TRƯỜNG ĐẠI HỌC
BÁCH KHOA HÀ NỘI

Nguyễn Công Phương

Engineering Electromagnetics

Poisson’s & Laplace’s Equations

Contents

Poisson's & Laplace's Equations

1. Poisson's Equation
2. Laplace's Equation
3. Uniqueness Theorem
4. Examples of the Solution of Laplace's Equation
5. Examples of the Solution of Poisson's Equation
6. Product Solution of Laplace's Equation
7. Numerical Methods

Poisson's Equation (1)

$$\left.\begin{aligned} &\text{Gauss's Law: } \nabla.\mathbf{D} = \rho_v \\ &\mathbf{D} = \varepsilon_0 \mathbf{E} \\ &\text{Gradient: } \mathbf{E} = -\nabla V \end{aligned}\right\} \begin{aligned} &\rightarrow \nabla.\mathbf{D} = \nabla.(\varepsilon\mathbf{E}) = -\nabla.(\varepsilon\nabla V) = \rho_v \\ &\rightarrow \boxed{\nabla.\nabla V = -\frac{\rho_v}{\varepsilon}} \end{aligned}$$

(Poisson's Equation)

$$\left.\begin{aligned} \nabla V &= \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \\ \nabla.\mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \end{aligned}\right\}$$

$$\rightarrow \nabla.\nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_y}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_z}{\partial z}\right) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Poisson's Equation (2)

$$\left.\begin{array}{l} \nabla . \nabla V = -\dfrac{\rho_v}{\varepsilon} \\ \nabla . \nabla V = \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} \\ \text{Define } \nabla . \nabla = \nabla^2 \end{array}\right\} \longrightarrow \boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}}$$

(rectangular)

$$\boxed{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}}$$

(cylindrical)

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2} = -\frac{\rho_v}{\varepsilon}}$$

(spherical)

Ex. Poisson's Equation (3)

Find the Laplacian of the following scalar fields:

$$a)\ A = 2xy^2z^3$$

$$b)\ B = \frac{\cos 2\varphi}{\rho}$$

$$c)\ C = \frac{20\sin\theta}{r^3}$$

Poisson's & Laplace's Equations

1. Poisson's Equation
2. **Laplace's Equation**
3. Uniqueness Theorem
4. Examples of the Solution of Laplace's Equation
5. Examples of the Solution of Poisson's Equation
6. Product Solution of Laplace's Equation
7. Numerical Methods

Laplace's Equation

Poisson's Equation: $$\left.\begin{aligned}\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon} \\ \rho_v = 0\end{aligned}\right\}$$

$$\rightarrow \boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0}$$ (Laplace's equation, rectangular)

$$\boxed{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0}$$ (cylindrical)

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2} = 0}$$

(spherical)

Poisson's & Laplace's Equations

1. Poisson's Equation
2. Laplace's Equation
3. **Uniqueness Theorem**
4. Examples of the Solution of Laplace's Equation
5. Examples of the Solution of Poisson's Equation
6. Product Solution of Laplace's Equation
7. Numerical Methods

Uniqueness Theorem (1)

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Assume two solutions V_1 & V_2, :

$$\left.\begin{array}{l} \nabla^2 V_1 = 0 \\ \nabla^2 V_2 = 0 \end{array}\right\} \rightarrow \nabla^2 (V_1 - V_2) = 0$$

Assume the boundary condition $V_b \quad \rightarrow V_{1b} = V_{2b} = V_b$

$$\left.\begin{array}{l} \nabla \cdot (V\mathbf{D}) = V(\nabla \cdot \mathbf{D}) + \mathbf{D} \cdot (\nabla V) \\ \qquad V = V_1 - V_2 \\ \qquad \mathbf{D} = \nabla (V_1 - V_2) \end{array}\right\}$$

$$\rightarrow \nabla \cdot [(V_1 - V_2)\nabla (V_1 - V_2)] = (V_1 - V_2)[\nabla \cdot \nabla (V_1 - V_2)] + $$
$$+ \nabla (V_1 - V_2) \cdot \nabla (V_1 - V_2)$$

Uniqueness Theorem (2)

$$\nabla.[(V_1-V_2)\nabla(V_1-V_2)]=(V_1-V_2)[\nabla.\nabla(V_1-V_2)]+\nabla(V_1-V_2).\nabla(V_1-V_2)$$

$$\rightarrow \boxed{\int_V \nabla.[(V_1-V_2)\nabla(V_1-V_2)]dv} = \int_V (V_1-V_2)[\nabla.\nabla(V_1-V_2)]dv +$$

$$+\int_V \nabla(V_1-V_2).\nabla(V_1-V_2)dv$$

Divergence theorem: $\oint_S \mathbf{D}.d\mathbf{S} = \int_V \nabla.\mathbf{D}dv$

$$\left.\begin{array}{r}\rightarrow \boxed{\int_V \nabla.[(V_1-V_2)\nabla(V_1-V_2)]dv} = \oint_S [(V_{1b}-V_{2b})\nabla(V_{1b}-V_{2b})].d\mathbf{S} \\ V_{1b}=V_{2b}=V_b\end{array}\right\}$$

$$\rightarrow \boxed{\int_V \nabla.[(V_1-V_2)\nabla(V_1-V_2)]dv} = 0$$

$$\rightarrow \boxed{0} = \int_V (V_1-V_2)[\nabla.\nabla(V_1-V_2)]dv + \int_V \nabla(V_1-V_2).\nabla(V_1-V_2)dv$$

Uniqueness Theorem (3)

$$\left.\begin{array}{r}\int_V (V_1 - V_2)[\nabla.\nabla(V_1 - V_2)]dv + \int_V \nabla(V_1 - V_2).\nabla(V_1 - V_2)dv = 0 \\ \nabla.\nabla(V_1 - V_2) = \nabla^2(V_1 - V_2) = 0\end{array}\right\}$$

$$\left.\begin{array}{r}\rightarrow \int_V \nabla(V_1 - V_2).\nabla(V_1 - V_2)dv = 0 = \int_V [\nabla(V_1 - V_2)]^2 dv \\ [\nabla(V_1 - V_2)]^2 \geq 0\end{array}\right\}$$

$$\left.\begin{array}{l}\left.\begin{array}{r}\rightarrow [\nabla(V_1 - V_2)]^2 = 0 \ \rightarrow \nabla(V_1 - V_2) = 0 \\ \nabla V = \dfrac{\partial V}{\partial x}\mathbf{a}_x + \dfrac{\partial V}{\partial y}\mathbf{a}_y + \dfrac{\partial V}{\partial z}\mathbf{a}_z\end{array}\right\} \rightarrow V_1 - V_2 = \text{const} \\ \left.\begin{array}{r}\text{At boundary } V_1 = V_{b1},\ V_2 = V_{b2} \\ V_{1b} = V_{2b} = V_b\end{array}\right\} \rightarrow \text{const} = V_{b1} - V_{b2} = 0\end{array}\right\} \rightarrow \boxed{V_1 = V_2}$$

Poisson’s & Laplace’s Equations

1. Poisson’s Equation
2. Laplace’s Equation
3. Uniqueness Theorem
4. **Examples of the Solution of Laplace’s Equation**
5. Examples of the Solution of Poisson’s Equation
6. Product Solution of Laplace’s Equation
7. Numerical Methods

Examples of the Solution of Laplace's Equation (1)

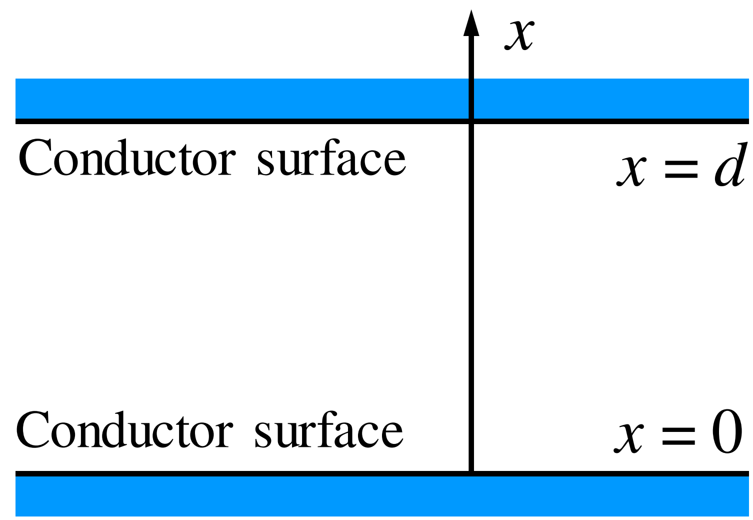
Ex. 1

$$\left.\begin{array}{l} \text{Assume } V = V(x) \\ \nabla^2 V = \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0 \end{array}\right\} \to \frac{d^2V}{dx^2} = 0 \to \left.\begin{array}{l} V = Ax + B \\ V\big|_{x=x_1} = V_1 \\ V\big|_{x=x_2} = V_2 \end{array}\right\}$$

$$\to \left\{\begin{array}{l} A = \dfrac{V_1 - V_2}{x_1 - x_2} \\ B = \dfrac{V_2 x_1 - V_1 x_2}{x_1 - x_2} \end{array}\right. \to \left.\begin{array}{l} V = \dfrac{V_1(x - x_2) - V_2(x - x_1)}{x_1 - x_2} \\ V\big|_{x=0} = 0 \\ V\big|_{x=d} = V_0 \end{array}\right\} \to \boxed{V = \frac{V_0 x}{d}}$$

Examples of the Solution of Laplace's Equation (2)

Ex. 1



$$\left.\begin{aligned} &V = V(x) \\ &V\big|_{x=0} = 0 \\ &V\big|_{x=d} = V_0 \end{aligned}\right\} \to \left.\begin{aligned} V &= \frac{V_0 x}{d} \\ \mathbf{E} &= -\nabla V \end{aligned}\right\}$$

$$\left.\begin{aligned} \to \mathbf{E} &= -\frac{V_0}{d}\mathbf{a}_x \\ \mathbf{D} &= \varepsilon\mathbf{E} \end{aligned}\right\} \to \mathbf{D} = -\varepsilon\frac{V_0}{d}\mathbf{a}_x$$

$$\to \mathbf{D}_S = \mathbf{D}\big|_{x=0} = -\varepsilon\frac{V_0}{d}\mathbf{a}_x \to D_N = -\varepsilon\frac{V_0}{d} \to \rho_S = D_N = -\varepsilon\frac{V_0}{d}$$

$$\to Q = \int_S \rho_S dS = \int_S \frac{-\varepsilon V_0}{d} dS = -\varepsilon\frac{V_0 S}{d} \to C = \frac{|Q|}{V_0} \boxed{= \frac{\varepsilon S}{d}}$$

Examples of the Solution of Laplace's Equation (3)

Ex. 2

$$\left.\begin{array}{l}\text{Assume } V = V(\rho) \text{ (cylindrical)} \\ \nabla^2 V = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial V}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 V}{\partial\varphi^2} + \dfrac{\partial^2 V}{\partial z^2} = 0\end{array}\right\} \rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = 0$$

$$\rightarrow \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0 \rightarrow \frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0 \rightarrow \rho\frac{dV}{d\rho} = A$$

$$\left.\begin{array}{l}\rightarrow V = A\ln\rho + B \\ \left.\begin{array}{l} V\big|_{\rho=a} = A\ln a + B = V_0 \\ V\big|_{\rho=b} = A\ln b + B = 0 \ (b > a)\end{array}\right\} \rightarrow \begin{cases} A = \dfrac{V_0}{\ln a - \ln b} \\ B = -\dfrac{V_0 \ln b}{\ln a - \ln b}\end{cases}\end{array}\right\} \rightarrow \boxed{V = V_0\frac{\ln(b/\rho)}{\ln(b/a)}}$$

Examples of the Solution of Laplace's Equation (4)

Ex. 2

$$\left.\begin{array}{l}\text{Assume } V = V(\rho) \text{ (cylindrical)} \\ \nabla^2 V = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial V}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 V}{\partial\varphi^2} + \dfrac{\partial^2 V}{\partial z^2} = 0\end{array}\right\} \rightarrow \boxed{V = V_0 \dfrac{\ln(b/\rho)}{\ln(b/a)}}$$

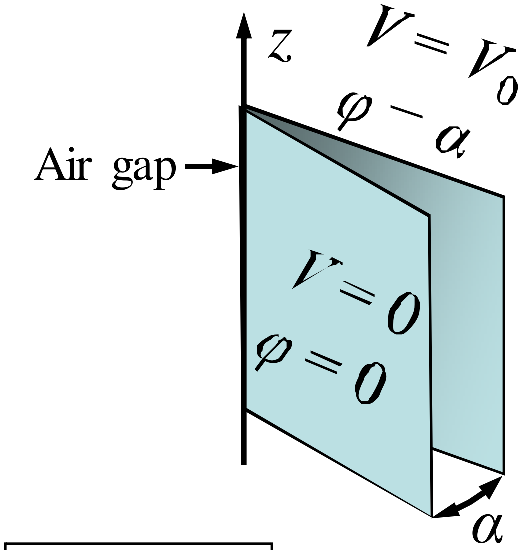
$$\rightarrow \mathbf{E} = -\nabla V = \frac{V_0}{\rho\ln(b/a)}\mathbf{a}_\rho$$

$$\rightarrow D_{N(\rho=a)} = \frac{\varepsilon V_0}{a\ln(b/a)} = \rho_S$$

$$\rightarrow Q = \int_S \rho_S dS = \frac{\varepsilon V_0 2\pi a L}{a\ln(b/a)} \rightarrow C = \frac{Q}{V_0} = \boxed{\frac{\varepsilon 2\pi L}{\ln(b/a)}}$$

Examples of the Solution of Laplace's Equation (5)

Ex. 3



$$\left.\begin{array}{l}\text{Assume } V = V(\varphi) \text{ (cylindrical)} \\ \nabla^2 V = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial V}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 V}{\partial\varphi^2} + \dfrac{\partial^2 V}{\partial z^2} = 0\end{array}\right\}$$

$$\rightarrow \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\varphi^2} = 0 \rightarrow \frac{\partial^2 V}{\partial\varphi^2} = 0 \rightarrow V = A\varphi + B$$

$$\left.\begin{array}{l} V\big|_{\varphi=0} = B = 0 \\ V\big|_{\varphi=\alpha} = A\alpha + B = V_0 \end{array}\right\} \rightarrow \begin{cases} B = 0 \\ A = \dfrac{V_0}{\alpha} \end{cases}$$

$$\rightarrow \boxed{V = V_0\frac{\varphi}{\alpha}}$$

$$\rightarrow \boxed{\mathbf{E} = -\nabla V = -\frac{V_0}{\alpha\rho}\mathbf{a}_\varphi}$$

