Nguyễn Công Phương

Engineering Electromagnetics

Electric Flux Density, Gauss's Law & Divergence

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Electric Flux Density, Gauss's Law & Divergence

1. Electric Flux Density
2. Gauss's Law
3. Divergence
4. Maxwell's First Equation
5. The Vector Operator ∇
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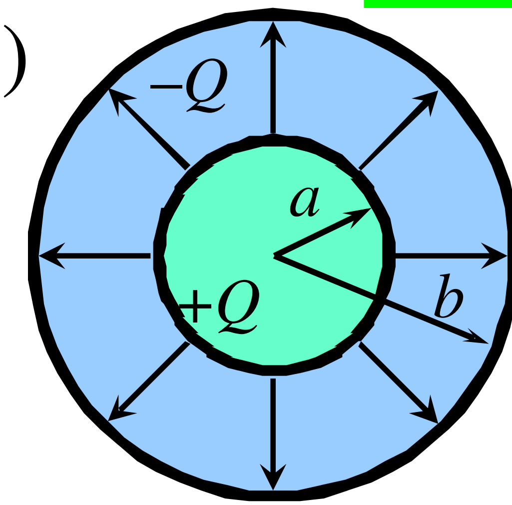
Electric Flux Density (1)

- M. Faraday (1837)
- *Phenomenon*: the total charge on the outer sphere was equal in *magnitude* to the original charge placed on the inner sphere, regardless of the dielectric material between the 2 spheres
- *Conclusion*: there was a "displacement" from the inner sphere to the outer, independent of the medium:

$$\Psi = Q$$

- Ψ: electric flux

Electric Flux Density (2)



$S_a = 4\pi a^2$ (m^2)

Density of the flux at the inner sphere:

$$\frac{\Psi}{4\pi a^2} = \frac{Q}{4\pi a^2}$$

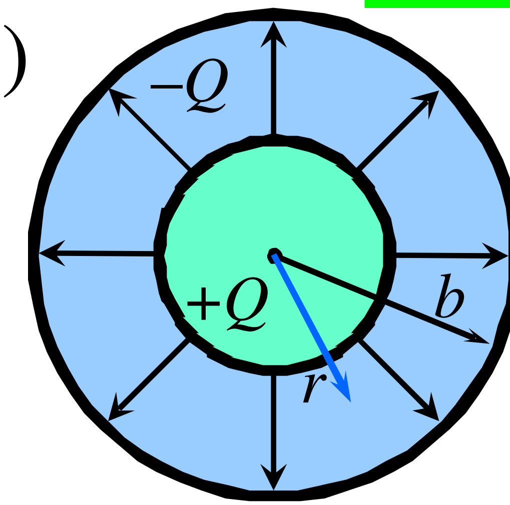
Electric flux density:

$$\mathbf{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\mathbf{a}_r$$

$$\mathbf{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\mathbf{a}_r$$

$$\mathbf{D}\Big|_{a\le r\le b} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

Electric Flux Density (3)



$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r \quad (a < r < b)$$

$$\left.\begin{array}{l} \boxed{\mathbf{D} = \dfrac{Q}{4\pi r^2}\mathbf{a}_r} \\ \mathbf{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r \\ \text{(in free space)} \end{array}\right\} \to \boxed{\mathbf{D} = \varepsilon_0 \mathbf{E}}$$

(in free space)

$$\left.\begin{array}{l} \mathbf{D} = \varepsilon_0 \mathbf{E} \\ \mathbf{E} = \displaystyle\int_V \frac{\rho_v dv}{4\pi\varepsilon_0 R^2}\mathbf{a}_r \end{array}\right\} \to \boxed{\mathbf{D} = \int_V \frac{\rho_v dv}{4\pi R^2}\mathbf{a}_r}$$

Electric Flux Density (4)

Ex. 1

Infinite uniform line charge of 10 nC/m lie along the x & y axes in free space. Find **D** at (0, 0, 3).

Electric Flux Density (5)

Ex. 2

The x & y axes are charged with uniform line charge of 10 nC/m. A point charge of 20nC is located at (3, 3, 0). The whole system is in free space. Find **D** at (0, 0, 3).

Electric Flux Density (6)

Ex. 3

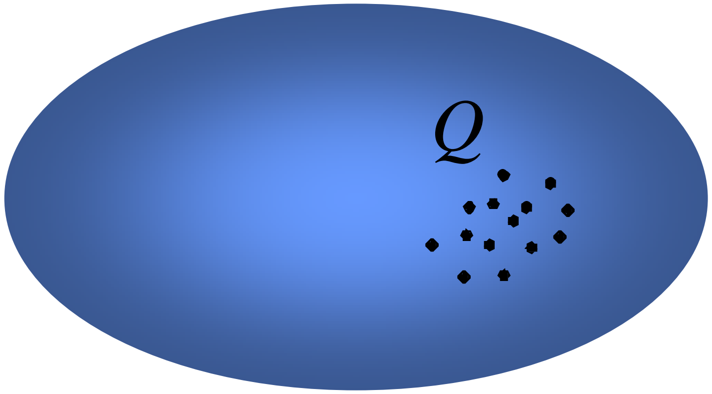
Given 3 infinite uniform sheets (all parallel to $x0y$) at $z = -3$, $z = 2$ & $z = 3$. Their surface charge density are 4 nC/m^2, 6 nC/m^2 & –9 nC/m^2 respectively. Find **D** at $P(5, 5, 5)$.

Electric Flux Density, Gauss's Law & Divergence

1. Electric Flux Density
2. **Gauss's Law**
3. Divergence
4. Maxwell's First Equation
5. The Vector Operator ∇
6. The Divergence Theorem

Gauss's Law (1)

- Generalization of Faraday's experiment
- Gauss's Law: *the electric flux passing through any closed surface is equal to the total charge enclosed by that surface*



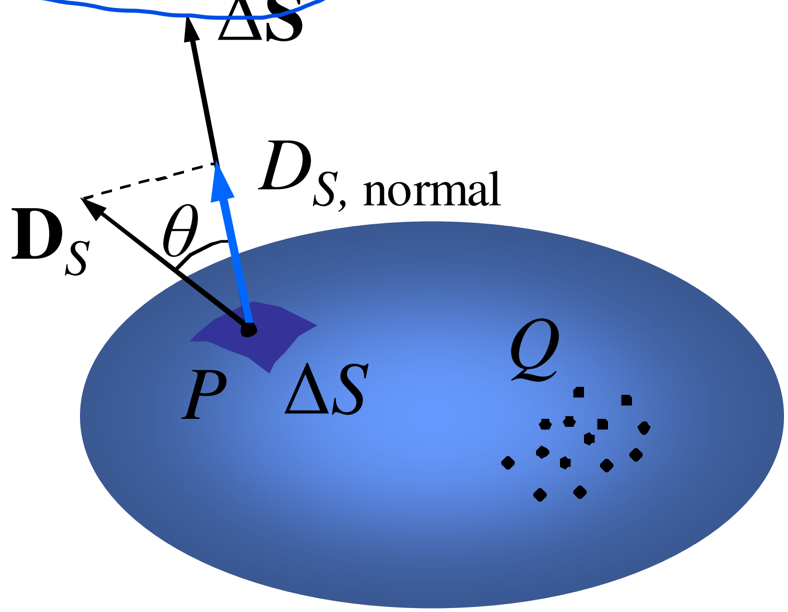
Gauss's Law (2)

$$\Delta\Psi = \text{flux crossing } \Delta\mathbf{S}$$

$$= D_S \cos\theta \Delta S$$

$$= \mathbf{D}_S . \Delta\mathbf{S}$$

$$\rightarrow \Psi = \int d\psi = \oint_{\substack{\text{closed}\\ \text{surface}}} \mathbf{D}_S . d\mathbf{S}$$



$$\boxed{\Psi = \oint_S \mathbf{D}_S . d\mathbf{S} = \text{charge enclosed} = Q}$$

Gauss's Law (3)

$$\boxed{\Psi = \oint_S \mathbf{D}_S . d\mathbf{S} = \text{charge enclosed} = Q}$$

$$Q = \sum Q_n$$

$$Q = \int \rho_L dL$$

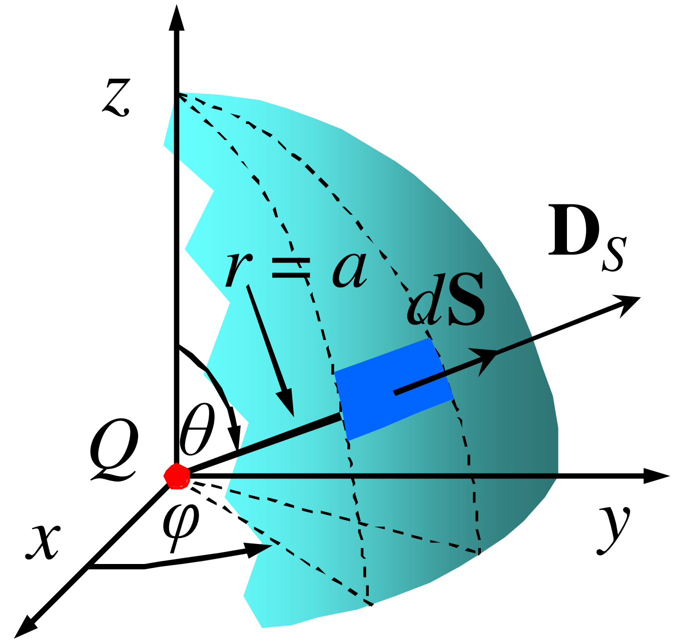
$$Q = \int_S \rho_S dS$$

$$Q = \int_V \rho_v dV$$

$$\boxed{\oint_S \mathbf{D}_S . d\mathbf{S} = \int_V \rho_v dv}$$

Gauss's Law (4)

$$\left.\begin{aligned}\mathbf{E} &= \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r \\ \mathbf{D} &= \varepsilon_0\mathbf{E}\end{aligned}\right\} \rightarrow \mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$



$$\rightarrow \mathbf{D}_S = \frac{Q}{4\pi a^2}\mathbf{a}_r \quad \text{(at the surface)}$$

$$\left.\begin{aligned}&\rightarrow \oint_S \mathbf{D}_S.d\mathbf{S} = \oint_S \frac{Q}{4\pi a^2}\mathbf{a}_r d\mathbf{S} \\ &dS = r^2\sin\theta d\theta d\varphi = a^2\sin\theta d\theta d\varphi \\ &\rightarrow d\mathbf{S} = a^2\sin\theta d\theta d\varphi\mathbf{a}_r\end{aligned}\right\} \rightarrow \oint_S \mathbf{D}_S.d\mathbf{S} = \oint_S \frac{Q}{4\pi}\sin\theta d\theta d\varphi$$

