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Engineering Electromagnetics

Magnetic Forces & Inductance

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Magnetic Forces & Inductance

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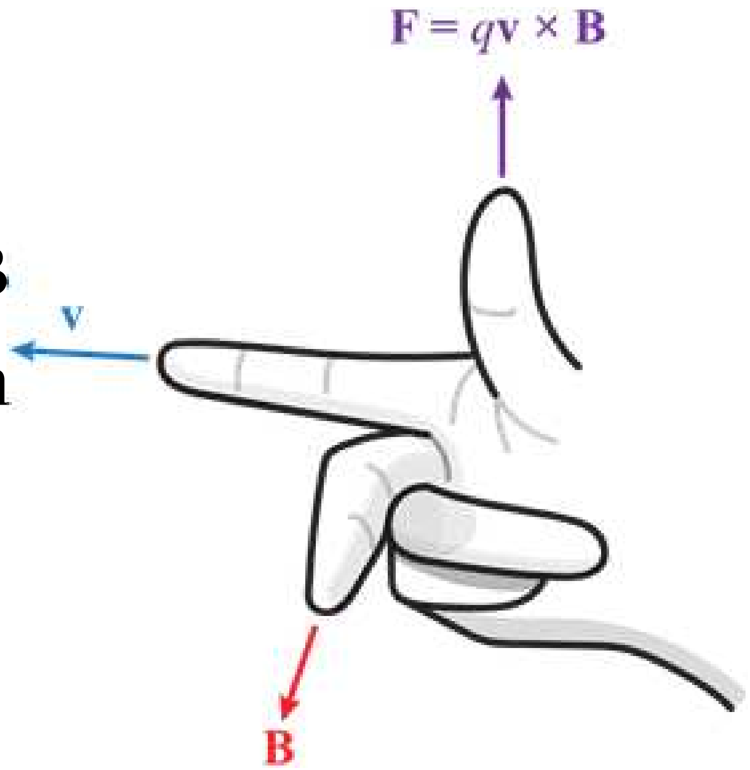
Force on a Moving Charge (1)

- In an electric field: $\mathbf{F} = Q\mathbf{E}$
- This force is in the same direction as the EFI (positive charge)
- In a magnetic field: $\mathbf{F} = Q\mathbf{v}\times\mathbf{B}$
- This force is perpendicular to both **v** & **B**
- In an electromagnetic field: $\mathbf{F} = Q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$
- (Lorentz force)



https://www.shmoop.com/electricity-magnetism/lorentz-force.html

Force on a Moving Charge (2)

Ex. 1

The point charge Q = 18 nC has a velocity of 5×10^6 m/s in the direction $\mathbf{a}_v = 0.04\mathbf{a}_x - 0.05\mathbf{a}_y + 0.2\mathbf{a}_z$. Find the magnitude of the force exerted on the charge by the following fields: a) $\mathbf{B} = -3\mathbf{a}_x + 4\mathbf{a}_y + 6\mathbf{a}_z$ mT; b) $\mathbf{E} = -3\mathbf{a}_x + 4\mathbf{a}_y + 6\mathbf{a}_z$ kV/m; c) **B** & **E** acting together.

$$\left.\begin{aligned}&\mathbf{F_B} = Q\mathbf{v}\times\mathbf{B}\\&\mathbf{v} = v\frac{\mathbf{a}_v}{|\mathbf{a}_v|} = 5\times10^6\frac{0.04\mathbf{a}_x - 0.05\mathbf{a}_y + 0.2\mathbf{a}_z}{\sqrt{0.04^2+0.05^2+0.2^2}}\\&\quad = 5\times10^6(0.19\mathbf{a}_x - 0.24\mathbf{a}_y + 0.95\mathbf{a}_z)\text{ m/s}\end{aligned}\right\}$$

$$\to \mathbf{F_B} = Q\mathbf{v}\times\mathbf{B} = Q\begin{vmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z\\ v_x & v_y & v_z\\ B_x & B_y & B_z\end{vmatrix} = 18\times10^{-9}\times5\times10^{3}\begin{vmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z\\ 0.19 & -0.24 & 0.95\\ -3 & 4 & 6\end{vmatrix}$$

$$= -0.47\mathbf{a}_x - 0.36\mathbf{a}_y + 0.0036\mathbf{a}_z\text{ mN}$$

$$\to F_{\mathbf{B}} = |\mathbf{F_B}| = \sqrt{0.47^2+0.36^2+0.0036^2} = \boxed{0.5928\text{ mN}}$$

Force on a Moving Charge (3)

Ex. 1

The point charge Q = 18 nC has a velocity of 5×10^6 m/s in the direction $\mathbf{a}_v = 0.04\mathbf{a}_x - 0.05\mathbf{a}_y + 0.2\mathbf{a}_z$. Find the magnitude of the force exerted on the charge by the following fields: a) $\mathbf{B} = -3\mathbf{a}_x + 4\mathbf{a}_y + 6\mathbf{a}_z$ mT; b) $\mathbf{E} = -3\mathbf{a}_x + 4\mathbf{a}_y + 6\mathbf{a}_z$ kV/m; c) **B** & **E** acting together.

$$\mathbf{F_E} = Q\mathbf{E} = 18\times10^{-9}(-3\mathbf{a}_x + 4\mathbf{a}_y + 6\mathbf{a}_z)\times10^3\ \mu\text{N}$$

$$\boxed{F_\mathbf{B} = 0.5928\ \text{mN}}$$

$$\to F_\mathbf{E} = |\mathbf{F_E}| = 18\times10^{-6}\sqrt{3^2+4^2+6^2} = \boxed{0.1406\ \text{mN}}$$

$$\mathbf{F_{EB}} = Q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = \mathbf{F_E} + \mathbf{F_B}$$

$$= 18.10^{-6}(-3\mathbf{a}_x + 4\mathbf{a}_y + 6\mathbf{a}_z) +$$

$$+ (-0.47\mathbf{a}_x - 0.36\mathbf{a}_y + 0.0036\mathbf{a}_z)\times10^{-3}$$

$$= -0.53\mathbf{a}_x - 0.29\mathbf{a}_y + 0.11\mathbf{a}_z\ \text{mN}$$

$$\to F_\mathbf{EB} = |\mathbf{F_{EB}}| = \sqrt{0.53^2 + 0.29^2 + 0.11^2} = \boxed{0.6141\ \text{mN}}$$

Force on a Moving Charge (4)

Ex. 2

A test charge Q C, moving with a velocity $\mathbf{v} = \mathbf{a}_x + \mathbf{a}_y$ m/s, experiences no force in a region of electric & magnetic fields. If the magnetic flux density $\mathbf{B} = \mathbf{a}_x - 2\mathbf{a}_z$ T, find $\mathbf{E}$.

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0$$

$$\rightarrow \mathbf{E} = -\mathbf{v} \times \mathbf{B}$$

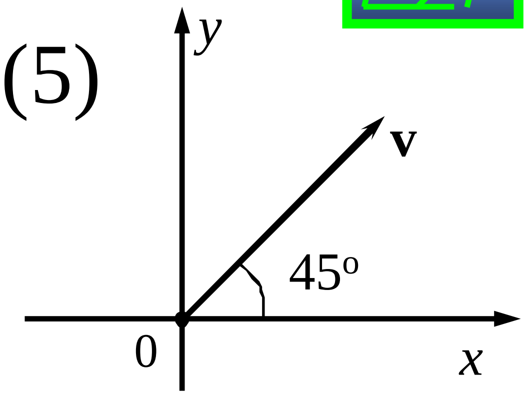
$$= -(\mathbf{a}_x + \mathbf{a}_y) \times (\mathbf{a}_x - 2\mathbf{a}_z)$$

$$= 2\mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z \text{ V/m}$$

Force on a Moving Charge (5)

Ex. 3

Given a magnetic flux density $\mathbf{B} = 10^{-2}\,\mathbf{a}_x$ T, find the force on an electron whose velocity is 10^7 m/s:
a) In the x direction, y direction, & z direction.
b) In the xy plane at 45° to the x axis.



$$\mathbf{F}_x = Q\mathbf{v}\times\mathbf{B} = Q(10^7\,\mathbf{a}_x \times 10^{-2}\,\mathbf{a}_x) = 0$$

$$\mathbf{F}_y = Q\mathbf{v}\times\mathbf{B} = -1.6\times10^{-19}(10^7\,\mathbf{a}_y \times 10^{-2}\,\mathbf{a}_x) = 1.6\times10^{-14}\,\mathbf{a}_z \text{ N}$$

$$\mathbf{F}_z = Q\mathbf{v}\times\mathbf{B} = -1.6\times10^{-19}(10^7\,\mathbf{a}_z \times 10^{-2}\,\mathbf{a}_x) = -1.6\times10^{-14}\,\mathbf{a}_y \text{ N}$$

$$\mathbf{v} = (\cos 45^\circ\,\mathbf{a}_x + \sin 45^\circ\,\mathbf{a}_y)10^7 \text{ m/s}$$

$$\mathbf{F} = Q\mathbf{v}\times\mathbf{B} = -1.6\times10^{-19}(\cos 45^\circ\,\mathbf{a}_x + \sin 45^\circ\,\mathbf{a}_y)10^7 \times 10^{-2}\,\mathbf{a}_x$$

$$= 1.13\times10^{-14}\,\mathbf{a}_z \text{ N}$$

Ex. 4

Force on a Moving Charge (6)

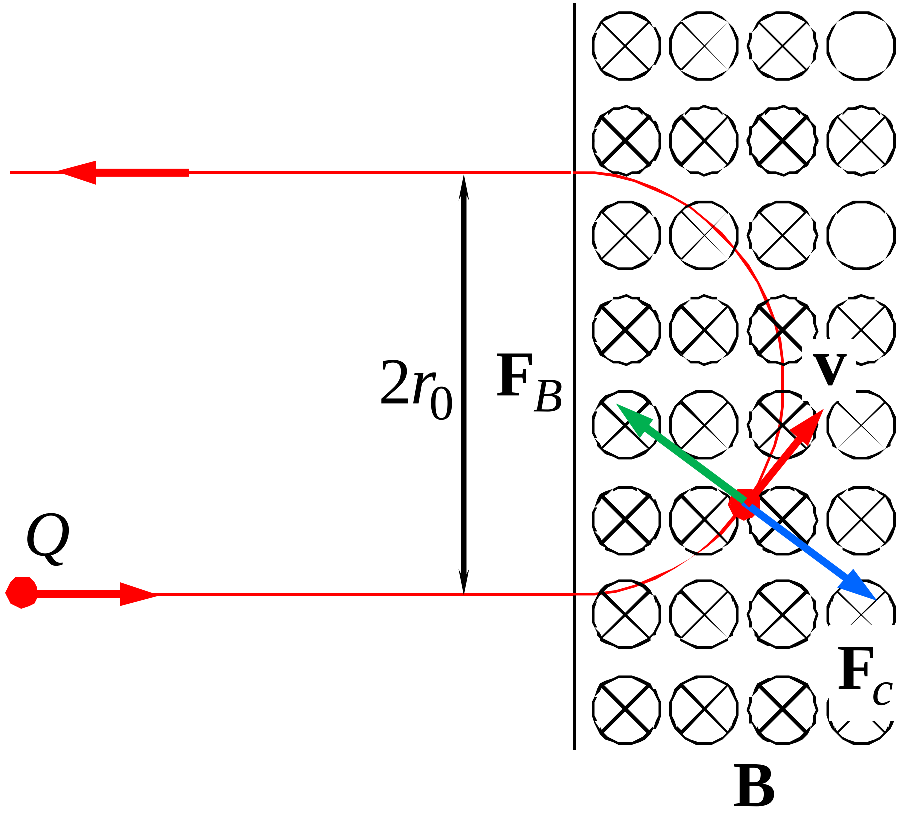
$$\mathbf{F}_B = Q\mathbf{v}\times\mathbf{B}$$

$$F_c = \frac{mv^2}{r_0}$$

$$F_c = F_B$$

$$\rightarrow \frac{mv^2}{r_0} = QvB$$

$$\rightarrow r_0 = \frac{mv}{QB}$$



Magnetic Forces & Inductance

1. Force on a Moving Charge
2. **Force on a Differential Current Element**
3. Force between Differential Current Elements
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Force on a Differential Current Element (1)

- Force on a differential current element:

$$d\mathbf{F} = dQ\mathbf{v}\times\mathbf{B}$$

- If charges are in motion in a conductor, the force is transferred to the conductor
- Consider only force on conductors carrying currents
- If $dQ = \rho_v dv$ (dv is an incremental volume)

$$\rightarrow d\mathbf{F} = \rho_v dv\mathbf{v}\times\mathbf{B}$$

- $$\mathbf{J} = \rho_v \mathbf{v}$$

$$\rightarrow \boxed{d\mathbf{F} = \mathbf{J}\times\mathbf{B}dv}$$

Force on a Differential Current Element (2)

$$\left.\begin{aligned} d\mathbf{F} &= \mathbf{J}\times\mathbf{B}dv \\ \mathbf{J}dv &= Id\mathbf{L} \end{aligned}\right\}$$

$$\rightarrow d\mathbf{F} = Id\mathbf{L}\times\mathbf{B}$$

$$\rightarrow \mathbf{F} = \int_V \mathbf{J}\times\mathbf{B}dv = \oint Id\mathbf{L}\times\mathbf{B} = -I\oint \mathbf{B}\times d\mathbf{L}$$

For a straight conductor in a uniform magnetic field:

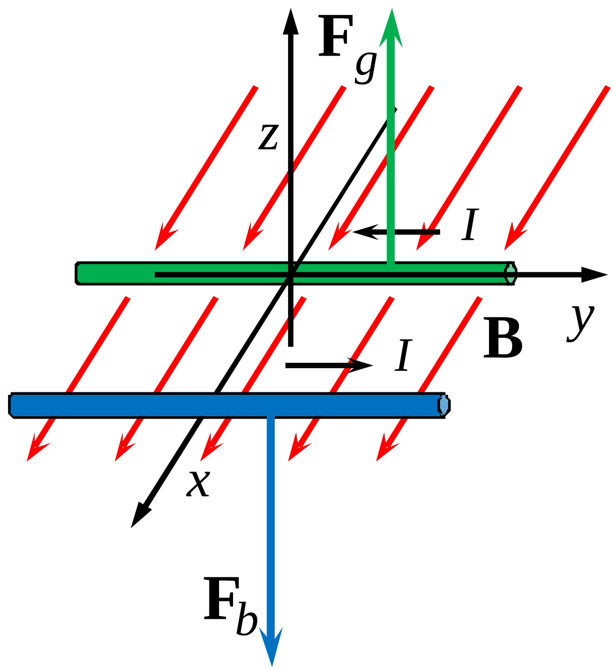
$$\mathbf{F} = I\mathbf{L}\times\mathbf{B}$$
$$F = BIL\sin\theta$$

Ex. 1 Force on a Differential Current Element (3)

Find the force on the current I in a uniform field $\mathbf{B}$.

$$\mathbf{F}_g = I\mathbf{L}\times\mathbf{B}$$
$$= I(-L\mathbf{a}_y)\times(B\mathbf{a}_x)$$
$$= BIL\mathbf{a}_z$$

$$\mathbf{F}_b = I\mathbf{L}\times\mathbf{B}$$
$$= I(L\mathbf{a}_y)\times(B\mathbf{a}_x)$$
$$= -BIL\mathbf{a}_z$$



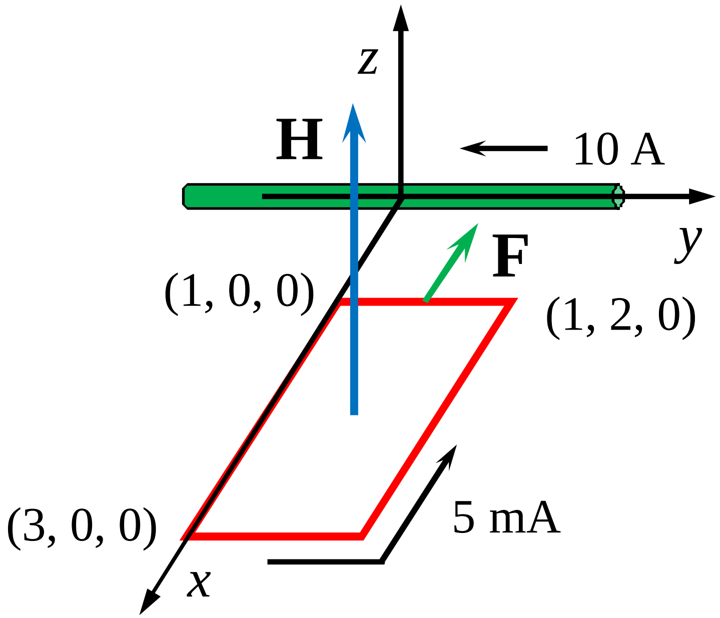
Ex. 2 Force on a Differential Current Element (4)

Find the force on the loop.

$$\mathbf{H} = \frac{I}{2\pi x}\mathbf{a}_z = \frac{10}{2\pi x}\mathbf{a}_z \ \text{A/m}$$

$$\mathbf{B} = \mu_0\mathbf{H} = 4\pi.10^{-7}\frac{10}{2\pi x}\mathbf{a}_z = \frac{2.10^{-6}}{x}\mathbf{a}_z \ \text{T}$$

$$\mathbf{F} = -I\oint \mathbf{B}\times d\mathbf{L} = -5.10^{-3}\oint \frac{2.10^{-6}}{x}\mathbf{a}_z \times d\mathbf{L}$$

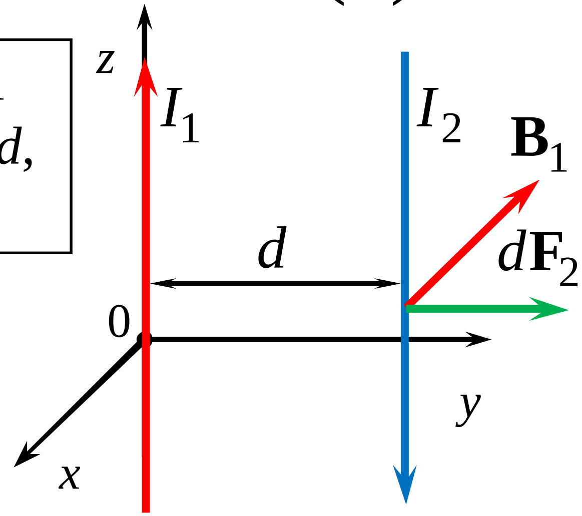


$$= -10^{-8}\left[\int_{x=1}^{3}\frac{\mathbf{a}_z}{x}\times dx\mathbf{a}_x + \int_{y=0}^{2}\frac{\mathbf{a}_z}{3}\times dy\mathbf{a}_y + \int_{x=3}^{1}\frac{\mathbf{a}_z}{x}\times dx\mathbf{a}_x + \int_{y=2}^{0}\frac{\mathbf{a}_z}{1}\times dy\mathbf{a}_y\right]$$

$$= -10^{-8}\left[\ln x\Big|_1^3\mathbf{a}_y + \frac{1}{3}y\Big|_0^2(-\mathbf{a}_x) + \ln x\Big|_3^1\mathbf{a}_y + y\Big|_2^0(-\mathbf{a}_x)\right] = \boxed{-1.33\times10^{-8}\mathbf{a}_x \ \text{N}}$$

Ex. 3 Force on a Differential Current Element (5)

Find the force per meter between two infinite and parallel filamentary current carrying conductor that are separated d, & carry a current I in opposite directions.



