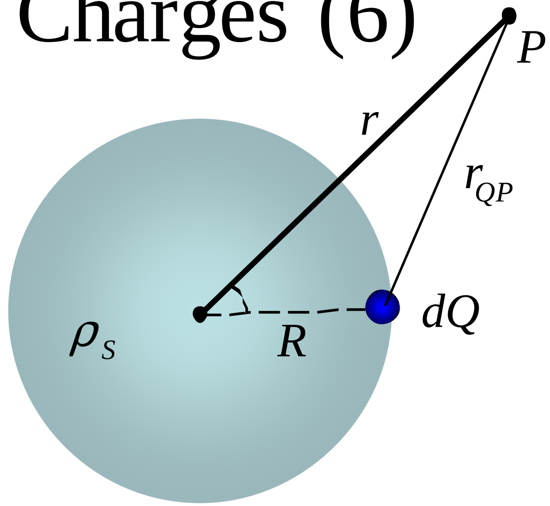
$$dQ = \rho_S dS = \rho_S R^2 \sin\theta d\theta d\varphi$$

$$\rightarrow dV = \frac{dQ}{4\pi\varepsilon_0 r_{QP}} = \frac{\rho_S R^2 \sin\theta d\theta d\varphi}{4\pi\varepsilon_0 r_{QP}}$$

$$r_{QP}^2 = R^2 + r^2 - 2rR\cos\varphi$$



The Potential Field of a System of Charges (6)



Ex. 3

Investigate a sphere of radius R has a uniform surface charge density ρ_S.

$$dQ = \rho_S dS = \rho_S R^2 \sin\theta d\theta d\varphi$$

$$\to dV = \frac{dQ}{4\pi\varepsilon_0 r_{QP}} = \frac{\rho_S R^2 \sin\theta d\theta d\varphi}{4\pi\varepsilon_0 r_{QP}}$$

$$r_{QP}^2 = R^2 + r^2 - 2rR\cos\theta \to 2r_{QP}dr_{QP} = 2rR\sin\theta d\theta \to r_{QP} = \frac{Rr\sin\theta d\theta}{dr_{QP}}$$

$$\to dV = \frac{\rho_S R dr_{QP} d\varphi}{4\pi\varepsilon_0 r}$$

$$\to V = \int_{r_{QP}=|r-R|}^{r+R}\int_{\varphi=0}^{2\pi}\frac{\rho_S R dr_{QP} d\varphi}{4\pi\varepsilon_0 r} = \begin{cases} \dfrac{\rho_S R^2}{\varepsilon_0 r}, r > R \\ \dfrac{\rho_S R}{\varepsilon_0}, r < R \end{cases}$$

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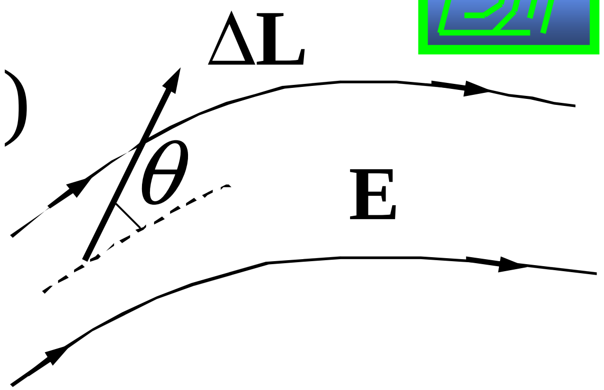
Potential Gradient (1)

- 2 methods to find potential: from electric field intensity & from charge distribution
- however **E** & $\rho_{v,\,S,\,L}$ are often not given
- → problem: finding EFI from potential
- solution: potential gradient

Potential Gradient (2)



