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Engineering Electromagnetics

Plane Wave Reflection & Dispersion

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Plane Wave Reflection & Dispersion

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Reflection of Uniform Plane Waves at Normal Incidence (1)

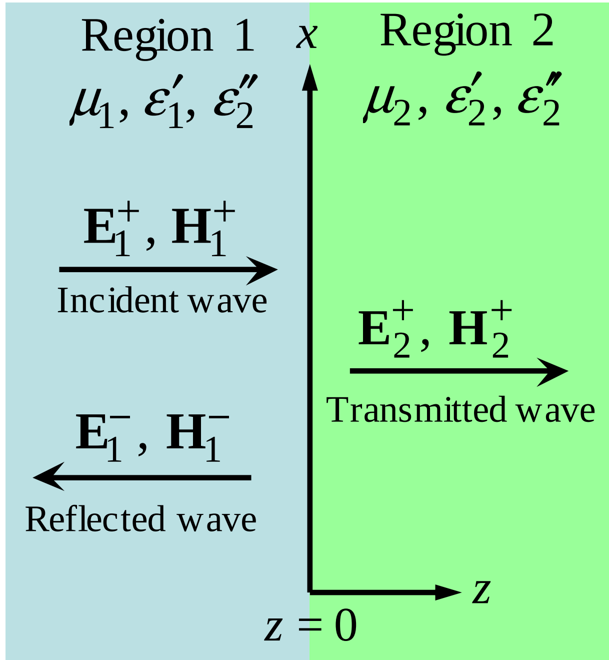
$$E_{x1}^{+}(z,t) = E_{x10}^{+} e^{-\alpha_1 z} \cos(\omega t - \beta_1 z)$$

$$E_{xs1}^{+} = E_{x10}^{+} e^{-jk_1 z}$$

$$H_{ys1}^{+} = \frac{1}{\eta_1} E_{x10}^{+} e^{-jk_1 z}$$

$$E_{xs2}^{+} = E_{x20}^{+} e^{-jk_2 z}$$

$$H_{ys2}^{+} = \frac{1}{\eta_2} E_{x20}^{+} e^{-jk_2 z}$$



Boundary c.: $E_{xs1}^{+}\Big|_{z=0} = E_{xs2}^{+}\Big|_{z=0} \rightarrow E_{x10}^{+} = E_{x20}^{+}$

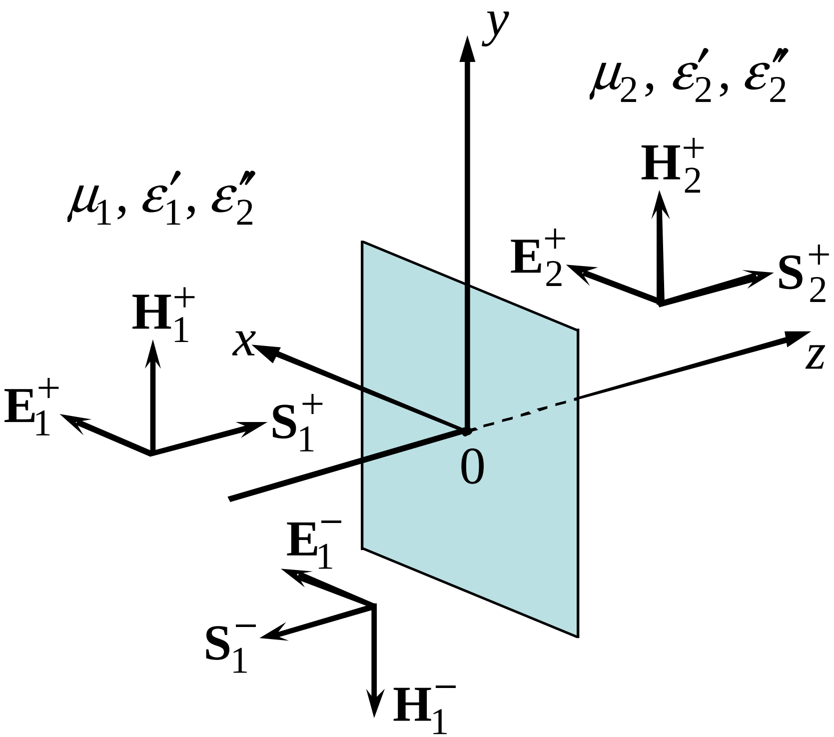
Boundary c.: $H_{xs1}^{+}\Big|_{z=0} = H_{xs2}^{+}\Big|_{z=0} \rightarrow \dfrac{E_{x10}^{+}}{\eta_1} = \dfrac{E_{x20}^{+}}{\eta_2}$

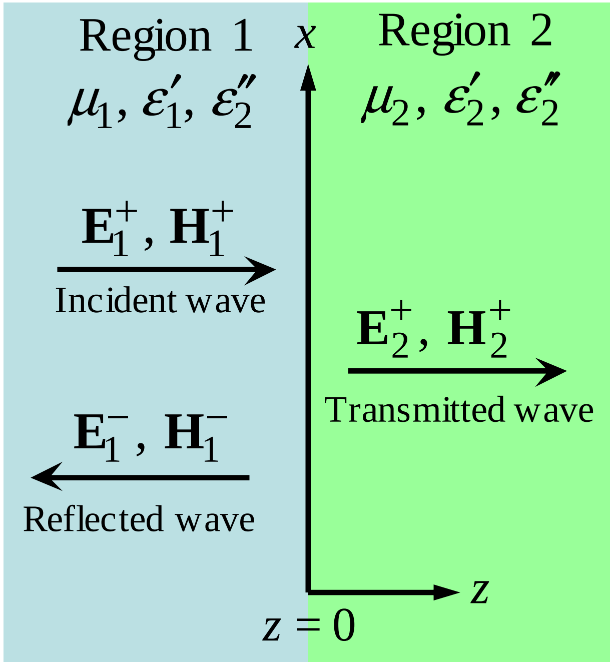
$\rightarrow \eta_1 = \eta_2$ (unreasonable)

$$E_{xs1}^{-} = E_{x10}^{-} e^{jk_1 z}$$

$$H_{ys1}^{-} = -\frac{1}{\eta_1} E_{x10}^{-} e^{jk_1 z}$$

Reflection of Uniform Plane Waves at Normal Incidence (2)





Reflection of Uniform Plane Waves at Normal Incidence (3)

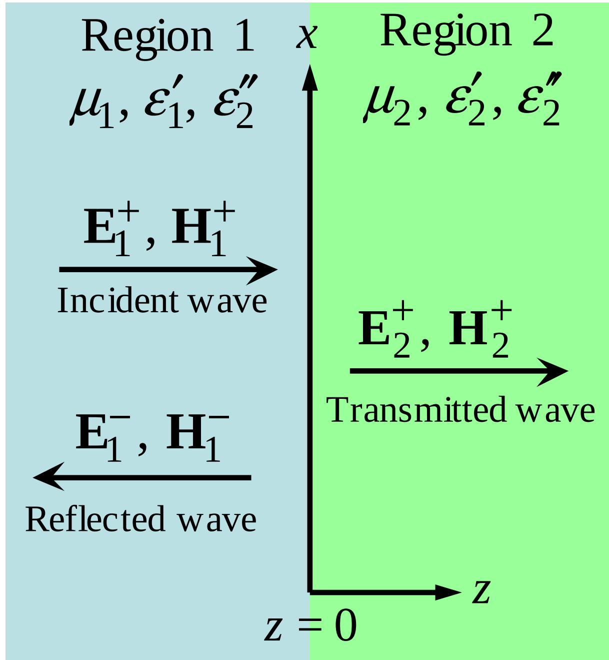
$$\left.\begin{aligned} E_{xs1} &= E_{xs2} \quad (z=0) \\ \rightarrow E_{xs1}^{+} + E_{xs1}^{-} &= E_{xs2}^{+} \quad (z=0) \end{aligned}\right\} \rightarrow E_{x10}^{+} + E_{x10}^{-} = E_{x20}^{+}$$

$$\left.\begin{aligned} H_{ys1} &= H_{ys2} \quad (z=0) \\ \rightarrow H_{ys1}^{+} + H_{ys1}^{-} &= H_{ys2}^{+} \quad (z=0) \end{aligned}\right\} \rightarrow \frac{E_{x10}^{+}}{\eta_1} - \frac{E_{x10}^{-}}{\eta_1} = \frac{E_{x20}^{+}}{\eta_2}$$

$$\rightarrow E_{x10}^{+} + E_{x10}^{-} = \frac{\eta_2}{\eta_1} E_{x10}^{+} - \frac{\eta_2}{\eta_1} E_{x10}^{-}$$

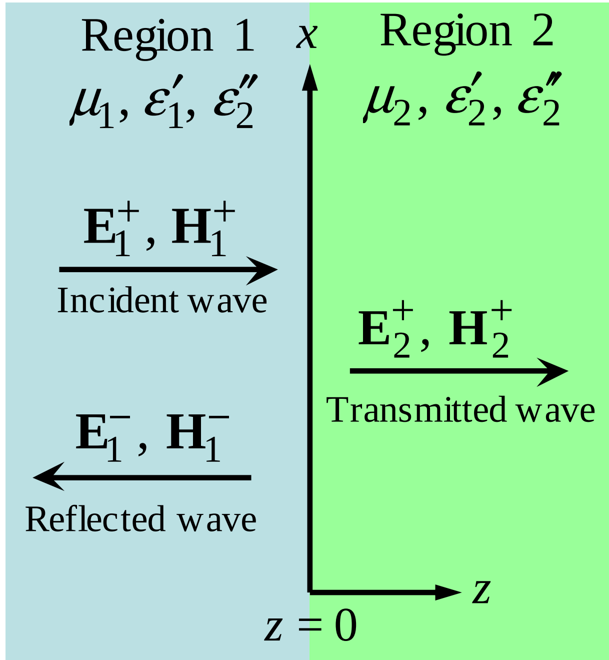
$$\rightarrow E_{x10}^{-} = E_{x10}^{+} \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$\left.\begin{aligned} &\rightarrow \boxed{\Gamma = \frac{E_{x10}^{-}}{E_{x10}^{+}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}} \\ &E_{x10}^{+} + E_{x10}^{-} = E_{x20}^{+} \end{aligned}\right\} \rightarrow \boxed{\tau = \frac{E_{x20}^{+}}{E_{x10}^{+}} = \frac{2\eta_2}{\eta_1 + \eta_2} = 1 + \Gamma}$$



Reflection of Uniform Plane Waves at Normal Incidence (4)

$$\Gamma = \frac{E_{x10}^{-}}{E_{x10}^{+}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \tau = \frac{E_{x20}^{+}}{E_{x10}^{+}} = \frac{2\eta_2}{\eta_1 + \eta_2} = 1 + \Gamma$$



Region 1 is dielectric, region 2 is conductor:

$$\eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\varepsilon_2'}} = 0 \rightarrow \tau = 0 \rightarrow E_{x20}^{+} = 0$$

$$\searrow \Gamma = -1 \rightarrow E_{x10}^{+} = -E_{x10}^{-}$$

$$E_{xs1} = E_{xs1}^{+} + E_{xs1}^{-} = E_{x10}^{+} e^{-j\beta_1 z} - E_{x10}^{+} e^{j\beta_1 z}$$

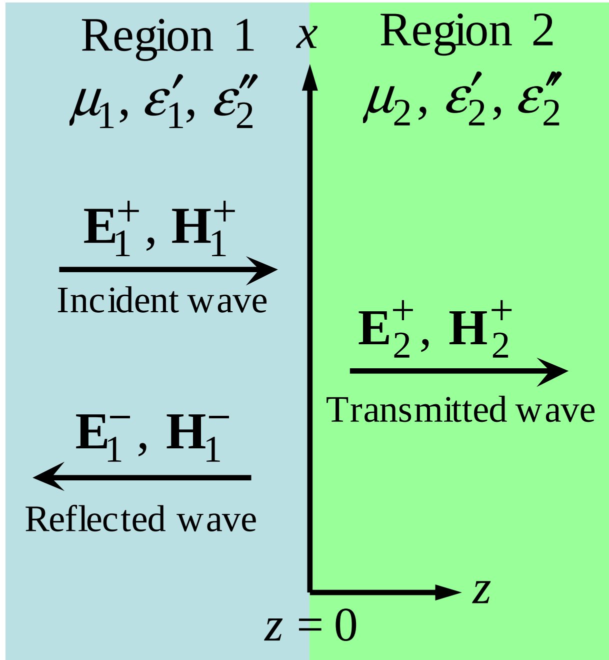
$$\text{Dielectric: } jk_1 = 0 + j\beta_1$$

$$\rightarrow E_{xs1} = (e^{-j\beta_1 z} - e^{j\beta_1 z}) E_{x10}^{+} = -j2\sin(\beta_1 z) E_{x10}^{+}$$

$$\rightarrow E_{x1}(z,t) = 2E_{x10}^{+} \sin(\beta_1 z)\sin(\omega t)$$

Reflection of Uniform Plane Waves at Normal Incidence (5)

$$\Gamma = \frac{E_{x10}^-}{E_{x10}^+} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \tau = \frac{E_{x20}^+}{E_{x10}^+} = \frac{2\eta_2}{\eta_1 + \eta_2} = 1 + \Gamma$$

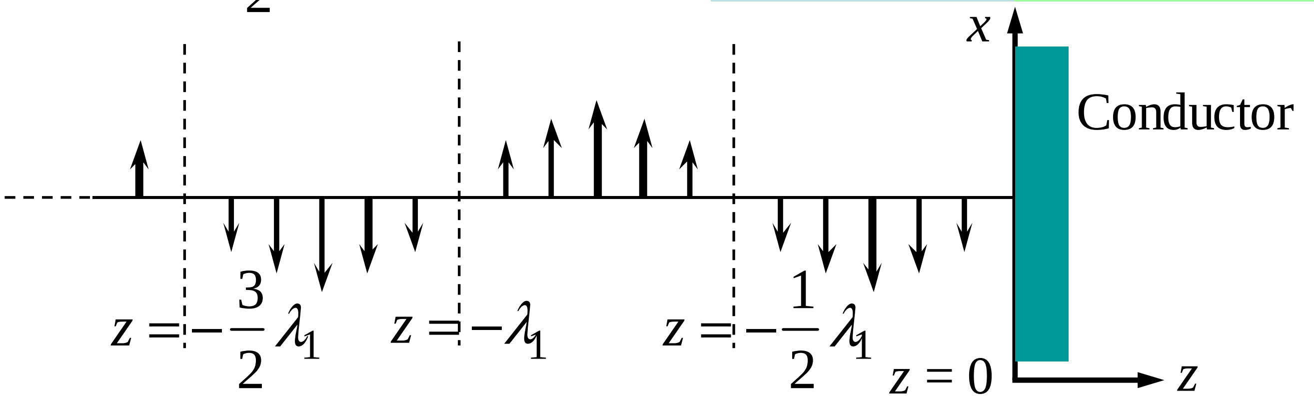


Region 1 is dielectric, region 2 is conductor:

$$E_{x1}(z,t) = 2E_{x10}^+ \sin(\beta_1 z)\sin(\omega t)$$

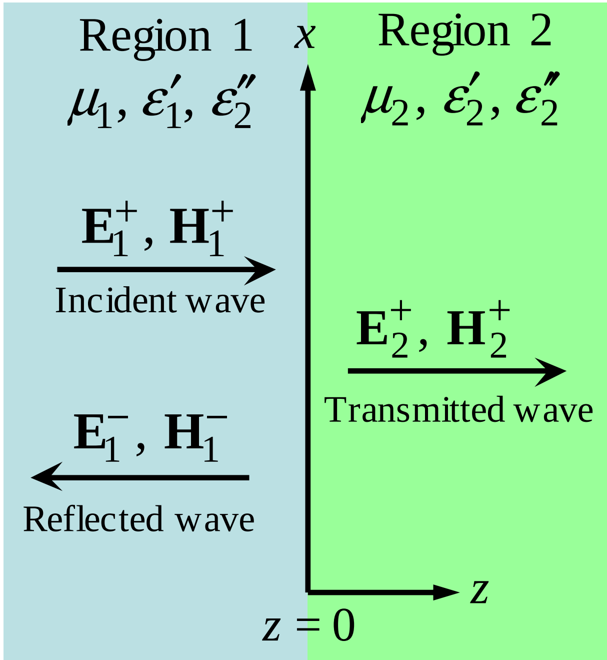
$$E_{x1} = 0 \rightarrow \beta_1 z = m\pi \quad (m = 0, \pm 1, \pm 2, \ldots)$$

$$\rightarrow \frac{2\pi}{\lambda_1} z = m\pi \rightarrow z = m\frac{\lambda_1}{2}$$



Reflection of Uniform Plane Waves at Normal Incidence (6)

$$\Gamma = \frac{E_{x10}^{-}}{E_{x10}^{+}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \tau = \frac{E_{x20}^{+}}{E_{x10}^{+}} = \frac{2\eta_2}{\eta_1 + \eta_2} = 1 + \Gamma$$



Region 1 is dielectric, region 2 is conductor:

$$\left.\begin{aligned} H_{ys1} &= H_{ys1}^{+} + H_{ys1}^{-} \\ H_{ys1}^{+} &= \frac{E_{xs1}^{+}}{\eta_1} \\ H_{ys1}^{-} &= -\frac{E_{xs1}^{-}}{\eta_1} \end{aligned}\right\}$$