Examples of the Solution of Laplace's Equation (6)

Ex. 4

$$\left.\begin{array}{l}\text{Assume } V = V(\theta) \text{ (spherical)} \\ \nabla^2 V = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial V}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial V}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 V}{\partial\varphi^2} = 0\end{array}\right\}$$

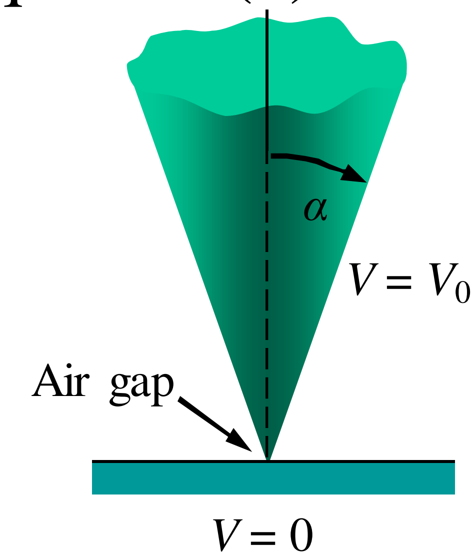
$$\left.\begin{array}{l}\rightarrow \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial V}{\partial\theta}\right) = 0 \\ \text{Assume } r \neq 0;\ \theta \neq 0;\ \theta \neq \pi\end{array}\right\} \rightarrow \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0 \rightarrow \sin\theta\frac{dV}{d\theta} = A$$

$$\rightarrow dV = A\frac{d\theta}{\sin\theta} \quad \rightarrow V = \int A\frac{d\theta}{\sin\theta} + B = A\ln\left(\tan\frac{\theta}{2}\right) + B$$

Examples of the Solution of Laplace's Equation (7)

Ex. 4

$$\left.\begin{aligned}&\text{Assume } V = V(\theta) \rightarrow V = A\ln\left(\tan\frac{\theta}{2}\right) + B\\&V\Big|_{\theta=\pi/2} = 0\\&V\Big|_{\theta=\alpha} = V_0 \quad (\alpha < \pi/2)\end{aligned}\right\}$$

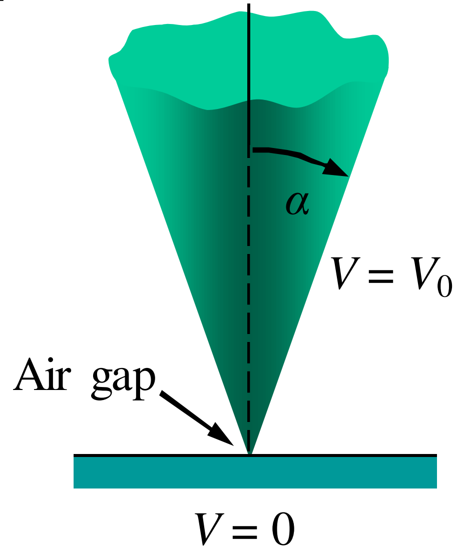


$$\rightarrow V = V_0 \frac{\ln\left(\tan\frac{\theta}{2}\right)}{\ln\left(\tan\frac{\alpha}{2}\right)} \rightarrow \mathbf{E} = -\nabla V = -\frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta = -\frac{V_0}{r\sin\theta\ln\left(\tan\frac{\alpha}{2}\right)}\mathbf{a}_\theta$$

$$\rightarrow \rho_S = D_N = \varepsilon E = -\frac{\varepsilon V_0}{r\sin\alpha\ln\left(\tan\frac{\alpha}{2}\right)}$$

Examples of the Solution of Laplace's Equation (8)

Ex. 4



Assume $V = V(\theta)$ $\quad \rightarrow \rho_S = -\dfrac{\varepsilon V_0}{r\sin\alpha\ln\left(\tan\dfrac{\alpha}{2}\right)}$

$$\left.\begin{array}{l} \rightarrow Q = \oint_S \rho_S dS = -\oint_S \dfrac{\varepsilon V_0}{r\sin\alpha\ln\left(\tan\dfrac{\alpha}{2}\right)} dS \\ \qquad dS = r\sin\alpha d\varphi dr \end{array}\right\}$$

$$\rightarrow Q = \frac{-\varepsilon V_0}{\sin\alpha\ln\left(\tan\dfrac{\alpha}{2}\right)}\int_0^\infty\int_0^{2\pi}\frac{r\sin\alpha d\varphi dr}{r} = \frac{-2\pi\varepsilon V_0}{\ln\left(\tan\dfrac{\alpha}{2}\right)}\int_0^\infty dr$$

$$\rightarrow C = \frac{Q}{V_0} \doteq \boxed{\frac{2\pi\varepsilon r_1}{\ln\left(\cot\dfrac{\alpha}{2}\right)}}$$

Poisson's & Laplace's Equations

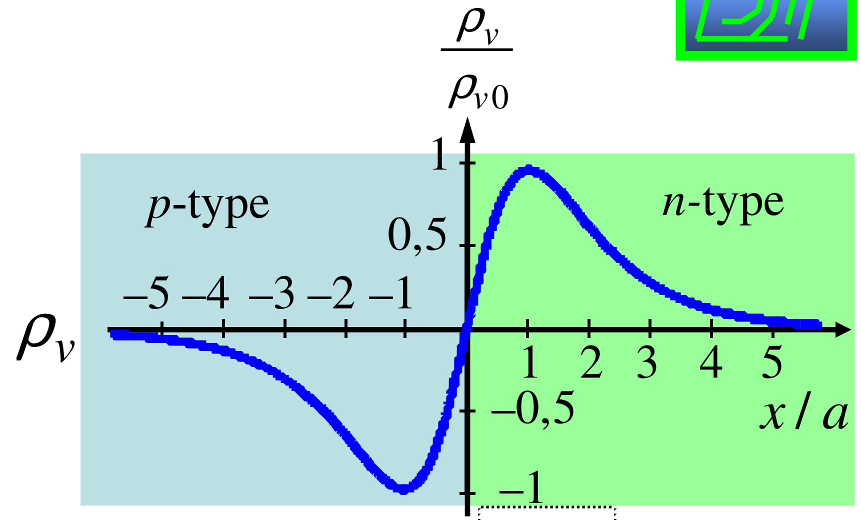
1. Poisson's Equation
2. Laplace's Equation
3. Uniqueness Theorem
4. Examples of the Solution of Laplace's Equation
5. **Examples of the Solution of Poisson's Equation**
6. Product Solution of Laplace's Equation
7. Numerical Methods

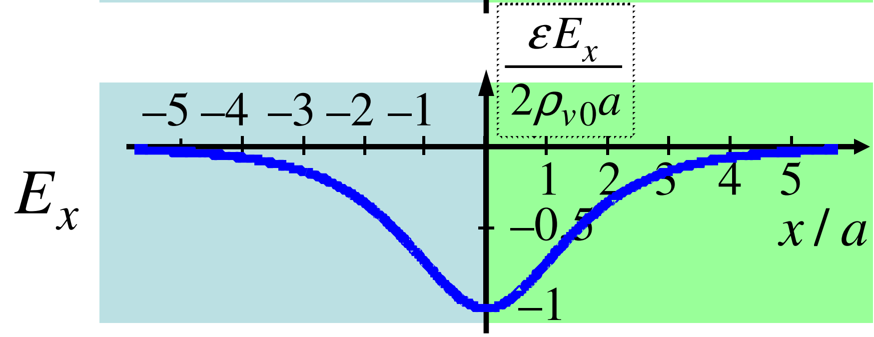
Examples of the Solution of Poisson's Equation (1)

$$\left.\begin{aligned} &\rho_v = 2\rho_{v0}\,\mathrm{sech}\,\frac{x}{a}\,\mathrm{th}\,\frac{x}{a} \\ &\left(\ \mathrm{sech}\,x = \frac{2}{e^x + e^{-x}};\ \mathrm{th}\,x = \frac{e^x - e^{-x}}{e^x + e^{-x}}\ \right) \\ &\text{Poisson's Equation :}\quad \nabla^2 V = -\frac{\rho_v}{\varepsilon} \end{aligned}\right\}$$

$$\rightarrow \frac{d^2V}{dx^2} = -\frac{2\rho_{v0}}{\varepsilon}\,\mathrm{sech}\,\frac{x}{a}\,\mathrm{th}\,\frac{x}{a}$$

$$\left.\begin{aligned} &\rightarrow \frac{dV}{dx} = \frac{2\rho_{v0}a}{\varepsilon}\,\mathrm{sech}\,\frac{x}{a} + C_1 \\ &E_x = -\frac{dV}{dx} \end{aligned}\right\} \rightarrow \left.\begin{aligned} &E_x = -\frac{2\rho_{v0}a}{\varepsilon}\,\mathrm{sech}\,\frac{x}{a} - C_1 \\ &\text{If } x \rightarrow \pm\infty \text{ then } E_x \rightarrow 0 \end{aligned}\right\} \rightarrow C_1 = 0$$





$$\rightarrow E_x = -\frac{2\rho_{v0}a}{\varepsilon}\,\mathrm{sech}\,\frac{x}{a}$$

Examples of the Solution of Poisson's Equation (2)

$$\rho_v = 2\rho_{v0}\,\mathrm{sech}\frac{x}{a}\,\mathrm{th}\frac{x}{a}$$

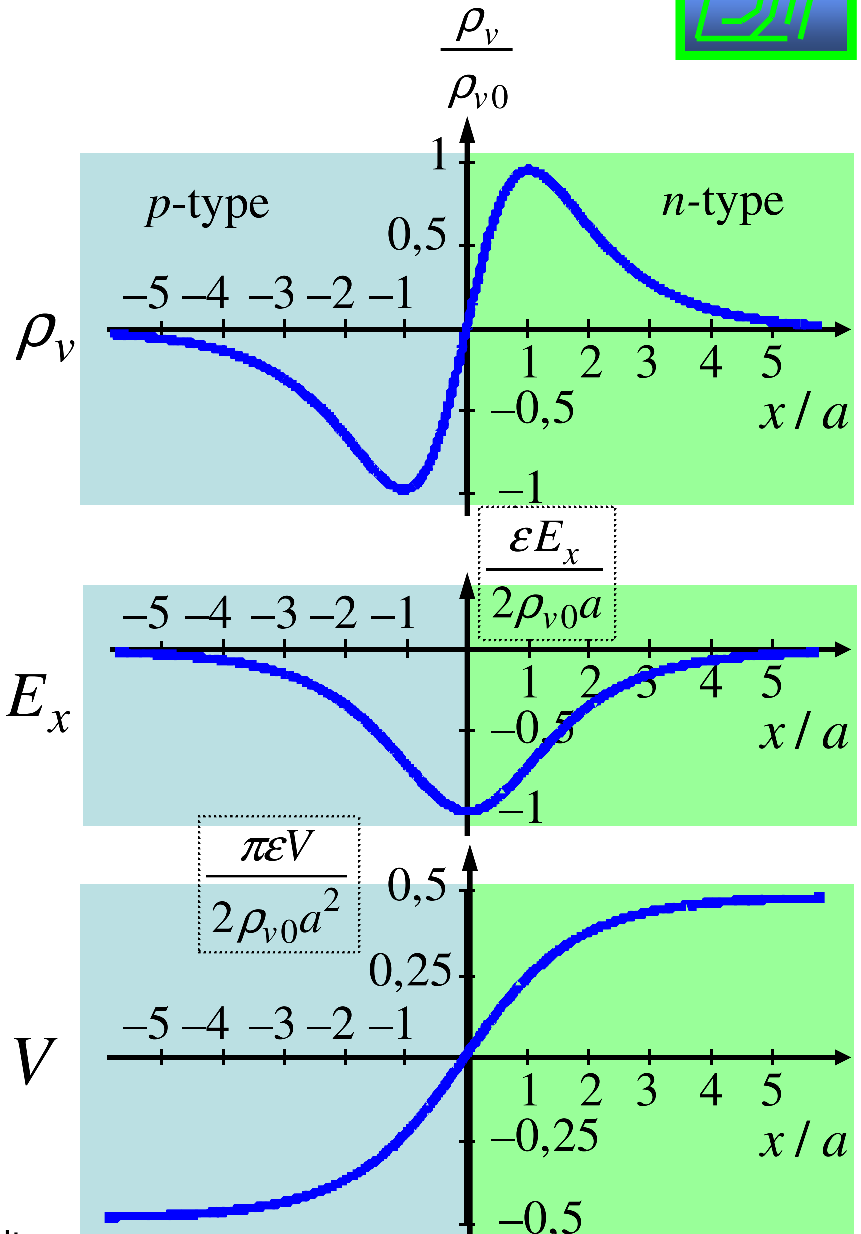
Poisson's equation : $\nabla^2 V = -\dfrac{\rho_v}{\varepsilon}$

$$\to E_x = -\frac{2\rho_{v0}a}{\varepsilon}\,\mathrm{sech}\frac{x}{a}$$

$$\to V = \frac{4\rho_{v0}a^2}{\varepsilon}\,\mathrm{arctg}\, e^{x/a} + C_2$$

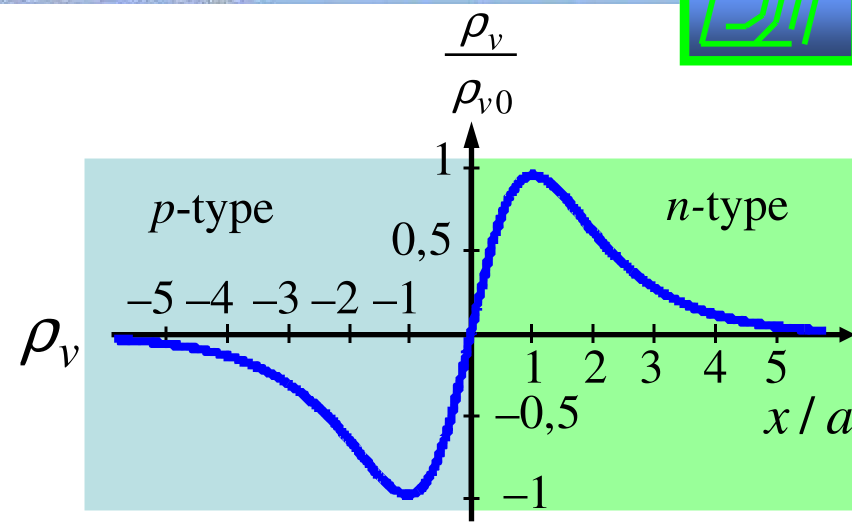
Supp. $V\big|_{x=0} = 0 \to 0 = \dfrac{4\rho_{v0}a^2}{\varepsilon}\dfrac{\pi}{4} + C_2$

$$\to V = \frac{4\rho_{v0}a^2}{\varepsilon}\left(\mathrm{arctg}\, e^{x/a} - \frac{\pi}{4}\right)$$