$I_1 = I_2 = I$

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\varphi$$

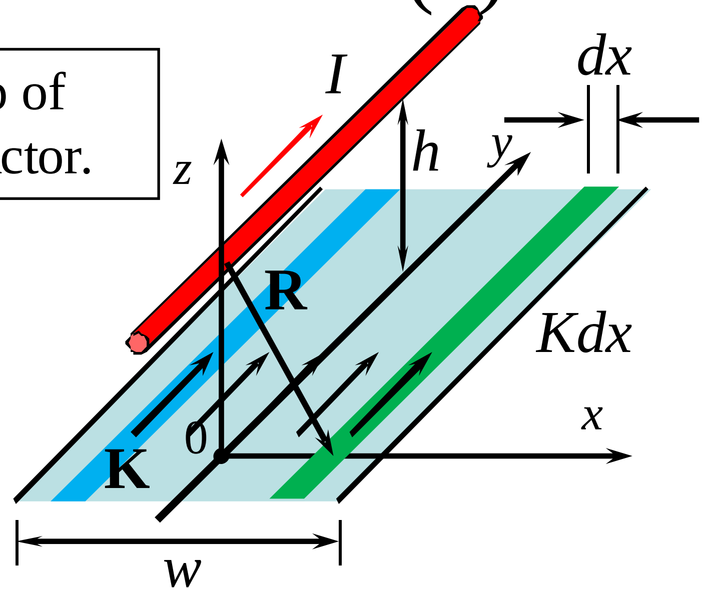
$$\mathbf{B}_1 = \mu_0 \mathbf{H}_1 = \mu_0 \frac{I_1}{2\pi\rho}\mathbf{a}_\varphi \Big|_{\rho=d,\ \varphi=\pi/2} = -\mu_0 \frac{I_1}{2\pi d}\mathbf{a}_x \text{ T}$$

$$d\mathbf{F}_2 = I_2 d\mathbf{L}_2 \times \mathbf{B}_1 = I_2(-dz_2\mathbf{a}_z) \times \left(-\mu_0 \frac{I_1}{2\pi d}\mathbf{a}_x\right) = \mu_0 \frac{I_1 I_2}{2\pi d} dz_2 \mathbf{a}_y$$

$$\rightarrow \mathbf{F}_2 = \int_{z_2=0}^{1} \mu_0 \frac{I_1 I_2}{2\pi d} dz_2 \mathbf{a}_y = \boxed{\mu_0 \frac{I^2}{2\pi d}\mathbf{a}_y \text{ N/m}}$$

Ex. 4 Force on a Differential Current Element (6)

A conductor carries current I parallel to a current strip of density **K**. Find the force per unit length on the conductor.



$$\mathbf{F} = \mu_0 \frac{I_1 I_2}{2\pi R}\mathbf{a}_R$$

$$\left.\begin{aligned} d\mathbf{F}_g &= \frac{\mu_0 IKdx}{2\pi\sqrt{x^2+h^2}} \cdot \frac{-h\mathbf{a}_z + x\mathbf{a}_x}{\sqrt{x^2+h^2}} \\ d\mathbf{F}_b &= \frac{\mu_0 IKdx}{2\pi\sqrt{x^2+h^2}} \cdot \frac{-h\mathbf{a}_z - x\mathbf{a}_x}{\sqrt{x^2+h^2}} \end{aligned}\right\}$$

$$\rightarrow d\mathbf{F} = d\mathbf{F}_g + d\mathbf{F}_b = \frac{-\mu_0 IKhdx}{\pi(x^2+h^2)}\mathbf{a}_z$$

$$\rightarrow \mathbf{F} = \int_0^{w/2} \frac{-\mu_0 IKhdx}{\pi(x^2+h^2)}\mathbf{a}_z = \boxed{-\frac{\mu_0 IK}{\pi}\operatorname{atan}\frac{w}{2h}\mathbf{a}_z \ \text{N/m}}$$

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Force between Differential Current Elements (1)

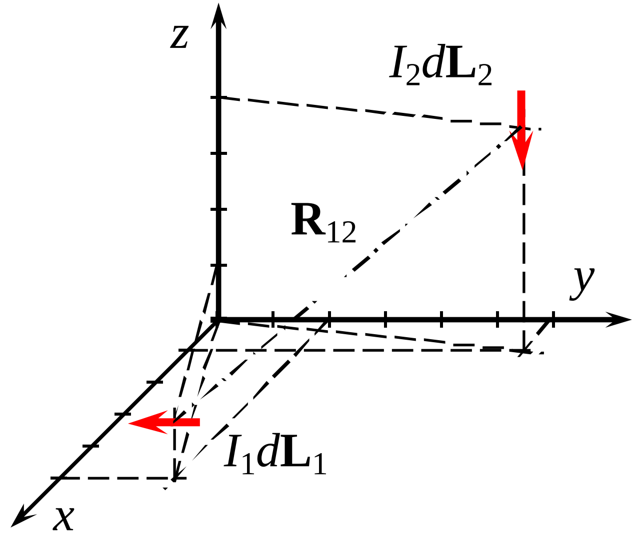
$$\left.\begin{aligned} d\mathbf{H}_2 &= \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2} \\ d\mathbf{F} = Id\mathbf{L}\times\mathbf{B} \;\rightarrow\; d(d\mathbf{F}_2) &= I_2 d\mathbf{L}_2 \times d\mathbf{B}_2 \\ d\mathbf{B}_2 &= \mu_0 d\mathbf{H}_2 \end{aligned}\right\}$$

$$\rightarrow d(d\mathbf{F}_2) = \mu_0 \frac{I_1 I_2}{4\pi R_{12}^2} d\mathbf{L}_2 \times (d\mathbf{L}_1 \times \mathbf{a}_{R12})$$

Force between Differential Current Elements (2)

Ex. 1

Given $I_1 d\mathbf{L}_1 = -3\mathbf{a}_y$ Am; $I_2 d\mathbf{L}_2 = -4\mathbf{a}_z$ Am.
Find the differential force on $d\mathbf{L}_2$.



$$d(d\mathbf{F}_2) = \mu_0 \frac{I_1 I_2}{4\pi R_{12}^2} d\mathbf{L}_2 \times (d\mathbf{L}_1 \times \mathbf{a}_{R12})$$

$$= \frac{4\pi \times 10^{-7}}{4\pi R_{12}^2} I_2 d\mathbf{L}_2 \times (I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12})$$

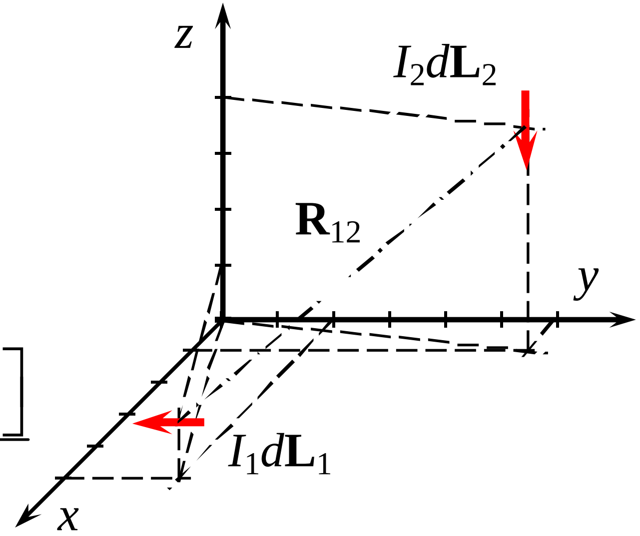
$$\mathbf{R}_{12} = (1-5)\mathbf{a}_x + (6-2)\mathbf{a}_y + (4-1)\mathbf{a}_z$$

$$= -4\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z \rightarrow \mathbf{a}_{R12} = \frac{-4\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z}{\sqrt{4^2 + 4^2 + 3^2}}; \quad R_{12} = \sqrt{4^2 + 4^2 + 3^2}$$

$$\rightarrow d(d\mathbf{F}_2) = \frac{4\pi \times 10^{-7}}{4\pi} (-4\mathbf{a}_z) \times \frac{\left[(-3\mathbf{a}_y) \times (-4\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z)\right]}{(4^2 + 4^2 + 3^2)^{3/2}}$$

Ex. 1 Force between Differential Current Elements (3)

Given $I_1 d\mathbf{L}_1 = -3\mathbf{a}_y$ Am; $I_2 d\mathbf{L}_2 = -4\mathbf{a}_z$ Am.
Find the differential force on $d\mathbf{L}_2$.



$$d(d\mathbf{F}_2) = \mu_0 \frac{I_1 I_2}{4\pi R_{12}^2} d\mathbf{L}_2 \times (d\mathbf{L}_1 \times \mathbf{a}_{R12})$$

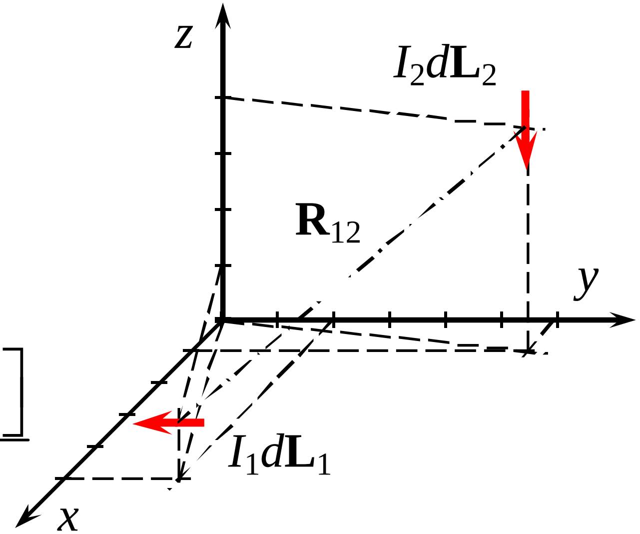
$$= \frac{4\pi \times 10^{-7}}{4\pi}(-4\mathbf{a}_z) \times \frac{\left[(-3\mathbf{a}_y) \times (-4\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z)\right]}{(4^2 + 4^2 + 3^2)^{3/2}}$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\rightarrow (-3\mathbf{a}_y) \times (-4\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z) = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & -3 & 0 \\ -4 & 4 & 3 \end{vmatrix} = -3(3\mathbf{a}_x + 4\mathbf{a}_z)$$

Ex. 1 Force between Differential Current Elements (4)

Given $I_1 d\mathbf{L}_1 = -3\mathbf{a}_y$ Am; $I_2 d\mathbf{L}_2 = -4\mathbf{a}_z$ Am.
Find the differential force on $d\mathbf{L}_2$.



$$d(d\mathbf{F}_2) = \mu_0 \frac{I_1 I_2}{4\pi R_{12}^2} d\mathbf{L}_2 \times (d\mathbf{L}_1 \times \mathbf{a}_{R12})$$

$$= \frac{4\pi \times 10^{-7}}{4\pi} (-4\mathbf{a}_z) \times \frac{\left[(-3\mathbf{a}_y) \times (-4\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z)\right]}{(4^2 + 4^2 + 3^2)^{3/2}}$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

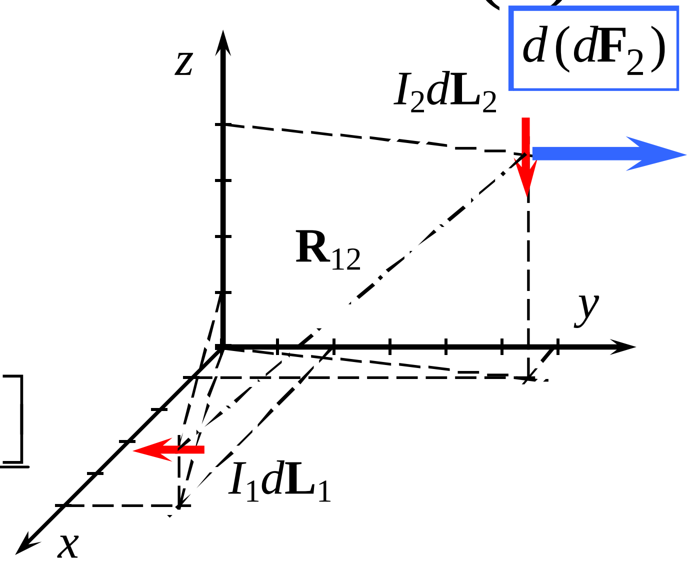
$$(-3\mathbf{a}_y) \times (-4\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z) = -3(3\mathbf{a}_x + 4\mathbf{a}_z)$$

$$\rightarrow (-4\mathbf{a}_z) \times \left[(-3\mathbf{a}_y) \times (-4\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z)\right] = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & -4 \\ -9 & 0 & -12 \end{vmatrix} = 36\mathbf{a}_y$$

Force between Differential Current Elements (5)

Ex. 1

Given $I_1 d\mathbf{L}_1 = -3\mathbf{a}_y$ Am; $I_2 d\mathbf{L}_2 = -4\mathbf{a}_z$ Am.
Find the differential force on $d\mathbf{L}_2$.



$$d(d\mathbf{F}_2) = \mu_0 \frac{I_1 I_2}{4\pi R_{12}^2} d\mathbf{L}_2 \times (d\mathbf{L}_1 \times \mathbf{a}_{R12})$$

$$= \frac{4\pi \times 10^{-7}}{4\pi}(-4\mathbf{a}_z) \times \frac{\left[(-3\mathbf{a}_y) \times (-4\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z)\right]}{(4^2 + 4^2 + 3^2)^{3/2}}$$

$$(-4\mathbf{a}_z) \times \left[(-3\mathbf{a}_y) \times (-4\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z)\right] = 36\mathbf{a}_y$$

$$\rightarrow d(d\mathbf{F}_2) = \frac{10^{-7}}{(4^2 + 4^2 + 3^2)^{3/2}} 36\mathbf{a}_y = \boxed{1.37 \times 10^{-8}\mathbf{a}_y \text{ N}}$$

Force between Differential Current Elements (6)

Ex. 2

Given $I_1 d\mathbf{L}_1 = -3\mathbf{a}_y$ Am; $I_2 d\mathbf{L}_2 = -4\mathbf{a}_z$ Am.
Find the differential force on $d\mathbf{L}_1$.

($d(d\mathbf{F}_2) = 1.37\times10^{-8}\,\mathbf{a}_y$ N was found in Ex.1)

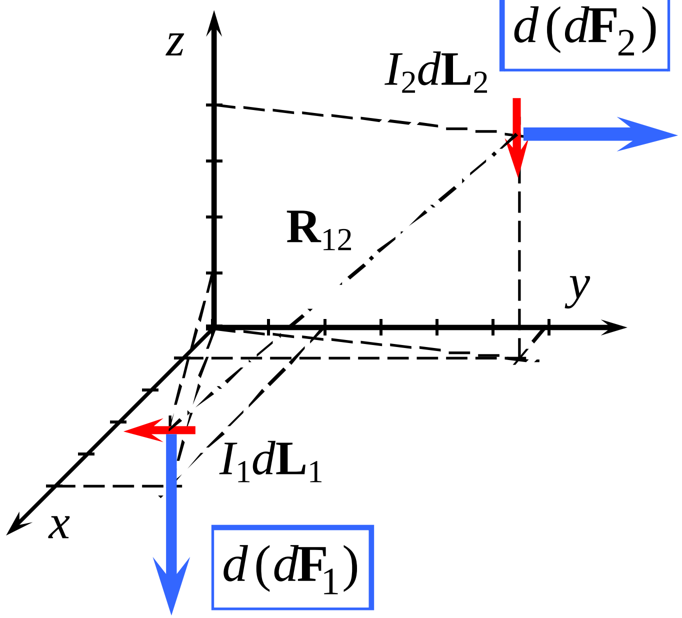
$$d(d\mathbf{F}_2) = \mu_0 \frac{I_1 I_2}{4\pi R_{12}^2} d\mathbf{L}_2 \times (d\mathbf{L}_1 \times \mathbf{a}_{R12})$$

$$d(d\mathbf{F}_1) = \mu_0 \frac{I_2 I_1}{4\pi R_{21}^2} d\mathbf{L}_1 \times (d\mathbf{L}_2 \times \mathbf{a}_{R21})$$

$$= \frac{4\pi\times10^{-7}}{4\pi R_{21}^2} I_1 d\mathbf{L}_1 \times (I_2 d\mathbf{L}_2 \times \mathbf{a}_{R21})$$

$$\mathbf{R}_{21} = (5-1)\mathbf{a}_x + (2-6)\mathbf{a}_y + (1-4)\mathbf{a}_z$$

$$\rightarrow d(d\mathbf{F}_1) = -1.83\times10^{-8}\,\mathbf{a}_z$$



Why $d(d\mathbf{F}_2) \neq d(d\mathbf{F}_1)$?

