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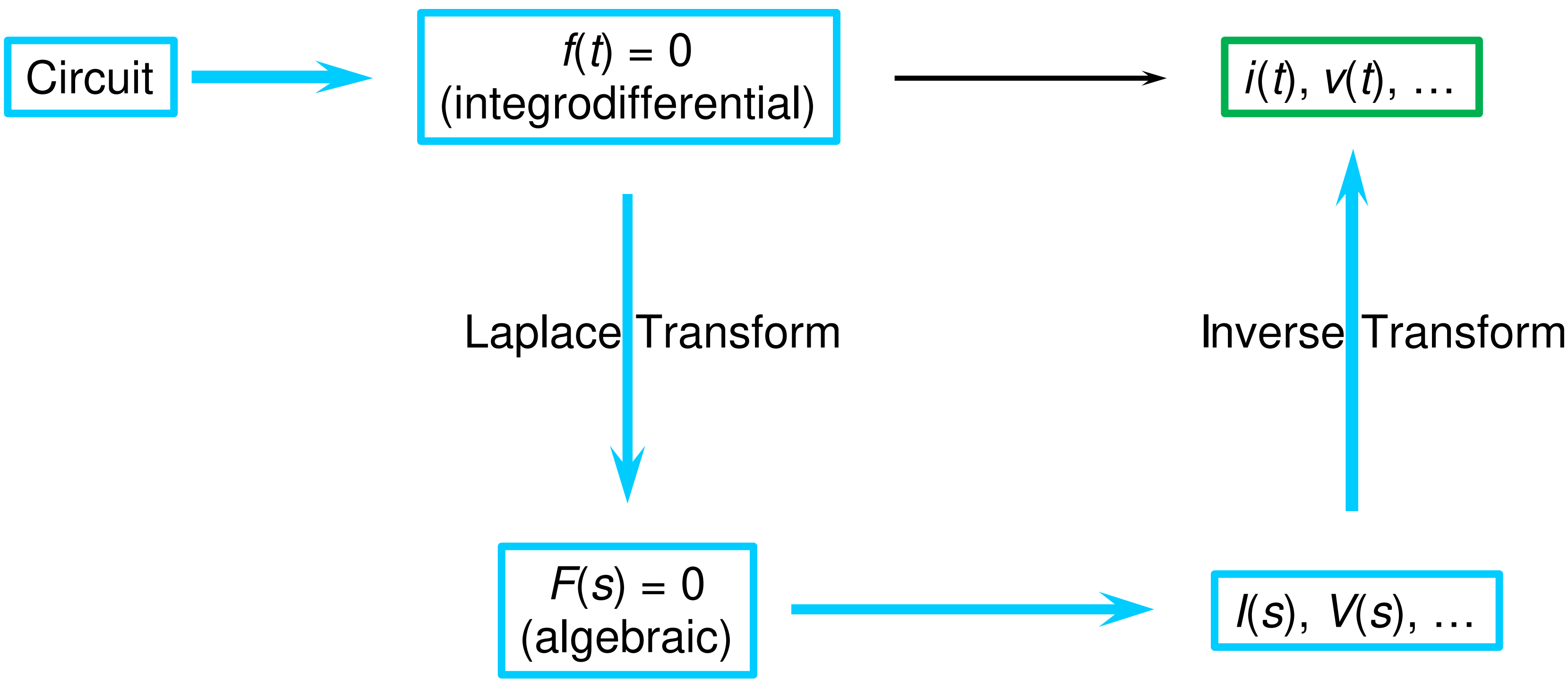
Electric Circuit Theory

The Laplace Transform

Contents

The Laplace Transform



The Laplace Transform

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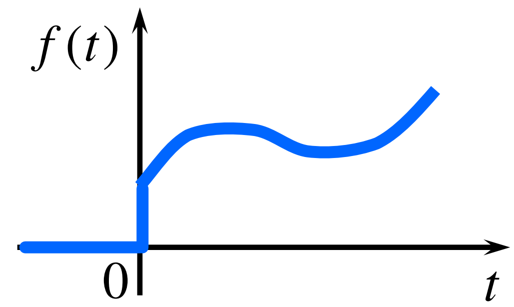
Definition



$$F(s) = L[f(t)] = \int_0^{\infty} f(t) e^{-st} dt$$

$$s = \sigma + j\omega$$

$$\int_0^{\infty} |f(t)| e^{-\sigma t} dt < \infty$$

$$f(t) = L^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s) e^{st} ds$$

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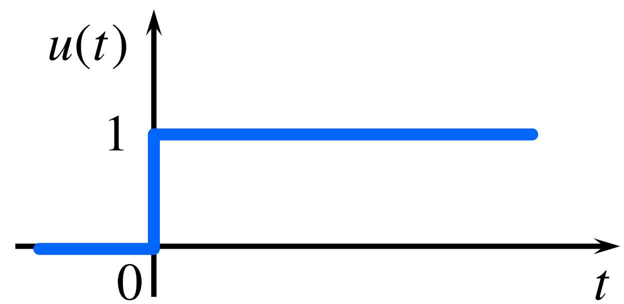
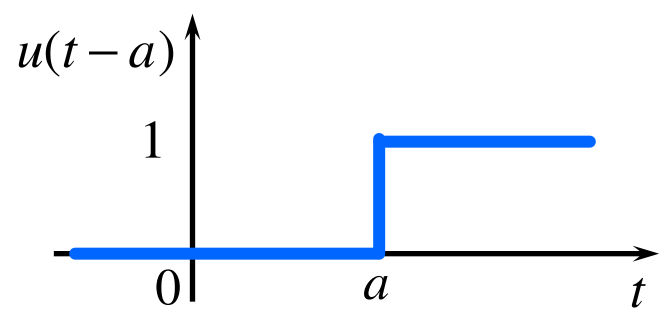
Two Important Singularity Functions (1)

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

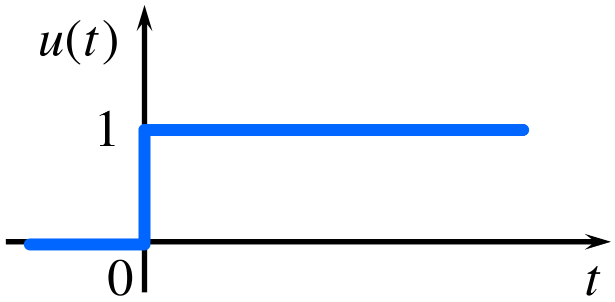


$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$



Ex. 1 Two Important Singularity Functions (2)

Determine the Laplace transform for the waveform?



$$F(s) = \int_0^\infty u(t)e^{-st}dt$$

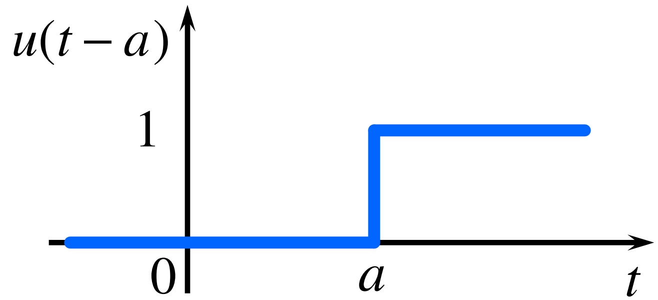
$$= \int_0^\infty 1e^{-st}dt$$

$$= -\frac{1}{s}e^{-st}\Big|_0^\infty$$

$$= \boxed{\frac{1}{s}}$$

Ex. 2 Two Important Singularity Functions (3)

Determine the Laplace transform for the waveform?



$$F(s) = \int_0^\infty u(t-a)e^{-st}dt$$

$$= \int_0^a 0dt + \int_a^\infty 1e^{-st}dt$$

$$= -\frac{1}{s}e^{-st}\Big|_a^\infty$$

$$= \boxed{\frac{e^{-as}}{s}}$$

Ex. 3 Two Important Singularity Functions (4)

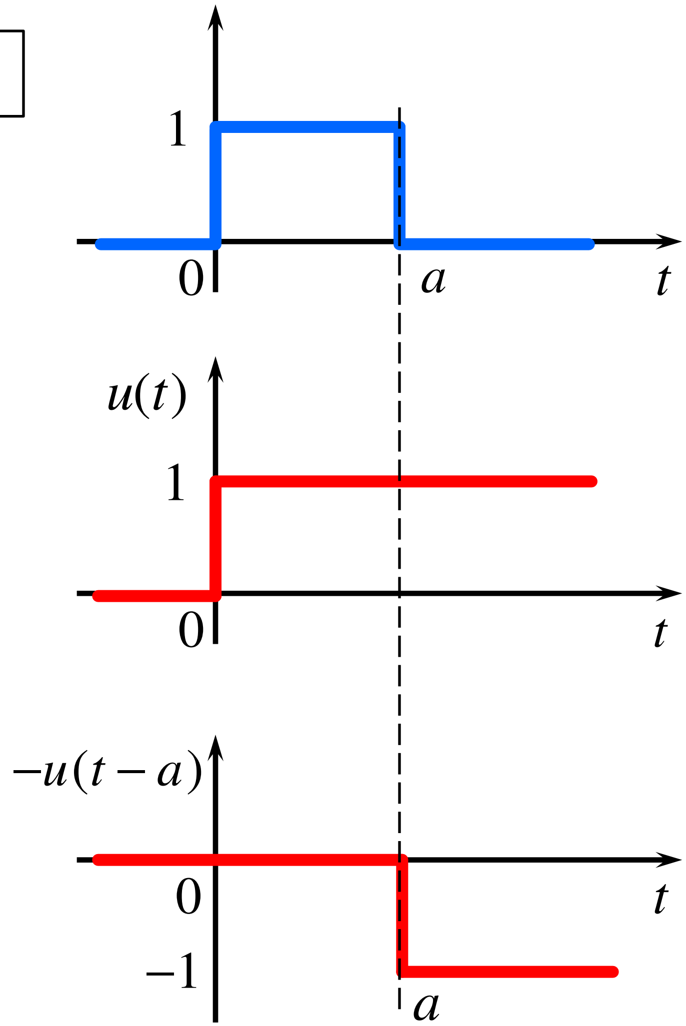
Determine the Laplace transform for the waveform?