Examples of the Solution of Poisson's Equation (3)

$$\rho_v = 2\rho_{v0}\,\mathrm{sech}\frac{x}{a}\,\mathrm{th}\frac{x}{a}$$



$$V = \frac{4\rho_{v0}a^2}{\varepsilon}\left(\mathrm{arctg}\, e^{x/a} - \frac{\pi}{4}\right)$$

$$V_0 = V_{x\to\infty} - V_{x\to-\infty} = \frac{2\pi\rho_{v0}a^2}{\varepsilon}$$

$$Q = \int_V \rho_v dv = \int_V 2\rho_{v0}\,\mathrm{sech}\frac{x}{a}\,\mathrm{th}\frac{x}{a}\,dv = S\int_0^\infty 2\rho_{v0}\,\mathrm{sech}\frac{x}{a}\,\mathrm{th}\frac{x}{a}\,dx = 2\rho_{v0}aS$$

$$\left.\begin{aligned} &\rightarrow Q = S\sqrt{\frac{2\rho_{v0}\varepsilon V_0}{\pi}} \\ &I = \frac{dQ}{dt} = C\frac{dV_0}{dt} \rightarrow C = \frac{dQ}{dV_0} \end{aligned}\right\} \rightarrow \boxed{C = S\sqrt{\frac{\rho_{v0}\varepsilon}{2\pi V_0}} = \frac{\varepsilon S}{2\pi a}}$$

Poisson's & Laplace's Equations

1. Poisson's Equation
2. Laplace's Equation
3. Uniqueness Theorem
4. Examples of the Solution of Laplace's Equation
5. Examples of the Solution of Poisson's Equation
6. **Product Solution of Laplace's Equation**
7. Numerical Methods

Product Solution of Laplace's Equation (1)

- Previous examples assumed that V varies with one of the three coordinates
- The product solution can be used to solve for $V(x, y)$

$$\left.\left.\begin{aligned}&\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0\\ &\quad V = V(x, y)\end{aligned}\right\} \rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0\quad\right\}$$

- Assume $V = XY,\ X = X(x),\ Y = Y(y)$

$$\rightarrow Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0 \qquad \rightarrow Y\frac{d^2 X}{dx^2} + X\frac{d^2 Y}{dy^2} = 0$$

Product Solution of Laplace's Equation (2)

$$Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}=0\to\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0\to\frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}$$

$$\left.\begin{array}{l}\dfrac{1}{X}\dfrac{d^2X}{dx^2}\ \text{ involves no } y\\ -\dfrac{1}{Y}\dfrac{d^2Y}{dy^2}\ \text{ involves no } x\end{array}\right\}$$

$$\to\begin{cases}\dfrac{1}{X}\dfrac{d^2X}{dx^2}=\alpha^2\\ -\dfrac{1}{Y}\dfrac{d^2Y}{dy^2}=\alpha^2\end{cases}$$

Product Solution of Laplace's Equation (3)

$$\left.\begin{aligned} &V = V(\rho,\varphi) = R(\rho)\Phi(\varphi) \\ &\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \end{aligned}\right\} \rightarrow \frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

$$\rightarrow \begin{cases} \dfrac{\rho}{R}\dfrac{d}{d\rho}\left(\rho\dfrac{dR}{d\rho}\right) = \alpha^2 \\ -\dfrac{1}{\Phi}\dfrac{d^2\Phi}{d\varphi^2} = \alpha^2 \end{cases}$$

Product Solution of Laplace's Equation (4)

$$\left.\begin{aligned} &V = V(\rho,\theta) = R(\rho)\Theta(\theta) \\ &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2} = 0 \end{aligned}\right\}$$

$$\to \left(\frac{\rho}{R^2}\frac{\partial^2 R}{\partial\rho^2} + \frac{2\rho}{R}\frac{\partial R}{\partial\rho}\right) + \left(\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} + \frac{1}{\Theta\,\mathrm{tg}\,\theta}\frac{\partial\Theta}{\partial\theta}\right) = 0$$

$$\to \begin{cases} \dfrac{\rho}{R^2}\dfrac{\partial^2 R}{\partial\rho^2} + \dfrac{2\rho}{R}\dfrac{\partial R}{\partial\rho} = n(n+1) \\ \dfrac{1}{\Theta}\dfrac{\partial^2\Theta}{\partial\theta^2} + \dfrac{1}{\Theta\,\mathrm{tg}\,\theta}\dfrac{\partial\Theta}{\partial\theta} = -n(n+1) \end{cases}$$

Product Solution of Laplace's Equation (5)

Ex.

Solve for the potential & EFI in the vicinity of a conducting cylinder (posses an infinite length) in a uniform EFI E_0. The permittivities of the environment & the cylinder are ε_1 & ε_2 respectively. EFI is perpendicular to the cylinder's axis.

$$V = V(\rho,\varphi) = R(\rho)\Phi(\varphi) \to \begin{cases} \dfrac{\rho}{R}\dfrac{d}{d\rho}\left(\rho\dfrac{dR}{d\rho}\right) = \alpha^2 \\ -\dfrac{1}{\Phi}\dfrac{d^2\Phi}{d\varphi^2} = \alpha^2 \end{cases}$$

$$\left.\begin{array}{l} \text{Assume } \Phi(\varphi) = \sum A_p \cos p\varphi + \sum B_p \sin p\varphi, \quad p = \pm\alpha \\ V(\varphi) = V(-\varphi); \quad V(\varphi) = -V(\pi - \varphi) \end{array}\right\}$$

$$\to \Phi(\varphi) = A_1 \cos\varphi, \qquad \alpha = 1$$

Product Solution of Laplace's Equation (6)

Ex.

Solve for the potential & EFI in the vicinity of a conducting cylinder (posses an infinite length) in a uniform EFI E_0. The permittivities of the environment & the cylinder are ε_1 & ε_2 respectively. EFI is perpendicular to the cylinder's axis.

$$\begin{cases} -\dfrac{1}{\Phi}\dfrac{d^2\Phi}{d\varphi^2} = \alpha^2 \quad \to \Phi(\varphi) = A_1 \cos\varphi, \qquad \alpha = 1 \\ \dfrac{\rho}{R}\dfrac{d}{d\rho}\left(\rho\dfrac{dR}{d\rho}\right) = \alpha^2 \end{cases}$$

$$\left.\begin{matrix} \dfrac{\rho}{R}\dfrac{d}{d\rho}\left(\rho\dfrac{dR}{d\rho}\right) = \alpha^2 \\ \text{Assume } R(\rho) = B_k\rho^k \end{matrix}\right\} \to \left.\begin{matrix} \dfrac{\rho}{R}\dfrac{d}{d\rho}\left(\rho\dfrac{dR}{d\rho}\right) = \dfrac{k^2 B_k \rho^k}{B_k \rho^k} = \alpha^2 \\ \alpha = 1 \end{matrix}\right\} \to k = \pm 1$$

$$\to R(\rho) = B_1^+\rho + B_1^-\rho^{-1}$$

Product Solution of Laplace's Equation (7)

Ex.

Solve for the potential & EFI in the vicinity of a conducting cylinder (posses an infinite length) in a uniform EFI E_0. The permittivities of the environment & the cylinder are ε_1 & ε_2 respectively. EFI is perpendicular to the cylinder's axis.

$$\left.\begin{array}{l}\begin{cases}-\dfrac{1}{\Phi}\dfrac{d^2\Phi}{d\varphi^2}=\alpha^2\rightarrow\Phi(\varphi)=A_1\cos\varphi\\[2ex]\dfrac{\rho}{R}\dfrac{d}{d\rho}\left(\rho\dfrac{dR}{d\rho}\right)=\alpha^2\rightarrow R(\rho)=B_1^+\rho+B_1^-\rho^{-1}\end{cases}\\[2ex]V=V(\rho,\varphi)=R(\rho)\Phi(\varphi)\end{array}\right\}$$

$$\rightarrow V=A_1B_1^+\rho\cos\varphi+A_1B_1^-\rho^{-1}\cos\varphi=C^+\rho\cos\varphi+C^-\rho^{-1}\cos\varphi$$

Product Solution of Laplace's Equation (8)

Ex.

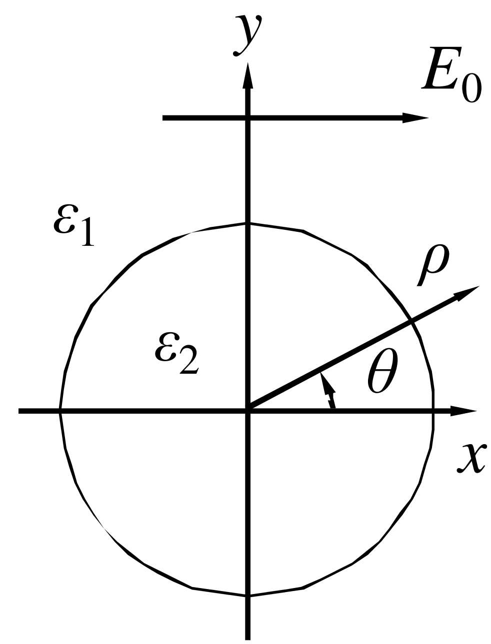
Solve for the potential & EFI in the vicinity of a conducting cylinder (posses an infinite length) in a uniform EFI E_0. The permittivities of the environment & the cylinder are ε_1 & ε_2 respectively. EFI is perpendicular to the cylinder's axis.

Exterior: $V_1 = C_1^+ \rho \cos\varphi + C_1^- \rho^{-1} \cos\varphi$

Interior: $V_2 = C_2^+ \rho \cos\varphi + C_2^- \rho^{-1} \cos\varphi$

$$\left.\begin{aligned} V\big|_{-\infty} &= E_0\, x\big|_{x\to-\infty} \\ V\big|_{-\infty} &= V_1\big|_{\theta=\pi,\ \rho\to-\infty} = -C_1^+ x\big|_{x\to-\infty} \end{aligned}\right\} \to C_1^+ = -E_0$$

$$\left.\begin{aligned} &V_{\text{gốc tọa độ}} = V_2\big|_{\rho\to 0} = \left.\frac{C_2^-}{\rho}\right|_{\rho\to 0} \to \infty \\ &\text{EFI is finite at the origin} \end{aligned}\right\} \to C_2^- = 0$$

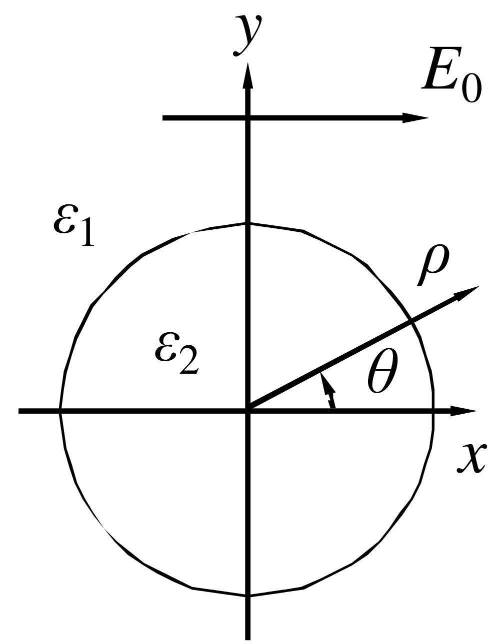


Product Solution of Laplace's Equation (9)

Ex.

Solve for the potential & EFI in the vicinity of a conducting cylinder (posses an infinite length) in a uniform EFI E_0. The permittivities of the environment & the cylinder are ε_1 & ε_2 respectively. EFI is perpendicular to the cylinder's axis.

$$\left.\begin{aligned}\text{Exterior: } & V_1 = C_1^+ \rho \cos\varphi + C_1^- \rho^{-1} \cos\varphi \\ \text{Interior: } & V_2 = C_2^+ \rho \cos\varphi + C_2^- \rho^{-1} \cos\varphi \\ & C_1^+ = -E_0, \quad C_2^- = 0 \end{aligned}\right\}$$



$$\rightarrow \begin{cases} V_1 = -E_0 \rho \cos\varphi + C_1^- \rho^{-1} \cos\varphi \\ V_2 = C_2^+ \rho \cos\varphi \end{cases}$$

Product Solution of Laplace's Equation (10)

Ex.

Solve for the potential & EFI in the vicinity of a conducting cylinder (posses an infinite length) in a uniform EFI E_0. The permittivities of the environment & the cylinder are ε_1 & ε_2 respectively. EFI is perpendicular to the cylinder's axis.

$$V_1 = -E_0\rho\cos\varphi + C_1^-\rho^{-1}\cos\varphi$$

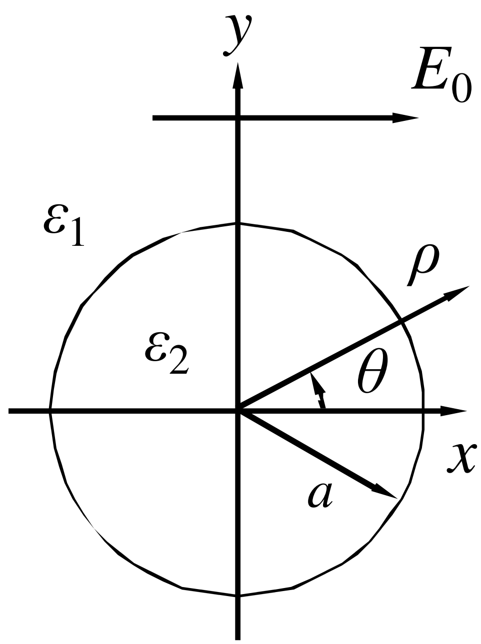
$$V_2 = C_2^+\rho\cos\varphi$$

$$V_1\big|_{\rho=a} = V_2\big|_{\rho=a}$$

$$\to (-E_0 a + C_1^- a^{-1})\cos\varphi = C_2^+ a\cos\varphi$$

$$\varepsilon_1 \frac{\partial V_1}{\partial\rho}\bigg|_{\rho=a} = \varepsilon_2 \frac{\partial V_2}{\partial\rho}\bigg|_{\rho=a}$$

$$\to \varepsilon_1(-E_0 - C_1^- a^{-2})\cos\varphi = \varepsilon_2 C_2^+ a\cos\varphi$$



Product Solution of Laplace's Equation (11)

Ex.

