Nguyễn Công Phương

Engineering Electromagnetics

The Steady Magnetic Field

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The Steady Magnetic Field (1)

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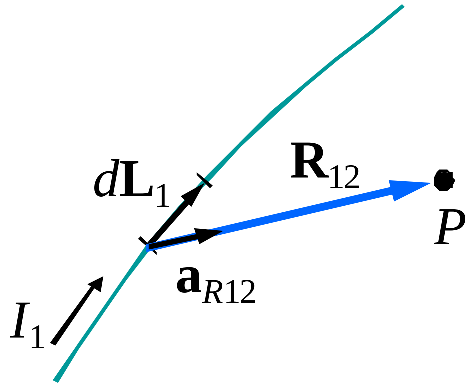
3. Curl
4. Stokes' Theorem
5. Magnetic Flux & Magnetic Flux Density
6. Magnetic Potential
7. Derivation of the Steady – Magnetic – Field Law

The Steady Magnetic Field (2)

- The source of the steady magnetic field may be:
  - Permanent magnet
  - Electric field changing linearly with time
  - Direct current
- Consider the field produced by a differential dc element in free space only

Biot – Savart Law (1)

$$\boxed{d\mathbf{H} = \frac{Id\mathbf{L}\times\mathbf{a}_R}{4\pi R^2} = \frac{Id\mathbf{L}\times\mathbf{R}}{4\pi R^3}}$$



H: magnetic field intensity (A/m)

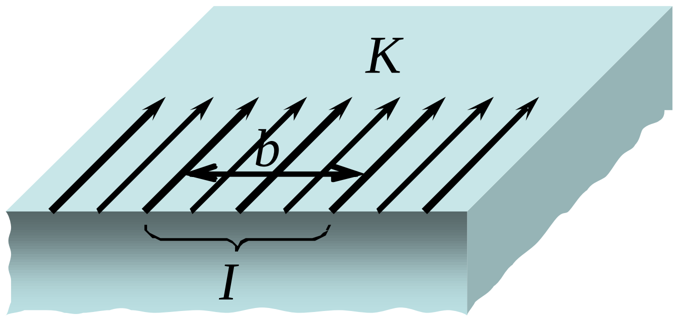
The direction of **H** is determined by the right-hand rule

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}$$

$$d\mathbf{H} = \frac{Id\mathbf{L}\times\mathbf{a}_R}{4\pi R^2} \rightarrow \mathbf{H} = \oint \frac{Id\mathbf{L}\times\mathbf{a}_R}{4\pi R^2}$$

Biot – Savart Law (2)

$$I = Kb$$

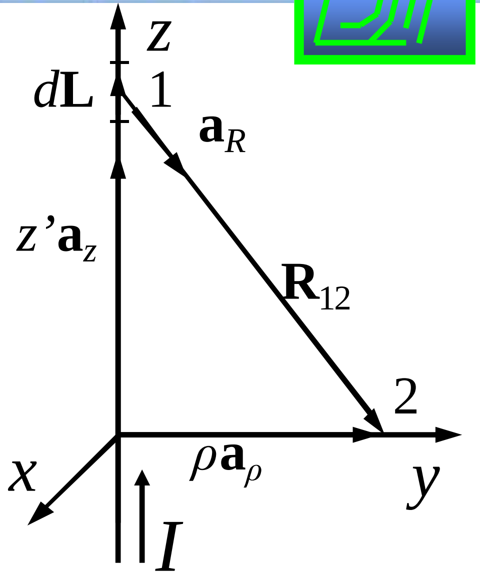


$$I = \int K dN$$

$$Id\mathbf{L} = \mathbf{K}dS$$

$$\mathbf{H} = \oint \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \int_S \frac{\mathbf{K} \times \mathbf{a}_R dS}{4\pi R^2}$$

Biot – Savart Law (3)



$$\left.\begin{aligned} d\mathbf{H}_2 &= \frac{Id\mathbf{L}_1\times\mathbf{a}_{R12}}{4\pi R_{12}^2} \\ d\mathbf{L}_1 &= dz'\mathbf{a}_z \\ \mathbf{R}_{12} = \rho\mathbf{a}_\rho - z'\mathbf{a}_z \rightarrow \mathbf{a}_{R12} &= \frac{\rho\mathbf{a}_\rho - z'\mathbf{a}_z}{\sqrt{\rho^2+z'^2}} \end{aligned}\right\}$$

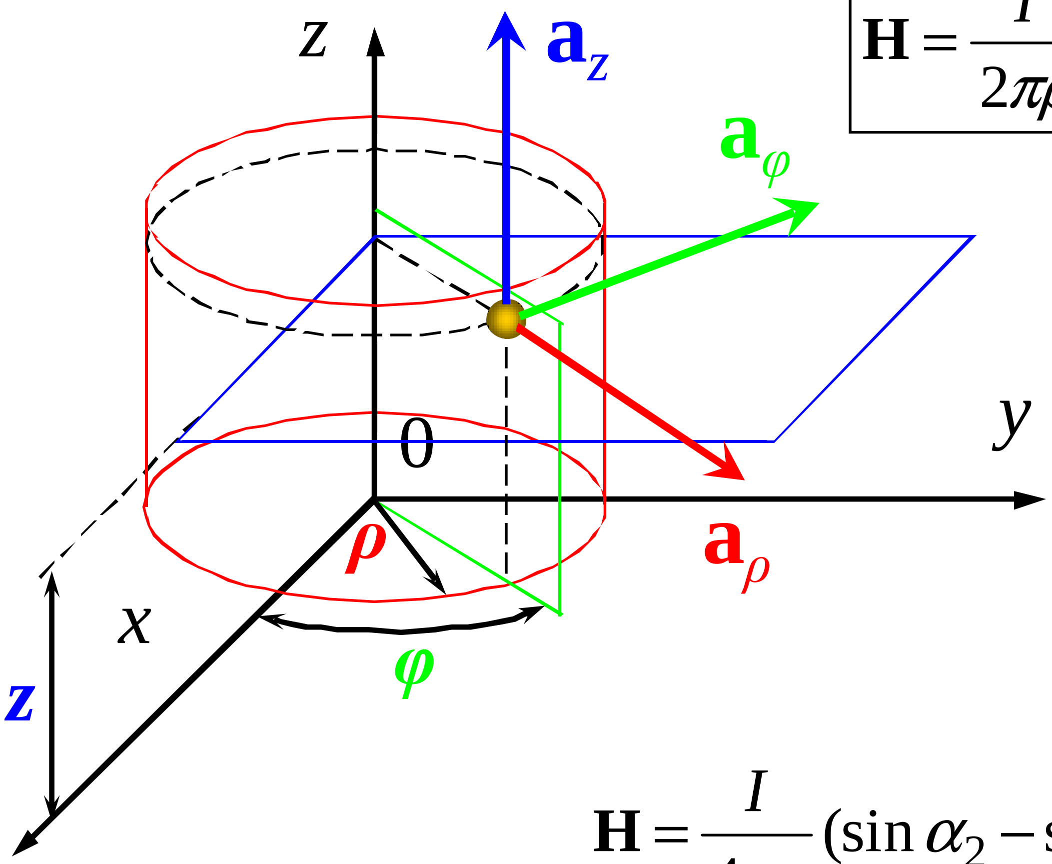
$$\rightarrow d\mathbf{H}_2 = \frac{Idz'\mathbf{a}_z\times(\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2+z'^2)^{3/2}} \quad \left.\begin{aligned} \rightarrow \mathbf{H}_2 = \int_{-\infty}^{\infty}\frac{Idz'\mathbf{a}_z\times(\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2+z'^2)^{3/2}} \\ \mathbf{a}_z\times\mathbf{a}_\rho = \mathbf{a}_\varphi;\ \mathbf{a}_z\times\mathbf{a}_z = 0 \end{aligned}\right\}$$

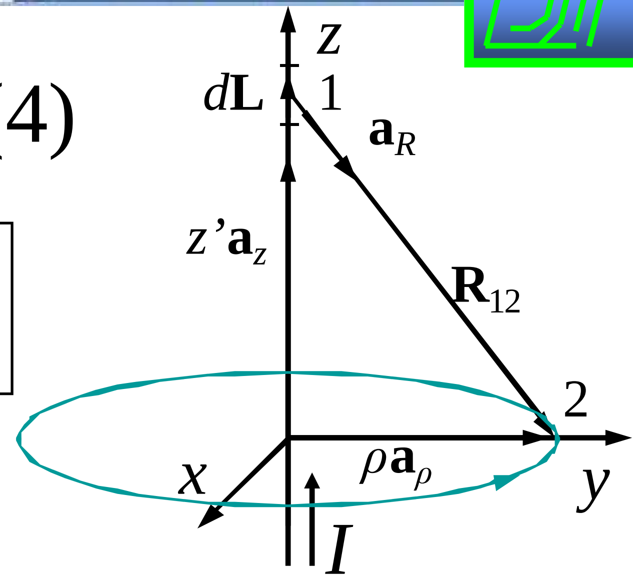
$$\rightarrow \mathbf{H}_2 = \frac{I}{4\pi}\int_{-\infty}^{\infty}\frac{\rho dz'\mathbf{a}_\varphi}{(\rho^2+z'^2)^{3/2}} = \frac{I\rho\mathbf{a}_\varphi}{4\pi}\int_{-\infty}^{\infty}\frac{dz'}{(\rho^2+z'^2)^{3/2}}$$

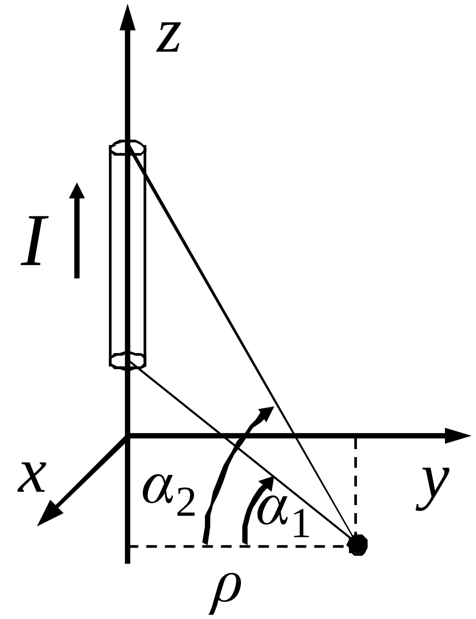
$$= \frac{I\rho\mathbf{a}_\varphi}{4\pi}\left.\frac{z'}{\rho^2\sqrt{\rho^2+z'^2}}\right|_{z'=-\infty}^{z'=\infty} = \boxed{\frac{I}{2\pi\rho}\mathbf{a}_\varphi}$$

Biot – Savart Law (4)

$$\boxed{\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\varphi}$$



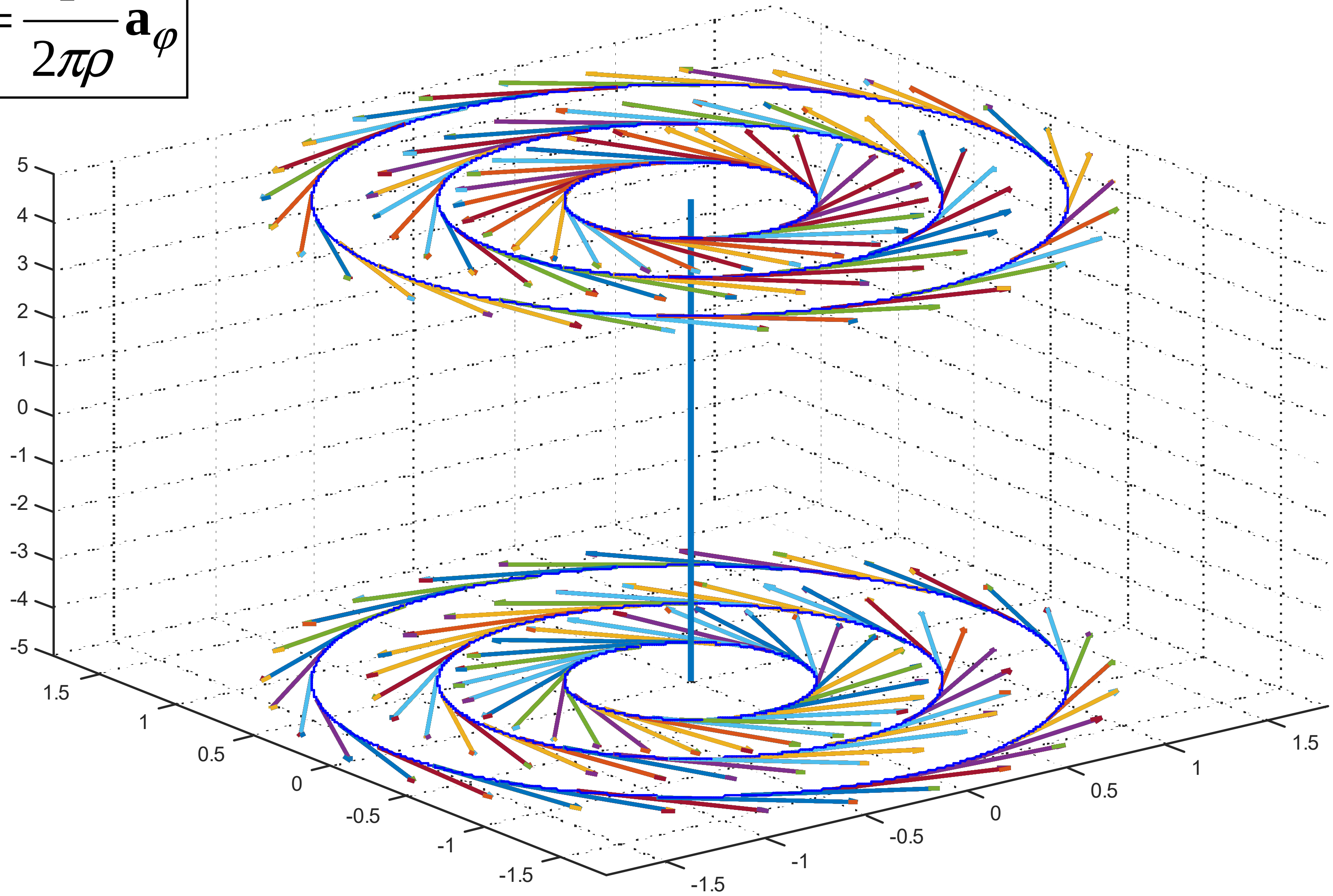




$$\mathbf{H} = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\varphi$$

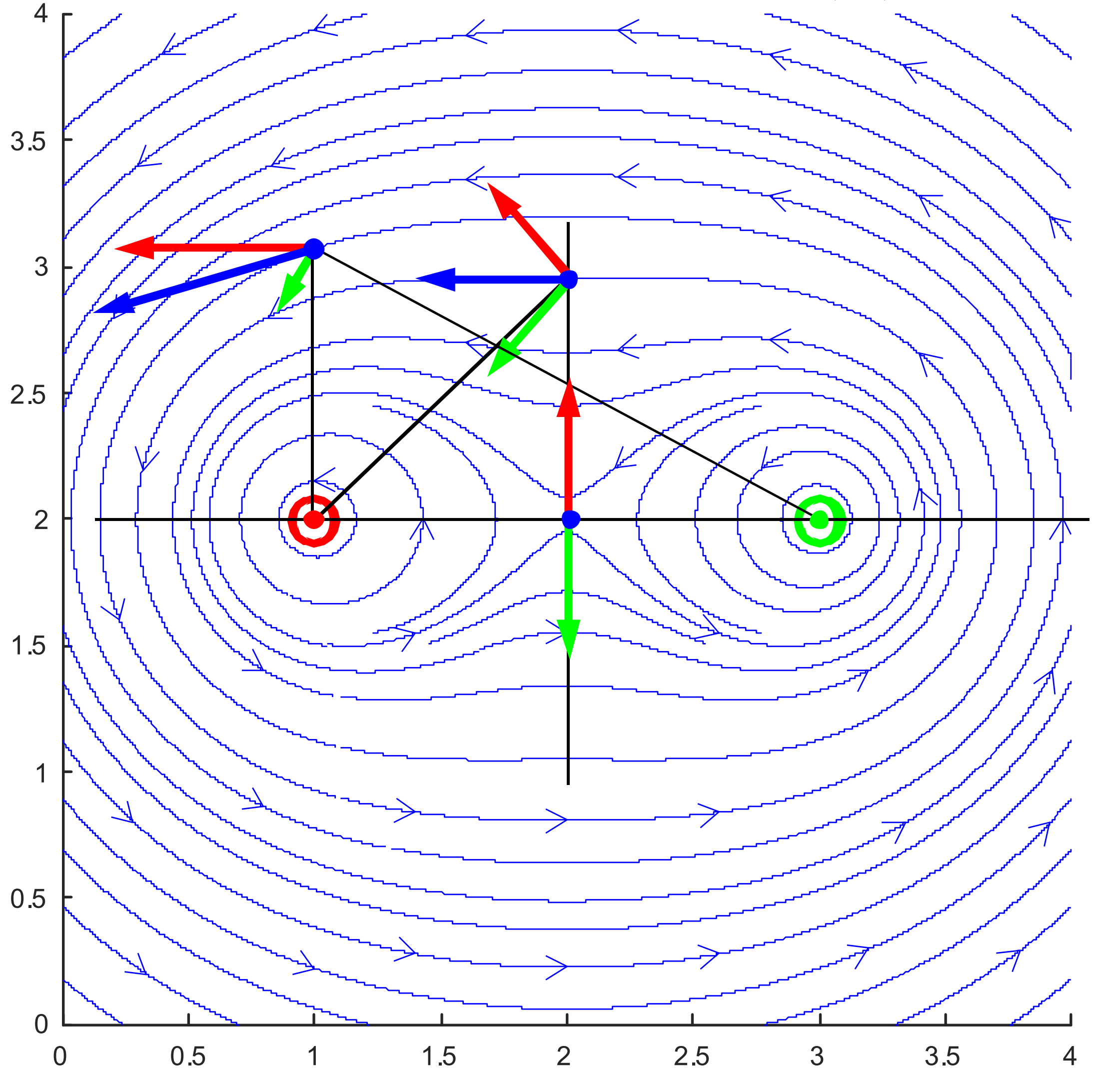
Biot – Savart Law (5)

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\varphi$$



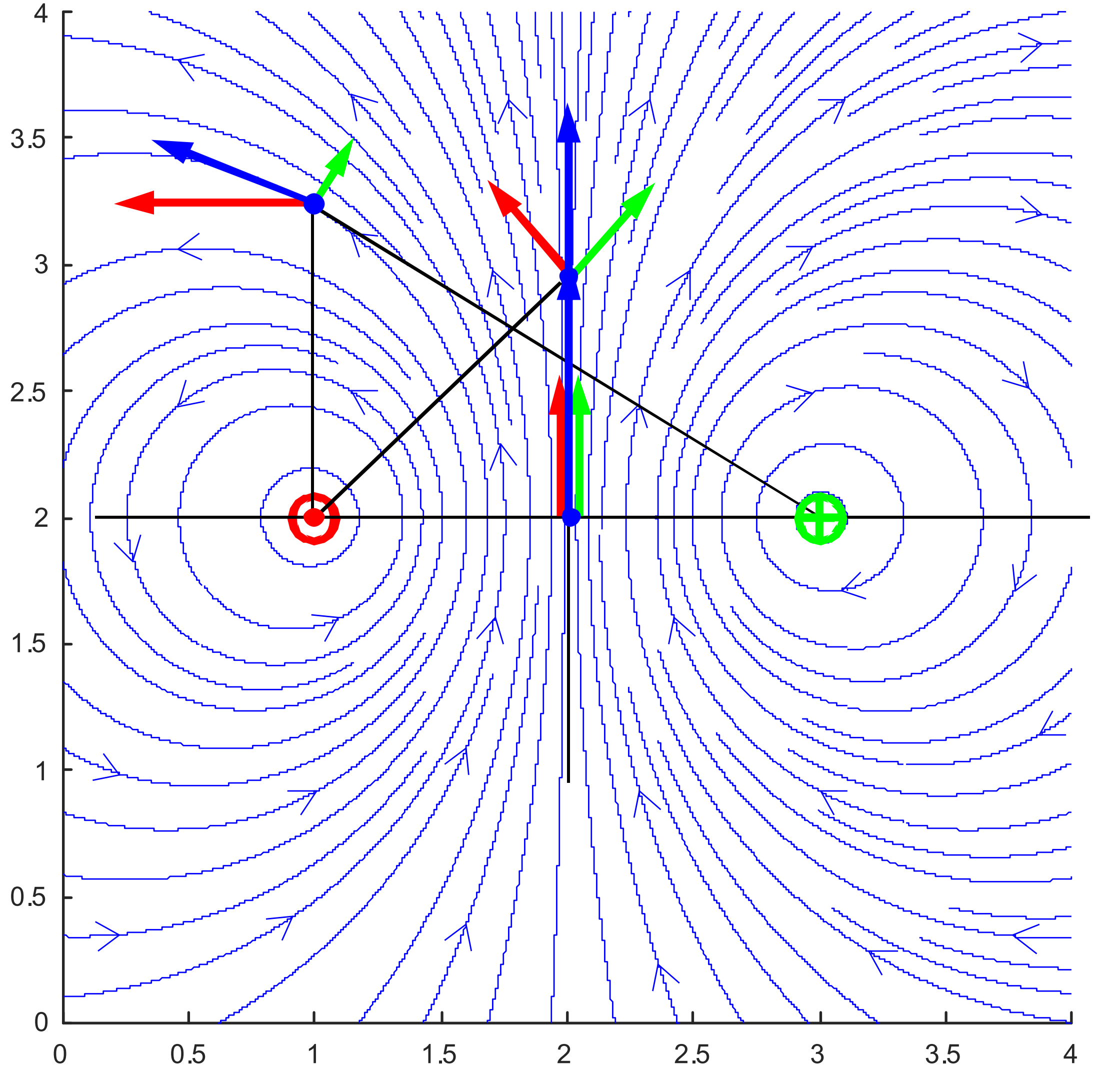
Biot – Savart Law (6)

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\varphi$$



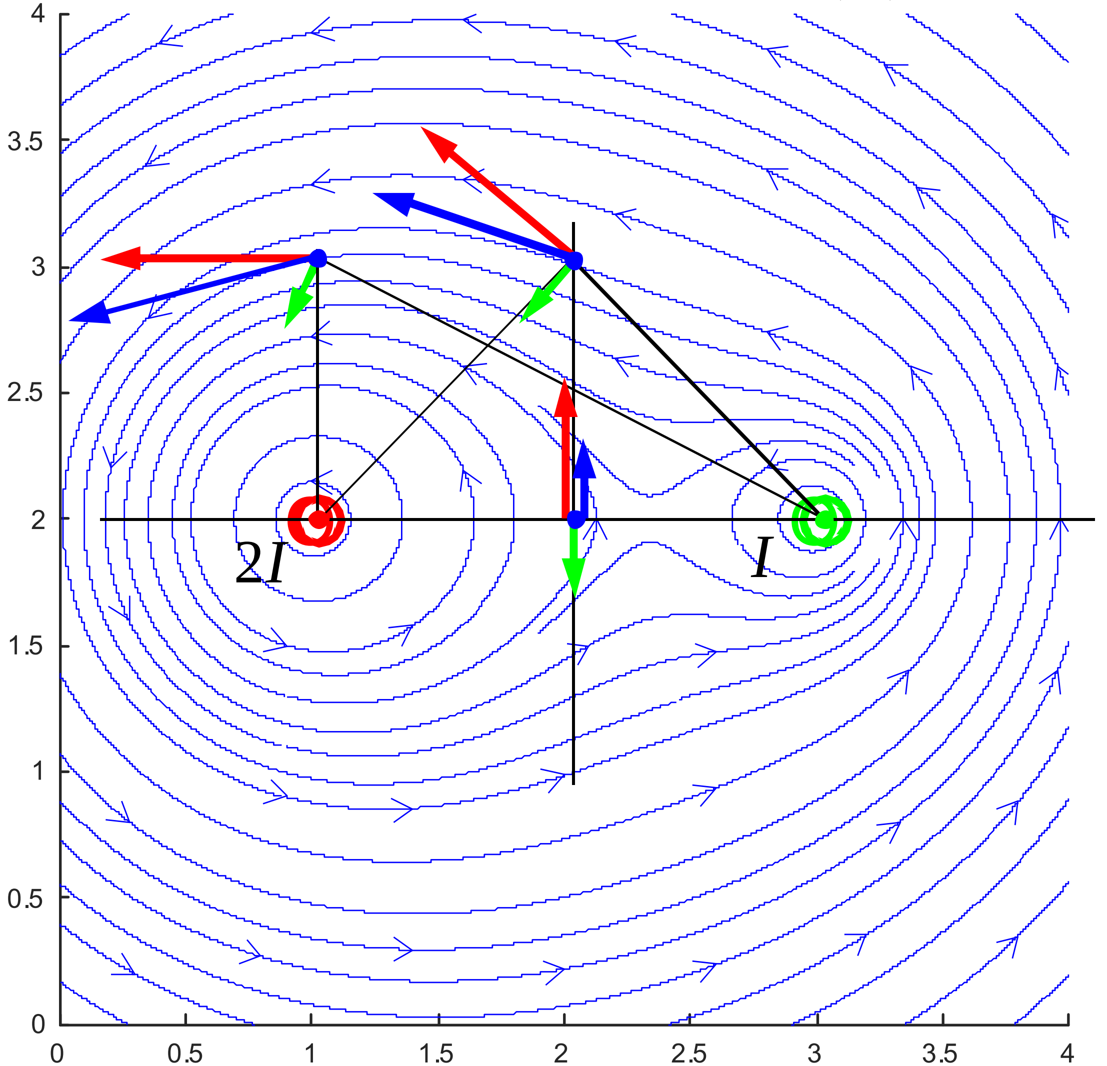
Biot – Savart Law (7)