$$F(s) = \int_0^\infty [u(t) - u(t-a)]e^{-st}dt$$

$$\int_0^\infty u(t)e^{-st}dt = \frac{1}{s}$$

$$\int_0^\infty u(t-a)e^{-st}dt = \frac{e^{-st}}{s}$$

$$\rightarrow F(s) = \frac{1}{s} - \frac{e^{-as}}{s} = \boxed{\frac{1-e^{-as}}{s}}$$



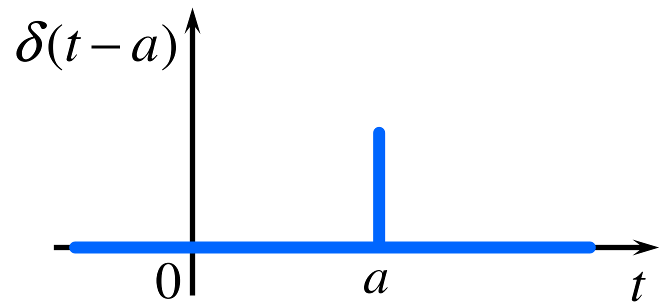
Two Important Singularity Functions (5)

$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)dt = 1 \qquad \varepsilon > 0$$

$$\delta(t-a) = 0 \qquad t \neq a$$

$$\int_{a-\varepsilon}^{a+\varepsilon} \delta(t-a)dt = 1 \qquad \varepsilon > 0$$



$$\int_{t_1}^{t_2} f(t)\delta(t-a)dt = \begin{cases} f(a) & t_1 < a < t_2 \\ 0 & a < t_1, a > t_2 \end{cases}$$

Ex. 4 Two Important Singularity Functions (6)

Determine the Laplace transform of an impulse function?

$$\left.\begin{aligned} &F(s) = \int_0^\infty \delta(t-a)e^{-st}dt \\ &\int_{t_1}^{t_2} f(t)\delta(t-a)dt = \begin{cases} f(a) & t_1 < a < t_2 \\ 0 & a < t_1, a > t_2 \end{cases} \end{aligned}\right\} \rightarrow F(s) = e^{-as}$$

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Transform Pairs (1)

Ex. 1

Find the Laplace transform of $f(t) = t$?

$$F(s) = \int_0^\infty t e^{-st} dt$$

$$\text{Let } u = t \ \ \& \ \ dv = e^{-st}dt \rightarrow du = dt \ \ \& \ \ v = \int e^{-st}dt = -\frac{1}{s}e^{-st}$$

$$\rightarrow F(s) = -\frac{t}{s}e^{-st}\Big|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}dt = 0 - \frac{e^{-st}}{s}\Big|_0^\infty = \boxed{\frac{1}{s^2}}$$

Ex. 2

Transform Pairs (2)

Find the Laplace transform of $f(t) = \cos\omega t$?

$$F(s) = \int_0^\infty \cos(\omega t) e^{-st} dt$$

$$= \int_0^\infty \frac{e^{j\omega t} + e^{-j\omega t}}{2} e^{-st} dt$$

$$= \int_0^\infty \frac{e^{-(s-j\omega)t} + e^{-(s+j\omega)t}}{2} dt$$

$$= \frac{1}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right)$$

$$= \boxed{\frac{s}{s^2+\omega^2}}$$

Transform Pairs (3)

Ex. 3

Find the Laplace transform of $f(t) = \sin\omega t$?

Transform Pairs (4)

$f(t)$	$\delta(t)$	$u(t)$	e^{-at}	t	te^{-at}	$\sin at$	$\cos at$
$F(s)$	1	$\frac{1}{s}$	$\frac{1}{s+a}$	$\frac{1}{s^2}$	$\frac{1}{(s+a)^2}$	$\frac{a}{s^2+a^2}$	$\frac{s}{s^2+a^2}$

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Properties of the Transform (1)

Property	$f(t)$	$F(s)$
1. Magnitude scaling	$Af(t)$	$AF(s)$
2. Addition/subtraction	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
3. Time scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
4. Time shifting	$f(t-a)u(t-a),\ a \geq 0$ $f(t)u(t-a),\ a \geq 0$	$e^{-as}F(s)$ $e^{-as}L[f(t+a)]$
5. Frequency shifting	$e^{-at}f(t)$	$F(s+a)$
6. Differentiation	$d^n f(t)/dt^n$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f^1(0) \ldots - s^o f^{n-1}(0)$
7. Multiplication by t	$t^n f(t)$	$(-1)^n d^n F(s)/ds^n$
8. Division by t	$f(t)/t$	$\int_s^\infty F(\lambda)d\lambda$
9. Integration	$\int_0^t f(\lambda)d\lambda$	$F(s)/s$
10. Convolution	$f_1(t) * f_2(t) = \int_0^t f_1(\lambda) f_2(t-\lambda)d\lambda$	$F_1(s)F_2(s)$

Ex. 1 Properties of the Transform (2)

Find the Laplace transform of $f(t) = 5 + e^{-10t} - \cos 20t$?

$$f_1(t) \pm f_2(t) \to F_1(s) \pm F_2(s)$$

$$\to F(s) = L[5] + L[e^{-10t}] - L[\cos 20t]$$

$$Af(t) \to AF(s)$$

$$\to L[5] = 5L[1]$$

$$L[1] = \frac{1}{s}$$

$$\to L[5] = \frac{5}{s}$$

$$L[e^{-10t}] = \frac{1}{s+10}$$

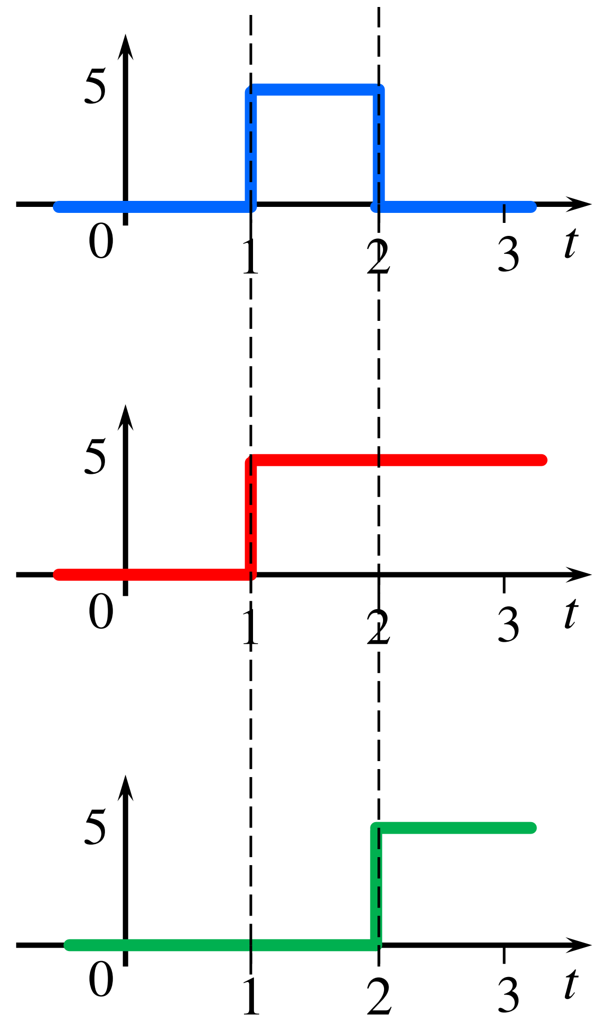
$$L[\cos 20t] = \frac{s}{s^2 + 20^2} = \frac{s}{s^2 + 400}$$

$$\to F(s) = \frac{5}{s} + \frac{1}{s+10} - \frac{s}{s^2+400} = \boxed{\frac{5s^3 + 2400s + 4000}{s(s+10)(s^2+400)}}$$

Properties of the Transform (3)

Ex. 2

Find the Laplace transform of the waveform?



$$f(t) = \boxed{5u(t-1)} - \boxed{5u(t-2)}$$

$$\rightarrow F(s) = 5\frac{e^{-s}}{s} - 5\frac{e^{-2s}}{s} = \boxed{\frac{5}{s}(e^{-s} - e^{-2s})}$$

Ex. 3 Properties of the Transform (4)

Find the Laplace transform of the waveform?