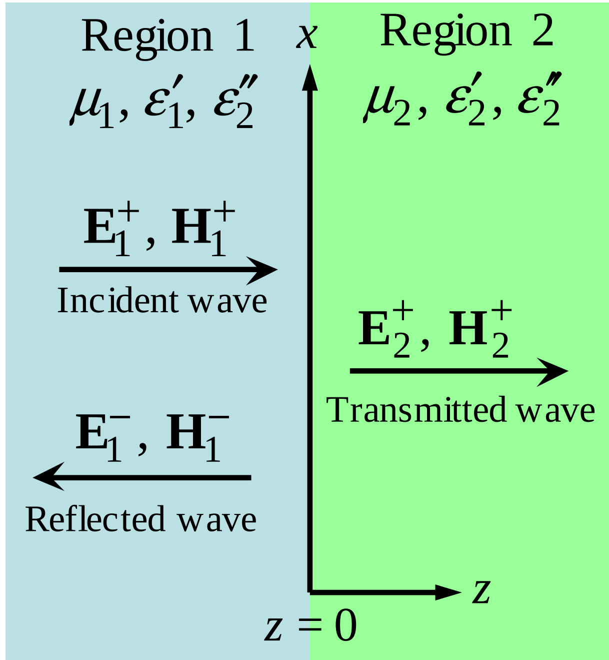
$$\rightarrow H_{ys1} = \frac{E_{x10}^{+}}{\eta_1}\left(e^{-j\beta_1 z} + e^{j\beta_1 z}\right) \quad \rightarrow H_{y1}(z,t) = 2\frac{E_{x10}^{+}}{\eta_1}\cos(\beta_1 z)\cos(\omega t)$$

Reflection of Uniform Plane Waves at Normal Incidence (7)

$$\Gamma = \frac{E_{x10}^{-}}{E_{x10}^{+}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \tau = \frac{E_{x20}^{+}}{E_{x10}^{+}} = \frac{2\eta_2}{\eta_1 + \eta_2} = 1 + \Gamma$$

Region 1 is dielectric, region 2 is dielectric:

η_1 & η_2 are positive real values,
$\alpha_1 = \alpha_2 = 0$



Ex. 1 Reflection of Uniform Plane Waves at Normal Incidence (8)

Given $\eta_1 = 100\Omega$, $\eta_2 = 300\Omega$, $E_{x10}^{+} = 100 \text{ V/m}$. Find the incident, reflected, and transmitted waves.

$$\Gamma = \frac{E_{x10}^{-}}{E_{x10}^{+}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{300 - 100}{300 + 100} = 0.5$$

$$E_{x10}^{-} = \Gamma E_{x10}^{+} = 0.5 \times 100 = 50 \text{ V/m}$$

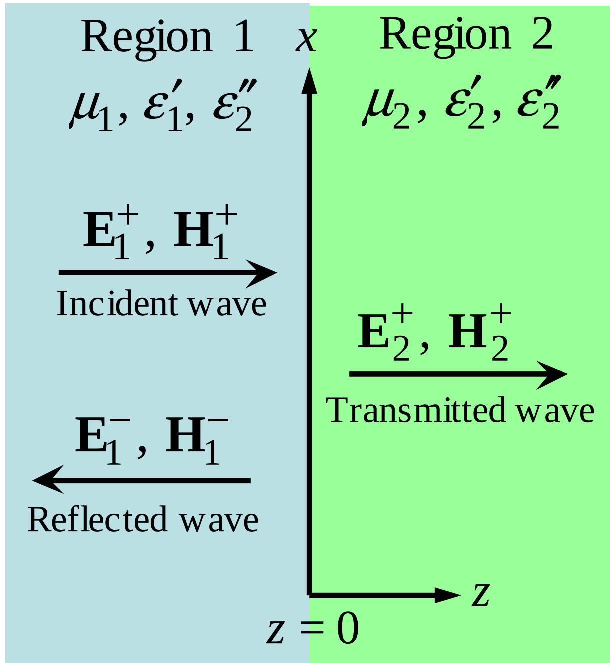
$$H_{y10}^{+} = \frac{E_{x10}^{+}}{\eta_1} = \frac{100}{100} = 1 \text{ A/m}$$

$$H_{y10}^{-} = -\frac{E_{x10}^{-}}{\eta_1} = -\frac{50}{100} = -0.5 \text{ A/m}$$

$$\tau = 1 + \Gamma = 1 + 0.5 = 1.5$$

$$E_{x20}^{+} = \tau E_{x10}^{+} = 1.5 \times 100 = 150 \text{ V/m}$$

$$H_{x20}^{+} = \frac{E_{x20}^{+}}{\eta_2} = \frac{150}{300} = 0.5 \text{ A/m}$$

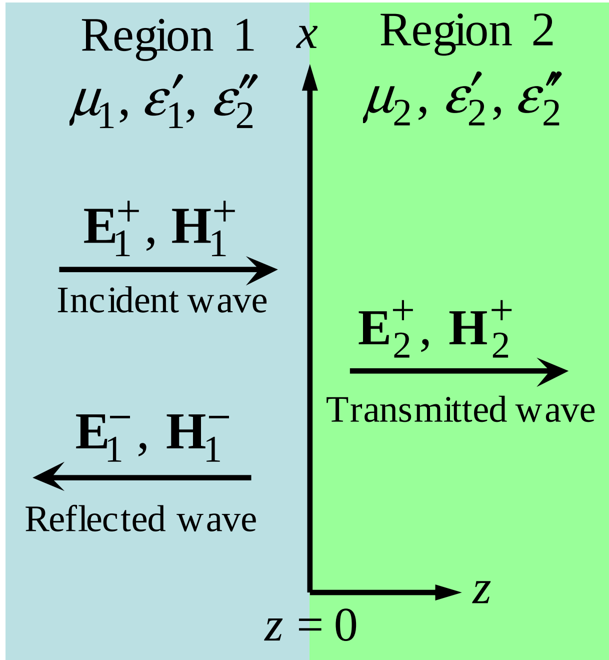


Reflection of Uniform Plane Waves at Normal Incidence (9)

$$S_{1,av}^{+} = \frac{1}{2}\mathrm{Re}[E_{x10}^{+}\hat{H}_{y10}^{+}] = \frac{1}{2}\mathrm{Re}[E_{x10}^{+}\frac{\hat{E}_{x10}^{+}}{\hat{\eta}_1}]$$
$$= \frac{1}{2}\mathrm{Re}\left[\frac{1}{\hat{\eta}_1}\right]\left|E_{x10}^{+}\right|^2$$

$$S_{1,av}^{-} = -\frac{1}{2}\mathrm{Re}[E_{x10}^{-}\hat{H}_{y10}^{-}] = \frac{1}{2}\mathrm{Re}[\Gamma E_{x10}^{+}\frac{\hat{\Gamma}\hat{E}_{x10}^{+}}{\hat{\eta}_1}]$$
$$= \frac{1}{2}\mathrm{Re}\left[\frac{1}{\hat{\eta}_1}\right]\left|E_{x10}^{+}\right|^2|\Gamma|^2$$

$$\rightarrow \boxed{S_{1,av}^{-} = |\Gamma|^2 S_{1,av}^{+}}$$



$$S_{2,av}^{+} = \frac{1}{2}\mathrm{Re}[E_{x20}^{+}\hat{H}_{y20}^{+}] = \frac{1}{2}\mathrm{Re}[\tau E_{x10}^{+}\frac{\hat{\tau}\hat{E}_{x10}^{+}}{\hat{\eta}_2}] = \frac{1}{2}\mathrm{Re}\left[\frac{1}{\hat{\eta}_2}\right]\left|E_{x10}^{+}\right|^2|\tau|^2$$

$$= \frac{\mathrm{Re}[1/\hat{\eta}_2]}{\mathrm{Re}[1/\hat{\eta}_1]}|\tau|^2 S_{1,av}^{+} = \left|\frac{\eta_1}{\eta_2}\right|^2 \frac{\eta_2+\hat{\eta}_2}{\eta_1+\hat{\eta}_1}|\tau|^2 S_{1,av}^{+} \quad \rightarrow \boxed{S_{2,av}^{+} = \left(1-|\Gamma|^2\right)S_{1,av}^{+}}$$

Ex. 2 Reflection of Uniform Plane Waves at Normal Incidence (10)

A 50-MHz uniform plane wave propagating in a medium ($\mu_r = 1$, $\varepsilon_r = 16$, $\sigma = 0.02$ S/m) strikes normally to the surface of another medium ($\mu_r = 1$, $\varepsilon_r = 25$, $\sigma = 0.2$ S/m). If the amplitude of the incident EFI at the interface is 10 V/m, find the average power density of the transmitted wave?

$$\left.\begin{array}{r} \varepsilon_1 = \varepsilon_1' - j\varepsilon_1'' \\ \varepsilon_1'' = \dfrac{\sigma_1}{\omega} \end{array}\right\} \rightarrow \varepsilon_1 = \varepsilon_{r1}\varepsilon_0 - j\frac{\sigma_1}{2\pi f} = 16\times 8.854\times 10^{-12} - j\frac{0.02}{2\pi\times 50\times 10^6}$$

$$= (14.17 - j6.366)10^{-11} \text{ F/m}$$

$$jk_1 = j\omega\sqrt{\mu_1\varepsilon_1} = j\omega\sqrt{\mu_{r1}\mu_0}\sqrt{\varepsilon_1' - j\varepsilon_2''}$$

$$= j2\pi\times 50\times 10^6\sqrt{4\pi\times 10^{-7}}\sqrt{(14.17 - j6.366)10^{-11}} = 0.92 + j4.29 \text{ 1/m}$$

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{4\pi\times 10^{-7}}{(14.17 - j6.366)10^{-11}}} = 87.95 + j18.85\ \Omega$$

Ex. 2 Reflection of Uniform Plane Waves at Normal Incidence (11)

A 50-MHz uniform plane wave propagating in a medium ($\mu_r = 1$, $\varepsilon_r = 16$, $\sigma = 0.02$ S/m) strikes normally to the surface of another medium ($\mu_r = 1$, $\varepsilon_r = 25$, $\sigma = 0.2$ S/m). If the amplitude of the incident EFI at the interface is 10 V/m, find the average power density of the transmitted wave?

$$\left.\begin{aligned} \varepsilon_2 &= \varepsilon_2' - j\varepsilon_2'' \\ \varepsilon_2'' &= \frac{\sigma_2}{\omega} \end{aligned}\right\} \rightarrow \varepsilon_2 = \varepsilon_{r2}\varepsilon_0 - j\frac{\sigma_2}{2\pi f} = 25\times 8.854\times 10^{-12} - j\frac{0.22}{2\pi\times 50\times 10^6}$$

$$= (2.21 - j6.366)10^{-10} \text{ F/m}$$

$$jk_2 = j\omega\sqrt{\mu_2\varepsilon_2} = j\omega\sqrt{\mu_{r2}\mu_0}\sqrt{\varepsilon_2' - j\varepsilon_2''}$$

$$= j2\pi\times 50\times 10^6\sqrt{4\pi\times 10^{-7}}\sqrt{(2.21 - j6.366)10^{-10}} = 5.30 + j7.45 \text{ 1/m}$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{4\pi\times 10^{-7}}{(2.21 - j6.366)10^{-10}}} = 35.19 + j25.02\ \Omega$$

Ex. 2 Reflection of Uniform Plane Waves at Normal Incidence (12)

A 50-MHz uniform wave plane propagating in a medium ($\mu_r = 1$, $\varepsilon_r = 16$, $\sigma = 0.02$ S/m) strikes normally to the surface of another medium ($\mu_r = 1$, $\varepsilon_r = 25$, $\sigma = 0.2$ S/m). If the amplitude of the incident EFI at the interface is 10 V/m, find the average power density of the transmitted wave?

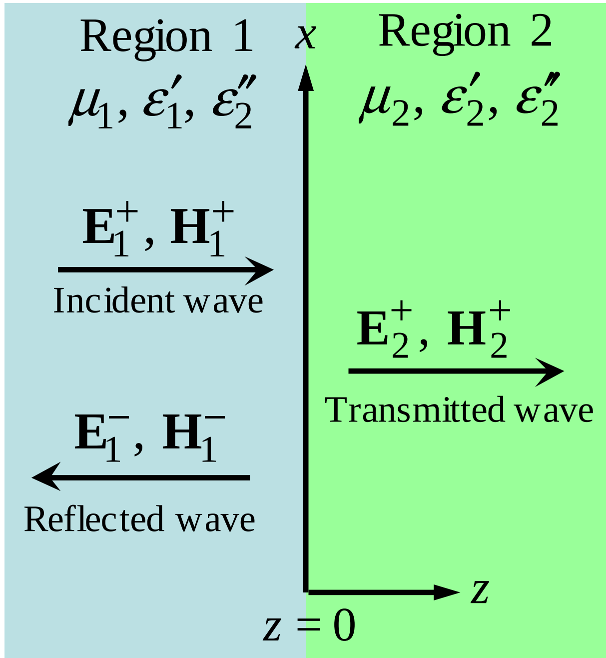
$$\eta_1 = 87.95 + j18.85\ \Omega;\ \eta_2 = 35.19 + j25.02\ \Omega$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{35.19 + j25.02 - (87.95 + j18.85)}{35.19 + j25.02 + (87.95 + j18.85)}$$

$$= -0.36 + j0.18$$

$$\tau = \Gamma + 1 = -0.36 + j0.18 + 1 = 0.63 + j0.18$$

$$\left.\begin{aligned} &\tau = \frac{E_{x20}^+}{E_{x10}^+} \rightarrow E_{x20}^+ = \tau E_{x10}^+ = 10\tau \\ &E_{x2s}^+ = E_{x20}^+ e^{-jk_2 z} \end{aligned}\right\} \rightarrow E_{x2s}^+ = 10\tau e^{-jk_2 z}$$



Ex. 2 Reflection of Uniform Plane Waves at Normal Incidence (13)

A 50-MHz uniform wave plane propagating in a medium ($\mu_r = 1$, $\varepsilon_r = 16$, $\sigma = 0.02$ S/m) strikes normally to the surface of another medium ($\mu_r = 1$, $\varepsilon_r = 25$, $\sigma = 0.2$ S/m). If the amplitude of the incident EFI at the interface is 10 V/m, find the average power density of the transmitted wave?

$$jk_2 = 5.30 + j7.45 \ 1/\text{m}; \quad \tau = 0.63 + j0.18; \ \eta_2 = 35.19 + j25.02 \ \Omega$$

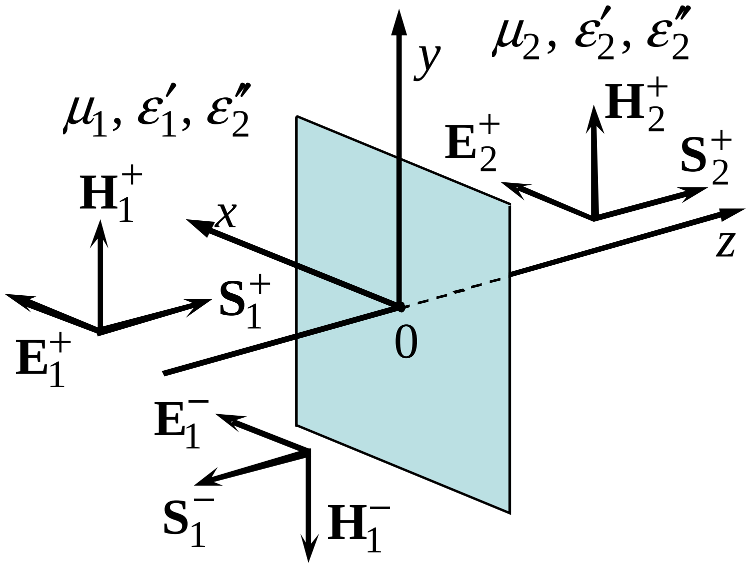
$$E_{x2s}^{+} = 10\tau e^{-jk_2 z} = 10(0.63 + j0.18)e^{-(5.30+j7.45)z}$$

$$= 10(0.6552e^{j15.9^\circ})e^{-(5.30+j7.45)z}$$

$$= 6.55e^{-5.30z}e^{-j7.45z}e^{j15.9^\circ} \ \text{V/m}$$

$$H_{y2s}^{+} = \frac{E_{x2s}^{+}}{\eta_2} = \frac{= 6.55e^{-5.30z}e^{-j7.45z}e^{j15.9^\circ}}{43.18e^{j35.4^\circ}}$$

$$= 0.152e^{-5.30z}e^{-j7.45z}e^{-j19.5^\circ} \ \text{A/m}$$

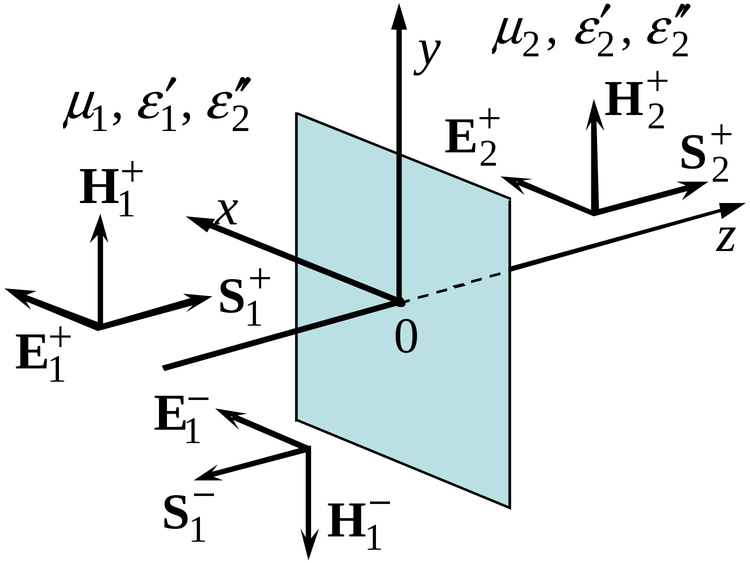


Ex. 2 Reflection of Uniform Plane Waves at Normal Incidence (14)