Solve for the potential & EFI in the vicinity of a conducting cylinder (posses an infinite length) in a uniform EFI E_0. The permittivities of the environment & the cylinder are ε_1 & ε_2 respectively. EFI is perpendicular to the cylinder's axis.

$$\left.\begin{array}{l} V_1 = -E_0\rho\cos\varphi + C_1^-\rho^{-1}\cos\varphi \\ V_2 = C_2^+\rho\cos\varphi \\ \left.\begin{array}{l} (-E_0 a + C_1^- a^{-1})\cos\varphi = C_2^+ a\cos\varphi \\ \varepsilon_1(-E_0 - C_1^- a^{-2})\cos\varphi = \varepsilon_2 C_2^+ a\cos\varphi \end{array}\right\} \to C_1^- = -E_0\dfrac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}a^2,\ C_2^+ = -E_0\dfrac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \end{array}\right\}$$

$$\to \boxed{\begin{cases} V_1 = -E_0\left(1 - \dfrac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\dfrac{a^2}{\rho^2}\right)\rho\cos\varphi, & \text{as } \rho \ge a \\ V_2 = -E_0\dfrac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\rho\cos\varphi & \text{as } \rho \le a \end{cases}}$$

Product Solution of Laplace's Equation (12)

Ex.

Solve for the potential & EFI in the vicinity of a conducting cylinder (posses an infinite length) in a uniform EFI E_0. The permittivities of the environment & the cylinder are ε_1 & ε_2 respectively. EFI is perpendicular to the cylinder's axis.

$$\begin{cases} V_1 = -E_0\left(1-\dfrac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\dfrac{a^2}{\rho^2}\right)\rho\cos\varphi, & \text{as } \rho \ge a \\ V_2 = -E_0\dfrac{2\varepsilon_1}{\varepsilon_1+\varepsilon_2}\rho\cos\varphi & \text{as } \rho \le a \end{cases}$$

$$\boxed{E_{1\rho} = -\frac{\partial V_1}{\partial \rho} = E_0\left(1-\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\frac{a^2}{\rho^2}\right)\cos\varphi, \qquad E_{1\varphi} = -\frac{\partial V_1}{\partial \varphi} = -E_0\left(1-\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\frac{a^2}{\rho^2}\right)\sin\varphi}$$

$$E_{2\rho} = -\frac{\partial V_2}{\partial \rho} = E_0\frac{2\varepsilon_1}{\varepsilon_1+\varepsilon_2}\cos\varphi, \qquad E_{2\varphi} = -\frac{\partial V_2}{\partial \varphi} = -E_0\frac{2\varepsilon_1}{\varepsilon_1+\varepsilon_2}\sin\varphi$$

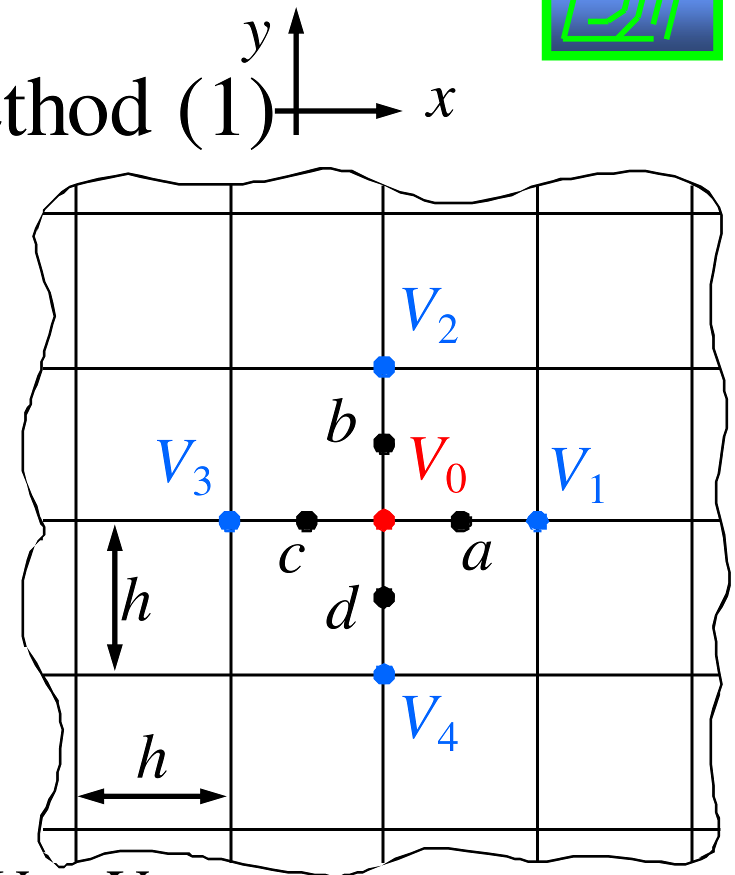
$$\rightarrow \boxed{E_2 = E_{2z} = E_0\frac{2\varepsilon_1}{\varepsilon_1+\varepsilon_2}}$$

Poisson's & Laplace's Equations

1. Poisson's Equation
2. Laplace's Equation
3. Uniqueness Theorem
4. Examples of the Solution of Laplace's Equation
5. Examples of the Solution of Poisson's Equation
6. Product Solution of Laplace's Equation
7. **Numerical Methods**
   a. **Finite Difference Method**
   b. **Finite Element Method**

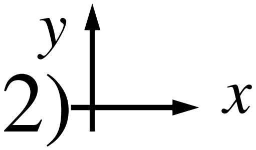
Finite Difference Method (1)

$$\left.\begin{aligned}\nabla^2 V &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \\ V &= V(x, y)\end{aligned}\right\} \rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

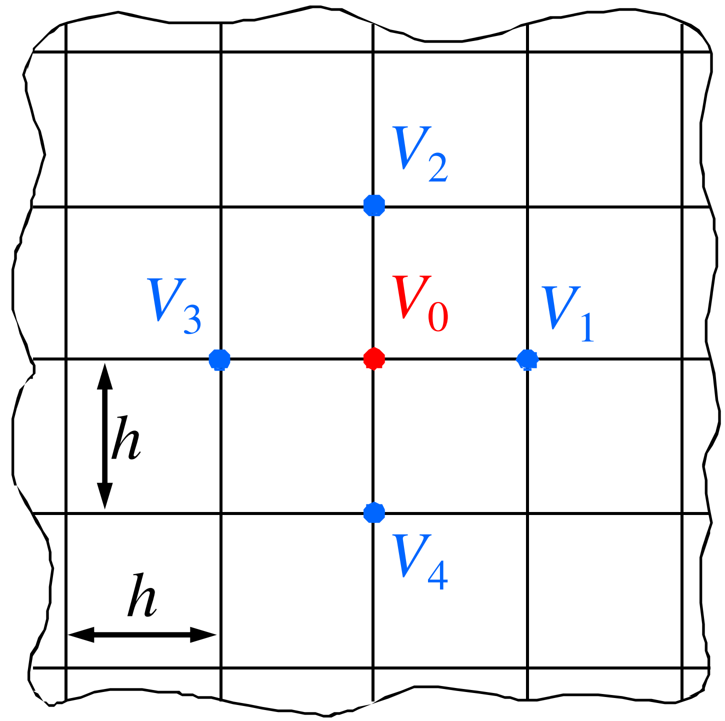


$$\left.\begin{aligned}\left.\frac{\partial V}{\partial x}\right|_a &\approx \frac{V_1 - V_0}{h} \\ \left.\frac{\partial V}{\partial x}\right|_c &\approx \frac{V_0 - V_3}{h} \\ \frac{\partial^2 V}{\partial x^2} &\approx \frac{\left.\frac{\partial V}{\partial x}\right|_a - \left.\frac{\partial V}{\partial x}\right|_c}{h}\end{aligned}\right\} \rightarrow \frac{\partial^2 V}{\partial x^2} \approx \frac{V_1 - V_0 - V_0 + V_3}{h^2}$$

Finite Difference Method (2)



$$\left.\begin{aligned}
&\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0\\
&\frac{\partial^2 V}{\partial x^2}\approx\frac{V_1-V_0-V_0+V_3}{h^2}\\
&\frac{\partial^2 V}{\partial y^2}\approx\frac{V_2-V_0-V_0+V_4}{h^2}
\end{aligned}\right\}$$



$$\rightarrow\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\approx\frac{V_1+V_2+V_3+V_4-4V_0}{h}=0$$

$$\rightarrow\boxed{V_0\approx\frac{1}{4}\left(V_1+V_2+V_3+V_4\right)}$$

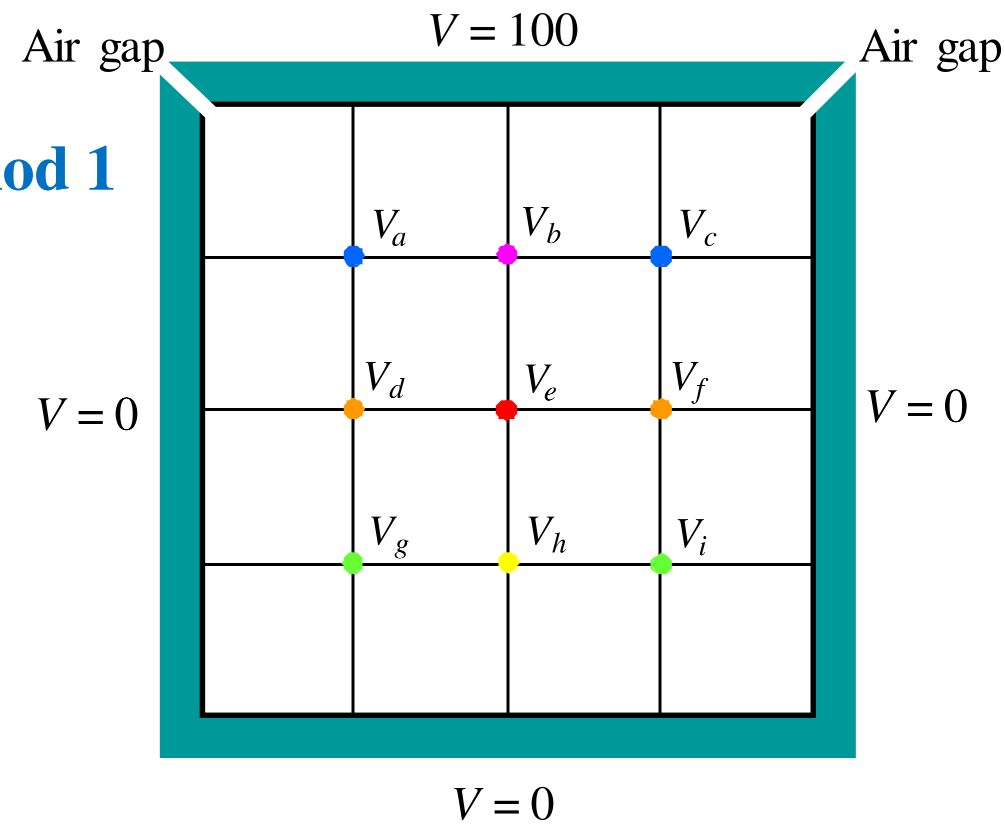
Finite Difference Method (3)

Ex. 1

$$V_0 = \frac{1}{4}\left(V_1 + V_2 + V_3 + V_4\right)$$

Method 1

$$\begin{cases} 4V_a = V_d + V_b + 100 + 0 \\ 4V_b = V_a + V_e + V_c + 100 \\ 4V_c = V_b + V_f + 0 + 100 \\ 4V_d = V_g + V_e + V_a + 0 \\ 4V_e = V_b + V_d + V_h + V_f \\ 4V_f = V_c + V_e + V_i + 0 \\ 4V_g = V_d + 0 + 0 + V_h \\ 4V_h = V_e + V_g + 0 + V_i \\ 4V_i = V_f + V_h + 0 + 0 \end{cases} \rightarrow \begin{cases} V_a = 42.86\,\text{V} \\ V_b = 52.68\,\text{V} \\ V_c = 42.86\,\text{V} \\ V_d = 18.75\,\text{V} \\ V_e = 25.00\,\text{V} \\ V_f = 18,75\,\text{V} \\ V_g = 7.14\,\text{V} \\ V_h = 9.82\,\text{V} \\ V_i = 7.14\,\text{V} \end{cases}$$

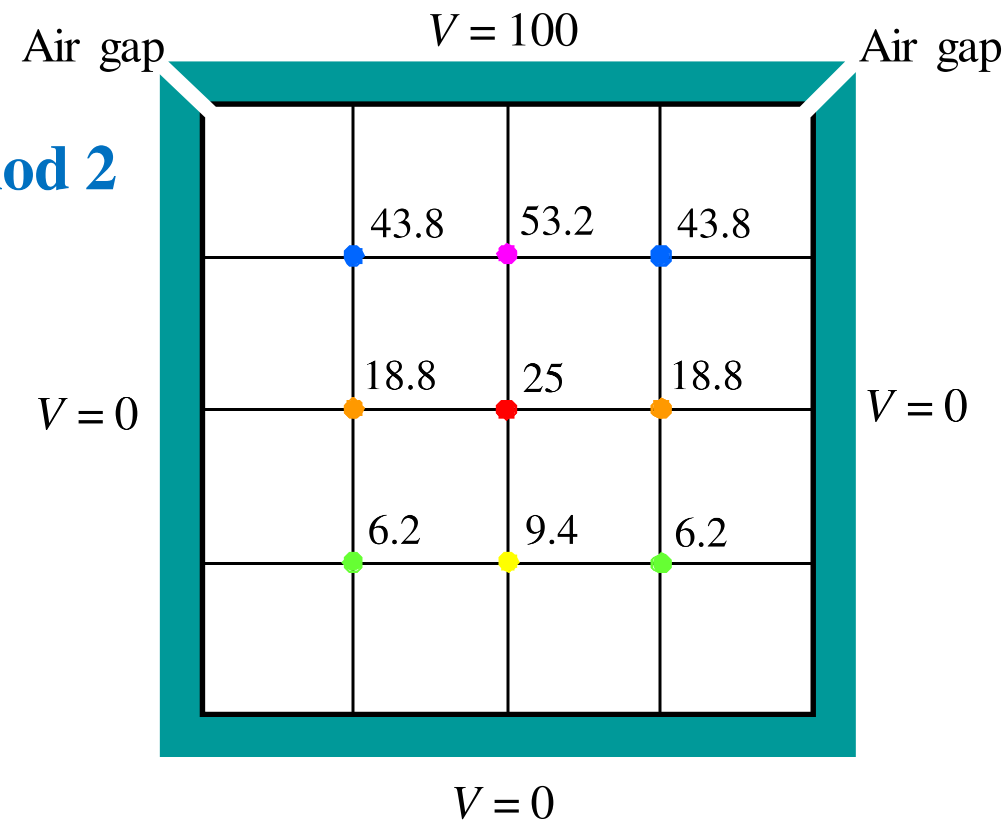


Finite Difference Method (4)

