Fundamentals of Electric Circuits

DC Circuits

Chapter 8. Second Order Circuits

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8.2. Finding initial and final values
8.3. The source-free series / parallel RLC circuit
8.4. Step response of a series / parallel RLC circuit
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8.1. Introduction

+ Previous chapter: considered circuits with a single storage element (C or L) → first-order circuits

+ In this chapter: consider circuits containing two (independent) storage elements → *second-order circuits*

+ A second-order circuit:

→ characterized by a second-order differential equation

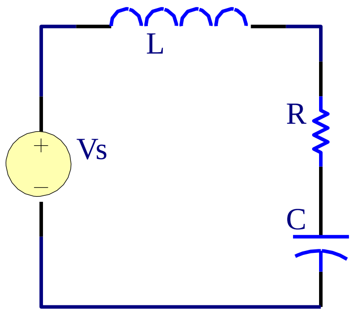
→ consists of resistors and the equivalence of 02 energy storage elements

+ Second-order circuit classification:

→ Two storage elements of different type: *L* and *C*

→ Two storage elements of one type: *L* or *C*

→ An op amp circuit with 02 storage elements



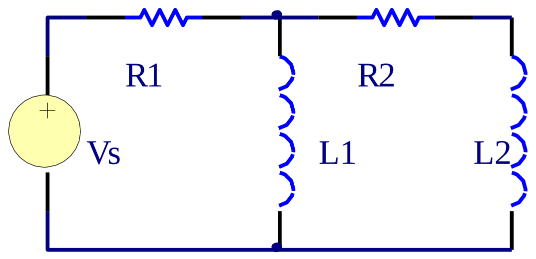
Series RLC circuit

Second Order Circuits

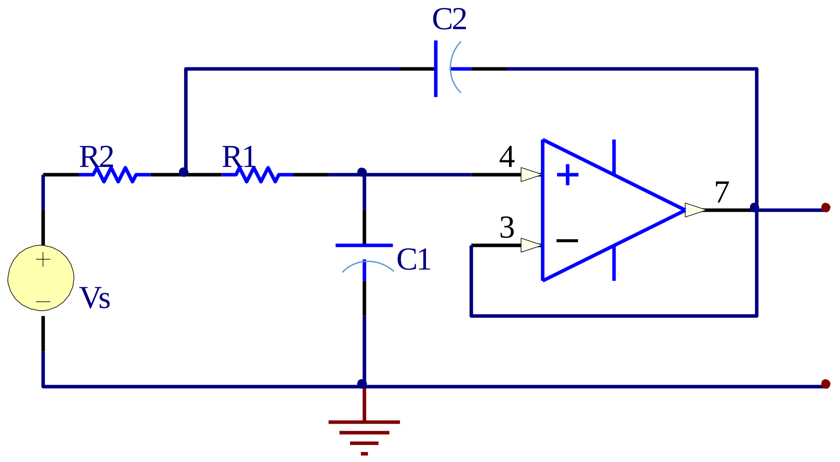
8.1. Introduction

+ Analysis of second-order circuit:

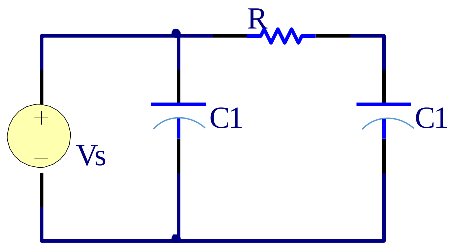
- First, consider circuits that are excited by the initial conditions of the storage elements → source free circuits give natural responses
- Second, with independent sources, circuits will give both the nature response and the forced response



RL circuit



Op amp with 2 storage element



RC circuit

8.2. Finding initial and final values

+ The major problem: finding the initial and final conditions on circuits variables

→ Easy to get the initial and final values of v and i

→ Difficult to find the initial values of their derivatives: dv/dt, di/dt

+ Key points in determining the initial conditions:

- First, carefully handle the *polarity of voltage* $v_C(t)$, and the directions of $i_L(t)$
- Second, keep in mind that $v_C(t)$, $i_L(t)$ *are always continuous*

$$\boxed{v_c(+0) = v_c(-0), i_L(+0) = i_L(-0)}$$

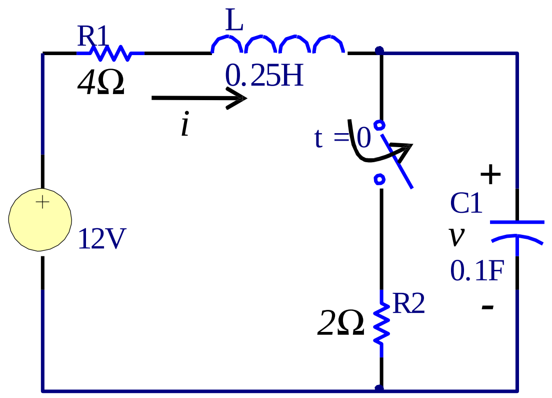
t = 0: the time that the switching event takes place

t = -0: the time just before a switching event

t = +0: the time just after a switching event

8.2. Finding initial and final values

+ Example 1: The switch has been closed for a long time, and opens at t = 0. Find $i(+0), v(+0), di(+0)/dt, dv(+0)/dt, i(\infty), v(\infty)$

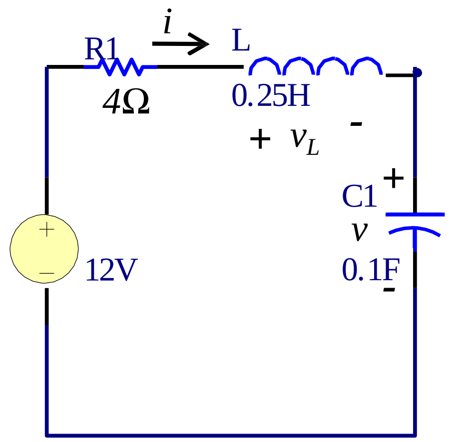


For t = -0: The circuit has reached DC steady state → L acts like a short circuit, C acts like an open circuit

$$i_L(-0) = \frac{E}{R_1 + R_2} = 2A \qquad v_c(-0) = R_2 i_L(-0) = 2.2 = 4V$$

As $v_C(t)$ and $i_L(t)$ cannot change abruptly, we have:

$$v_c(+0) = v_c(-0) = 4V \qquad i_L(+0) = i_L(-0) = 2A$$



8.2. Finding initial and final values

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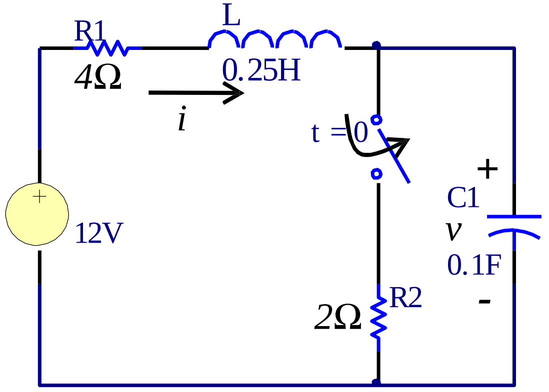
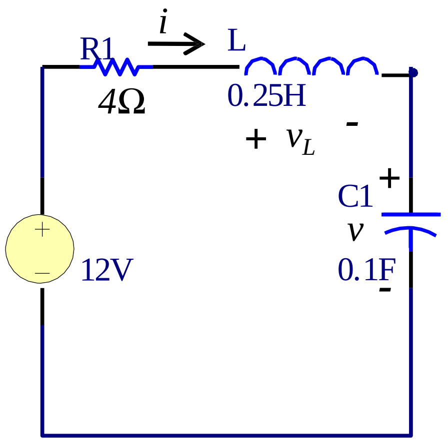


For t = +0:

$$i_c = C\frac{dv_c}{dt} \rightarrow \frac{dv_c(+0)}{dt} = \frac{i_c(+0)}{C} = \frac{2}{0.1} = 20V/s$$

For t > 0:



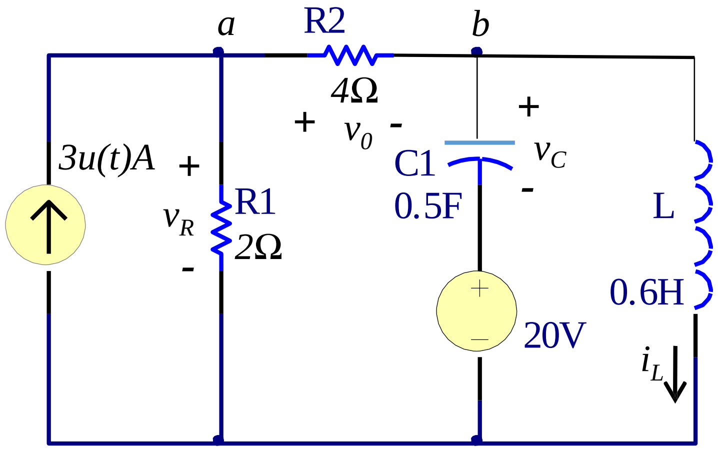
$$R_1 i + L\frac{di}{dt} + v_c = E \rightarrow R_1 i_L(+0) + L\frac{di_L(+0)}{dt} + v_c(+0) = E$$

$$L\frac{di_L(+0)}{dt} = E - R_1 i_L(+0) - v_c(+0) = 12 - 4.2 - 4 = 0V \rightarrow \frac{di_L(+0)}{dt} = 0A/s$$

For t → ∞: new DC steady state

$$i(\infty) = 0A \qquad v_c(\infty) = E = 12V$$

8.2. Finding initial and final values



+ Example 2: Find $i_L(+0), v_C(+0), v_R(+0), di_L(+0)/dt,\ dv_C(+0)/dt, i_L(\infty), v_C(\infty), v_R(\infty)$

For *t* = < 0 : *3.u(t) =0*

At *t* = *-0 (old DC steady state):* $i_L(-0)=0, v_R(-0)=0, v_C(-0)=-E=-20V$

For *t* > *0 (new DC steady state):*

$3u(t)=3$ $\boxed{i_L(+0)=i_L(-0)=0A,}\boxed{v_C(+0)=v_C(-0)=-20V}$

KCL at node *a:* $I=\dfrac{v_R(+0)}{R_1}+\dfrac{v_0(+0)}{R_2}$

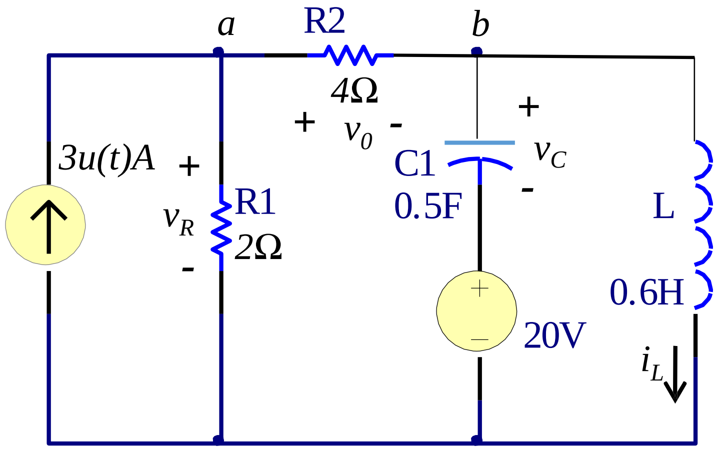
KVL to the middle mesh: $-v_R(+0)+v_0(+0)+v_c(+0)+E=0\rightarrow v_R(+0)=v_0(+0)$

$\boxed{v_R(+0)=v_0(+0)=4V}$

Applying KVL to the right mesh: $v_L(+0)=v_c(+0)+E=-20+20=0V\rightarrow \boxed{\dfrac{di_L(+0)}{dt}=0A/s}$

8.2. Finding initial and final values

+ Example 2: Find $i_L(+0), v_C(+0), v_R(+0), di_L(+0)/dt,\ dv_C(+0)/dt, i_L(\infty), v_C(\infty), v_R(\infty)$



KCL at node *b:*

$$\frac{v_0(+0)}{R_2} = i_c(+0) + i_L(+0) \rightarrow i_c(+0) = \frac{v_0(+0)}{R_2} - i_L(+0) = \frac{4}{4} - 0 = 1A$$

$$\rightarrow \boxed{\frac{dv_c(+0)}{dt} = \frac{i_c(+0)}{C} = \frac{1}{0.5} = 2V/s}$$