Force between Differential Current Elements (7)

$$d(d\mathbf{F}_2) = \mu_0 \frac{I_1 I_2}{4\pi R_{12}^2} d\mathbf{L}_2 \times (d\mathbf{L}_1 \times \mathbf{a}_{R12})$$

$$\rightarrow \mathbf{F}_2 = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[d\mathbf{L}_2 \times \oint \frac{d\mathbf{L}_1 \times \mathbf{a}_{R12}}{R_{12}^2} \right]$$

$$= \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[\oint \frac{d\mathbf{L}_1 \times \mathbf{a}_{R12}}{R_{12}^2} \right] \times d\mathbf{L}_2$$

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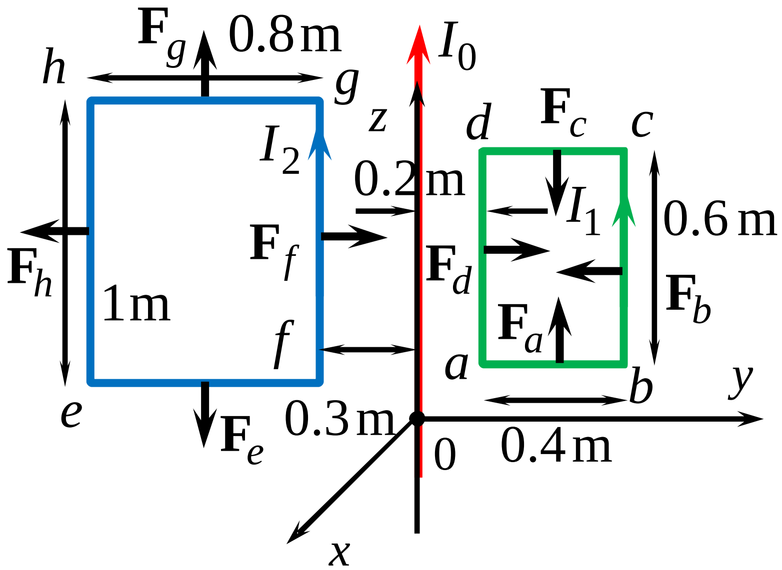
Force & Torque on a Closed Circuit (1)

- Force on a filamentary closed circuit: $\mathbf{F} = -I\oint \mathbf{B}\times d\mathbf{L}$
- If $\mathbf{B}$ = const $\rightarrow$ $\mathbf{F} = -I\mathbf{B}\times\oint d\mathbf{L}$
- In an electrostatic field: $\oint d\mathbf{L} = 0$
- $\rightarrow$ the force on a closed filamentary circuit in a uniform magnetic field is zero
- *General*: any real closed circuit carrying direct currents experiences a total vector force of zero in a uniform magnetic field

Ex. 2 Force & Torque on a Closed Circuit (3)

I_0 = 5A, I_1 = 3A, I_2 = 4A. Find the total force on the wire due to the two loops?



$$\mathbf{B}_+ = -\frac{\mu_0 I_0}{2\pi\rho}\mathbf{a}_x, \quad \mathbf{B}_- = \frac{\mu_0 I_0}{2\pi\rho}\mathbf{a}_x$$

$$\mathbf{F}_{green} = \oint_{C_{green}} I_1 d\mathbf{L} \times \mathbf{B}_+$$

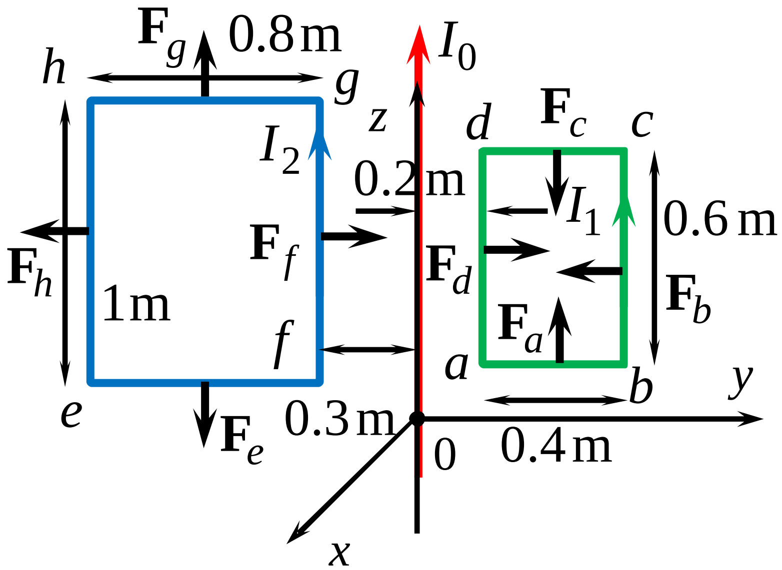
$$= \int_a^b I_1 d\mathbf{L}_a \times \mathbf{B}_+ + \int_b^c I_1 d\mathbf{L}_b \times \mathbf{B}_+ + \int_c^d I_1 d\mathbf{L}_c \times \mathbf{B}_+ + \int_d^a I_1 d\mathbf{L}_d \times \mathbf{B}_+$$

$$\mathbf{F}_{blue} = \oint_{C_{blue}} I_2 d\mathbf{L} \times \mathbf{B}_-$$

$$= \int_e^f I_2 d\mathbf{L}_e \times \mathbf{B}_- + \int_f^g I_2 d\mathbf{L}_f \times \mathbf{B}_- + \int_g^h I_2 d\mathbf{L}_g \times \mathbf{B}_- + \int_h^e I_2 d\mathbf{L}_d \times \mathbf{B}_-$$

Ex. 2 Force & Torque on a Closed Circuit (4)

$I_0 = 5\text{A}$, $I_1 = 3\text{A}$, $I_2 = 4\text{A}$. Find the total force on the wire due to the two loops?



$$\mathbf{F}_{green} = \int_a^b I_1 d\mathbf{L}_a \times \mathbf{B}_+ + \int_b^c I_1 d\mathbf{L}_b \times \mathbf{B}_+ + \int_c^d I_1 d\mathbf{L}_c \times \mathbf{B}_+ + \int_d^a I_1 d\mathbf{L}_d \times \mathbf{B}_+$$

$$\mathbf{F}_{bue} = \int_e^f I_2 d\mathbf{L}_e \times \mathbf{B}_- + \int_f^g I_2 d\mathbf{L}_f \times \mathbf{B}_- + \int_g^h I_2 d\mathbf{L}_g \times \mathbf{B}_- + \int_h^e I_2 d\mathbf{L}_d \times \mathbf{B}_-$$

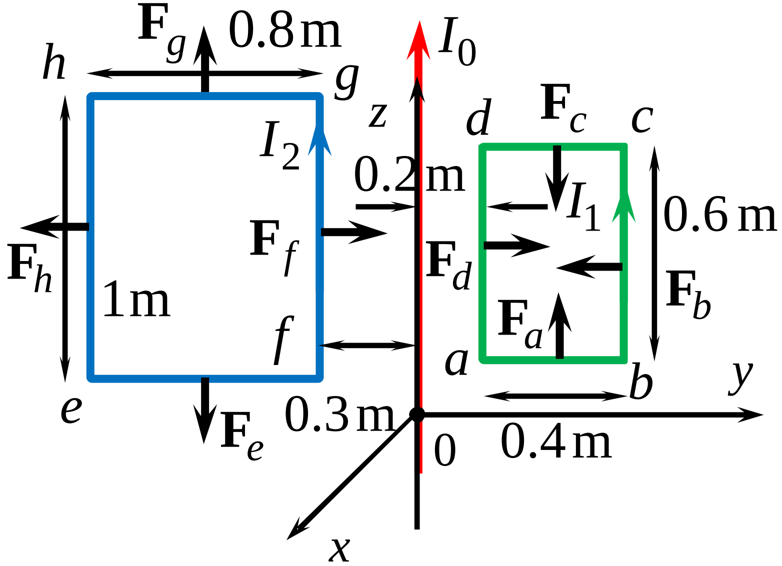
$$\mathbf{F}_a + \mathbf{F}_c = 0,\ \mathbf{F}_e + \mathbf{F}_g = 0$$

$$\rightarrow \mathbf{F}_{total} = \int_f^g I_2 d\mathbf{L}_f \times \mathbf{B}_- + \int_h^e I_2 d\mathbf{L}_d \times \mathbf{B}_- + \int_b^c I_1 d\mathbf{L}_b \times \mathbf{B}_+ + \int_d^a I_1 d\mathbf{L}_d \times \mathbf{B}_+$$

Force & Torque on a Closed Circuit (5)

Ex. 2

$I_0 = 5$A, $I_1 = 3$A, $I_2 = 4$A. Find the total force on the wire due to the two loops?



$$\mathbf{F}_{total} = \int_f^g I_2 d\mathbf{L}_f \times \mathbf{B}_- + \int_h^e I_2 d\mathbf{L}_d \times \mathbf{B}_-$$

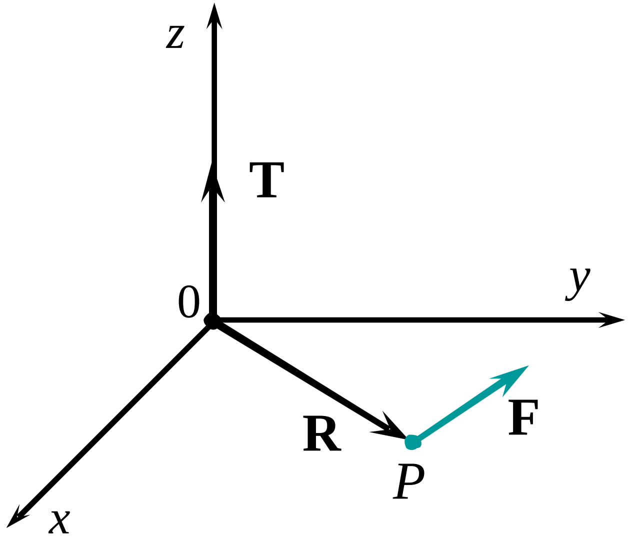
$$+ \int_b^c I_1 d\mathbf{L}_b \times \mathbf{B}_+ + \int_d^a I_1 d\mathbf{L}_d \times \mathbf{B}_+$$

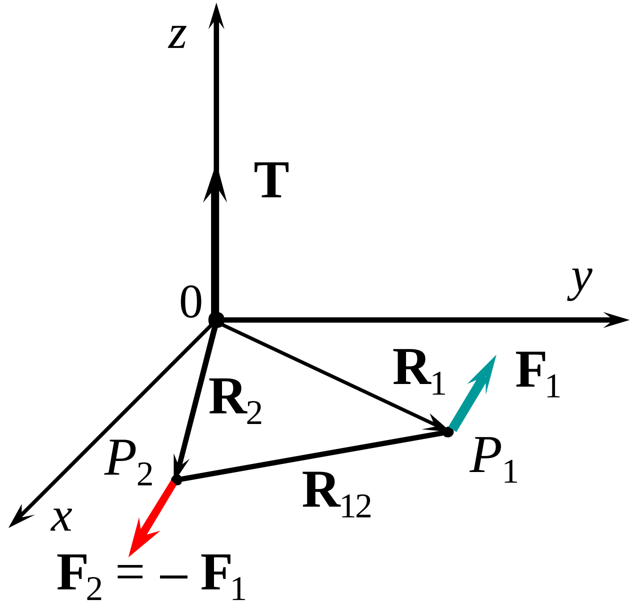
$$\mathbf{B}_+ = -\frac{\mu_0 I_0}{2\pi\rho}\mathbf{a}_x, \quad \mathbf{B}_- = \frac{\mu_0 I_0}{2\pi\rho}\mathbf{a}_x$$

$$\int_f^g I_2 d\mathbf{L}_f \times \mathbf{B}_- = \int_{z=0}^{1} I_2 (dz\mathbf{a}_z) \times \left. \left(\frac{\mu_0 I_0}{2\pi\rho}\mathbf{a}_x \right) \right|_{\rho=0.3} = \int_{z=0}^{1} \frac{\mu_0 I_0 I_2 dz}{2\pi(0.3)}\mathbf{a}_y$$

$$= \left(\left. \frac{\mu_0 I_0 I_2 z}{2\pi(0.3)} \right|_0^1 \right)\mathbf{a}_y = \frac{\mu_0 I_0 I_2}{2\pi(0.3)}\mathbf{a}_y$$

Force & Torque on a Closed Circuit (6)





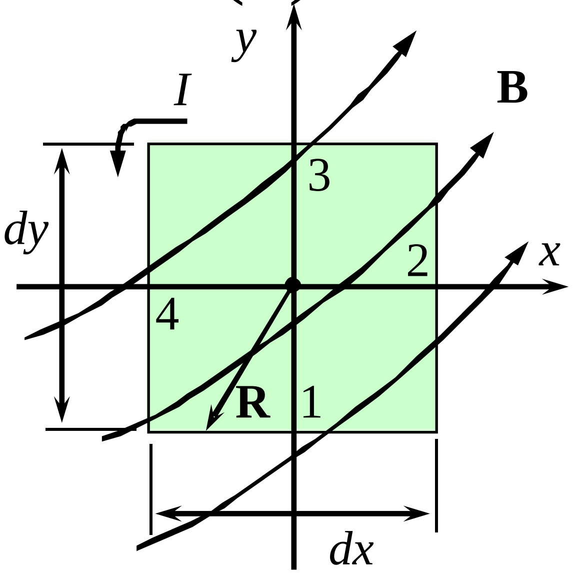
$$\mathbf{T} = \mathbf{R} \times \mathbf{F}$$

$$\mathbf{T} = \mathbf{R}_1 \times \mathbf{F}_1 + \mathbf{R}_2 \times \mathbf{F}_2$$

$$= (\mathbf{R}_1 - \mathbf{R}_2) \times \mathbf{F}_1$$

$$= \mathbf{R}_{21} \times \mathbf{F}_1$$

Force & Torque on a Closed Circuit (7)



$$\left.\begin{aligned} d\mathbf{T}_1 &= \mathbf{R}_1 \times d\mathbf{F}_1 \\ d\mathbf{F}_1 &= Idx\mathbf{a}_x \times \mathbf{B}_0 = Idx(B_{0y}\mathbf{a}_z - B_{0z}\mathbf{a}_y) \\ \mathbf{R}_1 &= -\frac{1}{2}dy\mathbf{a}_y \end{aligned}\right\}$$

$$\rightarrow d\mathbf{T}_1 = -\frac{1}{2}dy\mathbf{a}_y \times Idx(B_{0y}\mathbf{a}_z - B_{0z}\mathbf{a}_y)$$

$$\left.\begin{aligned} &= -\frac{1}{2}dxdyIB_{0y}\mathbf{a}_x \\ \text{Similarly: } d\mathbf{T}_3 &= -\frac{1}{2}dxdyIB_{0y}\mathbf{a}_x \end{aligned}\right\} \rightarrow d\mathbf{T}_1 + d\mathbf{T}_3 = -dxdyIB_{0y}\mathbf{a}_x$$

$$\text{Similarly: } d\mathbf{T}_2 + d\mathbf{T}_4 = dxdyIB_{0x}\mathbf{a}_y$$

$$\rightarrow d\mathbf{T} = dxdyI(B_{0x}\mathbf{a}_y - B_{0y}\mathbf{a}_x) = dxdyI\mathbf{a}_z \times \mathbf{B}_0 \boxed{= Id\mathbf{S} \times \mathbf{B}}$$

Force & Torque on a Closed Circuit (7)

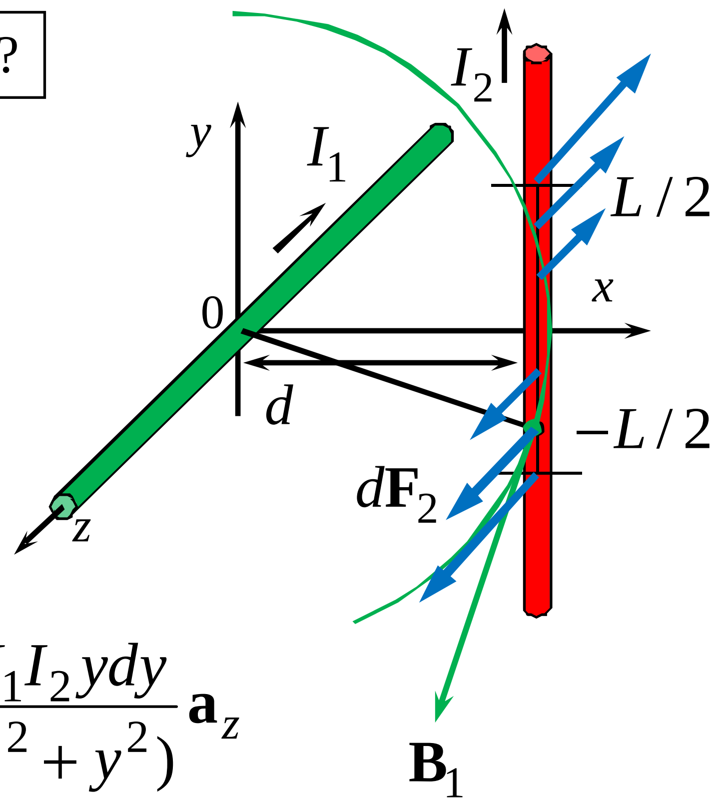
Ex. 3

Find the force & torque created by I_1 a segment of I_2?

$$\left.\begin{aligned} d\mathbf{F}_2 &= I_2 d\mathbf{L}_2 \times \mathbf{B}_1 \\ \mathbf{B}_1 &= \frac{\mu_0 I_1}{2\pi\rho}\mathbf{a}_\varphi = \frac{\mu_0 I_1}{2\pi\sqrt{d^2+y^2}} \cdot \frac{y\mathbf{a}_x - d\mathbf{a}_y}{\sqrt{d^2+y^2}} \\ d\mathbf{L}_2 &= dy\mathbf{a}_y \end{aligned}\right\}$$

$$\rightarrow d\mathbf{F}_2 = I_2(dy\mathbf{a}_y) \times \frac{\mu_0 I_1 (y\mathbf{a}_x - d\mathbf{a}_y)}{2\pi(d^2+y^2)} = \frac{-\mu_0 I_1 I_2 y dy}{2\pi(d^2+y^2)}\mathbf{a}_z$$

$$\rightarrow \mathbf{F}_2 = \int_{-L/2}^{L/2} d\mathbf{F}_2 = \int_{-L/2}^{L/2} \frac{-\mu_0 I_1 I_2 y dy}{2\pi(d^2+y^2)}\mathbf{a}_z = \boxed{0}$$



Force & Torque on a Closed Circuit (8)

Ex. 3

Find the force & torque created by I_1 a segment of I_2?