Ex. 1

$$V_0 = \frac{1}{4}(V_1 + V_2 + V_3 + V_4)$$

Method 2



$$\frac{1}{4}(0+100+0+0) = 25$$

$$\frac{1}{4}(100+50+0+25) = 43.8$$

$$\frac{1}{4}(0+25+0+0) = 6.2$$

$$\frac{1}{4}(43.8+100+43.8+25) = 53.2$$

$$\frac{1}{4}(25+43.8+0+6.2) = 18.8$$

$$\frac{1}{4}(6.2+25+6.2+0) = 9.4$$

Step 1

Finite Difference Method (5)

Ex. 1

$$V_0 = \frac{1}{4}(V_1 + V_2 + V_3 + V_4)$$

Method 2

$$\boxed{\frac{1}{4}(100+50+0+25) = 43.8}$$

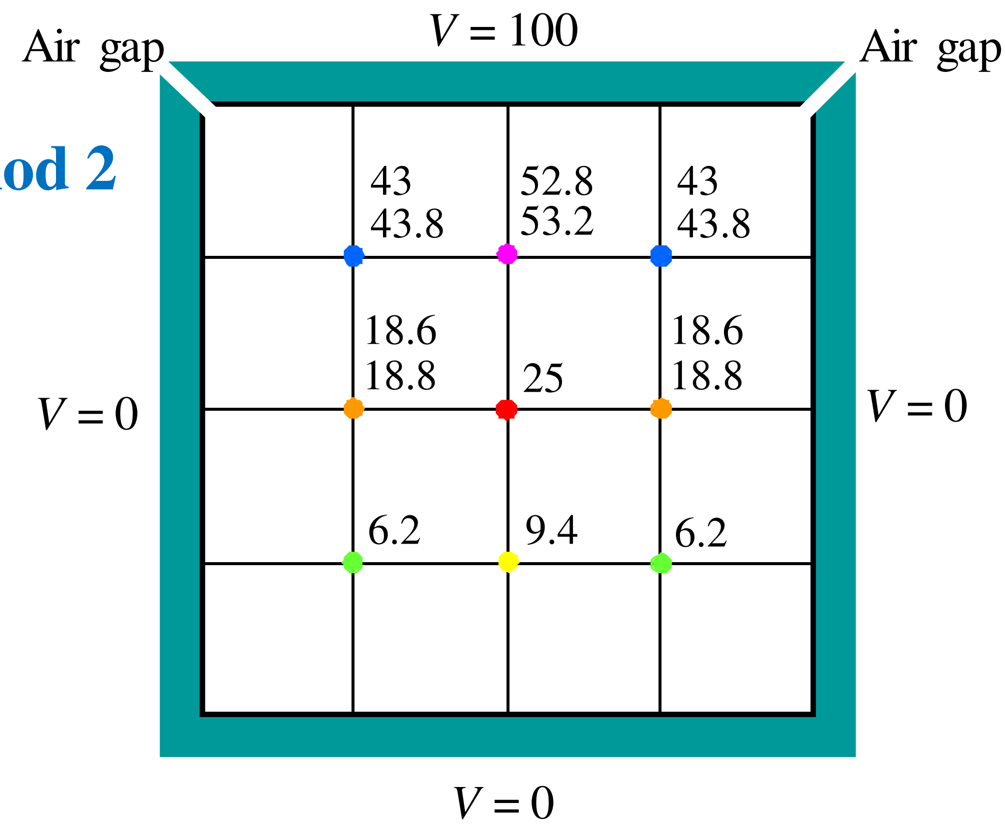
$$\boxed{\frac{1}{4}(53.2+100+0+18.8) = 43}$$

$$\boxed{\frac{1}{4}(43.8+100+43.8+25) = 53.2}$$

$$\boxed{\frac{1}{4}(43+100+43+25) = 52.8}$$

$$\boxed{\frac{1}{4}(25+43.8+0+6.2) = 18.8}$$

$$\boxed{\frac{1}{4}(25+43+0+6.2) = 18.6}$$



Step 2

Finite Difference Method (6)

Ex. 1

$$V_0 = \frac{1}{4}(V_1 + V_2 + V_3 + V_4)$$

Method 2

$$\frac{1}{4}(0+100+0+0) = 25$$

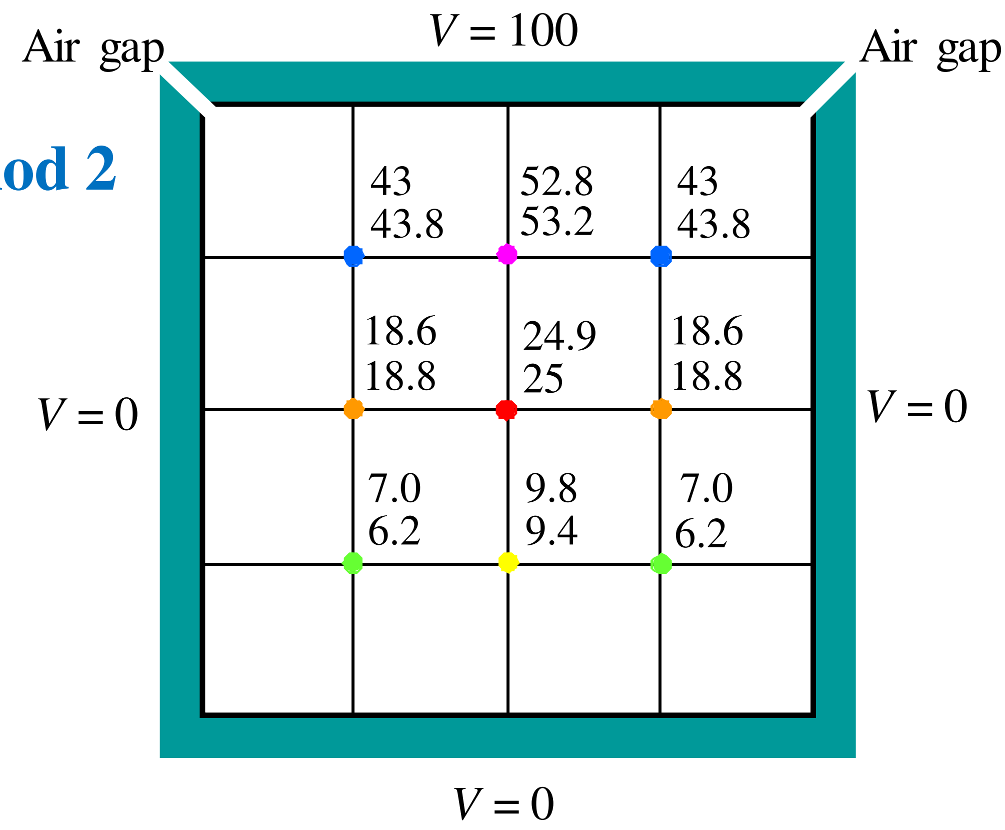
$$\frac{1}{4}(18.6+52.8+18.6+9.4) = 24.9$$

$$\frac{1}{4}(0+25+0+0) = 6.2$$

$$\frac{1}{4}(9.4+18.6+0+0) = 7.0$$

$$\frac{1}{4}(6.2+25+6.2+0) = 9.4$$

$$\frac{1}{4}(7.0+25+7.0+0) = 9.8$$



Step 2

Finite Difference Method (7)

Ex. 1

$$V_0 = \frac{1}{4}(V_1 + V_2 + V_3 + V_4)$$

Method 2

$$\frac{1}{4}(52.8 + 100 + 0 + 18.6) = 42.9$$

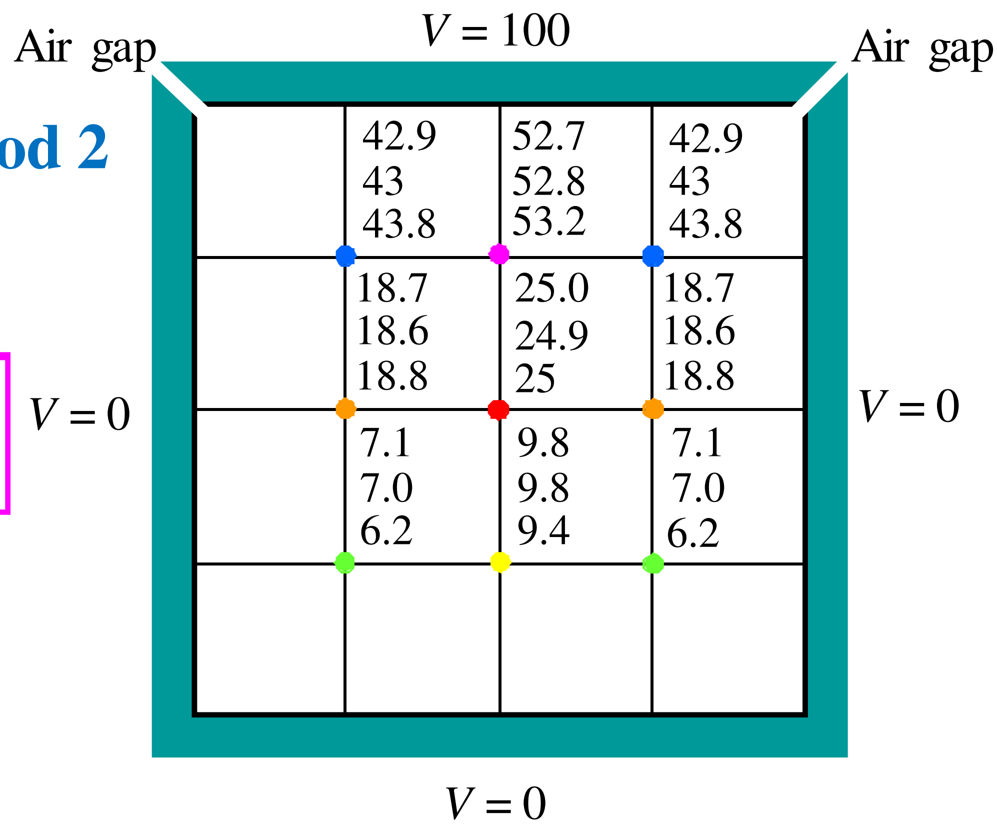
$$\frac{1}{4}(42.9 + 100 + 42.9 + 24.9) = 52.7$$

$$\frac{1}{4}(24.9 + 42.9 + 0 + 7.0) = 18.7$$

$$\frac{1}{4}(18.7 + 52.7 + 18.7 + 9.8) = 25.0$$

$$\frac{1}{4}(9.8 + 18.7 + 0 + 0) = 7.1$$

$$\frac{1}{4}(7.1 + 25 + 7.1 + 0) = 9.8$$

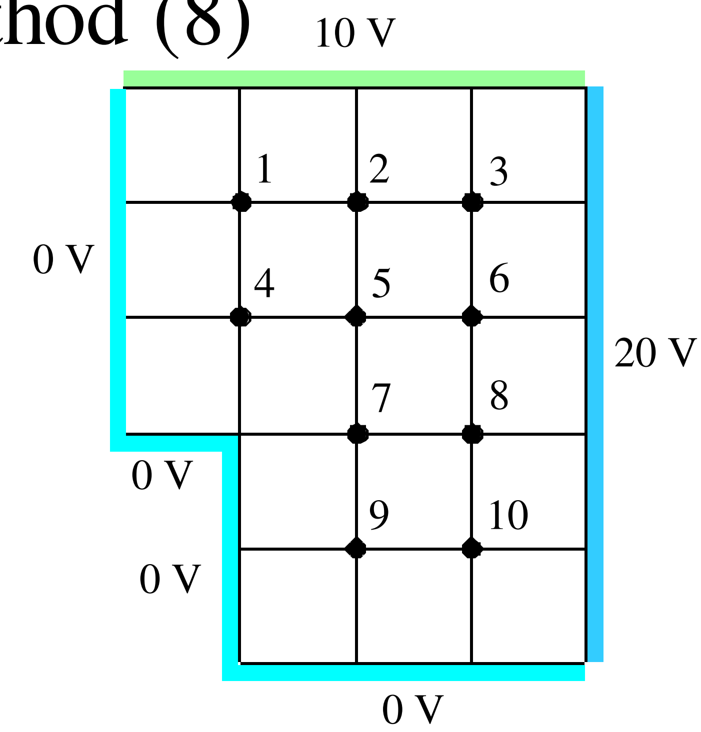


Step 3

Finite Difference Method (8)

Ex. 2

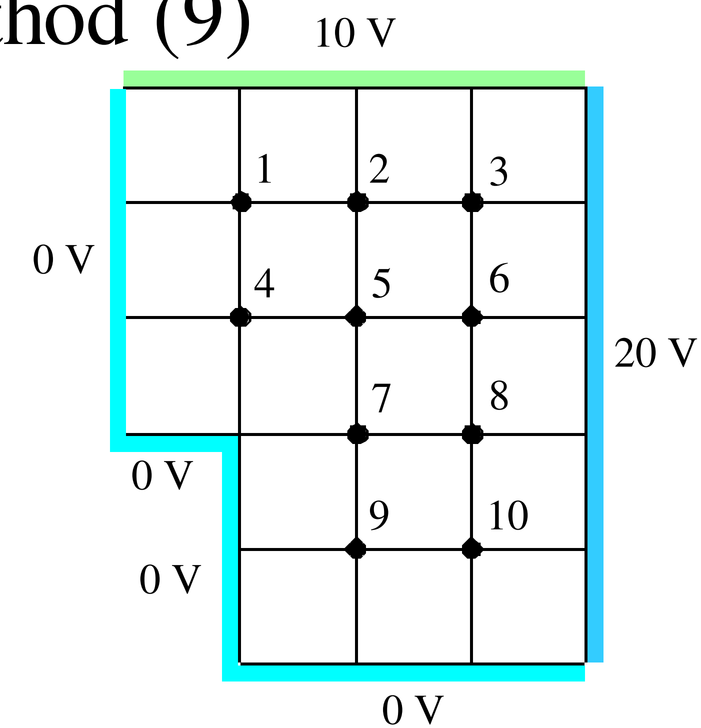
Method 1



$$\begin{cases} 4V_1 = V_2 + 10 + 0 + V_4 \\ 4V_2 = V_1 + V_5 + V_3 + 10 \\ 4V_3 = V_2 + V_6 + 20 + 10 \\ 4V_4 = V_1 + 0 + 0 + V_5 \\ 4V_5 = V_2 + V_4 + V_7 + V_6 \\ 4V_6 = V_3 + V_5 + V_8 + 20 \\ 4V_7 = V_5 + 0 + V_9 + V_8 \\ 4V_8 = V_6 + V_7 + V_{10} + 20 \\ 4V_9 = V_7 + 0 + 0 + V_{10} \\ 4V_{10} = V_8 + V_9 + 0 + 20 \end{cases} \rightarrow \begin{cases} V_1 = 5.6423\,\text{V} \\ V_2 = 9.1735\,\text{V} \\ V_3 = 13.1111\,\text{V} \\ V_4 = 3.3957\,\text{V} \\ V_5 = 7.9405\,\text{V} \\ V_6 = 13.2710\,\text{V} \\ V_7 = 5.9219\,\text{V} \\ V_8 = 12.0324\,\text{V} \\ V_9 = 3.7147\,\text{V} \\ V_{10} = 8.9368\,\text{V} \end{cases}$$