Gauss's Law (5)

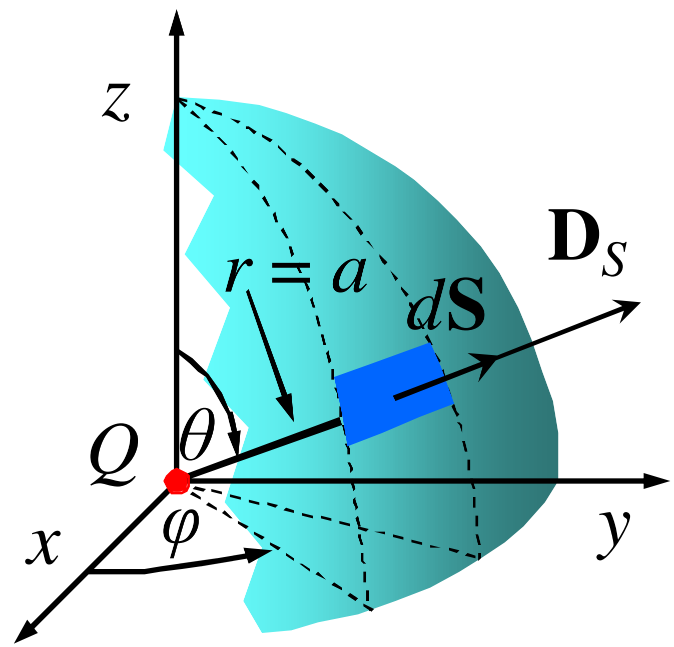
$$\oint_S \mathbf{D}_S . d\mathbf{S} = \oint_S \frac{Q}{4\pi} \sin\theta d\theta d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{Q}{4\pi} \sin\theta d\theta d\varphi$$

$$= \int_0^{2\pi} \frac{Q}{4\pi} (-\cos\theta)\Big|_0^{\pi} d\varphi$$

$$= \int_0^{2\pi} \frac{Q}{2\pi} d\varphi$$

$$= Q$$



Gauss's Law (6)

Ex. 1

Given a point charge 1 nC at (2, 0, 3) & another point charge 2 nC at (4, – 5, 6). Find the total electric flux leaving the enclosed surface formed by the six planes x, y, $z = \pm 8$.

Gauss's Law (7)

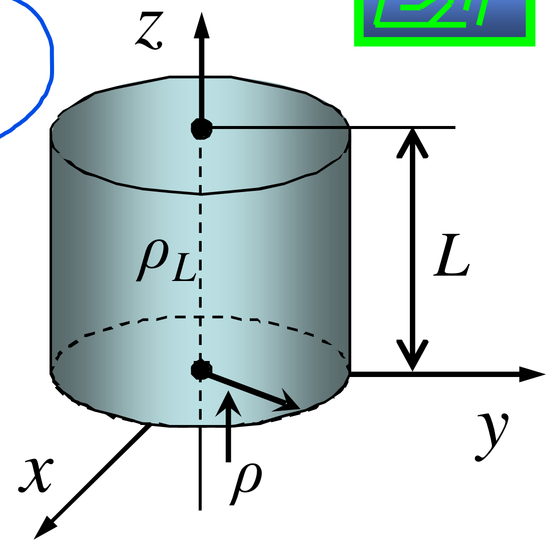
- Coulomb's law is to find **E** [$= f(Q)$]
- Sometimes it is difficult to find **E** using Coulomb's law
- Gauss may find **D** ($\rightarrow$ **E**) for a given Q

$$\boxed{Q = \oint_S \mathbf{D}_S . d\mathbf{S}}$$

- The solution is easy if we are able to find a closed surface satisfying 2 conditions:
  - $\mathbf{D}_S$ is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_S.d\mathbf{S}$ becomes $D_S dS$ or zero, respectively
  - On that portion of that surface for which $\mathbf{D}_S.d\mathbf{S} \neq 0$, $D_S = \text{const}$
- (Gaussian surface)

Gauss's Law (8)

$$\boxed{E = ?}$$



$$\mathbf{D} = D_\rho \mathbf{a}_\rho;\ D_\rho = f(\rho)$$

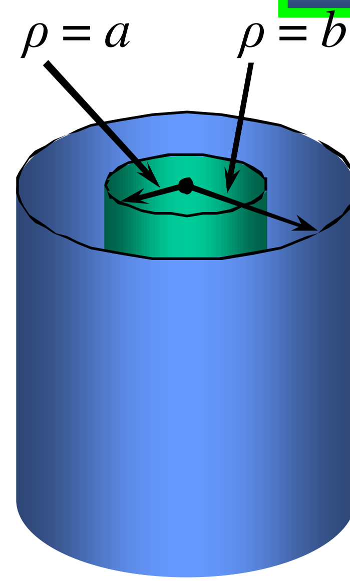
$$Q = \oint_{\text{cylinder}} \mathbf{D}_S . d\mathbf{S}$$

$$= D_S \int_{\text{sides}} dS + 0\int_{\text{top}} dS + 0\int_{\text{bottom}} dS$$

$$\left.\begin{array}{r} = D_S \displaystyle\int_{z=0}^{z=L} \int_{\varphi=0}^{\varphi=2\pi} \rho d\varphi dz \ = D_S 2\pi\rho L \ \rightarrow D_S = D_\rho = \dfrac{Q}{2\pi\rho L} \\ Q = \rho_L L \end{array}\right\}$$

$$\rightarrow D_\rho = \frac{\rho_L}{2\pi\rho} \qquad \rightarrow E_\rho = \frac{\rho_L}{2\pi\varepsilon_0\rho}$$

Gauss's Law (9)



2 coaxial cylindrical conductors. The outer surface of the inner cylinder has a ρ_S.

$$Q = D_S 2\pi\rho L$$

(total charge of a right circular cylinder of L & ρ $(a < \rho < b)$)

$$Q = \int_{z=0}^{z=L}\int_{\varphi=0}^{\varphi=2\pi} \rho_S a\, d\varphi\, dz = 2\pi a L \rho_S$$

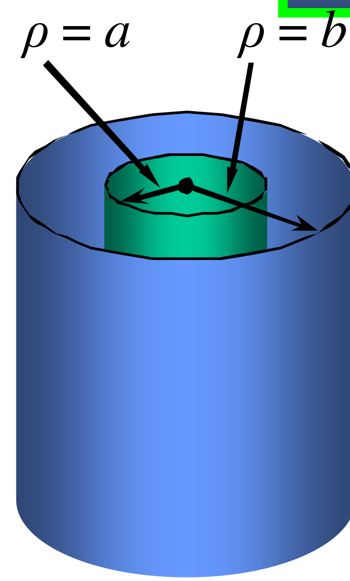
(total charge of the inner cylinder of length L)

$$\rightarrow D_S = \frac{a\rho_S}{\rho} \qquad \mathbf{D} = \frac{a\rho_S}{\rho}\mathbf{a}_\rho \quad (a < \rho < b)$$

$$\rho_L = Q\big|_{l=1} = \rho_S\, S\big|_{l=1} = \rho_S .2\pi a.1 = 2\pi a \rho_s$$

$$\rightarrow \boxed{\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho}$$

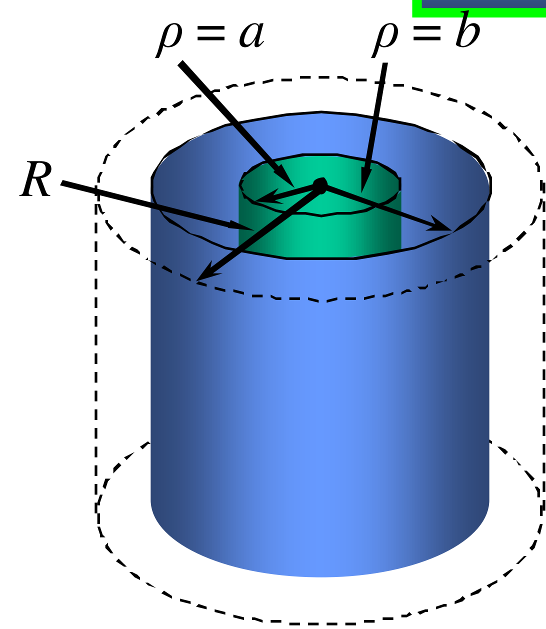
Gauss's Law (10)



$$\left.\begin{aligned} Q_{\text{outer cylinder}} &= -Q_{\text{inner cylinder}} \\ Q_{\text{outer cylinder}} &= 2\pi b L \rho_{S,\text{outer cylinder}} \\ Q_{\text{inner cylinder}} &= 2\pi a L \rho_{S,\text{inner cylinder}} \end{aligned}\right\}$$

$$\rightarrow \rho_{S,\text{outer cylinder}} = -\frac{a}{b}\rho_{S,\text{inner cylinder}}$$

Gauss's Law (11)



$$\Psi_{R,R>b} = Q_{\text{outer cylinder}} + Q_{\text{inner cylinder}} = 0 = D_{S,R} 2\pi RL$$

$$\rightarrow D_{S,R} = 0$$

The coaxial cable/capacitor has no external field & there is no field within the inner cylinder