A 50-MHz uniform wave plane propagating in a medium ($\mu_r = 1$, $\varepsilon_r = 16$, $\sigma = 0.02$ S/m) strikes normally to the surface of another medium ($\mu_r = 1$, $\varepsilon_r = 25$, $\sigma = 0.2$ S/m). If the amplitude of the incident EFI at the interface is 10 V/m, find the average power density of the transmitted wave?

$$\begin{cases} E_{x2s}^{+} = 6.55e^{-5.30z}e^{-j7.45z}e^{j15.9^\circ} \text{ V/m} \\ H_{y2s}^{+} = 0.152e^{-5.30z}e^{-j7.45z}e^{-j19.5^\circ} \text{ A/m} \end{cases}$$

$$\rightarrow \begin{cases} \mathbf{E}_{2s}^{+} = 6.55e^{-5.30z}e^{-j7.45z}e^{j15.9^\circ}\mathbf{a}_x \text{ V/m} \\ \mathbf{H}_{2s}^{+} = 0.152e^{-5.30z}e^{-j7.45z}e^{-j19.5^\circ}\mathbf{a}_y \text{ A/m} \end{cases}$$



$$\mathbf{S}_{2,av}^{+} = \frac{1}{2}\text{Re}\left[\mathbf{E}_s \times \hat{\mathbf{H}}_s\right]$$

$$= \frac{1}{2}\text{Re}\left[(6.55e^{-5.30z}e^{-j7.45z}e^{j15.9^\circ})(0.152e^{-5.30z}e^{j7.45z}e^{j19.5^\circ})\mathbf{a}_z\right]$$

Ex. 2 Reflection of Uniform Plane Waves at Normal Incidence (15)

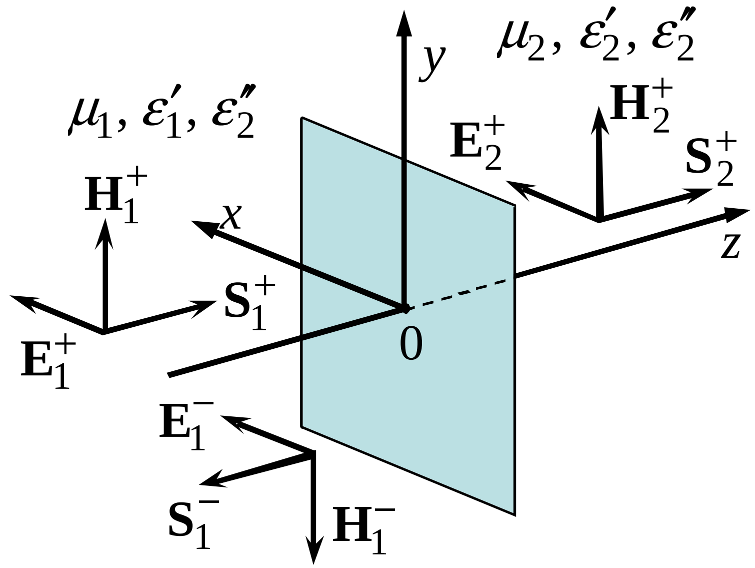
A 50-MHz uniform wave plane propagating in a medium ($\mu_r = 1$, $\varepsilon_r = 16$, $\sigma = 0.02$ S/m) strikes normally to the surface of another medium ($\mu_r = 1$, $\varepsilon_r = 25$, $\sigma = 0.2$ S/m). If the amplitude of the incident EFI at the interface is 10 V/m, find the average power density of the transmitted wave?

$$\mathbf{S}^{+}_{2,av} = \frac{1}{2}\mathrm{Re}\left[(6.55e^{-5.30z}e^{-j7.45z}e^{j15.9^{\circ}})(0.152e^{-5.30z}e^{j7.45z}e^{j19.5^{\circ}})\mathbf{a}_z\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[0.9956e^{-10.60z}e^{j35.4^{\circ}}\mathbf{a}_z\right]$$

$$= 0.4978e^{-10.60z}\cos 35.4^{\mathrm{o}}\,\mathbf{a}_z$$

$$= \boxed{0.4058e^{-10.60z}\mathbf{a}_z \ \ \mathrm{W/m^2}}$$



Plane Wave Reflection & Dispersion

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Standing Wave Ratio (1)

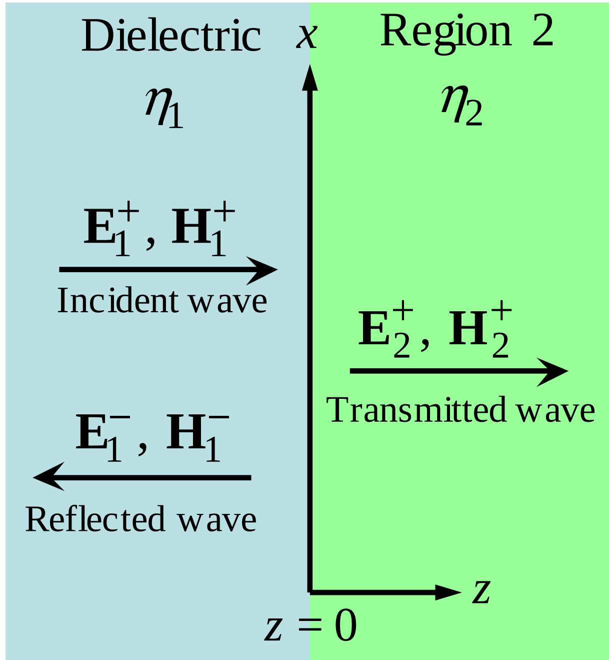
$$\left.\begin{array}{r} E_{xs1} = E_{x1}^{+} + E_{x1}^{-} = E_{x10}^{+} e^{-j\beta_1 z} + \Gamma E_{x10}^{+} e^{j\beta_1 z} \\ \Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = |\Gamma| e^{j\varphi} \end{array}\right\}$$

$$\to E_{xs1} = \left(e^{-j\beta_1 z} + |\Gamma| e^{j(\beta_1 z + \varphi)}\right) E_{x10}^{+}$$

$$E_{xs1,\max} = \left(1 + |\Gamma|\right) E_{x10}^{+}$$

$$\to -\beta_1 z = \beta_1 z + \varphi + 2m\pi \quad (m = 0, \pm 1, \pm 2, \ldots)$$

$$\to \boxed{z_{\max} = -\frac{1}{2\beta_1}(\varphi + 2m\pi)}$$

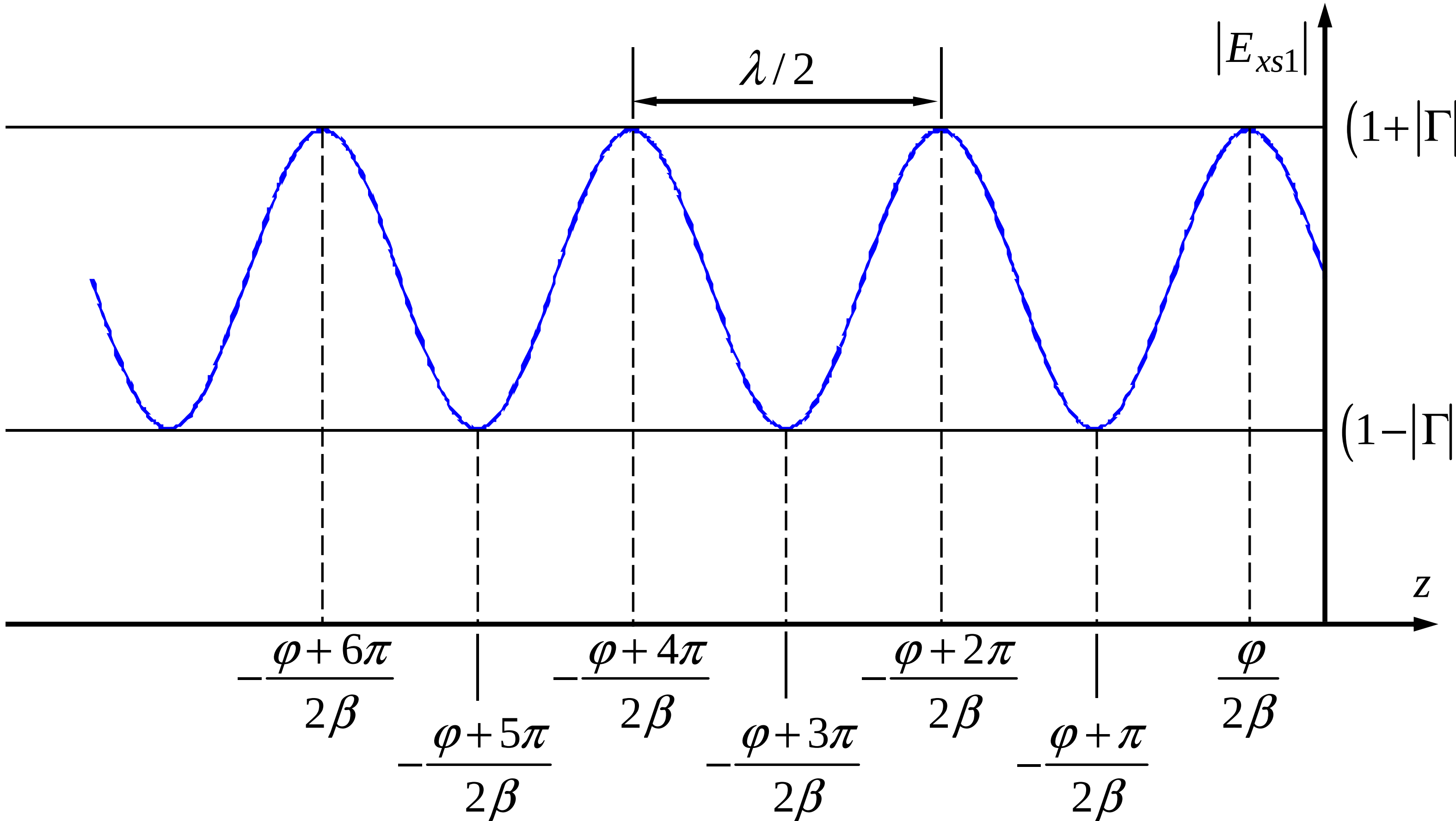


$$E_{xs1,\min} = \left(1 - |\Gamma|\right) E_{x10}^{+}$$

$$\to -\beta_1 z = \beta_1 z + \varphi + \pi + 2m\pi \quad (m = 0, \pm 1, \pm 2, \ldots) \to \boxed{z_{\min} = -\frac{1}{2\beta_1}[\varphi + (2m+1)\pi]}$$

Standing Wave Ratio (2)

$$E_{xs1} = \left(e^{-j\beta_1 z} + |\Gamma| e^{j(\beta_1 z + \varphi)}\right) E_{x10}^{+} \qquad z_{\max} = -\frac{1}{2\beta_1}(\varphi + 2m\pi) \qquad z_{\min} = -\frac{1}{2\beta_1}[\varphi + (2m+1)\pi]$$



Standing Wave Ratio (3)

$$E_{xs1} = \left(e^{-j\beta_1 z} + |\Gamma| e^{j(\beta_1 z+\varphi)}\right) E_{x10}^{+}$$

$$= E_{x10}^{+}\left(e^{-j\varphi/2} e^{-j\beta_1 z} + |\Gamma| e^{j\varphi/2} e^{j\beta_1 z}\right) e^{j\varphi/2}$$

$$= E_{x10}^{+}\left(e^{-j\varphi/2} e^{-j\beta_1 z} + |\Gamma| e^{j\varphi/2} e^{j\beta_1 z}\right) e^{j\varphi/2}$$

$$+\left(|\Gamma| E_{x10}^{+} e^{-j\varphi/2} e^{-j\beta_1 z}\right) - \left(|\Gamma| E_{x10}^{+} e^{-j\varphi/2} e^{-j\beta_1 z}\right)$$

$$= E_{x10}^{+}\left(1-|\Gamma|\right) e^{-j\beta_1 z} + E_{x10}^{+}|\Gamma|\left(e^{-j\varphi/2} e^{-j\beta_1 z} + e^{j\varphi/2} e^{j\beta_1 z}\right) e^{j\varphi/2}$$

$$= E_{x10}^{+}\left(1-|\Gamma|\right) e^{-j\beta_1 z} + 2|\Gamma| E_{x10}^{+} e^{j\varphi/2} \cos(\beta_1 z + \varphi/2)$$

$$\rightarrow E_{x1}(z,t) = \boxed{\left(1-|\Gamma|\right) E_{x10}^{+} \cos(\omega t - \beta_1 z)} + \boxed{2|\Gamma| E_{x10}^{+} \cos(\beta_1 z + \varphi/2) \cos(\omega t + \varphi/2)}$$

Standing Wave Ratio (4)

$$E_{x1}(z,t) = \left(1-|\Gamma|\right) E_{x10}^{+} \cos(\omega t - \beta_1 z) + 2|\Gamma| E_{x10}^{+} \cos(\beta_1 z + \varphi/2)\cos(\omega t + \varphi/2)$$

$$E_{xs1,\max} = 1+|\Gamma|$$

$$E_{xs1,\min} = 1-|\Gamma|$$

$$s = \frac{E_{xs1,\max}}{E_{xs1,\min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

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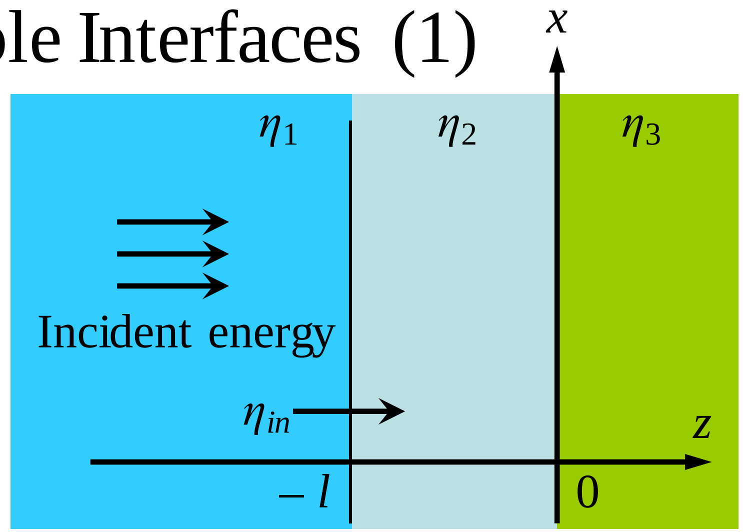
3. **Wave Reflection from Multiple Interfaces**
4. Plane Wave Propagation in General Directions
5. Plane Wave Reflection at Oblique Incidence Angles
6. Wave Propagation in Dispersive Media

Wave Reflection from Multiple Interfaces (1)

The steady – state has 5 waves:

- Incident wave in region 1
- Reflected wave in region 1
- Transmitted wave in region 3
- 2 opposite waves in region 2



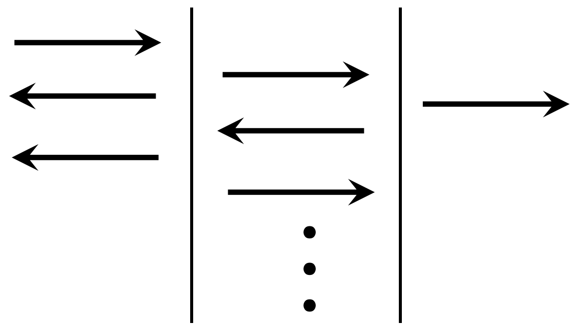
$E_{xs2} = E^{+}_{x20} e^{-j\beta_2 z} + E^{-}_{x20} e^{j\beta_2 z}$ where $\beta_2 = \omega\sqrt{\varepsilon_{r2}}/c$, E^{+}_{x20} & E^{-}_{x20} are complex

$$H_{ys2} = H^{+}_{y20} e^{-j\beta_2 z} + H^{-}_{y20} e^{j\beta_2 z}$$

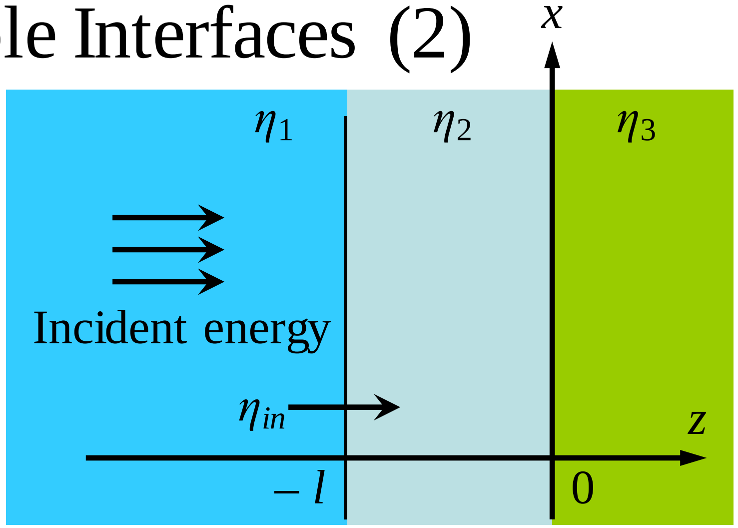
$$\Gamma_{23} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2}$$

$$E^{-}_{x20} = \Gamma_{23} E^{+}_{x20}$$

$$H^{+}_{y20} = \frac{E^{+}_{x20}}{\eta_2} \qquad H^{-}_{y20} = -\frac{E^{-}_{x20}}{\eta_2} = -\frac{\Gamma_{23} E^{+}_{x20}}{\eta_2}$$