Finite Difference Method (9)

Ex. 2



Method 2

$$V_1^{(0)} = V_2^{(0)} = \ldots = V_{10}^{(0)} = 0$$

$$V_1^{(1)} = \frac{1}{4}\left(V_2^{(0)} + 10 + 0 + V_4^{(0)}\right) = 2.5000\,\text{V}$$

$$V_2^{(1)} = \frac{1}{4}\left(V_3^{(0)} + 10 + V_1^{(1)} + V_5^{(0)}\right) = 3.1250\,\text{V}$$

...

$$V_7^{(1)} = \frac{1}{4}\left(V_8^{(0)} + V_5^{(1)} + 0 + V_9^{(0)}\right) = 0.2344\,\text{V}$$

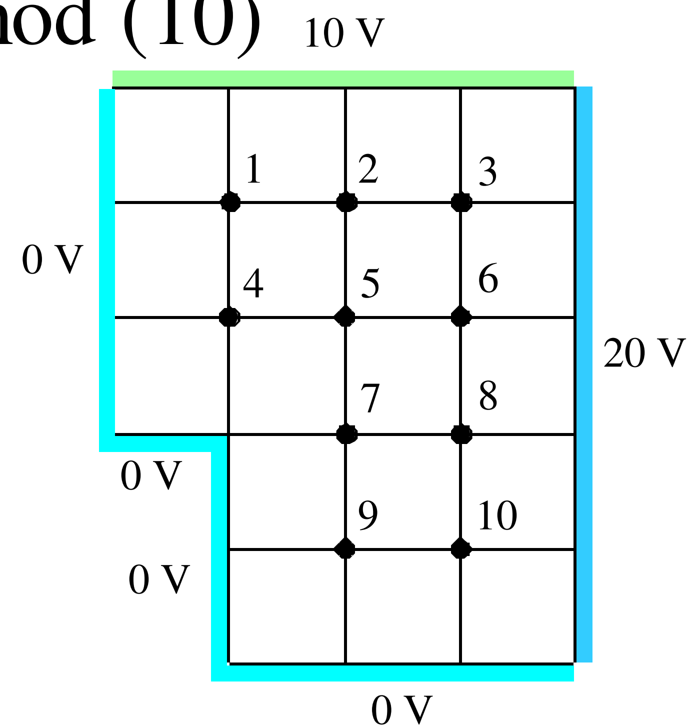
...

$$V_{10}^{(1)} = \frac{1}{4}\left(20 + V_8^{(1)} + V_9^{(1)} + 0\right) = 6.7358\,\text{V}$$

Finite Difference Method (10)

Ex. 2

k	0	1		23	24
$V_1^{(k)}$ (V)	0	2.5000		5.6429	5.6429
$V_2^{(k)}$ (V)	0	3.1250		9.1735	9.1735
$V_3^{(k)}$ (V)	0	8.2813		13.1111	13.1111
$V_4^{(k)}$ (V)	0	0.6250		3.3957	3.3957
$V_5^{(k)}$ (V)	0	0.9375		7.9405	7.9405
$V_6^{(k)}$ (V)	0	7.3047		13.2710	13.2710
$V_7^{(k)}$ (V)	0	0.2344		5.9219	5.9219
$V_8^{(k)}$ (V)	0	6.8848		13.0324	13.0324
$V_9^{(k)}$ (V)	0	0.0586		3.7147	3.7147
$V_{10}^{(k)}$ (V)	0	6.7358		8.9368	8.9368



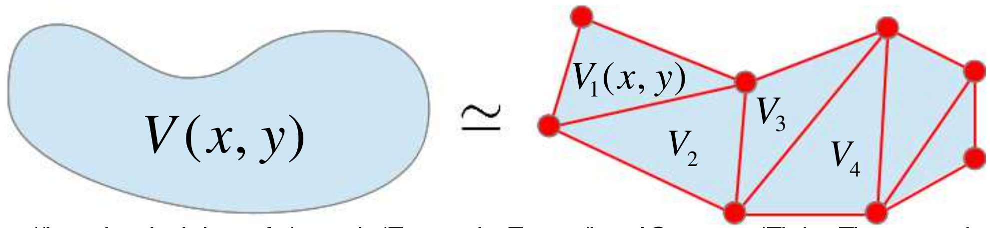
Method 2

Poisson's & Laplace's Equations

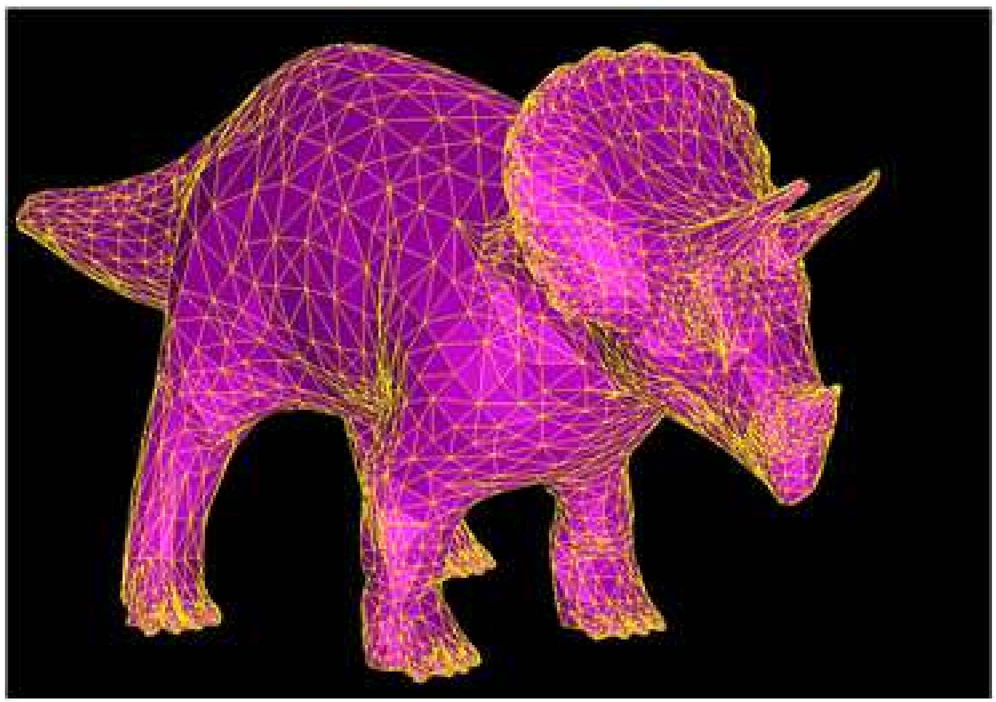
1. Poisson's Equation
2. Laplace's Equation
3. Uniqueness Theorem
4. Examples of the Solution of Laplace's Equation
5. Examples of the Solution of Poisson's Equation
6. Product Solution of Laplace's Equation
7. **Numerical Methods**
   a. Finite Difference Method
   b. **Finite Element Method**

Finite Element Method (1)

$$\nabla^2 V = 0$$



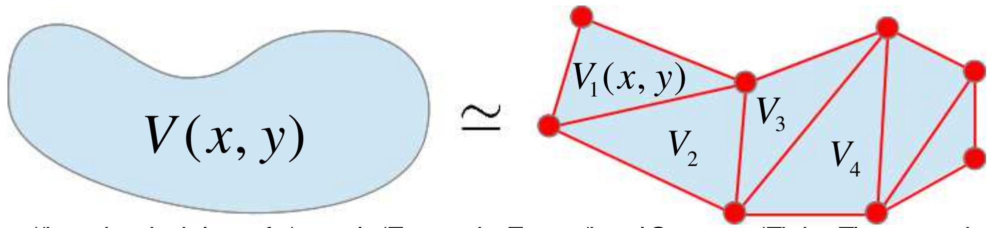
http://imagine.inrialpes.fr/people/Francois.Faure/htmlCourses/FiniteElements.html



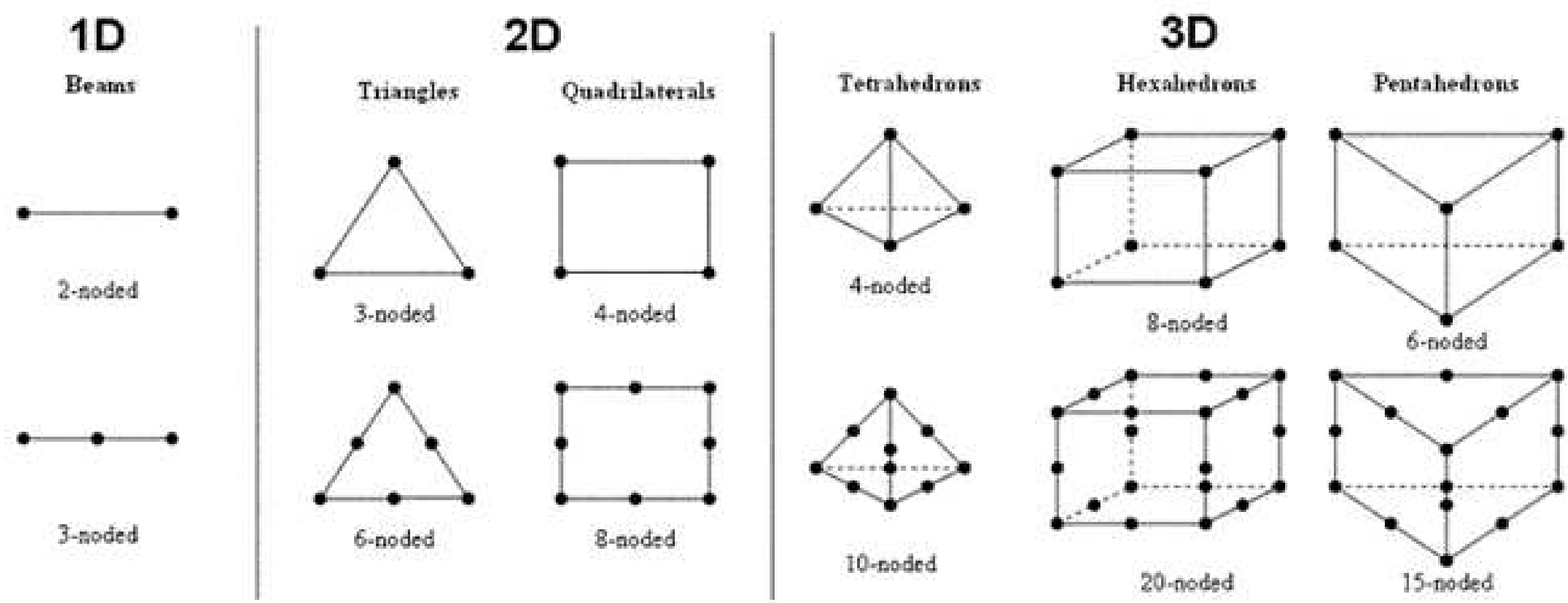
http://www.cosy.sbg.ac.at/~held/projects/mesh/mesh.html

Finite Element Method (2)

$$\nabla^2 V = 0$$



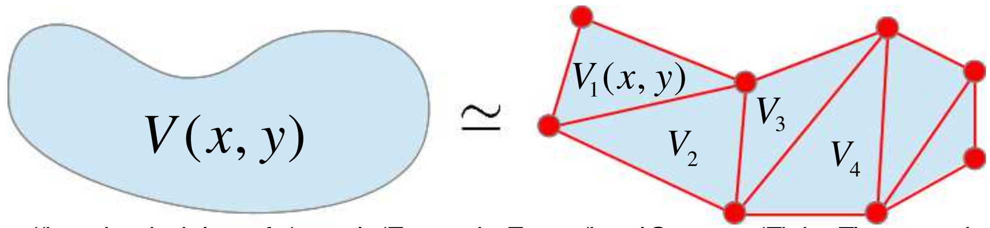
http://imagine.inrialpes.fr/people/Francois.Faure/htmlCourses/FiniteElements.html



http://illustrations.marin.ntnu.no/structures/analysis/FEM/theory/index.html

Finite Element Method (3)

$$\nabla^2 V = 0$$



http://imagine.inrialpes.fr/people/Francois.Faure/htmlCourses/FiniteElements.html

- Discretizing the solution region into a finite number of elements,
- Obtaining governing equations for a typical elements,
- Combining all elements in the solution, &
- Solving the system of equations obtained.