$$V = -\int \mathbf{E}.d\mathbf{L}$$

$$\Delta V \doteq -\mathbf{E}.\Delta\mathbf{L}$$

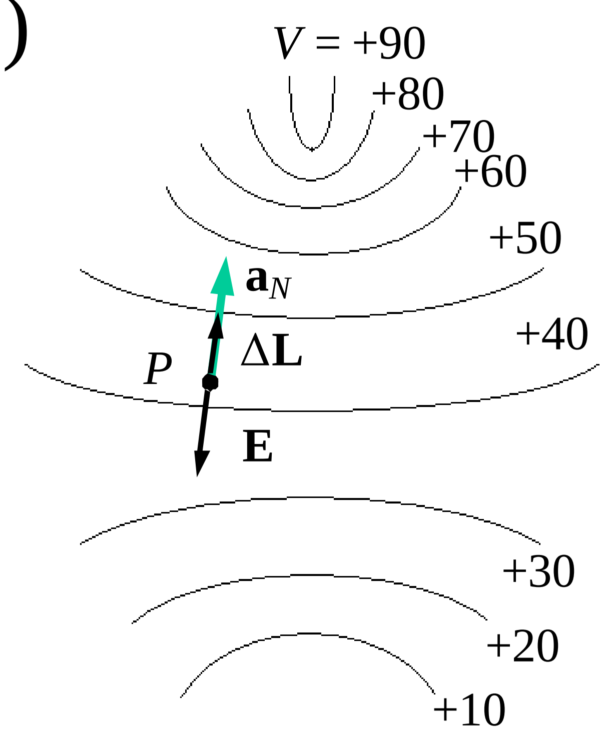
$$\Delta V \doteq -E\Delta L\cos\theta$$

$$\frac{dV}{dL} = -E\cos\theta$$

$$E = \left.\frac{dV}{dL}\right|_{\max} \qquad (\cos\theta = -1)$$

Potential Gradient (3)

$$E = \left.\frac{dV}{dL}\right|_{\max}$$



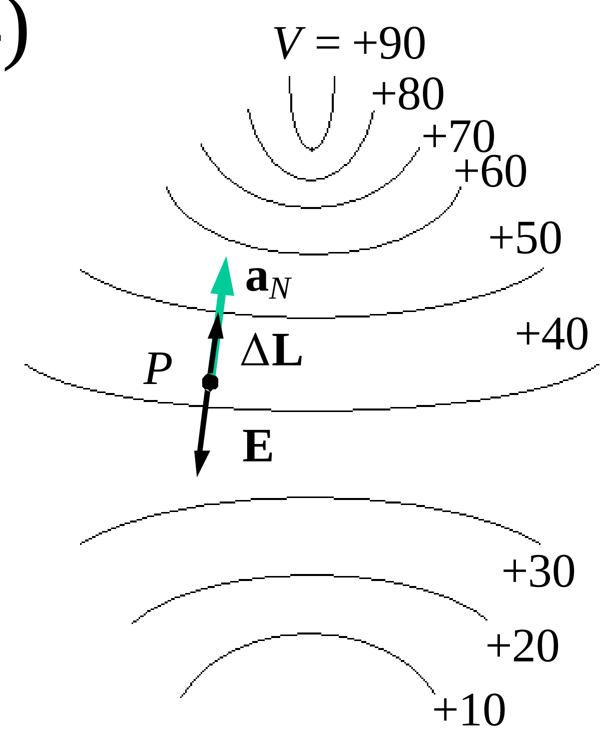
- The magnitude of **E** is given by the maximum value of the rate of change of potential with distance
- This maximum value is obtained when the direction of the distance increment is opposite to **E**, or, in other words, the direction of **E** is *opposite* to the direction in which the potential is *increasing* the most rapidly

$$\mathbf{E} = -\left(\left.\frac{dV}{dL}\right|_{\max}\right)\mathbf{a}_N$$

Potential Gradient (4)

$$\mathbf{E} = -\left(\left. \frac{dV}{dL} \right|_{\max} \right) \mathbf{a}_N$$

$$\left. \frac{dV}{dL} \right|_{\max} = \frac{dV}{dN} \quad \rightarrow \mathbf{E} = -\frac{dV}{dN} \mathbf{a}_N$$



$$\boxed{\text{Gradient of } T = \text{ grad } T = \frac{dT}{dN} \mathbf{a}_N}$$

$$\boxed{\mathbf{E} = -\text{grad } V}$$

Potential Gradient (5)

$$\mathbf{E} = -\,\text{grad}\ V$$

$$\left.\begin{aligned} V = V(x,y,z) \ \rightarrow dV &= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \\ dV = -\mathbf{E}.d\mathbf{L} &= -E_x dx - E_y dy - E_z dz \end{aligned}\right\} \rightarrow \begin{cases} E_x = -\dfrac{\partial V}{\partial x} \\ E_y = -\dfrac{\partial V}{\partial y} \\ E_z = -\dfrac{\partial V}{\partial z} \end{cases}$$

$$\rightarrow \mathbf{E} = -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right)$$

$$\rightarrow \boxed{\text{grad}\ V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z}$$