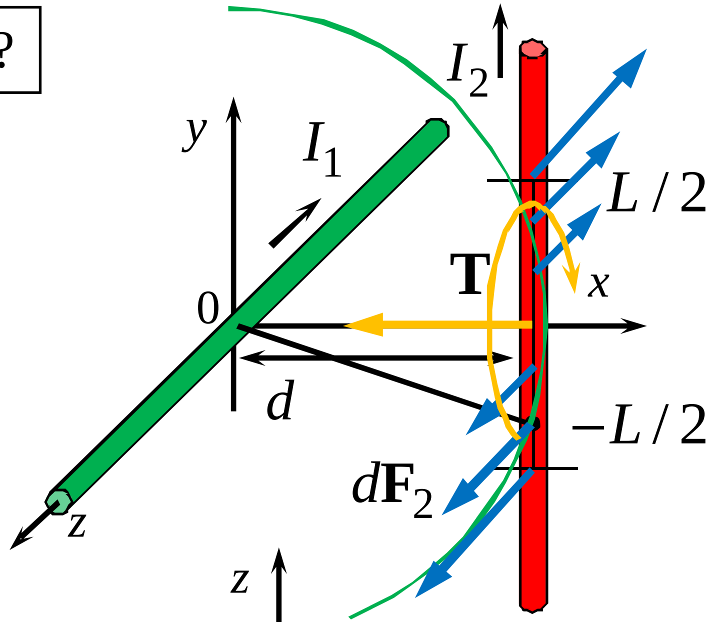
$$d\mathbf{F}_2 = \frac{-\mu_0 I_1 I_2 y dy}{2\pi(d^2 + y^2)}\mathbf{a}_z$$

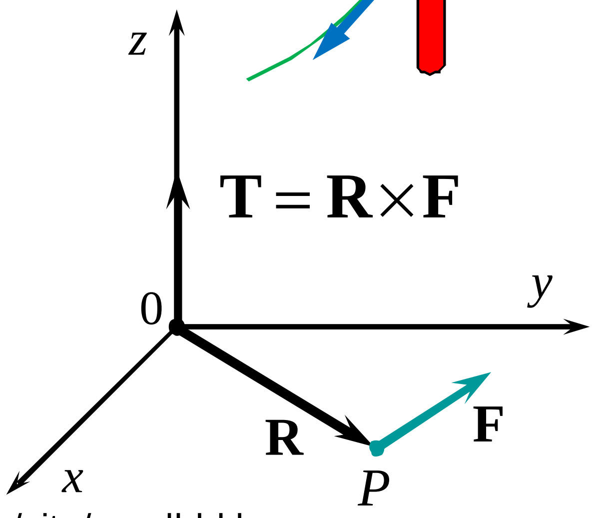
$$d\mathbf{T}_2 = \mathbf{R}_2 \times d\mathbf{F}_2 = (y\mathbf{a}_y)\times\left[\frac{-\mu_0 I_1 I_2 y dy}{2\pi(d^2 + y^2)}\mathbf{a}_z\right]$$

$$= \frac{-\mu_0 I_1 I_2 y^2 dy}{2\pi(d^2 + y^2)}\mathbf{a}_x$$

$$\rightarrow \mathbf{T}_2 = \int_{-L/2}^{L/2} d\mathbf{T}_2 = \int_{-L/2}^{L/2} \frac{-\mu_0 I_1 I_2 y^2 dy}{2\pi(d^2 + y^2)}\mathbf{a}_x$$

$$= \boxed{\frac{-\mu_0 I_1 I_2 L^3 dy}{24\pi d^2}\mathbf{a}_x}$$





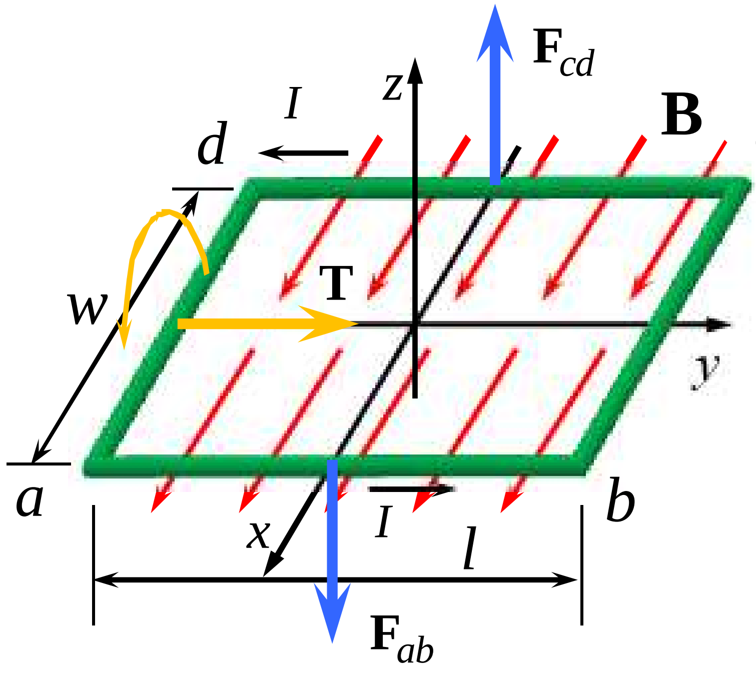
Force & Torque on a Closed Circuit (8)

- The differential magnetic dipole moment: $d\mathbf{m} = Id\mathbf{S}$
- Unit: Am^2
- $\rightarrow \quad d\mathbf{T} = d\mathbf{m}\times\mathbf{B}$
- Holds for differential loops of any shape
- In a uniform magnetic field: $\mathbf{T} = I\mathbf{S}\times\mathbf{B} = \mathbf{m}\times\mathbf{B}$

Force & Torque on a Closed Circuit (10)

Ex. 4

Find the torque on the closed circuit. **(Method 1)**



$$\mathbf{F} = I\mathbf{L} \times \mathbf{B}$$

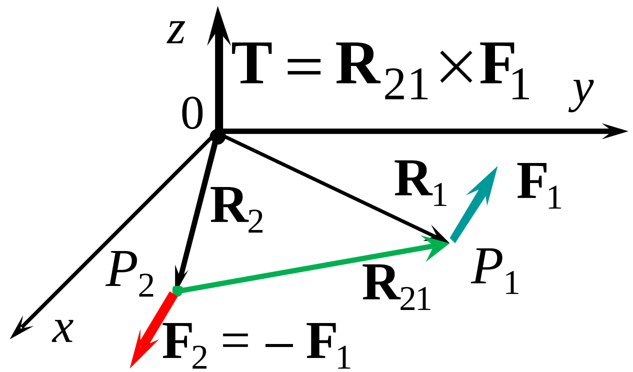
$$\mathbf{F}_{ab} = I\mathbf{L}_{ab} \times \mathbf{B} = I(l\mathbf{a}_y) \times (B\mathbf{a}_x) = -BIl\mathbf{a}_z$$

$$\mathbf{F}_{bc} = I\mathbf{L}_{bc} \times \mathbf{B} = I(-w\mathbf{a}_x) \times (B\mathbf{a}_x) = 0$$

$$\mathbf{F}_{cd} = I\mathbf{L}_{cd} \times \mathbf{B} = I(-l\mathbf{a}_y) \times (B\mathbf{a}_x) = BIl\mathbf{a}_z$$

$$\mathbf{F}_{da} = I\mathbf{L}_{da} \times \mathbf{B} = I(w\mathbf{a}_x) \times (B\mathbf{a}_x) = 0$$

$$\mathbf{T} = \mathbf{R}_{da} \times \mathbf{F}_{ab} = (w\mathbf{a}_x) \times (-BIl\mathbf{a}_z) = \boxed{BIlw\mathbf{a}_y}$$

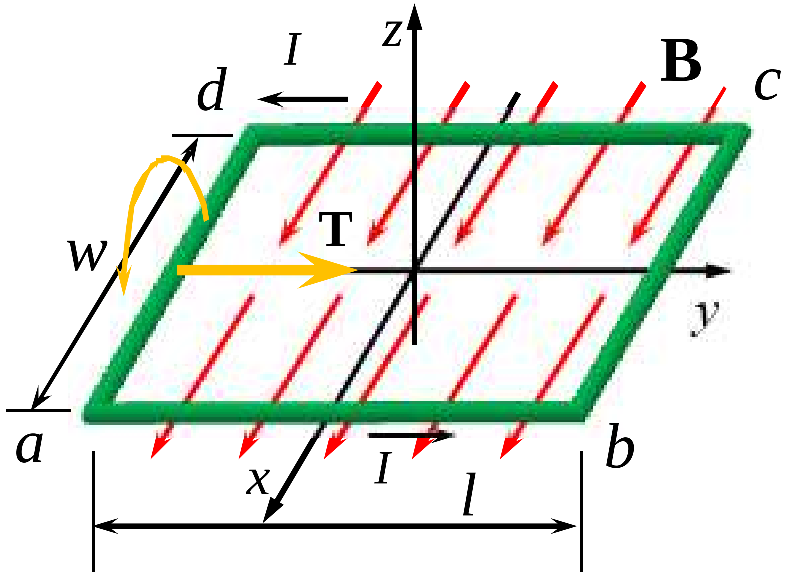


$$\mathbf{T} = \mathbf{R}_{21} \times \mathbf{F}_1$$

Force & Torque on a Closed Circuit (11)

Ex. 4

Find the torque on the closed circuit. **(Method 2)**



$$\mathbf{T} = I\mathbf{S} \times \mathbf{B}$$

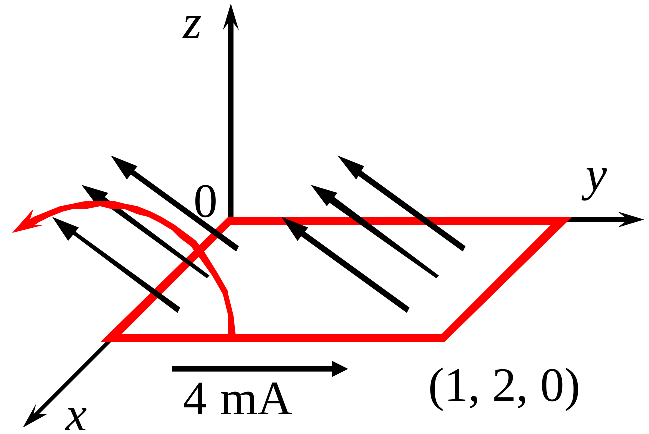
$$= I(lw\mathbf{a}_z) \times (B\mathbf{a}_x)$$

$$= \boxed{BIlw\mathbf{a}_y}$$

Force & Torque on a Closed Circuit (12)

Ex. 5

Given $\mathbf{B}_0 = -0{,}6\mathbf{a}_y + 0{,}8\mathbf{a}_z$ T. Find the torque on the closed circuit.



$$\mathbf{T} = I\mathbf{S}\times\mathbf{B}$$

$$\rightarrow \mathbf{T} = 4\times10^{-3}(1\times2\mathbf{a}_z)\times(-0.6\mathbf{a}_y + 0.8\mathbf{a}_z)$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \rightarrow 1\times2\mathbf{a}_z\times(-0.6\mathbf{a}_y + 0.8\mathbf{a}_z) = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & 2 \\ 0 & -0.6 & 0.8 \end{vmatrix} = 1.2\mathbf{a}_x$$

$$\rightarrow \mathbf{T} = 4.8\times10^{-3}\mathbf{a}_x \text{ Nm}$$

Force & Torque on a Closed Circuit (13)

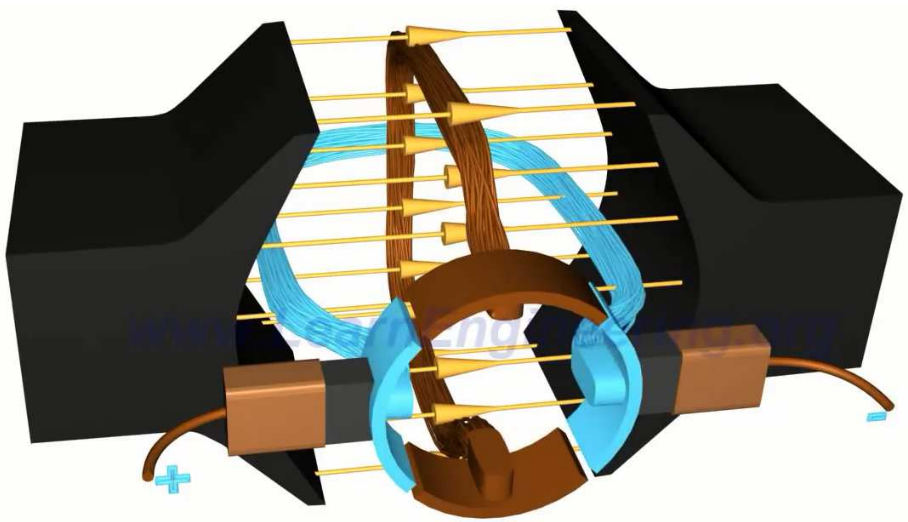
Ex. 6

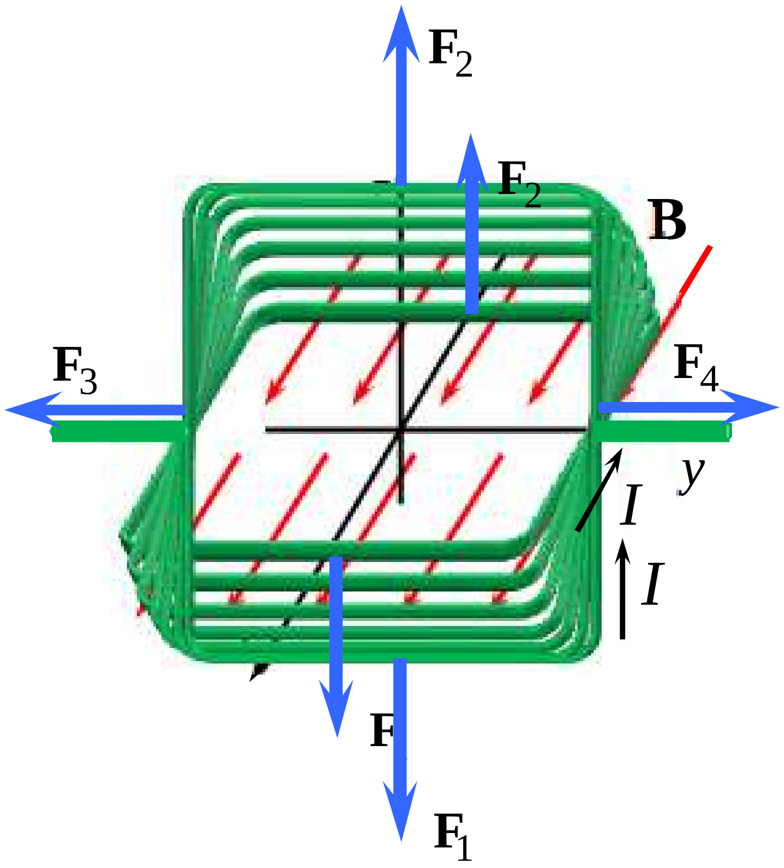
Find the torque on the closed circuit.

$$\mathbf{T} = I\mathbf{S} \times \mathbf{B}$$

$$= I(lw\mathbf{a}_x) \times (B\mathbf{a}_x)$$

$$= 0$$





Magnetic Forces & Inductance

1. Force on a Moving Charge
2. Force on a Differential Current Element
3. Force between Differential Current Elements
4. Force & Torque on a Closed Circuit
5. **Magnetization & Permeability**
6. Magnetic Boundary Conditions
7. The Magnetic Circuit
8. Potential Energy of Magnetic Fields
9. Inductance & Mutual Inductance

Magnetization & Permeability (1)