$$f(t) = \boxed{(-5t+10)}\boxed{[u(t-1)-u(t-2)]}$$

$$= -5tu(t-1) + 10u(t-1) +$$

$$+ 5tu(t-2) - 10u(t-2)$$

$$-5tu(t-1) = -5(t-1+1)u(t-1)$$

$$= -5(t-1)u(t-1) - 5u(t-1)$$

$$5tu(t-2) = 5(t-2+2)u(t-2)$$

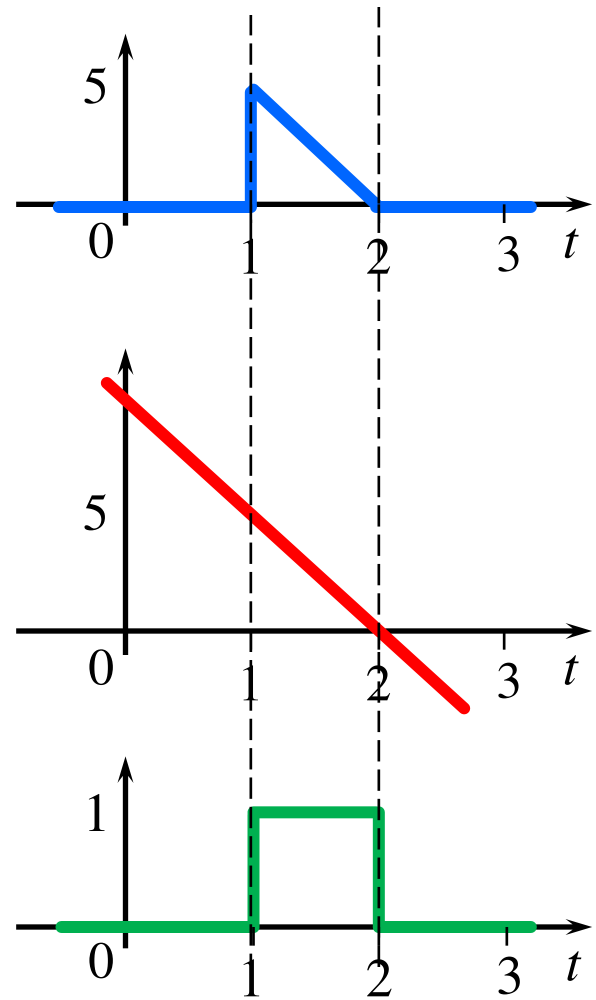
$$= 5(t-2)u(t-2) + 10u(t-2)$$

$$\rightarrow f(t) = -5(t-1)u(t-1) - 5u(t-1) +$$

$$+ 10u(t-1)$$

$$+ 5(t-2)u(t-2) + 10u(t-2)$$

$$- 10u(t-2)$$



Ex. 3 Properties of the Transform (5)

Find the Laplace transform of the waveform?

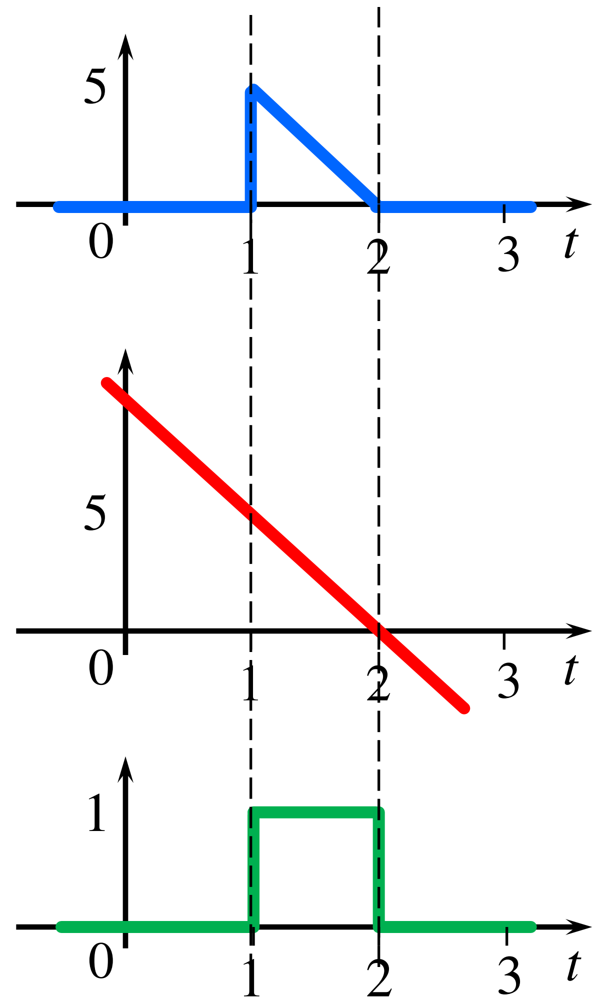
$$f(t) = (-5t+10)[u(t-1)-u(t-2)]$$

$$= -5(t-1)u(t-1) - 5u(t-1) +$$
$$+10u(t-1)$$
$$+5(t-2)u(t-2) + 10u(t-2)$$
$$-10u(t-2)$$

$$= -5(t-1)u(t-1) + 5u(t-1) +$$
$$+5(t-2)u(t-2)$$

$$\rightarrow F(s) = -5\frac{e^{-s}}{s^2} + \frac{5}{s}e^{-s} + 5\frac{e^{-2s}}{s^2}$$

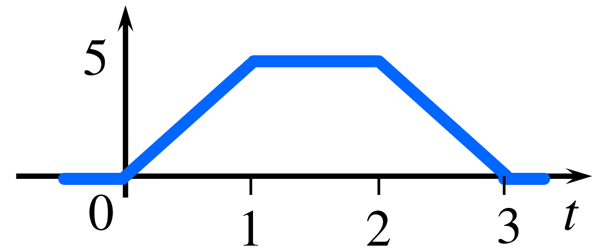
$$= \boxed{-\frac{5e^{-s}}{s^2}(1-s-e^{-s})}$$



Properties of the Transform (6)

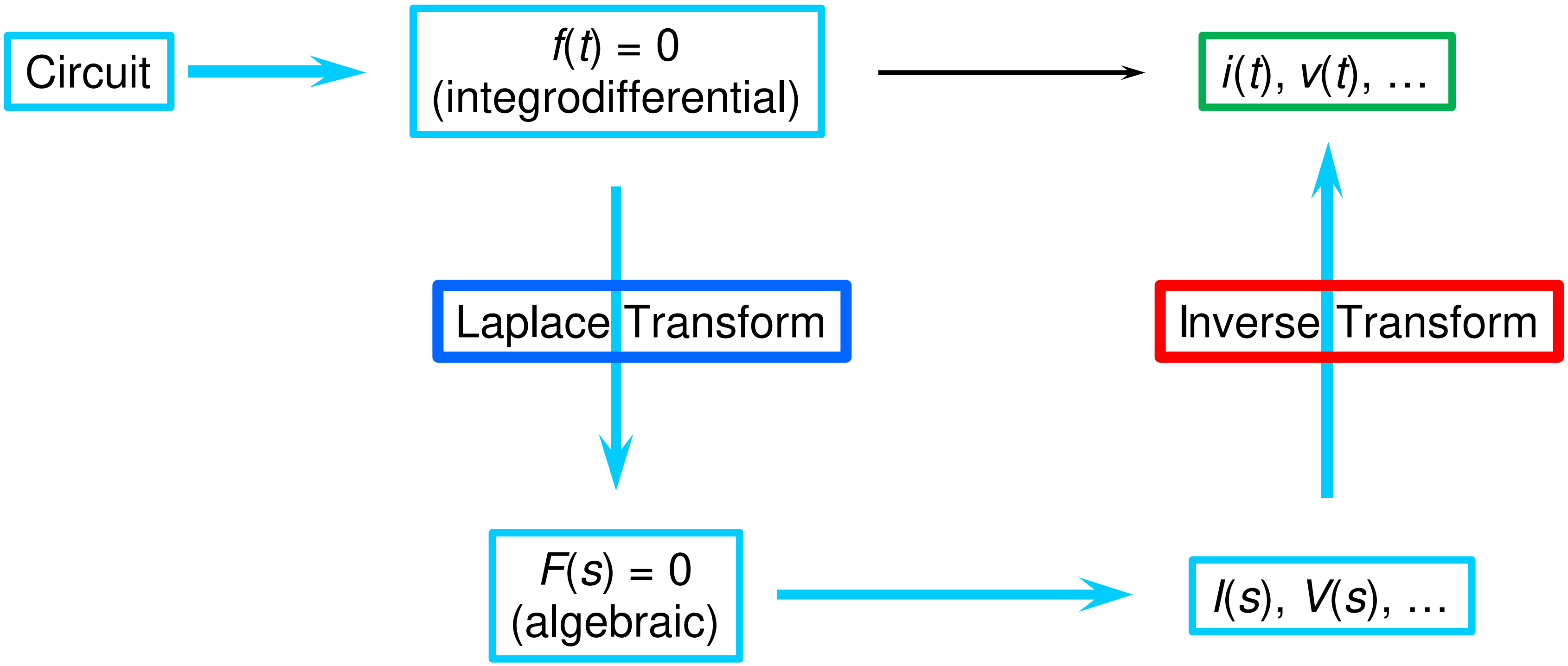
Ex. 4

Find the Laplace transform of the waveform?