Wave Reflection from Multiple Interfaces (2)



$$E_{xs2} = E^{+}_{x20}e^{-j\beta_2 z} + E^{-}_{x20}e^{j\beta_2 z}$$

$$H_{ys2} = H^{+}_{y20}e^{-j\beta_2 z} + H^{-}_{y20}e^{j\beta_2 z}$$

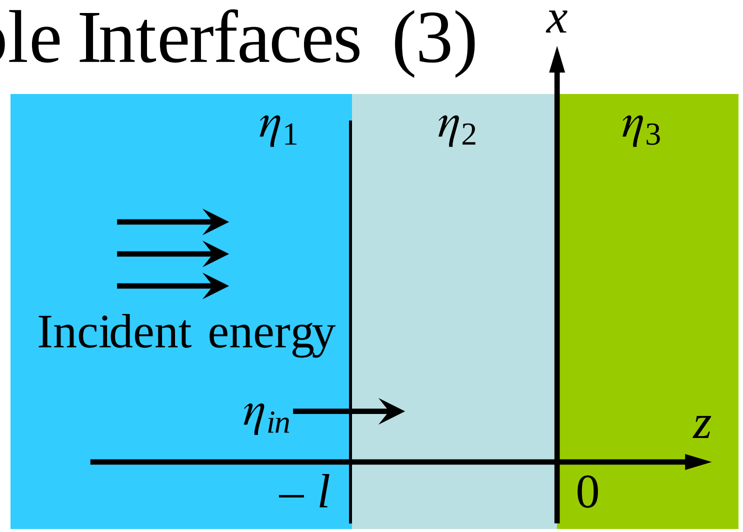
$$\left.\begin{array}{l}\text{Define } \eta_w(z) = \dfrac{E_{xs2}}{H_{ys2}} = \dfrac{E^{+}_{x20}e^{-j\beta_2 z} + E^{-}_{x20}e^{j\beta_2 z}}{H^{+}_{y20}e^{-j\beta_2 z} + H^{-}_{y20}e^{j\beta_2 z}} \\ E^{-}_{x20} = \Gamma_{23}E^{+}_{x20}, \quad H^{+}_{y20} = \dfrac{E^{+}_{x20}}{\eta_2}, \quad H^{-}_{y20} = -\dfrac{\Gamma_{23}E^{+}_{x20}}{\eta_2}\end{array}\right\}$$

$$\left.\begin{array}{l}\to \eta_w(z) = \eta_2 \dfrac{e^{-j\beta_2 z} + \Gamma_{23}e^{j\beta_2 z}}{e^{-j\beta_2 z} - \Gamma_{23}e^{j\beta_2 z}} \\ \Gamma_{23} = \dfrac{\eta_3 - \eta_2}{\eta_3 + \eta_2}, \qquad e^{j\varphi} = \cos\varphi + j\sin\varphi\end{array}\right\}$$

$$\to \eta_w(z) = \eta_2 \times \frac{(\eta_3+\eta_2)(\cos\beta_2 z - j\sin\beta_2 z) + (\eta_3-\eta_2)(\cos\beta_2 z + j\sin\beta_2 z)}{(\eta_3+\eta_2)(\cos\beta_2 z - j\sin\beta_2 z) - (\eta_3-\eta_2)(\cos\beta_2 z + j\sin\beta_2 z)}$$

$$= \eta_2 \frac{\eta_3\cos\beta_2 z - j\eta_2\sin\beta_2 z}{\eta_2\cos\beta_2 z - j\eta_3\sin\beta_2 z}$$

Wave Reflection from Multiple Interfaces (3)



$$E^{+}_{xs1} + E^{-}_{xs1} = E_{xs2} \quad (z = -l)$$

$$\left.\begin{aligned} &\rightarrow E^{+}_{x10} + E^{-}_{x10} = E_{xs2}(z=-l) \\ &H^{+}_{ys1} + H^{-}_{ys1} = H_{ys2} \quad (z=-l) \\ &\rightarrow \frac{E^{+}_{x10}}{\eta_1} - \frac{E^{-}_{x10}}{\eta_1} = \frac{E_{xs2}(z=-l)}{\eta_w(-l)} \end{aligned}\right\}$$

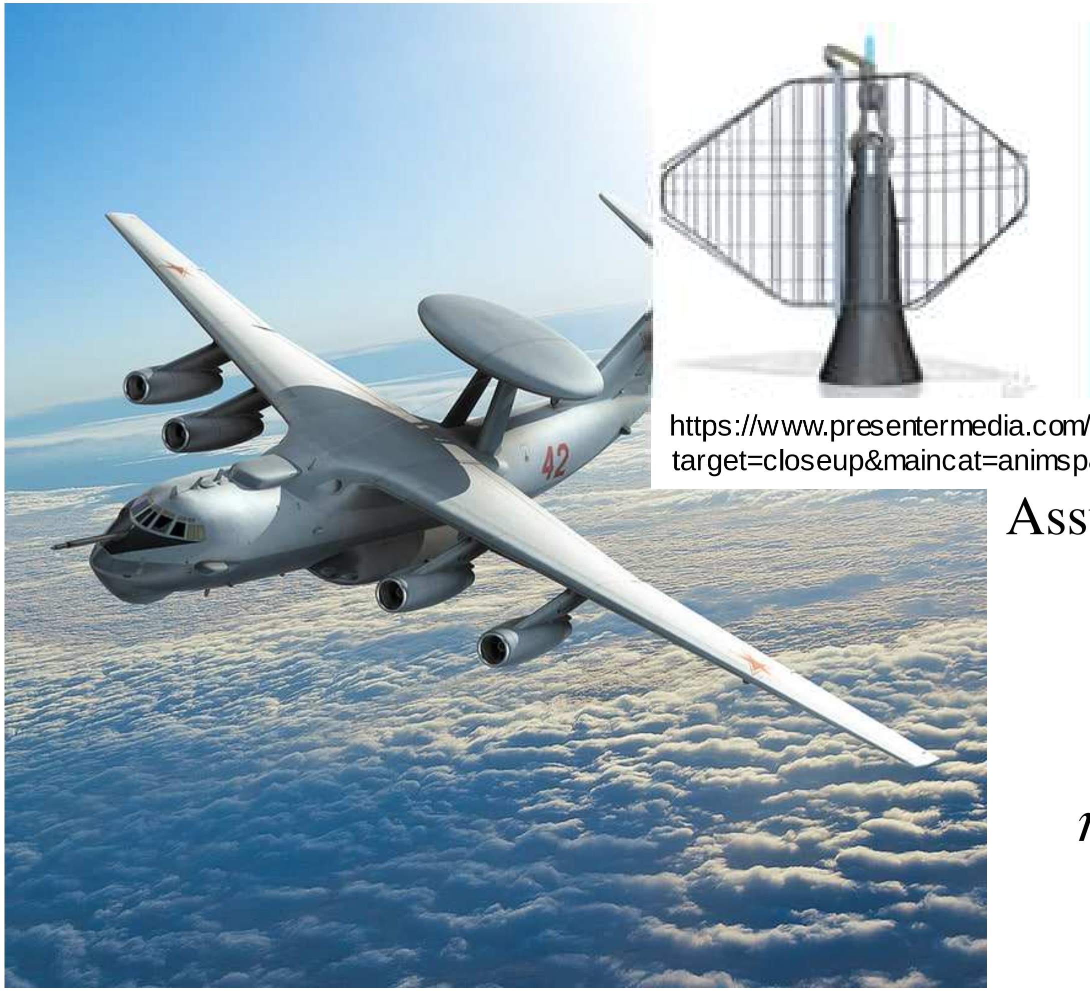
$$\rightarrow \boxed{\frac{E^{-}_{x10}}{E^{+}_{x10}} = \Gamma = \frac{\eta_{in} - \eta_1}{\eta_{in} + \eta_1}}, \text{ where } \eta_{in} = \eta_w\big|_{z=-l}$$

$$\eta_w(z) = \eta_2 \frac{\eta_3 \cos\beta_2 z - j\eta_2 \sin\beta_2 z}{\eta_2 \cos\beta_2 z - j\eta_3 \sin\beta_2 z}$$

$$\rightarrow \boxed{\eta_{in} = \eta_2 \frac{\eta_3 \cos\beta_2 l + j\eta_2 \sin\beta_2 l}{\eta_2 \cos\beta_2 l + j\eta_3 \sin\beta_2 l}}$$

$$\boxed{\eta_{in} = \eta_1 : \text{matched}}$$

Wave Reflection from Multiple Interfaces (4)



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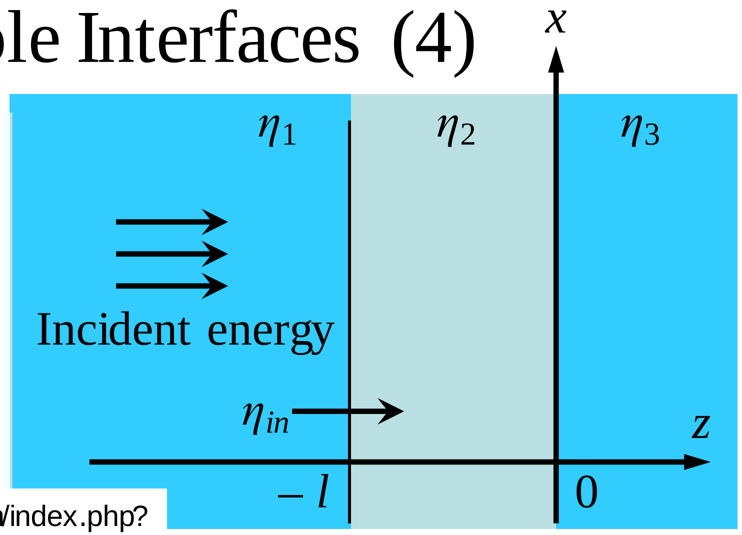


Assume: $\begin{cases} \eta_3 = \eta_1 \\ \beta_2 l = m\pi \end{cases}$

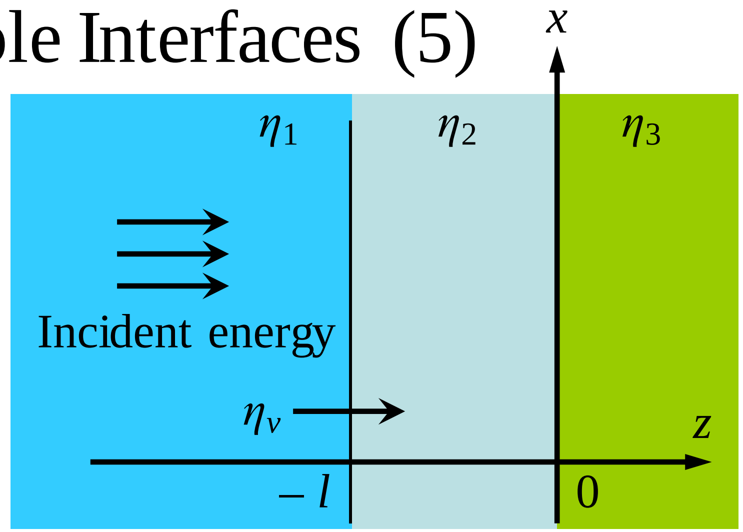
$$\left.\begin{array}{l} \beta_2 l = m\pi \\ \beta_2 = \dfrac{2\pi}{\lambda_2} \end{array}\right\} \rightarrow \boxed{l = m\frac{\lambda_2}{2}}$$

$$\eta_{in} = \eta_2 \frac{\eta_3 \cos\beta_2 l + j\eta_2 \sin\beta_2 l}{\eta_2 \cos\beta_2 l + j\eta_3 \sin\beta_2 l}$$

$$\rightarrow \eta_{in} = \eta_3$$

https://www.turbosquid.com/3d-models/russian-beriev-a-50-aircraft-3d-lwo/449074

Wave Reflection from Multiple Interfaces (5)



Assume: $\begin{cases} \eta_3 \neq \eta_1 \\ \beta_2 l = (2m-1)\dfrac{\pi}{2} \end{cases}$

$$\left.\begin{aligned} &\beta_2 l = (2m-1)\frac{\pi}{2} \\ &\beta_2 = \frac{2\pi}{\lambda_2} \end{aligned}\right\} \rightarrow \boxed{l = (2m-1)\frac{\lambda_2}{4}}$$

$$\eta_{in} = \eta_2 \frac{\eta_3 \cos\beta_2 l + j\eta_2 \sin\beta_2 l}{\eta_2 \cos\beta_2 l + j\eta_3 \sin\beta_2 l}$$

$$\rightarrow \eta_{in} = \frac{\eta_2^2}{\eta_3}$$

Total transmission: $\eta_v = \eta_1$

$$\rightarrow \boxed{\eta_2 = \sqrt{\eta_1 \eta_3}}$$

Wave Reflection from Multiple Interfaces (6)

Ex.

It is required to coat a glass surface with an appropriate dielectric layer to provide total transmission from air to the glass at a wavelength of 570 nm. The glass has dielectric constant, $\varepsilon_r = 2.1$. Find the required dielectric constant for the coating and its minimum thickness.

$$\eta_1 = \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\ \Omega$$

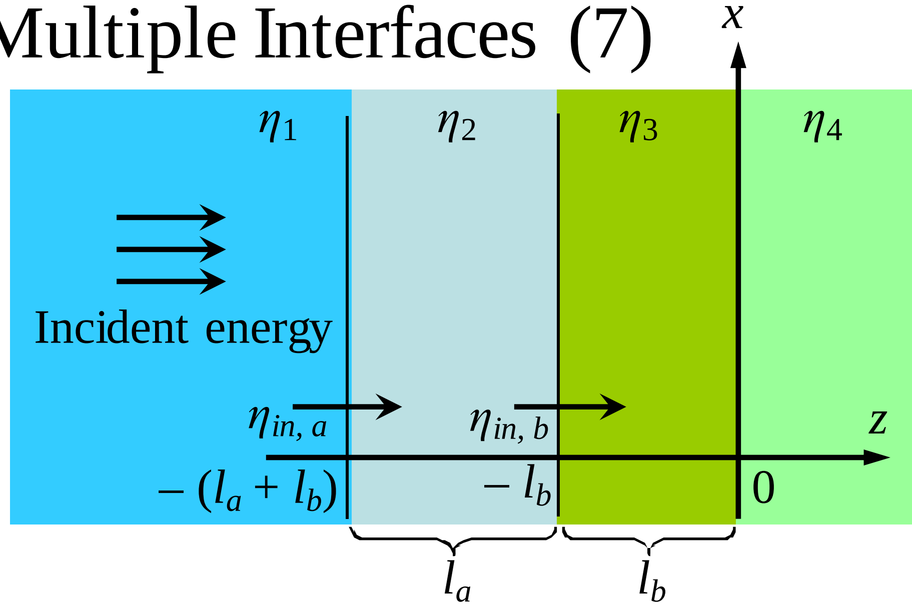
$$\eta_3 = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \sqrt{\frac{\mu_0 1}{\varepsilon_0\varepsilon_r}} = \frac{\eta_1}{\sqrt{\varepsilon_r}} = \frac{377}{\sqrt{2.1}} = 260\Omega$$

Total transmission: $\eta_2 = \sqrt{\eta_1\eta_3} = \sqrt{377\times 260} = 313\Omega$

$$\eta_2 = \frac{\eta_1}{\sqrt{\varepsilon_{r2}}} \rightarrow \varepsilon_{r2} = \left(\frac{\eta_1}{\eta_2}\right)^2 = \left(\frac{377}{313}\right)^2 = \boxed{1.45}$$

$$\lambda_2 = \frac{\lambda_1}{\sqrt{\mu_{r2}\varepsilon_{r2}}} = \frac{570}{\sqrt{1\times 1.45}} = 473\text{ nm} \rightarrow l_2 = \frac{\lambda_2}{4} = \frac{473}{4} = \boxed{118\text{ nm} = 0.118\ \mu\text{m}}$$