Potential Gradient (6)

$$\left.\begin{aligned}\text{grad } V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \\ \nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z \quad \to \nabla T = \frac{\partial T}{\partial x}\mathbf{a}_x + \frac{\partial T}{\partial y}\mathbf{a}_y + \frac{\partial T}{\partial z}\mathbf{a}_z\end{aligned}\right\}$$

$$\left.\begin{aligned}\to \nabla T = \text{grad } T \\ \mathbf{E} = -\text{grad } V\end{aligned}\right\}$$

$$\to \boxed{\mathbf{E} = -\nabla V}$$

Potential Gradient (7)

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \quad \text{(Descartes)}$$

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \varphi}\mathbf{a}_\varphi + \frac{\partial V}{\partial z}\mathbf{a}_z \quad \text{(Cylindrical)}$$

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\mathbf{a}_\varphi \quad \text{(Spherical)}$$

Potential Gradient (8)

Gradient: $$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

Divergence: $$\nabla .\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Potential Gradient (9)

Ex. 1

Find the gradient of each of the following functions:

$$a)\ f_1 = 2a^2y - 5y^3z$$

$$b)\ f_2 = 6\rho\sin\varphi + 4\rho z\cos 3\varphi$$

$$c)\ f_3 = \frac{1}{r} + 2r\sin\theta\cos\varphi$$

Potential Gradient (10)

Ex. 2

Given a potential field $V = x^2 - 10yz$ (V) & a point $P(1, 3, 1)$. Find several values at P: V_P, $\mathbf{E}_P$, the direction of $\mathbf{E}_P$, $\mathbf{D}_P$, & ρ_v.

$$V_P = 1^2 - 10 \times 3 \times 1 = -29 \text{ V}$$

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial x} \mathbf{a}_x + \frac{\partial V}{\partial y} \mathbf{a}_y + \frac{\partial V}{\partial z} \mathbf{a}_z \right) = -2x\mathbf{a}_x + 10z\mathbf{a}_y + 10y\mathbf{a}_z \text{ V/m}$$

$$\rightarrow \mathbf{E}_p = -2 \times 1\mathbf{a}_x + 10 \times 1\mathbf{a}_y + 10 \times 3\mathbf{a}_z = -2\mathbf{a}_x + 10\mathbf{a}_y + 30\mathbf{a}_z \text{ V/m}$$

$$\mathbf{a}_{E,P} = \frac{\mathbf{E}_p}{|\mathbf{E}_p|} = \frac{-2\mathbf{a}_x + 10\mathbf{a}_y + 30\mathbf{a}_z}{\sqrt{(-2)^2 + 10^2 + 30^2}} = -0.063\mathbf{a}_x + 0.32\mathbf{a}_y + 0.95\mathbf{a}_z$$

Potential Gradient (11)

Ex. 2

Given a potential field $V = x^2 - 10yz$ (V) & a point $P(1, 3, 1)$. Find several values at P: V_P, $\mathbf{E}_P$, the direction of $\mathbf{E}_P$, $\mathbf{D}_P$, & ρ_v.

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \quad = 8.854\times10^{-12}(-2x\mathbf{a}_x + 10z\mathbf{a}_y + 10y\mathbf{a}_z)$$

$$= -17.71x\mathbf{a}_x + 88.54z\mathbf{a}_y + 88.54y\mathbf{a}_z \ \text{ pC/m}^2$$

$$\rho_v = \nabla.\mathbf{D}$$

$$= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$= \frac{\partial(-17.71x)}{\partial x} + \frac{\partial(88.84z)}{\partial y} + \frac{\partial(88.84y)}{\partial z} \ = -17.71 \ \text{ pC/m}^3$$

Energy & Potential

1. Moving a Point Charge in an Electric Field
2. The Line Integral
3. Potential Difference & Potential
4. The Potential Field of a Point Charge
5. The Potential Field of a System of Charges
6. Potential Gradient
7. **The Dipole**
8. Energy Density in the Electrostatic Field

The Dipole (1)