Gauss's Law (12)

Ex. 2

Consider a coaxial cable of 1-m length, its inner radius is 1mm, the outer one is 4mm. Conductors are separated by air. The total charge on the inner cylinder is 40nC. Find charge density on each conductor, **E** & **D**.

$$\rho_{S,\text{inner cylinder}} = \frac{Q_{\text{inner cylinder}}}{2\pi a L} = \frac{40\times10^{-9}}{2\pi\times10^{-3}\times1} = 6.37\ \ \mu\text{C/m}^2$$

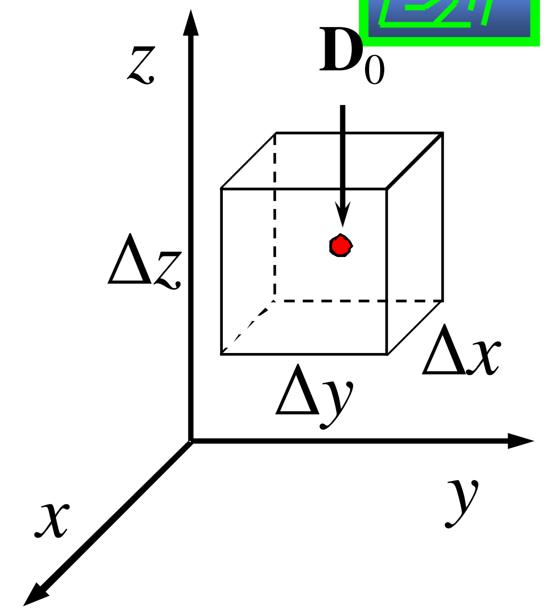
$$\rho_{S,\text{outer cylinder}} = \frac{Q_{\text{outer cylinder}}}{2\pi b L} = \frac{-40\times10^{-9}}{2\pi\times4\times10^{-3}\times1} = -1.59\ \ \mu\text{C/m}^2$$

$$D_\rho\Big|_{10^{-3}<\rho<4.10^{-3}} = \frac{a\rho_{S,inner\,cylinder}}{\rho} = \frac{1\times10^{-3}\times6.37\times10^{-6}}{\rho} = \frac{6.37}{\rho}\ \ \text{nC/m}^2$$

$$E_\rho\Big|_{10^{-3}<\rho<4.10^{-3}} = \frac{D_\rho}{\varepsilon_0} = \frac{6.37\times10^{-9}}{8.854\times10^{-12}\rho} = \frac{719}{\rho}\ \ \text{V/m}^2$$

Gauss's Law (13)

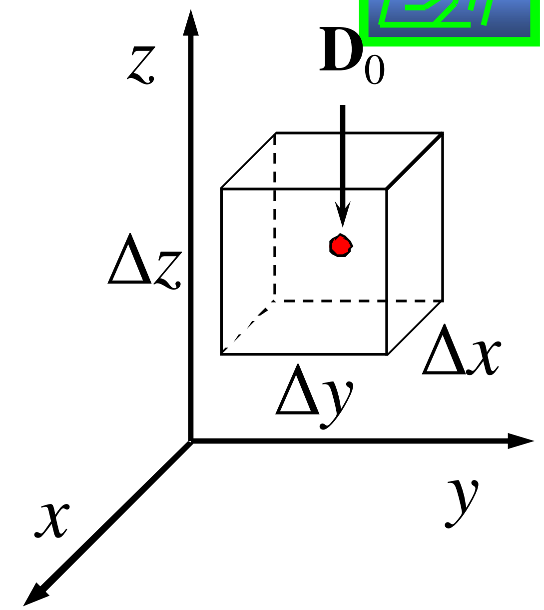
- The application of Gauss's law (to find **D**) needs a gaussian surface
- *Problem*: hard to find such surface
- *Solution*: choose a very small closed surface (approaching zero)



$$\mathbf{D} = \mathbf{D}_0 = D_{x0}\mathbf{a}_x + D_{y0}\mathbf{a}_y + D_{z0}\mathbf{a}_z$$

Gauss's Law (14)

$$\boxed{Q = \oint_S \mathbf{D}_S . d\mathbf{S}}$$



$$\oint_S \mathbf{D}.d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

Because the closed surface is very small, **D** is almost constant over the surface

$$\int_{\text{front}} \doteq \mathbf{D}_{\text{front}} . \Delta\mathbf{S}_{\text{front}} \doteq \mathbf{D}_{\text{front}} . \Delta y \Delta z \mathbf{a}_x \doteq D_{x,\text{front}} \Delta y \Delta z$$

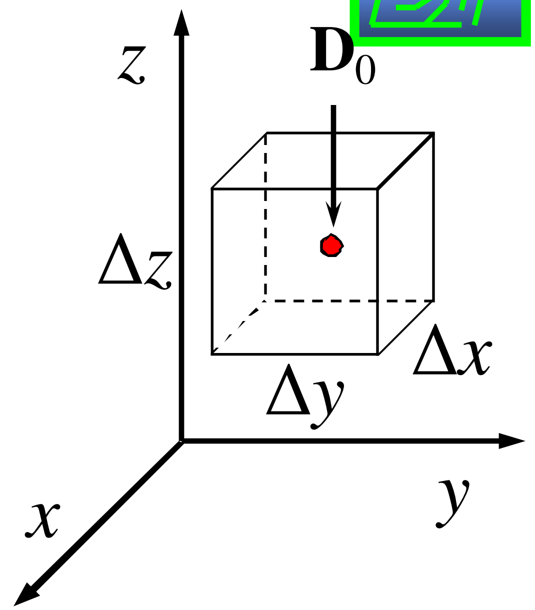
Gauss's Law (15)

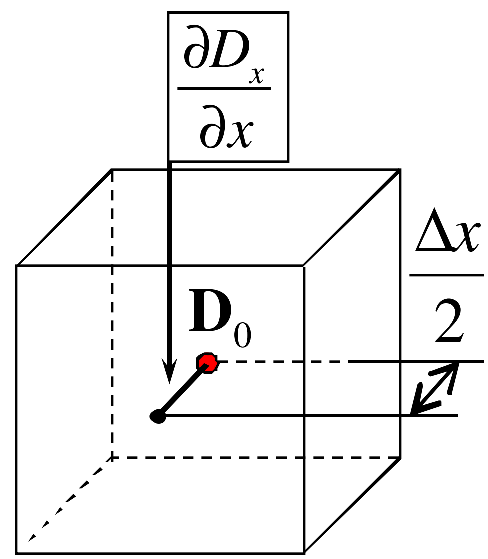
$$\int_{\text{front}} \doteq D_{x,\text{front}} \Delta y \Delta z$$

$$D_{x,\text{front}} \doteq D_{x0} + \frac{\Delta x}{2} \times (\text{rate of change of } D_x \text{ with } x)$$

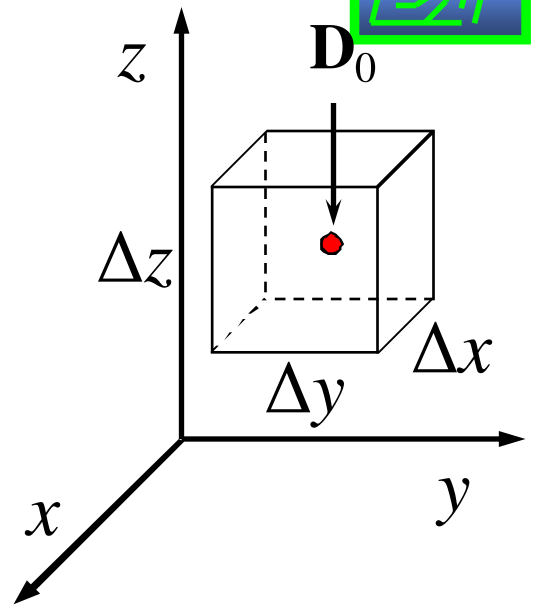
$$\doteq D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

$$\rightarrow \int_{\text{front}} \doteq \left(D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \Delta z$$





Gauss's Law (16)



$$\int_{\text{front}} \doteq \left(D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x} \right) \Delta y \Delta z$$

$$\left.\begin{aligned}
&\int_{\text{back}} \doteq \mathbf{D}_{\text{back}}.\Delta\mathbf{S}_{\text{back}} \doteq \mathbf{D}_{\text{back}}.(-\Delta y \Delta z \mathbf{a}_x) \\
&\qquad \doteq -D_{x,\text{back}} \Delta y \Delta z \\
&D_{x,\text{back}} \doteq D_{x0} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}
\end{aligned}\right\}$$

$$\rightarrow \int_{\text{back}} \doteq \left(-D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x} \right) \Delta y \Delta z$$

Gauss's Law (17)

$$\left.\begin{aligned} \int_{\text{front}} &\doteq \left(D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x} \right)\Delta y \Delta z \\ \int_{\text{back}} &\doteq \left(-D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x} \right)\Delta y \Delta z \end{aligned}\right\}$$

$$\rightarrow \int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x}\Delta x \Delta y \Delta z$$