The Laplace Transform

The Laplace Transform



Inverse Transform (1)

$$F(s)=\frac{P(s)}{Q(s)}=\frac{a_m s^m+a_{m-1}s^{m-1}+\ldots+a_1 s+a_0}{b_n s^n+b_{n-1}s^{n-1}+\ldots+b_1 s+b_0}$$

$$\text{Simple poles: }\quad F(s)=\frac{K_1}{s+p_1}+\frac{K_2}{s+p_2}+\ldots+\frac{K_n}{s+p_n}$$

$$\text{Complex-conjugate poles: }\quad F(s)=\frac{P_1(s)}{Q_1(s)(s+\alpha-j\beta)(s+\alpha+j\beta)}$$

$$=\frac{K_1}{s+\alpha-j\beta}+\frac{K_1^*}{s+\alpha+j\beta}+\ldots$$

$$\text{Multiple poles: }\quad F(s)=\frac{P_1(s)}{Q_1(s)(s+p_1)^n}$$

$$=\frac{K_{11}}{(s+p_1)}+\frac{K_{12}}{(s+p_1)^2}+\ldots+\frac{K_{1n}}{(s+p)^n}+\ldots$$

Inverse Transform (2)

$$\text{Simple poles}: \quad F(s) = \frac{P(s)}{Q(s)} = \frac{K_1}{s+p_1} + \frac{K_2}{s+p_2} + \ldots + \frac{K_n}{s+p_n}$$

$$(s+p_i)\left.\frac{P(s)}{Q(s)}\right|_{s=-p_i} = 0 + \ldots + 0 + K_i + 0 + \ldots + 0$$

$$L^{-1}\left[\frac{K_i}{s+p_i}\right] = K_i e^{-p_i t}$$

$$f(t) = K_1 e^{-p_1 t} + K_2 e^{-p_2 t} + \ldots + K_n e^{-p_n t}$$

Ex. 1

Inverse Transform (3)

Find the inverse Laplace transform of $F(s)=\dfrac{25s^2+300s+640}{s(s+4)(s+8)}$

$$F(s)=\frac{K_1}{s}+\frac{K_2}{s+4}+\frac{K_3}{s+8}; \qquad (s+p_i)\frac{P(s)}{Q(s)}\bigg|_{s=p_i}=0+\ldots+0+K_i+0+\ldots+0$$

$$K_1=sF(s)\big|_{s=0}=s\frac{25s^2+300s+640}{s(s+4)(s+8)}\bigg|_{s=0}=\frac{25s^2+300s+640}{(s+4)(s+8)}\bigg|_{s=0}=\frac{640}{4\times 8}=20$$

$$K_2=(s+4)F(s)\big|_{s=-4}=(s+4)\frac{25s^2+300s+640}{s(s+4)(s+8)}\bigg|_{s=-4}=\frac{25s^2+300s+640}{s(s+8)}\bigg|_{s=-4}=$$

$$=\frac{25(-4)^2+300(-4)+640}{(-4)(-4+8)}=10$$

Ex. 1

Inverse Transform (4)

Find the inverse Laplace transform of $F(s) = \dfrac{25s^2 + 300s + 640}{s(s+4)(s+8)}$

$$F(s) = \frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+8}; \qquad (s+p_i)\frac{P(s)}{Q(s)}\bigg|_{s=p_i} = 0 + \ldots + 0 + K_i + 0 + \ldots + 0$$

$$K_1 = 20; \quad K_2 = 10$$

$$K_3 = (s+8)F(s)\big|_{s=-8} = (s+8)\frac{25s^2 + 300s + 640}{s(s+4)(s+8)}\bigg|_{s=-8} = \frac{25s^2 + 300s + 640}{s(s+4)}\bigg|_{s=-8} =$$

$$= \frac{25(-8)^2 + 300(-8) + 640}{(-8)(-8+4)} = -5$$

$$\rightarrow F(s) = \frac{20}{s} + \frac{10}{s+4} - \frac{5}{s+8} \qquad \rightarrow \boxed{f(t) = 20 + 10e^{-4t} - 5e^{-8t}}$$

Ex. 1

Inverse Transform (5)

Find the inverse Laplace transform of $F(s) = \dfrac{25s^2 + 300s + 640}{s(s+4)(s+8)}$

$$F(s) = \frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+8}$$

$$K_1 = \left.\frac{25s^2 + 300s + 640}{\cancel{s}(s+4)(s+8)}\right|_{s=0} = \left.\frac{25s^2 + 300s + 640}{(s+4)(s+8)}\right|_{s=0} = \frac{640}{4\times 8} = 20$$

$$K_2 = \left.\frac{25s^2 + 300s + 640}{s\,\cancel{(s+4)}\,(s+8)}\right|_{s=-4} = \left.\frac{25s^2 + 300s + 640}{s(s+8)}\right|_{s=-4} = 10$$

$$K_3 = \left.\frac{25s^2 + 300s + 640}{s(s+4)\,\cancel{(s+8)}}\right|_{s=-8} = \left.\frac{25s^2 + 300s + 640}{s(s+4)}\right|_{s=-8} = -5$$

$$\to F(s) = \frac{20}{s} + \frac{10}{s+4} - \frac{5}{s+8} \qquad \to \boxed{f(t) = 20 + 10e^{-4t} - 5e^{-8t}}$$

Inverse Transform (6)

Ex. 2

Find the inverse Laplace transform of $F(s) = \dfrac{100(s+6)}{(s+1)(s+3)}$

Inverse Transform (7)

Complex-conjugate poles: $F(s) = \dfrac{P_1(s)}{Q_1(s)(s+\alpha-j\beta)(s+\alpha+j\beta)} = \dfrac{K_1}{s+\alpha-j\beta} + \dfrac{K_1^*}{s+\alpha+j\beta} + \ldots$

$$(s+\alpha-j\beta)\frac{P(s)}{Q(s)}\bigg|_{s=-\alpha+j\beta} = K_1 = |K_1|\underline{/\theta}$$

$$K_1^* = |K_1|\underline{/-\theta}$$

$$\to F(s) = \frac{|K_1|\underline{/\theta}}{s+\alpha-j\beta} + \frac{|K_1|\underline{/-\theta}}{s+\alpha+j\beta} + \ldots = \frac{|K_1|e^{j\theta}}{s+\alpha-j\beta} + \frac{|K_1|e^{-j\theta}}{s+\alpha+j\beta} + \ldots$$

$$\left.\begin{aligned}\to f(t) = |K_1|e^{j\theta}e^{-(\alpha-j\beta)t} + |K_1|e^{-j\theta}e^{-(\alpha+j\beta)t} + \ldots = |K_1|e^{-\alpha t}\left[e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}\right] + \ldots \\ e^{j\phi} = \cos\phi + j\sin\phi\end{aligned}\right\}$$

$$\to f(t) = |K_1|e^{-\alpha t}\left[\cos(\beta t+\theta) + j\sin(\beta t+\theta) + \cos(-\beta t-\theta) + j\sin(-\beta t-\theta)\right] + \ldots$$

$$= \boxed{2|K_1|e^{-\alpha t}\cos(\beta t+\theta) + \ldots}$$

Ex. 3

Inverse Transform (8)

Find the inverse Laplace transform of $F(s)=\dfrac{4s^2+76s}{(s+2)(s^2+6s+25)}$

$$F(s)=\frac{K_1}{s+3-j4}+\frac{K_2}{s+3+j4}+\frac{K_3}{s+2}$$

$$K_1=(s+\alpha-j\beta)\left.\frac{P(s)}{Q(s)}\right|_{s=-\alpha+j\beta}\;;\;\; f(t)=2\left|K_1\right|e^{-\alpha t}\cos(\beta t+\theta)+\ldots$$

$$K_3=(s+2)\left.\left(\frac{4s^2+76s}{(s+2)(s^2+6s+25)}\right)\right|_{s=-2}=\left.\frac{4s^2+76s}{s^2+6s+25}\right|_{s=-2}=-8$$

$$K_1=(s+3-j4)\left.\frac{4s^2+76s}{(s+2)(s^2+6s+25)}\right|_{s=-3+j4}=6-j8=10\underline{/-53.1^{\circ}}$$

$$\to f(t)=2\times 10e^{-3t}\cos(4t-53.1^{\circ})-8e^{-2t}=\boxed{20e^{-3t}\cos(4t-53.1^{\circ})-8e^{-2t}}$$

Ex. 3

Inverse Transform (9)

Find the inverse Laplace transform of $F(s) = \dfrac{4s^2 + 76s}{(s+2)(s^2+6s+25)}$

$$F(s) = \frac{K_1}{s+3-j4} + \frac{K_2}{s+3+j4} + \frac{K_3}{s+2}$$

$$K_3 = \left.\frac{4s^2+76s}{\cancel{(s+2)}(s^2+6s+25)}\right|_{s=-2} = \left.\frac{4s^2+76s}{s^2+6s+25}\right|_{s=-2} = -8$$

$$K_1 = \left.\frac{4s^2+76s}{(s+2)\cancel{(s+3-j4)}(s+3+j4)}\right|_{s=-3+j4} = 6-j8 = 10\underline{/-53.1^\circ}$$

$$\to f(t) = 2\times 10e^{-3t}\cos(4t-53.1^\circ) - 8e^{-2t} = \boxed{20e^{-3t}\cos(4t-53.1^\circ) - 8e^{-2t}}$$

Ex. 4

Inverse Transform (10)

Find the inverse Laplace transform of $F(s) = \dfrac{5(s+2)}{s(s^2+4s+5)}$

Inverse Transform (11)

$$\text{Multiple poles:}\quad F(s) = \frac{P_1(s)}{Q_1(s)(s+p_1)^n} = \frac{K_{11}}{(s+p_1)} + \frac{K_{12}}{(s+p_1)^2} + \ldots + \frac{K_{1n}}{(s+p)^n} + \ldots$$

$$(s+p_1)^n F(s)\Big|_{s=-p_1} = K_{1n}$$

$$\frac{d}{ds}[(s+p_1)^n F(s)]\Big|_{s=-p_1} = K_{1n-1}$$

$$\frac{d^2}{ds^2}[(s+p_1)^n F(s)]\Big|_{s=-p_1} = (2!)K_{1n-2}$$

$$K_{1j} = \frac{1}{(n-j)!}\frac{d^{n-j}}{ds^{n-j}}[(s+p_1)^n F(s)]\Big|_{s=-p_1}$$