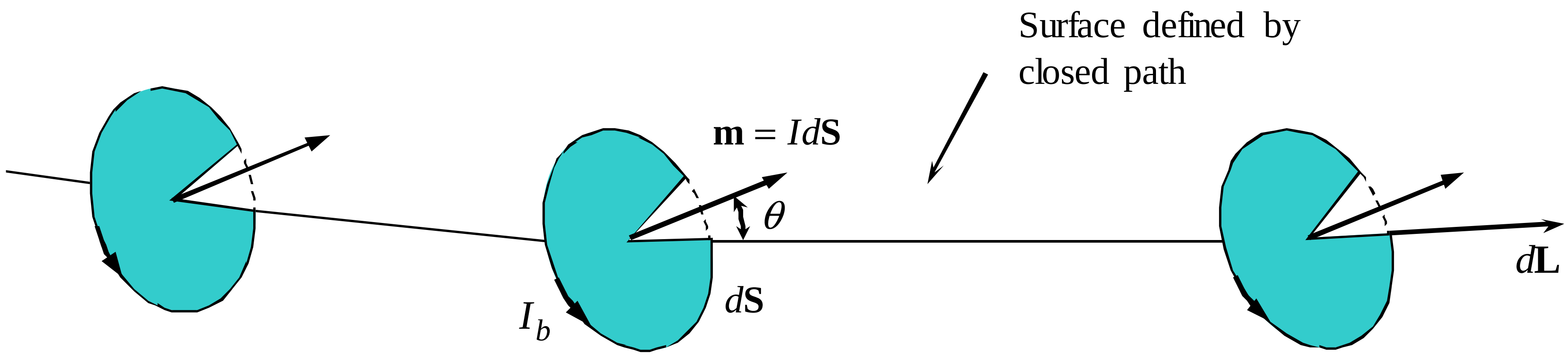
- The magnetization is defined basing on the magnetic dipole moment **m**
- $\mathbf{m} = I_b d\mathbf{S}$ (unit: Am^2)
- I_b: the bound current circulates about a path enclosing $d\mathbf{S}$
- For Δv, the total magnetic dipole moment: $\mathbf{m}_{total} = \sum_{i=1}^{n\Delta v} \mathbf{m}_i$
- n: number of magnetic dipole in a unit volume
- Definition of the magnetization: $\mathbf{M} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} \mathbf{m}_i$
- **M**: the (total) magnetic dipole moment per unit volume

Magnetization & Permeability (2)

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} \mathbf{m}_i \text{ : the (total) magnetic dipole moment per unit volume}$$

$$\left.\begin{matrix} d\mathbf{S}.d\mathbf{L} \\ \mathbf{m} = I_b d\mathbf{S} \end{matrix}\right\} \to dI_b = n\mathbf{m}.d\mathbf{L}$$

$$\to dI_b = \mathbf{M}.d\mathbf{L} \quad \to I_b = \oint \mathbf{M}.d\mathbf{L}$$



Magnetization & Permeability (3)

$$\left.\begin{aligned} &\oint \mathbf{H}.d\mathbf{L} = I_T \\ &\mathbf{H} = \frac{\mathbf{B}}{\mu_0} \\ &I_T = I_b + I \\ &I_b = \oint \mathbf{M}.d\mathbf{L} \end{aligned}\right\} \rightarrow I = I_T - I_b = \oint\left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right).d\mathbf{L}$$

$$\left.\begin{aligned} &\text{Redefine: } \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \quad \rightarrow \boxed{\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})} \\ &\text{(if } \mathbf{M} = 0 \text{ then } \mathbf{B} = \mu_0\mathbf{H}) \\ &\text{Definition of the magnetic susceptibility: } \chi_m = \frac{\mathbf{M}}{\mathbf{H}} \\ &\text{Defintion of the relative permeability: } \mu_R = 1 + \chi_m \\ &\text{Definition of the permeability: } \mu = \mu_0 \mu_R \end{aligned}\right\}$$

$$\rightarrow \boxed{\mathbf{B} = \mu\mathbf{H}}$$

Magnetization & Permeability (4)

$$I_b = \oint_S \mathbf{J}_b.d\mathbf{S}$$

$$I_T = \oint_S \mathbf{J}_T.d\mathbf{S}$$

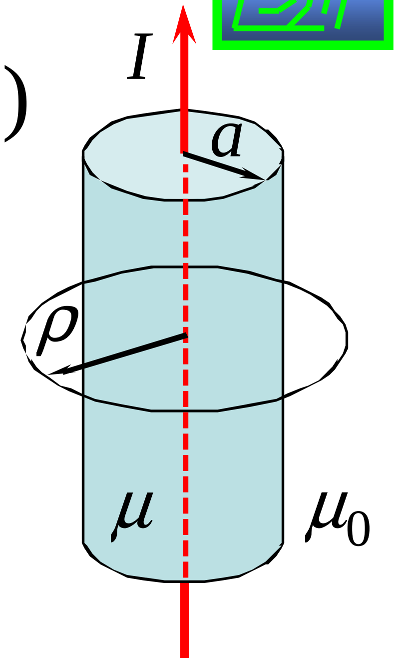
$$I = \oint_S \mathbf{J}.d\mathbf{S}$$

$$\left.\begin{array}{l} I_b = \oint \mathbf{M}.d\mathbf{L} \\ I_b = \oint_S \mathbf{J}_b.d\mathbf{S} \\ \oint \mathbf{M}.d\mathbf{L} = \int_S (\nabla\times\mathbf{M}).d\mathbf{S} \end{array}\right\} \rightarrow \left\{\begin{array}{l} \nabla\times\mathbf{M} = \mathbf{J}_b \\ \nabla\times\dfrac{\mathbf{B}}{\mu_0} = \mathbf{J}_T \\ \nabla\times\mathbf{H} = \mathbf{J} \end{array}\right.$$

Magnetization & Permeability (5)

Ex.

A line current I of infinite extent is within a cylinder of radius a that has permeability μ, the cylinder is surrounded by free space. Find **B**, **H**, & **M** everywhere, & the current density?



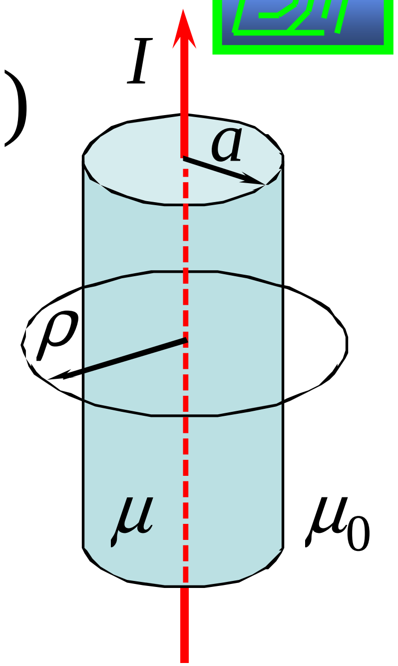
$$I = \oint \mathbf{H}.d\mathbf{L} = H_\varphi 2\pi\rho \rightarrow H_\varphi = \frac{I}{2\pi\rho}$$

$$\rightarrow B_\varphi = \begin{cases} \mu H_\varphi = \dfrac{\mu I}{2\pi\rho}, & 0 < \rho < a \\ \mu_0 H_\varphi = \dfrac{\mu_0 I}{2\pi\rho}, & \rho > a \end{cases}$$

Magnetization & Permeability (6)

Ex.

A line current I of infinite extent is within a cylinder of radius a that has permeability μ, the cylinder is surrounded by free space. Find **B**, **H**, & **M** everywhere, & the current density?



$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \quad \rightarrow \mathbf{M} = \frac{\mathbf{B}}{\mu_0} - \mathbf{H}$$

$$\rightarrow M_\varphi = \begin{cases} \left(\dfrac{\mu}{\mu_0} - 1\right) H_\varphi = \dfrac{(\mu - \mu_0)I}{2\pi\rho\mu_0}, & 0 < \rho < a \\ 0, & \rho > a \end{cases}$$

$$\mathbf{J}_b = \nabla \times \mathbf{M} = -\frac{\partial M_\varphi}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho M_\varphi)\mathbf{a}_z = 0, \quad 0 < \rho < a$$

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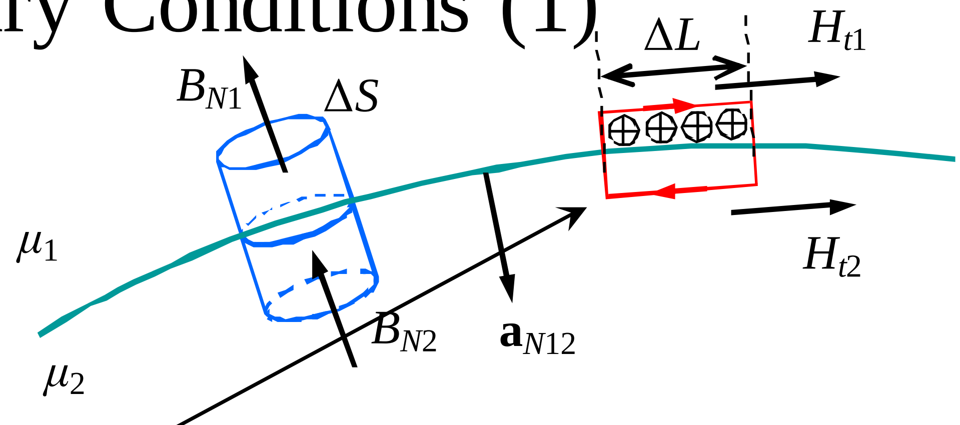
Magnetic Boundary Conditions (1)



$$\oint_S \mathbf{B}.d\mathbf{S} = 0$$

$$\rightarrow B_{N1}\Delta S - B_{N2}\Delta S = 0$$

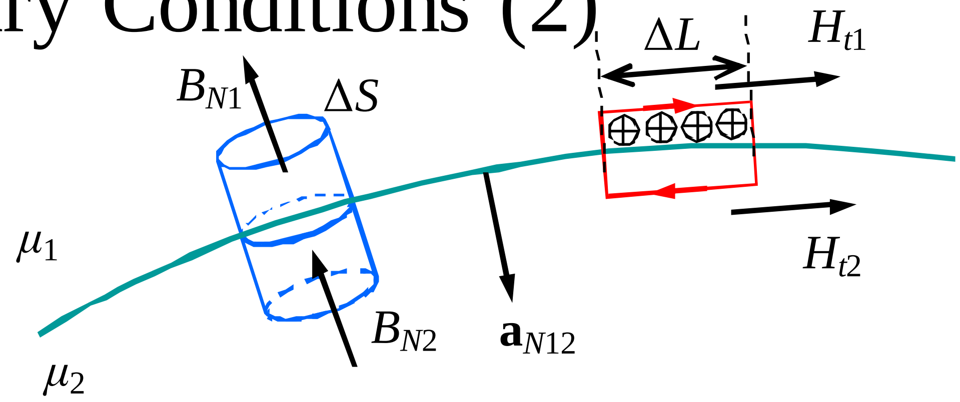
$$\rightarrow B_{N2} = B_{N1}$$

$$\rightarrow H_{N2} = \frac{\mu_1}{\mu_2} H_{N1} \quad \rightarrow M_{N2} = \chi_{m2} H_{N2} = \chi_{m2}\frac{\mu_1}{\mu_2} H_{N1} = \frac{\chi_{m2}\mu_1}{\chi_{m1}\mu_2} M_{N1}$$

$$\oint \mathbf{H}.d\mathbf{L} = I \quad \rightarrow H_{t1}\Delta L - H_{t2}\Delta L = K\Delta L$$

$$\rightarrow H_{t1} - H_{t2} = K \quad \rightarrow \frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = K \quad \rightarrow M_{t2} = \frac{\chi_{m2}}{\chi_{m1}} M_{t1} - \chi_{m2} K$$

Magnetic Boundary Conditions (2)



$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{N12} = \mathbf{K}$$

$$(\mathbf{H}_{t1} - \mathbf{H}_{t2}) = \mathbf{a}_{N12} \times \mathbf{K}$$

Normal	Tangential
$H_{N2} = \dfrac{\mu_1}{\mu_2} H_{N1}$	$H_{t1} - H_{t2} = K$
$B_{N2} = B_{N1}$	$\dfrac{B_{t1}}{\mu_1} - \dfrac{B_{t2}}{\mu_2} = K$
$M_{N2} = \dfrac{\chi_{m2}\mu_1}{\chi_{m1}\mu_2} M_{N1}$	$M_{t2} = \dfrac{\chi_{m2}}{\chi_{m1}} M_{t1} - \chi_{m2} K$

Magnetic Boundary Conditions (3)

Ex. 1

Where $z > 0$ (region 1), $\mu = \mu_1 = 4\ \mu$H/m; where $z < 0$ (region 2), $\mu_2 = 7\ \mu$H/m; at $z = 0$, given a surface current $\mathbf{K} = 80\mathbf{a}_x$ A/m. In region 1 there is a magnetic field $\mathbf{B}_1 = 2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$ mT. Find $\mathbf{B}_2$.

$$\mathbf{B}_{N1} = (\mathbf{B}_1.\mathbf{a}_{N12})\mathbf{a}_{N12} = [(2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z).(-\mathbf{a}_z)](-\mathbf{a}_z) = \mathbf{a}_z \text{ mT}$$

$$\left.\begin{aligned} &\to \mathbf{B}_{N2} = \mathbf{B}_{N1} = \mathbf{a}_z \text{ mT} \\ &\mathbf{B}_1 = \mathbf{B}_{N1} + \mathbf{B}_{t1} \to \mathbf{B}_{t1} = \mathbf{B}_1 - \mathbf{B}_{N1} \end{aligned}\right\}$$

$$\to \mathbf{B}_{t1} = (2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z) - (\mathbf{a}_z) = 2\mathbf{a}_x - 3\mathbf{a}_y \text{ mT}$$

$$\to \mathbf{H}_{t1} = \frac{\mathbf{B}_{t1}}{\mu_1} = \frac{(2\mathbf{a}_x - 3\mathbf{a}_y)10^{-3}}{4\times10^{-6}} = 500\mathbf{a}_x - 750\mathbf{a}_y \text{ A/ m}$$

Magnetic Boundary Conditions (4)

Ex. 1

Where $z > 0$ (region 1), $\mu = \mu_1 = 4\ \mu$H/m; where $z < 0$ (region 2), $\mu_2 = 7\ \mu$H/m; at $z = 0$, given a surface current $\mathbf{K} = 80\mathbf{a}_x$ A/m. In region 1 there is a magnetic field $\mathbf{B}_1 = 2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$ mT. Find $\mathbf{B}_2$.

$$\left.\begin{aligned}&\mathbf{H}_{t1} = 500\mathbf{a}_x - 750\mathbf{a}_y \text{ A/m}\\ &(\mathbf{H}_{t1} - \mathbf{H}_{t2}) = \mathbf{a}_{N12} \times \mathbf{K}\end{aligned}\right\}$$

$$\left.\begin{aligned}&\rightarrow \mathbf{H}_{t2} = \mathbf{H}_{t1} - \mathbf{a}_{N12} \times \mathbf{K} = 500\mathbf{a}_x - 750\mathbf{a}_y - (-\mathbf{a}_z) \times 80\mathbf{a}_x\\ &\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}\end{aligned}\right\}$$

$$\rightarrow \mathbf{H}_{t2} = 500\mathbf{a}_x - 750\mathbf{a}_y + 80\mathbf{a}_y = 500\mathbf{a}_x - 670\mathbf{a}_y \text{ A/m}$$

Magnetic Boundary Conditions (5)

Ex. 1

Where $z > 0$ (region 1), $\mu = \mu_1 = 4\ \mu$H/m; where $z < 0$ (region 2), $\mu_2 = 7\ \mu$H/m; at $z = 0$, given a surface current $\mathbf{K} = 80\mathbf{a}_x$ A/m. In region 1 there is a magnetic field $\mathbf{B}_1 = 2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$ mT. Find $\mathbf{B}_2$.

$$\mathbf{H}_{t2} = 500\mathbf{a}_x - 670\mathbf{a}_y \text{ A/m}$$

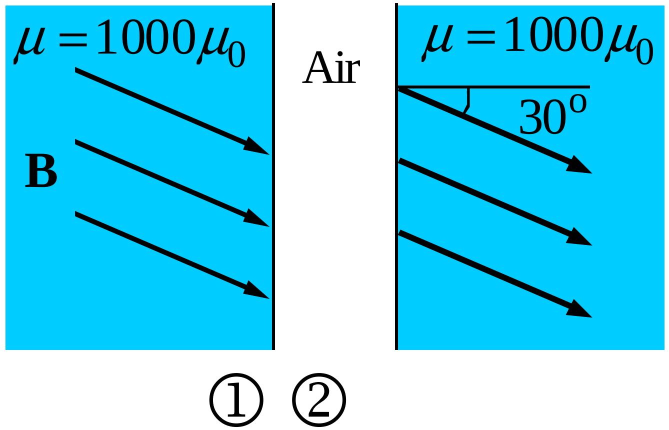
$$\left.\begin{aligned} &\rightarrow \mathbf{B}_{t2} = \mu_2\mathbf{H}_{t2} = 7\times10^{-6}(500\mathbf{a}_x - 670\mathbf{a}_y) = 3.5\mathbf{a}_x - 4.69\mathbf{a}_y \text{ mT} \\ &\mathbf{B}_2 = \mathbf{B}_{N2} + \mathbf{B}_{t2} \\ &\mathbf{B}_{N2} = \mathbf{a}_z \text{ mT} \end{aligned}\right\}$$

$$\rightarrow \mathbf{B}_2 = \mathbf{B}_{N2} + \mathbf{B}_{tt2} = \boxed{3.5\mathbf{a}_x - 4.69\mathbf{a}_y + \mathbf{a}_z \text{ mT}}$$

Magnetic Boundary Conditions (6)

Ex. 2

A uniform magnetic field of strength B = 1.2 T exists within an iron core. If an air gap is cut with the orientation shown, find the magnitude and direction of B in the gap.



$$B_{N2} = B_{N1} = B\cos 30^o = 1.2 \times 0.866 = 1.0 \text{ T}$$

$$H_{t1} = H_{t2} \rightarrow \frac{B_{t1}}{\mu} = \frac{B_{t2}}{\mu_0}$$

$$\rightarrow B_{t2} = \frac{\mu_0}{1000\mu_0} B_{t1} = \frac{1}{1000} \times 1.2 \times \sin 30^o = 0.06 \text{ mT}$$

$$\rightarrow B_2 = \sqrt{B_{N2}^2 + B_{t2}^2} = \sqrt{1.0^2 + (0.06 \times 10^{-3})^2} \approx \boxed{1 \text{ T}}$$