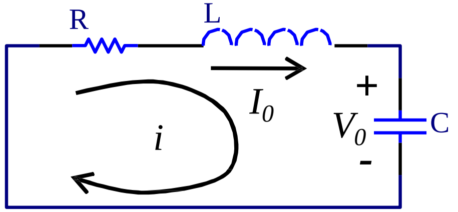
For new steady state:

$$I = \frac{v_a(\infty)}{R_1} + \frac{v_a(\infty)}{R_2} \rightarrow v_a(\infty) = \frac{R_1 R_2}{R_1 + R_2} I = \frac{2.4}{2+4} 3 = 4V$$

$$\rightarrow \boxed{v_R(\infty) = v_a(\infty) = 4V} \qquad \boxed{i_L(\infty) = \frac{v_a(\infty)}{R_2} = \frac{4}{4} = 1A} \qquad \boxed{v_c(\infty) = -E = -20V}$$

8.3. The source-free series / parallel RLC circuit

8.3.1. The source-free series RLC circuit



+ Understanding of the **natural response** of the series RLC circuit: → necessary background for future studies in *filter design* and *communications network*

+ Consider case: The series RLC circuit that is excited by the energy initially stored V_0, and I_0

At $t = 0$: $$v_c(0) = \frac{1}{C}\int_{-\infty}^{0} i(t)dt = V_0 \qquad i(0) = I_0$$

For t ≥ 0, applying KVL to the loop: $$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{0} idt = 0 \rightarrow \boxed{\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0}$$

+ To solve a second-order differential equation → it requires 2 initial conditions

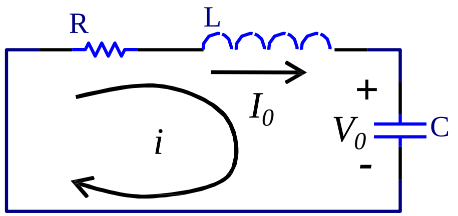
→ $i(+0)$ and $i'(+0)$, or

→ $i(+0)$ and $v(+0)$

8.3. The source-free series / parallel RLC circuit

8.3.1. The source-free series RLC circuit

+ Find $i'(+0)$: $Ri(0)+L\frac{di(0)}{dt}+V_0=0\rightarrow\frac{di(0)}{dt}=i'(0)=-\frac{1}{L}(RI_0+V_0)$



+ Solution of the second-order equation: $i(t)=Ae^{st}$

+ Substituting the solution into the equation, we have:

$$As^2e^{st}+\frac{AR}{L}se^{st}+\frac{A}{LC}e^{st}=0\rightarrow Ae^{st}\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)=0$$

$$\begin{cases} s_1=-\frac{R}{2L}+\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}=-\alpha+\sqrt{\alpha^2-\omega_0^2} & \alpha=\frac{R}{2L} \\ s_2=-\frac{R}{2L}-\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}=-\alpha-\sqrt{\alpha^2-\omega_0^2} & \omega_0=\frac{1}{\sqrt{LC}} \end{cases}$$

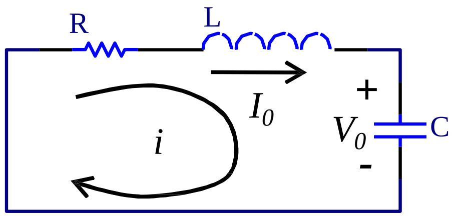
s_1 and s_2 are called *natural frequencies* [Np/s]

$\boldsymbol{\omega_0}$ *resonant frequency (un-damped natural frequency, or damping factor)*

8.3. The source-free series / parallel RLC circuit

8.3.1. The source-free series RLC circuit

$$\begin{cases} s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} & \alpha = \dfrac{R}{2L} \\ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} & \omega_0 = \dfrac{1}{\sqrt{LC}} \end{cases}$$



+ There are three types of solutions:

If $\alpha > \omega_0$: over damped case

If $\alpha = \omega_0$: critically damped case

If $\alpha < \omega_0$: under damped case

+ 2 values of *s:* → 2 possible solutions for *i:* $i_1 = A_1 e^{s_1 t}, i_2 = A_2 e^{s_2 t} \rightarrow \boxed{i = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$

A_1, A_2 → determined from the initial values $i(0)$ and $di(0)/dt$

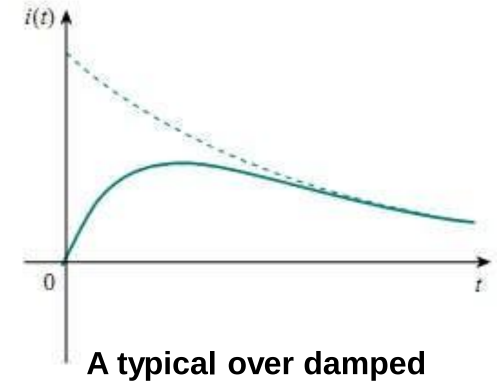
8.3. The source-free series / parallel RLC circuit

8.3.1. The source-free series RLC circuit

+ *Over damped* case $\alpha > \omega_0$ → s_1 and s_2 are *negative and real*

$$i = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

→ Response *decays* and *approaches* zero as *t* increases

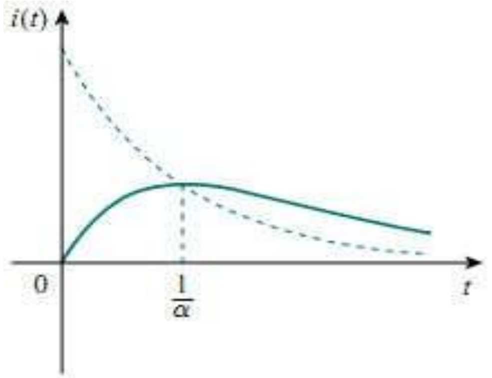


A typical over damped response

+ *Critically damped* case $\alpha = \omega_0$

$$i = A_1 t e^{\alpha t} + A_2 e^{\alpha t} = \left(A_1 t + A_2\right) e^{\alpha t}$$

→ Response reaches a maximum value at $t = 1/\alpha$, and then *decays* all the way to zero



A typical critically damped response

8.3. The source-free series / parallel RLC circuit

8.3.1. The source-free series RLC circuit

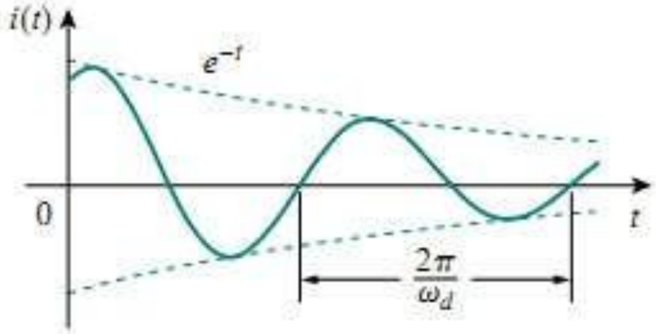
+ *Under damped* case $\alpha < \omega_0$

$$\begin{cases} s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha + j\omega_d \\ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha - j\omega_d \end{cases}$$

