Fundamentals of Electric Circuits

DC Circuits

Chapter 7. First Order Circuits

7.1. Introduction

+ In this chapter: → study 2 types of circuits consisting storage elements : R-C, R-L

+ R-C, R-L circuits: → first-order differential equations of KCL, KVL

+ A **first-order circuit** → characterized by a first-order differential equation

+ To excite first order circuits:

  - *Source-free circuit:* Energy is initially stored in the capacitive or inductive element. The energy (*dependent source*) causes current to flow in the circuit and is dissipated in the resistors

  - *Independent source:* DC source (sinusoidal source, exponential source)

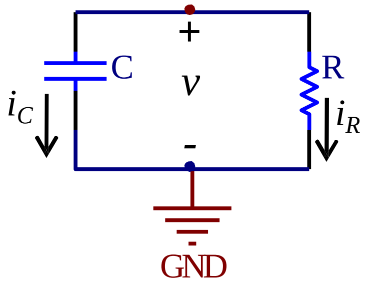
+ Applications of first order circuits: delay/relay circuit, photoflash unit, automobile ignition circuit,...

7.2. The source-free RC/RL circuit

7.2.1. R-C circuit

+ A source-free R-C circuit:

→ Occurs when its DC source is suddenly disconnected
→ Energy already stored in the capacitor is released to the resistors



+ Considered case: Series combination of a resistor and an initially charged capacitor with V_0 → determine the circuit response

Energy stored in the capacitor: $w = \frac{1}{2} C V_0^2$

KCL at the top node of the circuit: $i_C + i_R = 0 \to C\frac{dv}{dt} + \frac{v}{R} = 0 \to \frac{dv}{dt} + \frac{v}{RC} = 0$ *(First-order differential equation)*

$$\to \frac{dv}{v} = -\frac{1}{RC}dt \to \ln v = -\frac{t}{RC} + \ln A \to \ln\frac{v}{A} = -\frac{t}{RC} \to v(t) = Ae^{-\frac{t}{RC}}$$

$$A = v(0) = V_0 \to \boxed{v(t) = V_0 . e^{-\frac{t}{RC}}}$$

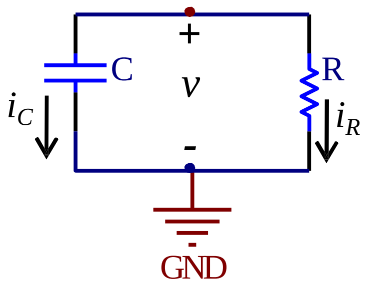
7.2. The source-free RC/RL circuit

7.2.1. R-C circuit

+ **Circuit response** is: Natural response

→ Due to the initial energy stored and the physical characteristics of the circuit

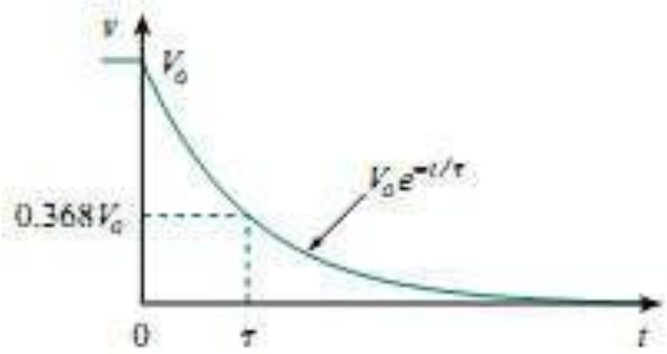
→ Not due to external voltage or current sources



+ **The natural response** of a circuit → the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation

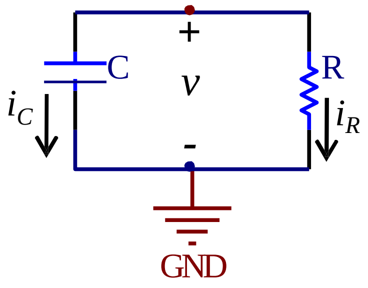
+ The **time constant** of a RC circuit: → the time required for the response to decay by a factor of 1/e or 36.8% of its initial value

$$\tau = RC \to v(t) = V_0.e^{-\frac{t}{\tau}}$$



7.2. The source-free RC/RL circuit

7.2.1. R-C circuit



+ After 5τ:

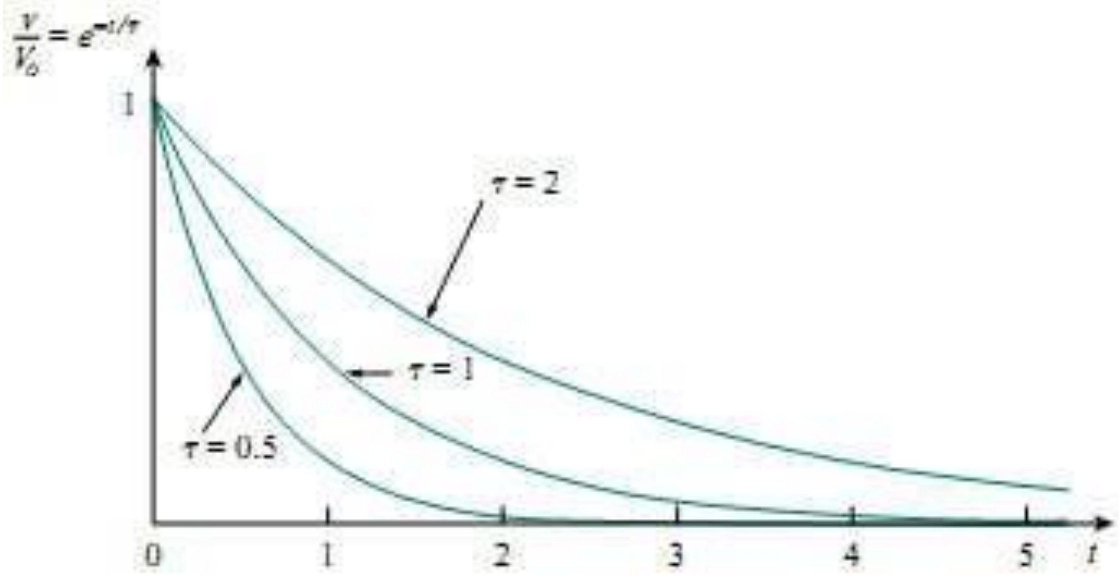
→ the voltage *v(t)* is less than *1%* of V_0 →the capacitor is fully discharged (or charged)

→ Circuit to reach its *final state (steady state)* when no changes take place with time

$$v(t) = V_0 . e^{-\frac{t}{\tau}}$$

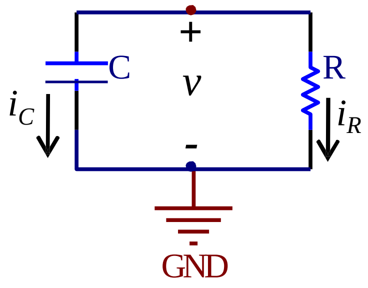
t	$v(t) / V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674

+ The greater the time constant
→ the slower the response decay



7.2. The source-free RC/RL circuit

7.2.1. R-C circuit



+ The **current** $i_R(t)$:

$$i_R(t)=\frac{v(t)}{R}=\boxed{\frac{V_0}{R}e^{-\frac{t}{\tau}}}$$

+ The **power** dissipated in the resistor:

$$p_R(t)=v(t)i_R(t)=\boxed{\frac{V_0^2}{R}e^{-\frac{2t}{\tau}}}$$

+ The **energy** absorbed by the resistor:

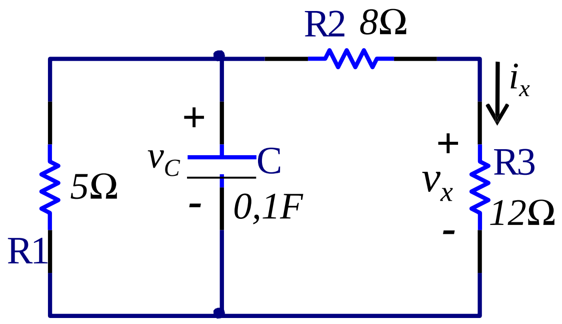
$$w_R(t)=\int_0^t pdt=\int_0^t \frac{V_0^2}{R}e^{-\frac{2t}{\tau}}dt=-\frac{\tau V_0^2}{2R}e^{-\frac{2t}{\tau}}\Big|_0^t=\boxed{\frac{1}{2}CV_0^2\left(1-e^{-\frac{2t}{\tau}}\right)}$$

+ Note: $t\to\infty\Rightarrow w_R(t)=\frac{1}{2}CV_0^2=w_C(0)$ *(the energy initially stored in the capacitor)*

7.2. The source-free RC/RL circuit

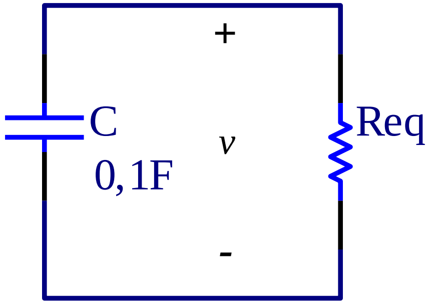
7.2.1. R-C circuit

+ Example 1: Find v_C, v_x and i_x for $t > 0$ if $v_C(0) = 15V$



→ equivalent resistance: $R_{eq} = R_1 // (R_2 + R_3) = \frac{5.(8+12)}{5+8+12} = 4\Omega$

→ time constant: $\tau = R_{eq}C = 4.0.1 = 0.4s$



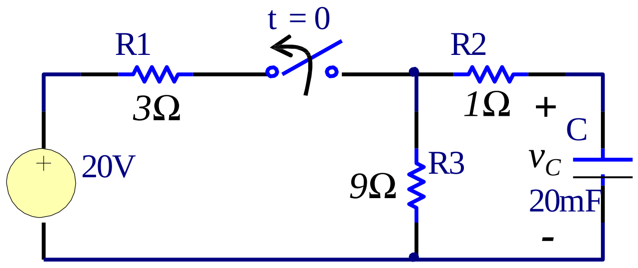
$$\rightarrow v_c(t) = v_0 e^{-\frac{t}{\tau}} = 15e^{-\frac{t}{0.4}} = 15e^{-2.5t}V$$

$$\rightarrow v_x(t) = \frac{R_3}{R_2 + R_3} v_C(t) = \frac{12}{12+8} 15e^{-2.5t} = 9e^{-2.5t}V$$

$$\rightarrow i_x(t) = \frac{v_x(t)}{R_3} = \frac{9e^{-2.5t}}{12} = 0.75e^{-2.5t}A$$