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_{\varphi}$$



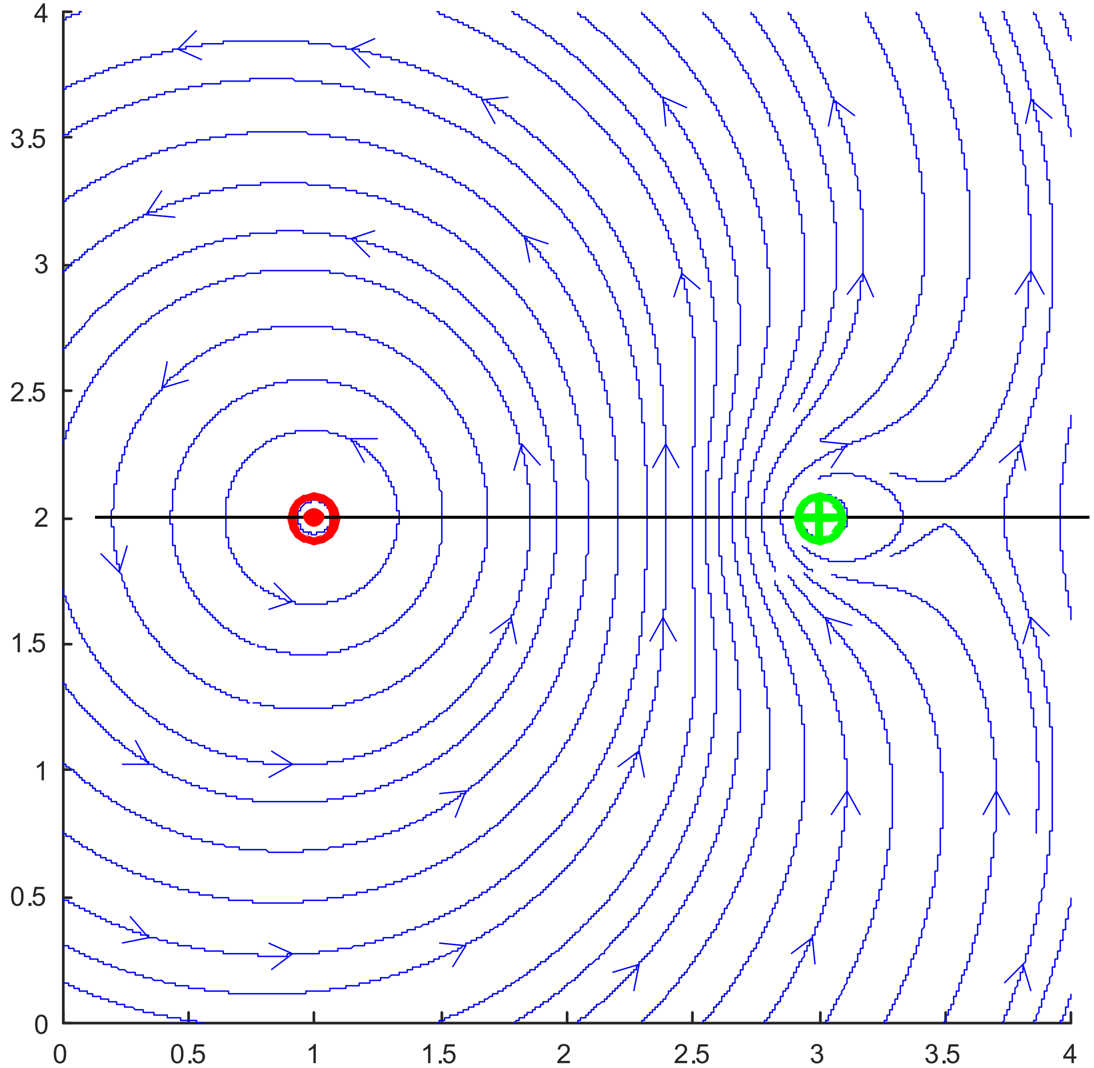
Biot – Savart Law (8)

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\varphi$$



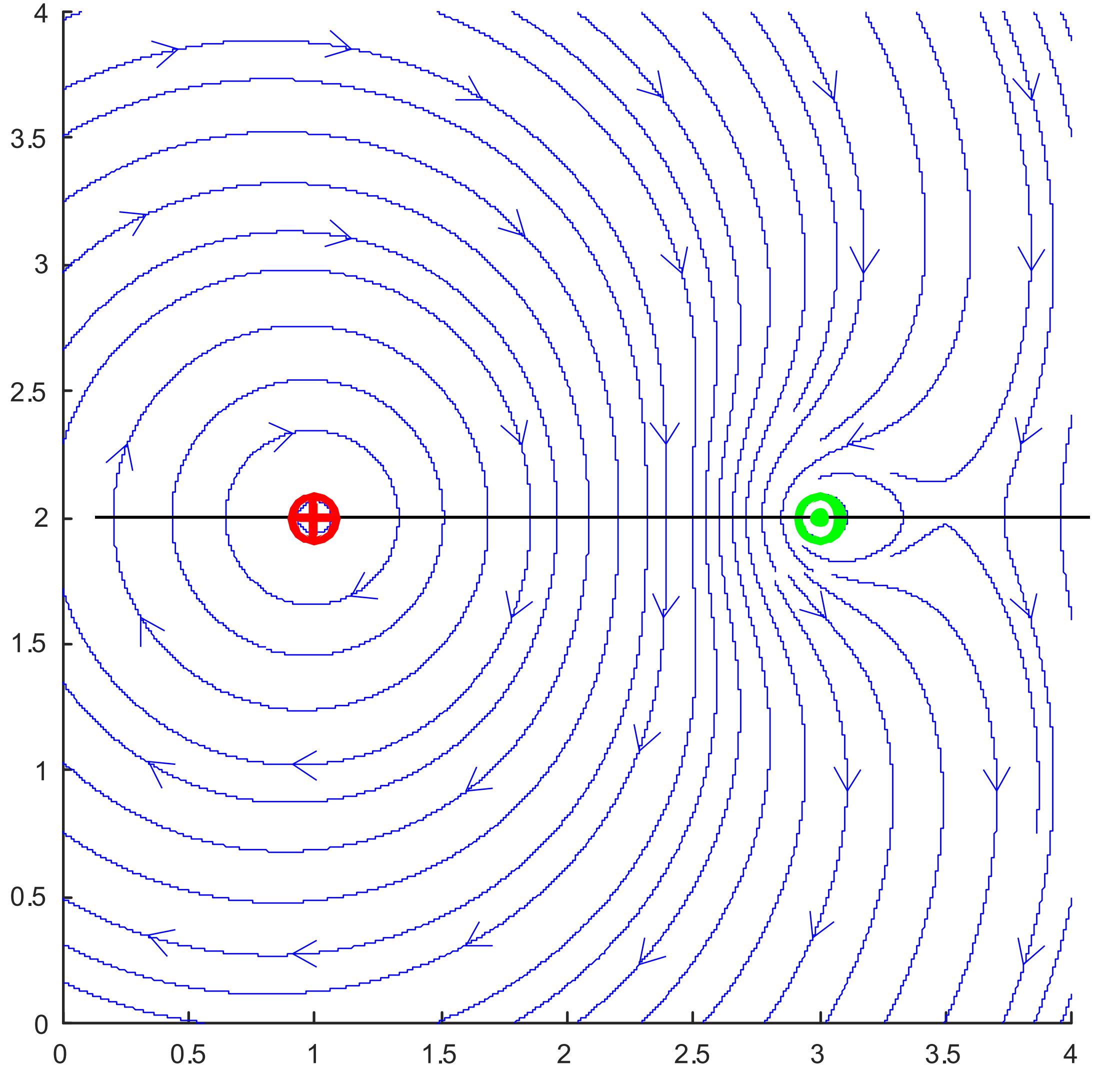
Biot – Savart Law (9)

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_{\varphi}$$

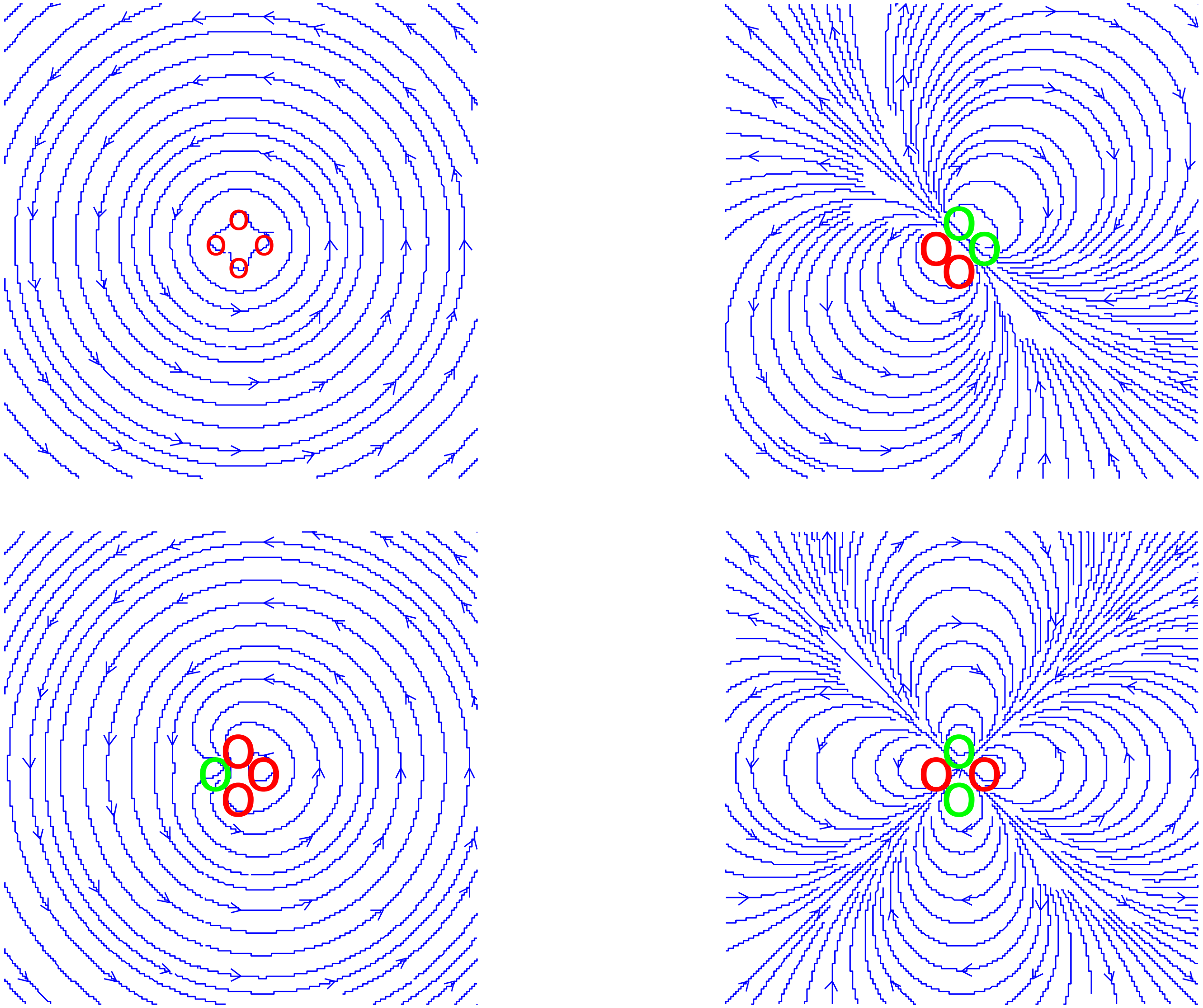


Biot – Savart Law (10)

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_{\varphi}$$



Biot – Savart Law (11)



Biot – Savart Law (12)

$$\mathbf{H} = \oint \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

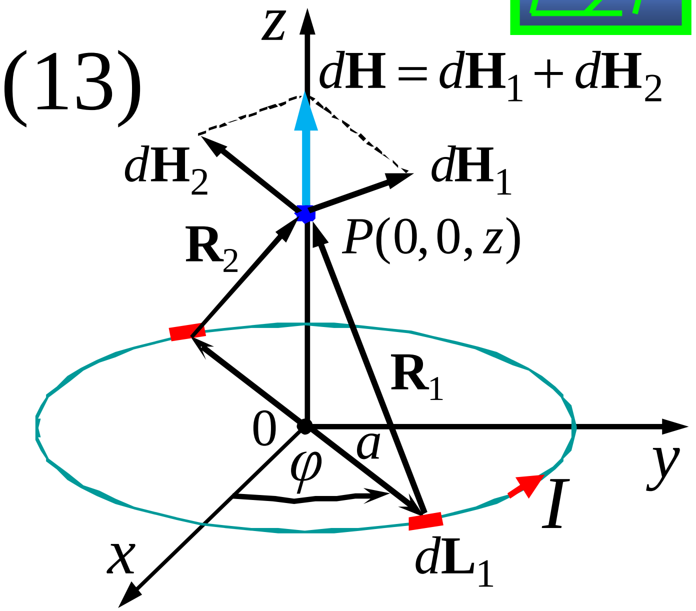
$$\mathbf{H} = \int_S \frac{\mathbf{K} \times \mathbf{a}_R dS}{4\pi R^2}$$

$$\mathbf{H} = \int_V \frac{\mathbf{J} \times \mathbf{a}_R dV}{4\pi R^2}$$

Biot – Savart Law (13)

Ex. 1

Given a circular hoop of radius a centered about the origin in the xy plane carries a constant current I. Find MFI at P?



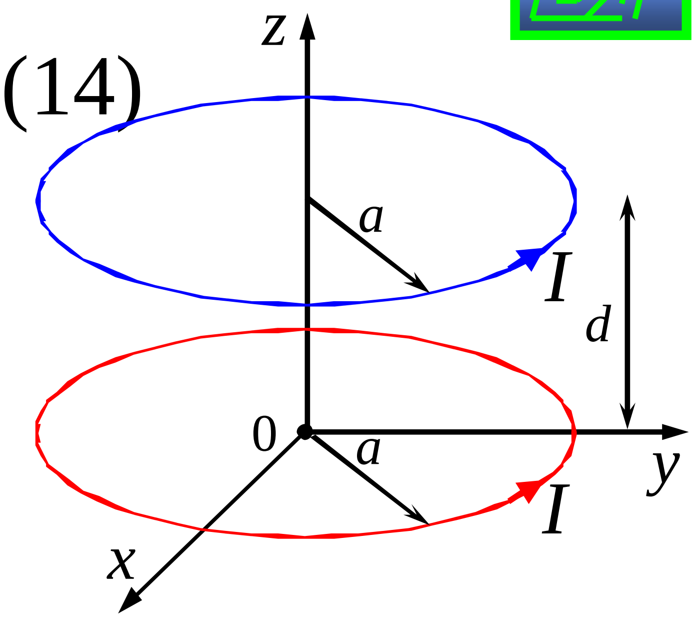
$$\left.\begin{aligned}
d\mathbf{H}_1 &= \frac{I d\mathbf{L}_1 \times \mathbf{a}_{R1}}{4\pi R_1^2} \\
d\mathbf{L}_1 &= a\, d\varphi\, \mathbf{a}_\varphi \\
\mathbf{R}_1 &= -a\mathbf{a}_\rho + z\mathbf{a}_z \\
R_1 &= \sqrt{z^2 + a^2} \\
\mathbf{a}_{R1} &= \frac{-a\mathbf{a}_\rho + z\mathbf{a}_z}{\sqrt{z^2 + a^2}}
\end{aligned}\right\} \rightarrow \left.\begin{aligned}
d\mathbf{H}_1 &= \frac{I a\, d\varphi}{4\pi (z^2 + a^2)^{3/2}} (a\mathbf{a}_z + z\mathbf{a}_\rho) \\
d\mathbf{H}_2 &= \frac{I a\, d\varphi}{4\pi (z^2 + a^2)^{3/2}} (a\mathbf{a}_z - z\mathbf{a}_\rho)
\end{aligned}\right\}$$

$$\rightarrow d\mathbf{H} = \frac{2 I a^2 d\varphi}{4\pi (z^2 + a^2)^{3/2}} \mathbf{a}_z \rightarrow \mathbf{H} = \int_0^\pi \frac{2 I a^2 d\varphi}{4\pi (z^2 + a^2)^{3/2}} \mathbf{a}_z = \frac{I a^2}{2 (z^2 + a^2)^{3/2}} \mathbf{a}_z$$

Biot – Savart Law (14)

Ex. 2

Find MFI on the z axis?



$$\left.\begin{aligned} H_{z,\,red} &= \frac{Ia^2}{2(z^2+a^2)^{3/2}} \\ H_{z,\,blue} &= \frac{Ia^2}{2[(z-d)^2+a^2]^{3/2}} \end{aligned}\right\}$$

$$\rightarrow H_z = \frac{Ia^2}{2}\left(\frac{1}{(z^2+a^2)^{3/2}} + \frac{1}{[(z-d)^2+a^2]^{3/2}}\right)$$

$$\left.\frac{\partial H_z}{\partial z}\right|_{z=d/2} = 0 \qquad \left.\frac{\partial^2 H_z}{\partial z^2}\right|_{z=d/2,d=a} = 0$$

Biot – Savart Law (15)

Ex. 3

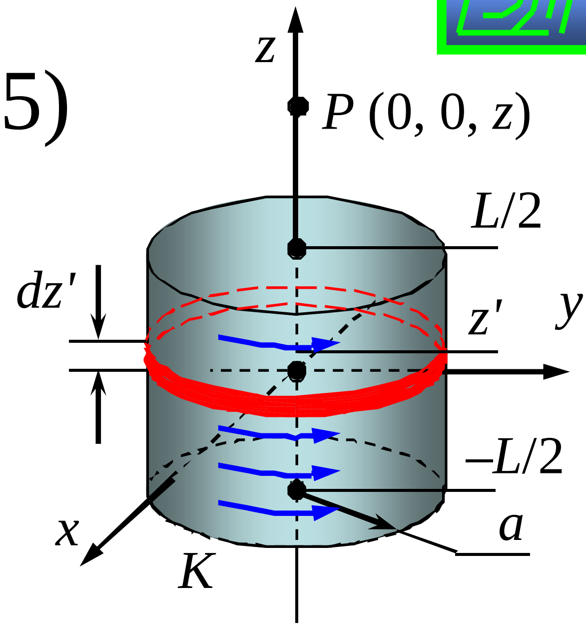
Find MFI on the z axis?

$$\left.\begin{array}{l}\mathbf{H} = \dfrac{Ia^2}{2(z^2+a^2)^{3/2}}\mathbf{a}_z \\ z = z - z', I = Kdz'\end{array}\right\}$$

$$\rightarrow dH_z = \frac{a^2 Kdz'}{2[(z-z')^2+a^2]^{3/2}}$$

$$\rightarrow H_z = \int_{z'=-L/2}^{L/2} \frac{a^2 Kdz'}{2[(z-z')^2+a^2]^{3/2}}$$

$$= \frac{K}{2}\left(\frac{-z+L/2}{\sqrt{(z-L/2)^2+a^2}} + \frac{z+L/2}{\sqrt{(z+L/2)^2+a^2}}\right)$$



$$\lim_{L\to\infty} H_z = K$$

The Steady Magnetic Field

1. Biot – Savart Law
2. **Ampere's Circuital Law**
3. Curl
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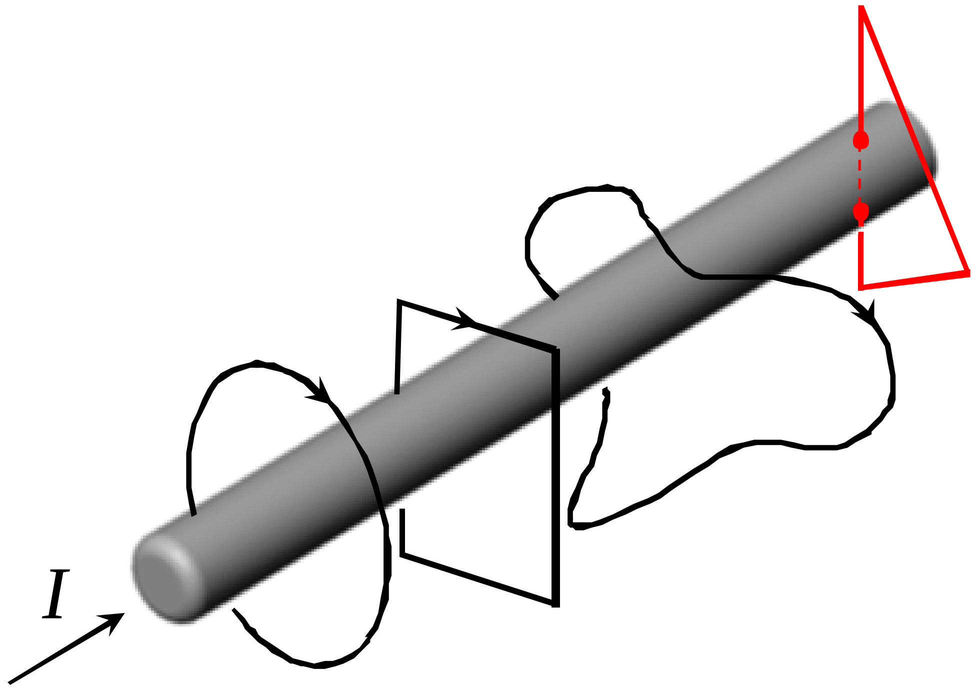
6. Magnetic Potential
7. Derivation of the Steady – Magnetic – Field Law

Ampere's Circuital Law (1)

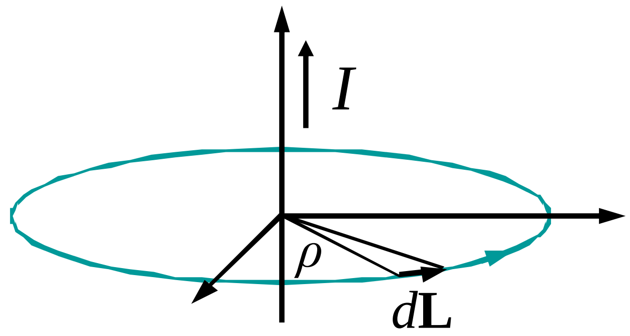
$$\oint \mathbf{H}.d\mathbf{L} = I$$

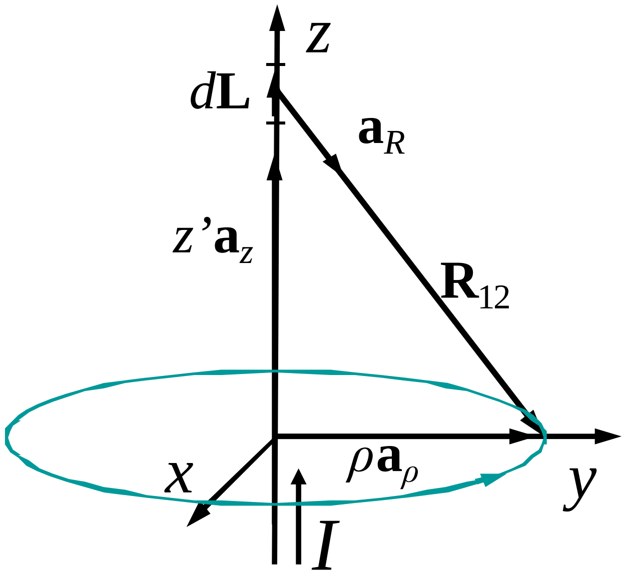


Ampere's Circuital Law (2)