ω_d: damped natural frequency

ω_0: un-damped natural frequency

$$\boxed{i = e^{-\alpha t}\left(B_1 \cos \omega_d t + B_2 \sin \omega_d t\right)}$$



A typical under damped response

→ The natural response is *exponentially damped* and *oscillatory* in nature

→ The response has a time constant of $1/\alpha$ and a period of $T = 2\pi/\omega_d$

8.3. The source-free series / parallel RLC circuit

8.3.1. The source-free series RLC circuit

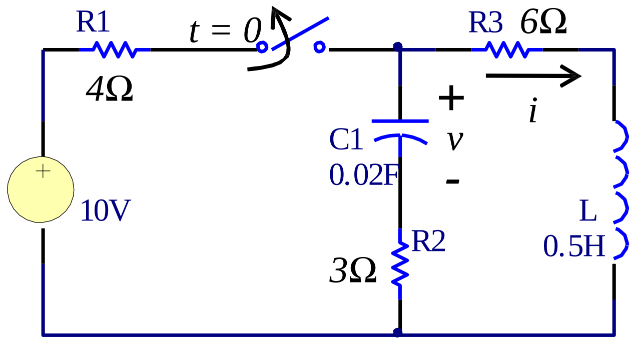
+ Notes:

- *Damping effect* → due to the presence of R
- *Damping factor* α → the rate at which the response is damped
  - $\alpha = 0$: having LC circuit with $1/LC$ as the un-damped natural frequency
  - $\alpha < \omega_0$: response is not only damped but also oscillatory
- *By adjusting R*, → response may be made un-damped: over damped, critically damped, or under damped
- *Oscillatory response* → possible due to the presence of L, C: They allows the flow of energy back and forth
- *Critically damped* case → the *borderline* between the under damped and over damped cases

8.3. The source-free series / parallel RLC circuit

8.3.1. The source-free series RLC circuit

+ Example 3: Find $i(t)$ in the circuit. Assume that the circuit has reached steady state at $t = -0$



For $t < 0$: $$i(0) = \frac{E}{R_1 + R_3} = 1A \qquad v(0) = R_3 i(0) = 6V$$

For $t > 0$: Source – free series RLC circuit

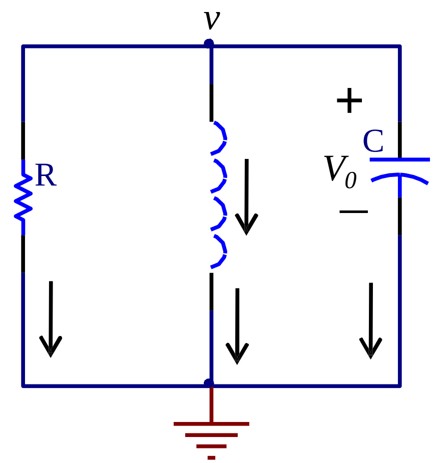
$$\alpha = \frac{R_{eq}}{2L} = \frac{R_2 + R_3}{2L} = 9 \qquad \omega_0 = \frac{1}{\sqrt{LC}} = 10 \qquad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -9 \pm j4.359$$

Response is under-damped:

$$i(t) = e^{-9t}\left(A_1 \cos 4.359t + A_2 \sin 4.359t\right)$$

8.3. The source-free series / parallel RLC circuit

8.3.2. The source-free parallel RLC circuit



Apply node voltage method:

$$\frac{v}{R}+\frac{1}{L}\int_{-\infty}^{t} v\,dt + C\frac{dv}{dt}=0 \rightarrow \boxed{\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0}$$

Characteristic equation: $\rightarrow s^2+\frac{1}{RC}s+\frac{1}{LC}=0$

$$s_{1,2}=-\frac{1}{2RC}\pm\sqrt{\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2} \qquad \alpha=\frac{1}{2RC} \quad \omega_0=\frac{1}{\sqrt{LC}}$$

Over damped $(\alpha>\omega_0)$: $\boxed{v(t)=A_1e^{s_1t}+A_2e^{s_2t}}$

A_i determined from the initial conditions:

v(0), dv(0)/dt

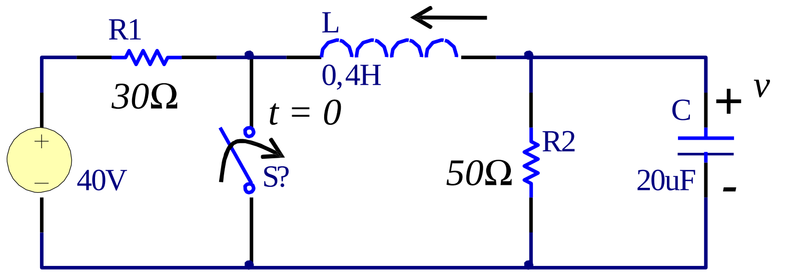
Critically damped $(\alpha=\omega_0)$: $\boxed{v(t)=(A_1+A_2t)e^{-\alpha t}}$

Under damped $(\alpha<\omega_0)$: $\boxed{v(t)=e^{-\alpha t}(A_1\cos\omega_d t+A_2\sin\omega_d t)}$ $\quad s_{1,2}=-\alpha\pm j\omega_d, \omega_d=\sqrt{\omega_0^2-\alpha^2}$

8.3. The source-free series / parallel RLC circuit

8.3.2. The source-free parallel RLC circuit

+ Example 4: Find v(t) for t > 0 in the RLC circuit



When $t < 0$: The switch is opened

$$v(0)=\frac{R_2}{R_1+R_2}E=\frac{50}{30+50}40=25V \qquad i(0)=-\frac{E}{R_1+R_2}=\frac{40}{30+50}=-0{,}5A$$

At $t = 0$, applying the KCL to the parallel RLC circuit:

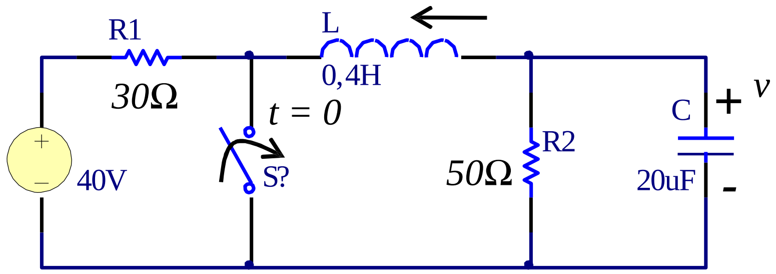
$$\frac{v(0)}{R_2}+\frac{1}{L}\int_{-\infty}^{0}vdt+C\frac{dv(0)}{dt}=0\rightarrow\frac{v(0)}{R_2}+i(0)+C\frac{dv(0)}{dt}=0$$

$$\rightarrow\frac{dv(0)}{dt}=-\frac{v(0)+R_2i(0)}{R_2C}=-\frac{25-50\times0.5}{50\times20.10^{-6}}=0V/s$$

8.3. The source-free series / parallel RLC circuit

8.3.2. The source-free parallel RLC circuit

+ Example 4: Find v(t) for t > 0 in the RLC circuit



When $t > 0$: The switch is closed

$$\alpha = \frac{1}{2R_2C} = 500 \qquad \omega_0 = \frac{1}{\sqrt{LC}} = 354$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -854 \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -146 \quad \rightarrow v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

At $t = 0$:

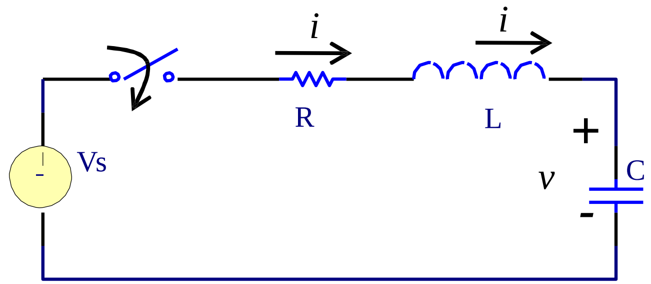
$$\begin{cases} v(0) = A_1 + A_2 = 25 \\ \dfrac{dv(0)}{dt} = -854A_1 - 146A_2 = 0 \end{cases} \rightarrow \begin{cases} A_1 = -5.16 \\ A_2 = 30.16 \end{cases}$$

The complete solution: $v(t) = -5.16e^{-854t} + 30.16e^{-146t}$

8.4. Step response of a series / parallel RLC circuit

8.4.1. Step response of a series RLC circuit

+ For $t > 0$: Applying KVL around the loop



$$\begin{cases} L\dfrac{di}{dt}+Ri+v=V_s \\ i=C\dfrac{dv}{dt} \end{cases} \rightarrow \frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{v}{LC}=\frac{V_s}{LC} \rightarrow v(t)=v_n(t)+v_f(t)$$

$v_n(t)$: natural response , $v_f(t)$: forced response

Over damped: $$\boxed{v(t)=V_s+A_1e^{s_1t}+A_2e^{s_2t}}$$

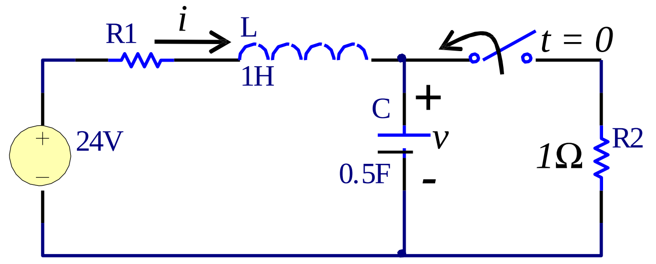
Critically damped: $$\boxed{v(t)=V_s+(A_1+A_2t)e^{-\alpha t}}$$

Under damped: $$\boxed{v(t)=V_s+(A_1\cos\omega_dt+A_2\sin\omega_dt)e^{-\alpha t}}$$

A_1, A_2 → determined from the initial values $v(0)$ and $dv(0)/dt$

8.4. Step response of a series / parallel RLC circuit

8.4.1. Step response of a series RLC circuit



+ Example 5: Find $v(t)$ and $i(t)$ for $t > 0$ in the case of the different values of $R_1 = 5\Omega, 4\Omega, 1\Omega$

- $R_1 = 5\Omega$

For $t < 0$: $i(0) = \dfrac{E}{R_1 + R_2} = \dfrac{24}{5+1} = 4A$ $\quad v(0) = R_2 i(0) = 4V$ $\quad \alpha = \dfrac{R_1}{2L} = 2.5$ $\quad \omega_0 = \dfrac{1}{\sqrt{LC}} = 2$

$$\rightarrow s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -1 \quad \rightarrow s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -4$$

For $t > 0$: $v(t) = E + A_1 e^{s_1 t} + A_2 e^{s_2 t} = 24 + A_1 e^{-t} + A_2 e^{-4t}$

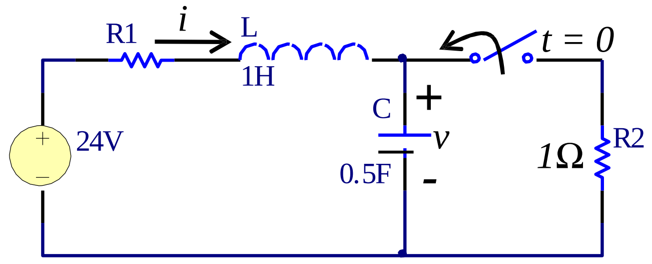
At $t = 0$: $v(0) = 24 + A_1 + A_2 = 4 \rightarrow \boxed{A_1 + A_2 = -20}$

$$i(t) = C\frac{dv}{dt} = C\left(-A_1 e^{-t} - 4A_2 e^{-4t}\right) \rightarrow i(0) = \boxed{C\left(-A_1 - 4A_2\right) = 4}$$

$$\rightarrow v(t) = 24 + \frac{4}{3}\left(-16e^{-t} + e^{-4t}\right)V \qquad \rightarrow i(t) = C\frac{dv}{dt} = 0.5\frac{4}{3}\left(16e^{-t} - 4e^{-4t}\right) = \frac{8}{3}\left(4e^{-t} - e^{-4t}\right)A$$