7.2. The source-free RC/RL circuit

7.2.1. R-C circuit



+ Example 2: The switch has been closed for a long time, it is opened at $t = 0$. Find $v_C(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor

→ For $t < 0$, the switch is closed → C is open circuit to DC source

$$v_c(t) = \frac{v(t)}{R_1 + R_3} R_3 = \frac{20}{3+9} 9 = 15V \rightarrow v_c(0) = V_0 = 15V$$

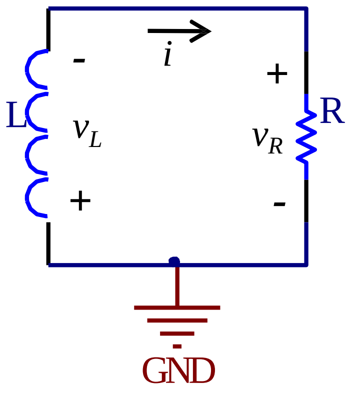
→ *For t* > *0*, time constant of the source free *R-C* circuit: $R_{eq} = R_2 + R_3 = 10\Omega \rightarrow \tau = R_{eq}C = 10.20.10^{-3} = 0.2s$

→ The voltage across the capacitor for $t \geq 0$: $v_c(t) = V_0 e^{-\frac{t}{\tau}} = 15e^{-5t}V$

→ The initial energy stored in the capacitor: $w_c(0) = \frac{1}{2} C V_0^2 = \frac{1}{2} 20.10^{-3}.15^2 = 2.25J$

7.2. The source-free RC/RL circuit

7.2.2. R-L circuit

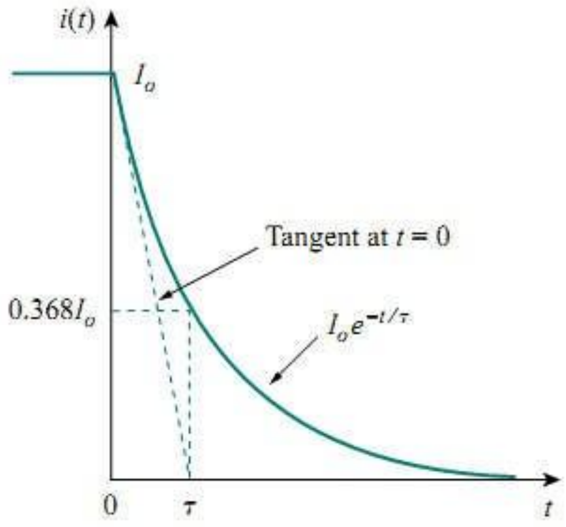


+ Consider case: $i_L(0) = I_0 \neq 0$

→ Energy stored in the inductor as: $w_L(0) = \frac{1}{2} L I_0^2$

+ Current through the resistor: $v_L(t) + v_R(t) = 0 \rightarrow L\frac{di}{dt} + Ri = 0 \rightarrow \int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt$

$$\rightarrow \ln\frac{i(t)}{I_0} = -\frac{R}{L}t \rightarrow \boxed{i(t) = I_0 e^{-\frac{R}{L}t} = I_0 e^{-\frac{t}{\tau}} \qquad \tau = \frac{L}{R}}$$



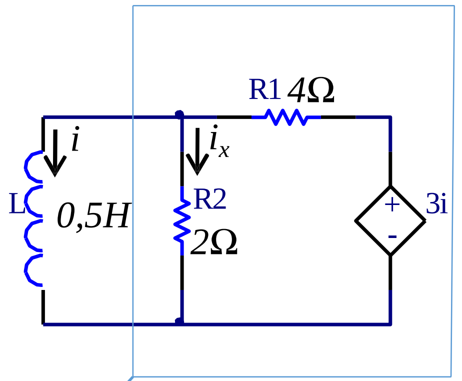
+ Energy absorbed by the resistor:

$$w_R(t) = \int_0^t p dt = \int_0^t R I_0^2 e^{-\frac{2t}{\tau}} dt = \boxed{\frac{1}{2} L I_0^2 \left(1 - e^{-\frac{2t}{\tau}}\right)} \qquad t \rightarrow \infty : w_R(t) = \frac{1}{2} L I_0^2 = w_L(0)$$

7.2. The source-free RC/RL circuit

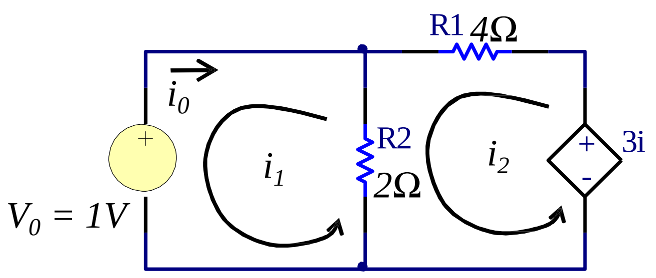
7.2.2. R-L circuit

+ Example 3: Assuming that $i(0) = 10A$, calculate $i(t)$ and $i_x(t)$



Solution 1: Apply the Thevenin's law

→ Connect V_0 = 1V to the gate then apply loop current method to calculate i_0



$$\begin{cases} R_2(i_1 - i_2) + V_0 = 0 \\ (R_1 + R_2)i_2 - R_2 i_1 - 3i_1 = 0 \end{cases} \rightarrow \begin{cases} i_1 - i_2 = -\dfrac{1}{2} \\ -5i_1 + 6i_2 = 0 \end{cases} \rightarrow \begin{cases} i_1 = -3A \\ i_0 = -i_1 = 3A \end{cases}$$

→ Hence: $R_{th} = \dfrac{V_0}{i_0} = \dfrac{1}{3}\Omega$ → Obtained an equivalent RL circuit

→ Time constant of the RL circuit: $\tau = \dfrac{L}{R_{th}} = \dfrac{3}{2} = 1.5s$

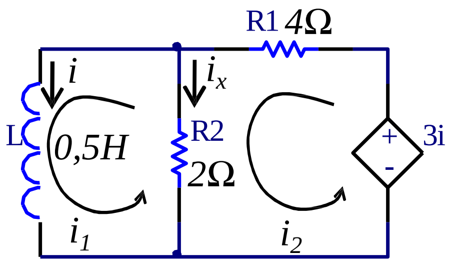
→ The current through the inductor is: $i(t) = i_0 e^{-\frac{t}{\tau}} = 10e^{-\frac{2}{3}t} A$

→ The current through the resistor R_2 is: $i_x(t) = \dfrac{v_L(t)}{R_2} = \dfrac{L}{R_2}\dfrac{di}{dt} = -\dfrac{0.5}{2}.10.\dfrac{2}{3}e^{-\frac{2}{3}t} = -1.67e^{-\frac{2}{3}t} A$

7.2. The source-free RC/RL circuit

7.2.2. R-L circuit

+ Example 3: Assuming that $i(0) = 10A$, calculate $i(t)$ and $i_x(t)$



***Solution 2**:* Apply directly the loop current method to the given circuit

$$\begin{cases} L\dfrac{di_1}{dt}+R_2(i_1-i_2)=0 \\ (R_1+R_2)i_2-R_2i_1-3i_1=0 \end{cases} \rightarrow \begin{cases} \dfrac{di_1}{dt}+4i_1-4i_2=0 \\ i_2=\dfrac{5}{6}i_1 \end{cases}$$

The current through the inductor is:

$$\rightarrow \frac{di_1}{dt}+\frac{2}{3}i_1=0 \rightarrow i_1(t)=\boxed{i(t)=i(0)e^{-\frac{2}{3}t}=10e^{-\frac{2}{3}t}}A, t>0$$

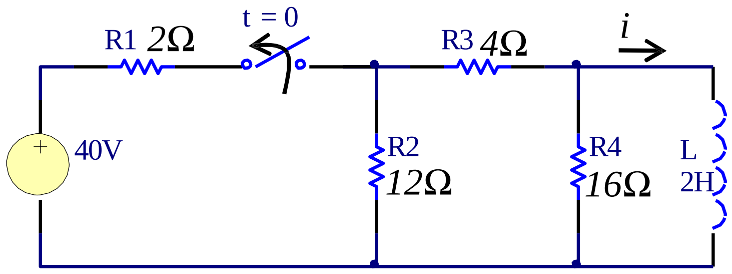
The current through the resistor R_2 is:

$$\boxed{i_x(t)=\frac{v_L(t)}{R_2}=\frac{L}{R_2}\frac{di}{dt}=-\frac{0.5}{2}.10.\frac{2}{3}e^{-\frac{2}{3}t}=-1.67e^{-\frac{2}{3}t}A}$$

7.2. The source-free RC/RL circuit

7.2.2. R-L circuit

+ Example 4: The switch has been closed for a long time. At *t = 0*, the switch is opened. Calculate $i(t)$ for $t > 0$



t < 0: the inductor acts as a short circuit

$$i_{R1} = \frac{E}{R_1 + \frac{R_2 R_3}{R_2 + R_3}} = \frac{40}{2 + \frac{12.4}{12+4}} = 8A \rightarrow i_L = i_{R1}\frac{R_2}{R_2+R_3} = 8.\frac{12}{12+4} = 6A \rightarrow i(0) = i_0 = 6A$$

t > 0: we have a R-L circuit

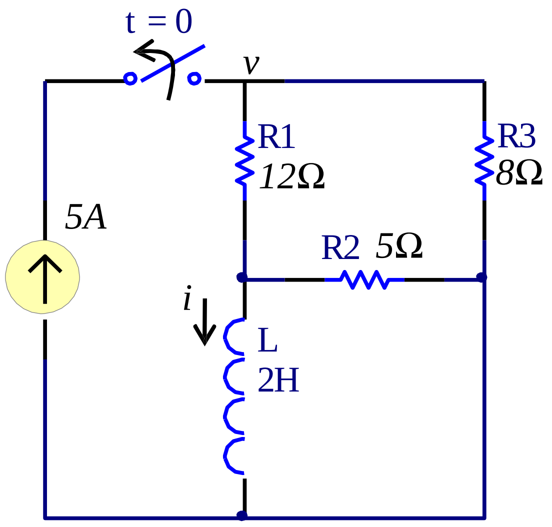
$$R_{eq} = \frac{R_4(R_2+R_3)}{R_4+R_2+R_3} = \frac{16(12+4)}{16+12+4} = 8\Omega$$