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{R_2 - R_1}{R_1 R_2}$$

$$R_1 \doteq R_2$$

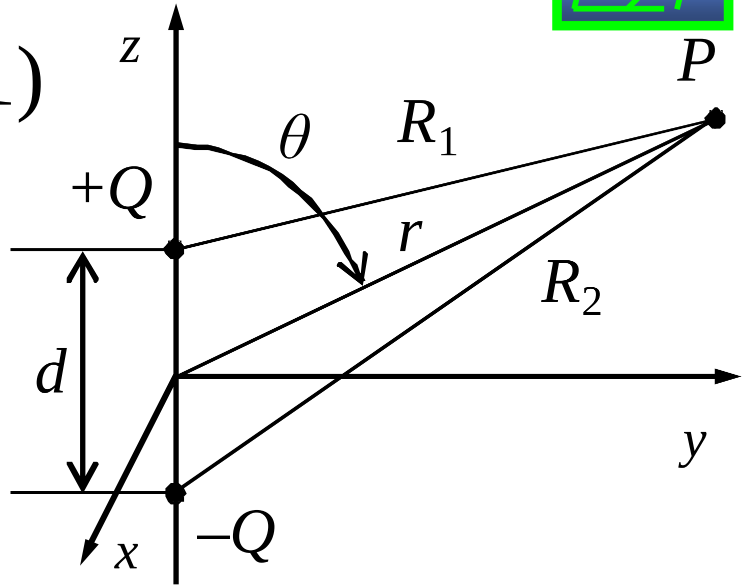
$$R_2 - R_1 \doteq d\cos\theta$$

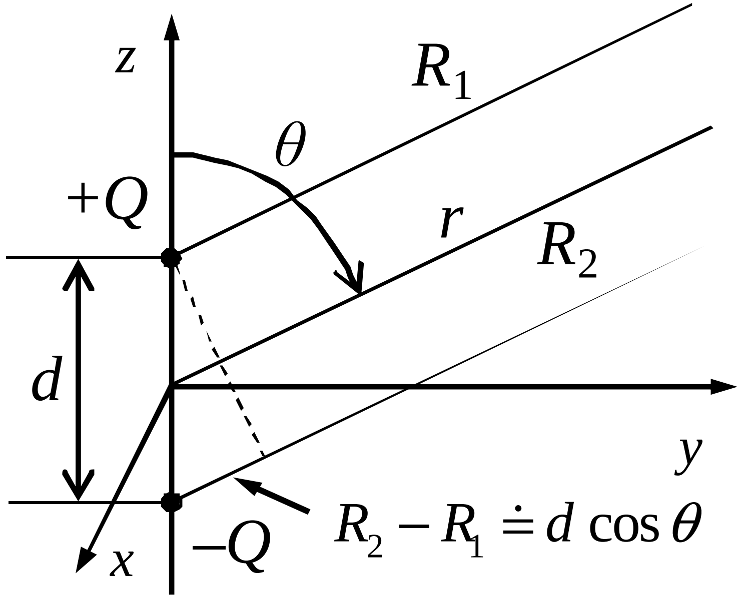
$$\rightarrow \boxed{V = \frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}}$$

$$\mathbf{E} = -\nabla V$$

$$= -\left(\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\mathbf{a}_\varphi\right)$$

$$\boxed{\mathbf{E} = \frac{Qd}{4\pi\varepsilon_0 r^3}(2\cos\theta\mathbf{a}_r + \sin\theta\mathbf{a}_\theta)}$$