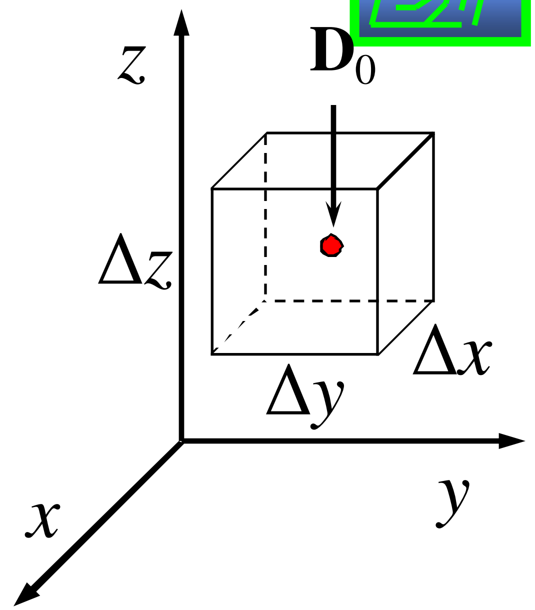
$$\text{Similarly:} \left\{\begin{aligned} \int_{\text{right}} + \int_{\text{left}} &\doteq \frac{\partial D_y}{\partial y}\Delta x \Delta y \Delta z \\ \int_{\text{top}} + \int_{\text{bottom}} &\doteq \frac{\partial D_z}{\partial z}\Delta x \Delta y \Delta z \end{aligned}\right.$$

Gauss's Law (18)

$$\left.\begin{aligned}
\int_{\text{front}} + \int_{\text{back}} &\doteq \frac{\partial D_x}{\partial x}\Delta x\Delta y\Delta z \\
\int_{\text{right}} + \int_{\text{left}} &\doteq \frac{\partial D_y}{\partial y}\Delta x\Delta y\Delta z \\
\int_{\text{top}} + \int_{\text{bottom}} &\doteq \frac{\partial D_z}{\partial z}\Delta x\Delta y\Delta z \\
\oint_S \mathbf{D}.d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} &+ \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}
\end{aligned}\right\}$$

$$\rightarrow \oint_S \mathbf{D}.d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x\Delta y\Delta z$$

Gauss's Law (19)



$$\left.\begin{aligned}\oint_S \mathbf{D}.d\mathbf{S} &\doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x \Delta y \Delta z \\ \Delta v &= \Delta x \Delta y \Delta z\end{aligned}\right\}$$

$$\rightarrow \oint_S \mathbf{D}.d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta v$$

$$\boxed{Q_{\text{enclosed in}\,\Delta v} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\times \Delta v}$$

Gauss's Law (20)

Ex. 3

Find the approximate value for the total charge inclosed in an incremental volume of 10^{-10} m^3 located at the origin. Given $\mathbf{D} = e^{-x}\sin y\,\mathbf{a}_x - e^{-x}\cos y\,\mathbf{a}_y + 2z\mathbf{a}_z$ C/m^2.

$$Q_{\text{enclosed in }\Delta v} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \times \Delta v$$

$$\frac{\partial D_x}{\partial x} = -e^{-x}\sin y \;\to\; \left.\frac{\partial D_x}{\partial x}\right|_{x=0} = 0$$

$$\frac{\partial D_y}{\partial y} = e^{-x}\sin y \;\to\; \left.\frac{\partial D_y}{\partial y}\right|_{y=0} = 0$$

$$\frac{\partial D_z}{\partial z} = 2$$

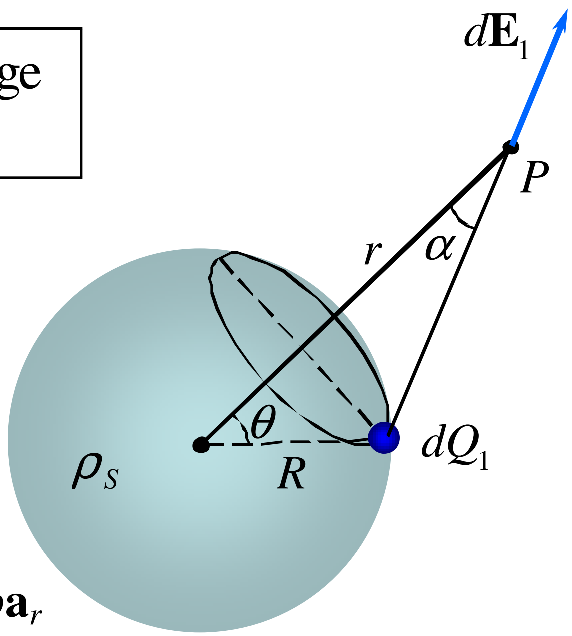
$$\to Q_{\text{enclosed in }\Delta v} \doteq (0+0+2)10^{-10} = 0.2 \text{ nC}$$

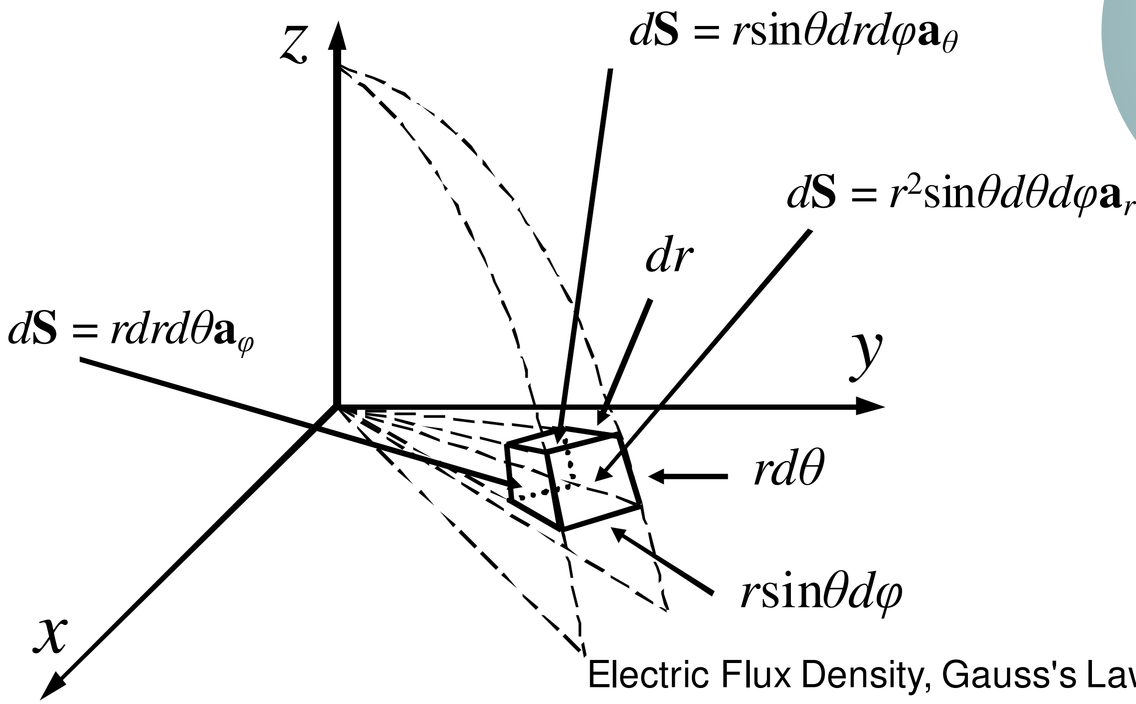
Gauss's Law (21)

Ex. 4

A sphere of radius R has a uniform surface charge density ρ_S. Find $\mathbf{E}$ at P?

$$dQ_1 = \rho_S dS_1 = \rho_S R^2 \sin\theta d\theta d\varphi$$



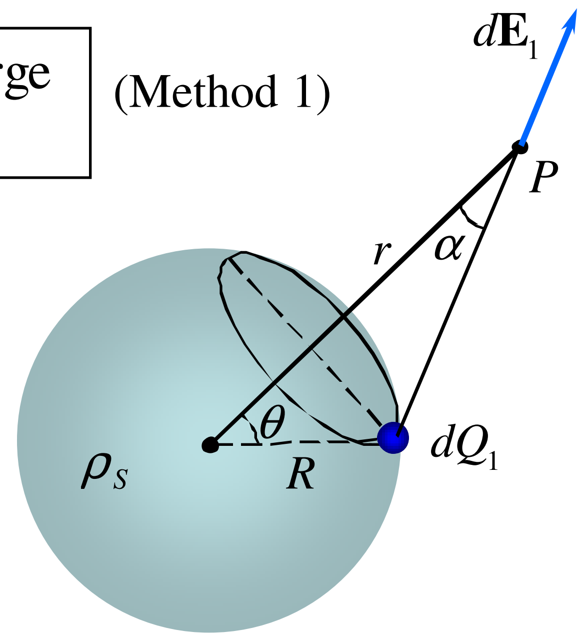


Gauss's Law (21)

Ex. 4

A sphere of radius R has a uniform surface charge density ρ_S. Find $\mathbf{E}$ at P?

(Method 1)



$$dQ_1 = \rho_S dS_1 = \rho_S R^2 \sin\theta d\theta d\varphi$$

$$dE_1 = \frac{dQ_1}{4\pi\varepsilon_0 R_{Q_1P}^2}\cos\alpha$$

$$= \frac{\rho_S R^2 \sin\theta d\theta d\varphi}{4\pi\varepsilon_0 R_{Q_1P}^2}\cos\alpha$$

$$= \frac{\rho_S R^2 \sin\theta d\theta d\varphi}{4\pi\varepsilon_0 R_{Q_1P}^2} \times \frac{r - R\cos\theta}{\sqrt{r^2 + R^2 - 2rR\cos\theta}}$$

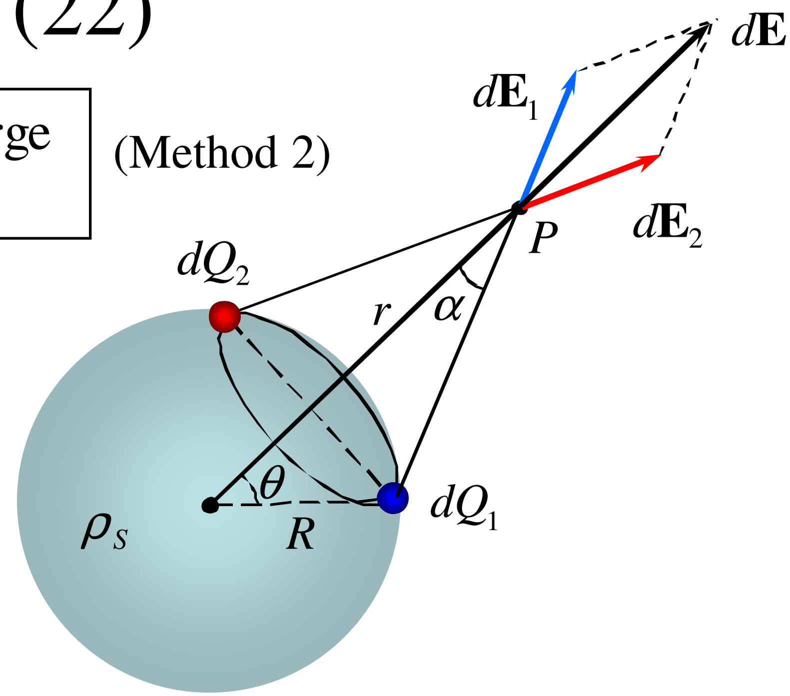
$$\rightarrow E_P = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{\rho_S R^2 \sin\theta(r - R\cos\theta)d\theta d\varphi}{4\pi\varepsilon_0 (r^2 + R^2 - 2rR\cos\theta)^{3/2}} = ???$$

Gauss's Law (22)

Ex. 4

A sphere of radius R has a uniform surface charge density ρ_S. Find $\mathbf{E}$ at P?