Ex. 1

$$\boxed{\oint \mathbf{H}.d\mathbf{L} = I}$$





$$\left.\begin{aligned}\mathbf{H} &= H_\varphi \mathbf{a}_\varphi \\ d\mathbf{L} &= \rho \tan(d\varphi)\mathbf{a}_\varphi \approx \rho d\varphi \mathbf{a}_\varphi\end{aligned}\right\}$$

$$\rightarrow \oint \mathbf{H}.d\mathbf{L} = \int_0^{2\pi} H_\varphi \rho d\varphi$$

$$= H_\varphi \rho \int_0^{2\pi} d\varphi$$

$$= H_\varphi 2\pi\rho = I \;\rightarrow H_\varphi = \frac{I}{2\pi\rho}$$

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\varphi$$

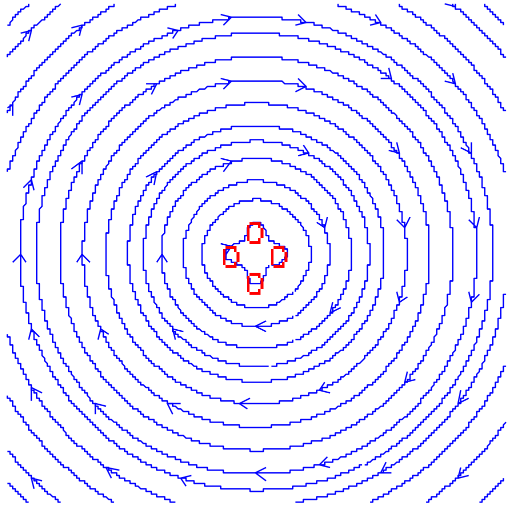
Ampere's Circuital Law (3)

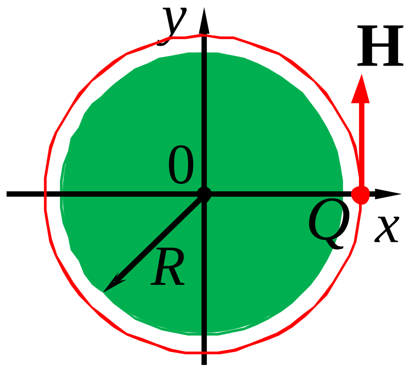
Ex. 2

$$\oint \mathbf{H}.d\mathbf{L} = I$$

$$\rightarrow H(2\pi\rho) = J(\pi R^2)$$

$$\rightarrow \mathbf{H} = \frac{JR^2}{2\rho}\mathbf{a}_\varphi, \quad \rho > R$$





Ampere's Circuital Law (4)

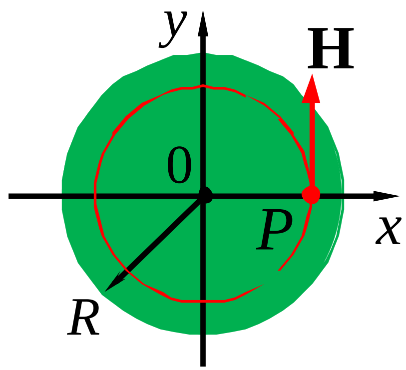
Ex. 2

$$\mathbf{H} = \frac{JR^2}{2\rho}\mathbf{a}_\varphi, \quad \rho > R$$

$$\oint \mathbf{H}.d\mathbf{L} = I$$

$$\rightarrow H(2\pi\rho) = J(\pi\rho^2)$$

$$\rightarrow \mathbf{H} = \frac{J\rho}{2}\mathbf{a}_\varphi, \quad \rho < R$$



Ampere's Circuital Law (5)

Ex. 3

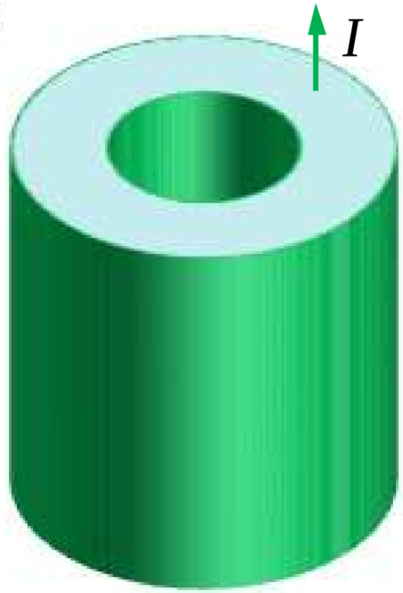
$$\oint \mathbf{H}_r.d\mathbf{L} = I_r = 0 \qquad \rightarrow H_r = 0$$

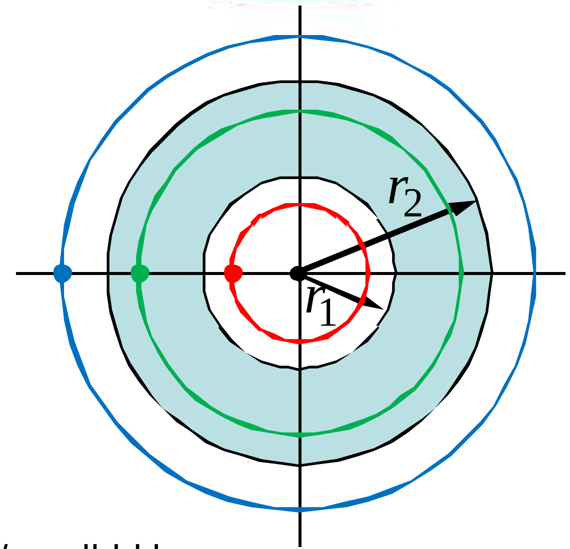
$$\oint \mathbf{H}_b.d\mathbf{L} = I_b = I$$

$$\rightarrow H_b(2\pi r_b) = I \quad \rightarrow H_b = \frac{I}{2\pi r_b} \quad \rightarrow \mathbf{H}_b = \frac{I}{2\pi r_b}\mathbf{a}_\varphi$$

$$\left.\begin{array}{l} \oint \mathbf{H}_g.d\mathbf{L} = I_g = JS_g \\ \left.\begin{array}{l} J = \dfrac{I}{\pi r_2^2 - \pi r_1^2} \\ S_g = \pi r_g^2 - \pi r_1^2 \end{array}\right\} \rightarrow I_g = I\dfrac{r_g^2 - r_1^2}{r_2^2 - r_1^2} \end{array}\right\}$$

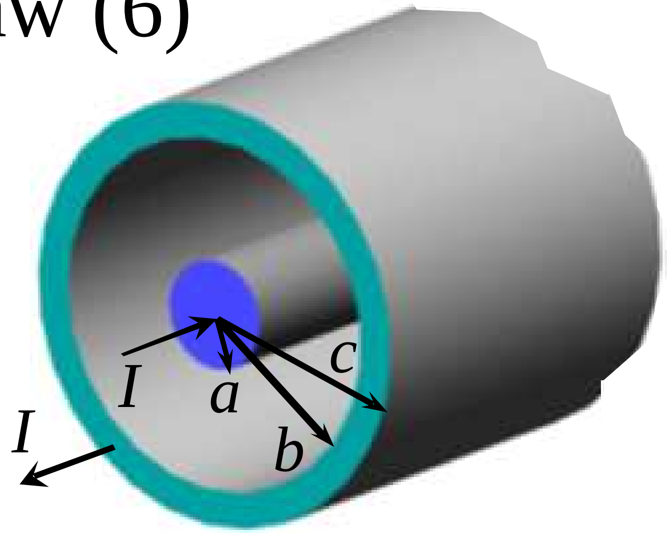
$$\rightarrow H_g(2\pi r_g) = I\frac{r_g^2 - r_1^2}{r_2^2 - r_1^2} \quad \rightarrow \mathbf{H}_g = \frac{I}{2\pi r_g}.\frac{r_g^2 - r_1^2}{r_2^2 - r_1^2}\mathbf{a}_\varphi$$





Ampere's Circuital Law (6)

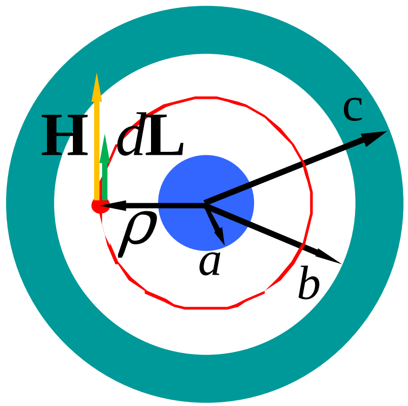
Ex. 4



$$\oint \mathbf{H}.d\mathbf{L} = I$$

$$\rightarrow H(2\pi\rho) = I$$

$$\rightarrow H(\rho) = \frac{I}{2\pi\rho}, \quad a < \rho < b$$



Ampere's Circuital Law (7)

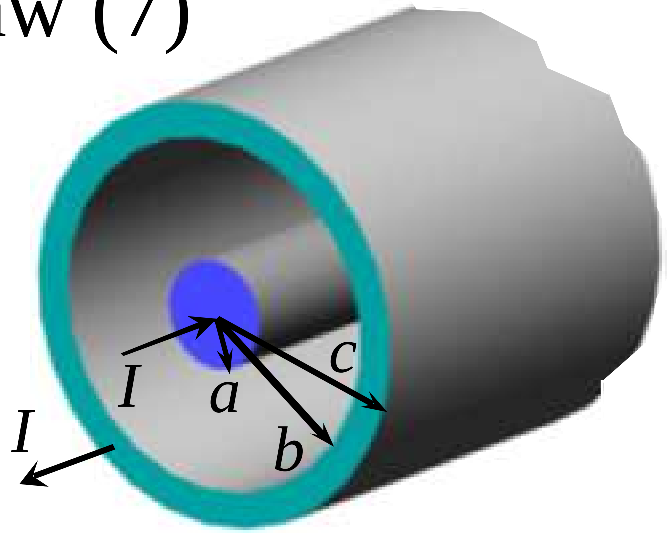
Ex. 4

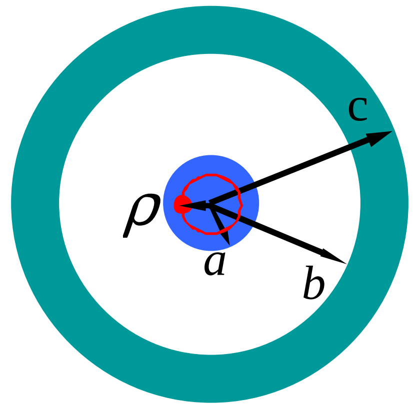
$$\oint \mathbf{H}.d\mathbf{L} = I'$$

$$\left.\begin{aligned} &\rightarrow H(2\pi\rho) = JS' \\ &J = \frac{I}{\pi a^2} \\ &S' = \pi\rho^2 \end{aligned}\right\}$$

$$\rightarrow H(2\pi\rho) = \frac{I}{\pi a^2}\pi\rho^2$$

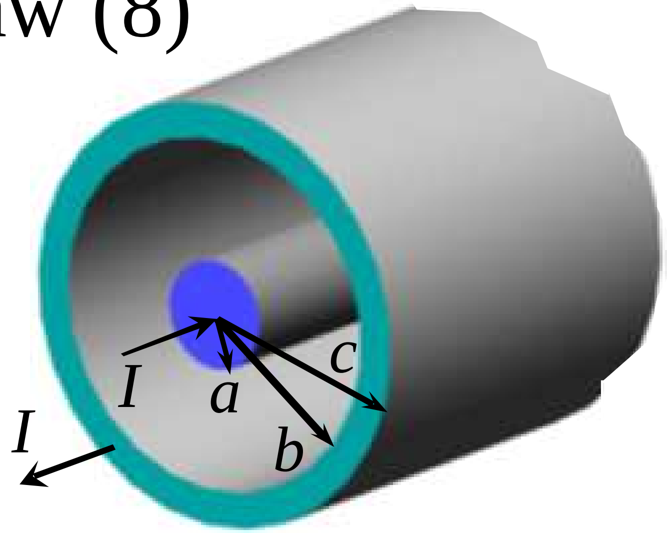
$$\rightarrow H = \frac{I}{2\pi a^2}\rho, \quad \rho < a$$





Ampere's Circuital Law (8)

Ex. 4

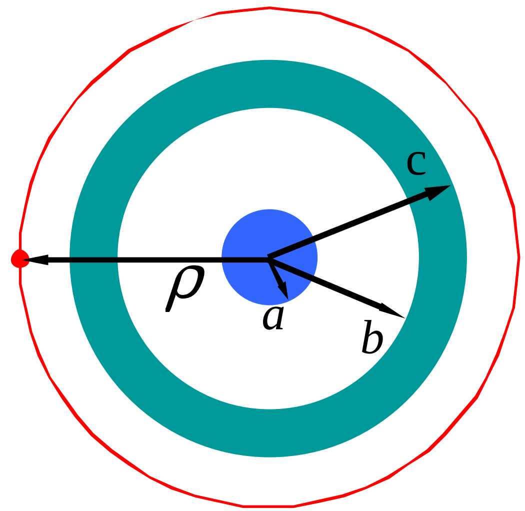


$$\oint \mathbf{H}.d\mathbf{L} = \sum I$$

$$\sum I = I_{\text{inner conductor}} + I_{\text{outer conductor}} = I - I = 0$$

$$\to H(2\pi\rho) = 0$$

$$\to H(\rho) = 0, \quad \rho > c$$



Ampere's Circuital Law (9)

Ex. 4

$$\oint \mathbf{H}.d\mathbf{L} = \sum I$$

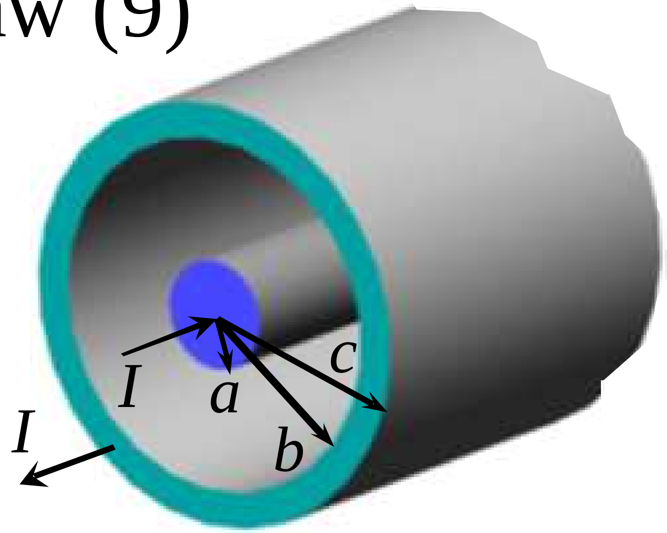
$$\sum I = I_{\text{inner conductor}} + I_{\text{partial outer conductor}}$$

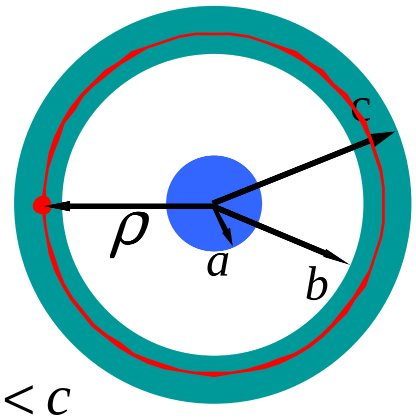
$$I_{\text{partial outer conductor}} = JS''$$

$$J = \frac{I}{\pi c^2 - \pi b^2}$$

$$S'' = \pi\rho^2 - \pi b^2$$

$$\rightarrow I_{\text{partial outer conductor}} = \frac{I\pi(\rho^2 - b^2)}{\pi(c^2 - b^2)}$$





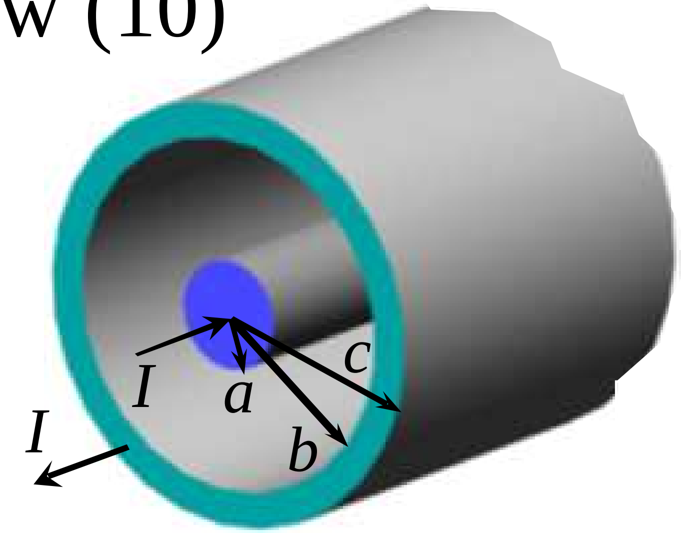
$$\rightarrow H(2\pi\rho) = I\frac{c^2 - \rho^2}{c^2 - b^2} \rightarrow H(\rho) = \frac{I}{2\pi\rho}.\frac{c^2 - \rho^2}{c^2 - b^2}, \quad b < \rho < c$$

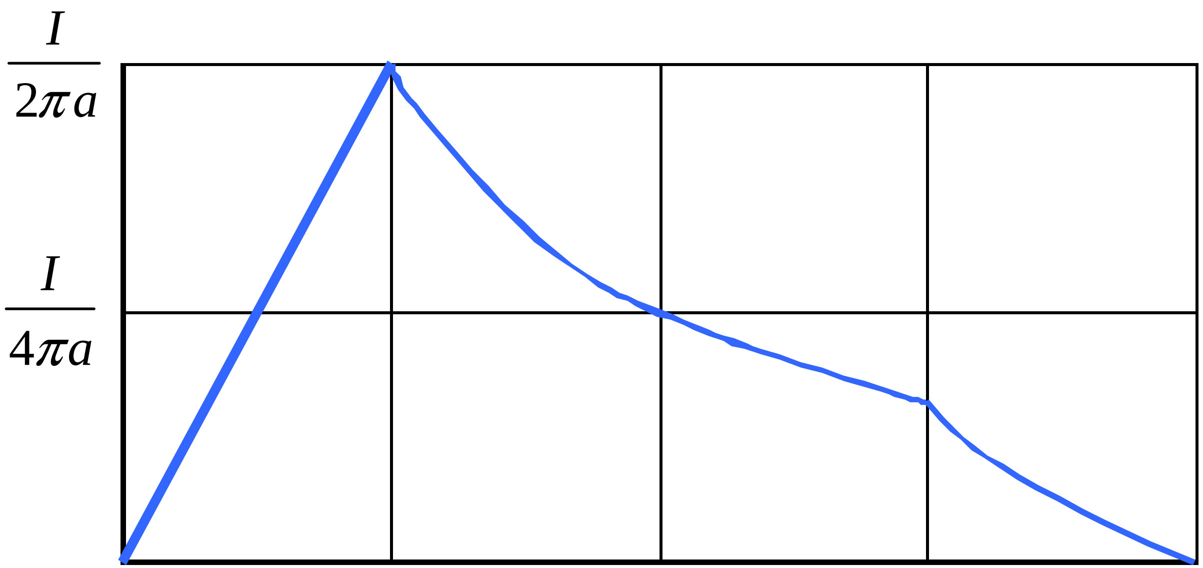
Ampere's Circuital Law (10)

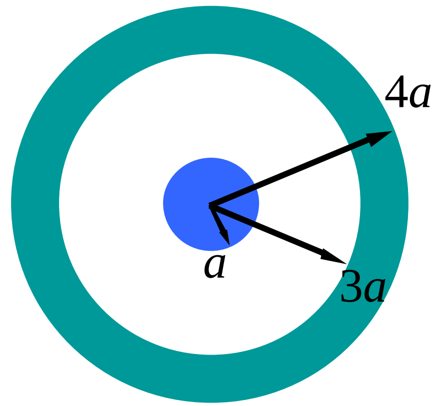
Ex. 4

$$H_\varphi = I\frac{\rho}{2\pi a^2}\ (\rho < a);\ H_\varphi = \frac{I}{2\pi\rho}\ (a < \rho < b)$$

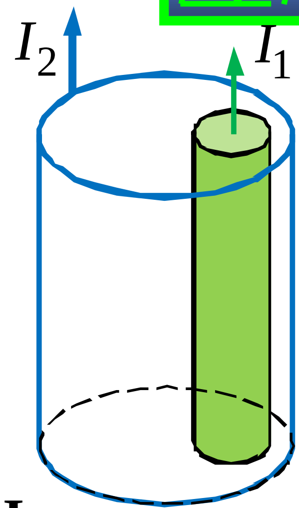
$$H_\varphi = \frac{I}{2\pi\rho}\frac{c^2-\rho^2}{c^2-b^2}\ (b < \rho < c);\ H_\varphi = 0\ (\rho > c)$$







Ampere's Circuital Law (11)



Ex. 5

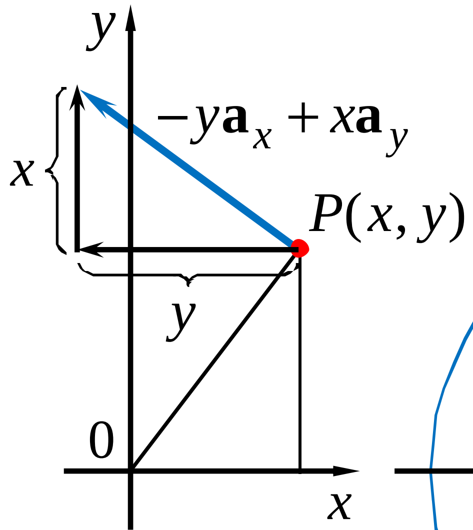
$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$$

$$\oint \mathbf{H}_2 . d\mathbf{L} = I_2$$

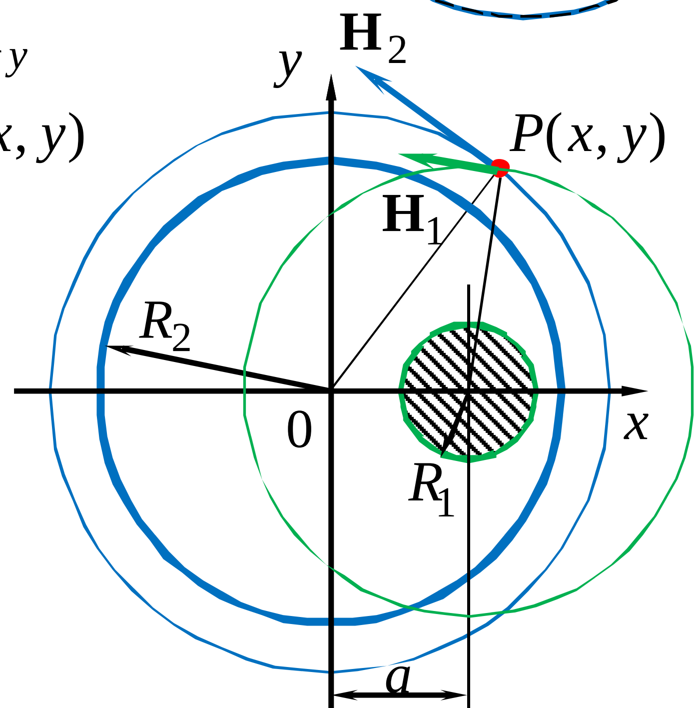
$$\rightarrow H_2 (2\pi\sqrt{x^2 + y^2}) = I_2$$

$$\rightarrow H_2 = \frac{I_2}{2\pi\sqrt{x^2 + y^2}}$$

$$\rightarrow \mathbf{H}_2 = \frac{I_2}{2\pi\sqrt{x^2 + y^2}} . \frac{-y\mathbf{a}_x + x\mathbf{a}_y}{\sqrt{x^2 + y^2}}$$