8.4. Step response of a series / parallel RLC circuit

8.4.1. Step response of a series RLC circuit



+ Example 5: Find $v(t)$ and $i(t)$ for $t > 0$ in the case of the different values of $R_1 = 5\Omega,\ 4\Omega,\ 1\Omega$

- $R_1 = 4\Omega$

For $t < 0$: $i(0) = \dfrac{E}{R_1 + R_2} = \dfrac{24}{4+1} = 4.5A \quad v(0) = R_2 i(0) = 4.5V \quad \alpha = \dfrac{R_1}{2L} = 2 \quad \omega_0 = \dfrac{1}{\sqrt{LC}} = 2$

$$\rightarrow s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -2 \quad \rightarrow s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -2$$

For $t > 0$: $\quad v(t) = E + (A_1 + A_2 t)e^{\alpha t} = 24 + (A_1 + A_2 t)e^{-2t}$

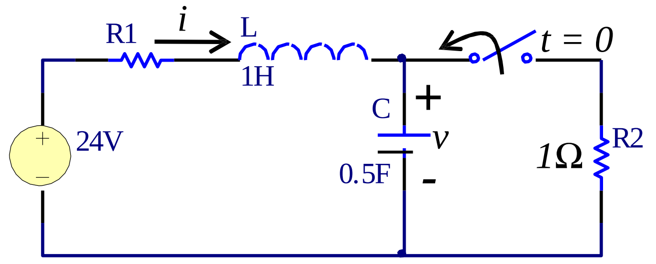
At $t = 0$: $\quad v(0) = 24 + A_1 = 4.5 \rightarrow \boxed{A_1 = -19.5}$

$$i(t) = C\frac{dv}{dt} = C(-2A_1 - 2tA_2 + A_2)e^{-2t} \rightarrow i(0) = C(-2A_1 + A_2) = 4.5 \rightarrow \boxed{A_2 = 57}$$

$$\rightarrow v(t) = 24 + (-19.5 + 57t)e^{-2t}V \quad \rightarrow i(t) = C\frac{dv}{dt} = 0.5 \times 2\left(19.5 - 57t + \frac{57}{2}\right)e^{-2t} = (48 - 57t)e^{-2t}A$$

8.4. Step response of a series / parallel RLC circuit

8.4.1. Step response of a series RLC circuit



+ Example 5: Find *v(t)* and *i(t)* for *t > 0* in the case of the different values of $R_1 = 5\Omega, 4\Omega, 1\Omega$

- $R_1 = 1\Omega$

For *t < 0*: $i(0) = \dfrac{E}{R_1 + R_2} = \dfrac{24}{1+1} = 12A$ $\quad v(0) = R_2 i(0) = 12V$ $\quad \alpha = \dfrac{R_1}{2L} = 0.5$ $\quad \omega_0 = \dfrac{1}{\sqrt{LC}} = 2$

$$\to s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -0.5 \pm j1.936$$

For *t > 0*: $v(t) = 24 + \left(A_1 \cos 1.936t + A_2 \sin 1.936t\right)e^{-0.5t} V$

At *t = 0*: $v(0) = 24 + A_1 = 12 \to \boxed{A_1 = -12}$

$$\frac{dv}{dt} = \left(-1.936 A_1 \sin 1.936t + 1.936 A_2 \cos 1.936t\right)e^{-0.5t} - 0.5e^{-0.5t}\left(A_1 \cos 1.936t + A_2 \sin 1.936t\right)$$

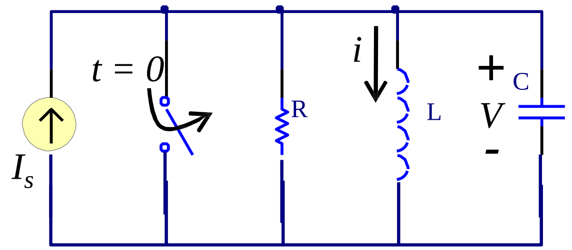
$$\to \frac{dv(0)}{dt} = 1.936 A_2 - 0.5 A_1 = \frac{i(0)}{C} = 24 \to \boxed{A_2 = 9.3} \qquad v(t) = 24 + \left(-12\cos 1.936t + 9.3 \sin 1.936t\right)e^{-0.5t} V$$

$$\to i(t) = C\frac{dv}{dt} = \left(9.291 \sin 1.936t + 12.002 \cos 1.936t\right)e^{-0.5t} A$$

8.4. Step response of a series / parallel RLC circuit

8.4.2. Step response of a parallel RLC circuit

+ Applying KCL at the top node for t > 0:



$$\left.\begin{array}{l} \frac{v}{R}+i+C\frac{dv}{dt}=I_s \\ v=L\frac{di}{dt} \end{array}\right\} \rightarrow \frac{d^2i}{dt^2}+\frac{1}{RC}\frac{di}{dt}+\frac{i}{RC}=\frac{I_s}{LC}$$

$\rightarrow i(t)=i_n(t)+i_f(t)$ $\quad$ $i_n(t)$ - Natural response $\quad$ $i_f(t)$ - Forced response

Over damped: $\boxed{i(t)=I_s+A_1e^{s_1t}+A_2e^{s_2t}}$

Critically damped: $\boxed{i(t)=I_s+(A_1+A_2t)e^{-\alpha t}}$

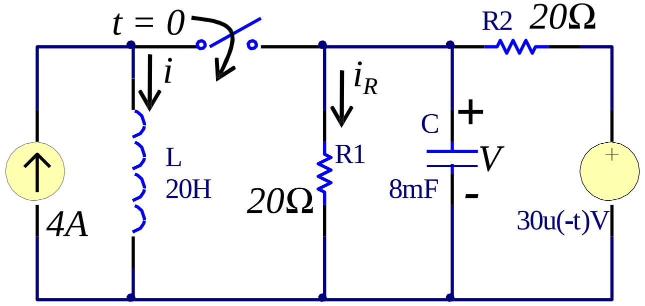
Under damped: $\boxed{i(t)=I_s+(A_1\cos\omega_d t+A_2\sin\omega_d t)e^{-\alpha t}}$

A_1, A_2 → determined from the initial values $i(0)$ and $di(0)/dt$

8.4. Step response of a series / parallel RLC circuit

8.4.2. Step response of a parallel RLC circuit

+ Example 6: Find $i(t)$, $i_R(t)$ for $t > 0$



For $t < 0$:

$$i(0) = 4A$$

$$v(0) = \frac{E}{R_1 + R_2} R_1 = \frac{30}{40} 20 = 15V$$

At $t = 0$:

$$v(0) = L\frac{di(0)}{dt} \rightarrow \frac{di(0)}{dt} = \frac{v(0)}{L} = \frac{15}{20} = 0.75A/s$$