→ time constant: $\tau = \frac{L}{R_{eq}} = \frac{2}{8} = 0.25s$

The current through the inductor is: $i(t) = i_0 e^{-\frac{t}{\tau}} = 6e^{-4t} A$

7.2. The source-free RC/RL circuit

7.2.2. R-L circuit



+ Example 5: Calculate $i(t)$ for $t > 0$

t < 0: the inductor acts as a short circuit

$$5=\frac{v}{R_1}+\frac{v}{R_3}=\frac{R_1+R_3}{R_1R_3}v\rightarrow v=\frac{R_1R_3}{R_1+R_3}5\rightarrow i(t)=\frac{v}{R_1}=\frac{R_3}{R_1+R_3}5=2A\rightarrow i(0)=i_0=2A$$

t > 0: we have a R-L circuit

$$R_{eq}=\frac{R_2(R_1+R_3)}{R_2+R_1+R_3}=\frac{5(12+8)}{5+12+8}=4\Omega$$

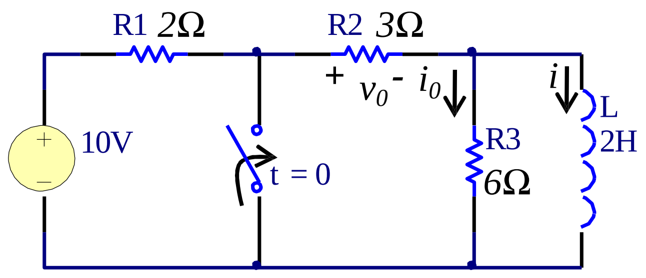
→ time constant: $$\tau=\frac{L}{R_{eq}}=\frac{2}{4}=0.5s$$

Hence, the current through the inductor is: $i(t)=i_0e^{-\frac{t}{\tau}}=2e^{-2t}A$

7.2. The source-free RC/RL circuit

7.2.2. R-L circuit

+ Example 6: Find i_0, v_0 and i for all time



t < 0: the inductor acts as a short circuit

$$i(t)=\frac{E}{R_1+R_2}=\frac{10}{2+3}=2A, t<0 \qquad i_0(t)=0A, t<0 \qquad v_0(t)=R_2 i(t)=3.2=6V, t<0$$

t > 0: voltage source is disconnected to the right side, and we have a R-L circuit which consists of R_2 , R_3 , and L

$$R_{eq}=\frac{R_2R_3}{R_2+R_3}=\frac{3.6}{3+6}=2\Omega$$ → time constant: $$\tau=\frac{L}{R_{eq}}=\frac{2}{2}=1s$$ → current through L: $$i(t)=i_0e^{-\frac{t}{\tau}}=2e^{-t}A$$

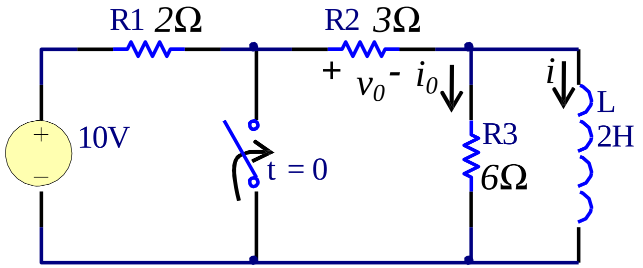
→ voltage across R_2 : $$v_0(t)=-v_L(t)=-L\frac{di}{dt}=-2.2.(-1)e^{-t}=4e^{-t}V$$

→ current through R_3 : $$i_0(t)=\frac{v_L(t)}{R_3}=\frac{L}{R_3}\frac{di}{dt}=\frac{2}{6}.2.(-1)e^{-t}=-0.67e^{-t}A$$

7.2. The source-free RC/RL circuit

7.2.2. R-L circuit

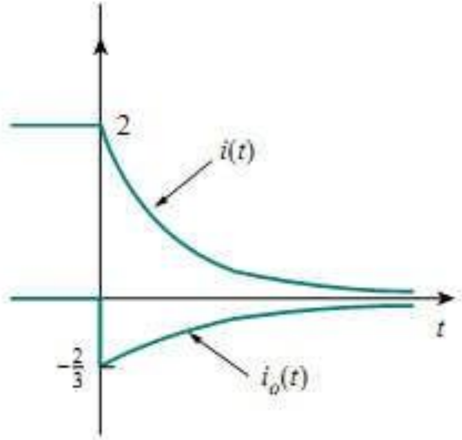
+ Example 6: Find i_0, v_0 and i for all time



For all time:

$$i(t)=\begin{cases} 2A & t<0 \\ 2e^{-t}A & t\geq 0 \end{cases} \qquad i_0(t)=\begin{cases} 0A & t<0 \\ -0.67e^{-t}A & t>0 \end{cases}$$

$$v_0(t)=\begin{cases} 6V & t<0 \\ 4e^{-t}V & t>0 \end{cases}$$



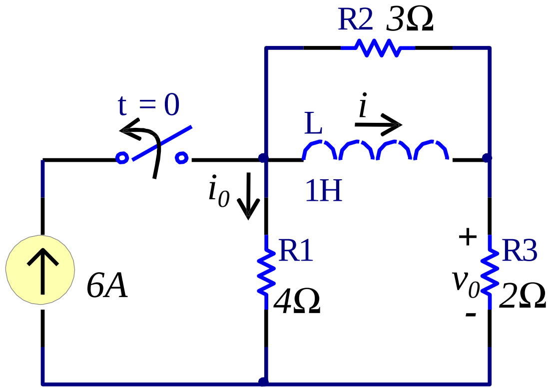
Note:

- The current through inductor is continuous at $t = 0$
- $i_0(t)$ drops from 0 to $-2/3A$, and $v_0(t)$ drops from 6 to 4 at $t = 0$
- τ is the same regardless of what the output is defined to be

7.2. The source-free RC/RL circuit

7.2.2. R-L circuit

+ Example 7: Find i_0, v_0 and i for all time



t < 0: the inductor acts as a short circuit

→ the current source and R_1 and R_3 are in parallel

$$6=\frac{v}{R_1}+\frac{v}{R_3}=\frac{R_1+R_3}{R_1R_3}v\rightarrow v=v_0(t)=\frac{R_1R_3}{R_1+R_3}6=8V\rightarrow i(t)=\frac{v}{R_3}=4A\rightarrow i_0(t)=2A$$

t > 0: current source is disconnected, and we have a R-L circuit which consists of R_1 , R_2 , R_3 , and L

$$R_{eq}=\frac{R_2(R_1+R_3)}{R_2+R_1+R_3}=\frac{3.6}{3+6}=2\Omega \qquad \tau=\frac{L}{R_{eq}}=\frac{1}{2}=0.5s \qquad i(t)=i(0)e^{-\frac{t}{\tau}}=4e^{-2t}A$$

$$i_2(t)=\frac{v_L(t)}{R_2}=\frac{L}{R_2}\frac{di}{dt}=\frac{1}{3}.4.(-2)e^{-2t}=-\frac{8}{3}e^{-2t}A\rightarrow i_0(t)=-(i+i_2)=-\left(4e^{-2t}-\frac{8}{3}e^{-2t}\right)=-\frac{4}{3}e^{-2t}=-1.33e^{-2t}A$$

$$v_0(t)=-R_3i_0(t)=2.67e^{-2t}V$$

For all time: → easy to find

7.3. Singularity functions

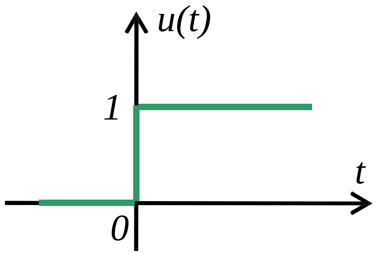
+ Singularity function (*switching function*):

→ functions that either are discontinuous or have discontinuous derivatives

→ Useful in circuit analysis: Serve as good approximations to the switching signals

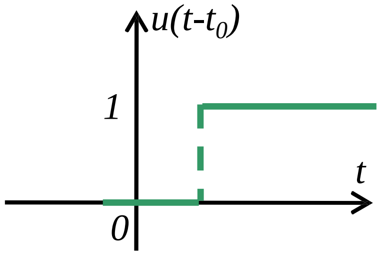
+ Widely used singularity functions: *unit step, unit impulse, unit ramp*

+ Unit step function u(t): equals to 0 for negative values of *t* and 1 for positive values of *t*



$$u(t)=\begin{cases} 0, & t<0 \\ 1, & t>0 \\ undefined, & t=0 \end{cases}$$

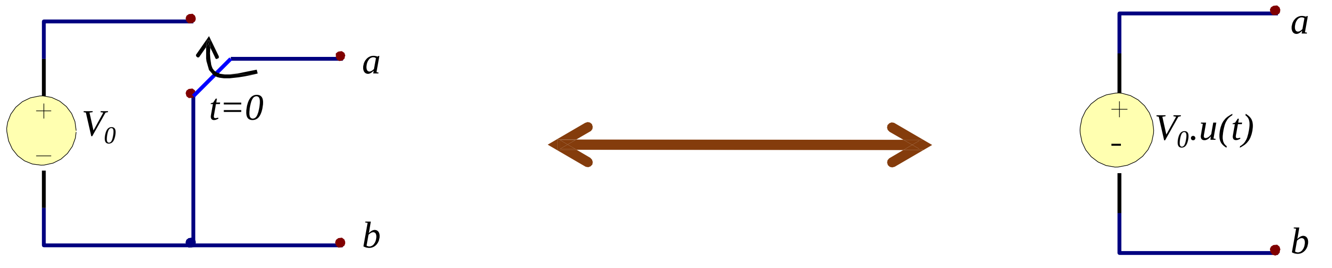
$$u(t-t_0)=\begin{cases} 0, & t<t_0 \\ 1, & t>t_0 \\ undefined, & t=t_0 \end{cases}$$