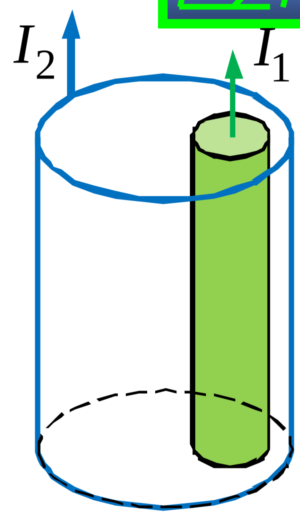


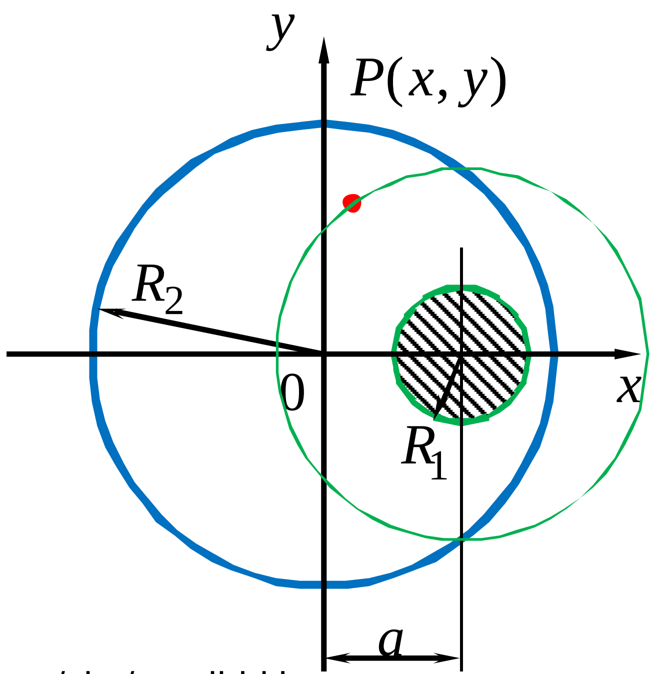
$$\mathbf{H}_1 = \frac{I_1}{2\pi[(x-a)^2 + y^2]}[-y\mathbf{a}_x + (x-a)\mathbf{a}_y]$$



Ampere's Circuital Law (12)

Ex. 5





Ampere's Circuital Law (13)

Ex. 6

$$\oint \mathbf{H}_{Pg}.d\mathbf{L} = I_{Pb}$$

$$\rightarrow H_{Pg}(2\pi x_P) = J(\pi x_P^2)$$

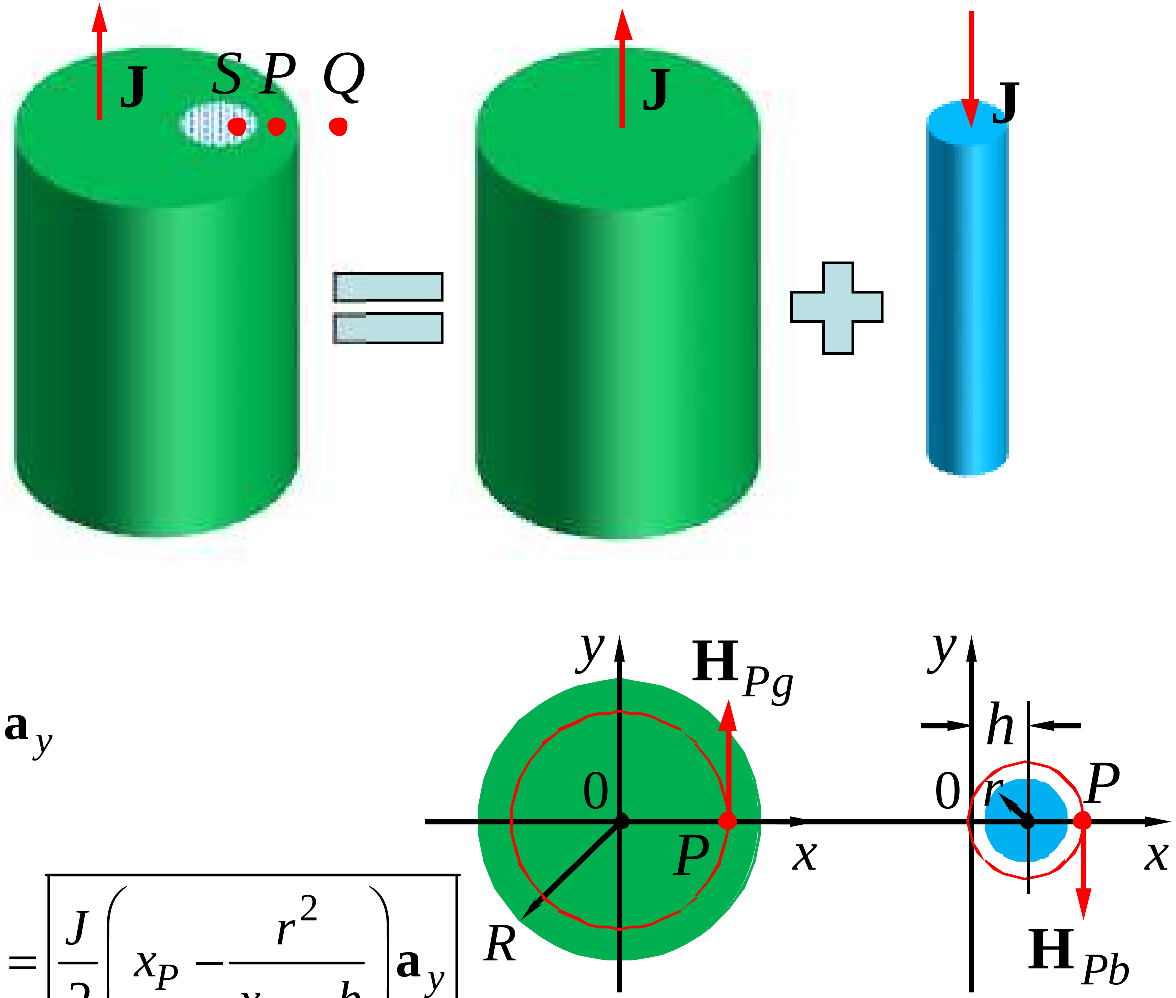
$$\rightarrow \mathbf{H}_{Pg} = \frac{Jx_P}{2}\mathbf{a}_y$$

$$\oint \mathbf{H}_{Pb}.d\mathbf{L} = I_{Pb}$$

$$\rightarrow H_{Pb}[2\pi(x_P - h)] = J(\pi r^2)$$

$$\rightarrow \mathbf{H}_{Pb} = -\frac{Jr^2}{2(x_P - h)}\mathbf{a}_y$$

$$\rightarrow \mathbf{H}_P = \mathbf{H}_{Pg} + \mathbf{H}_{Pb} = \boxed{\frac{J}{2}\left(x_P - \frac{r^2}{x_P - h}\right)\mathbf{a}_y}$$



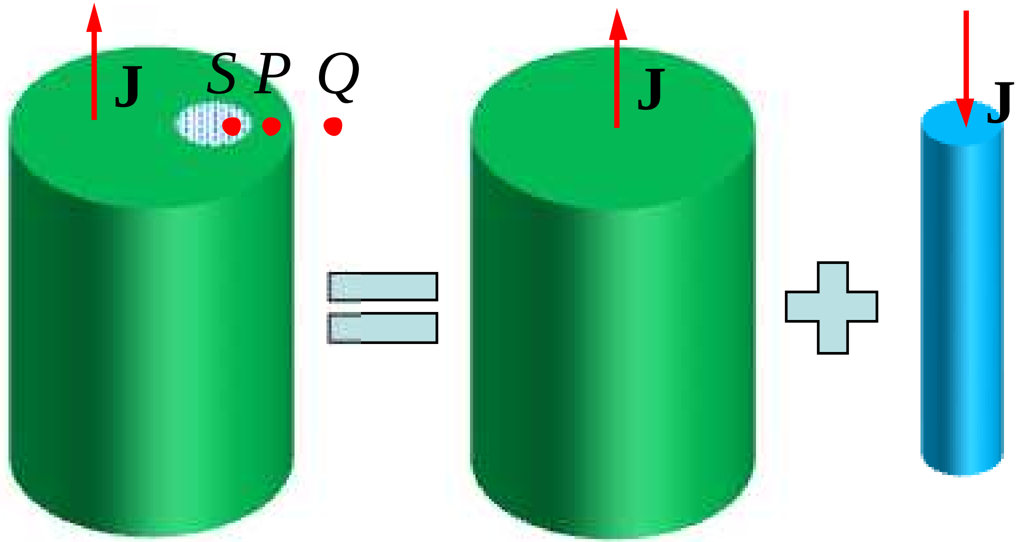
Ampere's Circuital Law (14)

Ex. 6

$$\oint \mathbf{H}_{Qg}.d\mathbf{L} = I_{Qb}$$

$$\rightarrow H_{Qg}(2\pi x_Q) = J(\pi R^2)$$

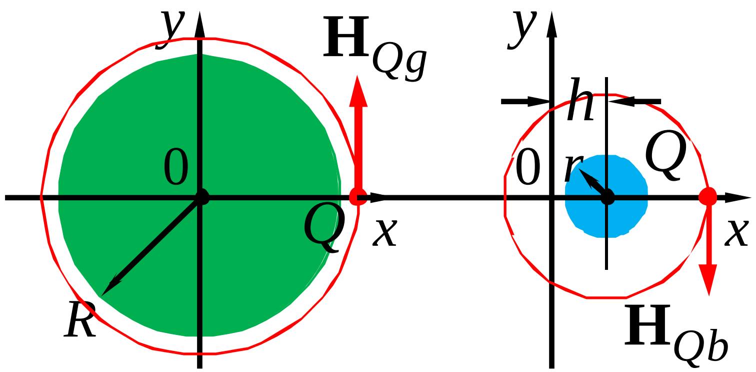
$$\rightarrow \mathbf{H}_{Qg} = \frac{JR^2}{2x_Q}\mathbf{a}_y$$



$$\oint \mathbf{H}_{Qb}.d\mathbf{L} = I_{Qb}$$

$$\rightarrow H_{Qb}[2\pi(x_Q - h)] = J(\pi r^2)$$

$$\rightarrow \mathbf{H}_{Qb} = -\frac{Jr^2}{2(x_Q - h)}\mathbf{a}_y$$



$$\rightarrow \mathbf{H}_P = \mathbf{H}_{Pg} + \mathbf{H}_{Pb} = \boxed{\frac{J}{2}\left(\frac{R^2}{x_Q} - \frac{r^2}{x_Q - h}\right)\mathbf{a}_y}$$

Ampere's Circuital Law (15)

Ex. 7

$$\oint_{abcd} \mathbf{H}.d\mathbf{L} = I = K_y L$$

$$\rightarrow \int_a^b \mathbf{H}.d\mathbf{L} + \int_b^c \mathbf{H}.d\mathbf{L} + \int_c^d \mathbf{H}.d\mathbf{L} + \int_d^a \mathbf{H}.d\mathbf{L} = KL$$

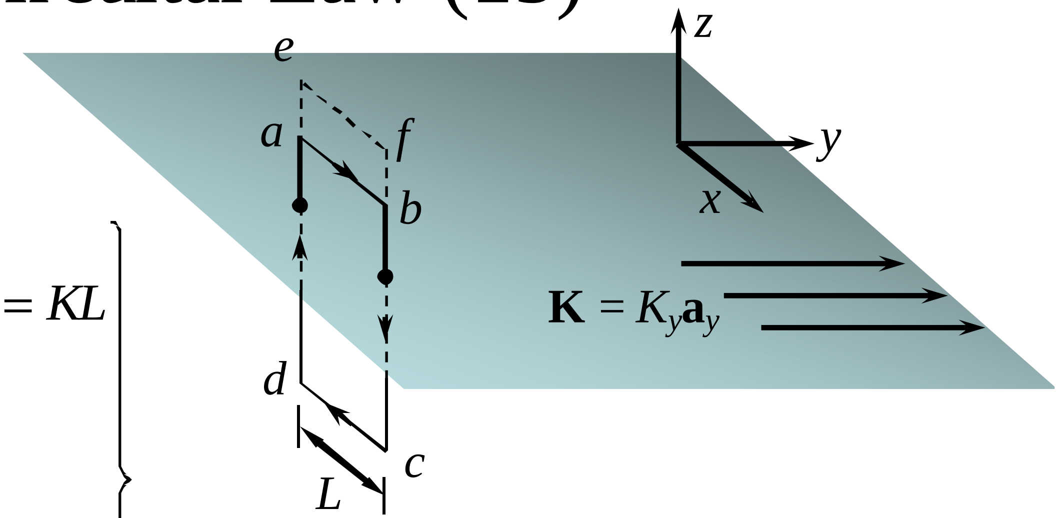
$$\int_b^c \mathbf{H}.d\mathbf{L} = 0, \quad \int_d^a \mathbf{H}.d\mathbf{L} = 0$$

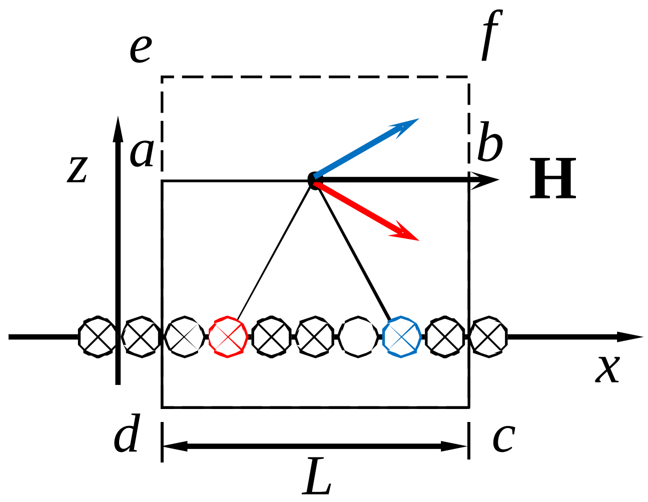
$$\int_a^b \mathbf{H}.d\mathbf{L} = H_{ab} L, \quad \int_c^d \mathbf{H}.d\mathbf{L} = -H_{cd} L$$

$$\rightarrow H_{ab} L - H_{cd} L = K_y L$$

$$\rightarrow H_{ab} - H_{cd} = K_y$$

$$\oint_{efcd} \mathbf{H}.d\mathbf{L} = I = K_y L \qquad \rightarrow H_{ef} - H_{cd} = K_y$$





Ampere's Circuital Law (16)

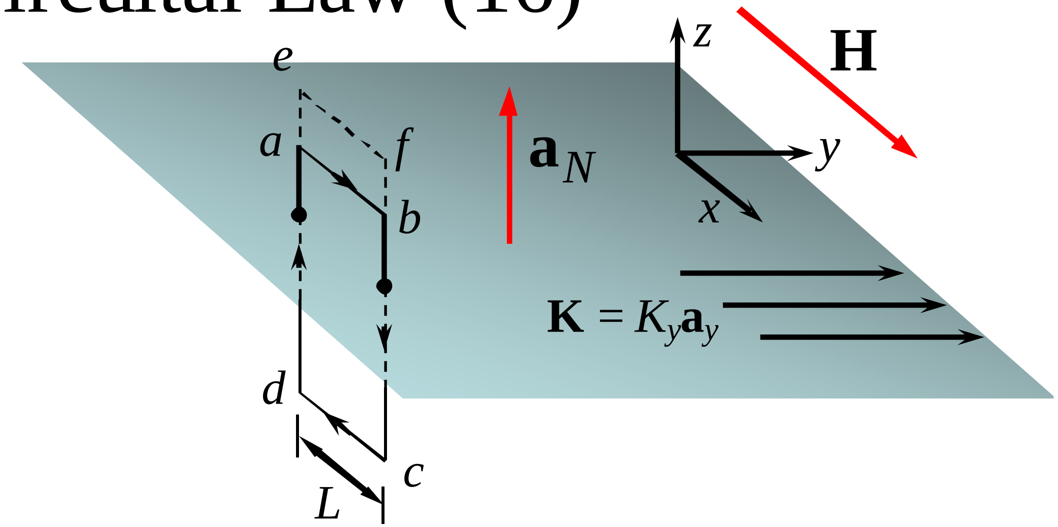
Ex. 7

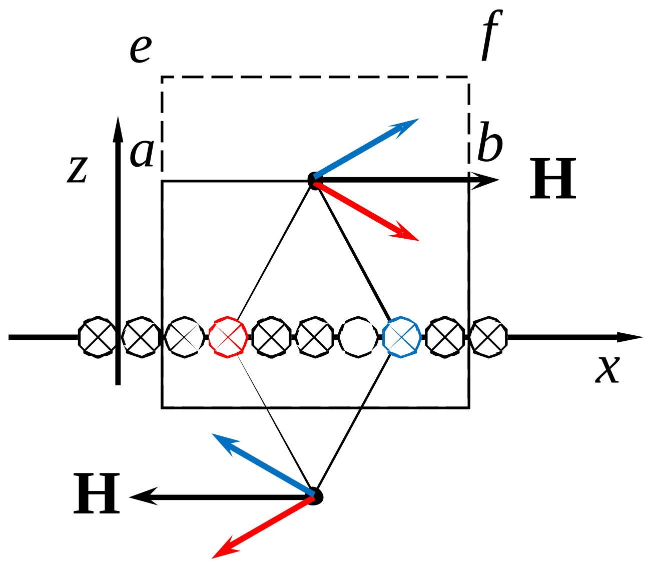
$$\left.\begin{array}{l} H_{ab} - H_{cd} = K_y \\ H_{ef} - H_{cd} = K_y \end{array}\right\}$$

$$\rightarrow H_{ab} = H_{ef}$$

$$\rightarrow \begin{cases} H_x = \dfrac{1}{2}K_y & (z > 0) \\ H_x = -\dfrac{1}{2}K_y & (z < 0) \end{cases}$$

$$\rightarrow \mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_N$$

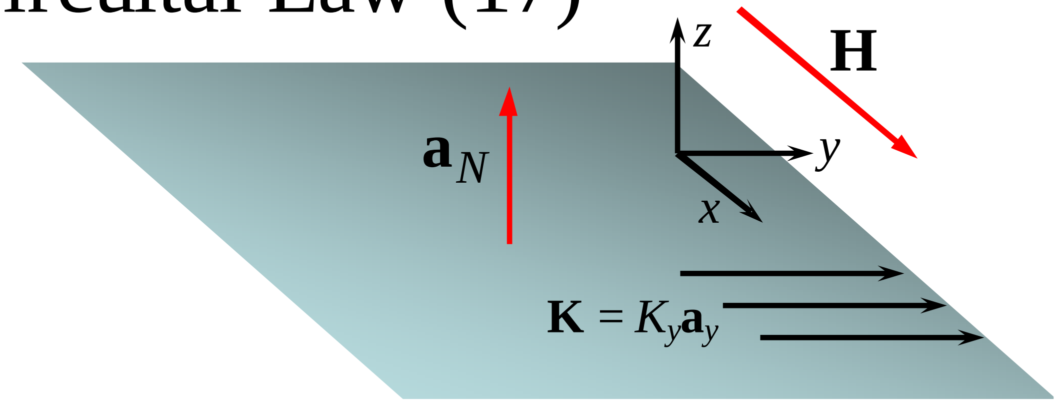




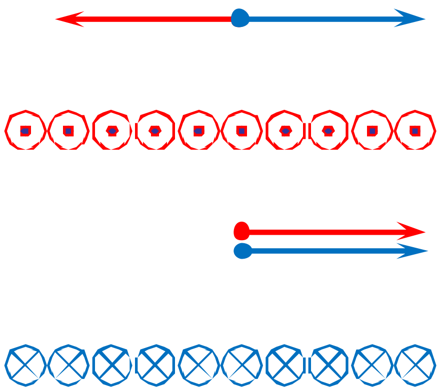
Ampere's Circuital Law (17)

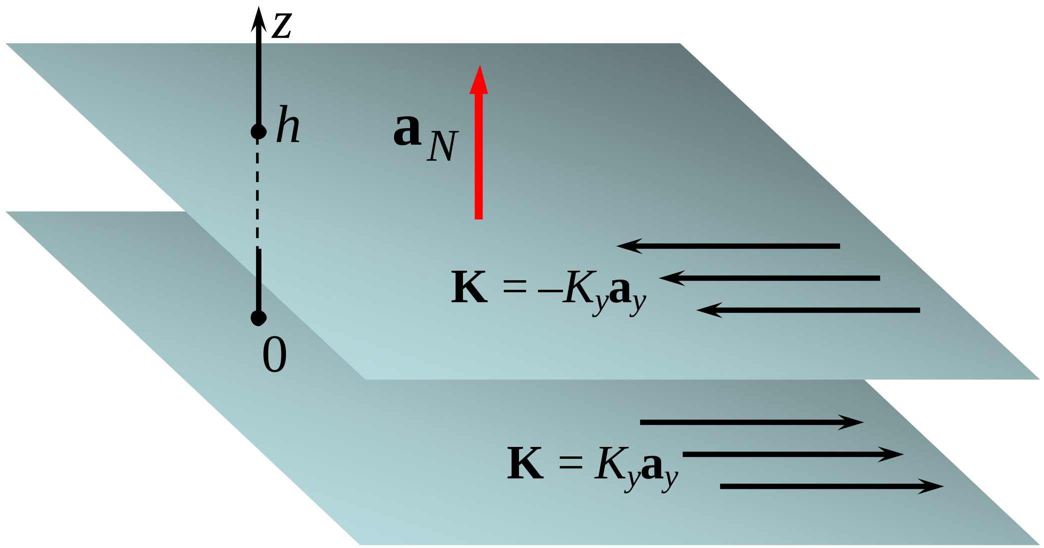
Ex. 8

$$\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_N$$



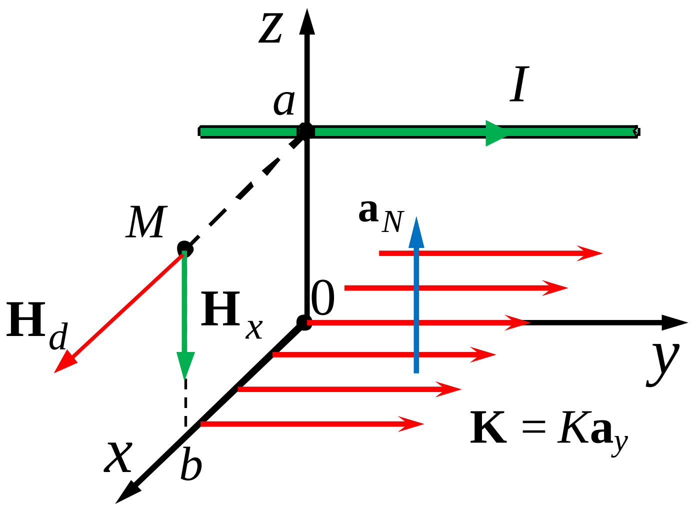
$$\begin{cases} \mathbf{H} = \mathbf{K} \times \mathbf{a}_N & (0 < z < h) \\ \mathbf{H} = 0 & (z < 0,\ z > h) \end{cases}$$





Ampere's Circuital Law (18)

Ex. 9



$$\mathbf{H}_x = \frac{I}{2\pi\rho}\mathbf{a}_\rho = \frac{I}{2\pi b}(-\mathbf{a}_z)$$

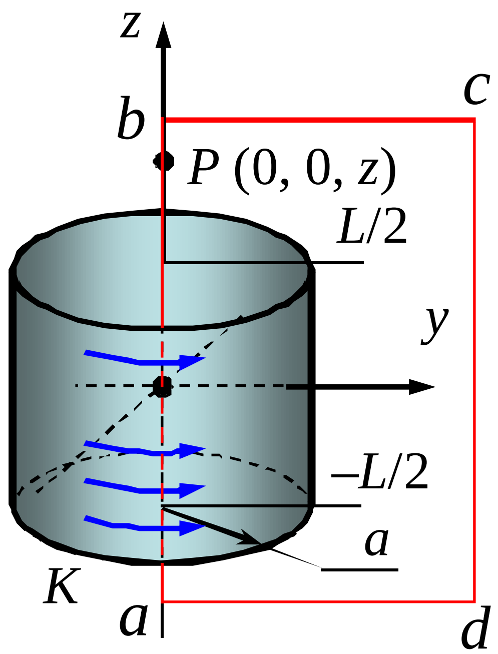
$$\mathbf{H}_d = \frac{1}{2}\mathbf{K}\times\mathbf{a}_N = \frac{1}{2}(K\mathbf{a}_y)\times\mathbf{a}_z = \frac{K}{2}\mathbf{a}_x$$

$$\mathbf{H}_M = \mathbf{H}_d + \mathbf{H}_x = \frac{K}{2}\mathbf{a}_x - \frac{I}{2\pi b}\mathbf{a}_z$$