(Method 2)



$$\left.\begin{aligned} &\oint_S \varepsilon_0 \mathbf{E}.d\mathbf{S} = Q \\ &\oint_S \varepsilon_0 \mathbf{E}.d\mathbf{S} = \varepsilon_0 E_{Pr}(4\pi r^2) \\ &Q = \rho_S (4\pi R^2) \end{aligned}\right\}$$

$$\to \varepsilon_0 E_{Pr}(4\pi r^2) = \rho_S (4\pi R^2)$$

$$\to \boxed{E_{Pr} = \frac{\rho_S R^2}{\varepsilon_0 r^2}}, \qquad r > R$$

Gauss's Law (23)

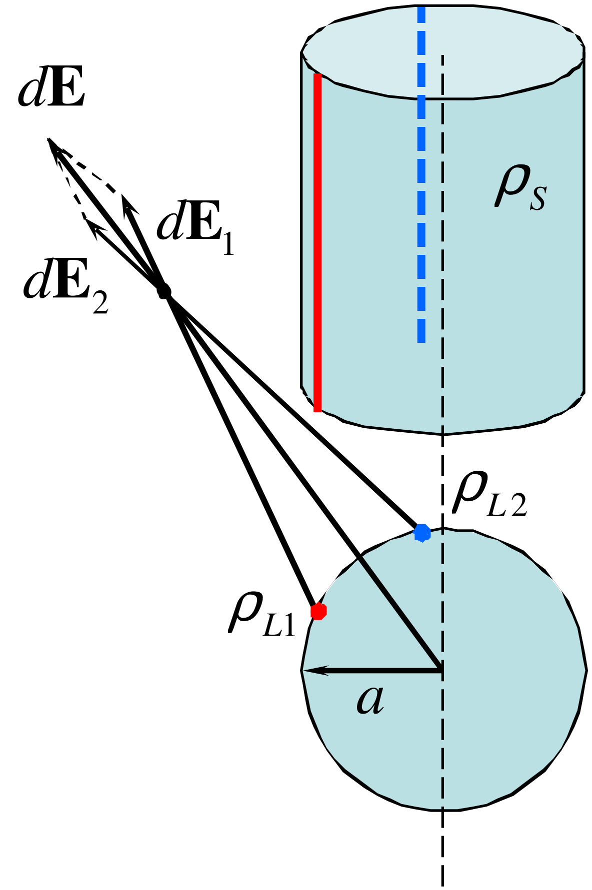
Ex. 5

An infinitely long cylinder of radius a has a uniform surface charge density ρ_S. Find $\mathbf{E}$?

$$\left.\begin{array}{l} \oint_S \varepsilon_0 \mathbf{E}.d\mathbf{S} = Q \\ \\ \oint_S \varepsilon_0 \mathbf{E}.d\mathbf{S} = \varepsilon_0 E_r(2\pi r L) \\ \\ Q = \rho_S(2\pi a L) \end{array}\right\}$$

$$\rightarrow \varepsilon_0 E_r(2\pi r L) = \rho_S(2\pi a L)$$

$$\rightarrow \boxed{E_r = \frac{\rho_S a}{\varepsilon_0 r}}, \qquad r > a$$



Electric Flux Density, Gauss's Law & Divergence

1. Electric Flux Density
2. Gauss's Law
3. **Divergence**
4. Maxwell's First Equation
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6. The Divergence Theorem

Divergence (1)

$$\oint_S \mathbf{D}.d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

$$\rightarrow \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \doteq \frac{\oint_S \mathbf{D}.d\mathbf{S}}{\Delta v} = \frac{Q}{\Delta v}$$

$$\rightarrow \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D}.d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

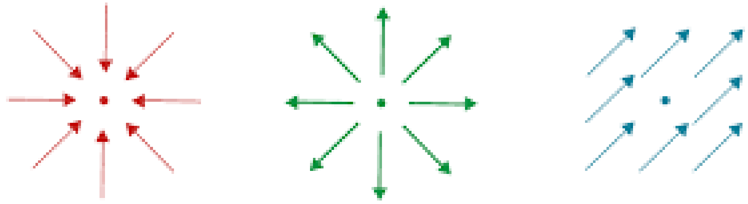
$$\rightarrow \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A}.d\mathbf{S}}{\Delta v}$$

$$\boxed{\text{Divergence of } \mathbf{A} = \text{div}\,\mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A}.d\mathbf{S}}{\Delta v}}$$

Divergence (2)

$$\text{Divergence of } \mathbf{A} = \text{div}\, \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A}.d\mathbf{S}}{\Delta v}$$

- *Definition*: the divergence of the vector flux density **A** is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero
- Divergence is an operation which is performed on a vector, but the result is a scalar
- Divergence only tells us *how much* flux is leaving a small volume (on a per-unit-volume basis), not *direction*



https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/divergence-and-curl-articles/a/divergence

Divergence (3)

$$\boxed{\text{Divergence of } \mathbf{A} = \operatorname{div} \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A}.d\mathbf{S}}{\Delta v}}$$

$$\operatorname{div} \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \qquad \text{(Descartes)}$$

$$\operatorname{div} \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\varphi}{\partial \varphi} + \frac{\partial D_z}{\partial z} \qquad \text{(Cylindrical)}$$

$$\operatorname{div} \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\varphi}{\partial \varphi} \qquad \text{(Spherical)}$$

Divergence (4)

Ex. 1

Find divergence at the origin, given $\mathbf{D} = e^{-x}\sin y\,\mathbf{a}_x - e^{-x}\cos y\,\mathbf{a}_y + 2z\mathbf{a}_z$ C/m^2.

Divergence (5)

Ex. 2

Find the divergence of the following vectors:

a) $\mathbf{A} = xy^2z^3(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)$

b) $\mathbf{A} = \rho\cos\varphi\mathbf{a}_\rho + \dfrac{z}{\rho}\sin\varphi\mathbf{a}_z$

c) $\mathbf{A} = r^2\sin\theta\cos\varphi(\mathbf{a}_r + \mathbf{a}_\theta + \mathbf{a}_\varphi)$

Electric Flux Density, Gauss's Law & Divergence

1. Electric Flux Density
2. Gauss's Law
3. Divergence
4. **Maxwell's First Equation**
5. The Vector Operator ∇
6. The Divergence Theorem

Maxwell's First Equation (1)

$$\oint_S \mathbf{D}.d\mathbf{S} = Q \rightarrow \frac{\oint_S \mathbf{D}.d\mathbf{S}}{\Delta v} = \frac{Q}{\Delta v}$$

$$\left.\begin{aligned} &\rightarrow \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D}.d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} \\ &\operatorname{div} \mathbf{D} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D}.d\mathbf{S}}{\Delta v} \qquad \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v \end{aligned}\right\}$$

$$\rightarrow \boxed{\operatorname{div} \mathbf{D} = \rho_v}$$ Maxwell's first equation

Maxwell's First Equation (2)

$$\boxed{\operatorname{div} \mathbf{D} = \rho_v}$$

- Apply to electrostatic & steady magnetic fields
- The electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there

Maxwell's First Equation (3)

Ex.

Given $\mathbf{D} = 4xy\mathbf{a}_x + z^2\mathbf{a}_y$ C/m², find ρ_v of the region about $P(1,1,1)$.

$$\rho_v = \operatorname{div}\mathbf{D}$$

$$\operatorname{div}\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \frac{\partial}{\partial x}4xy + \frac{\partial}{\partial y}z^2 + \frac{\partial}{\partial z}0 = 4y$$

$$\rightarrow \rho_v = 4y$$

$$\rightarrow \rho_{v,P} = \rho_v\big|_{x=y=z=1} = 4\times 1 = \boxed{4\ \text{C/m}^3}$$

Electric Flux Density, Gauss's Law & Divergence

1. Electric Flux Density
2. Gauss's Law
3. Divergence
4. Maxwell's First Equation
5. **The Vector Operator ∇**
6. The Divergence Theorem

$\nabla(1)$

$$\boxed{\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z}$$

$$\nabla.\mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right).\left(D_x\mathbf{a}_x + D_y\mathbf{a}_y + D_z\mathbf{a}_z\right)$$

$$= \frac{\partial}{\partial x}(D_x) + \frac{\partial}{\partial y}(D_y) + \frac{\partial}{\partial z}(D_z)$$

$$= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \mathrm{div}\,\mathbf{D}$$

$$\rightarrow \boxed{\mathrm{div}\,\mathbf{D} = \nabla.\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}}$$

∇(2)

Ex.