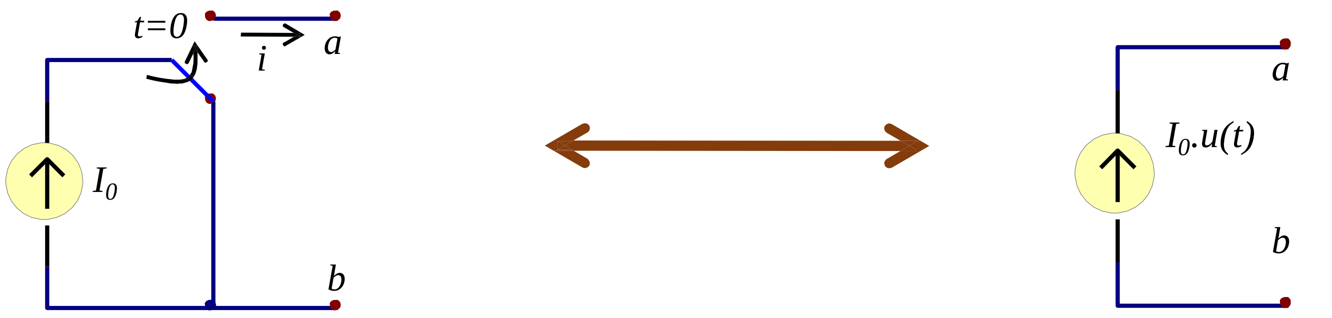


7.3. Singularity functions

+ Step function: → can be used to represent an abrupt change in voltage or current

$$v(t)=\begin{cases} 0, & t<t_0 \\ V_0, & t>t_0 \\ undefined, & t=t_0 \end{cases} \rightarrow v(t)=V_0.u(t-t_0)$$



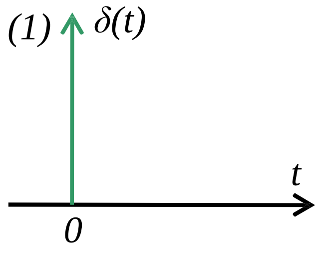


7.3. Singularity functions

+ Unit impulse function δ(t) (*delta function*): derivative of the unit step function *u(t)*

$$\delta(t) = \begin{cases} 0, & t < 0 \\ 0, & t > 0 \\ undefined, & t = 0 \end{cases}$$

$$\int_{-0}^{+0} \delta(t)dt = 1 \rightarrow \int_{-\infty}^{+\infty} \delta(t)dt = 1$$



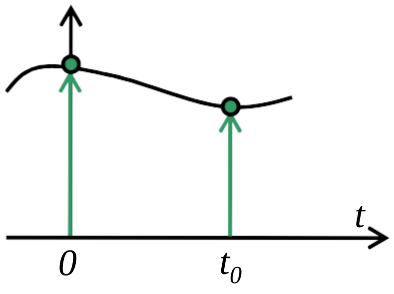
→ Unit impulse function δ(t) is zero everywhere except at *t* = *0*, where it is undefined

+ Impulsive currents and voltages: occur in electric circuits as a result of switching operations of impulsive sources

→ Not physically realizable, but useful mathematical tool

$$\int_a^b f(t)\delta(t)dt = f(0)$$

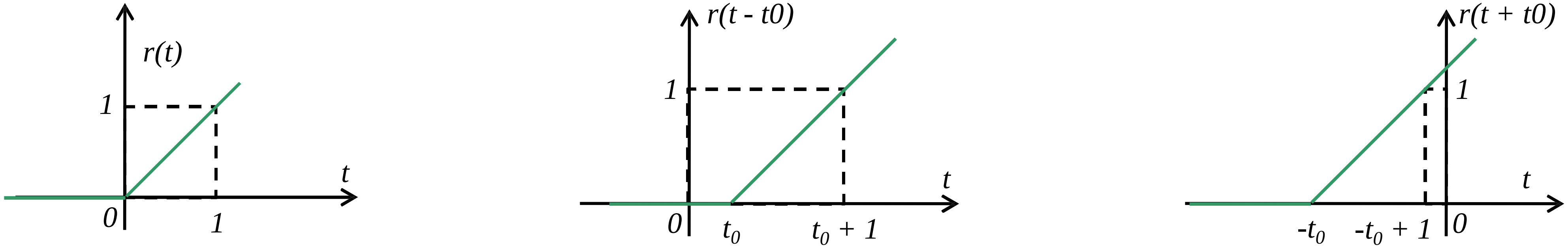
$$\int_a^b f(t)\delta(t-t_0)dt = f(t_0)$$



7.3. Singularity functions

+ Unit ramp function: zero for negative values of *t* and has a unit slope for positive values of *t*

$$r(t) = \int_{-\infty}^{t} u(t)dt = t.u(t)$$



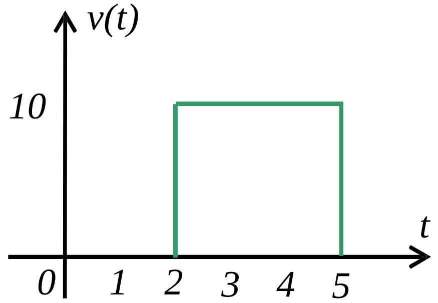
$$r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases}$$

$$r(t - t_0) = \begin{cases} 0, & t \le t_0 \\ t - t_0, & t \ge t_0 \end{cases}$$

$$r(t + t_0) = \begin{cases} 0, & t \le -t_0 \\ t - t_0, & t \ge -t_0 \end{cases}$$

7.3. Singularity functions

+ Example 8: Express the given voltage pulse (*gate function*) in terms of the unit step. Calculate its derivative and sketch it



The given function may be regarded as a step function which:

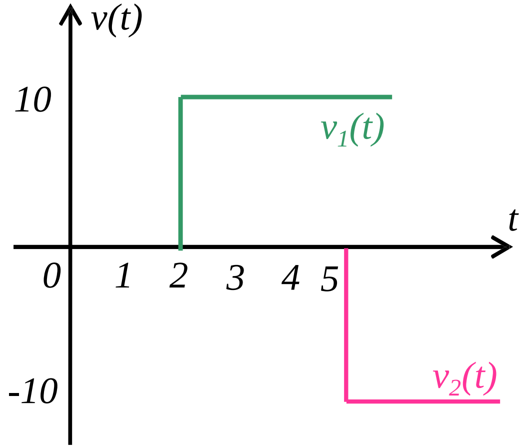
→ Switch on at one value of t $(t = 2s)$

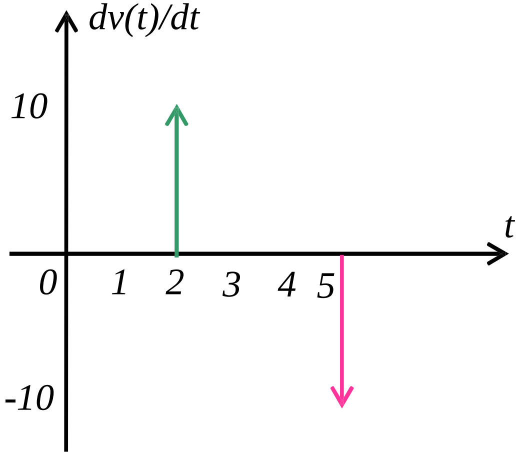
→ Switch off at another value of t $(t = 5s)$

$$v(t) = v_1(t) + v_2(t) = 10u(t-2) - 10u(t-5) V$$

Then, the derivative of the gate function:

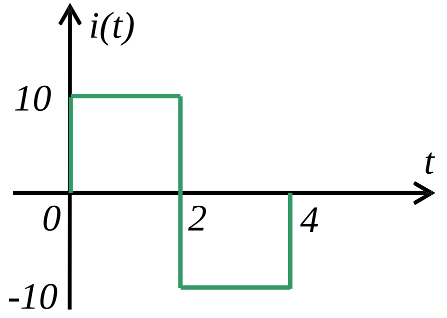
$$\frac{d}{dt}v(t) = 10\delta(t-2) - 10\delta(t-5)$$





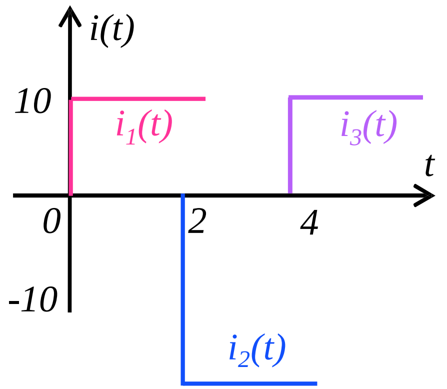
7.3. Singularity functions

+ Example 9: Express the current pulse in terms of the unit step. Find its integral and sketch it



The given current pulse may be regarded as:

$$i(t) = i_1(t) + i_2(t) + i_3(t) = 10u(t) - 20u(t-2) + 10u(t-4)A$$

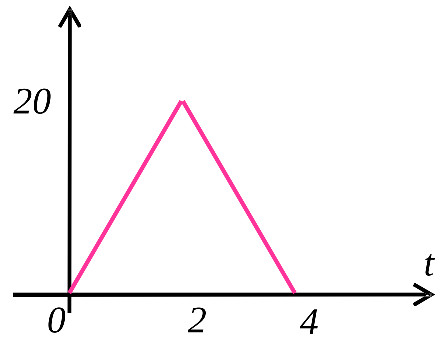


The integral of the current pulse:

$$\int_{-\infty}^{t} i(t)dt = \int_{-\infty}^{t} 10[u(t) - 2u(t-2) + u(t-4)]dt$$

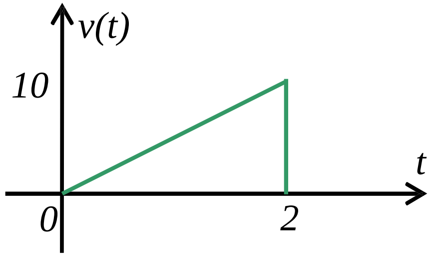
$$\rightarrow \int_{-\infty}^{t} i(t)dt = 10[t.u(t) - 2t.u(t-2) + t.u(t-4)]$$

$$\rightarrow \int_{-\infty}^{t} i(t)dt = 10[r(t) - 2r(t-2) + r(t-4)]$$



7.3. Singularity functions

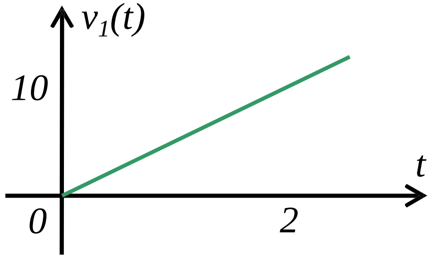
+ Example 10: Express the given *saw tooth* in terms of singularity functions