Ampere's Circuital Law (19)

Ex. 10

A hollow cylinder with a surface current density of K. Find MFI on the z-axiz?



$$\oint \mathbf{H}.d\mathbf{L} = I$$

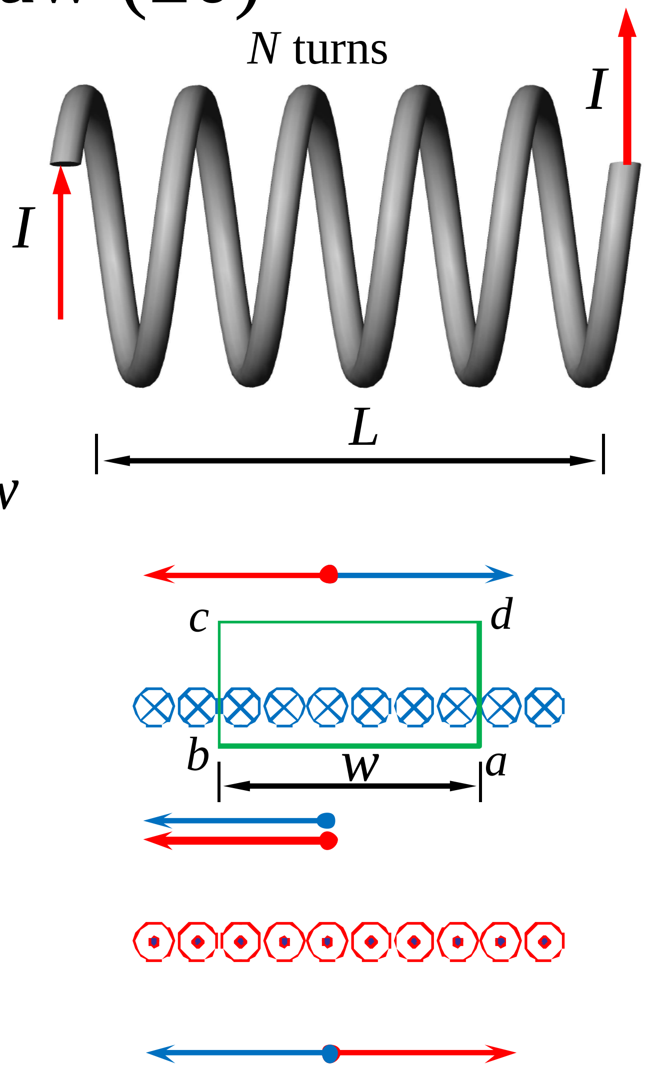
$$\rightarrow \int_a^b \mathbf{H}.d\mathbf{L} + \int_b^c \mathbf{H}.d\mathbf{L} + \int_c^d \mathbf{H}.d\mathbf{L} + \int_d^a \mathbf{H}.d\mathbf{L} = KL$$

$$\rightarrow H_{ab}L = KL$$

$$\rightarrow H_{ab} = K$$

Ampere's Circuital Law (20)

Ex. 11



$$\oint \mathbf{H}.d\mathbf{L} = I$$

$$\rightarrow \int_a^b \mathbf{H}.d\mathbf{L} + \int_b^c \mathbf{H}.d\mathbf{L} + \int_c^d \mathbf{H}.d\mathbf{L} + \int_d^a \mathbf{H}.d\mathbf{L} = Kw$$

$$\rightarrow H_{ab} w = I \frac{N}{L} w$$

$$\rightarrow H_{ab} = \frac{NI}{L}$$

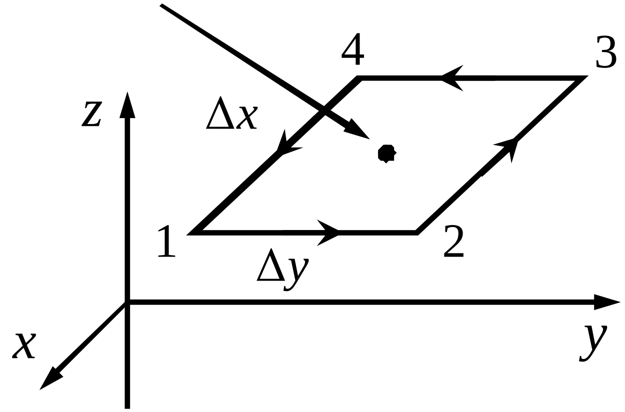
The Steady Magnetic Field

1. Biot – Savart Law
2. Ampere's Circuital Law
3. **Curl**
4. Stokes' Theorem
5. Magnetic Flux & Magnetic Flux Density
6. Magnetic Potential
7. Derivation of the Steady – Magnetic – Field Law

Curl (1)

$$\oint \mathbf{H}.d\mathbf{L} = I$$

$$\mathbf{H} = \mathbf{H}_0 = H_{x0}\mathbf{a}_x + H_{y0}\mathbf{a}_y + H_{z0}\mathbf{a}_z$$



$$\left.\begin{aligned} (\mathbf{H}.\Delta\mathbf{L})_{1-2} &= H_{y,1-2}\Delta y \\ H_{y,1-2} &\approx H_{y0} + \frac{\partial H_y}{\partial x}\left(\frac{1}{2}\Delta x\right) \end{aligned}\right\}$$

$$\rightarrow (\mathbf{H}.\Delta\mathbf{L})_{1-2} \approx \left(H_{y0} + \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)\Delta y$$

$$(\mathbf{H}.\Delta\mathbf{L})_{2-3} \approx H_{x,2-3}(-\Delta x) \approx -\left(H_{x0} + \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y\right)\Delta x$$

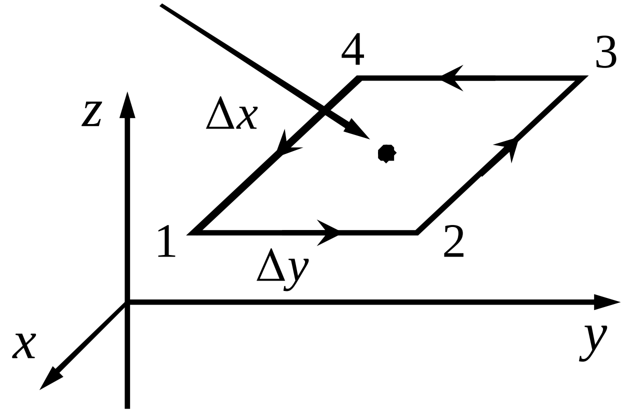
$$(\mathbf{H}.\Delta\mathbf{L})_{3-4} \approx -\left(H_{y0} - \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)\Delta y$$

$$(\mathbf{H}.\Delta\mathbf{L})_{4-1} \approx \left(H_{x0} - \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y\right)\Delta x$$

Curl (2)

$$\mathbf{H} = \mathbf{H}_0 = H_{x0}\mathbf{a}_x + H_{y0}\mathbf{a}_y + H_{z0}\mathbf{a}_z$$

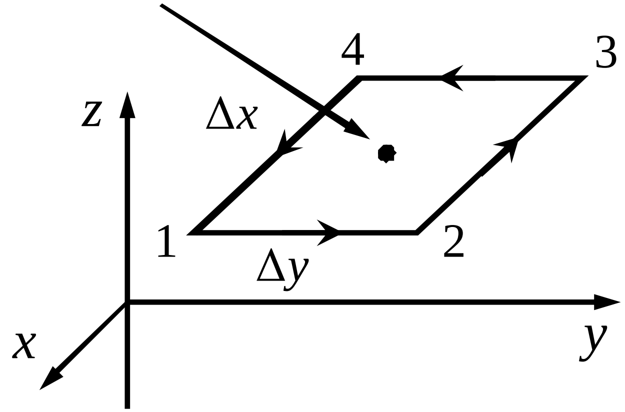
$$\oint \mathbf{H}.d\mathbf{L} = I$$



$$\left.\begin{aligned}
(\mathbf{H}.\Delta\mathbf{L})_{1-2} &\approx \left(H_{y0} + \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x \right)\Delta y \\
(\mathbf{H}.\Delta\mathbf{L})_{2-3} &\approx -\left(H_{x0} + \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y \right)\Delta x \\
(\mathbf{H}.\Delta\mathbf{L})_{3-4} &\approx -\left(H_{y0} - \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x \right)\Delta y \\
(\mathbf{H}.\Delta\mathbf{L})_{4-1} &\approx \left(H_{x0} - \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y \right)\Delta x \\
\oint \mathbf{H}.d\mathbf{L} &\approx (\mathbf{H}.\Delta\mathbf{L})_{1-2} + (\mathbf{H}.\Delta\mathbf{L})_{2-3} \\
&\quad + (\mathbf{H}.\Delta\mathbf{L})_{3-4} + (\mathbf{H}.\Delta\mathbf{L})_{4-1}
\end{aligned}\right\} \rightarrow \oint \mathbf{H}.d\mathbf{L} \approx \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right)\Delta x \Delta y$$

Curl (3)

$$\mathbf{H} = \mathbf{H}_0 = H_{x0}\mathbf{a}_x + H_{y0}\mathbf{a}_y + H_{z0}\mathbf{a}_z$$



$$\left.\begin{aligned} &\oint \mathbf{H}.d\mathbf{L} \approx \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x \Delta y \\ &\oint \mathbf{H}.d\mathbf{L} = \Delta I \\ &\Delta I \approx J_z \Delta x \Delta y \end{aligned}\right\}$$

$$\rightarrow \oint \mathbf{H}.d\mathbf{L} \approx \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x \Delta y \approx J_z \Delta x \Delta y$$

$$\rightarrow \frac{\oint \mathbf{H}.d\mathbf{L}}{\Delta x \Delta y} \approx \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \approx J_z \quad \rightarrow \lim_{\Delta x, \Delta y \to 0} \frac{\oint \mathbf{H}.d\mathbf{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z$$

$$\lim_{\Delta y, \Delta z \to 0} \frac{\oint \mathbf{H}.d\mathbf{L}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x$$

$$\lim_{\Delta z, \Delta x \to 0} \frac{\oint \mathbf{H}.d\mathbf{L}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y$$

Curl (4)

$$\lim_{\Delta x,\Delta y\to 0}\frac{\oint \mathbf{H}.d\mathbf{L}}{\Delta x\Delta y}=\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=J_z$$

$$\lim_{\Delta y,\Delta z\to 0}\frac{\oint \mathbf{H}.d\mathbf{L}}{\Delta y\Delta z}=\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=J_x$$

$$\lim_{\Delta z,\Delta x\to 0}\frac{\oint \mathbf{H}.d\mathbf{L}}{\Delta z\Delta x}=\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=J_y$$

Define $\boxed{(\text{rot}\,\mathbf{H})_N=\lim_{\Delta S_N\to 0}\frac{\oint \mathbf{H}.d\mathbf{L}}{\Delta S_N}}$

- S_N : planar area enclosed by the closed line integral
- $(\text{rot}\mathbf{H})_N$: the component (of rot**H**) perpendicular to S_N

Curl (5)

$$(\mathrm{rot}\,\mathbf{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \mathbf{H}.d\mathbf{L}}{\Delta S_N}$$

$$\mathrm{rot}\,\mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

$$\mathrm{rot}\,\mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\mathrm{rot}\,\mathbf{H} = \nabla \times \mathbf{H}$$

Curl (6)

$$\text{rot}\,\mathbf{H} = \nabla\times\mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

$$\nabla\times\mathbf{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\varphi + \left(\frac{1}{\rho}\frac{\partial(\rho H_\varphi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \varphi}\right)\mathbf{a}_z$$

$$\nabla\times\mathbf{H} = \frac{1}{r\sin\theta}\left(\frac{\partial(H_\varphi \sin\theta)}{\partial\theta} - \frac{\partial H_\theta}{\partial\varphi}\right)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\varphi} - \frac{\partial(rH_\varphi)}{\partial r}\right)\mathbf{a}_\theta$$
$$+\frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta}\right)\mathbf{a}_\varphi$$

Ex.

Curl (7)

Find the curl of the following vectors:

$$a)\ \mathbf{A} = x^2 y\mathbf{a}_x + y^2 z\mathbf{a}_y + xy\mathbf{a}_z$$

$$b)\ \mathbf{B} = \rho\cos\varphi\mathbf{a}_z + \frac{z\sin\varphi}{\rho}\mathbf{a}_\rho$$

$$c)\ \mathbf{C} = r^2\sin\theta\cos\varphi\mathbf{a}_r + \frac{\cos\theta\sin\varphi}{r^2}\mathbf{a}_\theta$$

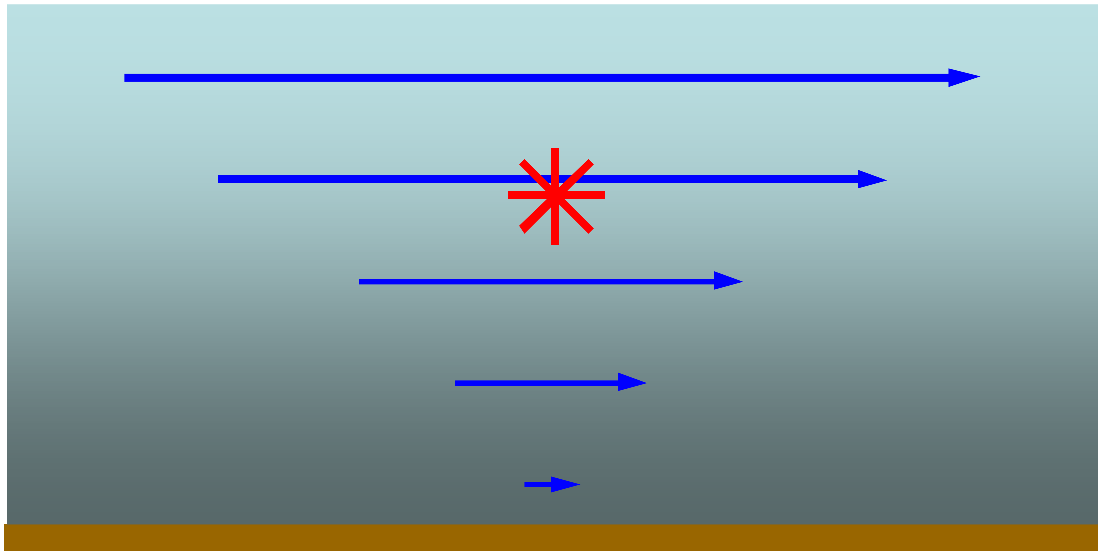
Curl (8)

$$\text{Curl: } \operatorname{rot}\mathbf{H} = \nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

$$\text{Gradient: } \nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

$$\text{Divergence: } \nabla.\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Curl (9)



$$\text{rot}\,\mathbf{H} = \nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \mathbf{a}_z$$

Curl (10)

$$\left.\begin{aligned}
&\text{rot}\,\mathbf{H}=\nabla\times\mathbf{H}=\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right)\mathbf{a}_x+\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)\mathbf{a}_y+\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)\mathbf{a}_z\\
&\lim_{\Delta x,\Delta y\to 0}\frac{\oint \mathbf{H}.d\mathbf{L}}{\Delta x\Delta y}=\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=J_z\\
&\lim_{\Delta y,\Delta z\to 0}\frac{\oint \mathbf{H}.d\mathbf{L}}{\Delta y\Delta z}=\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=J_x\\
&\lim_{\Delta z,\Delta x\to 0}\frac{\oint \mathbf{H}.d\mathbf{L}}{\Delta z\Delta x}=\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=J_y
\end{aligned}\right\}$$

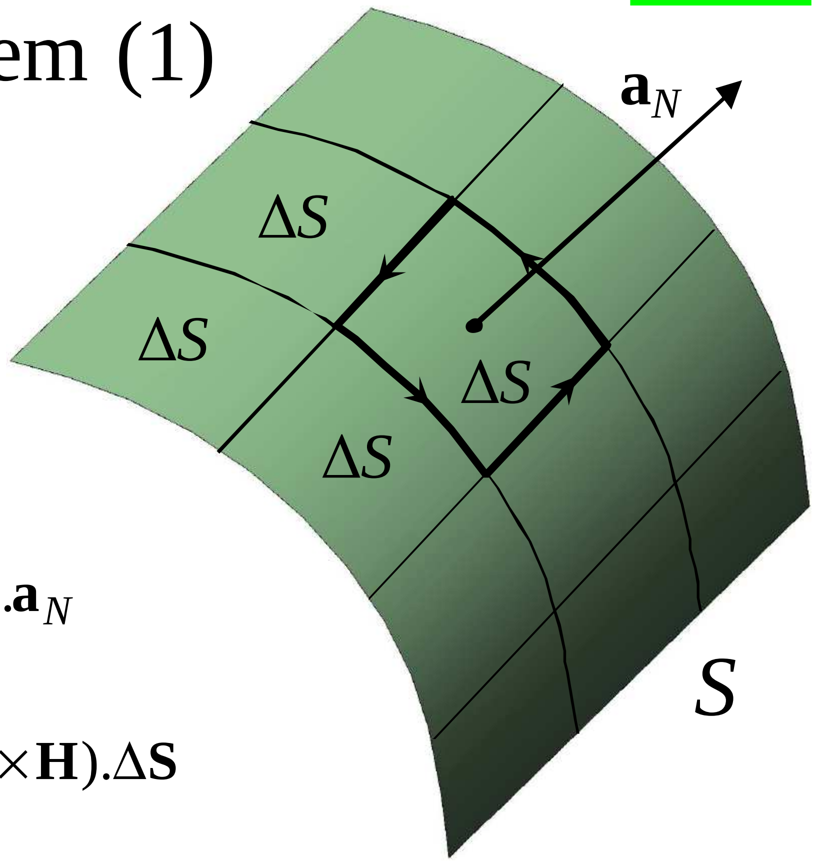
$$\longrightarrow \boxed{\nabla\times\mathbf{H}=\mathbf{J}}$$

(the second of Maxwell's four equations)

The Steady Magnetic Field

1. Biot – Savart Law
2. Ampere's Circuital Law
3. Curl
4. **Stokes' Theorem**
5. Magnetic Flux & Magnetic Flux Density
6. Magnetic Potential
7. Derivation of the Steady – Magnetic – Field Law

Stokes' Theorem (1)



$$\left.\begin{aligned} \mathbf{J}_N &\approx \frac{\mathbf{I}_N}{\Delta S} \\ \mathbf{I}_N &= \oint \mathbf{H}.d\mathbf{L}_{\Delta S} \end{aligned}\right\} \rightarrow \mathbf{J}_N \approx \frac{\oint \mathbf{H}.d\mathbf{L}_{\Delta S}}{\Delta S}$$

$$\left.\begin{aligned} \mathbf{J}_N &\approx \frac{\oint \mathbf{H}.d\mathbf{L}_{\Delta S}}{\Delta S} \\ \mathbf{J}_N &= (\nabla \times \mathbf{H})_N \end{aligned}\right\}$$

$$\rightarrow \frac{\oint \mathbf{H}.d\mathbf{L}_{\Delta S}}{\Delta S} \approx (\nabla \times \mathbf{H})_N = (\nabla \times \mathbf{H}).\mathbf{a}_N$$

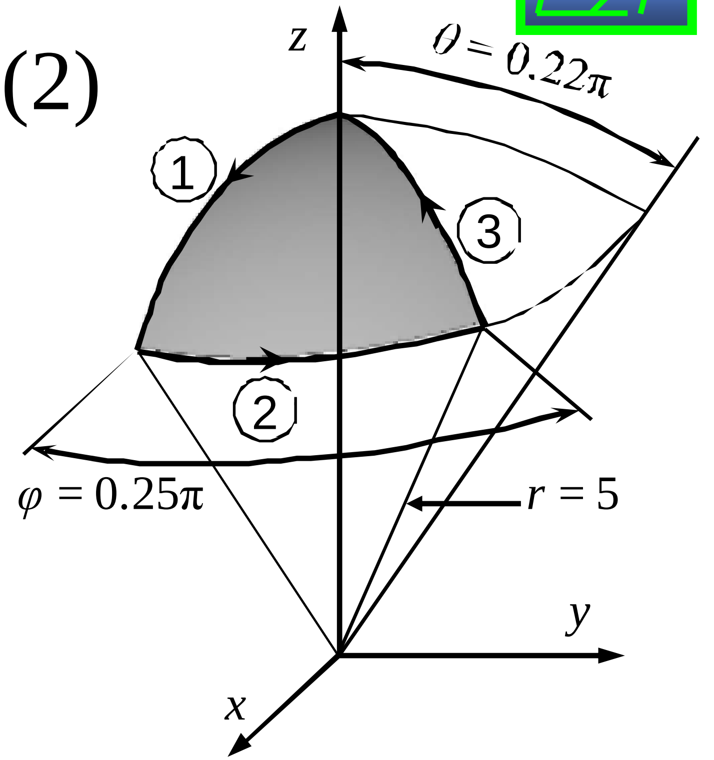
$$\rightarrow \oint \mathbf{H}.d\mathbf{L}_{\Delta S} \approx (\nabla \times \mathbf{H}).\mathbf{a}_N \Delta S = (\nabla \times \mathbf{H}).\Delta \mathbf{S}$$

$$\rightarrow \boxed{\oint \mathbf{H}.d\mathbf{L} = \int_S (\nabla \times \mathbf{H}).d\mathbf{S}}$$

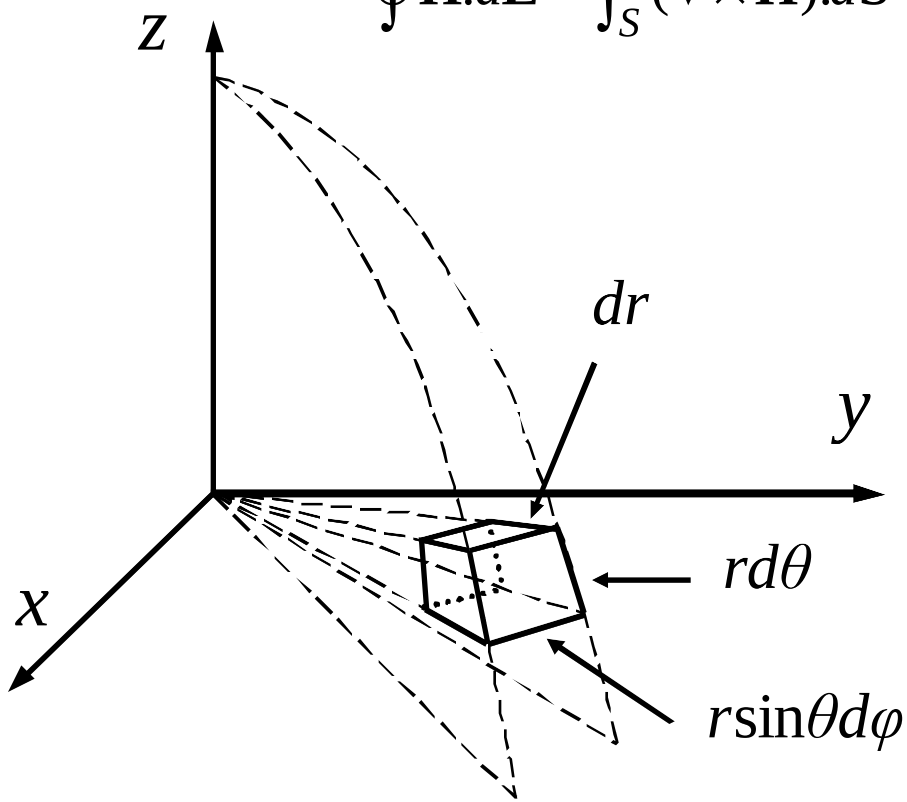
Ex. 1

Stokes' Theorem (2)

Given $\mathbf{H} = 6r\sin\varphi\mathbf{a}_r + 18r\sin\theta\cos\varphi\mathbf{a}_\varphi$ A/m.
Verify Stokes' theorem.