Given $\mathbf{D} = e^{-x}\sin y\,\mathbf{a}_x - e^{-x}\cos y\,\mathbf{a}_y + 2z\mathbf{a}_z$ C/m^2, find $\nabla.\mathbf{D}$?

Electric Flux Density, Gauss's Law & Divergence

1. Electric Flux Density
2. Gauss's Law
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5. The Vector Operator ∇
6. **The Divergence Theorem**

The Divergence Theorem (1)

- Applies to any vector field for which the appropriate partial derivatives exist

$$\left.\begin{aligned} &\oint_S \mathbf{D}.d\mathbf{S} = Q \\ &Q = \int_V \rho_v dv \\ &\nabla.\mathbf{D} = \rho_v \end{aligned}\right\} \rightarrow \oint_S \mathbf{D}.d\mathbf{S} = Q = \int_V \rho_v dv = \int_V \nabla.\mathbf{D} dv$$

$$\rightarrow \boxed{\oint_S \mathbf{D}.d\mathbf{S} = \int_V \nabla.\mathbf{D} dv}$$

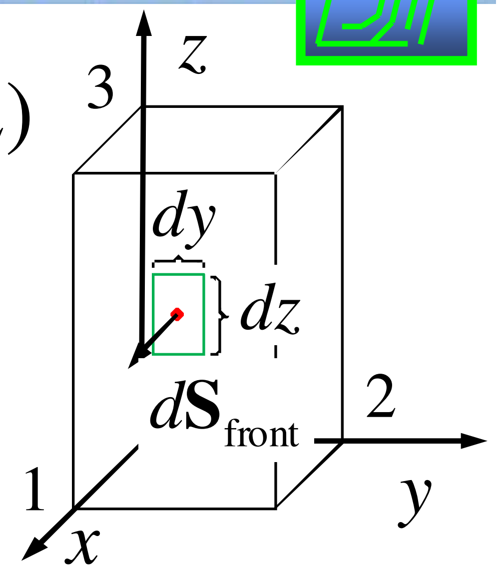
- *Theorem*: the integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface

The Divergence Theorem (2)

Ex.

Given $\mathbf{D} = 4xy\mathbf{a}_x + z^2\mathbf{a}_y$ C/m^2 & a rectangular parallelepiped. Verify the divergence theorem.

$$\oint_S \mathbf{D}.d\mathbf{S} = \int_V \nabla.\mathbf{D}dV \ (= Q)$$



Left side:

$$\oint_S \mathbf{D}.d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

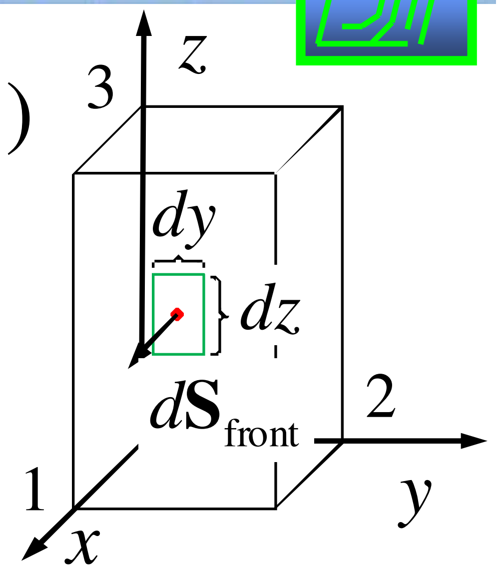
$$\left.\begin{aligned} \int_{\text{front}} &= \int_{z=0}^{z=3}\int_{y=0}^{y=2} \mathbf{D}_{\text{front}}.d\mathbf{S}_{\text{front}} \\ \mathbf{D}_{\text{front}} &= (4xy\mathbf{a}_x + z^2\mathbf{a}_y)\Big|_{x=1} = 4y\mathbf{a}_x + z^2\mathbf{a}_y \\ d\mathbf{S}_{\text{front}} &= dydz\mathbf{a}_x \end{aligned}\right\}$$

$$\rightarrow \int_{\text{front}} = \int_{z=0}^{z=3}\int_{y=0}^{y=2} (4y\mathbf{a}_x + z^2\mathbf{a}_y).(dydz\mathbf{a}_x) = \int_{z=0}^{z=3}\int_{y=0}^{y=2} 4ydydz$$

The Divergence Theorem (3)

Ex.

Given $\mathbf{D} = 4xy\mathbf{a}_x + z^2\mathbf{a}_y$ C/m^2 & a rectangular parallelepiped. Verify the divergence theorem.



$$\boxed{\oint_S \mathbf{D}.d\mathbf{S} = \int_V \nabla.\mathbf{D}dV \ (=Q)}$$

Left side:

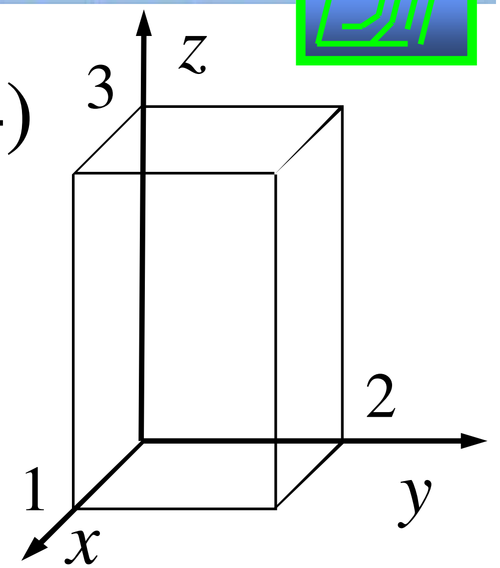
$$\oint_S \mathbf{D}.d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

$$\int_{\text{front}} = \int_{z=0}^{z=3}\int_{y=0}^{y=2}(4y\mathbf{a}_x + z^2\mathbf{a}_y).(dydz\mathbf{a}_x) = \int_{z=0}^{z=3}\int_{y=0}^{y=2} 4ydydz$$

$$= 12 \text{ C}$$

The Divergence Theorem (4)

Ex.



Given $\mathbf{D} = 4xy\mathbf{a}_x + z^2\mathbf{a}_y$ C/m^2 & a rectangular parallelepiped. Verify the divergence theorem.

$$\oint_S \mathbf{D}.d\mathbf{S} = \int_V \nabla.\mathbf{D}dV \ (= Q)$$

Left side:

$$\oint_S \mathbf{D}.d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

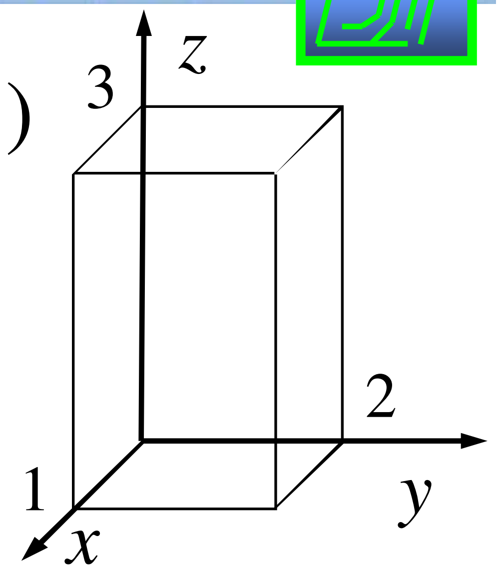
$$\int_{\text{back}} = \int_{z=0}^{z=3}\int_{y=0}^{y=2} \mathbf{D}\Big|_{x=0} .(-dydz\mathbf{a}_x) = -\int_{z=0}^{z=3}\int_{y=0}^{y=2} D_x\Big|_{x=0} dydz$$

$$D_x\Big|_{x=0} = (4xy)\Big|_{x=0} = 0$$

$$\rightarrow \int_{\text{back}} = 0$$

The Divergence Theorem (5)

Ex.



Given $\mathbf{D} = 4xy\mathbf{a}_x + z^2\mathbf{a}_y$ C/m^2 & a rectangular parallelepiped. Verify the divergence theorem.

$$\oint_S \mathbf{D}.d\mathbf{S} = \int_V \nabla.\mathbf{D}dV \ (=Q)$$

Left side:

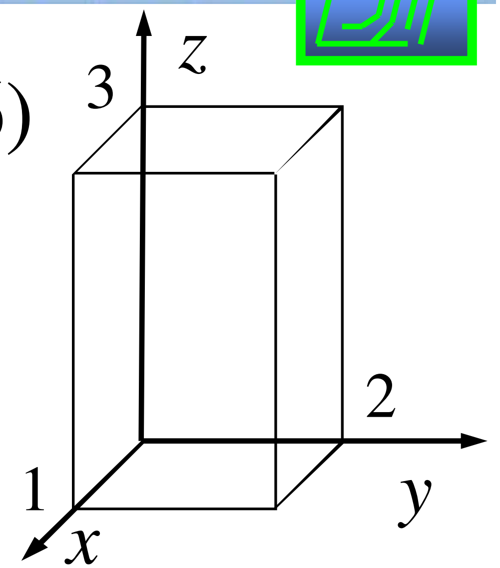
$$\oint_S \mathbf{D}.d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

$$\int_{\text{right}} = \int_{z=0}^{z=3}\int_{x=0}^{x=1} \mathbf{D}\Big|_{y=2}.(dxdz\mathbf{a}_y) = \int_{z=0}^{z=3}\int_{x=0}^{x=1} D_y\Big|_{y=2} dxdz$$

$$\left. \right\} \quad D_y\Big|_{y=2} = (z^2)\Big|_{y=2} = z^2$$

$$\to \int_{\text{right}} = \int_{z=0}^{z=3}\int_{x=0}^{x=1} z^2 dxdz$$

The Divergence Theorem (6)



Ex.