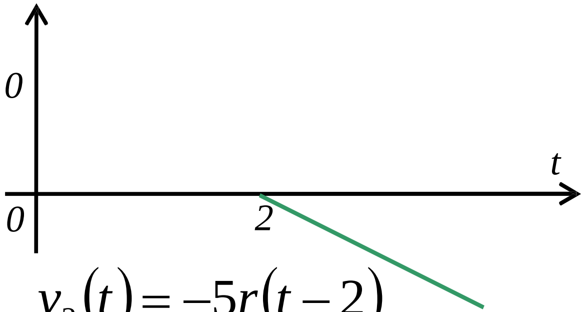
Solution 1: The saw tooth is expressed as a combination of unit functions

$$v(t) = 5t.u(t) - 5t.u(t-2) = 5t[u(t) - u(t-2)]$$

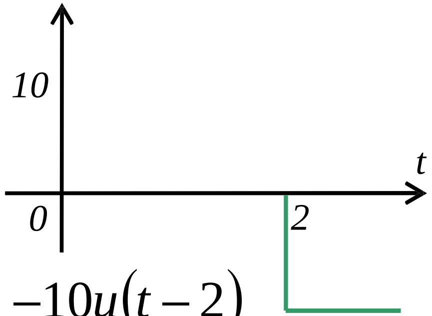
Solution 2: The saw tooth is expressed as a combination of 2 functions: *u(t)* & *r(t)*



$$v_1(t) = 5r(t)$$



$$v_2(t) = -5r(t-2)$$



$$v_3(t) = -10u(t-2)$$

$$v(t) = 5t.u(t) - 5t.u(t-2) = 5t.u(t) - 5(t-2+2).u(t-2) = \boxed{5t.u(t)} - 5\boxed{(t-2).u(t-2)} - 5.2.u(t-2)$$

$$\rightarrow v(t) = 5r(t) - 5r(t-2) - 10.u(t-2)$$

7.3. Singularity functions

+ Example 11: Express g(t) in terms of step and ramp functions $g(t)=\begin{cases} 3 & t<0 \\ -2 & 0<t<1 \\ 2t-4 & t>1 \end{cases}$

The signal g(t) can be expressed as the sum of three function specified within the three intervals: *$g_1(t)$, $g_2(t)$, $g_3(t)$*

with t < 0: $g_1(t)=3u(-t)$

with 0 < t < 1: $g_2(t)=-2[u(t)-u(t-1)]$

with t > 1: $g_3(t)=(2t-4)[u(t-1)]$

So:

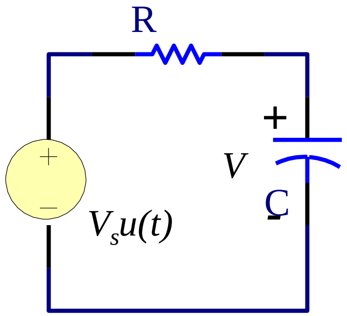
$$g(t)=3u(-t)-2[u(t)-u(t-1)]+(2t-4)[u(t-1)]=3u(-t)-2u(t)+(2t-2)u(t-1)$$

$$g(t)=3u(-t)-2u(t)+2r(t-1)=3[1-u(t)]-2u(t)+2r(t-1)$$

$$\rightarrow \boxed{g(t)=3-5u(t)+2r(t-1)}$$

7.4. Step response of a RC/RL circuit

7.4.1. Step response of a RC circuit



+ When the *DC* source of a RC circuit is suddenly applied:

→ voltage or current source can be modeled as a step function

→ the response of circuit is known as a *step response*

+ *Step response* of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source

+ Example: Consider the *RC* circuit, assume an initial voltage V0 on the capacitor

$$V(-0) = V(+0) = V_0$$

V(-0): Voltage across *C* just before switching

V(+0): Voltage across *C* immediately after switching

Apply KVL to the loop: $Ri(t) + v(t) = V_s u(t) \rightarrow RC\frac{dv}{dt} + v(t) = V_s u(t)$

With t > 0, we obtain: $\frac{dv}{dt} + \frac{1}{RC}v(t) = \frac{1}{RC}V_s$

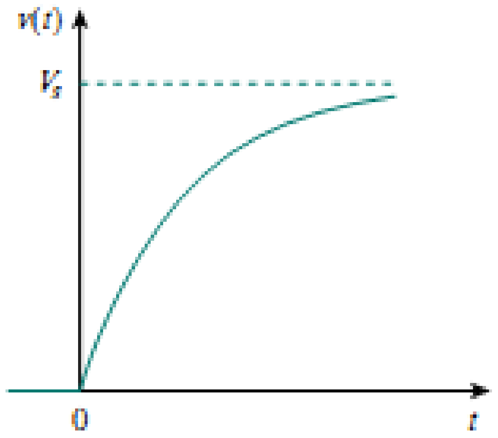
7.4. Step response of a RC/RL circuit

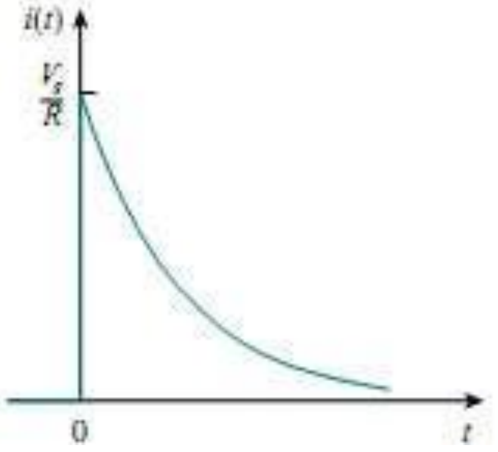
7.4.1. Step response of a RC circuit

$$\frac{dv}{dt}+\frac{1}{RC}\left(v-V_s\right)=0 \rightarrow \frac{dv}{v-V_s}=-\frac{dt}{RC} \rightarrow \ln\left(v-V_s\right)\Big|_{V_0}^{v(t)}=-\frac{t}{RC}\Big|_0^t$$

$$\rightarrow \ln\left[v(t)-V_s\right]-\ln\left(V_0-V_s\right)=-\frac{t}{RC}$$

$$\rightarrow \ln\frac{v-V_s}{V_0-V_s}=-\frac{t}{RC} \rightarrow \frac{v-V_s}{V_0-V_s}=e^{-\frac{t}{\tau}} \rightarrow v(t)=\begin{cases} V_0 & t<0 \\ V_s+\left(V_0-V_s\right)e^{-\frac{t}{\tau}} & t>0 \end{cases}$$





+ If C is uncharged initially: $V_0 = 0$

$$\boxed{v(t)=V_s\left(1-e^{-\frac{t}{\tau}}\right)u(t)V}$$

$$\boxed{i(t)=C\frac{dv}{dt}=\frac{V_s}{R}e^{-\frac{t}{\tau}}.u(t)A}$$

7.4. Step response of a RC/RL circuit

7.4.1. Step response of a RC circuit

+ In general, for t > 0:

$$v(t) = V_s + (V_0 - V_s)e^{-\frac{t}{\tau}} = v_n + v_f \qquad \begin{cases} v_n = (V_0 - V_s)e^{-\frac{t}{\tau}} \\ v_f = V_s \end{cases}$$

- v_n: Natural response (*not remained by excitation source*) of circuit, will decay to zero after five time constants
- v_f: Forced response (*remained by excitation source*) of circuit, will represents what the circuit is forced to do by the input excitation, and remains a long time

+ Thus, to find the step response of an RC circuit requires 3 things:

$$\boxed{v(t) = v(0)e^{-\frac{t}{\tau}} + v(\infty)\left(1 - e^{-\frac{t}{\tau}}\right)}$$

$v(0)$ is the initial voltage at $t = +0$.

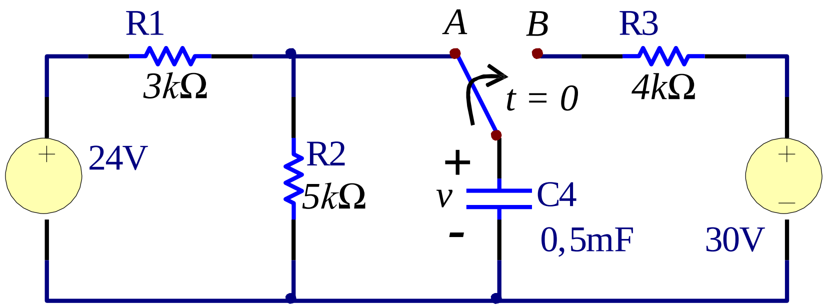
$v(\infty)$ is the final or steady state value

τ is the time constant of the circuit

7.4. Step response of a RC/RL circuit

7.4.1. Step response of a RC circuit

+ Example 12: The switch has been in position *A* for a long time. At *t = 0*, the switch moves to *B*. Determine *v(t)* for *t > 0* and calculate its value at *t = 1s* and *4s*



The voltage across the capacitor at t = -0: $v(-0) = \frac{24}{R_1 + R_2} R_2 = 15V$

From charge conservation law at node A: $v(+0) = v(-0) = 15V$

For t > 0, the time constant is: $\tau = R_3 C_4 = 4.10^3.0.5.10^{-3} = 2s$

Since the capacitor acts like an open circuit to *DC* at steady state: $v(\infty) = E_2 = 30V$

The voltage across C_4 is: $v(t) = v(+0)e^{-\frac{t}{\tau}} + v(\infty)\left(1 - e^{-\frac{t}{\tau}}\right) = 15e^{-0.5t} + 30\left(1 - e^{-0.5t}\right) = 30 - 15e^{-0.5t}V$