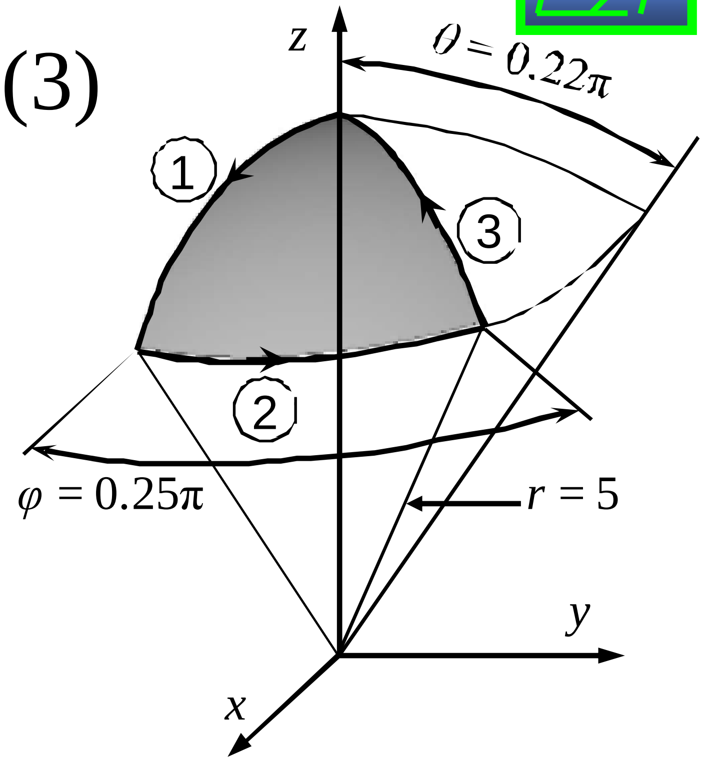
$$\oint \mathbf{H}.d\mathbf{L} = \int_S (\nabla \times \mathbf{H}).d\mathbf{S}$$



$$d\mathbf{L} = dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta d\varphi\mathbf{a}_\varphi$$

Ex. 1

Stokes' Theorem (3)



Given $\mathbf{H} = 6r\sin\varphi\mathbf{a}_r + 18r\sin\theta\cos\varphi\mathbf{a}_\varphi$ A/m.
Verify Stokes' theorem.

$$\oint \mathbf{H}.d\mathbf{L} = \int_S (\nabla\times\mathbf{H}).d\mathbf{S}$$

$$d\mathbf{L} = dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta d\varphi\mathbf{a}_\varphi$$

$$\rightarrow \oint \mathbf{H}.d\mathbf{L} = \oint \mathbf{H}.(dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta d\varphi\mathbf{a}_\varphi)$$

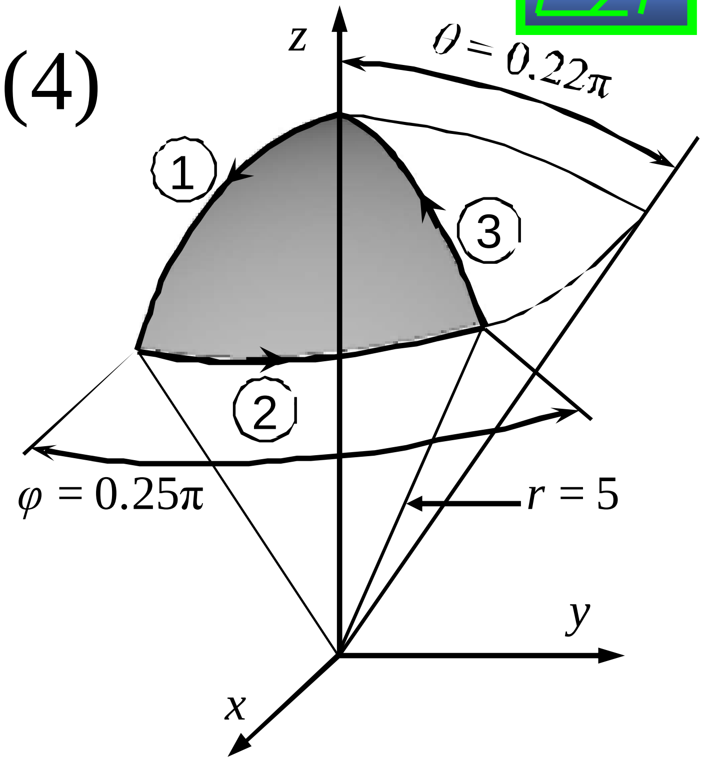
$$= \oint H_r dr + \oint H_\theta r d\theta + \oint H_\varphi r\sin\theta d\varphi$$

$$\left.\begin{array}{l} \oint H_r dr = \int_1 H_r dr + \int_2 H_r dr + \int_3 H_r dr \\ 1,2,3: r = 5 \rightarrow dr\big|_{1,2,3} = 0 \end{array}\right\} \rightarrow \oint H_r dr = 0$$

$$\left.\begin{array}{r} \oint H_\theta r d\theta \\ H_\theta = 0 \end{array}\right\} \rightarrow \oint H_\theta r d\theta = 0$$

Ex. 1 Stokes' Theorem (4)

Given **H** = 6$r$sin$\varphi$$\mathbf{a}_r$ + 18$r$sinθcos$\varphi$$\mathbf{a}_\varphi$ A/m.
Verify Stokes' theorem.



$$\oint \mathbf{H}.d\mathbf{L} = \int_S (\nabla \times \mathbf{H}).d\mathbf{S}$$

$$\oint \mathbf{H}.d\mathbf{L} = \oint H_r dr + \oint H_\theta r d\theta + \oint H_\varphi r \sin\theta d\varphi$$

$$\oint H_r dr = 0$$

$$\oint H_\theta r d\theta = 0$$

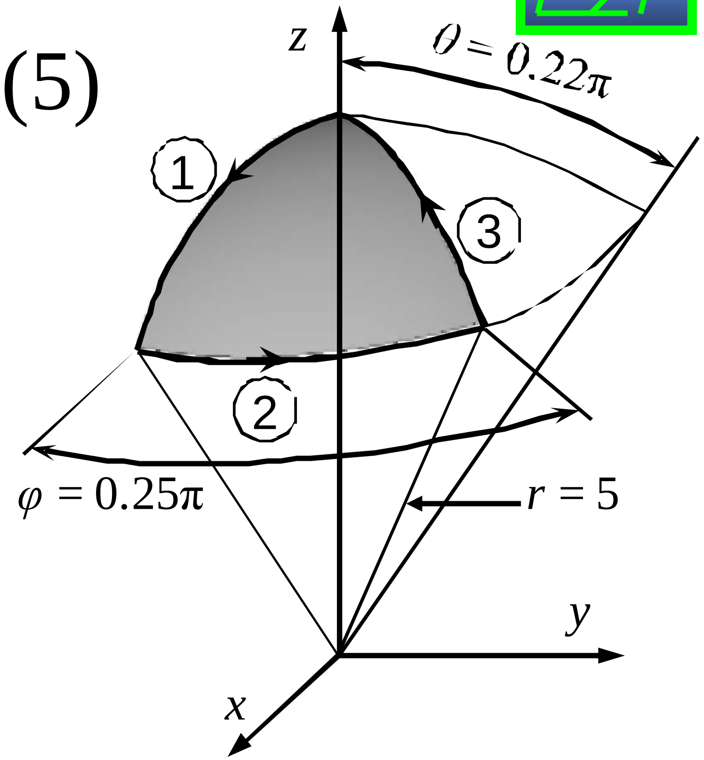
$$\left.\begin{array}{r} \oint H_\varphi r\sin\theta d\varphi = \int_1 H_\varphi r\sin\theta d\varphi + \int_2 H_\varphi r\sin\theta d\varphi + \int_3 H_\varphi r\sin\theta d\varphi \\ 1,3:\ \varphi = \text{const} \rightarrow d\varphi\big|_{1,3} = 0 \end{array}\right\}$$

$$\rightarrow \oint H_\varphi r\sin\theta d\varphi = \int_2 H_\varphi r\sin\theta d\varphi$$

Ex. 1

Stokes' Theorem (5)

Given $\mathbf{H} = 6r\sin\varphi\mathbf{a}_r + 18r\sin\theta\cos\varphi\mathbf{a}_\varphi$ A/m.
Verify Stokes' theorem.



$$\oint \mathbf{H}.d\mathbf{L} = \int_S (\nabla \times \mathbf{H}).d\mathbf{S}$$

$$\left.\begin{aligned} \oint \mathbf{H}.d\mathbf{L} &= \oint H_r dr + \oint H_\theta r d\theta + \oint H_\varphi r\sin\theta d\varphi \\ \oint H_r dr &= 0 \\ \oint H_\theta r d\theta &= 0 \\ \oint H_\varphi r\sin\theta d\varphi &= \int_2 H_\varphi r\sin\theta d\varphi \end{aligned}\right\}$$

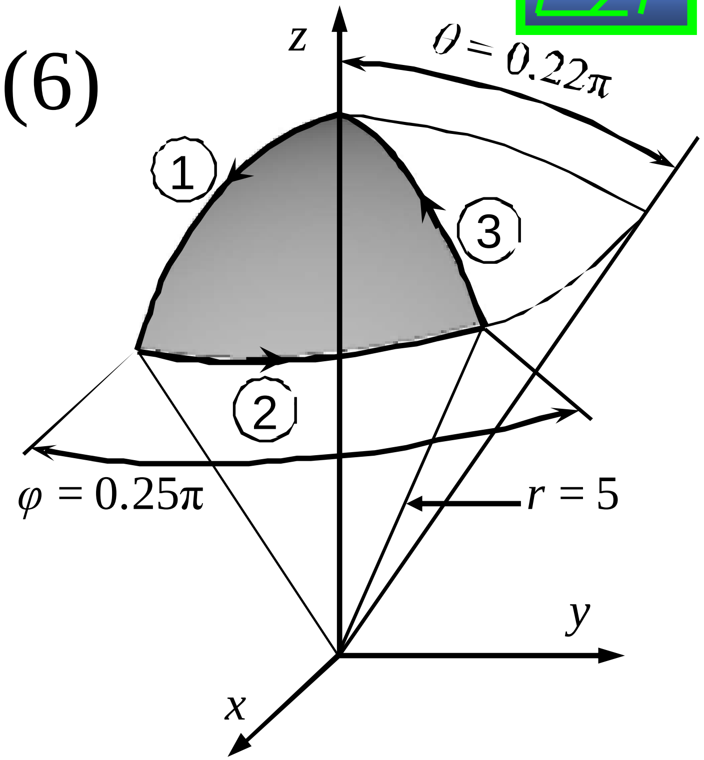
$$\rightarrow \oint \mathbf{H}.d\mathbf{L} = \int_2 H_\varphi r\sin\theta d\varphi = \int_0^{0.25\pi} H_\varphi r\sin\theta d\varphi = \int_0^{0.25\pi} H_\varphi\Big|_2 5\sin(0.22\pi)d\varphi$$

$$= \int_0^{0.25\pi} 3.19 H_\varphi\Big|_2 d\varphi$$

Stokes' Theorem (6)

Ex. 1

Given $\mathbf{H} = 6r\sin\varphi\,\mathbf{a}_r + 18r\sin\theta\cos\varphi\,\mathbf{a}_\varphi$ A/m.
Verify Stokes' theorem.



$$\oint \mathbf{H}.d\mathbf{L} = \int_S (\nabla\times\mathbf{H}).d\mathbf{S}$$

$$\oint \mathbf{H}.d\mathbf{L} = \oint H_r dr + \oint H_\theta r d\theta + \oint H_\varphi r\sin\theta d\varphi$$

$$\left.\begin{aligned} &= \int_0^{0.25\pi} 3.19\, H_\varphi\big|_2\, d\varphi \\ H_\varphi\big|_2 &= 18\times 5\sin(0.22\pi)\cos\varphi = 57.37\cos\varphi \end{aligned}\right\}$$

$$\rightarrow \oint \mathbf{H}.d\mathbf{L} = \int_0^{0.25\pi} 3.19\times 57.37\cos\varphi d\varphi = \int_0^{0.25\pi} 182.84\cos\varphi d\varphi$$

$$= 182.84\sin\varphi\Big|_0^{0.25\pi} = 182.84\sin(0.25\pi) = \boxed{129.27\text{ A}}$$

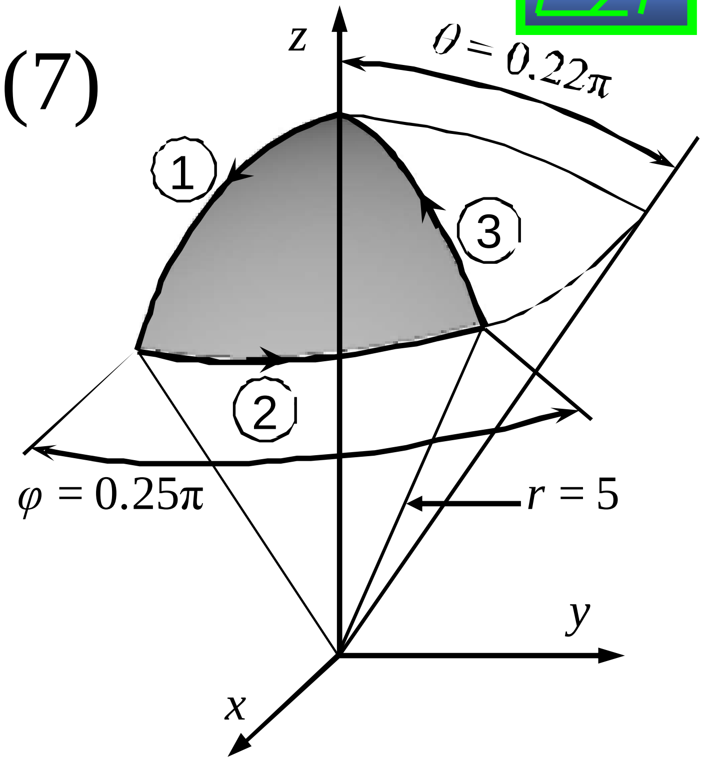
Ex. 1

Stokes' Theorem (7)

Given $\mathbf{H} = 6r\sin\varphi\,\mathbf{a}_r + 18r\sin\theta\cos\varphi\,\mathbf{a}_\varphi$ A/m.
Verify Stokes' theorem.

$$\oint \mathbf{H}.d\mathbf{L} = \int_S (\nabla\times\mathbf{H}).d\mathbf{S}$$

$$\boxed{\oint \mathbf{H}.d\mathbf{L} = 129.27\text{ A}}$$



$$\int_S (\nabla\times\mathbf{H}).d\mathbf{S}$$

$$\nabla\times\mathbf{H} = \frac{1}{r\sin\theta}\left(\frac{\partial(H_\varphi\sin\theta)}{\partial\theta} - \frac{\partial H_\theta}{\partial\varphi}\right)\mathbf{a}_r$$

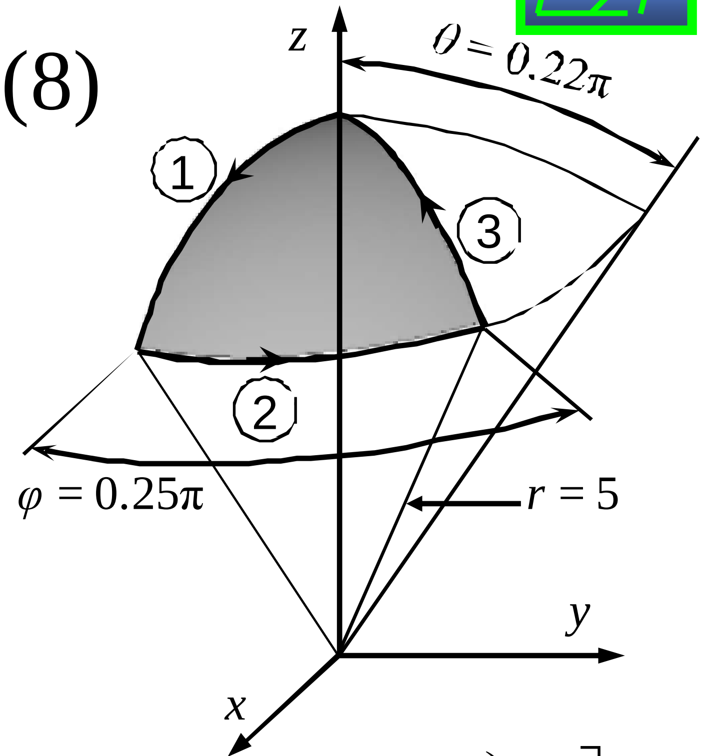
$$+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\varphi} - \frac{\partial(rH_\varphi)}{\partial r}\right)\mathbf{a}_\theta + \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta}\right)\mathbf{a}_\varphi$$

$$= \frac{1}{r\sin\theta}(36r\sin\theta\cos\theta\cos\varphi)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}6r\cos\varphi - 36r\sin\theta\cos\varphi\right)\mathbf{a}_\theta$$

Ex. 1

Stokes' Theorem (8)

Given $\mathbf{H} = 6r\sin\varphi\,\mathbf{a}_r + 18r\sin\theta\cos\varphi\,\mathbf{a}_\varphi$ A/m.
Verify Stokes' theorem.



$$\oint \mathbf{H}.d\mathbf{L} = \int_S (\nabla\times\mathbf{H}).d\mathbf{S}$$

$$\oint \mathbf{H}.d\mathbf{L} = 129.27 \text{ A}$$

$$\int_S (\nabla\times\mathbf{H}).d\mathbf{S}$$

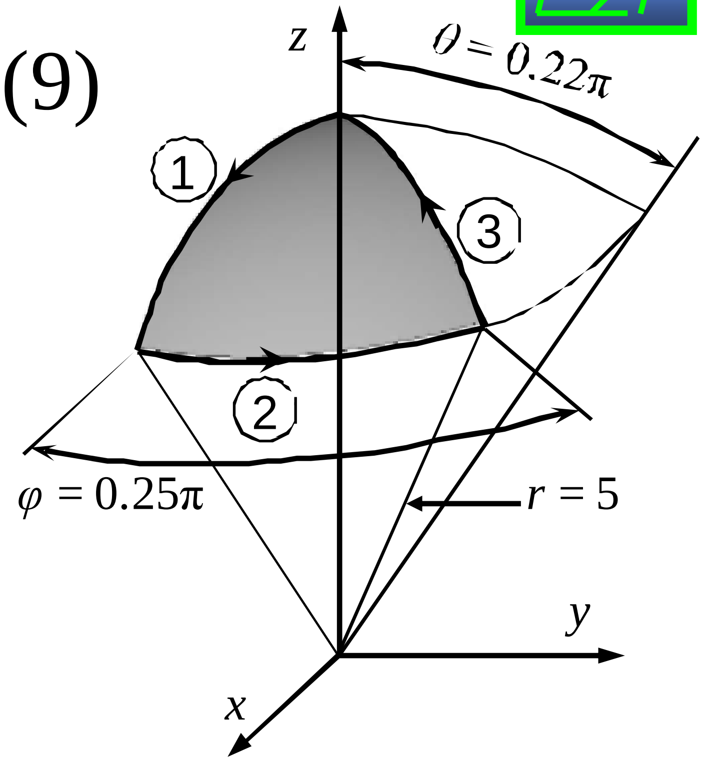
$$= \int_S \left[\frac{1}{r\sin\theta}(36r\sin\theta\cos\theta\cos\varphi)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}6r\cos\varphi - 36r\sin\theta\cos\varphi\right)\mathbf{a}_\theta \right] d\mathbf{S}$$

$$= \int_S \left[(36\cos\theta\cos\varphi)\mathbf{a}_r + \left(\frac{1}{\sin\theta}6\cos\varphi - 36\sin\theta\cos\varphi\right)\mathbf{a}_\theta \right] d\mathbf{S}$$

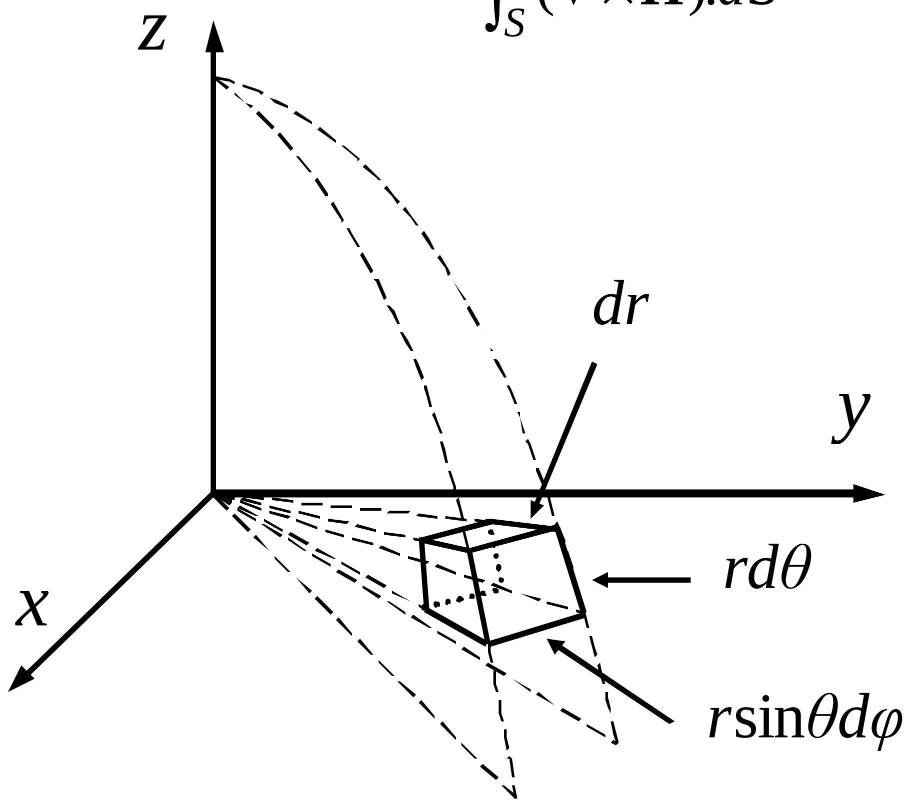
Stokes' Theorem (9)

Ex. 1

Given $\mathbf{H} = 6r\sin\varphi\mathbf{a}_r + 18r\sin\theta\cos\varphi\mathbf{a}_\varphi$ A/m.
Verify Stokes' theorem.



$$\int_S (\nabla \times \mathbf{H}).d\mathbf{S}$$

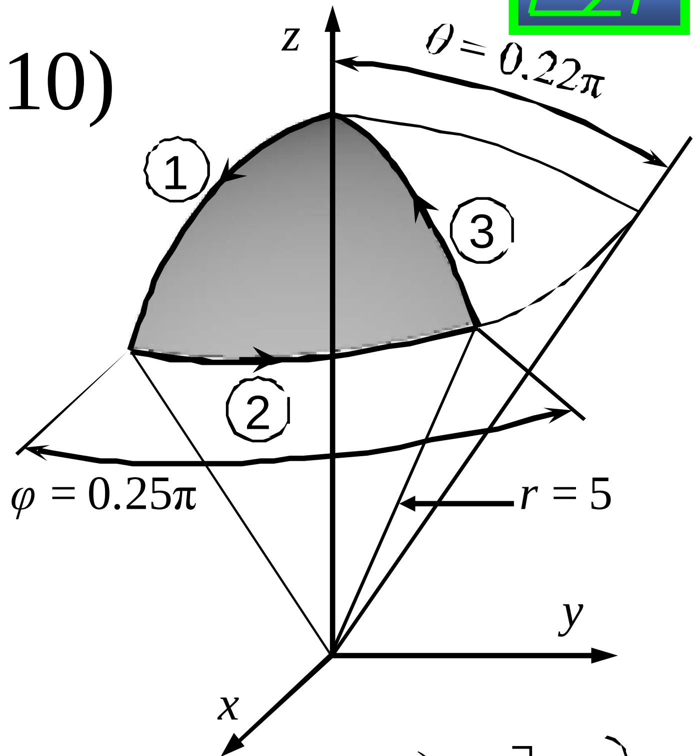


$$d\mathbf{S} = r^2 \sin\theta d\theta d\varphi \mathbf{a}_r$$

Ex. 1

Stokes' Theorem (10)

Given $\mathbf{H} = 6r\sin\varphi\,\mathbf{a}_r + 18r\sin\theta\cos\varphi\,\mathbf{a}_\varphi$ A/m.
Verify Stokes' theorem.