Wave Reflection from Multiple Interfaces (7)



$$\eta_{in,\,b} = \eta_3 \frac{\eta_4 \cos\beta_3 l_b + j\eta_3 \sin\beta_3 l_b}{\eta_3 \cos\beta_3 l_b + j\eta_4 \sin\beta_3 l_b}$$

$$\eta_{in,\,a} = \eta_2 \frac{\eta_{v,\,b} \cos\beta_2 l_a + j\eta_2 \sin\beta_2 l_a}{\eta_2 \cos\beta_2 l_a + j\eta_{v,\,b} \sin\beta_2 l_a}$$

$$\Gamma = \frac{\eta_{in,\,a} - \eta_1}{\eta_{in,\,a} + \eta_1}$$

The reflected power fraction: $|\Gamma|^2$

The fraction of the power transmitted into region 4: $1 - |\Gamma|^2$

Plane Wave Reflection & Dispersion

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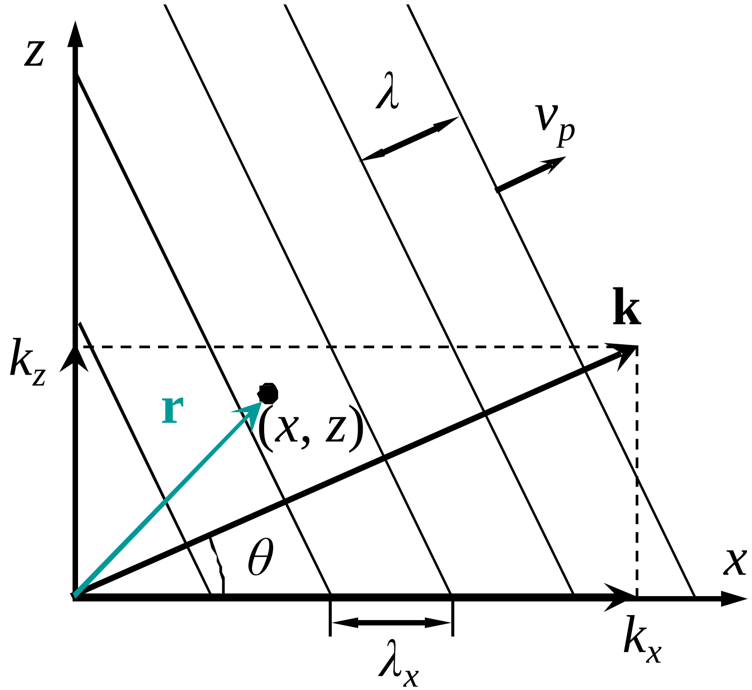
Plane Wave Propagation in General Directions (1)

Phase: **k.r**

$$\left.\begin{array}{l} \mathbf{E}_s = \mathbf{E}_0 e^{-j\mathbf{k}.\mathbf{r}} \\ \left.\begin{array}{l}\mathbf{k} = k_x\mathbf{a}_x + k_z\mathbf{a}_z \\ \mathbf{r} = x\mathbf{a}_x + z\mathbf{a}_z\end{array}\right\} \\ \rightarrow \mathbf{k}.\mathbf{r} = k_x x + k_z z \end{array}\right\}$$

$$\rightarrow \mathbf{E}_s = \mathbf{E}_0 e^{-j(k_x x + k_z z)}$$



$$\theta = \mathrm{atan}\left(\frac{k_z}{k_x}\right) \qquad \lambda = \frac{2\pi}{k} = \frac{2\pi}{\sqrt{k_x^2 + k_z^2}} \qquad v_p = \frac{\omega}{k} = \frac{\omega}{\sqrt{k_x^2 + k_z^2}}$$

Plane Wave Propagation in General Directions (2)

Ex. 1

Given a 50 MHz uniform wave, it has electric field amplitude 10 V/m. The medium is lossless, $\varepsilon_r = \varepsilon'_r = 9.0$; $\mu_r = 1.0$. The wave propagates in the x, y plane at a 30° angle to the x axis, & is linearly polarized along z. Find the phasor expression of the electric field.

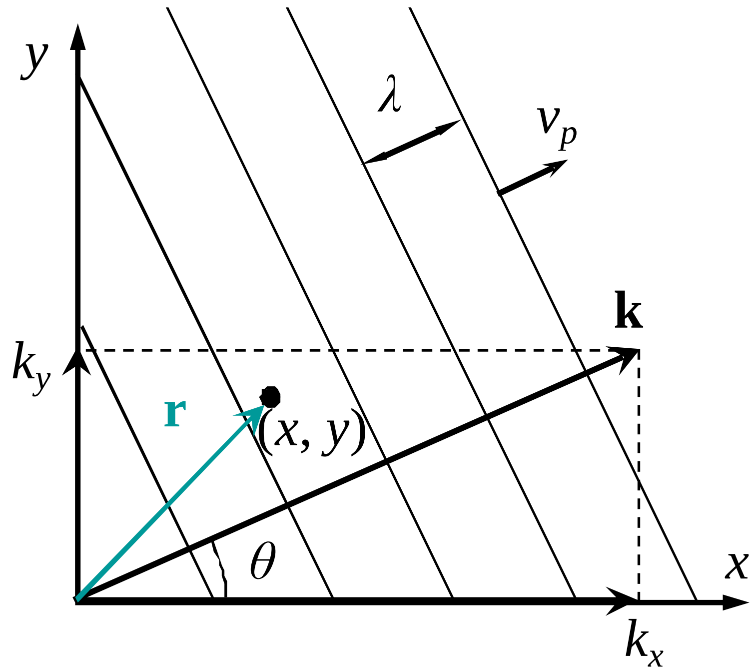
$$k = \omega\sqrt{\mu\varepsilon} = \frac{\omega\sqrt{\varepsilon_r}}{c} = \frac{2\pi \times 50 \times 10^6 \sqrt{9}}{3 \times 10^8}$$

$$= 3.14 \text{ m}^{-1}$$

$$\mathbf{k} = 3.14\cos 30^o \mathbf{a}_x + 3.14 \sin 30^o \mathbf{a}_y$$

$$= 2.72\mathbf{a}_x + 1.57\mathbf{a}_y$$

$$\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y$$



$$\mathbf{E}_s = E_0 e^{-j\mathbf{k}.\mathbf{r}}\mathbf{a}_z = E_0 e^{-j(k_x x + k_y y)}\mathbf{a}_z = 10e^{-j(2.72x + 1.57y)}\mathbf{a}_z \text{ V/m}$$

Plane Wave Propagation in General Directions (3)

Ex. 2

The EFI of a uniform plane wave is $377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x$ V/m. Find the direction of propagation?

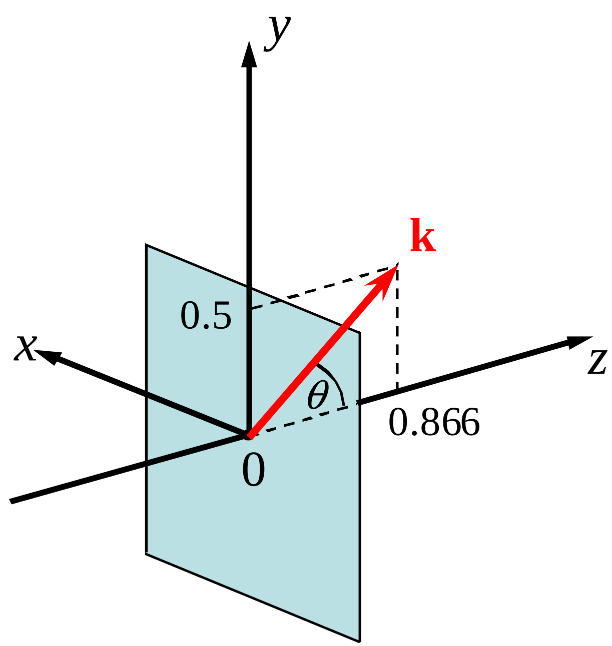
$$\mathbf{E}_s = 377e^{-j(0.866z+0.5y)}\mathbf{a}_x = E_0 e^{-j\mathbf{k}.\mathbf{r}}\mathbf{a}_x$$

$$\mathbf{k} = k\cos\theta\mathbf{a}_z + k\sin\theta\mathbf{a}_y$$

$$\mathbf{r} = z\mathbf{a}_z + y\mathbf{a}_y$$

$$\mathbf{k.r} = (k\cos\theta)z + (k\sin\theta)y = 0.866z + 0.5y$$

$$\rightarrow \begin{cases} k\cos\theta = 0.866 \\ k\sin\theta = 0.5 \end{cases} \rightarrow \theta = \text{atan}\frac{0.5}{0.866} = 30^{\text{o}}$$



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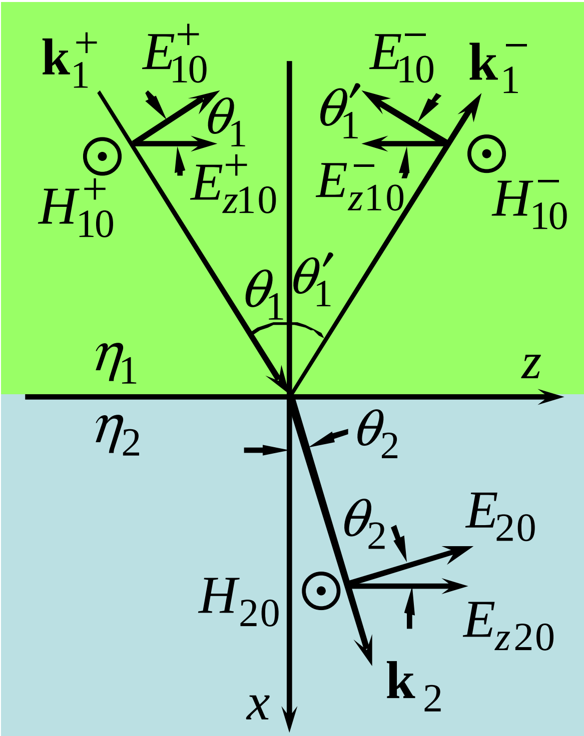
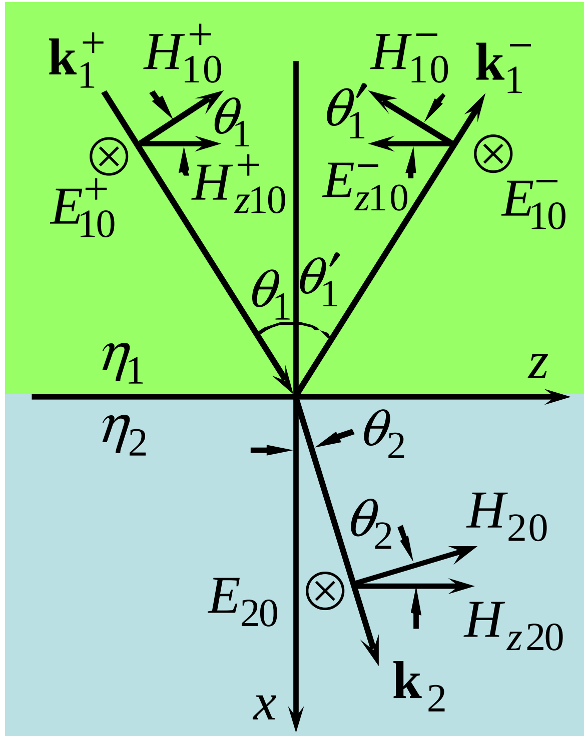
Plane Wave Reflection at Oblique Incidence Angles (1)

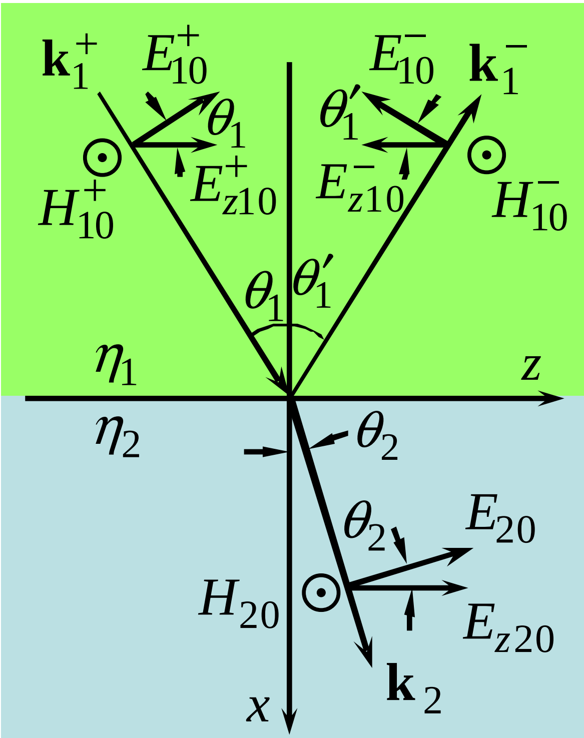


p – polarization, TM



s – polarization, TE

Plane Wave Reflection at Oblique Incidence Angles (2)



p – polarization, TM

$$\mathbf{E}_{s1}^+ = \mathbf{E}_{10}^+ e^{-j\mathbf{k}_1^+ . \mathbf{r}}$$

$$\mathbf{E}_{s1}^- = \mathbf{E}_{10}^- e^{-j\mathbf{k}_1^- . \mathbf{r}}$$

$$\mathbf{E}_{s2} = \mathbf{E}_{20} e^{-j\mathbf{k}_2 . \mathbf{r}}$$

$$\mathbf{k}_1^+ = k_1(\cos\theta_1 \mathbf{a}_x + \sin\theta_1 \mathbf{a}_z)$$

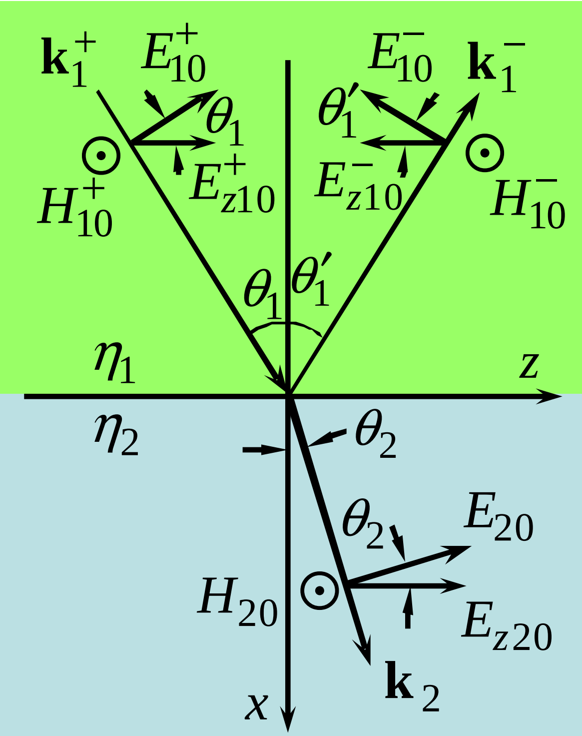
$$\mathbf{k}_1^- = k_1(-\cos\theta_1' \mathbf{a}_x + \sin\theta_1' \mathbf{a}_z)$$

$$\mathbf{k}_2 = k_2(\cos\theta_2 \mathbf{a}_x + \sin\theta_2 \mathbf{a}_z)$$

$$\mathbf{r} = x\mathbf{a}_x + z\mathbf{a}_z$$

$$k_1 = \omega\sqrt{\mu_1\varepsilon_1} = k_0\sqrt{\mu_{r1}\varepsilon_{r1}} = \frac{\omega}{c}\sqrt{\varepsilon_{r1}} = \frac{n_1\omega}{c} \qquad k_2 = \frac{\omega\sqrt{\varepsilon_{r2}}}{c} = \frac{n_2\omega}{c}$$

Plane Wave Reflection at Oblique Incidence Angles (3)



p – polarization, TM

$$\mathbf{E}_{s1}^+ = \mathbf{E}_{10}^+ e^{-j\mathbf{k}_1^+ . \mathbf{r}}$$

$$\mathbf{E}_{s1}^- = \mathbf{E}_{10}^- e^{-j\mathbf{k}_1^- . \mathbf{r}}$$

$$\mathbf{E}_{s2} = \mathbf{E}_{20} e^{-j\mathbf{k}_2 . \mathbf{r}}$$

$$E_{zs1}^+ = E_{z10}^+ e^{-j\mathbf{k}_1^+ . \mathbf{r}} = E_{10}^+ \cos\theta_1 e^{-jk_1(x\cos\theta_1 + z\sin\theta_1)}$$

$$E_{zs1}^- = E_{z10}^- e^{-j\mathbf{k}_1^- . \mathbf{r}} = E_{10}^- \cos\theta_1' e^{-jk_1(x\cos\theta_1' - z\sin\theta_1')}$$