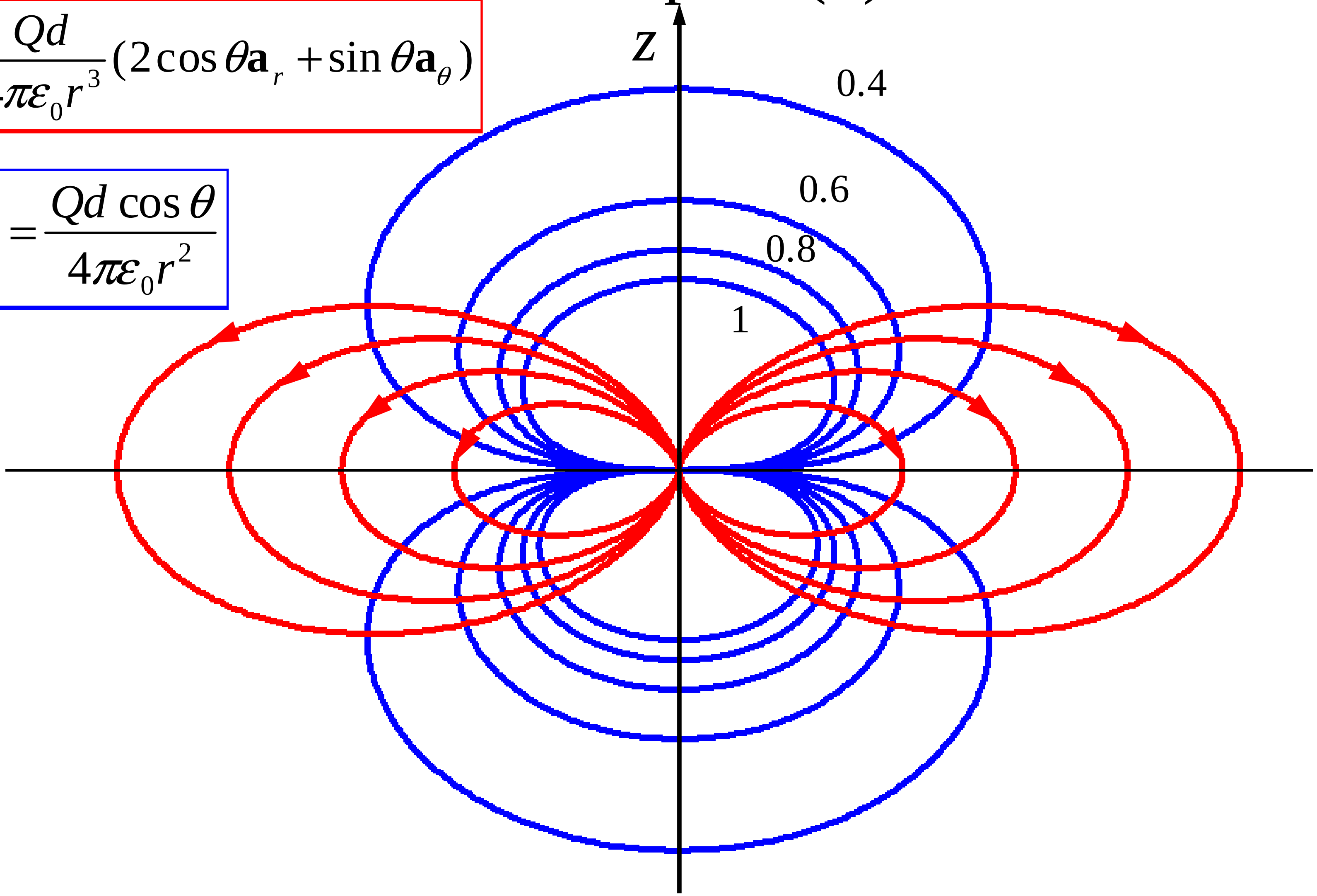




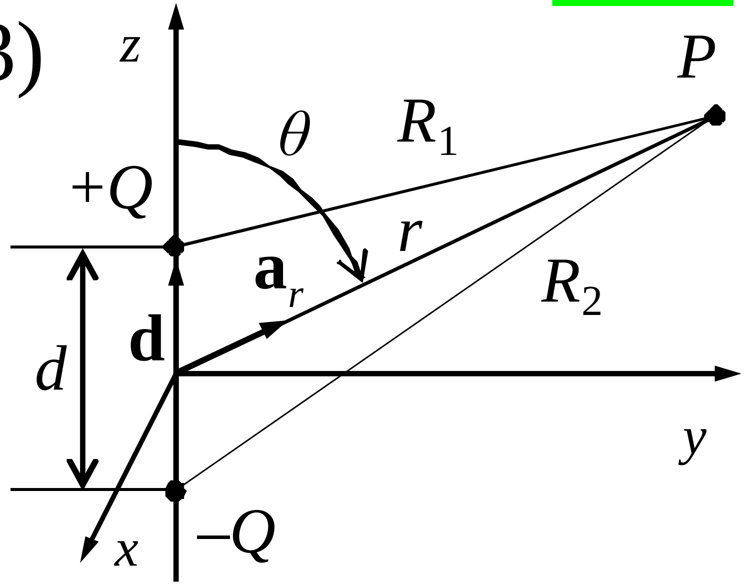
The Dipole (2)

$$\mathbf{E} = \frac{Qd}{4\pi\varepsilon_0 r^3}(2\cos\theta\mathbf{a}_r + \sin\theta\mathbf{a}_\theta)$$

$$V = \frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}$$



The Dipole (3)



The dipole moment $\mathbf{p} = Q\mathbf{d}$

$$\mathbf{d}.\mathbf{a}_r = d\cos\theta$$

$$V = \frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$\rightarrow V = \frac{\mathbf{p}.\mathbf{a}_r}{4\pi\varepsilon_0 r^2} = \frac{1}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|^2}\mathbf{p}.\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

r : locates *P*

r': locates the dipole center

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8. **Energy Density in the Electrostatic Field**