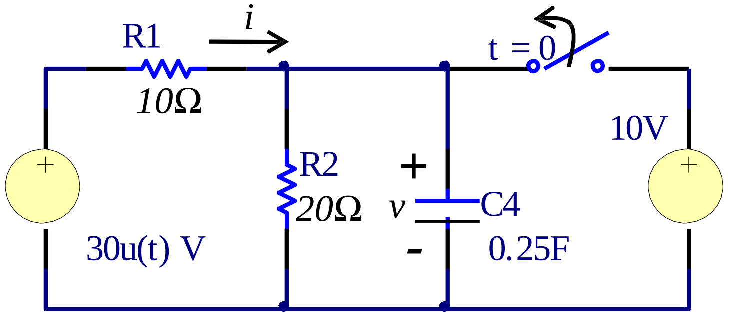
At *t = 1s:* $v(1s) = 30 - 15e^{-0.5.1} = 20.902V$

At *t = 4s:* $v(4s) = 30 - 15e^{-0.5.4} = 27.970V$

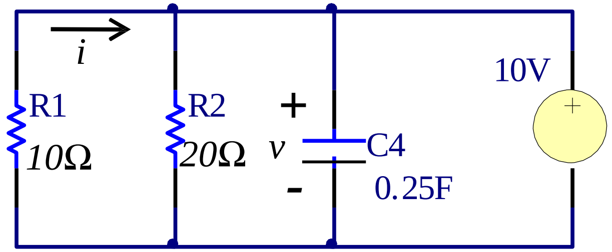
7.4. Step response of a RC/RL circuit

7.4.1. Step response of a RC circuit

+ Example 13: The switch has been closed for a long time and is opened at $t = 0$. Find i, v for all time

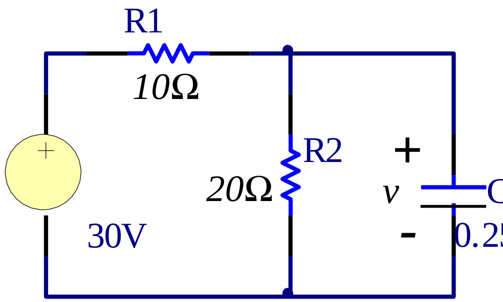


For t < 0: $v(0) = v(-0) = 10V$ $\quad i = -\frac{v}{R_1} = -1A$



For t > 0: $v(\infty) = \frac{30}{R_1 + R_2} R_2 = 20V$

$$\tau = R_{eq} C_4 = \frac{R_1 R_2}{R_1 + R_2} . C_4 = \frac{10.20}{10+20} .0.25 = \frac{5}{3} s$$

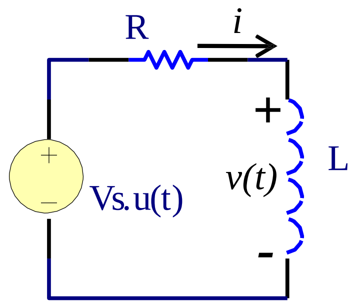


$$\rightarrow v(t) = v(0) e^{-\frac{t}{\tau}} + v(\infty)\left(1 - e^{-\frac{t}{\tau}}\right) = 10e^{-0.6t} + 20\left(1 - e^{-0.6t}\right) = 20 - 10e^{-0.6t} V$$

$$\rightarrow i(t) = \frac{v}{R_2} + C\frac{dv}{dt} = 1 - 0.5e^{-0.6t} + 0.25.(-10)(-\frac{3}{5})e^{-0.6t} = 1 + e^{-0.6t} A$$

7.4. Step response of a RC/RL circuit

7.4.2. Step response of a RL circuit



+ Connect a *R-L* circuit to a DC source → find *i* as the circuit response right after the connection

→ *i* be the sum of the natural response and the forced response $i = i_n + i_f$

The natural response $i_n = Ae^{-\frac{R}{L}t}, \tau = \frac{L}{R}$

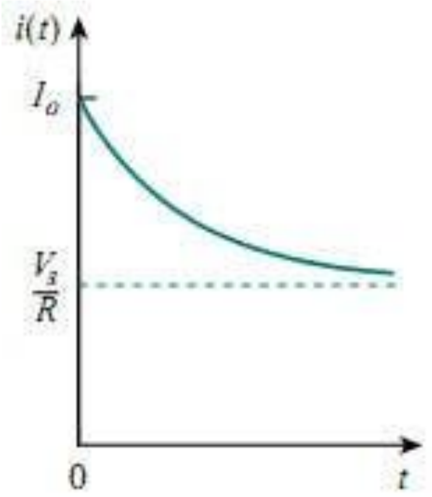
The steady-state response: $i_f = \frac{V_s}{R}$

$$\rightarrow i(t) = Ae^{-\frac{t}{\tau}} + \frac{V_s}{R}$$

At t = 0: $i(0) = i(-0) = I_0 \rightarrow A = I_0 - \frac{V_s}{R}$

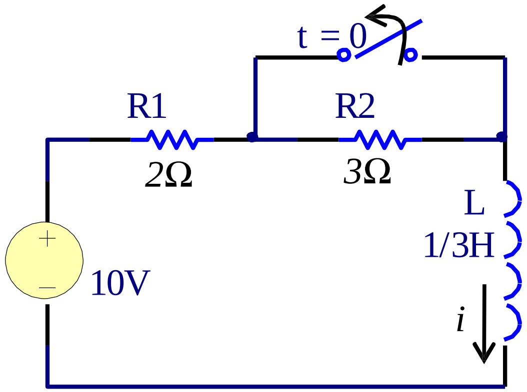
Thus: $i(t) = Ae^{-\frac{t}{\tau}} + \frac{V_s}{R} = \left(I_0 - \frac{V_s}{R}\right)e^{-\frac{t}{\tau}} + \frac{V_s}{R}$

$$\boxed{i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-\frac{t}{\tau}} = i(\infty) + [i(0) - i(\infty)]e^{-\frac{t}{\tau}}}$$



7.4. Step response of a RC/RL circuit

7.4.2. Step response of a RL circuit



+ Example 14: Find i(t) for t > 0. Assume that the switch has been closed for a long time

For t < 0: $i(-0)=\dfrac{E}{R_1}=\dfrac{10}{2}=5A \rightarrow i(0)=i(-0)=5A$

For t > 0: $$i(\infty)=\frac{E}{R_1+R_2}=\frac{10}{2+3}=2A$$

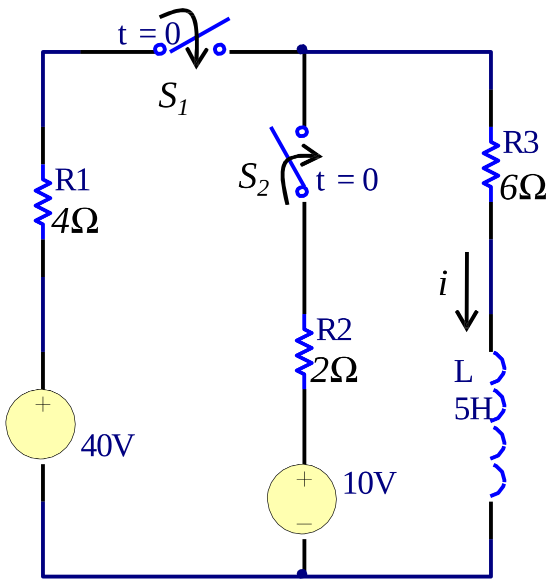
$$\tau=\frac{L}{R_1+R_2}=\frac{1}{3(2+3)}=\frac{1}{15}s$$

Thus, the current through inductor L is: $i(t)=i(\infty)+[i(0)-i(\infty)]e^{-\frac{t}{\tau}}=2+(5-2)e^{-15t}=2+3e^{-15t}A$

7.4. Step response of a RC/RL circuit

7.4.2. Step response of a RL circuit

+ Example 15: At $t = 0$, switch 1 is closed, and switch 2 is closed $4s$ later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2s$ and $t = 5s$



For $t < 0$: Two switches are open $\quad i(-0) = i(0) = 0A$

For $0 \le t \le 4$: The switch S_1 is closed, S2 is open

$$i(\infty) = \frac{E_1}{R_1 + R_3} = \frac{40}{4+6} = 4A$$

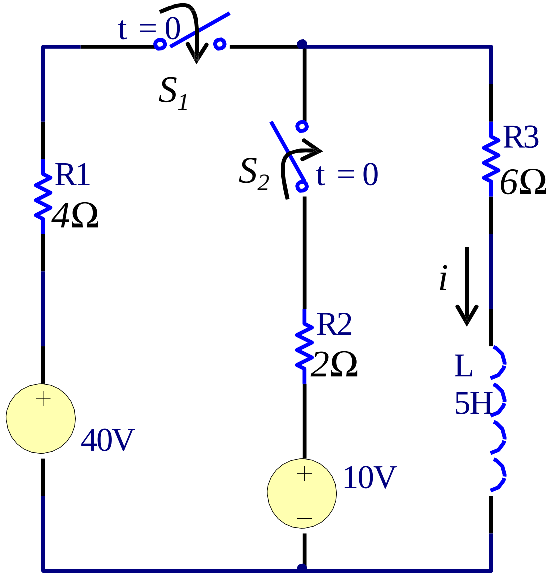
$$\tau_1 = \frac{L}{R_{eq1}} = \frac{5}{4+6} = 0.5s$$

$$\rightarrow i(t) = i(\infty) + [i(0) - i(\infty)]e^{-\frac{t}{\tau_1}} = 4 - 4e^{-2t}A$$

7.4. Step response of a RC/RL circuit

7.4.2. Step response of a RL circuit

+ Example 15: At *t = 0*, switch 1 is closed, and switch 2 is closed *4s* later. Find $i(t)$ for *t > 0*. Calculate i for *t = 2s* and *t = 5s*



For *t > 4:* The switch S2 is closed

$$i(4)=4-4e^{-2.4}\approx 4A$$

Using nodal analysis to find I in steady state

$$\frac{40-v}{R_1}+\frac{10-v}{R_2}=\frac{v}{R_3}\rightarrow v=\frac{180}{11}V\rightarrow i(\infty)=\frac{v}{R_3}=2.72A$$