Magnetic Forces & Inductance

1. Force on a Moving Charge
2. Force on a Differential Current Element
3. Force between Differential Current Elements
4. Force & Torque on a Closed Circuit
5. Magnetization & Permeability
6. Magnetic Boundary Conditions
7. **The Magnetic Circuit**
8. Potential Energy of Magnetic Fields
9. Inductance & Mutual Inductance

The Magnetic Circuit (1)

$$\mathbf{E} = -\nabla V$$

$$V_{AB} = \int_A^B \mathbf{E}.d\mathbf{L}$$

$$\mathbf{J} = \sigma \mathbf{E}$$

$$I = \int_S \mathbf{J}.d\mathbf{S}$$

$$V = IR$$

$$R = \frac{d}{\sigma S}$$

$$\oint \mathbf{E}.d\mathbf{L} = 0$$

$$\mathbf{H} = -\nabla V_m$$

$$V_{mAB} = \int_A^B \mathbf{H}.d\mathbf{L}$$

$$\mathbf{B} = \mu \mathbf{H}$$

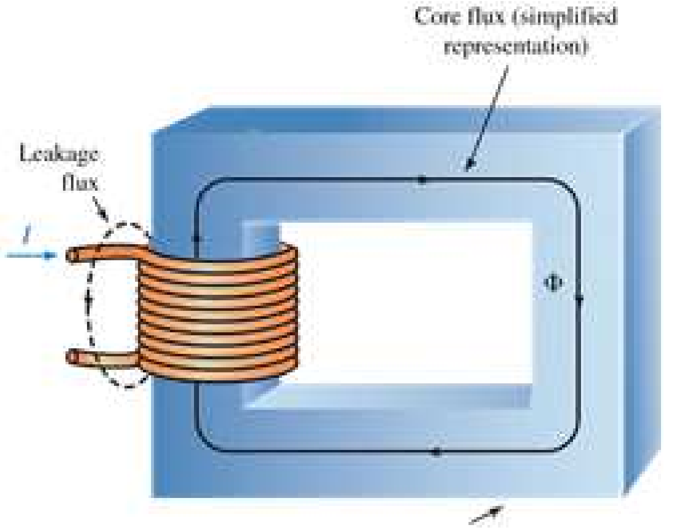
$$\Phi = \int_S \mathbf{B}.d\mathbf{S}$$

$$V_m = \Phi R_m$$

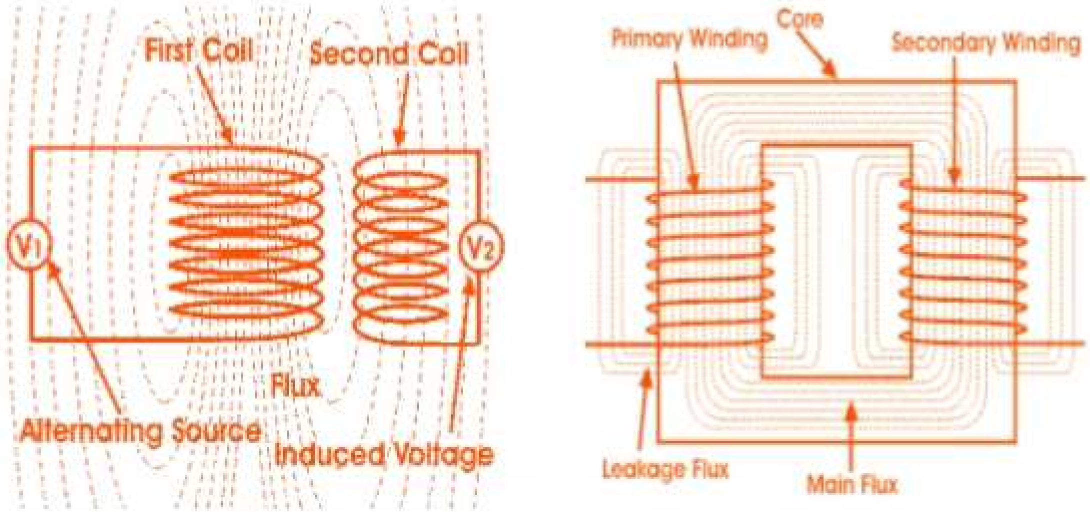
$$R_m = \frac{d}{\mu S}$$

$$\oint \mathbf{H}.d\mathbf{L} = I_{\text{total}}$$

The Magnetic Circuit (2)

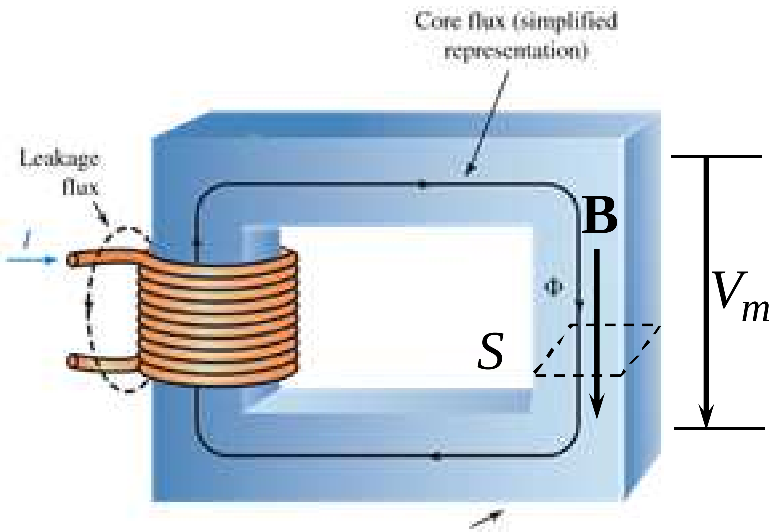


https://www.kullabs.com/classes/subjects/units/lessons/notes/note-detail/2817



https://www.slideshare.net/prodipdasdurjoy/presentation-of-manufacturing-of-distribution-transformer-prodip

The Magnetic Circuit (3)



https://www.kullabs.com/classes/subjects/units/lessons/notes/note-detail/2817

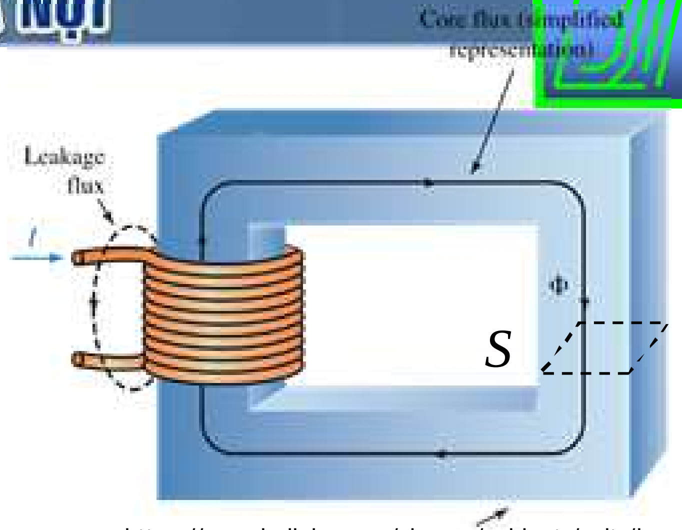
$$V_{mAB} = \int_A^B \mathbf{H}.d\mathbf{L} \approx HL_{AB}$$

$$\Phi = \int_S \mathbf{B}.d\mathbf{S} \approx BS$$

$$\oint \mathbf{H}.d\mathbf{L} = I_{\text{total}} = NI$$

Ex. 1 The Magnetic Circuit (4)

The core has a total average length of 0.6 m & a cross-sectional area of 16 cm². The coil has 500 turns. Find the current to produce a flux of 1.6 mWb in the core?

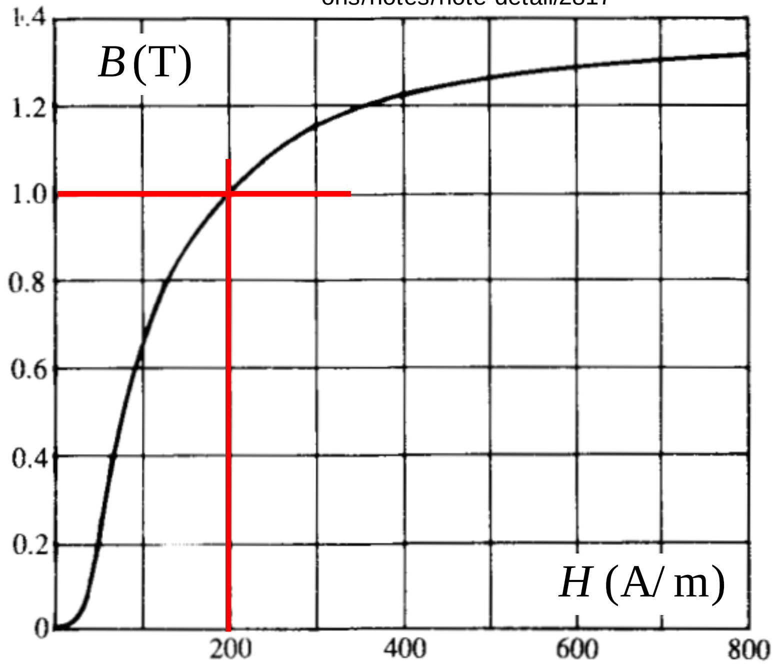


https://www.kullabs.com/classes/subjects/units/lessons/notes/note-detail/2817

$$\Phi = BS$$

$$\rightarrow B = \frac{\Phi}{S} = \frac{1.6\times10^{-3}}{16\times10^{-4}} = 1\text{T}$$

$$\rightarrow H = 200\text{ A/ m}$$



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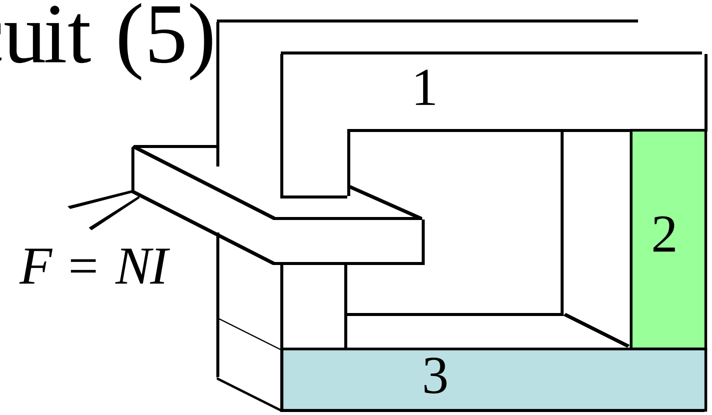
$$\oint \mathbf{H}.d\mathbf{L} = I_{\text{total}} = NI$$

$$\rightarrow H\ell = NI$$

$$\rightarrow I = \frac{H\ell}{N} = \frac{200\times0.6}{500} = \boxed{0.24\text{ A}}$$

Ex. 2

The Magnetic Circuit (5)



N = 500 turns, ℓ_1 = 40cm, $S_1 = S_3$ = 10cm^2, ℓ_2 = 20cm, S_2 = 16cm^2, ℓ_3 = 30cm. Find the current to produce a flux of 1 mWb in the core?

$$B_1 = B_3 = \frac{\Phi}{S_1} = \frac{1\times10^{-3}}{10\times10^{-4}} = 1\,\text{T} \rightarrow H_1 = H_3 = 200\ \text{A/m}$$

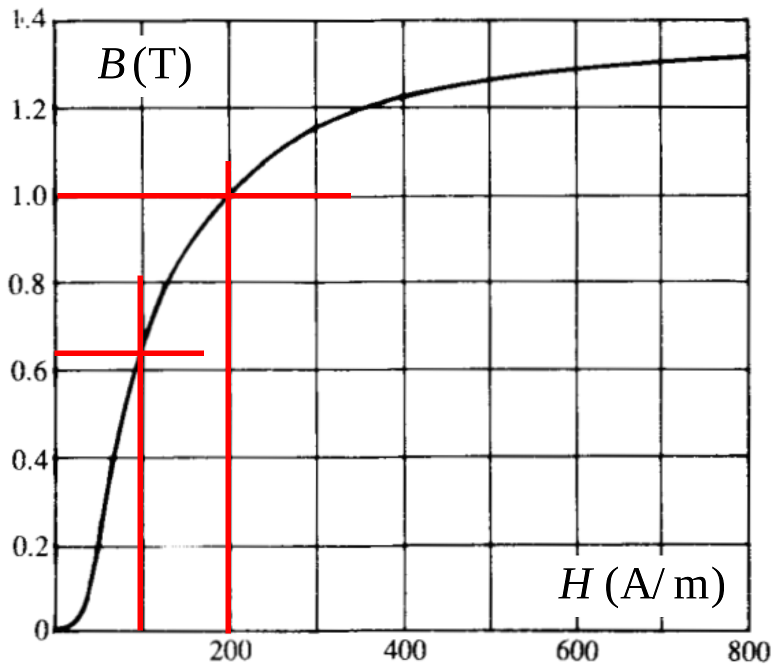
$$B_2 = \frac{\Phi}{S_2} = \frac{1\times10^{-3}}{16\times10^{-4}} = 0.625\,\text{T} \rightarrow H_2 = 95\,\text{A/m}$$

$$\oint \mathbf{H}.d\mathbf{L} = NI$$

$$\rightarrow H_1\ell_1 + H_2\ell_2 + H_3\ell_3 = NI$$

$$\rightarrow I = \frac{H_1\ell_1 + H_2\ell_2 + H_3\ell_3}{N}$$

$$= \frac{(200\times40 + 95\times20 + 200\times30)10^{-2}}{500} = \boxed{0.318\,\text{A}}$$



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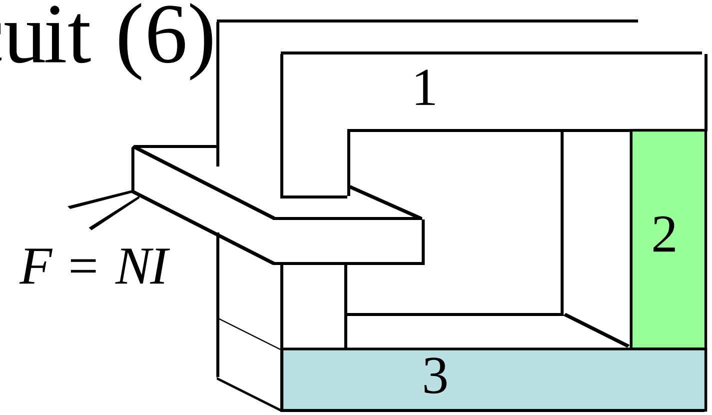
Ex. 3 The Magnetic Circuit (6)

N = 500 turns, ℓ_1 = 40cm, $S_1 = S_3$ = 10cm², ℓ_2 = 20cm, S_2 = 16cm², ℓ_3 = 30cm, I = 0.5A. Find the flux in the core?



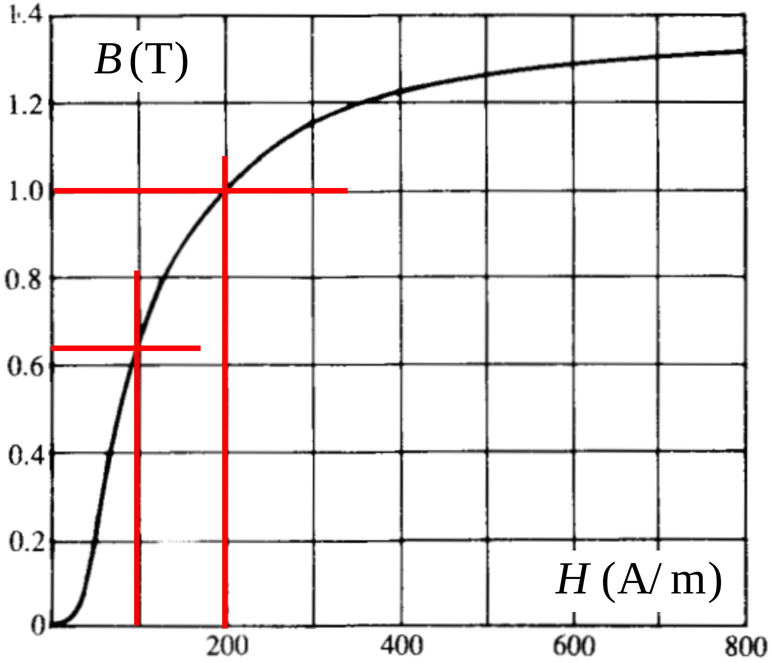
Suppose Φ = 1mWb

$$B_1 = B_3 = \frac{\Phi}{S_1} = \frac{1\times10^{-3}}{10\times10^{-4}} = 1\,\mathrm{T} \quad \rightarrow H_1 = H_3 = 200\,\mathrm{A/m}$$

$$B_2 = \frac{\Phi}{S_2} = \frac{1\times10^{-3}}{16\times10^{-4}} = 0.625\,\mathrm{T} \quad \rightarrow H_2 = 95\,\mathrm{A/m}$$

$$H_1\ell_1 + H_2\ell_2 + H_3\ell_3 = NI \rightarrow I = \frac{H_1\ell_1 + H_2\ell_2 + H_3\ell_3}{N}$$

$$I = \frac{(200\times40 + 95\times20 + 200\times30)10^{-2}}{500} = 0.318\,\mathrm{A}$$



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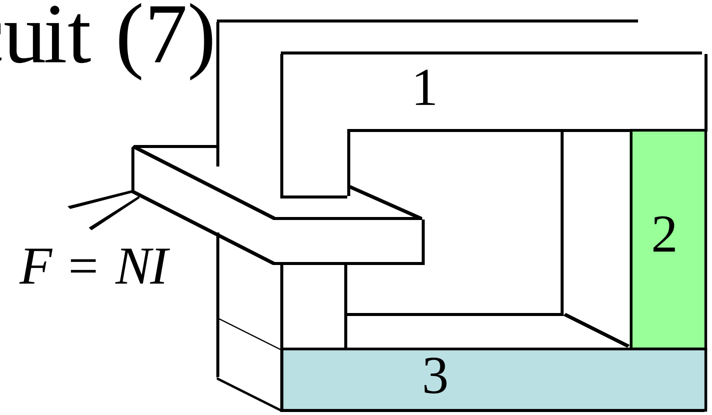
Ex. 3 The Magnetic Circuit (7)

N = 500 turns, ℓ_1 = 40cm, $S_1 = S_3$ = 10cm^2, ℓ_2 = 20cm, S_2 = 16cm^2, ℓ_3 = 30cm, I = 0.5A. Find the flux in the core?