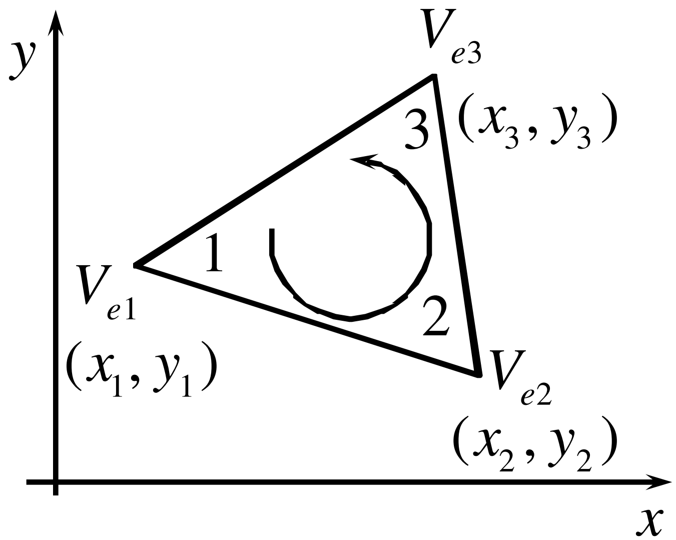
Finite Element Method (4)

$$V(x,y)=\sum_{e=1}^{N}V_e(x,y)$$

$$V_e(x,y)=a+bx+cy$$

$$\begin{bmatrix}V_{e1}\\V_{e2}\\V_{e3}\end{bmatrix}=\begin{bmatrix}1 & x_1 & y_1\\1 & x_2 & y_2\\1 & x_3 & y_3\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}1 & x_1 & y_1\\1 & x_2 & y_2\\1 & x_3 & y_3\end{bmatrix}^{-1}\begin{bmatrix}V_{e1}\\V_{e2}\\V_{e3}\end{bmatrix}$$



$$\rightarrow V_e=\begin{bmatrix}1 & x & y\end{bmatrix}\frac{1}{\begin{vmatrix}1 & x_1 & y_1\\1 & x_2 & y_2\\1 & x_3 & y_3\end{vmatrix}}\begin{bmatrix}(x_2y_3-x_3y_2) & (x_3y_1-x_1y_3) & (x_1y_2-x_2y_1)\\(y_2-y_3) & (y_3-y_1) & (y_1-y_2)\\(x_3-x_2) & (x_1-x_3) & (x_2-x_1)\end{bmatrix}^{-1}\begin{bmatrix}V_{e1}\\V_{e2}\\V_{e3}\end{bmatrix}$$

Finite Element Method (5)

$$V_e = \begin{bmatrix}1 & x & y\end{bmatrix} \frac{1}{\begin{vmatrix}1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3\end{vmatrix}} \begin{bmatrix}(x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1)\end{bmatrix}^{-1} \begin{bmatrix}V_{e1} \\ V_{e2} \\ V_{e3}\end{bmatrix}$$

$$\boxed{V_e = \sum_{i=1}^{3} \alpha_i(x, y) V_{ei}}$$

$$\alpha_1 = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y],$$

$$\alpha_2 = \frac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y],$$

$$\alpha_3 = \frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y],$$

$$A = \frac{1}{2}[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)]$$

Finite Element Method (6)

$$\nabla^2 V = 0$$

$$\left.\begin{array}{l} W_e = \dfrac{1}{2}\int_S \varepsilon E_e^2 dS \\ \mathbf{E} = -\nabla V \end{array}\right\}$$

$$ax + b = 0$$

$$\frac{d}{dx}\left(\frac{ax^2}{2} + bx + c\right) = 0$$

$$\left.\begin{array}{l} \rightarrow W_e = \dfrac{1}{2}\int_S \varepsilon \left|\nabla V_e\right|^2 dS \\ V_e = \displaystyle\sum_{i=1}^{3} \alpha_i(x, y) V_{ei} \\ \rightarrow \nabla V_e = \displaystyle\sum_{i=1}^{3} V_{ei} \nabla \alpha_i \end{array}\right\}$$

$$\rightarrow W_e = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon V_{ei}\left[\int_S (\nabla \alpha_i)(\nabla \alpha_j) dS\right] V_{ej}$$

Finite Element Method (7)

$$\left.\begin{aligned}
&W_e = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon V_{ei}\left[\int_S (\nabla\alpha_i)(\nabla\alpha_j)dS\right]V_{ej} \\
&C_{ij}^{(e)} = \int_S (\nabla\alpha_i)(\nabla\alpha_j)dS \\
&[V_e] = \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} \\
&\left[C^{(e)}\right] = \begin{bmatrix} C_{11}^{(e)} & C_{12}^{(e)} & C_{13}^{(e)} \\ C_{21}^{(e)} & C_{22}^{(e)} & C_{23}^{(e)} \\ C_{31}^{(e)} & C_{32}^{(e)} & C_{33}^{(e)} \end{bmatrix}
\end{aligned}\right\}$$

$$\rightarrow W_e = \frac{1}{2}\varepsilon[V_e]^T\left[C^{(e)}\right][V_e]$$

Finite Element Method (8)

$$C_{ij}^{(e)} = \int_S (\nabla\alpha_i)(\nabla\alpha_j)dS$$

$$\left.\begin{array}{l} C_{12}^{(e)} = \int_S (\nabla\alpha_1)(\nabla\alpha_2)dS \\ \\ \alpha_1 = \dfrac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y], \\ \alpha_2 = \dfrac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y], \\ A = \dfrac{1}{2}[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)] \end{array}\right\}$$

$$\to C_{12}^{(e)} = \frac{1}{4A}[(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)]$$

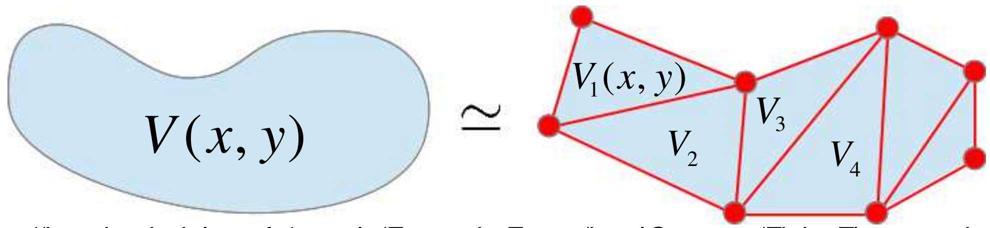
Finite Element Method (9)

$$\left.\begin{aligned}
C_{12}^{(e)} &= \frac{1}{4A}[(y_2-y_3)(y_3-y_1)+(x_3-x_2)(x_1-x_3)]\\
C_{13}^{(e)} &= \frac{1}{4A}[(y_2-y_3)(y_1-y_2)+(x_3-x_2)(x_2-x_1)]\\
C_{23}^{(e)} &= \frac{1}{4A}[(y_3-y_1)(y_1-y_2)+(x_1-x_3)(x_2-x_1)]\\
C_{11}^{(e)} &= \frac{1}{4A}[(y_2-y_3)^2+(x_3-x_2)^2]\\
C_{22}^{(e)} &= \frac{1}{4A}[(y_3-y_1)^2+(x_1-x_3)^2]\\
C_{33}^{(e)} &= \frac{1}{4A}[(y_1-y_2)^2+(x_2-x_1)^2]\\
C_{21}^{(e)} &= C_{12}^{(e)}, \quad C_{31}^{(e)} = C_{13}^{(e)}, \quad C_{32}^{(e)} = C_{23}^{(e)}\\
&P_1 = y_2-y_3,\ P_2 = y_3-y_1,\ P_3 = y_1-y_2\\
&Q_1 = x_3-x_2,\ Q_2 = x_1-x_3,\ Q_3 = x_2-x_1
\end{aligned}\right\} \rightarrow C_{ij}^{(e)} = \frac{1}{4A}(P_iP_j+Q_iQ_j)$$

$$A = \frac{P_2Q_3-P_3Q_2}{2}$$

Finite Element Method (3)

$$\nabla^2 V = 0$$



http://imagine.inrialpes.fr/people/Francois.Faure/htmlCourses/FiniteElements.html

- Discretizing the solution region into a finite number of elements,
- Obtaining governing equations for a typical elements,
- **Combining all elements in the solution,** &
- Solving the system of equations obtained.

Finite Element Method (10)

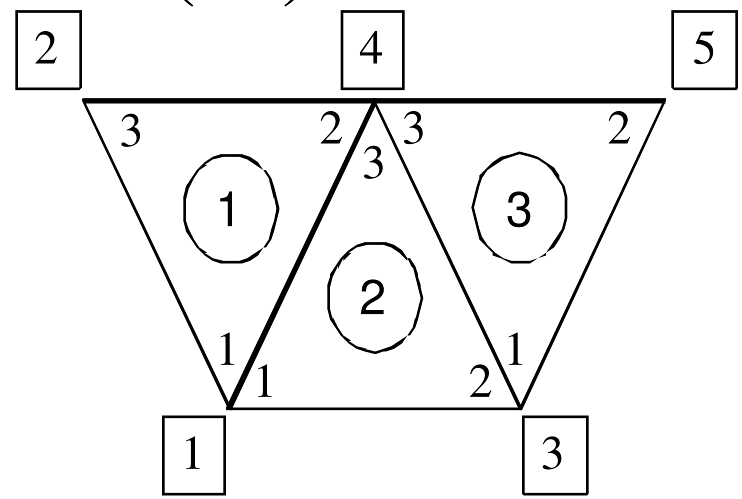
$$W_e = \frac{1}{2}\varepsilon [V_e]^T \left[C^{(e)}\right][V_e]$$

$$W = \sum_{e=1}^{N} W_e = \frac{1}{2}\varepsilon [V]^T [C][V]$$

$$[V] = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_n \end{bmatrix}$$

Finite Element Method (11)

$$W = \frac{1}{2}\varepsilon[V]^T[C][V]$$



$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} \end{bmatrix}$$

$$[C] = \begin{bmatrix} C_{11}^{(1)} + C_{11}^{(2)} & C_{13}^{(1)} & C_{12}^{(2)} & C_{12}^{(1)} + C_{13}^{(2)} & 0 \\ C_{31}^{(1)} & C_{33}^{(1)} & 0 & C_{32}^{(1)} & 0 \\ C_{21}^{(2)} & 0 & C_{22}^{(2)} + C_{11}^{(3)} & C_{23}^{(2)} + C_{13}^{(3)} & C_{12}^{(3)} \\ C_{21}^{(1)} + C_{31}^{(2)} & C_{23}^{(1)} & C_{32}^{(2)} + C_{31}^{(3)} & C_{22}^{(1)} + C_{33}^{(2)} + C_{33}^{(3)} & C_{32}^{(3)} \\ 0 & 0 & C_{21}^{(3)} & C_{23}^{(3)} & C_{22}^{(3)} \end{bmatrix}$$

$$C_{11} = C_{11}^{(1)} + C_{11}^{(2)}$$

$$C_{22} = C_{33}^{(1)}$$

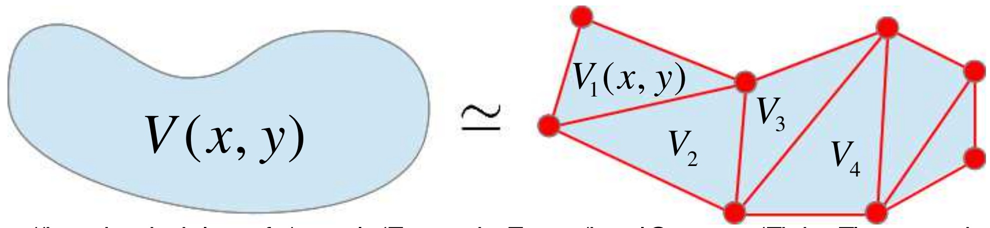
$$C_{44} = C_{22}^{(1)} + C_{33}^{(2)} + C_{33}^{(3)}$$

$$C_{14} = C_{41} = C_{12}^{(1)} + C_{13}^{(2)}$$

$$C_{23} = C_{32} = 0$$

Finite Element Method (3)

$\nabla^2 V = 0$



http://imagine.inrialpes.fr/people/Francois.Faure/htmlCourses/FiniteElements.html

- Discretizing the solution region into a finite number of elements,
- Obtaining governing equations for a typical elements,
- Combining all elements in the solution, &
- **Solving the system of equations obtained**.

Finite Element Method (12)

$$\nabla^2 V = 0$$

$$W = \frac{1}{2}\varepsilon [V]^T [C][V]$$

$$ax + b = 0$$

$$\frac{d}{dx}\left(\frac{ax^2}{2} + bx + c\right) = 0$$

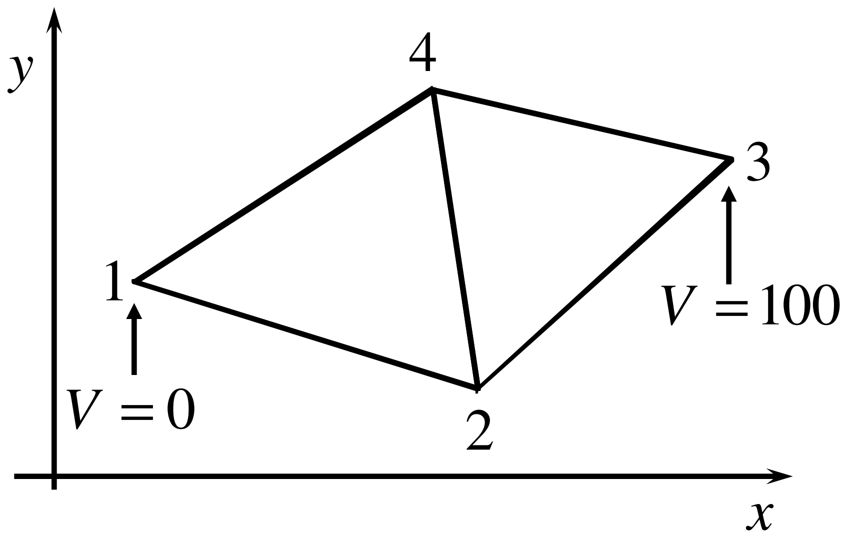
$$\frac{\partial W}{\partial V_1} = \frac{\partial W}{\partial V_2} = \cdots = \frac{\partial W}{\partial V_n} = 0 \leftrightarrow \frac{\partial W}{\partial V_k} = 0, \quad k = 1, 2, \ldots, n$$

$$\rightarrow \boxed{V_k = -\frac{1}{C_{kk}} \sum_{i=1, i \neq k}^{n} V_i C_{ki}}$$

Finite Element Method (13)

Ex.