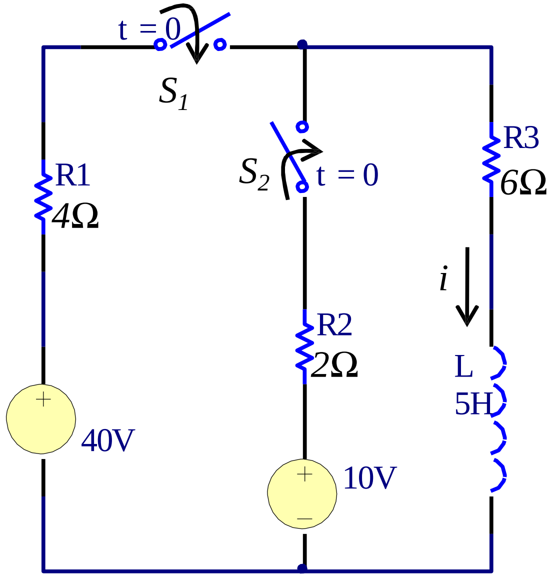
$$\tau=\frac{L}{R_{eq}}=\frac{L}{R_3+\dfrac{R_1R_2}{R_1+R_2}}=\frac{15}{22}s$$

$$\rightarrow i(t)=i(\infty)+\left[i(4)-i(\infty)\right]e^{-\frac{t-4}{\tau}}=2.72+1.28e^{-1.467(t-4)}A$$

7.4. Step response of a RC/RL circuit

7.4.2. Step response of a RL circuit

+ Example 15: At $t = 0$, switch 1 is closed, and switch 2 is closed $4s$ later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2s$ and $t = 5s$



For all time, we have:

$$i(t) = \begin{cases} 0 & t \le 0 \\ 4\left(1 - e^{-2t}\right) & 0 \le t \le 4 \\ 2.72 + 1.28e^{-1.467(t-4)} & t \ge 4 \end{cases}$$

At $t = 2s$:

$$i(2) = 4\left(1 - e^{-2.2}\right) = 3.93A$$

At $t = 5s$:

$$i(5) = 2.72 + 1.28e^{-1.467(5-4)} = 3.02A$$

7.5. First order op-amp circuits

+ Op amp circuit containing a storage element: → exhibit first-order behaviors

+ Examples of first-order op amp circuits: differentiators and integrators (in chapter 5)

+ For practical reasons, inductors are hardly ever used in op amp circuits → the op amp circuits considered here are of the RC type

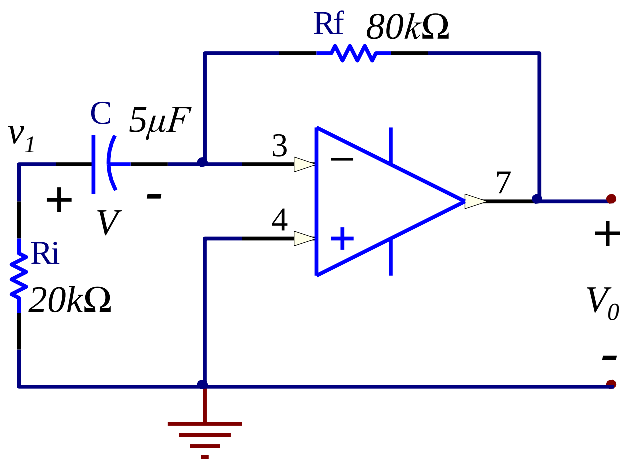
+ Methods to analyze op amp circuits:

→ Using nodal analysis
→ Using the Thevenin equivalent circuit to simpler the op amp circuit

7.5. First order op-amp circuits

+ Example 16: Find v_0 for $t > 0$, give that $v(0) = 3V$

Solution 1:



We have: $-\frac{v_1}{R_i} = C\frac{dv}{dt}$

Potentials at node 3 and node 4 are equal to zero:

$$v_1 - 0 = V \rightarrow v_1 = V \rightarrow \frac{dv}{dt} + \frac{v}{R_i C} = 0$$

$$\rightarrow v(t) = V_0 e^{-\frac{t}{\tau}}, \tau = R_i C \rightarrow v(t) = 3e^{-10t}V$$

At $t > 0$, applying KCL at node 3 gives:

$$C\frac{dv}{dt} = \frac{-V_0}{R_f} \rightarrow V_0 = -R_f C\frac{dv}{dt} = -80.10^3.5.10^{-6}\left(-30e^{-10t}\right) = 12e^{-10t}V$$

7.5. First order op-amp circuits

+ Example 16: Find v_0 for $t > 0$, give that $v(0) = 3V$

Solution 2:

Voltage across the capacitor *C*: $v(+0) = v(-0) = 3V$

Apply KCL at node 3: $\frac{3}{R_i} + \frac{-V_0(+0)}{R_f} = 0 \rightarrow V_0(+0) = 12V$

The circuit is source free: $V_0(\infty) = 0V$

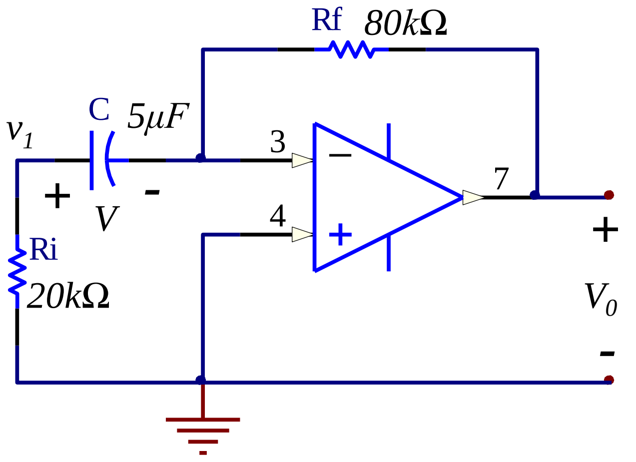
To find τ, need calculate R_{eq} across C:

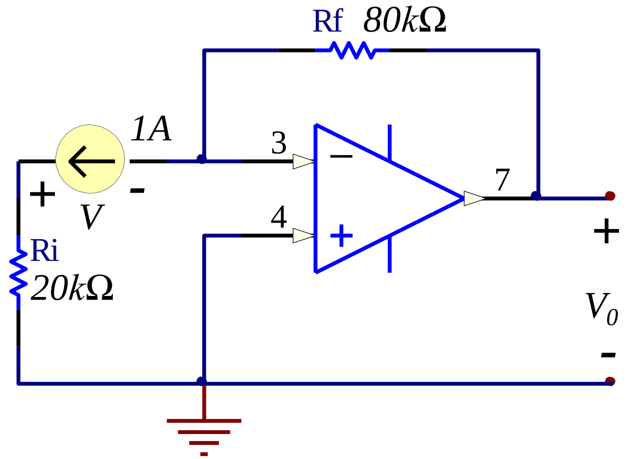
→ Replace C by a 1-A current source

→ Applying KVL to the input loop:

$$R_i i_s - v = 0 \rightarrow 20.10^3.1 - v = 0 \rightarrow v = 20kV \rightarrow R_{eq} = \frac{v}{i_s} = 20k\Omega$$

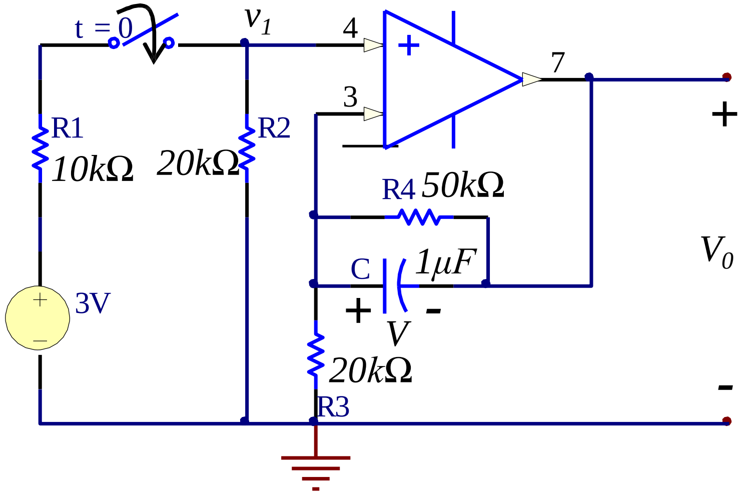
$$\rightarrow \tau = R_{eq}C = 0.1s \rightarrow V_0 = V_0(\infty) + [V_0(0) - V_0(\infty)]e^{-\frac{t}{\tau}} = 12e^{-10t}V, t > 0$$





7.5. First order op-amp circuits

+ Example 17: Determine *v(t) and v₀(t)*



V(t) is the step response:

$$v(t)=v(\infty)+[v(0)-v(\infty)]e^{-\frac{t}{\tau}},t>0$$

No current enters the op amp → the elements on the feedback loop constitute an *RC* circuit

$$\tau=R_4C=50.10^3.10^{-6}=0.05s$$

For *t < 0:* The switch is opened $V_0(0)=0V$

For *t > 0:* $V_1=\frac{R_2}{R_1+R_2}E=2V$

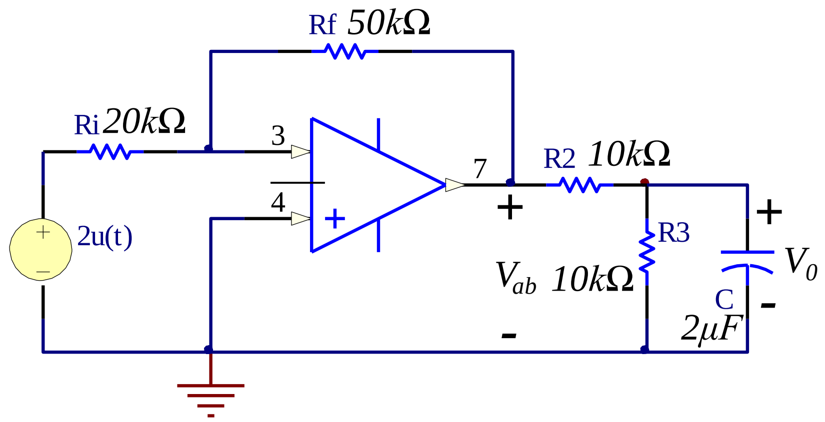
At steady state: C acts like an open circuit → op amp is a non-inverting amplifier

$$V_0(\infty)=\left(1=\frac{R_4}{R_3}\right)v_1=3.5\times2=7V \qquad V_1(\infty)=v(\infty)+V_0(\infty)\rightarrow v(\infty)=V_1(\infty)-V_0(\infty)=2-7=-5V$$

$$\rightarrow v(t)=v(\infty)+[v(0)-v(\infty)]e^{-\frac{t}{\tau}}=-5+5e^{-20t} \qquad \rightarrow V_0(t)=V_1(t)-v(t)=2-2e^{-20t}-\left(-5+5e^{-20t}\right)=7-7e^{-20t}V,t>0$$