Energy Density in the Electrostatic Field (1)

- Carrying a positive charge (1) from infinity into the field of another fixed positive charge (2) needs work
- If the charge 1 is held near the charge 2, it has a potential energy
- If then the charge 1 is released, it will accelerate away from the charge 2, acquiring kinetic energy
- Problem: find the potential energy present in a system of charges

Energy Density in the Electrostatic Field (2)

- (Work to position Q_2) = $Q_2V_{2,\,1}$
- $V_{2,\,1}$: the potential at Q_2 due to Q_1
- An additional charge Q_3:
- (Work to position Q_3) = $Q_3V_{3,\,1} + Q_3V_{3,\,2}$
- (Work to position Q_4) = $Q_4V_{4,\,1} + Q_4V_{4,\,2} + Q_4V_{4,\,3}$
- Total positioning work = potential energy of field =

$$= W_E = Q_2V_{2,\,1} + Q_3V_{3,\,1} + Q_3V_{3,\,2} + Q_4V_{4,\,1} + Q_4V_{4,\,2} + \quad + Q_4V_{4,\,3} + \ldots$$

Energy Density in the Electrostatic Field (3)

$$W_E = Q_2V_{2,1} + Q_3V_{3,1} + Q_3V_{3,2} + Q_4V_{4,1} + Q_4V_{4,2} + Q_4V_{4,3} + \ldots$$

$$\left.\begin{array}{r} Q_3V_{3,1} = Q_3\dfrac{Q_1}{4\pi\varepsilon_0 R_{13}} \\ R_{13} = R_{31} \end{array}\right\} \rightarrow Q_3V_{3,1} = Q_1\frac{Q_3}{4\pi\varepsilon_0 R_{31}} = Q_1V_{1,3}$$

$$W_E = Q_1V_{1,2} + Q_1V_{1,3} + Q_2V_{2,3} + Q_1V_{1,4} + Q_2V_{2,4} + Q_3V_{3,4} + \ldots$$

$$+\; W_E = Q_2V_{2,1} + Q_3V_{3,1} + Q_3V_{3,2} + Q_4V_{4,1} + Q_4V_{4,2} + Q_4V_{4,3} + \ldots$$

$$\begin{aligned} 2W_E = Q_1(V_{1,2} + V_{1,3} + V_{1,4} + \ldots) + \\ + Q_2(V_{2,1} + V_{2,3} + V_{2,4} + \ldots) + \\ + Q_3(V_{3,1} + V_{3,2} + V_{3,4} + \ldots) + \\ + \ldots \end{aligned}$$

Energy Density in the Electrostatic Field (4)

$$2W_E = Q_1(V_{1,2} + V_{1,3} + V_{1,4} + \ldots) + Q_2(V_{2,1} + V_{2,3} + V_{2,4} + \ldots) +$$

$$+ Q_2(V_{3,1} + V_{3,2} + V_{3,4} + \ldots) + \ldots$$

$$V_{1,2} + V_{1,3} + V_{1,4} + \ldots = V_1$$

$$V_{2,1} + V_{2,3} + V_{2,4} + \ldots = V_2$$

$$V_{3,1} + V_{3,2} + V_{3,4} + \ldots = V_3$$

$$\rightarrow W_E = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3 + \ldots) = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k$$

$$Q_k = \rho_v dv$$

$$\rightarrow \boxed{W_E = \frac{1}{2}\int_V \rho_v V dv}$$

Energy Density in the Electrostatic Field (5)

$$\left.\begin{aligned} W_E &= \frac{1}{2}\int_V \rho_v V dv \\ \text{Maxwell's 1st equation:}\quad \nabla.\mathbf{D} &= \rho_v \end{aligned}\right\}$$

$$\left.\begin{aligned} \to W_E &= \frac{1}{2}\int_V (\nabla.\mathbf{D}) V dv \\ \nabla.(V\mathbf{D}) &\equiv V(\nabla.\mathbf{D}) + \mathbf{D}.(\nabla V) \end{aligned}\right\}$$

$$\to W_E = \frac{1}{2}\int_V \left[\nabla.(V\mathbf{D}) - \mathbf{D}.(\nabla V)\right] dv$$