Inverse Transform (12)

Ex. 5

Find the inverse Laplace transform of $F(s) = \dfrac{10s^2 + 34s + 27}{s(s+3)^2}$

$$F(s) = \frac{K_{11}}{s+3} + \frac{K_{12}}{(s+3)^2} + \frac{K_2}{s}; \qquad K_{1j} = \frac{1}{(n-j)!}\frac{d^{n-j}}{ds^{n-j}}\left[(s+p_1)^n F(s)\right]\Big|_{s=-p_1}$$

$$K_{12} = (s+3)^2 F(s)\Big|_{s=-3} = (s+3)^2 \frac{10s^2 + 34s + 27}{s(s+3)^2}\bigg|_{s=-3} = \frac{10s^2 + 34s + 27}{s}\bigg|_{s=-3} = -5$$

$$K_{11} = \frac{d}{ds}\left[(s+3)^2 F(s)\right]\Big|_{s=-3} = \frac{d}{ds}\left(\frac{10s^2 + 34s + 27}{s}\right)\bigg|_{s=-3} =$$

$$= \frac{s(20s+34) - (10s^2 + 34s + 27)}{s^2}\bigg|_{s=-3} = 7$$

$$K_2 = sF(s)\Big|_{s=0} = s\frac{10s^2 + 34s + 27}{s(s+3)^2}\bigg|_{s=0} = 3$$

Ex. 5

Inverse Transform (13)

Find the inverse Laplace transform of $F(s) = \dfrac{10s^2 + 34s + 27}{s(s+3)^2}$

$$F(s) = \frac{K_{11}}{s+3} + \frac{K_{12}}{(s+3)^2} + \frac{K_2}{s}; \qquad K_{1j} = \frac{1}{(n-j)!}\frac{d^{n-j}}{ds^{n-j}}[(s+p_1)^n F(s)]\Big|_{s=-p_1}$$

$$K_{11} = 7; K_{12} = -5; K_2 = 3$$

$$\rightarrow F(s) = \frac{7}{s+3} - \frac{5}{(s+3)^2} + \frac{3}{s}$$

$$\rightarrow \boxed{f(t) = 7e^{-3t} - 5te^{-3t} + 3}$$

Ex. 5

Inverse Transform (14)

Find the inverse Laplace transform of $F(s) = \dfrac{10s^2 + 34s + 27}{s(s+3)^2}$

$$F(s) = \frac{K_{11}}{s+3} + \frac{K_{12}}{(s+3)^2} + \frac{K_2}{s} = \frac{7}{s+3} - \frac{5}{(s+3)^2} + \frac{3}{s} \rightarrow \boxed{f(t) = 7e^{-3t} - 5te^{-3t} + 3}$$

$$K_2 = \left.\frac{10s^2 + 34s + 27}{\cancel{s}(s+3)^2}\right|_{s=0} = \left.\frac{10s^2 + 34s + 27}{(s+3)^2}\right|_{s=0} = 3$$

$$K_{12} = \left.\frac{10s^2 + 34s + 27}{s\,\cancel{(s+3)^2}}\right|_{s=-3} = \left.\frac{10s^2 + 34s + 27}{s}\right|_{s=-3} = -5$$

$$K_{11} = \left.\left[\frac{d}{ds}\left(\frac{10s^2 + 34s + 27}{\cancel{(s+3)^2}\, s}\right)\right]\right|_{s=-3} = \left.\left[\frac{d}{ds}\left(\frac{10s^2 + 34s + 27}{s}\right)\right]\right|_{s=-3}$$

$$= \left.\frac{s(20s + 34) - (10s^2 + 34s + 27)}{s^2}\right|_{s=-3} = 7$$

Inverse Transform (15)

Ex. 6

Find the inverse Laplace transform of $F(s) = \dfrac{5(s+3)}{(s+1)(s+2)^2}$

The Laplace Transform

1. Definition
2. Two Important Singularity Functions
3. Transform Pairs
4. Properties of the Transform
5. Inverse Transform
6. **Initial-Value & Final-Value Theorems**
7. Laplace Circuit Solutions
8. Circuit Element Models
9. Analysis Techniques
10. Convolution Integral
11. Transfer Function

Initial-Value & Final-Value Theorems (1)

$$\text{Initial-value theorem}: \quad \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$\text{Final-value theorem}: \quad \lim_{t\to \infty} f(t) = \lim_{s\to 0} sF(s)$$

Ex. Initial-Value & Final-Value Theorems (2)

Find the initial and final values of $F(s) = \dfrac{5(s+1)}{s(s^2+2s+2)}$

$$f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{5(s+1)}{s^2+2s+2} = 0$$

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{5(s+1)}{s^2+2s+2} = 2.5$$

The Laplace Transform

1. Definition
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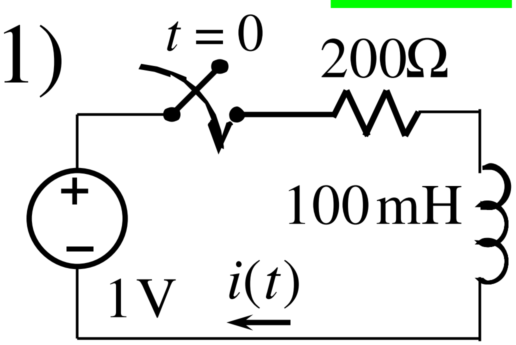
10. Convolution Integral
11. Transfer Function

Laplace Circuit Solutions (1)

Ex.



Find the current $i(t)$?

Method 1

$$v_L + v_R = e \rightarrow L\frac{di}{dt} + Ri = 1$$

$$L\frac{di_n}{dt} + Ri_n = 0 \rightarrow i_n = Ke^{-\alpha t} \rightarrow -LK\alpha e^{-\alpha t} + RKe^{-\alpha t} = 0 \rightarrow -L\alpha + R = 0$$

$$\rightarrow \alpha = \frac{R}{L} = \frac{200}{100\times 10^{-3}} = 2000 \rightarrow i_n = Ke^{-2000t}$$

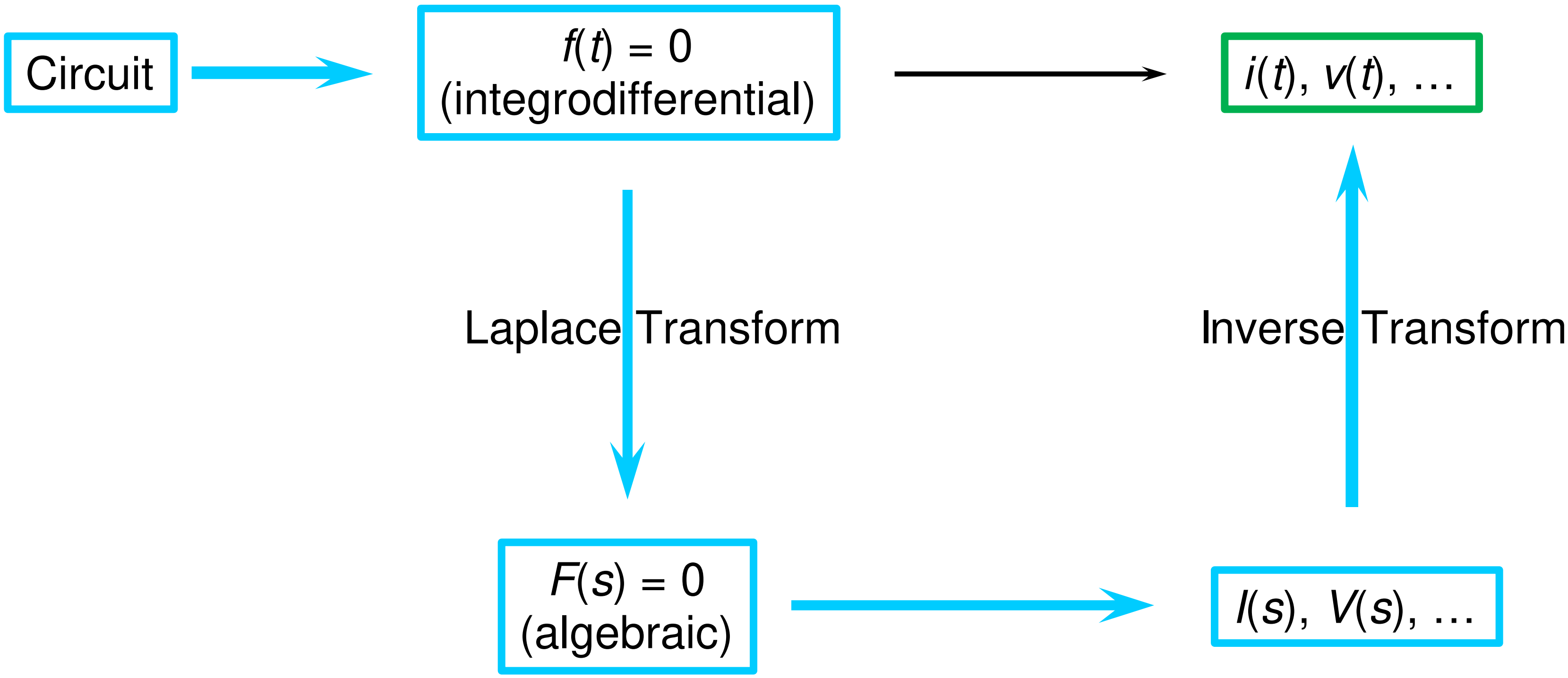
$$i_f = \frac{e}{R} = \frac{1}{200} = 0.005\,\text{A}$$

$$i = i_f + i_n = 0.005 + Ke^{-2000t}$$

$$i(0) = 0.005 + Ke^{-2000\times 0} = 0.005 + K = 0 \rightarrow K = -0.005$$

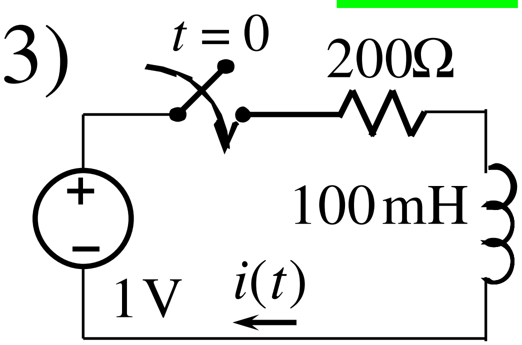
$$\rightarrow \boxed{i(t) = 0.005(1 - e^{-2000t})\,\text{A}}$$

Laplace Circuit Solutions (2)



Laplace Circuit Solutions (3)

Ex.