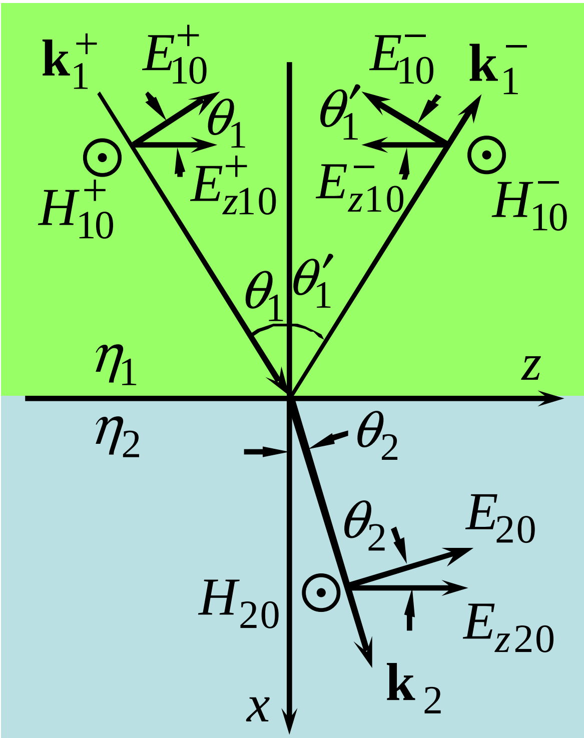
$$E_{zs2} = E_{z20} e^{-j\mathbf{k}_2 . \mathbf{r}} = E_{20} \cos\theta_2 e^{-jk_2(x\cos\theta_2 + z\sin\theta_2)}$$

$$E_{zs1}^+ + E_{zs1}^- = E_{zs2} \quad (\text{at } x = 0)$$

$$\rightarrow E_{10}^+ \cos\theta_1 e^{-jk_1 z\sin\theta_1} + E_{10}^- \cos\theta_1' e^{-jk_1 z\sin\theta_1'} = E_{20} \cos\theta_2 e^{-jk_2 z\sin\theta_2}$$

$$\rightarrow k_1 z\sin\theta_1 = k_1 z\sin\theta_1' = k_2 z\sin\theta_2 \rightarrow \begin{cases} \theta_1' = \theta_1 \\ \boxed{k_1\sin\theta_1 = k_2\sin\theta_2} \end{cases} \rightarrow n_1\sin\theta_1 = n_2\sin\theta_2$$

Plane Wave Reflection at Oblique Incidence Angles (4)



p – polarization, TM

$$\left.\begin{aligned} &\theta_1' = \theta_1 \\ &k_1 \sin\theta_1 = k_2 \sin\theta_2 \\ &E_{10}^+ \cos\theta_1 e^{-jk_1 z\sin\theta_1} + E_{10}^- \cos\theta_1' e^{-jk_1 z \sin\theta_1'} = \\ &= E_{20}\cos\theta_2 e^{-jk_2 z\sin\theta_2} \end{aligned}\right\}$$

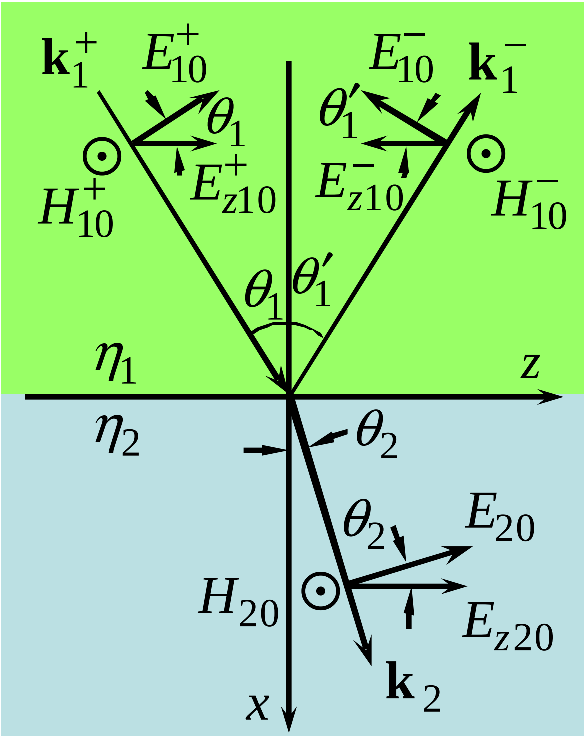
$$\rightarrow E_{10}^+ \cos\theta_1 + E_{10}^- \cos\theta_1 = E_{20}\cos\theta_2$$

$$H_{10}^+ + H_{10}^- = H_{20} \quad (\text{at } x = 0)$$

$$\rightarrow \frac{E_{10}^+ \cos\theta_1}{\eta_{1p}} - \frac{E_{10}^- \cos\theta_1}{\eta_{1p}} = \frac{E_{20}\cos\theta_2}{\eta_{2p}}$$

where $\eta_{1p} = \eta_1 \cos\theta_1, \quad \eta_{2p} = \eta_2 \cos\theta_2$

Plane Wave Reflection at Oblique Incidence Angles (5)



p – polarization, TM

$$\left.\begin{aligned} E_{10}^+\cos\theta_1 + E_{10}^-\cos\theta_1 &= E_{20}\cos\theta_2 \\ \frac{E_{10}^+\cos\theta_1}{\eta_{1p}} - \frac{E_{10}^-\cos\theta_1}{\eta_{1p}} &= \frac{E_{20}\cos\theta_2}{\eta_{2p}} \end{aligned}\right\}$$

$$\rightarrow \begin{cases} \Gamma_p = \dfrac{E_{10}^-}{E_{10}^+} = \dfrac{\eta_{2p} - \eta_{1p}}{\eta_{2p} + \eta_{1p}} \\[2ex] \tau_p = \dfrac{E_{20}}{E_{10}^+} = \dfrac{2\eta_{2p}}{\eta_{2p} + \eta_{1p}} \end{cases}$$

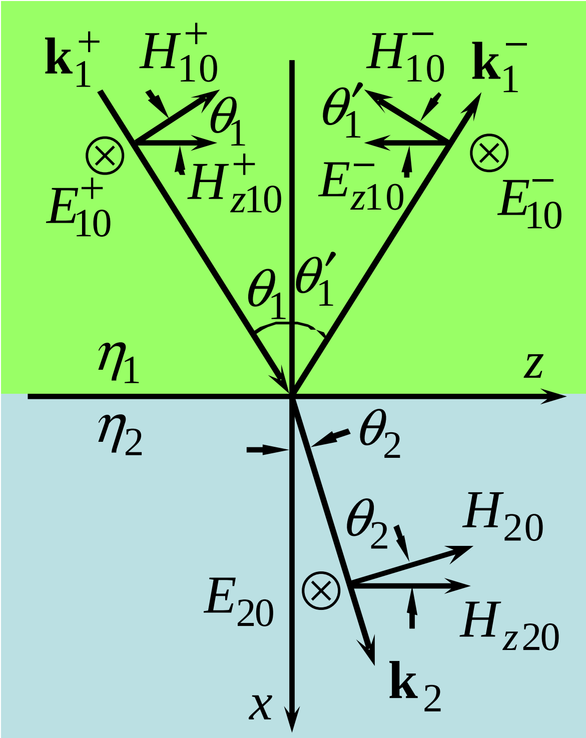
Plane Wave Reflection at Oblique Incidence Angles (6)

$$\Gamma_s = \frac{E_{y10}^-}{E_{y10}^+} = \frac{\eta_{2s} - \eta_{1s}}{\eta_{2s} + \eta_{1s}}$$

$$\tau_s = \frac{E_{y20}}{E_{y10}^+} = \frac{2\eta_{2s}}{\eta_{2s} + \eta_{1s}}$$

$$\eta_{1s} = \frac{\eta_1}{\cos\theta_1}$$

$$\eta_{2s} = \frac{\eta_2}{\cos\theta_2}$$



s – polarization, TE

Plane Wave Reflection at Oblique Incidence Angles (7)

Ex. 1

A uniform plane wave is incident from air onto glass at an angle of 30° from the normal. Find the fraction of the incident power that is reflected and transmitted for (a) *p* – polarization, & (b) *s* – polarization. Given glass refractive index n_2 = 1.45.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \rightarrow \theta_2 = \text{asin}\frac{\sin 30^o}{1.45} = 20.2^o$$

$$\eta_{1p} = \eta_1 \cos 30^o = 377 \times 0.866 = 326\Omega$$

$$\left.\begin{array}{l} \eta_1 = \sqrt{\dfrac{\mu_1}{\varepsilon_1}} = \sqrt{\dfrac{\mu_{r1}\mu_0}{\varepsilon_{r1}\varepsilon_0}} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \\ \eta_2 = \sqrt{\dfrac{\mu_2}{\varepsilon_2}} = \sqrt{\dfrac{\mu_{r2}\mu_0}{\varepsilon_{r2}\varepsilon_0}} = \sqrt{\dfrac{\mu_0}{\varepsilon_{r2}\varepsilon_0}} \end{array}\right\} \rightarrow \left.\begin{array}{l} \dfrac{\eta_1}{\eta_2} = \sqrt{\varepsilon_{r2}} \\ n_2 = \sqrt{\varepsilon_{r2}} \end{array}\right\} \rightarrow \frac{\eta_1}{\eta_2} = n_2$$

$$\rightarrow \eta_2 = \frac{\eta_1}{n_2} = \frac{377}{1.45} = 260\ \Omega$$

$$\rightarrow \eta_{2p} = \eta_2 \cos\theta_2 = 260\cos 20.2^o = 244\ \Omega$$

Plane Wave Reflection at Oblique Incidence Angles (8)

Ex. 1

A uniform plane wave is incident from air onto glass at an angle of 30° from the normal. Find the fraction of the incident power that is reflected and transmitted for (a) *p* – polarization, & (b) *s* – polarization. Given glass refractive index n_2 = 1.45.

$$\eta_{1p} = 326\,\Omega, \quad \eta_{2p} = 244\ \Omega$$

$$\Gamma_p = \frac{\eta_{2p} - \eta_{1p}}{\eta_{2p} + \eta_{1p}} = \frac{244 - 326}{244 + 326} = -0.144$$

$$\frac{P_{reflected}}{P_{incident}} = \left|\Gamma_p\right|^2 = (-0.144)^2 = 0.021$$

$$\frac{P_{transmitted}}{P_{incident}} = 1 - \left|\Gamma_p\right|^2 = 1 - (-0.144)^2 = 0.979$$

Plane Wave Reflection at Oblique Incidence Angles (9)

Ex. 1

A uniform plane wave is incident from air onto glass at an angle of 30° from the normal. Find the fraction of the incident power that is reflected and transmitted for (a) p – polarization, & (b) s – polarization. Given glass refractive index n_2 = 1.45.

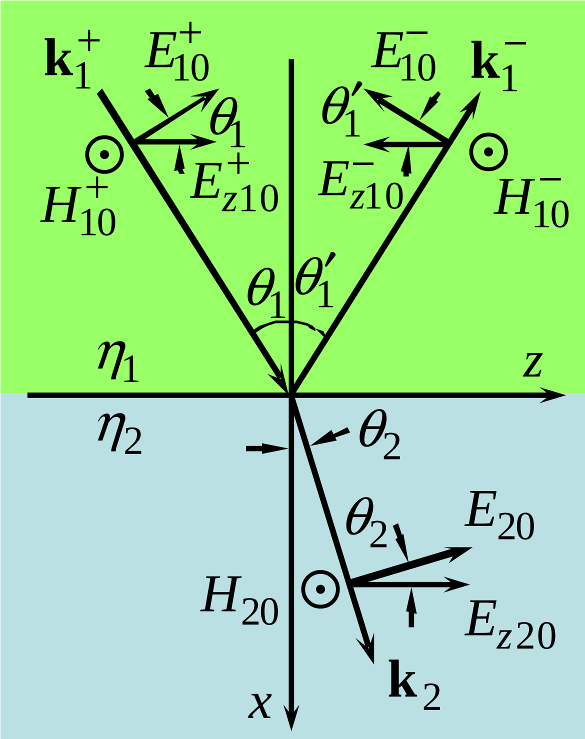
$$\eta_{1s} = \frac{\eta_1}{\cos\theta_1} = \frac{377}{\cos 30^o} = 435\,\Omega$$

$$\eta_{2s} = \frac{\eta_2}{\cos\theta_2} = \frac{260}{\cos 20.2^o} = 277\,\Omega$$

$$\Gamma_s = \frac{\eta_{2s}-\eta_{1s}}{\eta_{2s}+\eta_{1s}} = \frac{277-435}{277+435} = -0.222$$

$$\frac{P_{reflected}}{P_{incident}} = |\Gamma_s|^2 = (-0.222)^2 = 0.049$$

$$\frac{P_{transmitted}}{P_{incident}} = 1-|\Gamma_s|^2 = 1-(-0.222)^2 = 0.951$$



p – polarization, TM

Plane Wave Reflection at Oblique Incidence Angles (10)

Ex. 2

A uniform plane wave is incident from air onto a dielectric medium ($\varepsilon_r = 9$) at 30° with respect to the normal to the plane interface. Its EFI in air is $377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x$ V/m. Find the angular frequency, expressions for the EFI & MFI fields in both media, & the average power density of the wave in the dielectric medium. Assume that μ of the medium is the same as that of free space.

$$E_{xs} = 377e^{-j0.866z}e^{-j0.5y} = 377e^{-j[k_1^+\cos(30^\circ)z + k_1^+\sin(30^\circ)y]}$$

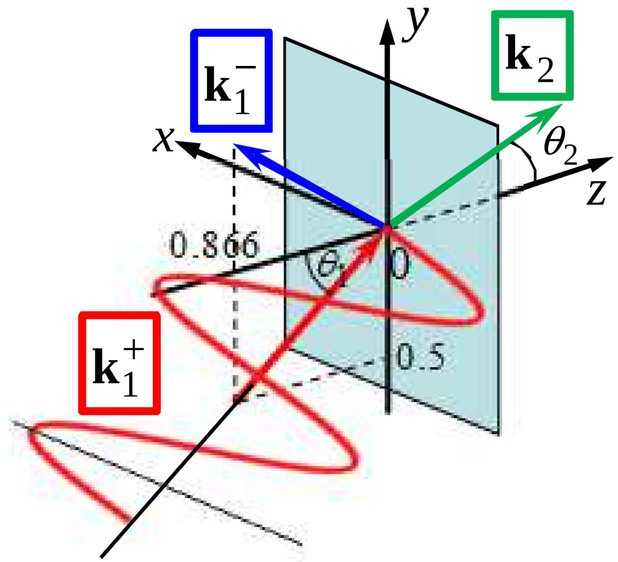
$$k_1^+ = \frac{0.866}{\cos 30^\circ} = 1 \text{ rad/m}$$

$$k_1^+ = \frac{\omega}{c}\sqrt{\varepsilon_{r1}} \to \omega = ck_1^+ = 3\times10^8 \times 1 = 3\times10^8 \text{ rad/s}$$

$$k_2 = \frac{\omega}{c}\sqrt{\varepsilon_{r2}} = \frac{3\times10^8}{3\times10^8}\sqrt{9} = 3 \text{ rad/m}$$

$$\eta_1 = \eta_0 = \sqrt{\mu_0/\varepsilon_0} = 377 = 120\pi\ \Omega$$

$$\eta_2 = \sqrt{\mu_2/\varepsilon_2} = \sqrt{\mu_0/\varepsilon_{r2}\varepsilon_0} = \eta_0/\sqrt{\varepsilon_{r2}} = 120\pi/\sqrt{9} = 40\pi\ \Omega$$



Plane Wave Reflection at Oblique Incidence Angles (11)

Ex. 2

A uniform plane wave is incident from air onto a dielectric medium ($\varepsilon_r = 9$) at 30° with respect to the normal to the plane interface. Its EFI in air is $377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x$ V/m. Find the angular frequency, expressions for the EFI & MFI fields in both media, & the average power density of the wave in the dielectric medium. Assume that μ of the medium is the same as that of free space.

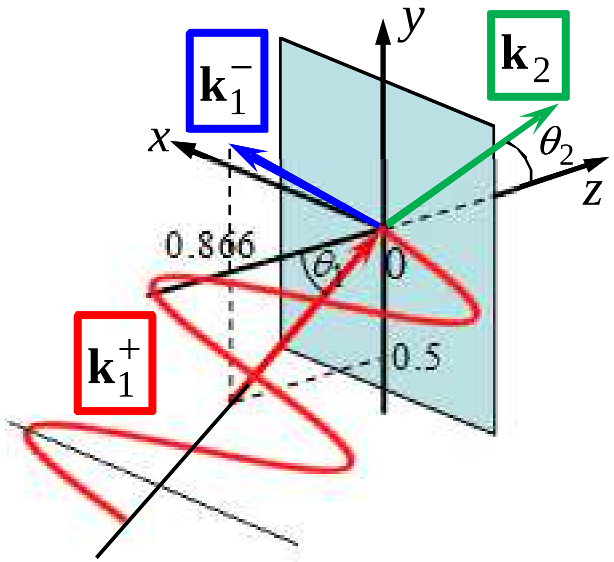
$$k_1^+ \sin\theta_1 = k_2 \sin\theta_2$$

$$\theta_2 = \operatorname{asin}\left(\frac{k_1^+}{k_2}\sin\theta_1\right) = \operatorname{asin}\left(\frac{1}{3}\sin 30^o\right) = 9.6^o$$

$$\Gamma_s = \frac{\eta_{2s} - \eta_{1s}}{\eta_{2s} + \eta_{1s}} = \frac{\dfrac{\eta_2}{\cos\theta_2} - \dfrac{\eta_1}{\cos\theta_1}}{\dfrac{\eta_2}{\cos\theta_2} - \dfrac{\eta_1}{\cos\theta_1}} = \frac{\eta_2\cos\theta_1 - \eta_2\cos\theta_1}{\eta_2\cos\theta_1 + \eta_2\cos\theta_1}$$

$$= \frac{40\pi\cos 30^o - 120\pi\cos 9.6^o}{40\pi\cos 30^o + 120\pi\cos 9.6^o} = -0.547$$

$$\tau_s = 1 + \Gamma_s = 1 - 0.547 = 0.453$$



Plane Wave Reflection at Oblique Incidence Angles (12)

Ex. 2

A uniform plane wave is incident from air onto a dielectric medium ($\varepsilon_r = 9$) at 30° with respect to the normal to the plane interface. Its EFI in air is $377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x$ V/m. Find the angular frequency, expressions for the EFI & MFI fields in both media, & the average power density of the wave in the dielectric medium. Assume that μ of the medium is the same as that of free space.