Given $\mathbf{D} = 4xy\mathbf{a}_x + z^2\mathbf{a}_y$ C/m^2 & a rectangular parallelepiped. Verify the divergence theorem.

$$\oint_S \mathbf{D}.d\mathbf{S} = \int_V \nabla.\mathbf{D}dV \ (= Q)$$

Left side:

$$\oint_S \mathbf{D}.d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

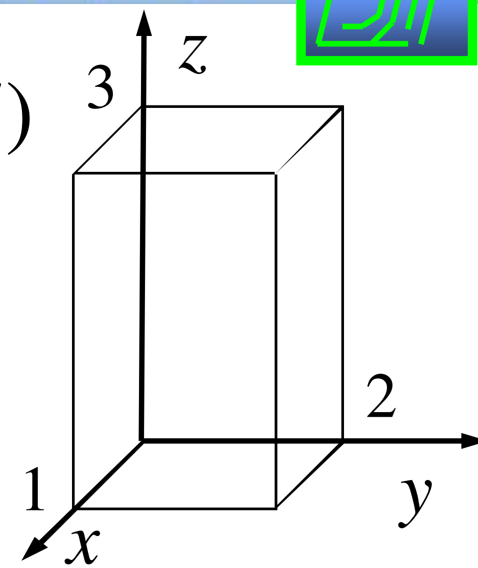
$$\int_{\text{left}} = \int_{z=0}^{z=3}\int_{x=0}^{x=1} \mathbf{D}\Big|_{y=0}.(-dxdz\mathbf{a}_y) = -\int_{z=0}^{z=3}\int_{x=0}^{x=1} D_y\Big|_{y=0} dxdz$$

$$D_y\Big|_{y=0} = (z^2)\Big|_{y=0} = z^2$$

$$\rightarrow \int_{\text{left}} = -\int_{z=0}^{z=3}\int_{x=0}^{x=1} z^2 dxdz$$

The Divergence Theorem (7)

Ex.



Given $\mathbf{D} = 4xy\mathbf{a}_x + z^2\mathbf{a}_y$ C/m² & a rectangular parallelepiped. Verify the divergence theorem.

$$\oint_S \mathbf{D}.d\mathbf{S} = \int_V \nabla.\mathbf{D}dV \ (= Q)$$

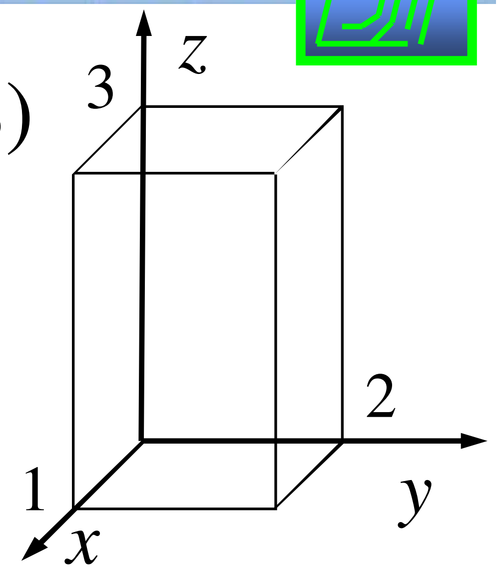
Left side:

$$\oint_S \mathbf{D}.d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

$$\int_{\text{top}} = \int_{x=0}^{x=1}\int_{y=0}^{y=2}(4xy\mathbf{a}_x + z^2\mathbf{a}_y).(dxdy\mathbf{a}_z) = \int_{x=0}^{x=1}\int_{y=0}^{y=2} 0 = 0$$

$$\int_{\text{bottom}} = \int_{x=0}^{x=1}\int_{y=0}^{y=2}(4xy\mathbf{a}_x + z^2\mathbf{a}_y).(dxdy\mathbf{a}_z) = \int_{x=0}^{x=1}\int_{y=0}^{y=2} 0 = 0$$

The Divergence Theorem (8)



Ex.

Given $\mathbf{D} = 4xy\mathbf{a}_x + z^2\mathbf{a}_y$ C/m² & a rectangular parallelepiped. Verify the divergence theorem.

$$\oint_S \mathbf{D}.d\mathbf{S} = \int_V \nabla.\mathbf{D}dV \ (=Q)$$

Left side:

$$\oint_S \mathbf{D}.d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

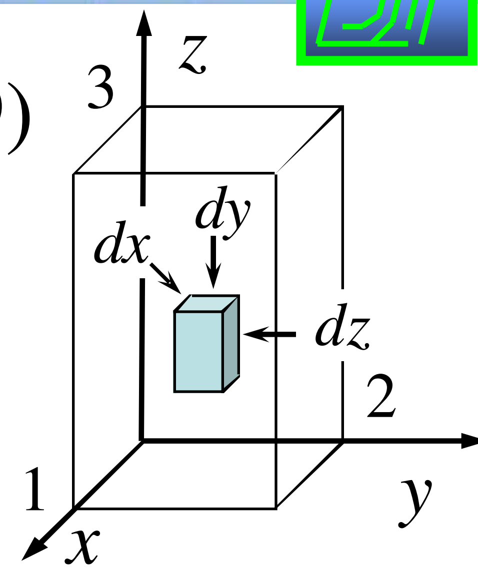
$$= 24 + 0 + \int_{z=0}^{z=3}\int_{x=0}^{x=1} z^2 dxdz - \int_{z=0}^{z=3}\int_{x=0}^{x=1} z^2 dxdz + 0 + 0$$

$$= \boxed{24\text{C}}$$

The Divergence Theorem (9)

Ex.

Given $\mathbf{D} = 4xy\mathbf{a}_x + z^2\mathbf{a}_y$ C/m^2 & a rectangular parallelepiped. Verify the divergence theorem.



$$\oint_S \mathbf{D}.d\mathbf{S} = \int_V \nabla.\mathbf{D}dV \ (= Q)$$

Right side:

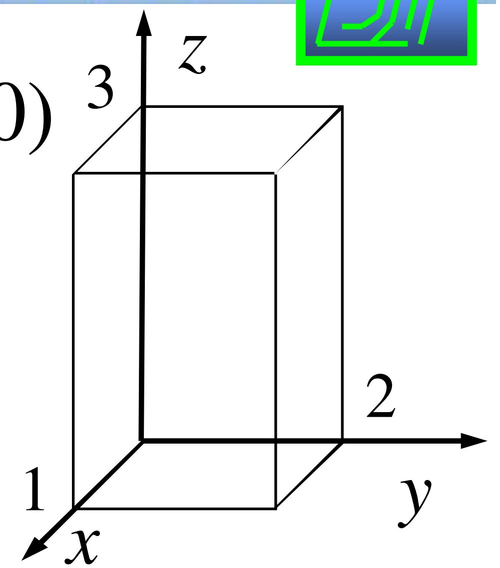
$$\int_V \nabla.\mathbf{D}dV$$

$$\nabla.\mathbf{D} = \frac{\partial}{\partial x}4xy + \frac{\partial}{\partial y}z^2 + \frac{\partial}{\partial z}0 = 4y$$

$$dV = dxdydz$$

$$\rightarrow \int_V \nabla.\mathbf{D}dV = \int_{z=0}^{z=3}\int_{y=0}^{y=2}\int_{x=0}^{x=1} 4y\,dxdydz = \boxed{24 \text{ C}}$$

The Divergence Theorem (10)



Ex.

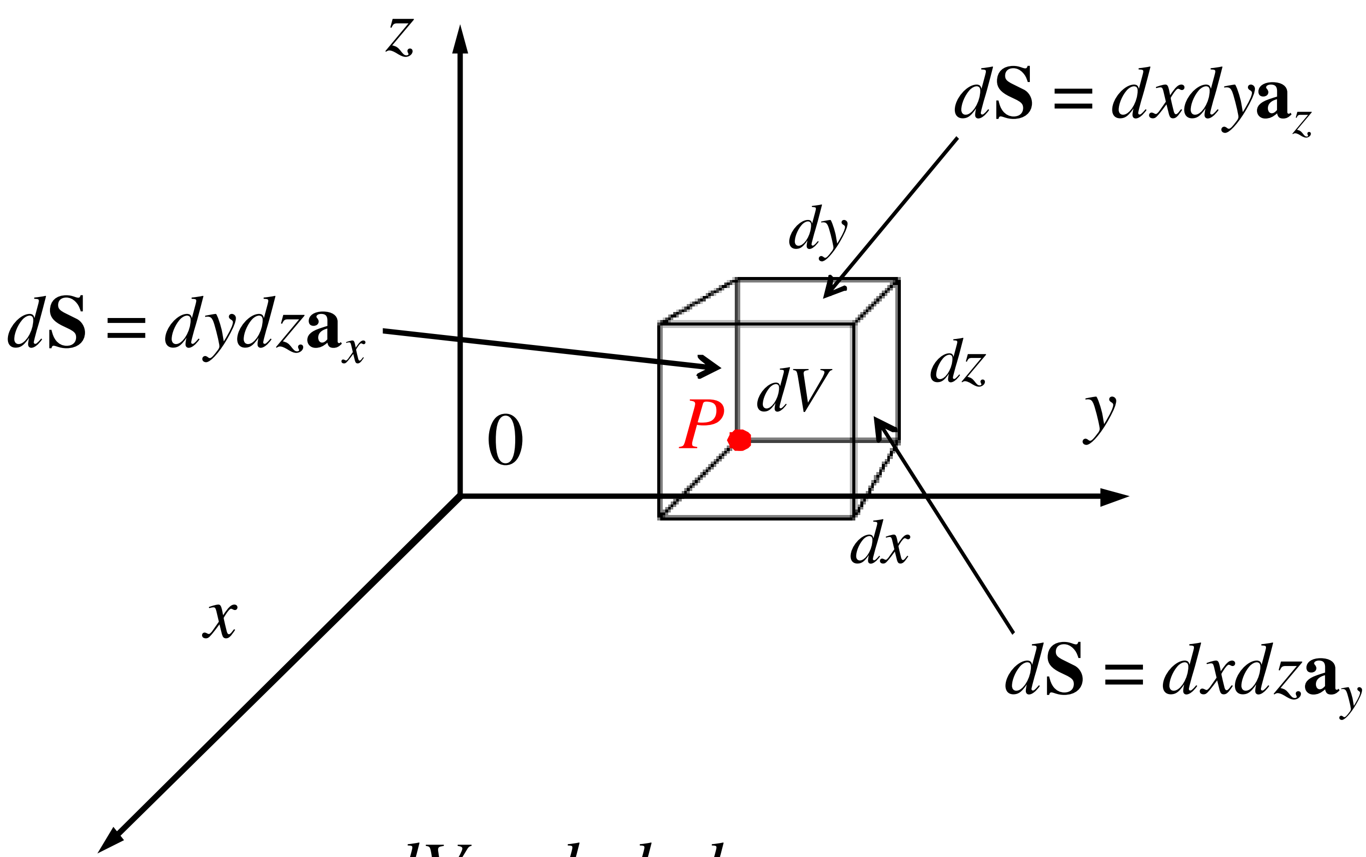
Given $\mathbf{D} = 4xy\mathbf{a}_x + z^2\mathbf{a}_y$ C/m^2 & a rectangular parallelepiped. Verify the divergence theorem.