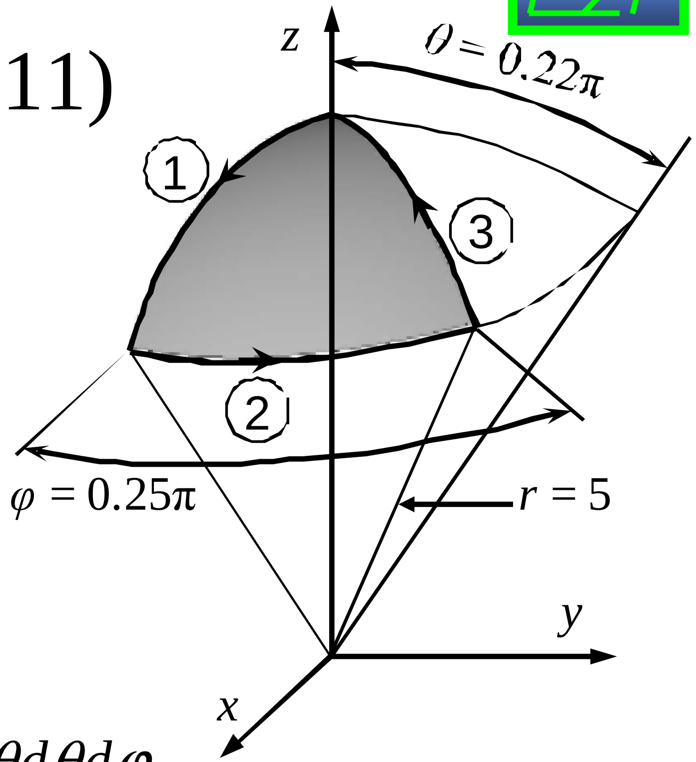
$$\oint \mathbf{H}.d\mathbf{L} = \int_S (\nabla\times\mathbf{H}).d\mathbf{S}$$

$$\boxed{\oint \mathbf{H}.d\mathbf{L} = 129.27\text{ A}}$$

$$\left.\begin{aligned}\int_S (\nabla\times\mathbf{H}).d\mathbf{S} &= \int_S \left[(36\cos\theta\cos\varphi)\mathbf{a}_r + \left(\frac{1}{\sin\theta}6\cos\varphi - 36\sin\theta\cos\varphi\right)\mathbf{a}_\theta\right]d\mathbf{S} \\ d\mathbf{S} &= r^2\sin\theta d\theta d\varphi\mathbf{a}_r\end{aligned}\right\}$$

$$\to \int_S (\nabla\times\mathbf{H}).d\mathbf{S} = \int_S 36\cos\theta\cos\varphi\mathbf{a}_r d\mathbf{S} = \int_S (36\cos\theta\cos\varphi)(5)^2\sin\theta d\theta d\varphi$$

Ex. 1 Stokes' Theorem (11)



Given $\mathbf{H} = 6r\sin\varphi\mathbf{a}_r + 18r\sin\theta\cos\varphi\mathbf{a}_\varphi$ A/m.
Verify Stokes' theorem.

$$\oint \mathbf{H}.d\mathbf{L} = \int_S (\nabla \times \mathbf{H}).d\mathbf{S}$$

$$\boxed{\oint \mathbf{H}.d\mathbf{L} = 129.27 \text{ A}}$$

$$\int_S (\nabla \times \mathbf{H}).d\mathbf{S} = \int_S (36\cos\theta\cos\varphi)(5)^2 \sin\theta d\theta d\varphi$$

$$= \int_0^{0.25\pi}\int_0^{0.22\pi} (36\cos\theta\cos\varphi)(5)^2 \sin\theta d\theta d\varphi$$

$$= \int_0^{0.25\pi} 900\left(\frac{1}{2}\sin^2\theta\right)\Bigg|_0^{0.22\pi} \cos\varphi d\varphi = \int_0^{0.25\pi} 182.84\cos\varphi d\varphi$$

$$= \int_0^{0.25\pi} 182.84\cos\varphi d\varphi = 182.84\sin\varphi\Big|_0^{0.25\pi} = \boxed{129.27 \text{ A}}$$

Ex. 2

Stokes' Theorem (12)

Extract $\oint \mathbf{H}.d\mathbf{L} = I$ from $\nabla \times \mathbf{H} = \mathbf{J}$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\rightarrow (\nabla \times \mathbf{H}).d\mathbf{S} = \mathbf{J}.d\mathbf{S}$$

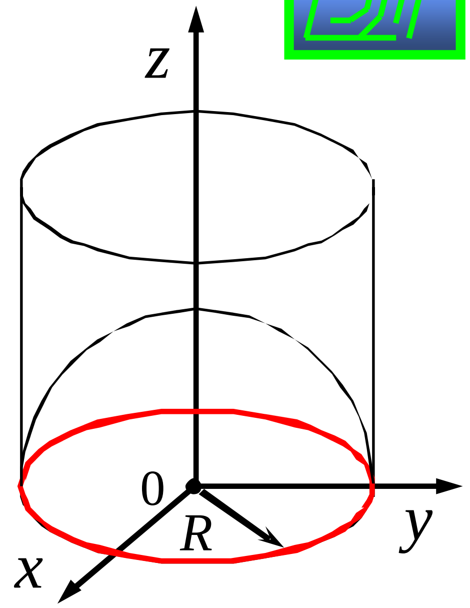
$$\left.\begin{aligned} &\rightarrow \int_S (\nabla \times \mathbf{H}).d\mathbf{S} = \int_S \mathbf{J}.d\mathbf{S} = I \\ &\oint \mathbf{H}.d\mathbf{L} = \int_S (\nabla \times \mathbf{H}).d\mathbf{S} \end{aligned}\right\}$$

$$\rightarrow \oint \mathbf{H}.d\mathbf{L} = I$$

Ex. 3

Stokes' Theorem (13)

Given $\mathbf{A} = -y\mathbf{a}_x + x\mathbf{a}_y - z\mathbf{a}_z = \rho\mathbf{a}_\varphi - z\mathbf{a}_z$. Verify Stokes' theorem for the circular bounding contour in the *xy* plane; check the result for:
a) The flat circular surface in the *xy* plane,
b) The hemispherical surface bounded by the contour,
c) The cylindrical surface bounded by the contour.



$$\oint \mathbf{A}.d\mathbf{L} = \int_S (\nabla\times\mathbf{A}).d\mathbf{S}$$

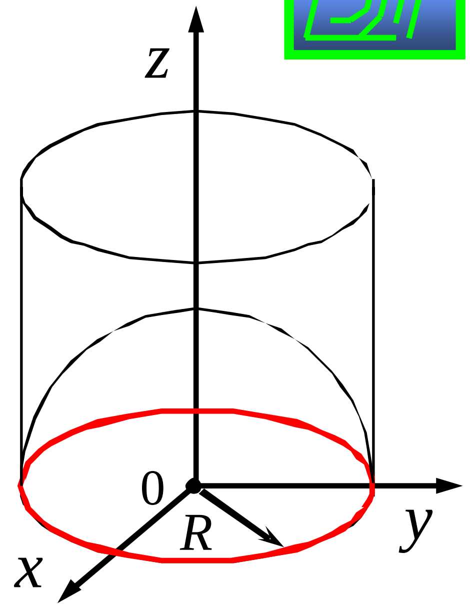
$$d\mathbf{L} = Rd\varphi\mathbf{a}_\varphi \to \mathbf{A}.d\mathbf{L} = R^2 d\varphi \to \oint \mathbf{A}.d\mathbf{L} = \int_0^{2\pi} R^2 d\varphi = \boxed{2\pi R^2}$$

$$\nabla\times\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{a}_z = 2\mathbf{a}_z$$

$$\int_{flat} (\nabla\times\mathbf{A}).d\mathbf{S} = \int_{flat} (2\mathbf{a}_z).dS\mathbf{a}_z = 2\int_{flat} dS = \boxed{2\pi R^2}$$

Ex. 3

Stokes' Theorem (14)



Given $\mathbf{A} = -y\mathbf{a}_x + x\mathbf{a}_y - z\mathbf{a}_z = \rho\mathbf{a}_\varphi - z\mathbf{a}_z$. Verify Stokes' theorem for the circular bounding contour in the *xy* plane; check the result for:
a) The flat circular surface in the *xy* plane,
b) The hemispherical surface bounded by the contour,
c) The cylindrical surface bounded by the contour.

$$\boxed{2\pi R^2} = \oint \mathbf{A}.d\mathbf{L} = \int_S (\nabla \times \mathbf{A}).d\mathbf{S}$$

$$\left.\begin{aligned} &\int_{hemi} (\nabla \times \mathbf{A}).d\mathbf{S} = \int_{hemi} (2\mathbf{a}_z).(R^2 \sin\theta d\theta d\varphi)\mathbf{a}_r \\ &\mathbf{a}_z.\mathbf{a}_r = \cos\theta \end{aligned}\right\}$$

$$\rightarrow \int_{hemi} (\nabla \times \mathbf{A}).d\mathbf{S} = \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{2\pi} R^2 \sin 2\theta d\theta d\varphi \quad = -2\pi R^2 \left.\frac{\cos 2\theta}{2}\right|_{\theta=0}^{\pi/2} = \boxed{2\pi R^2}$$

$$\int_{cy} (\nabla \times \mathbf{A}).d\mathbf{S} = ?$$

The Steady Magnetic Field

1. Biot – Savart Law
2. Ampere's Circuital Law
3. Curl
4. Stokes' Theorem
5. **Magnetic Flux & Magnetic Flux Density**
6. Magnetic Potential
7. Derivation of the Steady – Magnetic – Field Law

Magnetic Flux & Magnetic Flux Density (1)

- The magnetic flux density **B** is defined in free space:

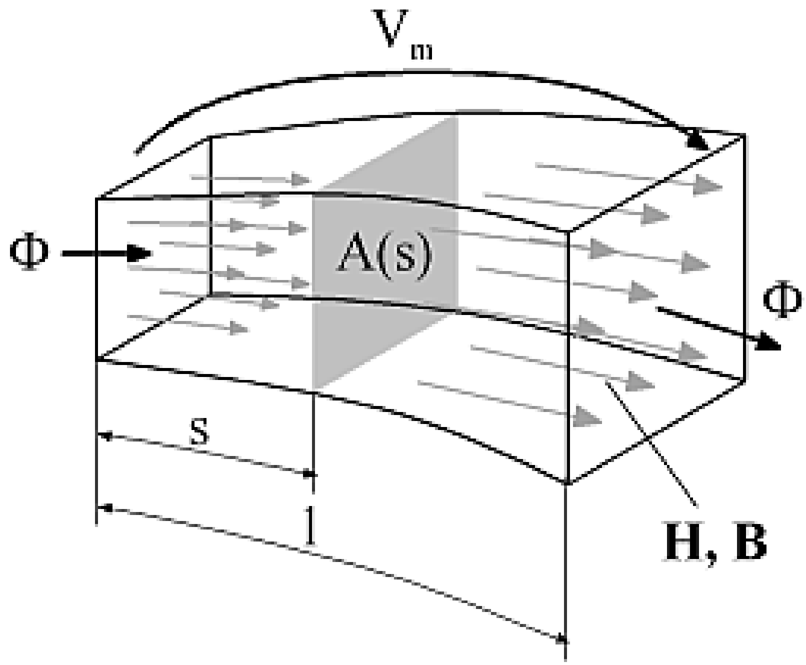
$$\boxed{\mathbf{B} = \mu_0 \mathbf{H}}$$

- Unit: Wb/m^2 or T or G
  (1T = 10000G)
- Permeability $\mu_0 = 4\pi \times 10^{-7}$ H/m
- Definition of magnetic flux:

$$\boxed{\Phi = \int_S \mathbf{B}.d\mathbf{S}}$$

- Electric flux:

$$\Psi = \oint_S \mathbf{D}.d\mathbf{S} = Q$$



https://www.maplesoft.com/documentation_center/online_manuals/modelica/Modelica_Magnetic_FluxTubes_UsersGuide.html

Magnetic Flux & Magnetic Flux Density (2)

- Gauss' law for the magnetic field:

$$\oint_S \mathbf{B}.d\mathbf{S} = 0$$

- The last of Maxwell's four equations:

$$\nabla.\mathbf{B} = 0$$

- Maxwell's four equations:

$$\boxed{\begin{aligned} &\nabla.\mathbf{D} = \rho_v \\ &\nabla\times\mathbf{E} = 0 \\ &\nabla\times\mathbf{H} = \mathbf{J} \\ &\nabla.\mathbf{B} = 0 \end{aligned}} \Rightarrow \boxed{\begin{aligned} &\oint_S \mathbf{D}.d\mathbf{S} = Q = \int_V \rho_v dv \\ &\oint \mathbf{E}.d\mathbf{L} = 0 \\ &\oint \mathbf{H}.d\mathbf{L} = I = \oint_S \mathbf{J}.d\mathbf{S} \\ &\oint_S \mathbf{B}.d\mathbf{S} = 0 \end{aligned}}$$

Ex. 1 Magnetic Flux & Magnetic Flux Density (3)

Given the magnetic field **B** = $3xy^2\mathbf{a}_z$ Wb/m². Determine the magnetic flux crossing the portion of the *xy* plane lying between $x = 0$, $x = 1$, $y = 0$, & $y = 1$.

$$\left.\begin{aligned} \Phi &= \int_S \mathbf{B}.d\mathbf{S} \\ \mathbf{B} &= 3xy^2\mathbf{a}_z \\ d\mathbf{S} &= dxdy\mathbf{a}_z \end{aligned}\right\} \rightarrow \Phi = \int_{x=0}^{1}\int_{y=0}^{1} 3xy^2 dxdy = 0.5 \text{ Wb}$$

Ex. 2 Magnetic Flux & Magnetic Flux Density (4)

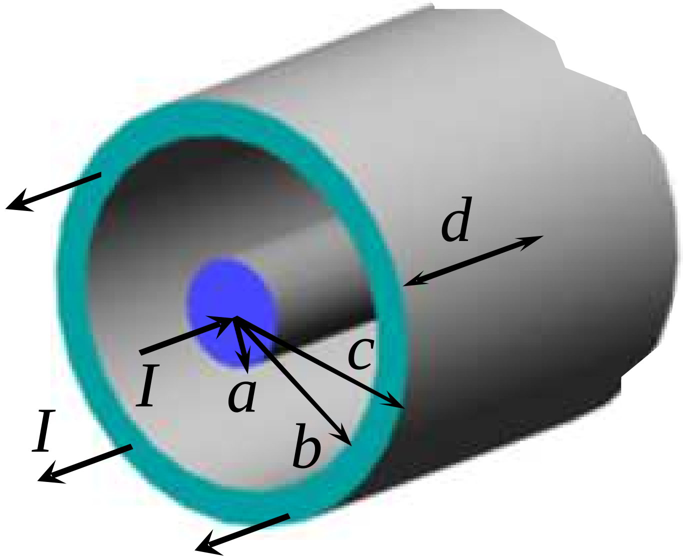
Find the flux between the conductors of the coaxial line.

$$H_\varphi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$$

$$\to \mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\varphi$$

$$\Phi = \int_S \mathbf{B}.d\mathbf{S}$$

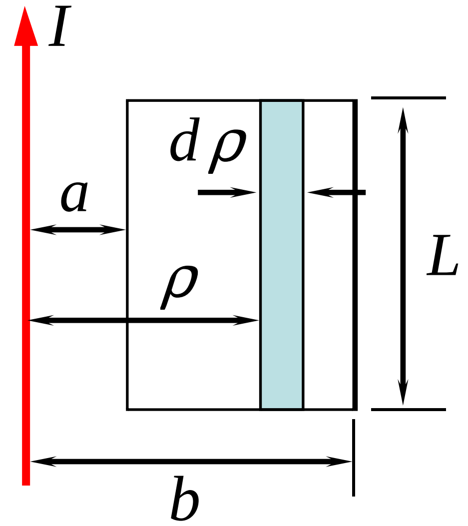
$$d\mathbf{S} = d\rho dz \mathbf{a}_\varphi$$



$$\to \Phi = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\varphi.d\rho dz \mathbf{a}_\varphi = \frac{\mu_0 Id}{2\pi}\ln\frac{b}{a}$$

Ex. 3 Magnetic Flux & Magnetic Flux Density (5)

Find the total flux through the rectangular circuit.

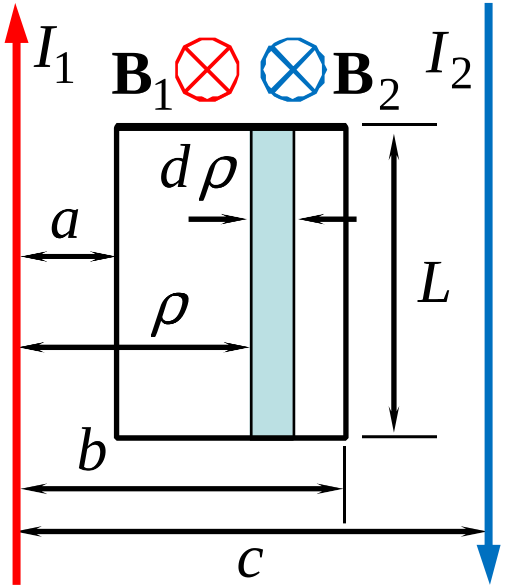


$$\left.\begin{aligned}\Phi &= \int_S \mathbf{B}.d\mathbf{S}\\ \mathbf{B} &= \mu_0\mathbf{H} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\varphi\\ d\mathbf{S} &= Ld\rho\mathbf{a}_\varphi\end{aligned}\right\}$$

$$\rightarrow \Phi = \int_{\rho=a}^{\rho=b} \frac{\mu_0 I}{2\pi\rho} Ld\rho = \frac{\mu_0 IL}{2\pi}\ln\frac{b}{a}$$

Ex. 4 Magnetic Flux & Magnetic Flux Density (6)

Find the total flux through the rectangular circuit.



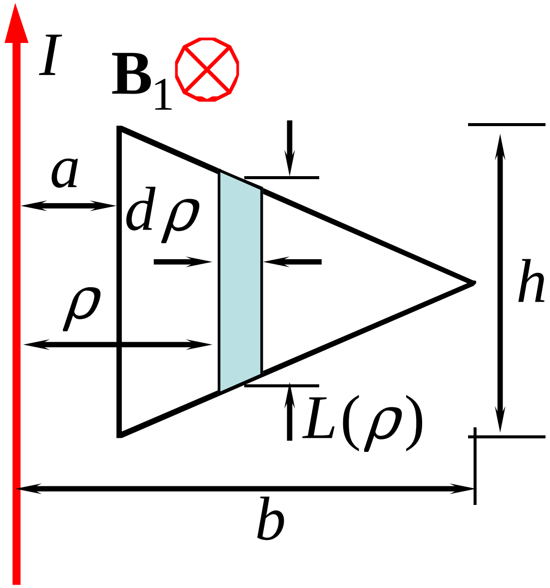
$$\left.\begin{aligned}
\Phi &= \int_S \mathbf{B}.d\mathbf{S} = \int_S (\mathbf{B}_1 + \mathbf{B}_2).d\mathbf{S} \\
\mathbf{B}_1 &= \mu_0 \mathbf{H}_1 = \frac{\mu_0 I_1}{2\pi\rho}\mathbf{a}_\varphi \\
\mathbf{B}_2 &= \mu_0 \mathbf{H}_2 = \frac{\mu_0 I_2}{2\pi(c-\rho)}\mathbf{a}_\varphi \\
d\mathbf{S} &= Ld\rho\mathbf{a}_\varphi
\end{aligned}\right\}$$

$$\rightarrow \Phi = \int_{\rho=a}^{\rho=b}\left[\frac{\mu_0 I_1}{2\pi\rho} + \frac{\mu_0 I_2}{2\pi(c-\rho)}\right]Ld\rho = \frac{\mu_0 L}{2\pi}\left(I_1 \ln\frac{b}{a} + I_2 \ln\frac{c-a}{c-b}\right)$$

Ex. 5 Magnetic Flux & Magnetic Flux Density (7)

Find the total flux through the triangular circuit.

$$\left.\begin{aligned}
\Phi &= \int_S \mathbf{B}.d\mathbf{S} \\
\mathbf{B} &= \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\varphi \\
d\mathbf{S} &= L(\rho)d\rho\mathbf{a}_\varphi \\
\frac{h}{b-a} = \frac{L(\rho)}{b-\rho} &\rightarrow L(\rho) = \frac{h(b-\rho)}{b-a}
\end{aligned}\right\}$$

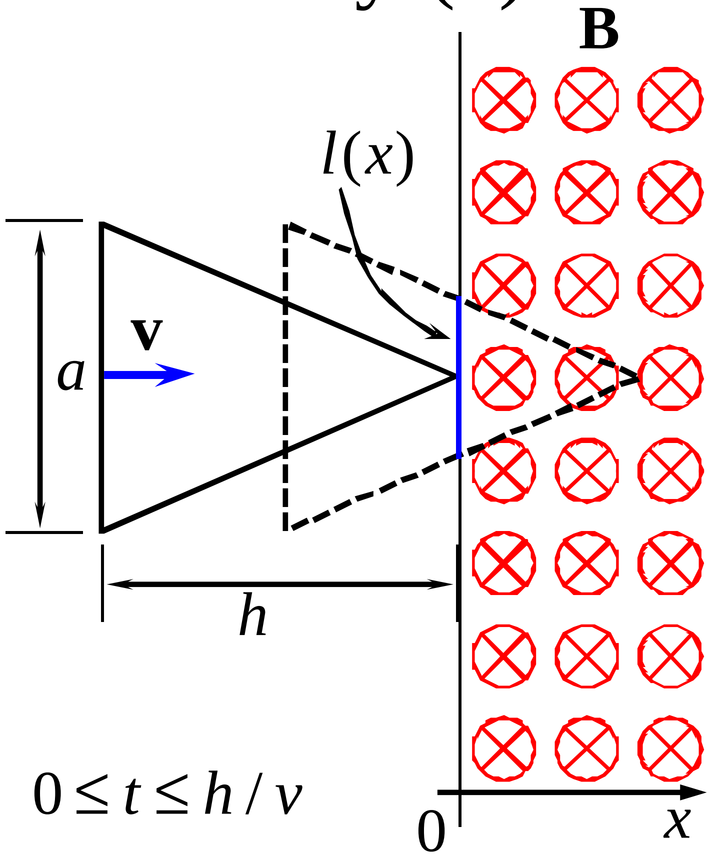


$$\rightarrow \Phi = \int_{\rho=a}^{\rho=b} \left(\frac{\mu_0 I}{2\pi\rho}\right)\left[\frac{h(b-\rho)}{b-a}d\rho\right]$$

Ex. 6 Magnetic Flux & Magnetic Flux Density (8)

Find the total flux through the triangular circuit when it enters a uniform field **B** with a speed of **v**?

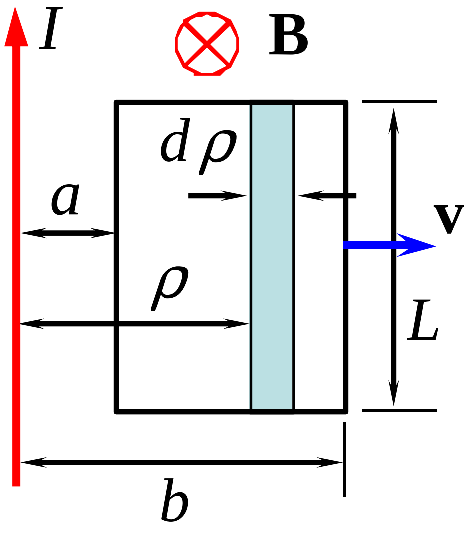
$$\left.\begin{aligned} &\Phi = \int_S \mathbf{B}.d\mathbf{S} = BS(x) \\ &S(x) = \frac{1}{2} l(x) x \\ &\frac{a}{h} = \frac{l(x)}{x} \rightarrow l(x) = \frac{a}{h} x \end{aligned}\right\} \rightarrow \left.\begin{aligned} &\Phi = B\frac{a}{2h} x^2 \\ &x = vt \end{aligned}\right\}$$



$$\rightarrow \Phi(t) = \left| \begin{array}{ll} \dfrac{aBv^2}{2h} t^2, & 0 \le t \le h/v \\ \dfrac{ahB}{2}, & t > h/v \end{array} \right.$$

VD7 Magnetic Flux & Magnetic Flux Density (9)

Find the total flux through the rectangular circuit if it moves with a speed of **v**?