Find the current $i(t)$?

Method 2

$$v_L + v_R = e \rightarrow 0.1\frac{di}{dt} + 200i = 1$$

$$L\left[0.1\frac{di}{dt} + 200i\right] = L[1] = L\left[0.1\frac{di}{dt}\right] + L[200i]$$

$$L[1] = \frac{1}{s}$$

$$L[200i] = 200I(s)$$

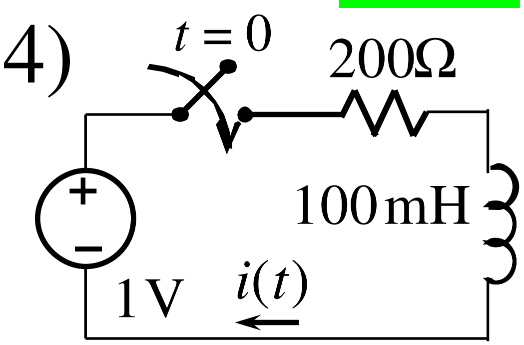
$$\frac{d^n f(t)}{dt^n} \rightarrow s^n F(s) - s^{n-1} f(0) - s^{n-2} f^1(0) \ldots - s^o f^{n-1}(0)$$

$$\rightarrow L\left[0.1\frac{di}{dt}\right] = 0.1[sI(s) - i(0)] = 0.1sI(s)$$

$$\rightarrow 0.1sI(s) + 200I(s) = \frac{1}{s}$$

Laplace Circuit Solutions (4)

Ex.



Find the current $i(t)$?

Method 2

$$v_L + v_R = e \rightarrow 0.1\frac{di}{dt} + 200i = 1$$

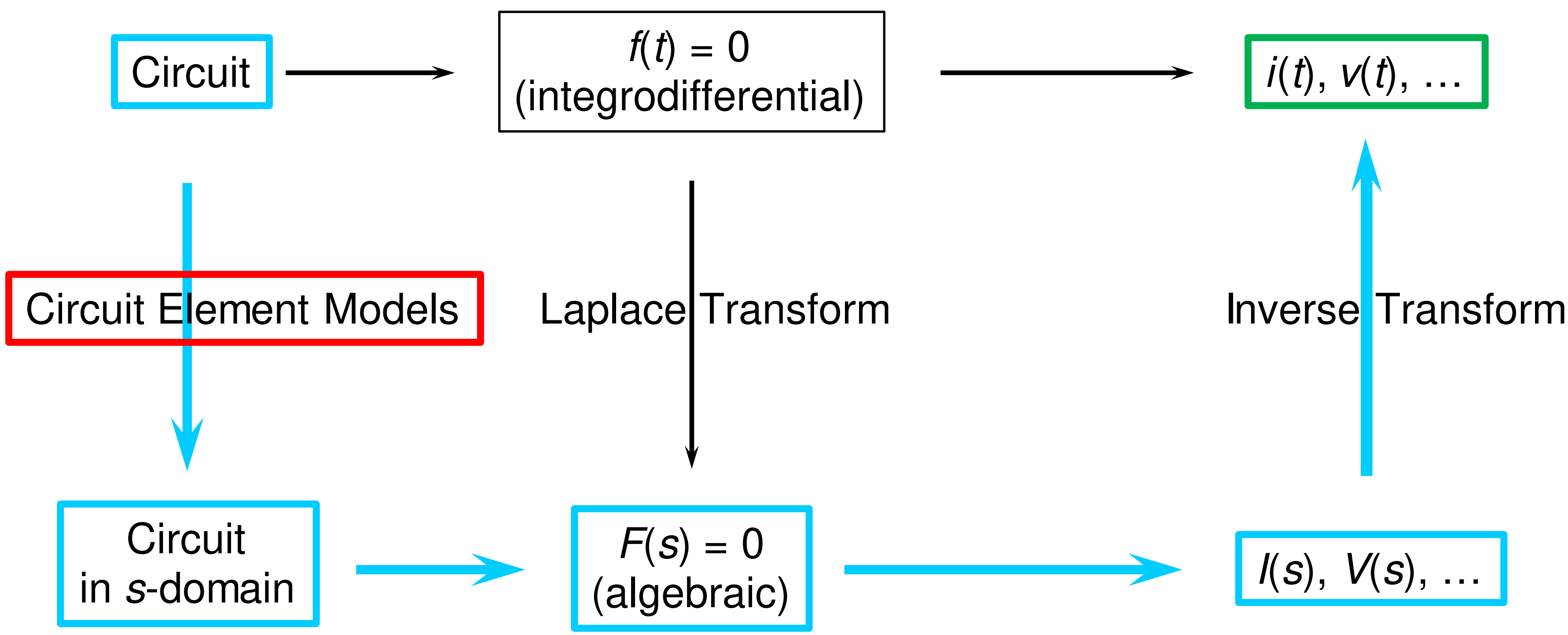
$$\rightarrow 0.1sI(s) + 200I(s) = \frac{1}{s}$$

$$\rightarrow I(s) = \frac{1}{s(0.1s+200)} = \frac{10}{s(s+2000)} = \frac{K_1}{s} + \frac{K_2}{s+2000}$$

$$K_1 = \left.\frac{10}{s+2000}\right|_{s=0} = 0.005$$

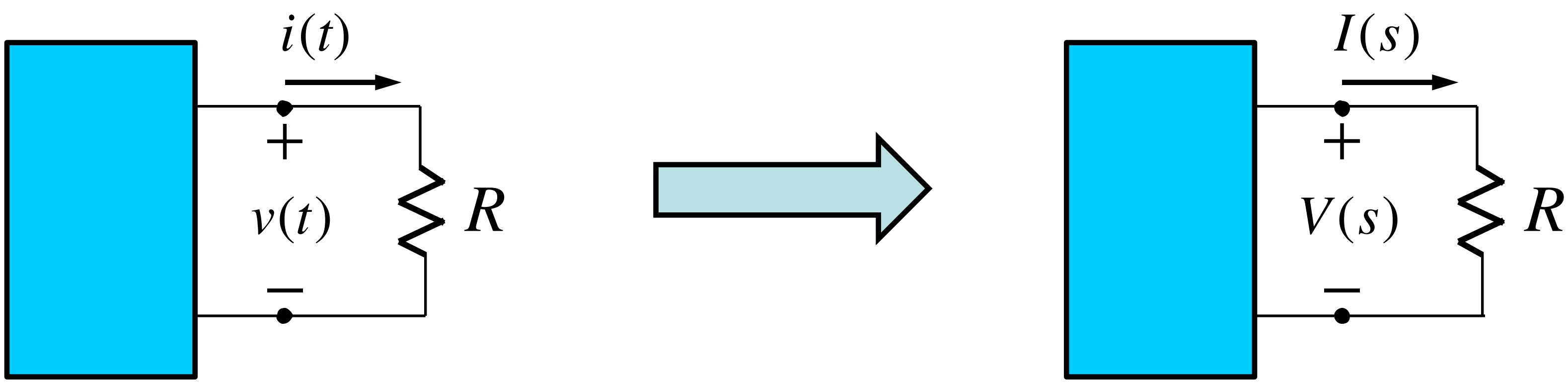
$$K_2 = \left.\frac{10}{s}\right|_{s=-2000} = -0.005$$

$$\rightarrow I(s) = \frac{0.005}{s} - \frac{0.005}{s+2000} \rightarrow \boxed{i(t) = 0.005(1 - e^{-2000t})\ \text{A}}$$


Circuit
f(t) = 0
(integrodifferential)
i(t), v(t), …
Circuit Element Models
Laplace Transform
Inverse Transform
Circuit
in s-domain
F(s) = 0
(algebraic)
I(s), V(s), …