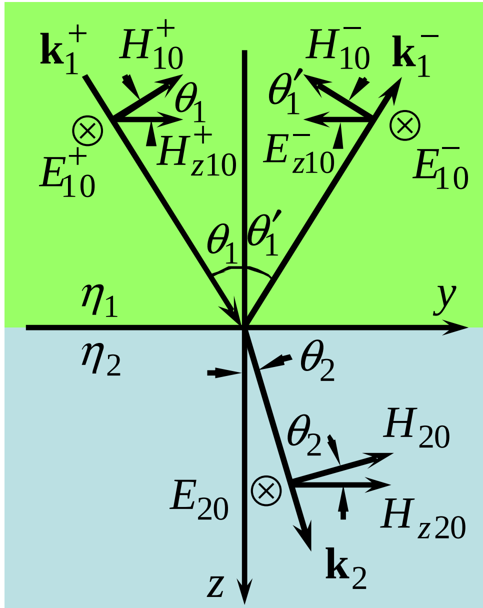
$$\mathbf{E}_{1s}^{+} = 377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x \text{ V/m}$$

$$\nabla\times\mathbf{E}_{1s}^{+} = -j\omega\mu_1\mathbf{H}_{1s}^{+} \rightarrow \frac{\partial E_{1xs}^{+}}{\partial z}\mathbf{a}_y - \frac{\partial E_{1xs}^{+}}{\partial y}\mathbf{a}_z = -j\omega\mu_1\mathbf{H}_{1s}^{+}$$

$$\rightarrow (-j0.866\mathbf{a}_y + j0.5\mathbf{a}_z)377e^{-j0.866z}e^{-j0.5y} = -j\omega\mu_1\mathbf{H}_{1s}^{+}$$

$$\rightarrow \mathbf{H}_{1s}^{+} = \frac{(-j0.866\mathbf{a}_y + j0.5\mathbf{a}_z)377e^{-j0.866z}e^{-j0.5y}}{-j3\times10^{8}\times4\pi\times10^{-7}}$$

$$= e^{-j0.866z}e^{-j0.5y}(0.866\mathbf{a}_y - 0.5\mathbf{a}_z) \text{ A/m}$$



s – polarization, TE

Plane Wave Reflection at Oblique Incidence Angles (13)

Ex. 2

A uniform plane wave is incident from air onto a dielectric medium ($\varepsilon_r = 9$) at 30° with respect to the normal to the plane interface. Its EFI in air is $377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x$ V/m. Find the angular frequency, expressions for the EFI & MFI fields in both media, & the average power density of the wave in the dielectric medium. Assume that μ of the medium is the same as that of free space.

$$\mathbf{E}_{1s}^{+} = 377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x = \mathbf{E}_{10}^{+}e^{-j\mathbf{k}_1^{+}.\mathbf{r}}\mathbf{a}_x \text{ V/m}$$

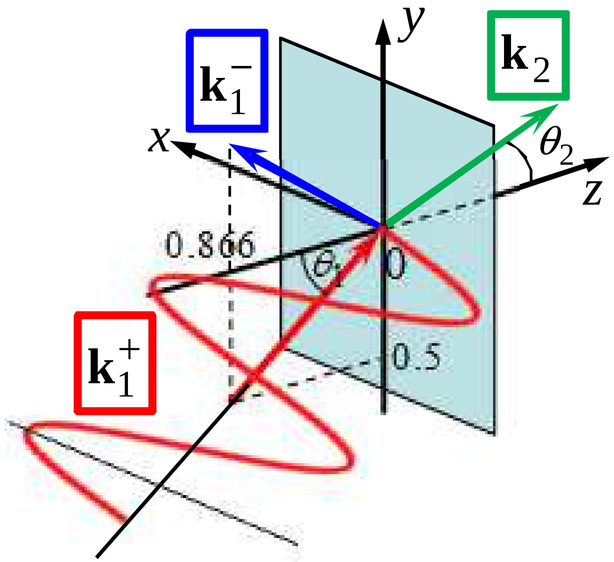
$$\mathbf{H}_{1s}^{+} = e^{-j0.866z}e^{-j0.5y}(0.866\mathbf{a}_y - 0.5\mathbf{a}_z) \text{ A/m}$$

$$\mathbf{k}_1^{+} = k_1^{+}(\cos\theta_1\mathbf{a}_z + \sin\theta_1 a_y)$$

$$= 1(\cos 30^{\circ}\mathbf{a}_z + \sin 30^{\circ}\mathbf{a}_y) = 0.866\mathbf{a}_z + 0.5\mathbf{a}_y \text{ rad/m}$$

$$\mathbf{r} = z\mathbf{a}_z + y\mathbf{a}_y \text{ m}$$

$$\mathbf{k}_1^{-} = -0.866\mathbf{a}_z + 0.5\mathbf{a}_y \text{ rad/m}$$



$$\mathbf{k}_2 = k_2(\cos\theta_2\mathbf{a}_z + \sin\theta_2)\mathbf{a}_y = 3(\cos 9.6^{\circ}\mathbf{a}_z + \sin 9.6^{\circ}\mathbf{a}_y) = 2.958\mathbf{a}_z + 0.5\mathbf{a}_y \text{ rad/m}$$

Plane Wave Reflection at Oblique Incidence Angles (14)

Ex. 2

A uniform plane wave is incident from air onto a dielectric medium ($\varepsilon_r = 9$) at 30° with respect to the normal to the plane interface. Its EFI in air is $377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x$ V/m. Find the angular frequency, expressions for the EFI & MFI fields in both media, & the average power density of the wave in the dielectric medium. Assume that μ of the medium is the same as that of free space.

$$\mathbf{E}_{1s}^{+} = 377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x = E_{10}^{+}e^{-j\mathbf{k}_1^{+}.\mathbf{r}}\mathbf{a}_x \text{ V/m}$$

$$\mathbf{H}_{1s}^{+} = e^{-j0.866z}e^{-j0.5y}(0.866\mathbf{a}_y - 0.5\mathbf{a}_z) \text{ A/m}$$

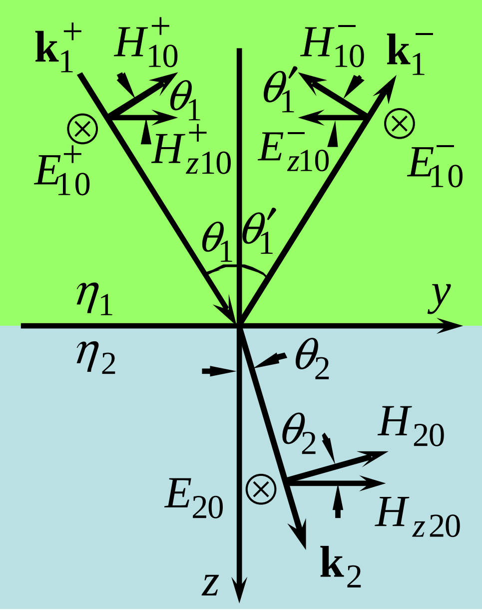
$$\mathbf{k}_1^{+} = 0.866\mathbf{a}_z + 0.5\mathbf{a}_y \text{ rad/m}; \ \mathbf{k}_1^{-} = -0.866\mathbf{a}_z + 0.5\mathbf{a}_y \text{ rad/m}$$

$$\mathbf{k}_2 = 2.958\mathbf{a}_z + 0.5\mathbf{a}_y \text{ rad/m}; \ \mathbf{r} = z\mathbf{a}_z + y\mathbf{a}_y \text{ m}$$

$$\Gamma_s = \frac{E_{10}^{-}}{E_{10}^{+}} \rightarrow E_{10}^{-} = \Gamma_s E_{10}^{+} = -0.547 \times 377 = -206.22 \text{ V/m}$$

$$\mathbf{E}_{1s}^{-} = E_{10}^{-}e^{-j\mathbf{k}_1^{-}.\mathbf{r}}\mathbf{a}_x = -206.22e^{-j(-0.866\mathbf{a}_z+0.5\mathbf{a}_y).(z\mathbf{a}_z+y\mathbf{a}_y)}\mathbf{a}_x$$

$$= -206.22e^{-j(-0.866z+0.5y)}\mathbf{a}_x = -206.22e^{j0.866z}e^{-j0.5y}\mathbf{a}_x \text{ V/m}$$



s – polarization, TE

Plane Wave Reflection at Oblique Incidence Angles (15)

Ex. 2

A uniform plane wave is incident from air onto a dielectric medium ($\varepsilon_r = 9$) at 30° with respect to the normal to the plane interface. Its EFI in air is $377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x$ V/m. Find the angular frequency, expressions for the EFI & MFI fields in both media, & the average power density of the wave in the dielectric medium. Assume that μ of the medium is the same as that of free space.

$$\mathbf{E}_{1s}^{+} = 377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x = E_{10}^{+}e^{-j\mathbf{k}_1^{+}.\mathbf{r}}\mathbf{a}_x \text{ V/m}$$

$$\mathbf{H}_{1s}^{+} = e^{-j0.866z}e^{-j0.5y}(0.866\mathbf{a}_y - 0.5\mathbf{a}_z) \text{ A/m}$$

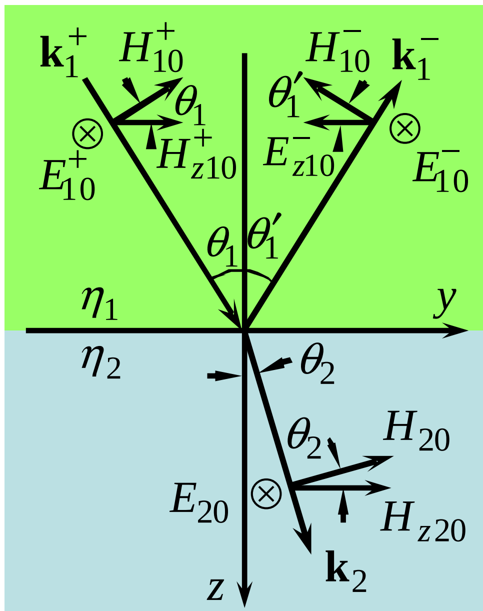
$$\mathbf{k}_1^{+} = 0.866\mathbf{a}_z + 0.5\mathbf{a}_y \text{ rad/m; } \mathbf{k}_1^{-} = -0.866\mathbf{a}_z + 0.5\mathbf{a}_y \text{ rad/m}$$

$$\mathbf{k}_2 = 2.958\mathbf{a}_z + 0.5\mathbf{a}_y \text{ rad/m; } \mathbf{r} = z\mathbf{a}_z + y\mathbf{a}_y \text{ m}$$

$$\tau_s = \frac{E_{20}}{E_{10}^{+}} \rightarrow E_{20} = \tau_s E_{10}^{+} = 0.453\times 377 = 170.78 \text{ V/m}$$

$$\mathbf{E}_{2s} = E_{20}e^{-j\mathbf{k}_2.\mathbf{r}}\mathbf{a}_x = 170.78e^{-j(2.958\mathbf{a}_z+0.5\mathbf{a}_y).(z\mathbf{a}_z+y\mathbf{a}_y)}\mathbf{a}_x$$

$$= 170.78e^{-j(2.958z+0.5y)}\mathbf{a}_x = 170.78e^{-j2.958z}e^{-j0.5y}\mathbf{a}_x \text{ V/m}$$



s – polarization, TE

Plane Wave Reflection at Oblique Incidence Angles (15)

Ex. 2

A uniform plane wave is incident from air onto a dielectric medium ($\varepsilon_r = 9$) at 30° with respect to the normal to the plane interface. Its EFI in air is $377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x$ V/m. Find the angular frequency, expressions for the EFI & MFI fields in both media, & the average power density of the wave in the dielectric medium. Assume that μ of the medium is the same as that of free space.

$$\mathbf{E}_{1s}^{+} = 377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x \text{ V/m}; \quad \mathbf{E}_{1s}^{-} = -206.22e^{j0.866z}e^{-j0.5y}\mathbf{a}_x \text{ V/m}$$

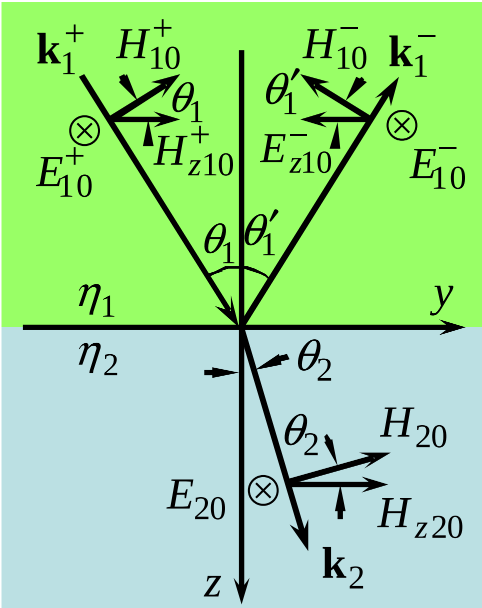
$$\mathbf{E}_{2s} = 170.78e^{-j2.958z}e^{-j0.5y}\mathbf{a}_x \text{ V/m}$$

$$\nabla\times\mathbf{E}_{1s}^{-} = \frac{\partial E_{1xs}^{-}}{\partial z}\mathbf{a}_y - \frac{\partial E_{1xs}^{-}}{\partial y}\mathbf{a}_z = -j\omega\mu_1\mathbf{H}_{1s}^{-}$$

$$\rightarrow (j0.866\mathbf{a}_y + j0.5\mathbf{a}_z)(-206.22e^{j0.866z}e^{-j0.5y}) = -j\omega\mu_1\mathbf{H}_{1s}^{-}$$

$$\rightarrow \mathbf{H}_{1s}^{-} = \frac{(-j0.866\mathbf{a}_y - j0.5\mathbf{a}_z)206.22e^{j0.866z}e^{-j0.5y}}{-j3\times10^{8}\times4\pi\times10^{-7}}$$

$$= e^{j0.866z}e^{-j0.5y}(0.474\mathbf{a}_y + 0.274\mathbf{a}_z) \text{ A/m}$$



s – polarization, TE

Plane Wave Reflection at Oblique Incidence Angles (16)

Ex. 2

A uniform plane wave is incident from air onto a dielectric medium ($\varepsilon_r = 9$) at 30° with respect to the normal to the plane interface. Its EFI in air is $377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x$ V/m. Find the angular frequency, expressions for the EFI & MFI fields in both media, & the average power density of the wave in the dielectric medium. Assume that μ of the medium is the same as that of free space.

$$\mathbf{E}_{1s}^{+} = 377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x \text{ V/m}; \quad \mathbf{E}_{1s}^{-} = -206.22e^{j0.866z}e^{-j0.5y}\mathbf{a}_x \text{ V/m}$$

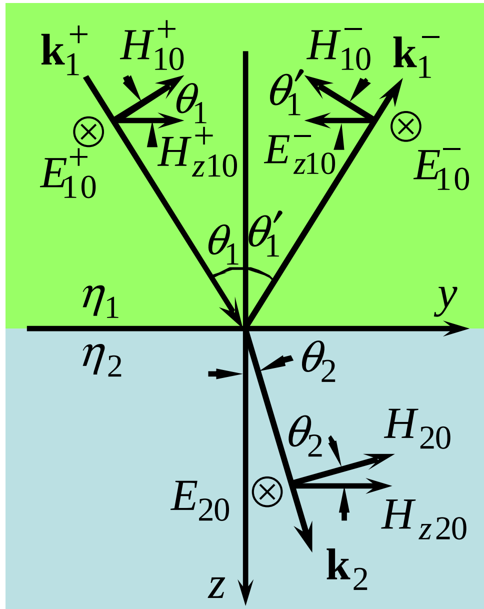
$$\mathbf{E}_{2s} = 170.78e^{-j2.958z}e^{-j0.5y}\mathbf{a}_x \text{ V/m}$$

$$\nabla\times\mathbf{E}_{2s} = \frac{\partial E_{2xs}}{\partial z}\mathbf{a}_y - \frac{\partial E_{2xs}}{\partial y}\mathbf{a}_z = -j\omega\mu_2\mathbf{H}_{2s}$$

$$\rightarrow (-j2.958\mathbf{a}_y - j0.5\mathbf{a}_z)170.78e^{-j2.958z}e^{-j0.5y} = -j\omega\mu_1\mathbf{H}_{2s}$$

$$\rightarrow \mathbf{H}_{2s} = \frac{(-j2.958\mathbf{a}_y + j0.5\mathbf{a}_z)170.78e^{-j2.958z}e^{-j0.5y}}{-j3\times10^{8}\times4\pi\times10^{-7}}$$

$$= e^{-j2.958z}e^{-j0.5y}(1.34\mathbf{a}_y - 0.227\mathbf{a}_z) \text{ A/m}$$



s – polarization, TE

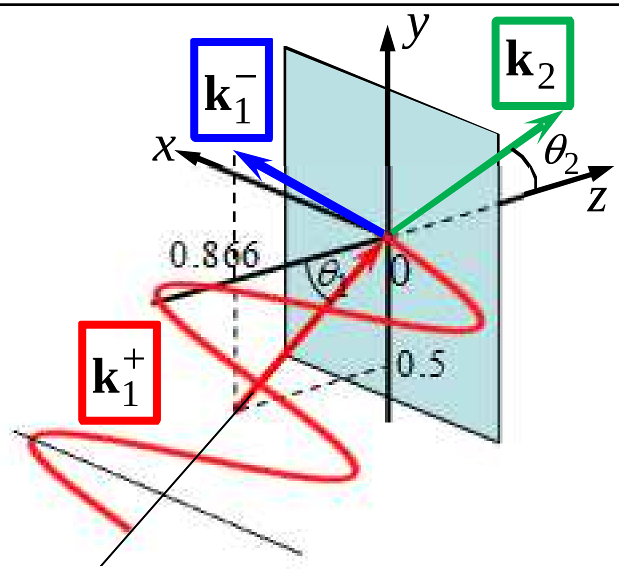
Plane Wave Reflection at Oblique Incidence Angles (17)