Node	**1**	**2**	**3**	**4**
x	0.5	3.1	5.0	2.8
y	1.0	0.4	1.7	2.0



$$C_{ij}^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j), \qquad A = \frac{P_2Q_3 - P_3Q_2}{2}$$

$$P_1 = y_2 - y_3,\ P_2 = y_3 - y_1,\ P_3 = y_1 - y_2$$

$$Q_1 = x_3 - x_2,\ Q_2 = x_1 - x_3,\ Q_3 = x_2 - x_1$$

Element 1:

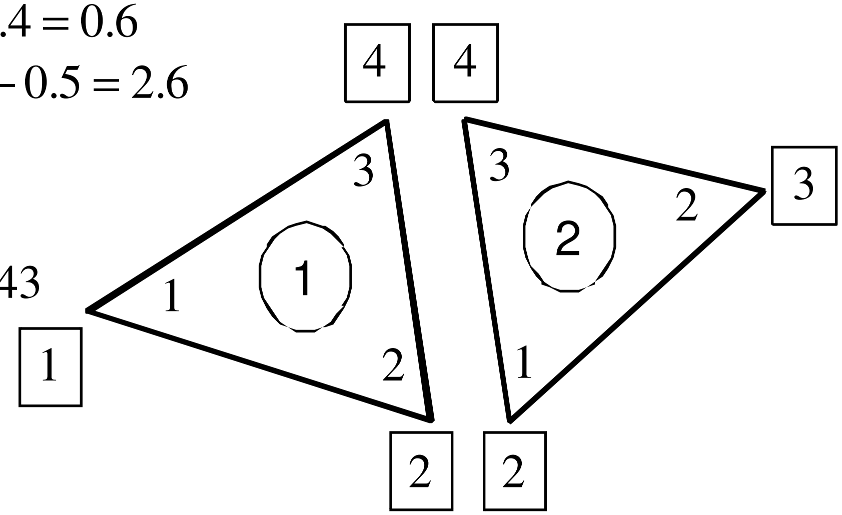
$$P_1 = 0.4 - 2.0 = -1.6;\ P_2 = 2.0 - 1.0 = 1.0;\ P_3 = 1.0 - 0.4 = 0.6$$

$$Q_1 = 2.8 - 3.1 = -0.3;\ Q_2 = 0.5 - 2.8 = -2.3;\ Q_3 = 3.1 - 0.5 = 2.6$$

$$A = \frac{1.0\times 2.6 - 0.6(-2.3)}{2} = 1.99$$

$$C_{12}^{(1)} = \frac{P_1P_2 + Q_1Q_2}{4\times 1.99} = \frac{(-1.6)1.0 + (-0.3)(-2.3)}{4\times 1.99} = -0.1143$$

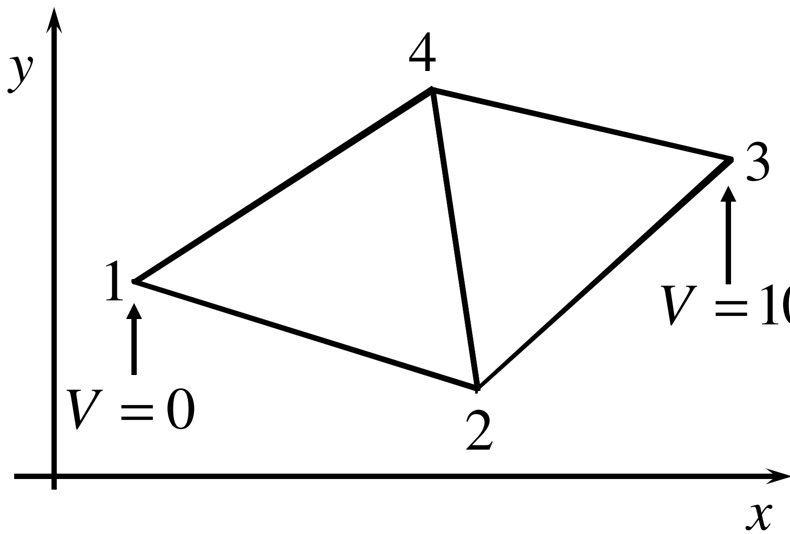
$$\left[C^{(1)}\right] = \begin{bmatrix} 0.3329 & -0.1143 & -0.2186 \\ -0.1143 & 0.7902 & -0.6759 \\ -0.2186 & -0.6759 & 0.8945 \end{bmatrix}$$



Finite Element Method (14)

Ex.

Node	1	2	3	4
x	0.5	3.1	5.0	2.8
y	1.0	0.4	1.7	2.0



$$C_{ij}^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j), \qquad A = \frac{P_2Q_3 - P_3Q_2}{2}$$

$$P_1 = y_2 - y_3,\ P_2 = y_3 - y_1,\ P_3 = y_1 - y_2$$

$$Q_1 = x_3 - x_2,\ Q_2 = x_1 - x_3,\ Q_3 = x_2 - x_1$$

Element 2:

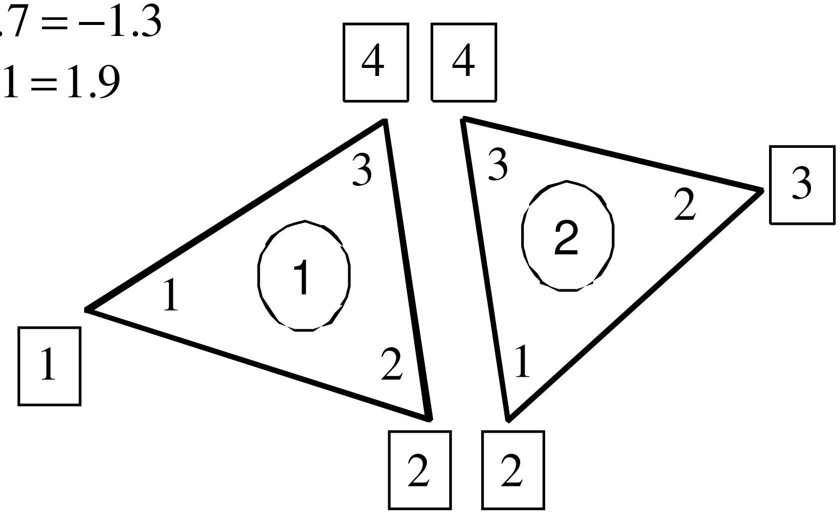
$$P_1 = 1.7 - 2.0 = -0.3;\ \ P_2 = 2.0 - 0.4 = 1.6;\ \ P_3 = 0.4 - 1.7 = -1.3$$

$$Q_1 = 2.8 - 5 = -2.2;\ \ Q_2 = 3.1 - 2.8 = 0.3;\ \ Q_3 = 5.0 - 3.1 = 1.9$$

$$A = \frac{1.6\times1.9 - (-1.3)0.3}{2} = 1.715$$

$$C_{22}^{(2)} = \frac{P_2P_2 + Q_2Q_2}{4\times1.715} = \frac{1.6\times1.6 + 0.3\times0.3}{4\times1.715} = 0.3863$$

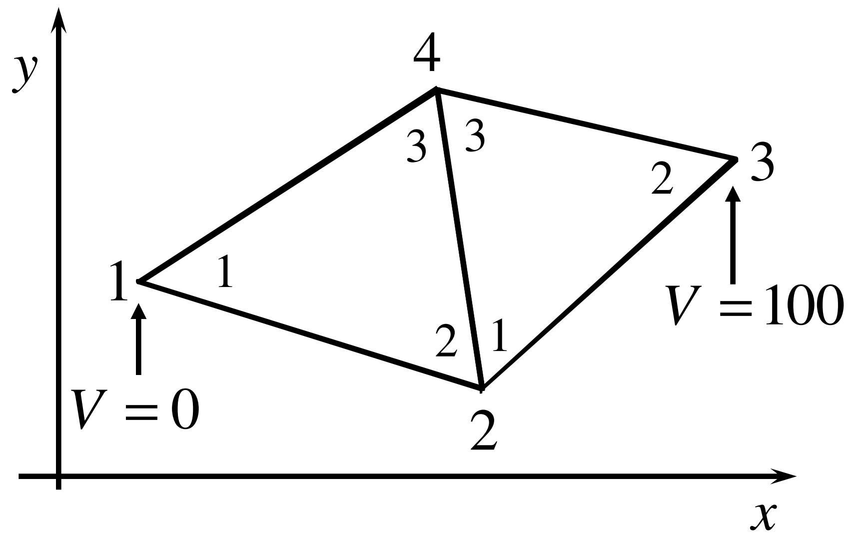
$$\left[C^{(2)}\right] = \begin{bmatrix} 0.7187 & -0.1662 & -0.5525 \\ -0.1662 & 0.3863 & -0.2201 \\ -0.5525 & -0.2201 & 0.7726 \end{bmatrix}$$



Ex. Finite Element Method (15)

$$\left[C^{(1)}\right] = \begin{bmatrix} 0.3329 & -0.1143 & -0.2186 \\ -0.1143 & 0.7902 & -0.6759 \\ -0.2186 & -0.6759 & 0.8945 \end{bmatrix}$$

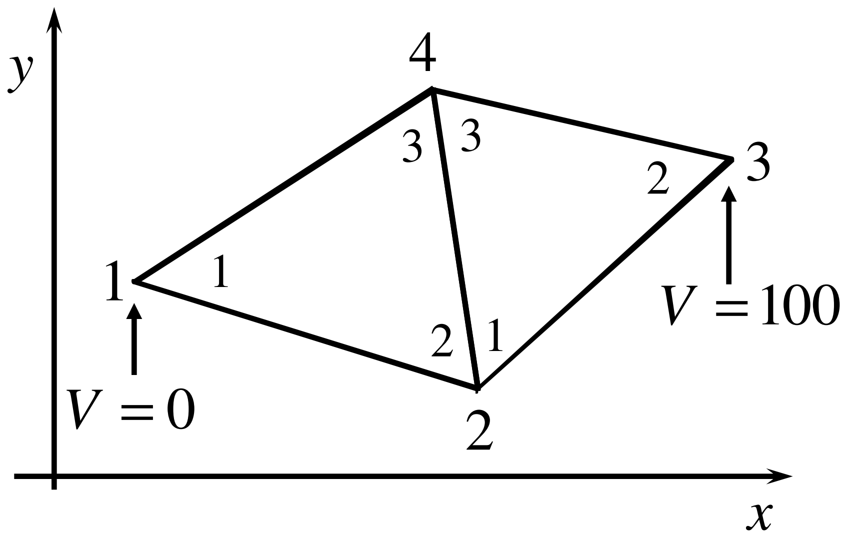
$$\left[C^{(2)}\right] = \begin{bmatrix} 0.7187 & -0.1662 & -0.5525 \\ -0.1662 & 0.3863 & -0.2201 \\ -0.5525 & -0.2201 & 0.7726 \end{bmatrix}$$



$$[C] = \begin{bmatrix} C_{11}^{(1)} & C_{12}^{(1)} & 0 & C_{13}^{(1)} \\ C_{21}^{(1)} & C_{22}^{(1)} + C_{11}^{(2)} & C_{12}^{(2)} & C_{23}^{(1)} + C_{13}^{(2)} \\ 0 & C_{21}^{(2)} & C_{22}^{(2)} & C_{23}^{(2)} \\ C_{31}^{(1)} & C_{32}^{(1)} + C_{31}^{(2)} & C_{32}^{(2)} & C_{33}^{(1)} + C_{33}^{(2)} \end{bmatrix} = \begin{bmatrix} 0.3329 & -0.1143 & 0 & -0.2186 \\ -0.1143 & 1.5089 & -0.1662 & -1.2284 \\ 0 & -0.1662 & 0.3863 & -0.2201 \\ -0.2186 & -1.2284 & -0.2201 & 1.6671 \end{bmatrix}$$

Ex. Finite Element Method (16)

$$[C]=\begin{bmatrix} 0.3329 & -0.1143 & 0 & -0.2186 \\ -0.1143 & 1.5089 & -0.1662 & -1.2284 \\ 0 & -0.1662 & 0.3863 & -0.2201 \\ -0.2186 & -1.2284 & -0.2201 & 1.6671 \end{bmatrix}$$



$$V_k = -\frac{1}{C_{kk}} \sum_{i=1, i\neq k}^{n} V_i C_{ki}$$

$$\rightarrow \begin{cases} V_2 = -\dfrac{1}{C_{22}}(V_1C_{12} + V_3C_{32} + V_4C_{42}) \\ V_4 = -\dfrac{1}{C_{44}}(V_1C_{14} + V_2C_{24} + V_3C_{34}) \end{cases}$$

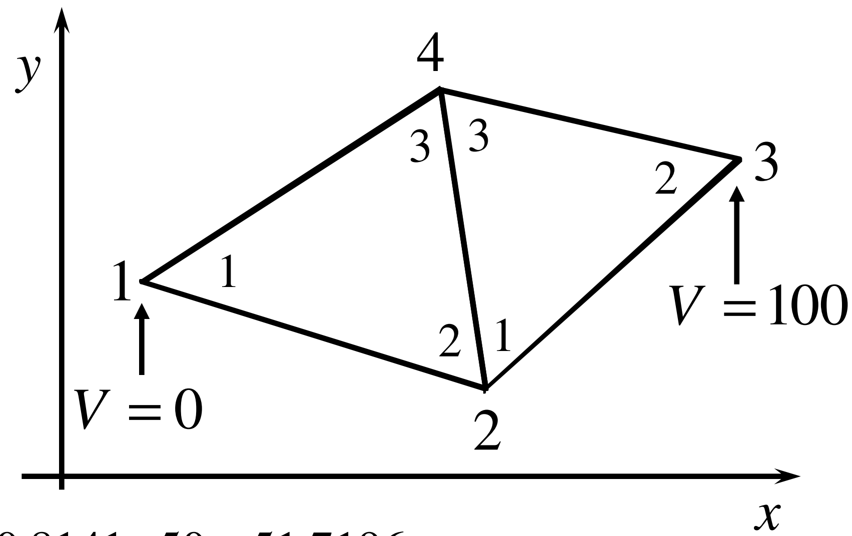
$$\rightarrow \begin{cases} V_2^{(k+1)} = -\dfrac{1}{1.5089}[(0(-0.1143) + 100(-0.1662) + V_4^{(k)}(-1.2284)] = 11.0146 + 0.8141V_4^{(k)} \\ V_4^{(k+1)} = -\dfrac{1}{1.6671}[0(-0.2186) + V_2^{(k)}(-1.2284) + 100(-0.2201)] \ = 13.2026 + 0.7368V_2^{(k)} \end{cases}$$

Ex.

Finite Element Method (17)

$$\begin{cases} V_2^{(k+1)} = 11.0146 + 0.8141 V_4^{(k)} \\ V_4^{(k+1)} = 13.2026 + 0.7368 V_2^{(k)} \end{cases}$$

$$V_2^{(0)} = V_4^{(0)} = \frac{0+100}{2} = 50$$



$$\begin{cases} V_2^{(1)} = 11.0146 + 0.8141 V_4^{(0)} = 11.0146 + 0.8141 \times 50 = 51.7196 \\ V_4^{(1)} = 13.2026 + 0.7368 V_2^{(0)} = 13.2026 + 0.7368 \times 50 = 50.0426 \end{cases}$$

$$\begin{cases} V_2^{(2)} = 11.0146 + 0.8141 V_4^{(1)} = 11.0146 + 0.8141 \times 50.0426 = 51.7543 \\ V_4^{(2)} = 13.2026 + 0.7368 V_2^{(1)} = 13.2026 + 0.7368 \times 51.7196 = 51.3096 \end{cases}$$

$$\begin{cases} V_2^{(3)} = 11.0146 + 0.8141 V_4^{(2)} = 11.0146 + 0.8141 \times 51.3096 = 52.7857 \\ V_4^{(3)} = 13.2026 + 0.7368 V_2^{(2)} = 13.2026 + 0.7368 \times 51.7543 = 51.3352 \end{cases}$$

$$
\begin{tikzcd}
Q \arrow[r] \arrow[dd] & \mathbf{F} = \dfrac{Q_1 Q_2}{4\pi\varepsilon R^2}\mathbf{a}_R \arrow[r] \arrow[d] & \mathbf{E} = \dfrac{Q}{4\pi\varepsilon R^2}\mathbf{a}_R \arrow[r] & \mathbf{D} = \varepsilon\mathbf{E} \\
 & W = -Q\int \mathbf{E}.d\mathbf{L} \arrow[r] & V = -\int \mathbf{E}.d\mathbf{L} \arrow[r] \arrow[d] & C = \dfrac{Q}{V} \\
I = \dfrac{dQ}{dt} \arrow[r] & R = \dfrac{V}{I} & \nabla^2 V = -\dfrac{\rho_v}{\varepsilon} &
\end{tikzcd}
$$