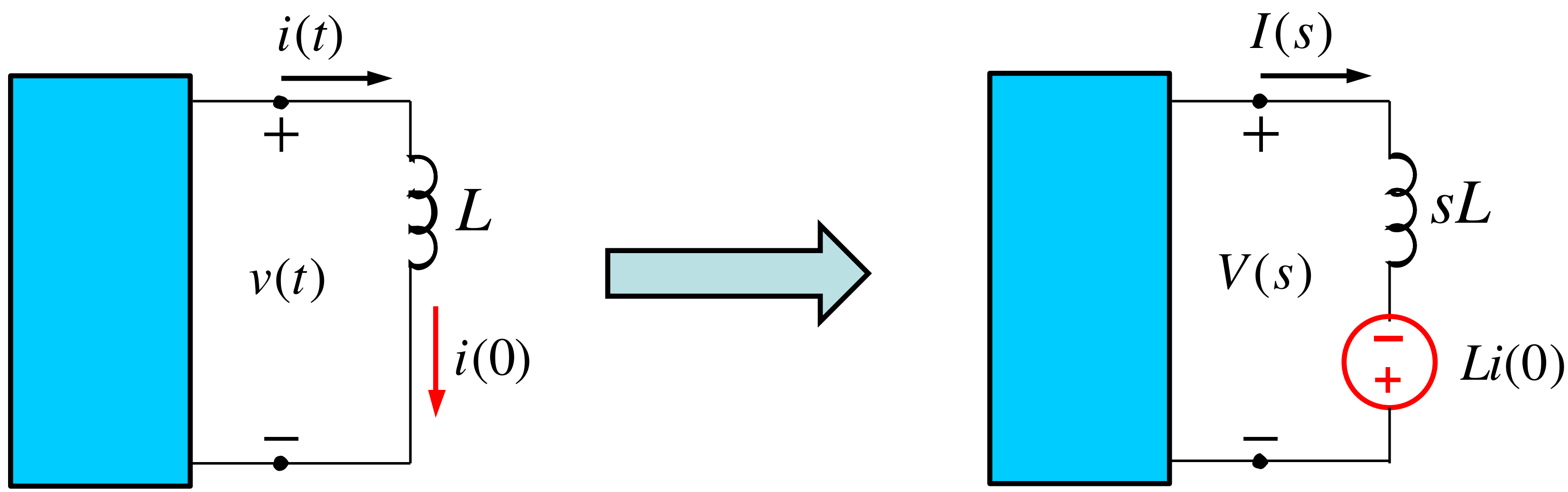
The Laplace Transform

Circuit Element Models (1)



$$\left.\begin{array}{c} v = Ri \\ Af(t) \to AF(s) \end{array}\right\} \to V(s) = RI(s)$$

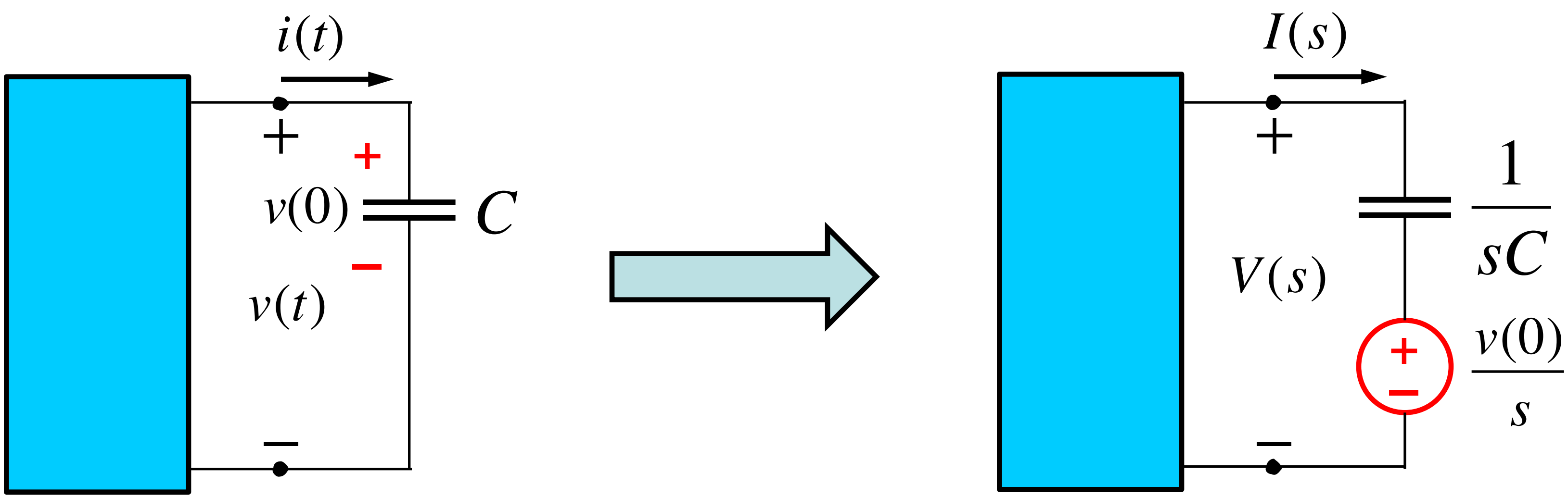
Circuit Element Models (2)



$$\left.\begin{array}{r} v = L\dfrac{di}{dt} \\ A\dfrac{df(t)}{dt} \to A[sF(s) - f(0)] \end{array}\right\} \to V(s) = L[sI(s) - i(0)]$$

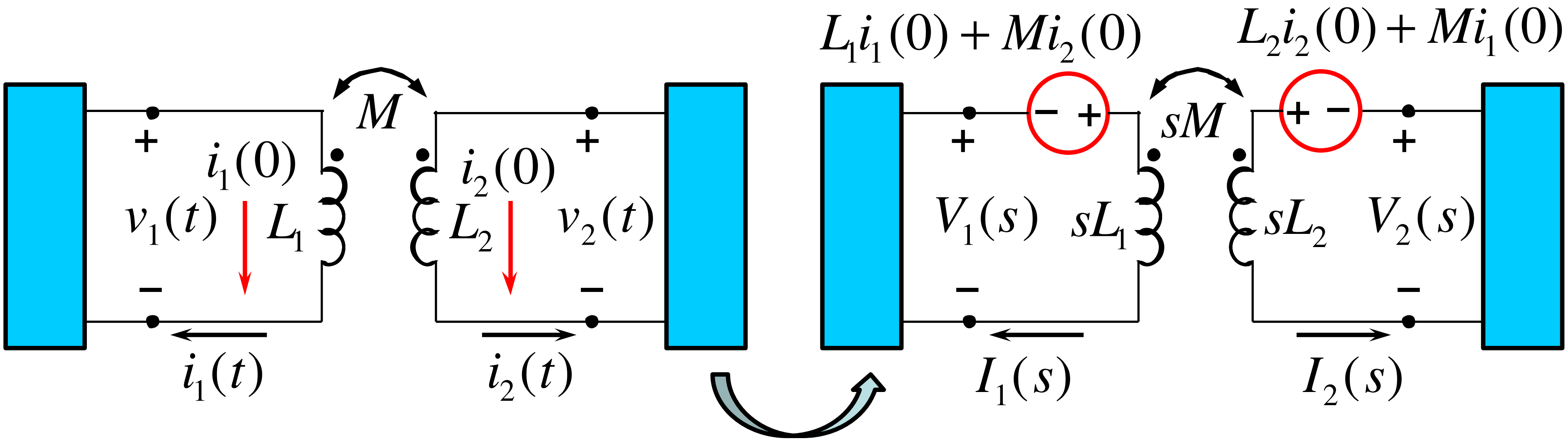
$$= sLI(s) - Li(0)$$

Circuit Element Models (3)



$$\left.\begin{aligned} v &= \frac{1}{C}\int_0^t i(x)dx + v(0) \\ \int_0^t f(\lambda)d\lambda &\to \frac{F(s)}{s} \\ v(0) &\to \frac{v(0)}{s} \end{aligned}\right\} \to V(s) = \frac{1}{sC}I(s) + \frac{v(0)}{s}$$

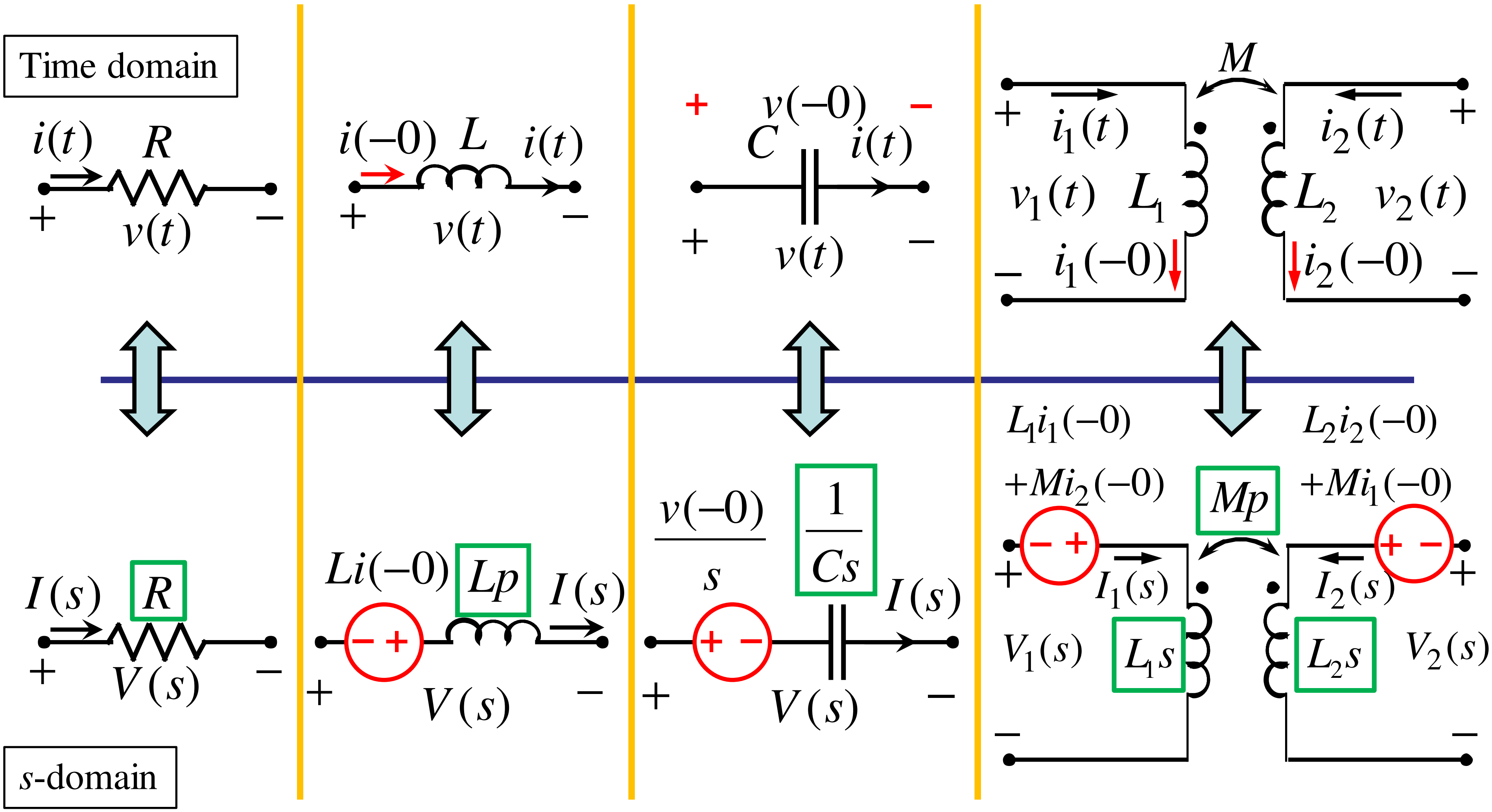
Circuit Element Models (4)