For $t > 0$: parallel RLC circuit with current source

$$R = \frac{R_1 R_2}{R_1 + R_2} = 10\Omega \qquad \alpha = \frac{1}{2RC} = 6.25 \qquad \omega_0 = \frac{1}{\sqrt{LC}} = 2.5 \qquad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \rightarrow s_1 = -11.978, s_2 = -0.522$$

$$i(t) = I_f + A_1 e^{-11.978t} + A_2 e^{-0.522t}$$

At $t = 0$:

$$\begin{cases} i(0) = 4 \\ \dfrac{di(0)}{dt} = 0.75 \end{cases} \rightarrow \begin{cases} A_1 + A_2 + 4 = 4 \\ -11.978A_1 - 0.522A_2 = 0.75 \end{cases} \rightarrow \begin{cases} A_1 = -0.0655 \\ A_2 = 0.0655 \end{cases}$$

$$\rightarrow i(t) = 4 - 0.0655e^{-11.978t} + 0.0655e^{-0.522t} A \qquad i_R(t) = \frac{u_L}{R_1} = \frac{1}{R_1} L\frac{di}{dt} = 0.785e^{-11.978t} - 0.0342e^{-0.522t} A$$

8.5. General second order circuits

+ Give a second order circuit: → the step response $x(t)$ (current or voltage) can be determined by taking the following 5 steps

→ Determine the initial conditions $x(0)$ and $dx(0)/dt$ and the final value $x(\infty)$

→ Find the natural response $x_n(t)$ (with 2 unknown constants) by turning off independent sources and applying KCL and KVL

→ Obtain the forced response as: $x_f(t) = x(\infty)$

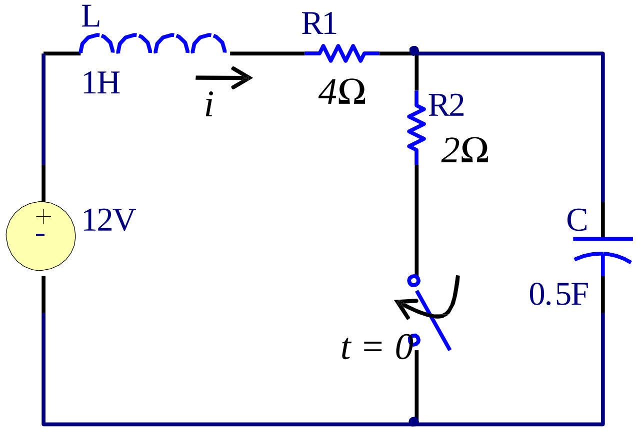
→ Otain the total response: sum of the natural response and forced response

$$x(t) = x_n(t) + x_f(t)$$

→ Determine 2 unknown constants by imposing the initial conditions $x(0)$ and $dx(0)/dt$

8.5. General second order circuits

+ Example 7: Find the complete response v and i for $t > 0$



Initial and final values

$$\begin{cases} v(0) = 12V \\ i(0) = 0A \end{cases} \qquad \begin{matrix} v(\infty) = 4V \\ i(\infty) = 2A \end{matrix}$$

Natural response: t > 0, turn off independent sources

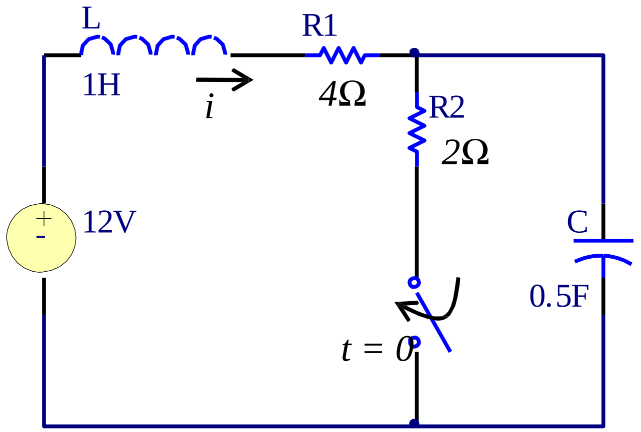
$$\begin{cases} i = \dfrac{v}{R_2} + C\dfrac{dv}{dt} \\ R_1 i + L\dfrac{di}{dt} + v = 0 \end{cases} \rightarrow \frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 0 \rightarrow v_n(t) = A_1e^{-2t} + Be^{-3t}$$

Complete response:

$$v(t) = v_n(t) + v_f(t) = 4 + A_1e^{-2t} + A_2e^{-3t}V$$

8.5. General second order circuits

+ Example 7: Find the complete response v and i for $t > 0$



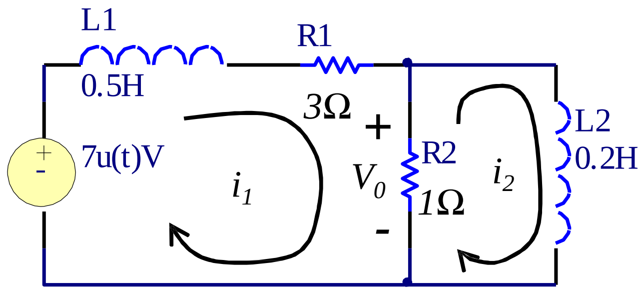
Imposing the initial condition:

$$\begin{cases} A_1 + A_2 = 8 \\ \dfrac{dv(0)}{dt} = -2A_1 - 2A_2 = -12 \end{cases} \rightarrow \begin{cases} A_1 = 12 \\ A_2 = -4 \end{cases} \rightarrow v(t) = 4 + 12e^{-2t} - 4e^{-3t} V$$

$$i = \frac{v}{R_2} + C\frac{dv}{dt} = 2 + 6e^{-2t} - 2e^{-3t} - 12e^{-2t} + 6e^{-3t} = 2 - 6e^{-2t} + 4e^{-3t} A$$