7.5. First order op-amp circuits

+ Example 18: Determine *step response* $v_0(t)$



Remove C, and find the Thevenin equivalent at its terminal

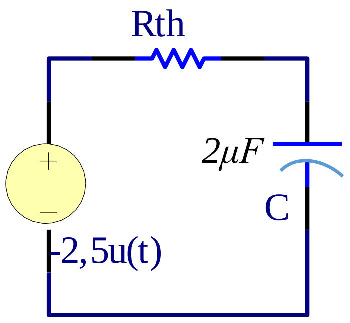
Open voltage at the terminal:

$$V_{ab} = -\frac{R_f}{R_i}V_i \rightarrow V_{th} = \frac{R_3}{R_2+R_3}V_{ab} = -\frac{R_3}{R_2+R_3}\frac{R_f}{R_i}V_i = -2.5u(t)$$

Equivalent resistor at the terminal:

$$R_{th} = R_3 // (R_0+R_2) \rightarrow R_{th} = \frac{(R_0+R_2)R_3}{R_0+R_2+R_3} = \frac{R_2R_3}{R_2+R_3} = 5k\Omega$$

We obtain the Thevenin equivalent circuit:



$$V_0(t) = -2.5\left(1-e^{-\frac{t}{\tau}}\right)u(t), \tau = R_{th}C = 5.10^3.2.10^{-6} = 0.01s$$

$$\rightarrow V_0(t) = -2.5\left(1-e^{-100t}\right)u(t)V, t>0$$

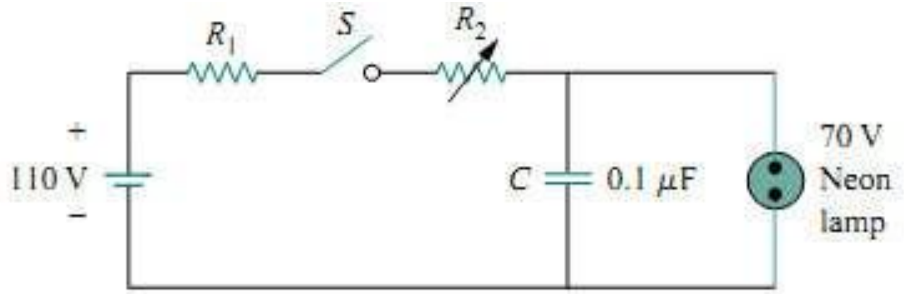
7.6. Applications

7.6.1. Delay circuits

+ As an example → consider an RC circuit:

→ Capacitor connected in parallel with a neon lamp

→ Voltage source can provide enough voltage to fire the lamp



+ Close the switch: V_C increases to *110V* with the time constant $(R_1 + R_2)C$

→ The lamp will act as an open-circuit and not emit light until the voltage across it exceeds a particular level (*70V*)

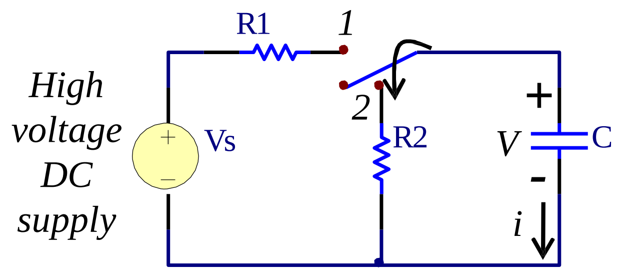
→ When V_C reaches, the lamp fires and the capacitor discharges through it → V_C drops and the lamp turn off

→ The lamp acts again as an open-circuit and *C* recharges

→ Adjusting R_2, we can introduce either short or long time delays

7.6. Applications

7.6.2. Photoflash unit



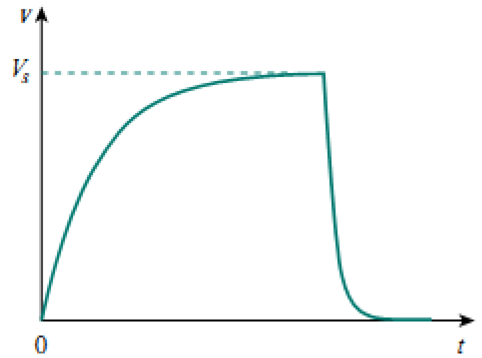
+ This application exploits the ability of the capacitor to oppose any abrupt change in voltage

+ Principle:

Switch is in 1: C charges slowly due to the large time constant $\tau = R_1C$

→ V_C rises gradually from zero to V_S

→ I_C *decreases from* I_1 to 0

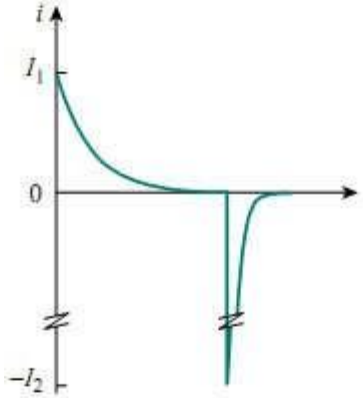


Switch is in 2: *C discharges*

→ Low resistance R_2 of the photo-lamp permits a high discharge current with peak I_2 in a short duration

$$t_{discharge} = 5R_2C$$

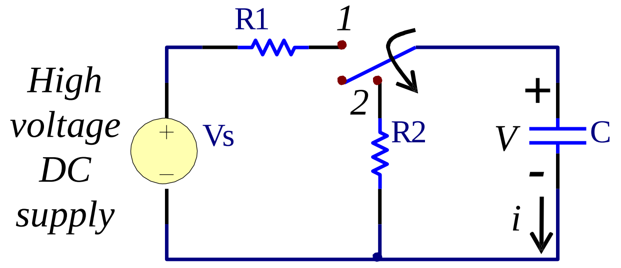
→ The circuit provides a short-duration, high current pulse



7.6. Applications

7.6.2. Photoflash unit

+ Example 19: An electronic flashgun has a current limiting $R_1 = 6k\Omega$, and $C = 2000\mu F$ charged to 240V. If the lamp resistance R_2 is 12Ω, we have:



→ *Peak charging current:* $I_1 = \frac{V_s}{R_1} = \frac{240}{6.10^3} = 40mA$

→ *Time required for the C to fully charge:* $t_{charge} = 5R_1C = 5.6.10^3.2000.10^{-6} = 60s$

→ *Peak discharging current:* $I_1 = \frac{V_s}{R_2} = \frac{240}{12} = 20A$

→ *Energy stored:* $w = \frac{1}{2}CV_s^2 = \frac{1}{2}2000.10^{-6}.240^2 = 57.6J$

→ *Energy stored in C is dissipated across the lamp during the discharging period:*

$$t_{discharge} = 5R_2C = 5.12.2000.10^{-6} = 0.12s \rightarrow p = \frac{w}{t_{discharge}} = \frac{57.6}{0.12} = 480W$$

7.6. Applications

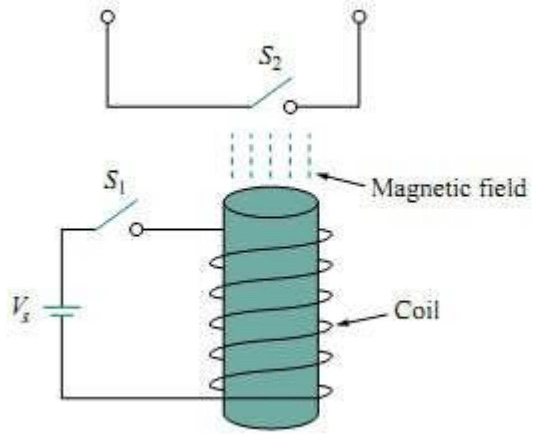
7.6.3. Relay circuit

+ Relay: → an electromagnetic device used to open or close a switch that controls another circuit

+ Operation principle of a relay: *RL* circuit

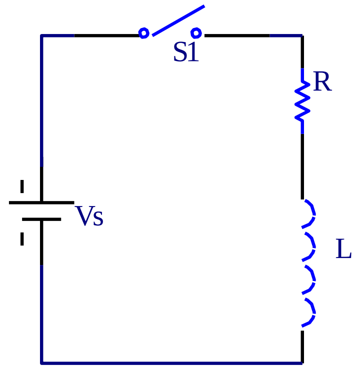
When S_1 is closed → i_L increases, produces a magnetic field

The magnetic field → pull the movable contact in the other circuit and close switch S_2



+ Relay delay time: time interval t_d between the closure of switches S_1 and S_2

+ Application of Relays: in the earliest *digital circuits and* are still used for *switching high power circuits*



7.6. Applications

7.6.3. Relay circuit

+ Example 20: The coil of a certain relay is operated by a 12V battery. If the coil has a resistance of 150Ω and an inductance of 30mH and the current needed to pull in is 50mA. Calculate the relay delay time

The current through the coil: $i(t) = i(\infty) + [i(0) - i(\infty)]e^{-\frac{R}{L}t}$

$$i(0) = 0, i(\infty) = \frac{v_s}{R} = \frac{12}{150} = 80mA, \tau = \frac{L}{R} = \frac{30.10^{-3}}{150} = 0.2ms$$

Thus: $i(t) = 80 - 80e^{-\frac{t}{\tau}}mA$

At t_d: $i(t_d) = 80 - 80e^{-\frac{t_d}{\tau}} = 50 \to e^{-\frac{t_d}{\tau}} = \frac{3}{8} \to t_d = \tau \ln\frac{8}{3} = 0.2\ln\frac{8}{3} = 0.1962ms$