Energy Density in the Electrostatic Field (6)

$$W_E = \frac{1}{2}\int_V \left[\nabla.(V\mathbf{D}) - \mathbf{D}.(\nabla V)\right]dv$$

$$= \frac{1}{2}\int_V \nabla.(V\mathbf{D})dv - \frac{1}{2}\int_V \mathbf{D}.(\nabla V)dv$$

$$\frac{1}{2}\int_V \nabla.(V\mathbf{D})dv$$

Div. theorem: $\oint_S \mathbf{D}.d\mathbf{S} = \int_V \nabla.\mathbf{D}dv$

$$\rightarrow \frac{1}{2}\int_V \nabla.(V\mathbf{D})dv = \frac{1}{2}\oint_S (V\mathbf{D}).d\mathbf{S}$$

$$\rightarrow W_E = \frac{1}{2}\oint_S (V\mathbf{D}).d\mathbf{S} - \frac{1}{2}\int_V \mathbf{D}.(\nabla V)dv$$

Energy Density in the Electrostatic Field (7)

$$W_E = \frac{1}{2}\oint_S (V\mathbf{D}).d\mathbf{S} - \frac{1}{2}\int_V \mathbf{D}.(\nabla V)dv$$

$V = \dfrac{Q}{4\pi\varepsilon_0 r}$: $\rightarrow$ 0 with $1/r$

$\mathbf{D} = \dfrac{Q}{4\pi r^2}\mathbf{a}_r$: $\rightarrow$ 0 with $1/r^2$

$d\mathbf{S}$: increases with r^2

$$\rightarrow \frac{1}{2}\oint_S (V\mathbf{D}).d\mathbf{S} = 0$$

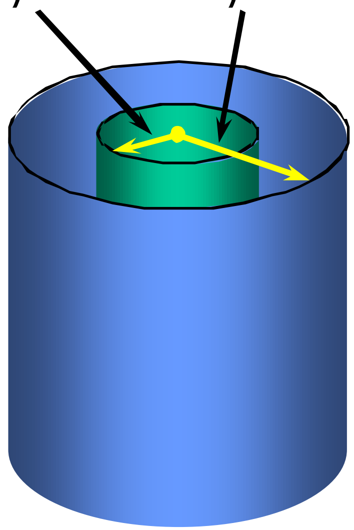
$$\rightarrow W_E = -\frac{1}{2}\int_V \mathbf{D}.(\nabla V)dv$$

$\mathbf{E} = -\nabla V$ (pot. grad)

$$\rightarrow \boxed{W_E = \frac{1}{2}\int_V \mathbf{D}.\mathbf{E}dv = \frac{1}{2}\int_V \varepsilon_0 E^2 dv}$$

Ex. 1 Energy Density in the Electrostatic Field (8)

Given a coaxial cable, the surface charge density of the outer surface of the inner cylinder is ρ_S. Find its potential energy?



<u>Method 1:</u>

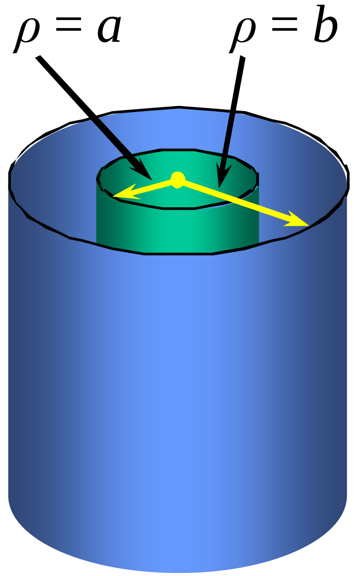
$$\left.\begin{array}{l} W_E = \dfrac{1}{2}\int_V \varepsilon_0 E^2 dv \\ D_\rho = \dfrac{a\rho_S}{\rho} \ (a < \rho < b) \ \rightarrow E = \dfrac{a\rho_S}{\varepsilon_0 \rho} \end{array}\right\}$$

$$\left.\begin{array}{r} \rightarrow W_E = \dfrac{1}{2}\int_{z=0}^{z=L}\int_{\varphi=0}^{\varphi=2\pi}\int_{\rho=a}^{\rho=b} \varepsilon_0 \left(\dfrac{a\rho_S}{\varepsilon_0\rho}\right)^2 dv \\ dv = \rho d\rho d\varphi dz \end{array}\right\}$$

$$\rightarrow W_E = \frac{1}{2}\int_{z=0}^{z=L}\int_{\varphi=0}^{\varphi=2\pi}\int_{\rho=a}^{\rho=b} \varepsilon_0 \frac{a^2\rho_S^2}{\varepsilon_0^2\rho^2}\rho d\rho d\varphi dz \boxed{= \frac{\pi L a^2 \rho_S^2}{\varepsilon_0}\ln\frac{b}{a}}$$

Ex. 1 Energy Density in the Electrostatic Field (9)

Given a coaxial cable, the surface charge density of the outer surface of the inner cylinder is ρ_S. Find its potential energy?