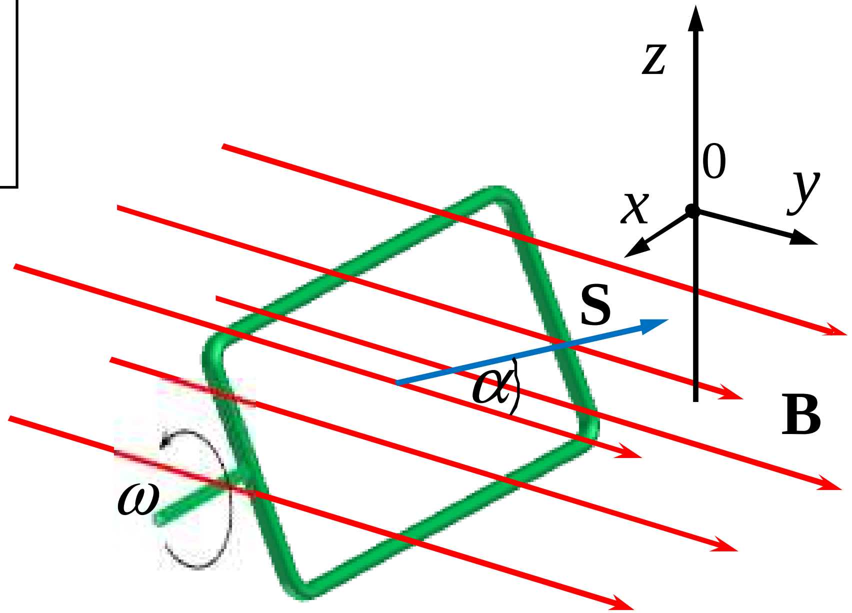


$$\left.\begin{aligned} \Phi &= \frac{\mu_0 IL}{2\pi}\ln\frac{b}{a} \\ a &= a_0 + vt \\ b &= b_0 + vt \end{aligned}\right\}$$

$$\rightarrow \Phi(t) = \frac{\mu_0 IL}{2\pi}\ln\frac{b_0 + vt}{a_0 + vt}$$

Ex. 8 Magnetic Flux & Magnetic Flux Density (10)

Find the total flux through the rectangular circuit having an area of S when it rotates with an angular speed of ω in a uniform field **B**?



$$\left.\begin{aligned}\Phi &= \int_S \mathbf{B}.d\mathbf{S} = \mathbf{B}.\mathbf{S} \\ &= BS\cos\alpha \\ \alpha &= \omega t\end{aligned}\right\}$$

$$\rightarrow \Phi = BS\cos\omega t$$

The Steady Magnetic Field

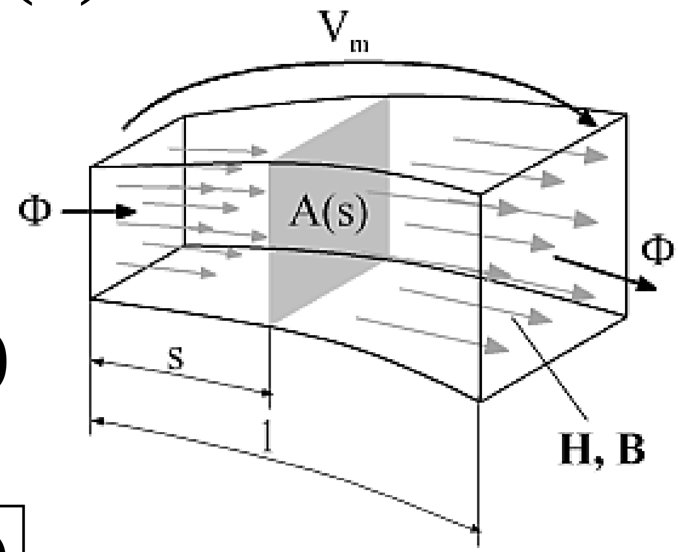
1. Biot – Savart Law
2. Ampere's Circuital Law
3. Curl
4. Stokes' Theorem
5. Magnetic Flux & Magnetic Flux Density
6. **Magnetic Potential**
7. Derivation of the Steady – Magnetic – Field Law

Magnetic Potential (1)

- The scalar magnetic potential V_m is defined by:

$$\mathbf{H} = -\nabla V_m$$

- $\nabla \times \mathbf{H} = \mathbf{J} \rightarrow \nabla \times \mathbf{H} = \mathbf{J} = \nabla \times (-\nabla V_m)$
- Curl of gradient of a scalar field is zero, therefore: $\boxed{\mathbf{H} = -\nabla V_m \quad (\mathbf{J} = 0)}$



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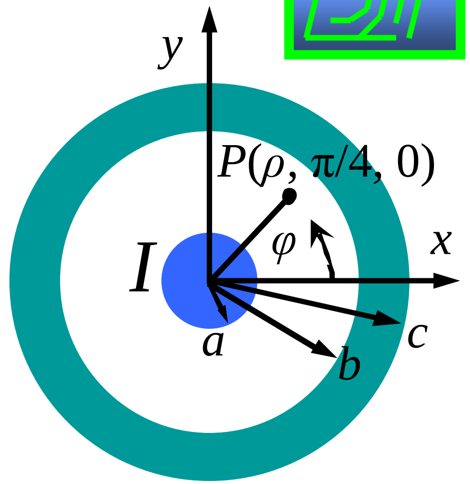
$$\nabla.\mathbf{B} = 0 \rightarrow \nabla.\mathbf{B} = \mu_0 \nabla.\mathbf{H} = 0$$

$$\rightarrow \mu_0 \nabla.(-\nabla V_m) = 0 \rightarrow \boxed{\nabla^2 V_m = 0 \quad (\mathbf{J} = 0)}$$

$$\boxed{V_{m,ab} = -\int_b^a \mathbf{H}.d\mathbf{L}}$$

Magnetic Potential (2)



$$a < \rho < b: \quad \mathbf{J} = 0$$

$$\left.\begin{array}{l} \mathbf{H} = \dfrac{I}{2\pi\rho}\mathbf{a}_\varphi \quad (a < \rho < b) \\ \mathbf{H} = -\nabla V_m \ (\mathbf{J} = 0) \end{array}\right\} \rightarrow \frac{I}{2\pi\rho} = -\nabla V_m\Big|_\varphi = -\frac{1}{\rho}\frac{\partial V_m}{\partial \varphi}$$

$$\rightarrow \frac{\partial V_m}{\partial \varphi} = -\frac{I}{2\pi} \rightarrow V_m = -\frac{I}{2\pi}\varphi$$

Assume $V_m\Big|_{\varphi=0} = 0 \rightarrow V_{mP} = \dfrac{I}{2\pi}\left(2n - \dfrac{1}{4}\right)\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$

$$= I\left(n - \frac{1}{8}\right) \quad (n = 0, \pm 1, \pm 2, \ldots)$$

Magnetic Potential (3)

- Definition of a vector magnetic potential **A**:

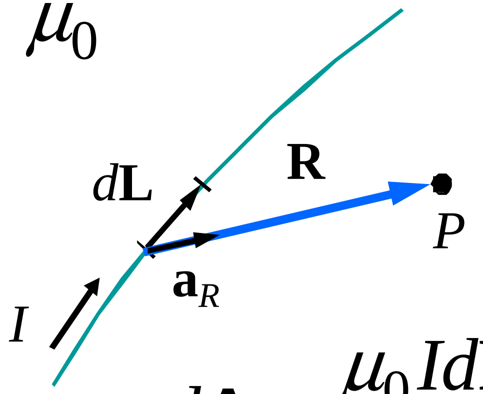
$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

- Unit: Wb/m

- $$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} = \frac{1}{\mu_0}\nabla \times \mathbf{A} \quad \rightarrow \nabla \times \mathbf{H} = \mathbf{J} = \frac{1}{\mu_0}\nabla \times \nabla \times \mathbf{A}$$

- **A** may be determined by:

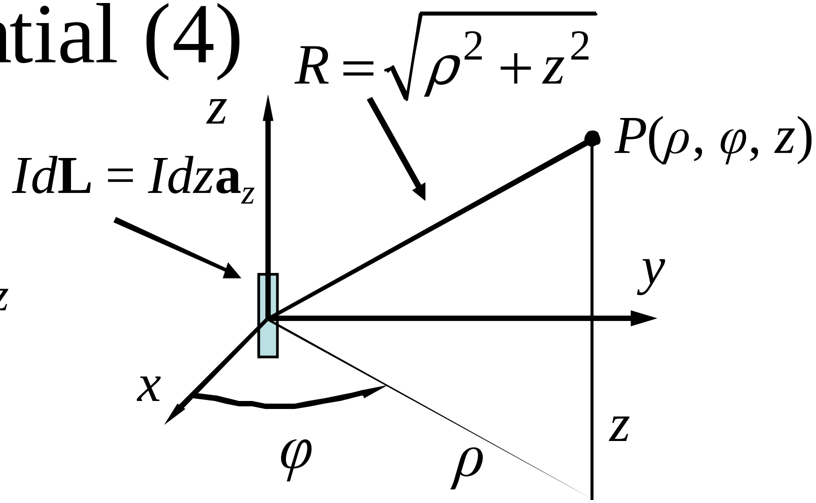
$$\boxed{\mathbf{A} = \oint \frac{\mu_0 I d\mathbf{L}}{4\pi R}}$$



$$d\mathbf{A} = \frac{\mu_0 I d\mathbf{L}}{4\pi R}$$

Magnetic Potential (4)

$$\left.\begin{aligned} d\mathbf{A} &= \frac{\mu_0 Id\mathbf{L}}{4\pi R} \\ d\mathbf{L} &= dz\mathbf{a}_z \\ R &= \sqrt{\rho^2 + z^2} \end{aligned}\right\} \rightarrow d\mathbf{A} = \frac{\mu_0 Idz}{4\pi\sqrt{\rho^2 + z^2}}\mathbf{a}_z$$



$$\left.\begin{aligned} &\rightarrow d\mathbf{A}_z = \frac{\mu_0 Idz\mathbf{a}_z}{4\pi\sqrt{\rho^2 + z^2}} \quad d\mathbf{A}_\varphi = 0 \quad d\mathbf{A}_\rho = 0 \\ &\mathbf{H} = \frac{1}{\mu_0}\nabla \times \mathbf{A} \end{aligned}\right\} \rightarrow d\mathbf{H} = \frac{1}{\mu_0}\nabla \times d\mathbf{A}$$

$$= \frac{1}{\mu_0}\left(-\frac{\partial dA_z}{\partial \rho}\right)\mathbf{a}_\varphi$$

$$\rightarrow d\mathbf{H} = \frac{Idz}{4\pi}\frac{\rho}{(\rho^2 + z^2)^{3/2}}\mathbf{a}_\varphi$$

Magnetic Potential (5)

$$d\mathbf{A} = \frac{\mu_0 I d\mathbf{L}}{4\pi R}$$

- In the case of current flow throughout a volume with a density **J**, then:

$$Id\mathbf{L} = \mathbf{J}dv$$

$$\rightarrow \mathbf{A} = \int_V \frac{\mu_0 \mathbf{J}dv}{4\pi R}$$

Magnetic Potential (6)

Ex. 1

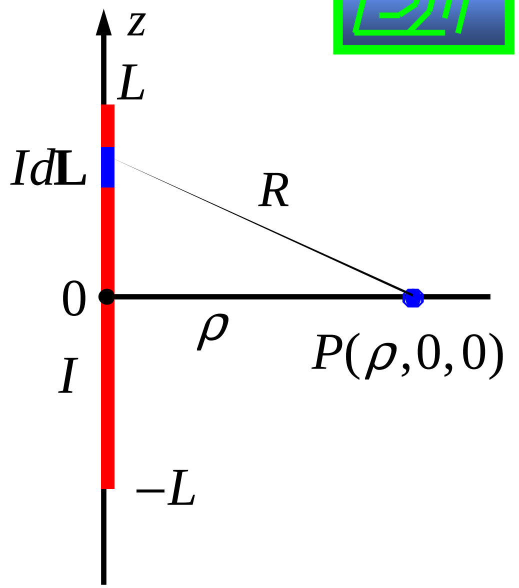
A very long, straight conductor lies along the *z* axis, carrying a uniform current *I* in the *z* direction. Find the magnetic potential difference between two points in space?

$$\left.\begin{aligned} V_{m,ab} &= -\int_b^a \mathbf{H}.d\mathbf{L} \\ d\mathbf{L} &= d\rho\mathbf{a}_\rho + \rho d\varphi\mathbf{a}_\varphi + dz\mathbf{a}_z \\ \mathbf{H} &= \frac{I}{2\pi\rho}\mathbf{a}_\varphi \end{aligned}\right\} \rightarrow V_{m,ab} = -\int_b^a \frac{I}{2\pi} d\varphi = \frac{I}{2\pi}(\varphi_b - \varphi_a)$$

Magnetic Potential (7)

Ex. 2

Find the vector magnetic potential in the plane bisecting a straight piece of thin wire of finite length $2L$ in free space.



$$\left.\begin{aligned} d\mathbf{A} &= \frac{\mu_0 I d\mathbf{L}}{4\pi R} \\ I d\mathbf{L} &= I dz' \mathbf{a}_z \\ R &= \sqrt{\rho^2 + (z')^2} \end{aligned}\right\}$$

$$\rightarrow \mathbf{A}(\rho,0,0) = \int_{z'=-L}^{L} \frac{\mu_0 I dz'}{4\pi\sqrt{\rho^2 + (z')^2}} \mathbf{a}_z \quad = \frac{\mu_0 I}{4\pi} \ln \frac{\sqrt{L^2 + \rho^2} + L}{\sqrt{L^2 + \rho^2} - L} \mathbf{a}_z$$

The Steady Magnetic Field

1. Biot – Savart Law
2. Ampere's Circuital Law
3. Curl
4. Stokes' Theorem
5. Magnetic Flux & Magnetic Flux Density
6. Magnetic Potential
7. **Derivation of the Steady – Magnetic – Field Law**

(1)

- Use formulae/definitions:

$$\mathbf{H} = \oint \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} \qquad \mathbf{B} = \mu_0 \mathbf{H} \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

- to show that

$$\mathbf{A} = \int_V \frac{\mu_0 \mathbf{J} dv}{4\pi R}$$

$$\boxed{\mathbf{A} = \int_V \frac{\mu_0 \mathbf{J} dv}{4\pi R} \rightarrow \mathbf{H} = \oint \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}}$$

(2)

$$\boxed{\mathbf{A} = \int_V \frac{\mu_0 \mathbf{J} dv}{4\pi R} \rightarrow \mathbf{H} = \oint \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}}$$

Current element at (x_1, y_1, z_1), $\mathbf{A}$ at (x_2, y_2, z_2) $\rightarrow \mathbf{A}_2 = \int_V \frac{\mu_0 \mathbf{J}_1 dv_1}{4\pi R_{12}}$

$$\left.\begin{array}{l} \mathbf{B} = \mu_0 \mathbf{H} \\ \mathbf{B} = \nabla \times \mathbf{A} \end{array}\right\} \rightarrow \mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{\nabla \times \mathbf{A}}{\mu_0}$$

$$\rightarrow \mathbf{H}_2 = \frac{\nabla_2 \times \mathbf{A}_2}{\mu_0} = \frac{\nabla_2}{\mu_0} \times \int_V \frac{\mu_0 \mathbf{J}_1 dv_1}{4\pi R_{12}} = \frac{1}{4\pi} \int_V \nabla_2 \times \frac{\mathbf{J}_1 dv_1}{R_{12}} = \frac{1}{4\pi} \int_V \left(\nabla_2 \times \frac{\mathbf{J}_1}{R_{12}} \right) dv_1$$

$$\nabla \times (S\mathbf{V}) = (\nabla S) \times \mathbf{V} + S(\nabla \times \mathbf{V})$$

$$\rightarrow H_2 = \frac{1}{4\pi} \int_V \left[\left(\nabla_2 \frac{1}{R_{12}} \right) \times \mathbf{J}_1 + \frac{1}{R_{12}} (\nabla_2 \times \mathbf{J}_1) \right] dv_1$$

(3)

$$\boxed{\mathbf{A} = \int_V \frac{\mu_0 \mathbf{J} dv}{4\pi R} \to \mathbf{H} = \oint \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}}$$

Current element at (x_1, y_1, z_1), **A** at (x_2, y_2, z_2)

$$\left.\begin{array}{r} \to H_2 = \dfrac{1}{4\pi}\displaystyle\int_V \left[\left(\nabla_2 \dfrac{1}{R_{12}}\right) \times \mathbf{J}_1 + \dfrac{1}{R_{12}}(\nabla_2 \times \mathbf{J}_1)\right] dv_1 \\ \nabla_2 \times \mathbf{J}_1 = 0 \end{array}\right\}$$

$$\left.\begin{array}{r} \to H_2 = \dfrac{1}{4\pi}\displaystyle\int_V \left[\left(\nabla_2 \dfrac{1}{R_{12}}\right) \times \mathbf{J}_1\right] dv_1 \\ R_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad \to \nabla_2 \dfrac{1}{R_{12}} = -\dfrac{\mathbf{R}_{12}}{R_{12}^3} = -\dfrac{\mathbf{a}_{R12}}{R_{12}^2} \end{array}\right\}$$

$$\to \mathbf{H}_2 = \frac{1}{4\pi}\int_V \frac{\mathbf{a}_{R12} \times \mathbf{J}_1}{R_{12}^2} dv_1 = \frac{1}{4\pi}\int_V \frac{\mathbf{J}_1 \times \mathbf{a}_{R12}}{R_{12}^2} dv_1$$

(4)

$$\boxed{\mathbf{A} = \int_V \frac{\mu_0 \mathbf{J} dv}{4\pi R} \rightarrow \mathbf{H} = \oint \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}}$$

Current element at (x_1, y_1, z_1), **A** at (x_2, y_2, z_2)

$$\left.\begin{aligned} &\rightarrow \mathbf{H}_2 = \frac{1}{4\pi} \int_V \frac{\mathbf{J}_1 \times \mathbf{a}_{R12}}{R_{12}^2} dv_1 \\ &\mathbf{J}_1 dv_1 = I_1 d\mathbf{L}_1 \end{aligned}\right\}$$

$$\rightarrow \boxed{\mathbf{H}_2 = \oint \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}}$$

(5)

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

$$\left.\begin{array}{l} \nabla \times \mathbf{H} = \mathbf{J} \\ \mathbf{B} = \mu_0 \mathbf{H} \\ \mathbf{B} = \nabla \times \mathbf{A} \end{array}\right\} \rightarrow \nabla \times \mathbf{H} = \nabla \times \frac{\mathbf{B}}{\mu_0} = \frac{1}{\mu_0} \nabla \times \nabla \times \mathbf{A}$$

$$\left.\begin{array}{r} \nabla \times \mathbf{H} = \frac{1}{\mu_0} \nabla \times \nabla \times \mathbf{A} \\ \nabla \times \nabla \times \mathbf{A} = \nabla(\nabla . \mathbf{A}) - \nabla^2 \mathbf{A} \\ \nabla^2 \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z \end{array}\right\}$$

$$\rightarrow \nabla \times \mathbf{H} = \frac{\nabla(\nabla . \mathbf{A}) - \nabla^2 \mathbf{A}}{\mu_0}$$

(6)

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

$$\left.\begin{array}{l} \rightarrow \nabla \times \mathbf{H} = \dfrac{\nabla(\nabla.\mathbf{A}) - \nabla^2 \mathbf{A}}{\mu_0} \\ \mathbf{A}_2 = \displaystyle\int_V \dfrac{\mu_0 \mathbf{J}_1 dv_1}{4\pi R_{12}} \\ \nabla.(S\mathbf{A}) = \mathbf{A}.(\nabla S) + S(\nabla.\mathbf{A}) \end{array}\right\}$$

$$\rightarrow \nabla_2 .\mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_V \left[\mathbf{J}_1 . \left(\nabla_2 \frac{1}{R_{12}} \right) + \frac{1}{R_{12}} (\nabla_2 .\mathbf{J}_1) \right] dv_1$$

(7)

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

$$\left.\begin{aligned}
&\rightarrow \nabla_2 .\mathbf{A}_2 = \frac{\mu_0}{4\pi}\int_V \left[\mathbf{J}_1 .\left(\nabla_2 \frac{1}{R_{12}} \right) + \frac{1}{R_{12}} (\nabla_2 .\mathbf{J}_1) \right] dv_1 \\
&\int_V \frac{1}{R_{12}} (\nabla_2 .\mathbf{J}_1) dv_1 = 0 \\
&\nabla_1 \frac{1}{R_{12}} = \frac{\mathbf{R}_{12}}{R_{12}^3} = -\nabla_2 \frac{1}{R_{12}}
\end{aligned}\right\}$$

$$\rightarrow \nabla_2 .\mathbf{A}_2 = \frac{\mu_0}{4\pi}\int_V \left[-\mathbf{J}_1 .\left(\nabla_1 \frac{1}{R_{12}} \right) \right] dv_1$$

(8)

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

$$\left.\begin{aligned} &\rightarrow \nabla_2 . \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_V \left[-\mathbf{J}_1 . \left(\nabla_1 \frac{1}{R_{12}} \right) \right] dv_1 \\ &\nabla.(S\mathbf{A}) = \mathbf{A}.(\nabla S) + S(\nabla . \mathbf{A}) \end{aligned}\right\}$$

$$\rightarrow \nabla_2 . \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_V \left[\frac{1}{R_{12}} (\nabla_1 . \mathbf{J}_1) - \nabla_1 . (\frac{\mathbf{J}_1}{R_{12}}) \right] dv_1$$

(9)

$$\boxed{\nabla\times\mathbf{H}=\mathbf{J}}$$

$$\rightarrow\nabla_2.\mathbf{A}_2=\frac{\mu_0}{4\pi}\int_V\left[\frac{1}{R_{12}}(\nabla_1.\mathbf{J}_1)-\nabla_1.(\frac{\mathbf{J}_1}{R_{12}})\right]dv_1$$

$$\nabla_1.\mathbf{J}_1=-\frac{\partial\rho_v}{\partial t}=0$$

$$\oint_S\mathbf{J}.d\mathbf{S}=\int_V\nabla.\mathbf{J}dv$$

$$\rightarrow\nabla_2.\mathbf{A}_2=-\frac{\mu_0}{4\pi}\oint_{S_1}\frac{\mathbf{J}_1}{R_{12}}d\mathbf{S}_1=0$$

(10)

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

$$\left.\begin{array}{l} \rightarrow \nabla \times \mathbf{H} = \dfrac{\nabla(\nabla.\mathbf{A}) - \nabla^2 \mathbf{A}}{\mu_0} \\ \nabla.\mathbf{A} = 0 \\ \left.\begin{array}{l} A_x = \int_V \dfrac{\mu_0 J_x dv}{4\pi R} \\ V = \int_V \dfrac{\rho_v dv}{4\pi\varepsilon_0 R} \\ \nabla^2 V = -\dfrac{\rho_v}{\varepsilon_0} \end{array}\right\} \rightarrow \left\{\begin{array}{l} \nabla^2 A_x = -\mu_0 J_x \\ \nabla^2 A_y = -\mu_0 J_y \\ \nabla^2 A_z = -\mu_0 J_z \end{array}\right. \rightarrow \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \end{array}\right\}$$

$$\rightarrow \boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

$$
\begin{tikzcd}
Q \arrow[r] \arrow[dd] & \mathbf{F} = \dfrac{Q_1 Q_2}{4\pi\varepsilon R^2}\mathbf{a}_R \arrow[r] \arrow[d] & \mathbf{E} = \dfrac{Q}{4\pi\varepsilon R^2}\mathbf{a}_R \arrow[r] & \mathbf{D} = \varepsilon\mathbf{E} \\
 & W = -Q\displaystyle\int \mathbf{E}.d\mathbf{L} \arrow[r] & V = -\displaystyle\int \mathbf{E}.d\mathbf{L} \arrow[r] & C = \dfrac{Q}{V} \\
I = \dfrac{dQ}{dt} \arrow[r] \arrow[d] & R = \dfrac{V}{I} & & \\
\mathbf{H} = \dfrac{I}{2\pi\rho}\mathbf{a}_\varphi \arrow[r] & \mathbf{B} = \mu\mathbf{H} \arrow[r] & \Phi = \displaystyle\int \mathbf{B}.d\mathbf{S} &
\end{tikzcd}
$$