Suppose Φ = 1mWb → I = 0.318 A

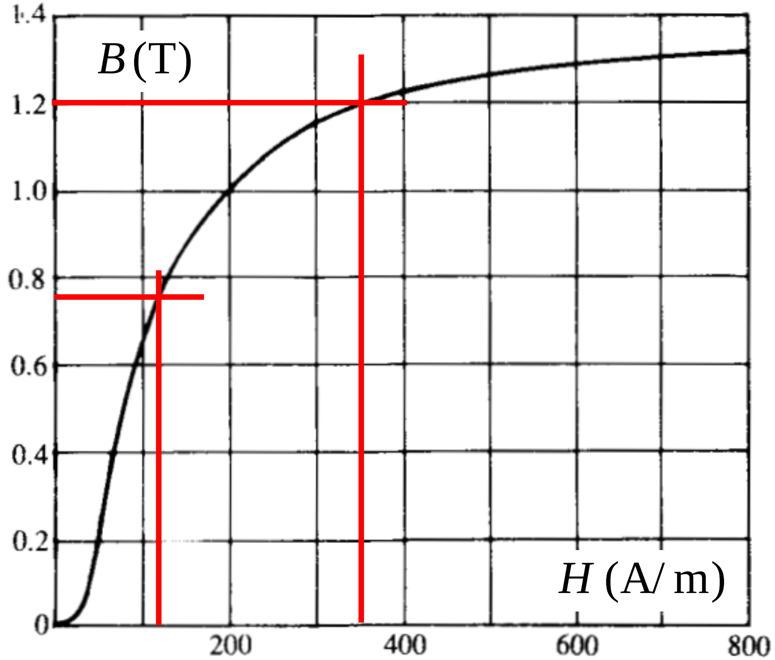
Suppose Φ = 1.2mWb

$$B_1 = B_3 = \frac{\Phi}{S_1} = \frac{1.2\times10^{-3}}{10\times10^{-4}} = 1.2\,\mathrm{T} \quad \to H_1 = H_3 = 350\ \mathrm{A/m}$$

$$B_2 = \frac{\Phi}{S_2} = \frac{1.2\times10^{-3}}{16\times10^{-4}} = 0.75\,\mathrm{T} \quad \to H_2 = 120\ \mathrm{A/m}$$

$$H_1\ell_1 + H_2\ell_2 + H_3\ell_3 = NI \to I = \frac{H_1\ell_1 + H_2\ell_2 + H_3\ell_3}{N}$$

$$I = \frac{(350\times40 + 120\times20 + 350\times30)10^{-2}}{500} = 0.538\ \mathrm{A}$$



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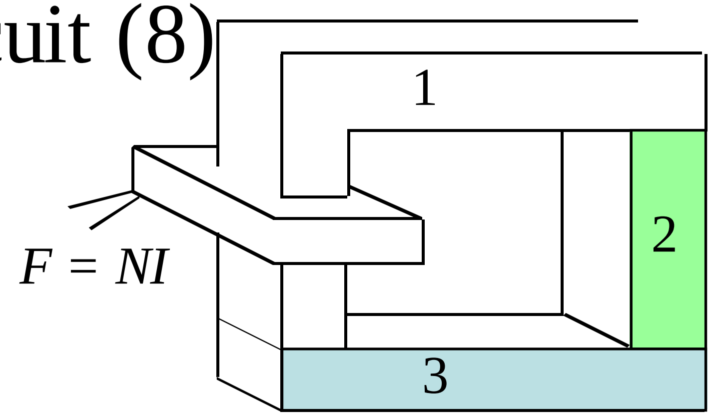
Ex. 3

The Magnetic Circuit (8)



N = 500 turns, ℓ_1 = 40cm, $S_1 = S_3$ = 10cm^2, ℓ_2 = 20cm, S_2 = 16cm^2, ℓ_3 = 30cm, I = 0.5A. Find the flux in the core?

Suppose Φ = 1mWb → I = 0.318 A

Suppose Φ = 1.2mWb → I = 0.538 A

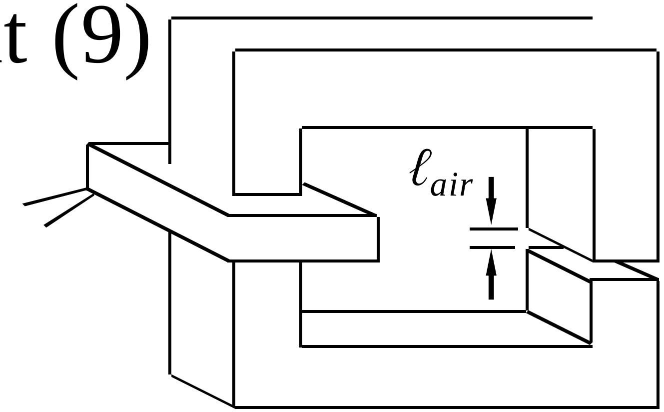
$$\Phi = aI + b \quad \rightarrow \begin{cases} 0.001 = 0.318a + b \\ 0.0012 = 0.538a + b \end{cases} \rightarrow \begin{cases} a = 0.9091\times10^{-3} \\ b = 0.7109\times10^{-3} \end{cases}$$

$$\rightarrow \Phi = (0.9091I + 0.7109)10^{-3}$$

$$= 0.9091\times0.5 + 0.7109 = \boxed{1.1654\,\text{mWb}}$$

Ex. 4

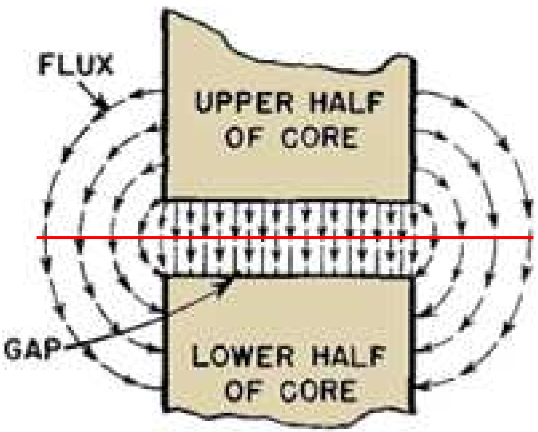
The Magnetic Circuit (9)



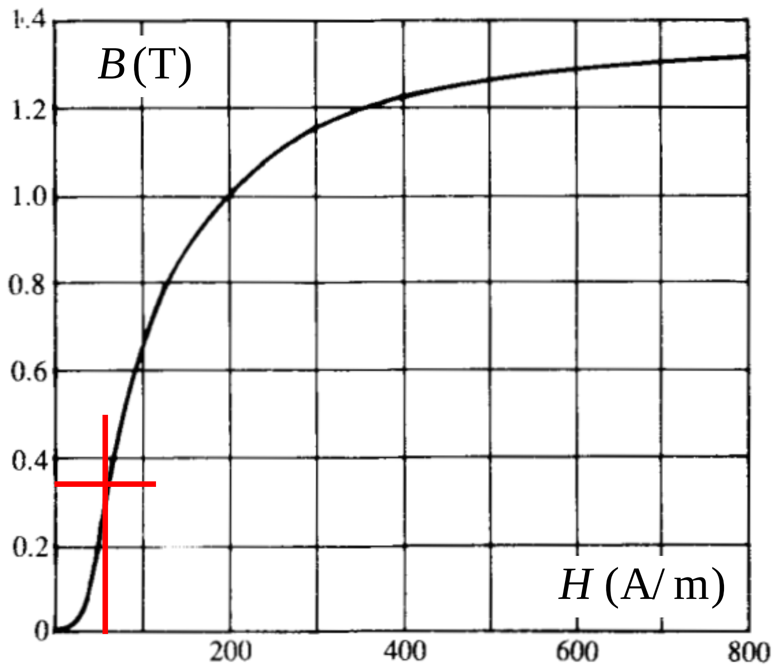
The core has a total average length of 0.44 m & a cross-sectional area of (0.02)(0.02) m². The air gap is 2 mm. The coil has 400 turns. Find the current to produce a flux of 0.14 mWb in the air gap?

$$B_c = \frac{\Phi}{S_c} = \frac{0.141\times10^{-3}}{4\times10^{-4}} = 0.35\,\mathrm{T} \rightarrow H_c = 60\ \mathrm{A/m}$$

$$B_a = \frac{\Phi}{S_a} = \frac{0.14\times10^{-3}}{(2\times10^{-2}\times110\%)^2} = 0.29\,\mathrm{T}$$



$$H_a = \frac{B_a}{\mu_0} = \frac{0.29}{4\pi\times10^{-7}} = 2.31\times10^{5}\ \mathrm{A/m}$$



http://www.vias.org/eltransformers/lee_electronic_transformers_07_07.html

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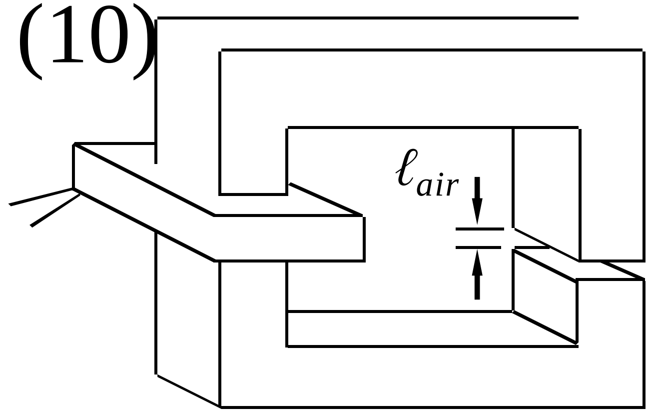
Ex. 4 The Magnetic Circuit (10)



The core has a total average length of 0.44 m & a cross-sectional area of (0.02)(0.02) m². The air gap is 2 mm. The coil has 400 turns. Find the current to produce a flux of 0.14 mWb in the air gap?

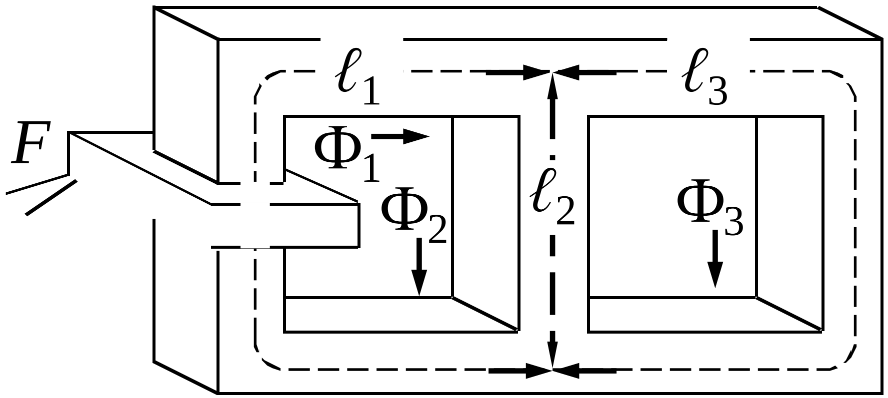
$$B_c = \frac{\Phi}{S_c} = \frac{0.141\times10^{-3}}{4\times10^{-4}} = 0.35\,\mathrm{T} \rightarrow H_c = 60\ \mathrm{A/m}$$

$$B_c = \frac{\Phi}{S_c} = \frac{0.14\times10^{-3}}{(2\times10^{-2}\times110\%)^2} = 0.29\,\mathrm{T} \quad \rightarrow H_a = 2.31\times10^5\ \mathrm{A/m}$$

$$H_c\ell_c + H_a\ell_a = NI \rightarrow I = \frac{H_c\ell_c + H_a\ell_a}{N} = \frac{60\times0.44 + (2.31\times10^5)(2\times10^{-3})}{400} = \boxed{1.22\ \mathrm{A}}$$

Ex. 5

The Magnetic Circuit (11)



$$F - H_1\ell_1 = H_2\ell_2 = H_3\ell_3$$

$$\Phi_1 = \Phi_2 + \Phi_3$$

Magnetic Forces & Inductance

1. Force on a Moving Charge
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5. Magnetization & Permeability
6. Magnetic Boundary Conditions
7. The Magnetic Circuit
8. **Potential Energy of Magnetic Fields**
9. Inductance & Mutual Inductance

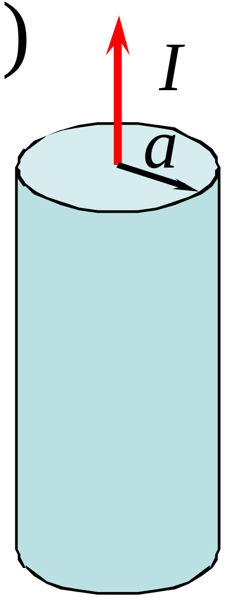
Potential Energy of Magnetic Fields (1)

$$W_H = \frac{1}{2}\int_V \mathbf{B.H}dv$$

$$= \frac{1}{2}\int_V \mu H^2 dv$$

$$= \frac{1}{2}\int_V \frac{B^2}{\mu} dv$$

Ex. Potential Energy of Magnetic Fields (2)



Find the magnetic energy associated with unit length of an infinitely long straight wire of radius *a* carrying a current *I*.

$$\mathbf{H}_{inside} = \frac{I}{2\pi a^2}\rho\mathbf{a}_\varphi$$

$$W_{inside} = \frac{1}{2}\int_V \mu H^2 dv$$

$$= \frac{1}{2}\int_0^a \left(\mu\frac{I^2}{4\pi^2 a^4}\rho^2\right)(1\times 2\pi\rho\times d\rho) = \frac{\mu I^2}{16\pi}$$

$$\mathbf{H}_{outside} = \frac{I}{2\pi\rho}\mathbf{a}_\varphi$$

$$W_{outside} = \frac{1}{2}\int_V \mu_0 H^2 dv = \frac{1}{2}\int_0^\infty \left(\mu_0\frac{I^2}{4\pi^2\rho^2}\right)(1\times 2\pi\rho\times d\rho) = \infty$$

Magnetic Forces & Inductance

1. Force on a Moving Charge
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3. Force between Differential Current Elements
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7. The Magnetic Circuit
8. Potential Energy of Magnetic Fields
9. **Inductance & Mutual Inductance**

Inductance & Mutual Inductance (1)

$$\left.\begin{aligned} &\Phi = \int_S \mathbf{B}.d\mathbf{S} = \int_S \mu_r \mu_0 \mathbf{H}.d\mathbf{S} \\ &\oint \mathbf{H}.d\mathbf{L} = I \end{aligned}\right\} \rightarrow \Phi \sim I$$

$$\boxed{L = \frac{\Phi}{I}}$$

$$L = N\frac{\Phi}{I}$$

Ex. 1 Inductance & Mutual Inductance (2)

$$L = N\frac{\Phi}{I}$$

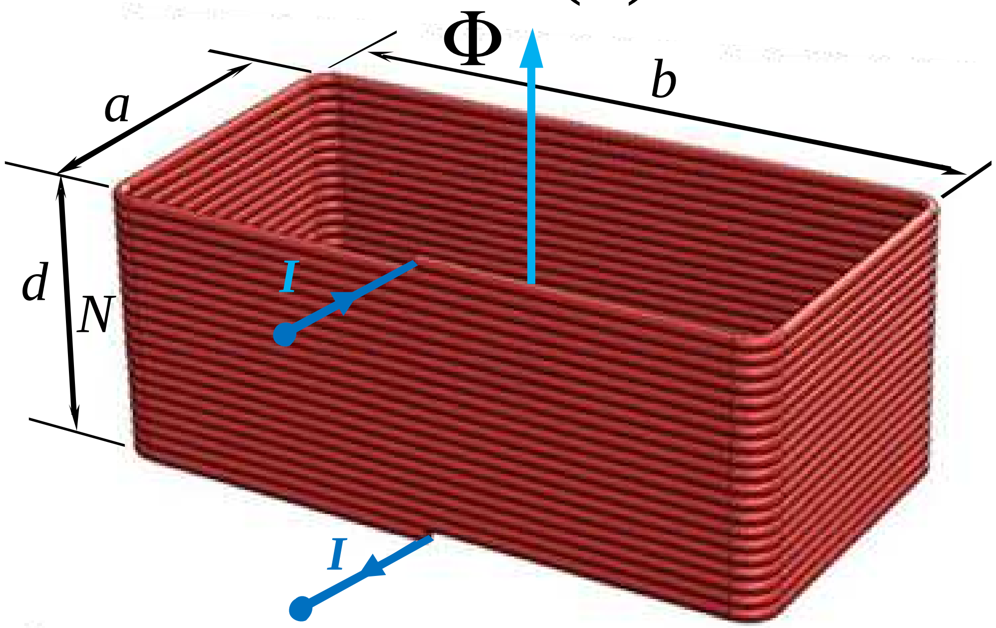
$$\Phi = N\int_S \mathbf{B}.d\mathbf{S} = N\int_S \mu_r\mu_0\mathbf{H}.d\mathbf{S}$$

$$= N\mu_r\mu_0 HS$$

$$\oint \mathbf{H}.d\mathbf{L} = NI$$

$$\rightarrow Hd = NI \rightarrow H = \frac{NI}{d}$$

$$\rightarrow \Phi = N^2\mu_r\mu_0\frac{I}{d}S$$

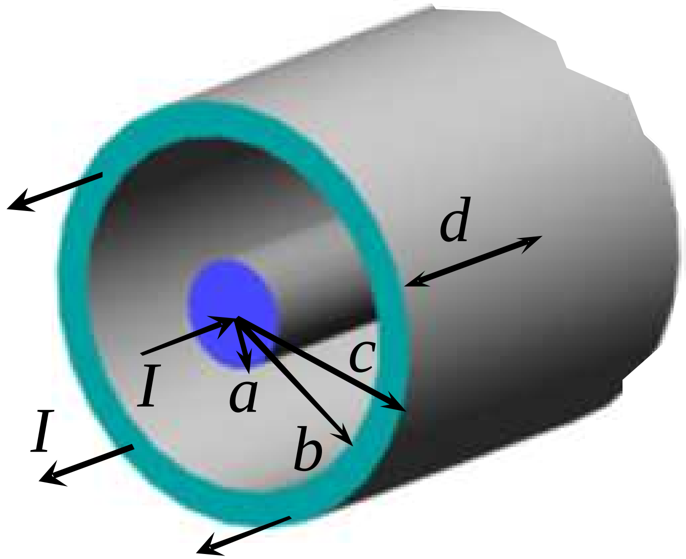


https://www.stlfinder.com/model/continuous-rectangular-coil/1975752

$$\rightarrow L = \frac{N^2\mu_r\mu_0\frac{I}{d}S}{I} = \boxed{\mu_r\mu_0\frac{N^2S}{d}}$$