$$\boxed{\oint_S \mathbf{D}.d\mathbf{S} = \int_V \nabla.\mathbf{D}dV \ (= Q)}$$

<u>Left side:</u> $\oint_S \mathbf{D}.d\mathbf{S} = 24 \text{ C}$

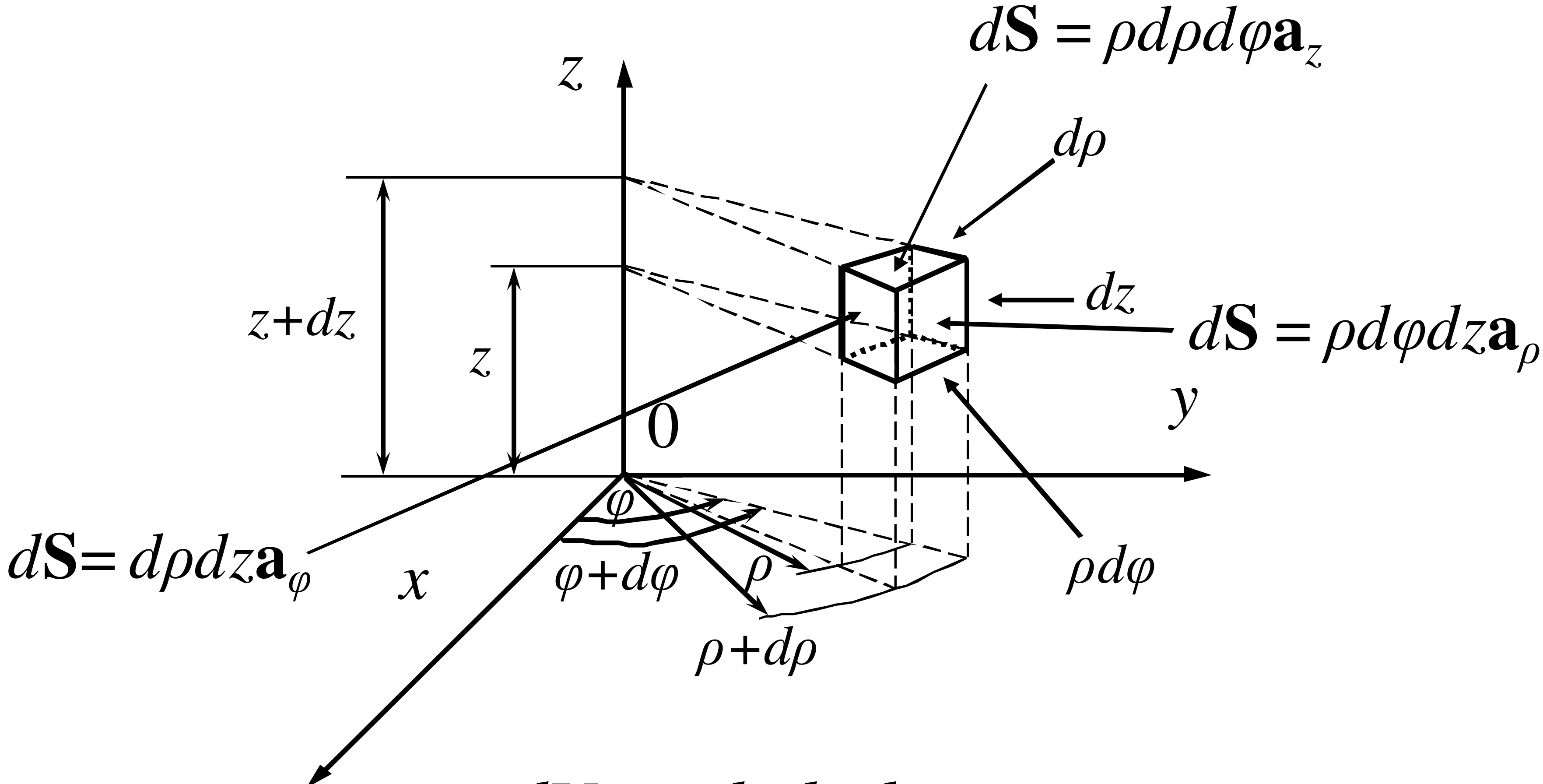
<u>Right side:</u> $\int_V \nabla.\mathbf{D}dV = 24 \text{ C}$

The Rectangular Coordinate System



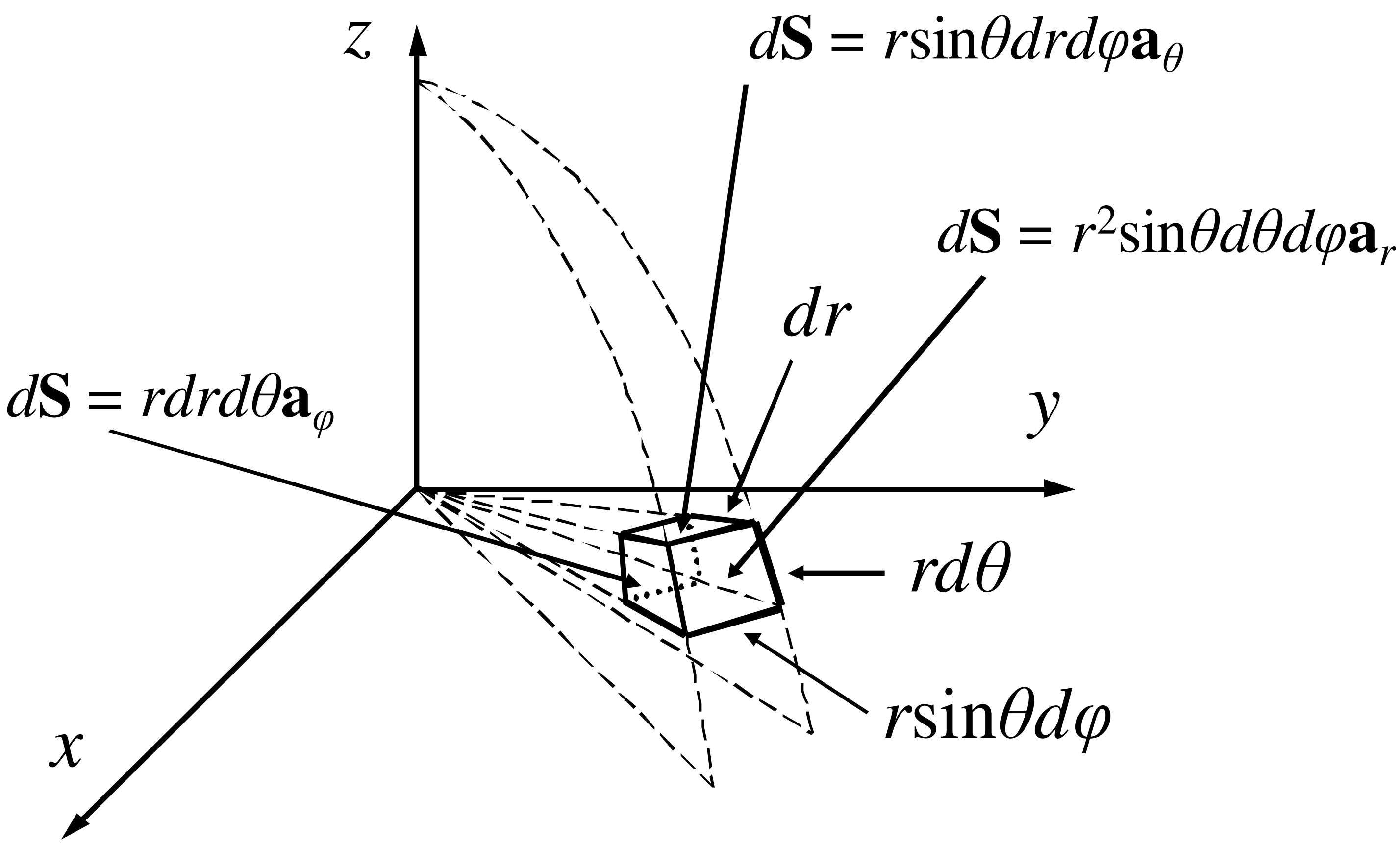
$$dV = dxdydz$$

The Circular Cylindrical Coordinate System



$$dV = \rho d\rho d\varphi dz$$

The Spherical Coordinate System



$$dV = r^2\sin\theta dr d\theta d\varphi$$

$$Q \longrightarrow \mathbf{F} = \frac{Q_1 Q_2}{4\pi\varepsilon R^2}\mathbf{a}_R \longrightarrow \mathbf{E} = \frac{Q}{4\pi\varepsilon R^2}\mathbf{a}_R \longrightarrow \mathbf{D} = \varepsilon\mathbf{E}$$