8.5. General second order circuits

+ Example 8: Find $v_0(t)$ for $t > 0$



Initial and final values

$$\begin{cases} i_1(0) = 0A \\ i_2(0) = 0A \\ v_0(t) = 0V \end{cases}$$

Natural response: t > 0, turn off independent source

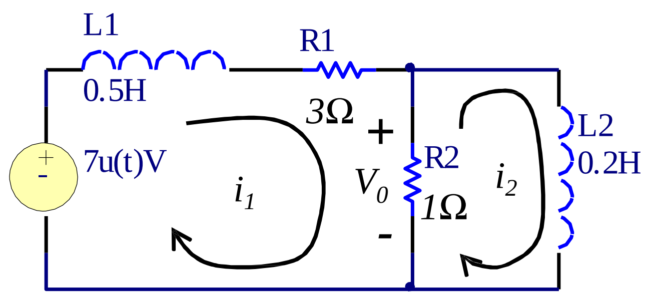
$$\begin{cases} (R_1 + R_2)i_1 - R_2 i_2 + L_1 \dfrac{di_1}{dt} = 0 \\ R_2(i_2 - i_1) + L_2 \dfrac{di_2}{dt} = 0 \end{cases} \rightarrow \frac{d^2 i_1}{dt^2} + 13\frac{di_1}{dt} + 30 i_1 = 0 \rightarrow i_{1n} = Ae^{-3t} + Be^{-10t} A$$

Force response:

$$i_{L1}(\infty) = i_{L2}(\infty) = \frac{E}{R_1} = \frac{7}{3} = 2.33A \qquad v_0(\infty) = 0V$$

8.5. General second order circuits

+ Example 8: Find $v_0(t)$ for $t > 0$



Complete response:

$$i_1(t) = i_{1f} + i_{1n} = 2.33 + Ae^{-3t} + Be^{-10t}\,A$$

Imposing the initial condition

$$\begin{cases} A + B + 2.33 = 0 \\ -3A - 10B = 14 \end{cases} \rightarrow \begin{cases} A = -1.33 \\ B = -1 \end{cases} \rightarrow i_1(t) = 2.33 - 1.33e^{-3t} - e^{-10t}\,A$$

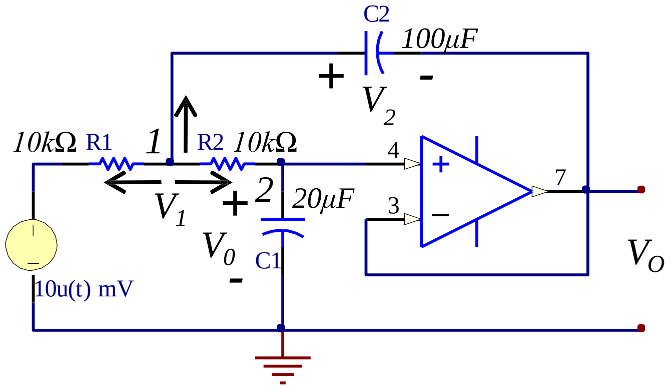
To find $v_0(t)$

$$L_1\frac{di_1}{dt} + (R_1 + R_2)i_1 - R_2 i_2 = E \rightarrow i_2(t) = 2.33 - 3.33e^{-3t} + e^{-10t}\,A$$

$$v_0(t) = R_2[i_1(t) - i_2(t)] = 2(e^{-3t} - e^{-10t})V$$

8.5. General second order circuits

+ Example 9: Find $v_0(t)$ for $t > 0$



$$At\ node\ 1: \frac{v_S - v_1}{R_1} = C_2\frac{dv_2}{dt} + \frac{v_1 - v_0}{R_2}$$

$$At\ node\ 2: \frac{v_1 - v_0}{R_2} = C_1\frac{dv_0}{dt} \quad \rightarrow v_1 = v_0 + R_2C_1\frac{dv_0}{dt}$$

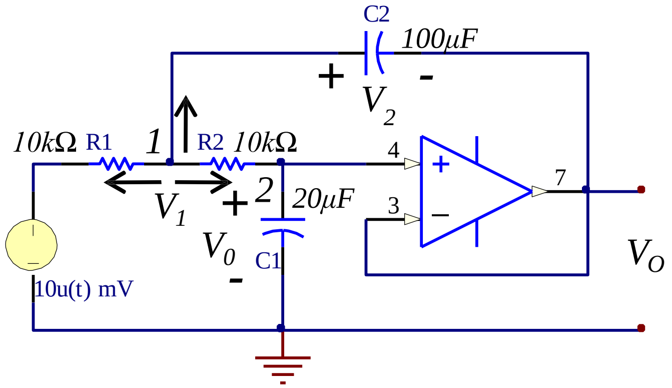
$$\rightarrow \frac{v_s - v_o}{R_1} = C_2\frac{dv_2}{dt} + C_1\frac{dv_o}{dt}$$

$$\rightarrow \frac{v_s - v_o}{R_1} = C_2\frac{dv_2}{dt} + C_1\frac{dv_o}{dt}$$

$$\rightarrow \frac{d^2v_0}{dt^2} + \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right)\frac{dv_0}{dt} + \frac{v_0}{R_1R_2C_1C_2} = \frac{v_s}{R_1R_2C_1C_2} \rightarrow \frac{d^2v_0}{dt^2} + 2\frac{dv_0}{dt} + 5v_0 = 5v_s$$

8.5. General second order circuits

+ Example 9: Find $v_0(t)$ for $t > 0$



Natural response: turn off the source

$$s^2+2s+5=0 \to s_{1,2}=-1\pm j2 \quad \to v_{0n}(t)=e^{-t}\left(A\cos 2t+B\sin 2t\right)$$

Force response: $v_0(\infty)=v_1(\infty)=v_s \to v_{0f}=v_0(\infty)=10mV$

Complete response: $v_0(t)=10+e^{-t}\left(A\cos 2t+B\sin 2t\right)$

Initial conditions: $v_0(+0)=0V \quad \dfrac{dv_0(+0)}{dt}=\dfrac{v_1-v_0}{R_2C_1}=0V$ (Eq. at node 2)

$$\to \begin{cases} v_0(+0)=10+A=0 \to A=-10 \\ \dfrac{dv_0(+0)}{dt}=-A+2B=0 \to B=-5 \end{cases}$$

$$\to v(t)=10-e^{-t}\left(10\cos 2t+5\sin 2t\right)mV$$

8.6. Applications

+ Practical applications of *RLC* circuits are found in control and communications circuits:

Ringing circuits

Peaking circuits

Resonant circuits

Filters

Smoothing circuit

Automobile ignition

→ Most of the circuits cannot be covered until we treat AC sources