Ex. 2 Inductance & Mutual Inductance (3)

$$\left.\begin{aligned} \Phi &= \frac{\mu_0 I d}{2\pi} \ln \frac{b}{a} \\ L &= \frac{\Phi}{I} \end{aligned}\right\}$$



$$\rightarrow L = \frac{\mu_0 d}{2\pi} \ln \frac{b}{a} \text{ H}$$

$$\rightarrow \text{per-meter inductance: } L = \frac{\mu_0}{2\pi} \ln \frac{b}{a} \text{ H/m}$$

Ex. 3 Inductance & Mutual Inductance (4)

$$\Phi = N\int_S \mathbf{B}.d\mathbf{S}$$

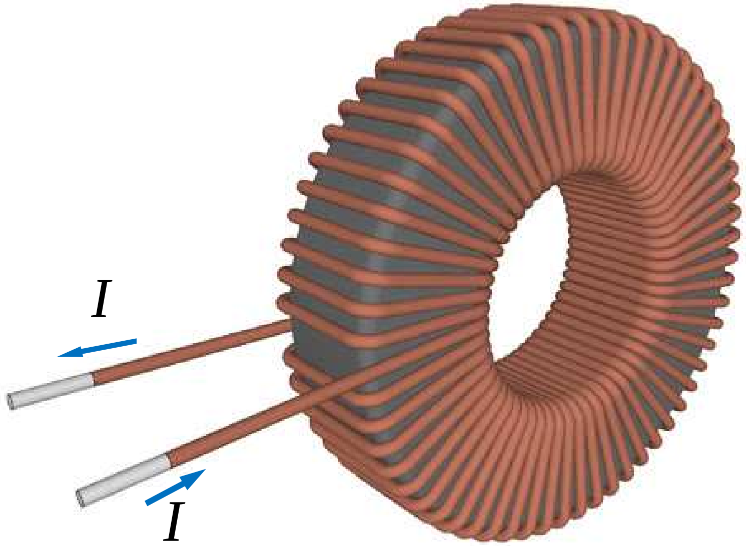
$$= N\int_S \mu_r\mu_0\mathbf{H}.d\mathbf{S}$$

$$NI = \oint \mathbf{H}.d\mathbf{L}$$

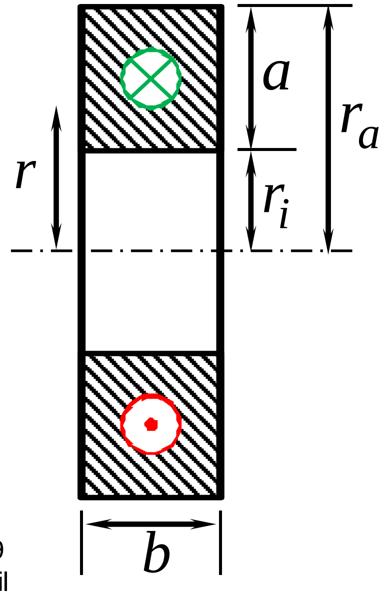
$$= \oint H dL$$

$$= H(2\pi r)$$

$$\to H = \frac{NI}{2\pi r}$$



https://3dwarehouse.sketchup.com/model/ec8884f9 04c69cbb92e83e251d26ee96/Toroidal-Inductor-Coil



$$\to \Phi = N\int_{r_i}^{r_a} \mu_r\mu_0\left(\frac{NI}{2\pi r}\right)(bdr) = \mu_r\mu_0\frac{N^2 Ib}{2\pi}\ln\frac{r_a}{r_i}$$

$$\to L = \frac{\Phi}{I} = \boxed{\mu_r\mu_0\frac{N^2 b}{2\pi}\ln\frac{r_a}{r_i}}$$

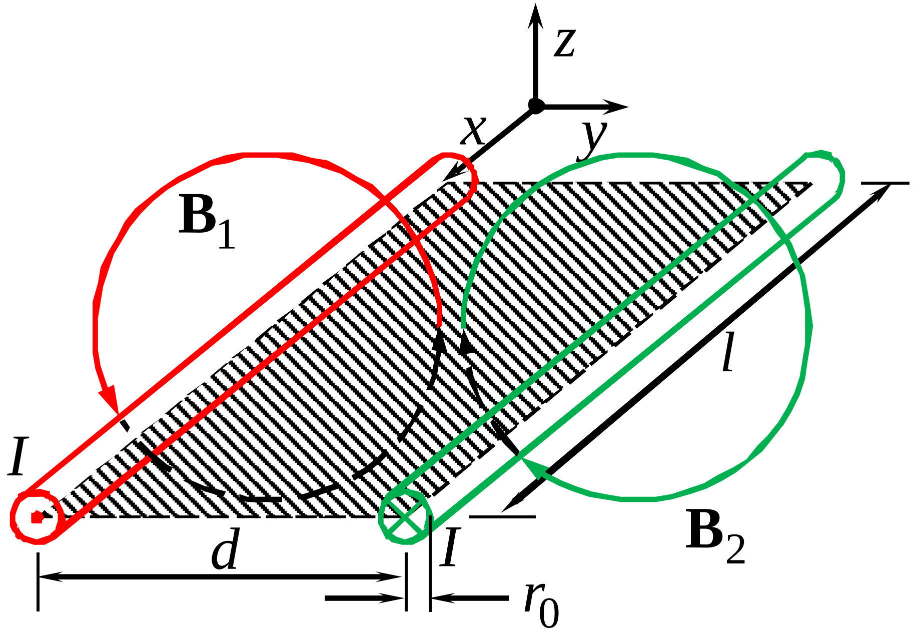
Ex. 4 Inductance & Mutual Inductance (5)

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$$

$$= \frac{I}{2\pi y}\mathbf{a}_z + \frac{I}{2\pi(d-y)}\mathbf{a}_z$$

$$\Phi = \int_S \mu_0 \mathbf{H}.d\mathbf{S}$$

$$= \int_S \mu_0 \left[\frac{I}{2\pi y}\mathbf{a}_z + \frac{I}{2\pi(d-y)}\mathbf{a}_z\right].d\mathbf{S}$$



$$= \frac{\mu_0 I l}{2\pi}\int_{r_0}^{d-r_0}\left(\frac{1}{y} + \frac{1}{d-y}\right)dy = \frac{\mu_0 I l}{2\pi}\ln\frac{d-r_0}{r_0} \quad \rightarrow L = \frac{\Phi}{I} = \boxed{\frac{\mu_0 l}{2\pi}\ln\frac{d-r_0}{r_0}}$$

Inductance & Mutual Inductance (6)

$$\boxed{L = \frac{2W_H}{I^2} \leftrightarrow L = \frac{N\Phi}{I}}$$

$$\left.\begin{aligned} L = \frac{1}{I^2}\int_V \mathbf{A}.\mathbf{J}dv \\ \mathbf{J}dv \approx Id\mathbf{L} \end{aligned}\right\} \rightarrow L = \frac{1}{I}\oint \mathbf{A}.d\mathbf{L}$$

$$\text{Stokes' theorem: } \oint \mathbf{A}.d\mathbf{L} = \int_S (\nabla \times \mathbf{A}).d\mathbf{S}$$

$$\rightarrow L = \frac{1}{I}\int_S (\nabla \times \mathbf{A}).d\mathbf{S}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\left.\begin{aligned} \rightarrow L = \frac{1}{I}\int_S \mathbf{B}.d\mathbf{S} \\ \Phi = \int_S \mathbf{B}.d\mathbf{S} \end{aligned}\right\} \rightarrow L = \frac{\Phi}{I}$$

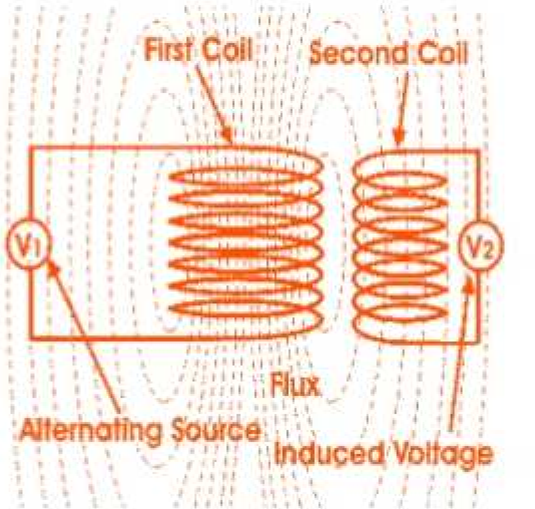
With N turns: $L = \frac{N\Phi}{I}$

Inductance & Mutual Inductance (7)

- Definition of *mutual inductance*:

$$M_{12} = \frac{N_2 \Phi_{12}}{I_1}$$

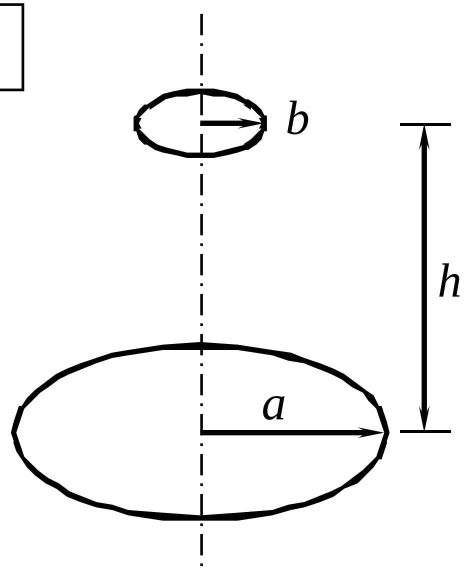
- Φ_{12}: flux linking two circuit
- N_2: number of turns in circuit 2
- Unit: H



https://www.slideshare.net/prodipdasdurjoy/presentation-of-manufacturing-of-distribution-transformer-prodip

Ex. 5 Inductance & Mutual Inductance (8)

Find the mutual inductance between the two coils if $b << a$.



$$\mathbf{H} = \frac{Ia^2}{2(h^2 + a^2)^{3/2}} \mathbf{a}_z$$

$$\Phi_{12} = BS_b = \left(\frac{\mu_0 I a^2}{2(h^2 + a^2)^{3/2}} \right) \left(\pi b^2 \right)$$

$$M_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{\Phi_{12}}{I} = \boxed{\frac{\mu_0 \pi a^2 b^2}{2(h^2 + a^2)^{3/2}}}$$

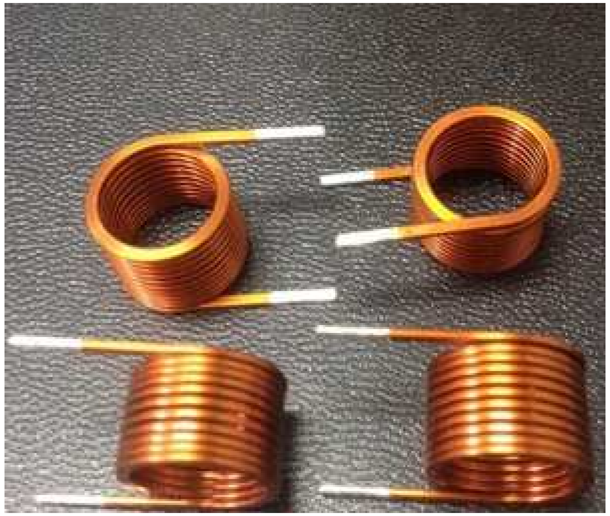
Ex. 6 Inductance & Mutual Inductance (9)

A uniform cylindrical coil in vacuum has R_1, L_1, & N_1 turns. Coaxial and at the center of this coil is a smaller coil of R_2, L_2, & N_2. $R_1 >> R_2$, $L_1 >> L_2$. Calculate the mutual inductance of the two coils.

$$N_1 I_1 = \oint \mathbf{H}_1 . d\mathbf{L} = \oint H_1 dL = H_1 L_1 \rightarrow H_1 = \frac{N_1 I_1}{L_1}$$

$$\Phi_{12} = B_1 S_2 = \mu_0 \frac{N_1 I_1}{L_1} \pi R_2^2$$

$$M_{12} = \frac{N_2 \Phi_{12}}{I_1} = \boxed{\mu_0 \frac{N_1 N_2}{L_1} \pi R_2^2}$$



https://www.alibaba.com/showroom /cylindrical-inductor.html

Ex. 7 Inductance & Mutual Inductance (10)

A toroidal coil of 2000 turns is wound over a magnetic ring with inner radius of 10 mm, outer radius of 15 mm, height of 10 mm, and relative permeability of 500. A very long, straight conductor passing through the center of the toroid carries a time-varying current. Determine the mutual inductance between the toroid and the straight conductor.

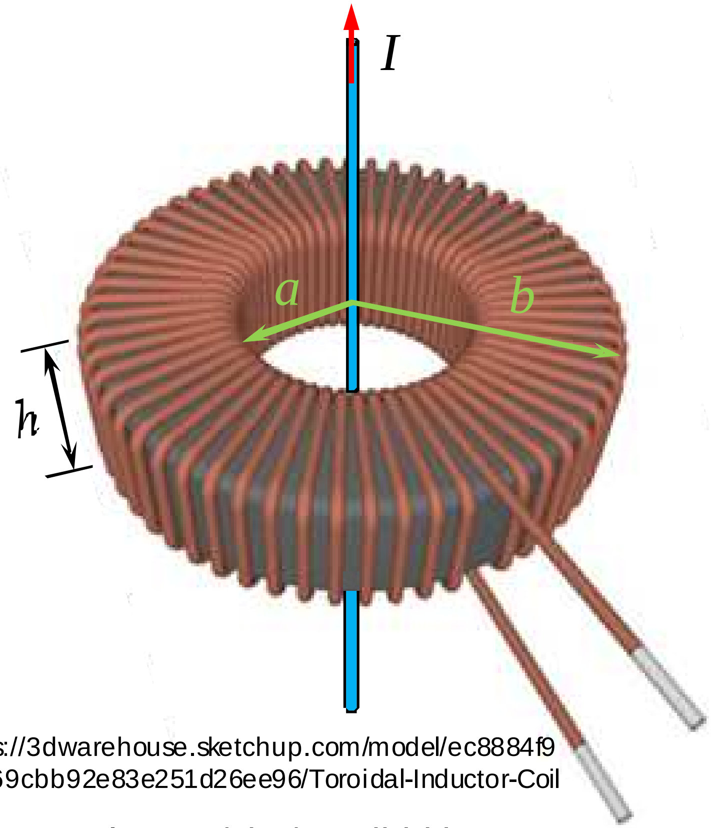
$$\mathbf{B}_1 = \mu \frac{I}{2\pi\rho}\mathbf{a}_\varphi$$

$$\Phi_{12} = \int_{S_2} \mathbf{B}_1 . d\mathbf{S}_2$$

$$= \int_{S_2} \left(\mu \frac{I}{2\pi\rho}\mathbf{a}_\varphi \right) . \left(h d\rho \mathbf{a}_\varphi \right)$$

$$= \frac{\mu I h}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$M_{12} = \frac{N_2 \Phi_{12}}{I_1} = \boxed{\frac{\mu N_2 h}{2\pi} \ln\left(\frac{b}{a}\right)}$$



https://3dwarehouse.sketchup.com/model/ec8884f904c69cbb92e83e251d26ee96/Toroidal-Inductor-Coil

$$
\begin{tikzcd}
Q \arrow[r] \arrow[dd] & \mathbf{F} = \dfrac{Q_1 Q_2}{4\pi\varepsilon R^2}\mathbf{a}_R \arrow[r] \arrow[d] & \mathbf{E} = \dfrac{Q}{4\pi\varepsilon R^2}\mathbf{a}_R \arrow[r] & \mathbf{D} = \varepsilon\mathbf{E} \\
 & W = -Q\int \mathbf{E}.d\mathbf{L} \arrow[r] & V = -\int \mathbf{E}.d\mathbf{L} \arrow[r] & C = \dfrac{Q}{V} \\
I = \dfrac{dQ}{dt} \arrow[r] \arrow[d] & R = \dfrac{V}{I} & & \\
\mathbf{H} = \dfrac{I}{2\pi\rho}\mathbf{a}_\varphi \arrow[r] & \mathbf{B} = \mu\mathbf{H} \arrow[r] \arrow[d] & \Phi = \int \mathbf{B}.d\mathbf{S} \arrow[r] & L = \dfrac{\Phi}{I} \\
 & \mathbf{F} = -I\oint \mathbf{B}\times d\mathbf{L} & &
\end{tikzcd}
$$