Ex. 2

A uniform plane wave is incident from air onto a dielectric medium ($\varepsilon_r = 9$) at 30° with respect to the normal to the plane interface. Its EFI in air is $377e^{-j0.866z}e^{-j0.5y}\mathbf{a}_x$ V/m. Find the angular frequency, expressions for the EFI & MFI fields in both media, & the average power density of the wave in the dielectric medium. Assume that μ of the medium is the same as that of free space.

$$\mathbf{E}_{2s} = 170.78e^{-j2.958z}e^{-j0.5y}\mathbf{a}_x \ \text{V/m}$$

$$\mathbf{H}_{2s} = e^{-j2.958z}e^{-j0.5y}(1.34\mathbf{a}_y - 0.227\mathbf{a}_z) \ \text{A/m}$$



$$\mathbf{S}_2 = \frac{1}{2}\text{Re}\left[\mathbf{E}_{2s} \times \hat{\mathbf{H}}_{2s}\right]$$

$$= \frac{1}{2}\text{Re}\left\{(170.78e^{-j2.958z}e^{-j0.5y}\mathbf{a}_x) \times \left[e^{j2.958z}e^{j0.5y}(1.34\mathbf{a}_y - 0.227\mathbf{a}_z)\right]\right\}$$

$$= 114.42\mathbf{a}_z + 19.38\mathbf{a}_y \ \text{W/m}^2$$

Plane Wave Reflection at Oblique Incidence Angles (18)

Total reflection: $|\Gamma|^2 = \Gamma\hat{\Gamma} = 1$

$$\left.\begin{array}{l}\cos\theta_2 = \sqrt{1-\sin^2\theta_2} \\ n_1\sin\theta_1 = n_2\sin\theta_2\end{array}\right\} \rightarrow \cos\theta_2 = \sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_1}$$

$$\eta_{2p} = \eta_2\cos\theta_2$$

$$\text{If } \sin\theta_1 > \frac{n_2}{n_1}$$

$$\rightarrow \eta_{2p} = j|\eta_{2p}|$$

$$\left.\begin{array}{l}\eta_{1p} = \eta_1\cos\theta_1 \\ \eta_1 > 0\end{array}\right\} \rightarrow \eta_{1p} > 0$$

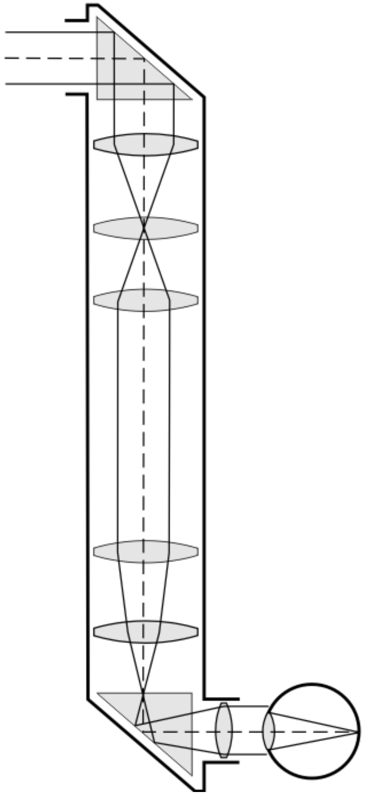
$$\rightarrow \Gamma_p = \frac{\eta_{2p}-\eta_{1p}}{\eta_{2p}+\eta_{1p}} = \frac{j|\eta_{2p}|-\eta_{1p}}{j|\eta_{2p}|+\eta_{1p}} = -\frac{\eta_{1p}-j|\eta_{2p}|}{\eta_{1p}+j|\eta_{2p}|} = -\frac{Z}{\hat{Z}} \rightarrow \Gamma_p\hat{\Gamma}_p = 1$$

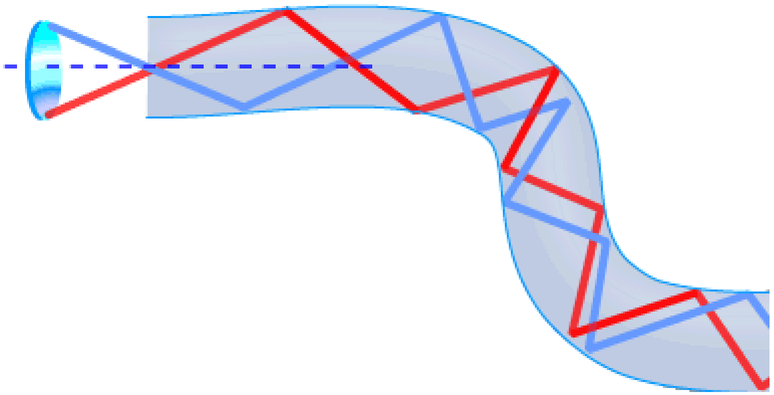
$$\rightarrow \boxed{\text{If } \sin\theta_1 \ge \frac{n_2}{n_1} \text{ then total reflection}} \qquad \rightarrow \boxed{\theta_1 \ge \theta_c = \text{asin}\frac{n_2}{n_1}}$$

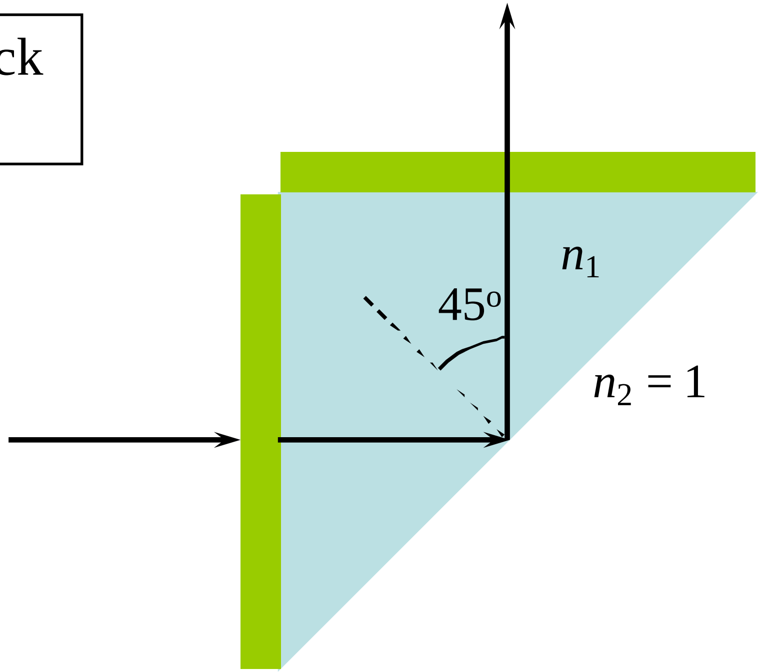
Plane Wave Reflection at Oblique Incidence Angles (19)

Ex. 3

Compute n_1 so that total reflection occurs at the back surface.







Plane Wave Reflection at Oblique Incidence Angles (20)

Total transmission: $\Gamma = 0$

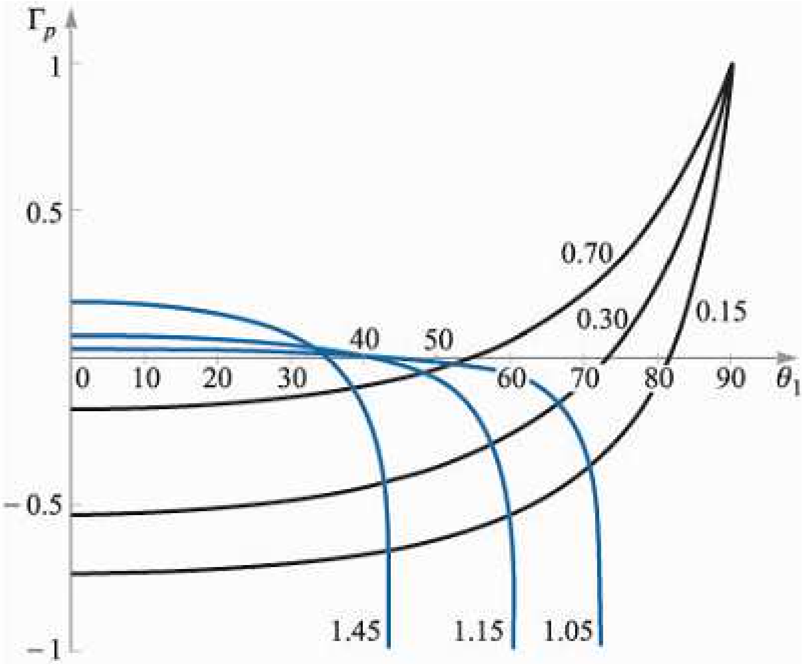
$$\left.\begin{array}{l}\Gamma_s = 0 \\ \Gamma_s = \dfrac{\eta_{2s} - \eta_{1s}}{\eta_{2s} + \eta_{1s}}\end{array}\right\} \to \eta_{2s} = \eta_{1s}$$

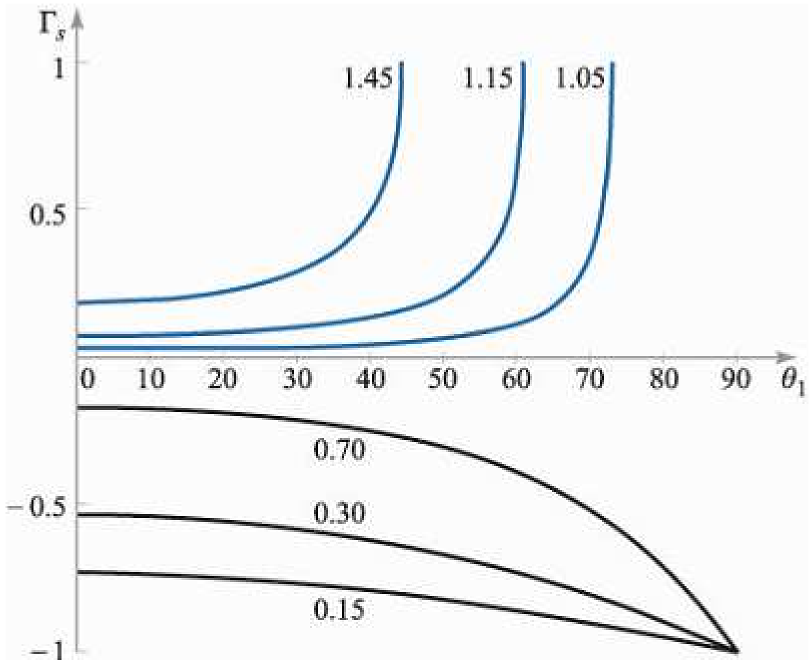
$$\left.\begin{array}{l}\eta_{2s} = \eta_{1s} \\ \eta_{1s} = \dfrac{\eta_1}{\cos\theta_1} \\ \eta_{2s} = \dfrac{\eta_2}{\cos\theta_2}\end{array}\right\} \to \left.\begin{array}{l}\dfrac{\eta_2}{\cos\theta_2} = \dfrac{\eta_1}{\cos\theta_1} \\ n_1 \sin\theta_1 = n_2 \sin\theta_2\end{array}\right\}$$

$$\to \eta_2 \left[1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1\right]^{-\frac{1}{2}} = \eta_1 \left[1 - \sin^2\theta_1\right]^{-\frac{1}{2}}$$

$$\Gamma_p = 0 \to \eta_2 \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1} = \eta_1 \sqrt{1 - \sin^2\theta_1} \to \boxed{\sin\theta_1 = \sin\theta_B = \frac{n_2}{\sqrt{n_1^2 + n_2^2}}}$$

Plane Wave Reflection at Oblique Incidence Angles (21)





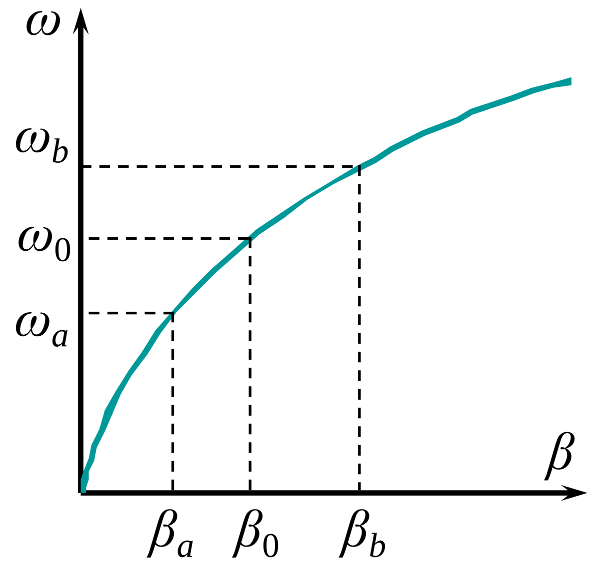
Plane Wave Reflection & Dispersion

1. Reflection of Uniform Plane Waves at Normal Incidence
2. Standing Wave Ratio
3. Wave Reflection from Multiple Interfaces
4. Plane Wave Propagation in General Directions
5. Plane Wave Reflection at Oblique Incidence Angles
6. **Wave Propagation in Dispersive Media**

Wave Propagation in Dispersive Media (1)

Wave Propagation in Dispersive Media (2)

$$\beta(\omega) = k = \omega\sqrt{\mu_0\varepsilon(\omega)} = n(\omega)\frac{\omega}{c}$$



$$\left.\begin{aligned} &E_{c,net}(z,t) = E_0\left(e^{-j\beta_a z}e^{-j\omega_a t} + e^{-j\beta_b z}e^{-j\omega_b t}\right) \\ &\Delta\omega = \omega_0 - \omega_a = \omega_b - \omega_0 \\ &\Delta\beta = \beta_0 - \beta_a = \beta_b - \beta_0 \end{aligned}\right\}$$

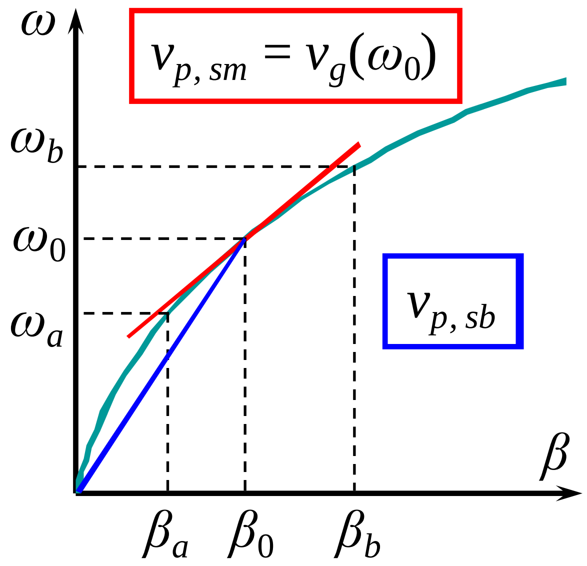
$$\rightarrow E_{c,net}(z,t) = E_0 e^{-j\beta_0 z}e^{j\omega_0 t}\left(e^{j\Delta\beta z}e^{-j\Delta\omega t} + e^{-j\Delta\beta z}e^{j\Delta\omega t}\right)$$

$$= 2E_0 e^{-j\beta_0 z}e^{j\omega_0 t}\cos(\Delta\omega t - \Delta\beta z)$$

$$\rightarrow E_{net}(z,t) = \mathrm{Re}[E_{c,net}] = 2E_0\cos(\Delta\omega t - \Delta\beta t)\cos(\omega_0 t - \beta_0 t)$$

Wave Propagation in Dispersive Media (3)

$$E_{net}(z,t) = 2E_0 \cos(\Delta\omega t - \Delta\beta t)\cos(\omega_0 t - \beta_0 t)$$



$$v_{p,carrier} = \frac{\omega_0}{\beta_0}$$

$$v_{p,envelope} = \frac{\Delta\omega}{\Delta\beta}$$

$$\lim_{\Delta\omega \to 0} \frac{\Delta\omega}{\Delta\beta} = \left.\frac{d\omega}{d\beta}\right|_{\omega_0} = v_g(\omega_0)$$