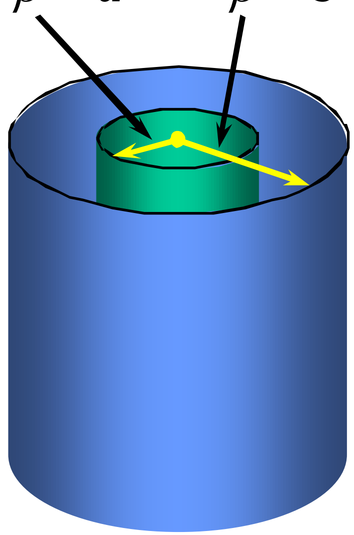
Method 2: $W_E = \frac{1}{2}\int_V \rho_v V dv$

$$\left.\begin{aligned} V_{AB} &= -\int_{\text{init}}^{\text{final}} \mathbf{E}.d\mathbf{L} \\ V_b &= 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} V_a &= -\int_b^a E_\rho d\rho \\ E_\rho &= \frac{a\rho_S}{\varepsilon_0 \rho} \end{aligned}\right\}$$

$$\left.\begin{aligned} \rightarrow V_a &= -\int_b^a \frac{a\rho_S}{\varepsilon_0\rho} d\rho = \frac{a\rho_S}{\varepsilon_0}\ln\frac{b}{a} \\ W_E &= \frac{1}{2}\int_V \rho_v V dv \end{aligned}\right\} \rightarrow W_E = \frac{1}{2}\int_V \rho_v \frac{a\rho_S}{\varepsilon_0}\ln\frac{b}{a} dv$$

Ex. 1 Energy Density in the Electrostatic Field (10)

Given a coaxial cable, the surface charge density of the outer surface of the inner cylinder is ρ_S. Find its potential energy?



Method 2: $$W_E = \frac{1}{2}\int_V \rho_v V dv$$

$$= \frac{1}{2}\int_v \rho_v \frac{a\rho_S}{\varepsilon_0}\ln\frac{b}{a}dv$$

$$\rho_v = \frac{\rho_S}{t}, \quad a - \frac{t}{2} \le \rho \le a + \frac{t}{2}, \quad t \ll a$$

$$\to W_E = \frac{1}{2}\int_{z=0}^{z=L}\int_{\varphi=0}^{\varphi=2\pi}\int_{\rho=a-t/2}^{\rho=a+t/2}\frac{\rho_S}{t}a\frac{\rho_S}{\varepsilon_0}\ln\frac{b}{a}\rho d\rho d\varphi dz$$

$$\boxed{= \frac{\pi L a^2 \rho_S^2}{\varepsilon_0}\ln\frac{b}{a}}$$

Ex. 2 Energy Density in the Electrostatic Field (11)

A metallic sphere of radius 10cm has a surface charge density of 10nC/m². Calculate the electric energy stored in the system.

Method 1: $W_E = \frac{1}{2}\int_V \varepsilon_0 E^2 dv$

$$\int_S \mathbf{D}.d\mathbf{S} = Q_{total} \to D(4\pi r^2) = \rho_S(4\pi R^2) \to D = \frac{\rho_S R^2}{r^2} \to E = \frac{\rho_S R^2}{\varepsilon_0 r^2}$$

$$\to W_E = \frac{1}{2}\int_V \varepsilon_0 \left(\frac{\rho_S R^2}{\varepsilon_0 r^2}\right)^2 dv$$

$$= \frac{1}{2}\int_{r=0.1}^{\infty}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \frac{(0.1)^2 \times 10^{-18}}{\varepsilon_0 r^4} r^2 \sin\theta dr d\theta d\varphi$$

$$= \boxed{71.06\,\text{nJ}}$$

Ex. 2 Energy Density in the Electrostatic Field (12)

A metallic sphere of radius 10cm has a surface charge density of 10nC/m^2. Calculate the electric energy stored in the system.

<u>Method 2:</u> ?