$$v_1(t) = L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt} \rightarrow V_1(s) = sL_1 I_1(s) - L_1 i_1(0) + sMI_2(s) - Mi_2(0)$$

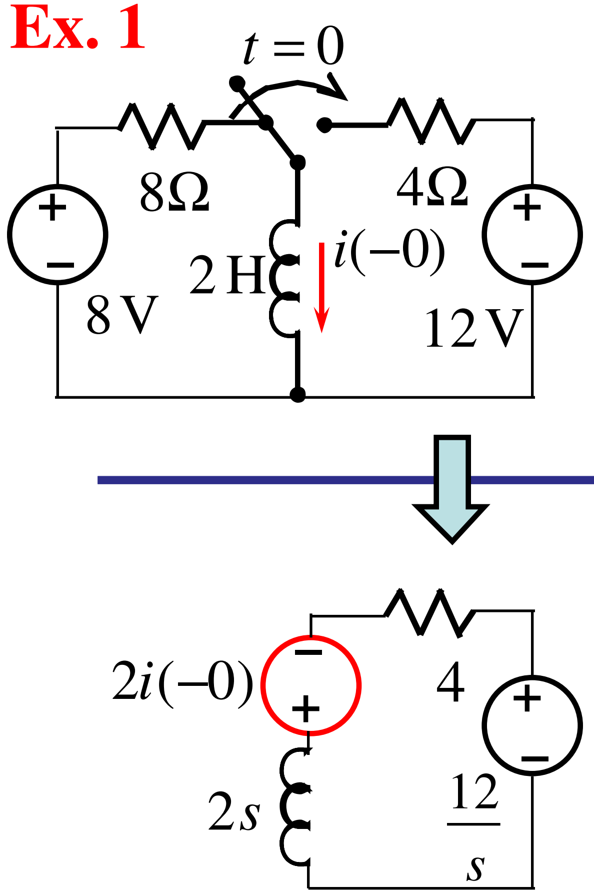
$$v_2(t) = L_2 \frac{di_2(t)}{dt} + M \frac{di_1(t)}{dt} \rightarrow V_2(s) = sL_2 I_2(s) - L_2 i_2(0) + sMI_1(s) - Mi_1(0)$$

Circuit Element Models (5)

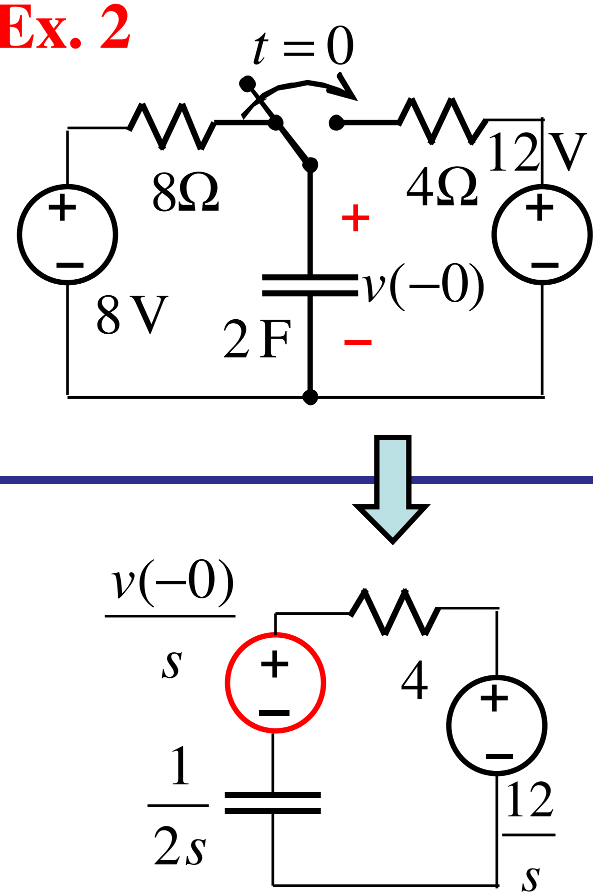


Circuit Element Models (6)

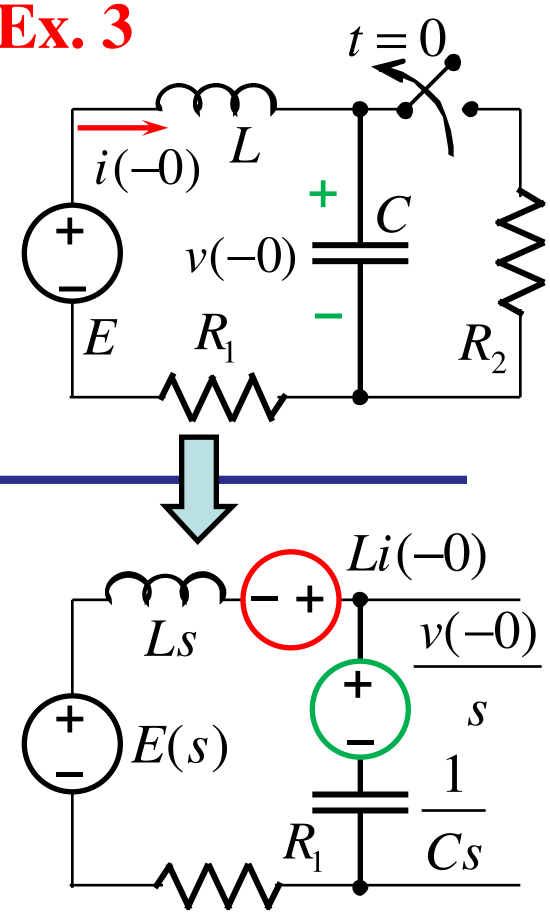
Ex. 1



Ex. 2



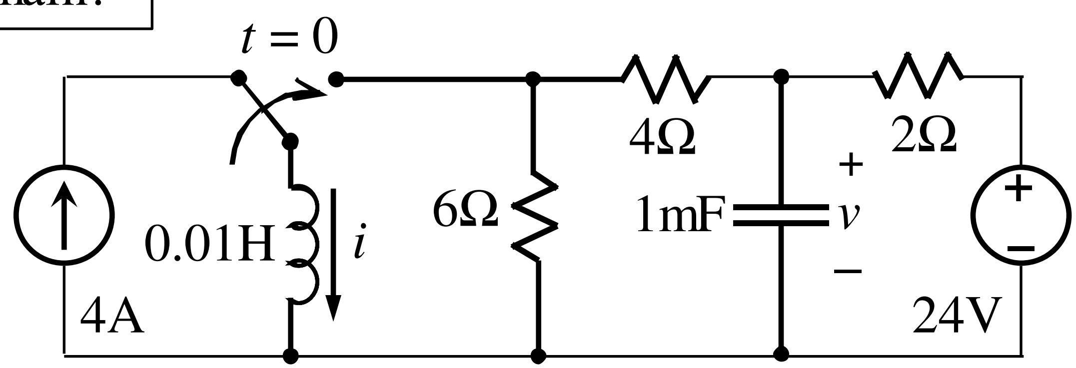
Ex. 3



Ex. 4

Circuit Element Models (7)

Transfer the circuit into Laplace domain?



The Laplace Transform

1. Definition
2. Two Important Singularity Functions
3. Transform Pairs
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8. Circuit Element Models
9. **Analysis Techniques**
10. Convolution Integral
11. Transfer Function

Analysis Techniques (1)

$$\text{KVL/KCL}: \quad x_1(t) + x_2(t) + \ldots + x_n(t) = 0$$

$$\downarrow$$

$$\text{KVL/KCL}: \quad X_1(s) + X_2(s) + \ldots + X_n(s) = 0$$

Circuit

Circuit Element Models

Circuit
in *s*-domain

DC circuit analysis techniques (KVL, KCL, nodal analysis, mesh analysis, source transformation, superposition, Thevenin/Norton equivalent, …)

I(*s*), *V*(*s*), …

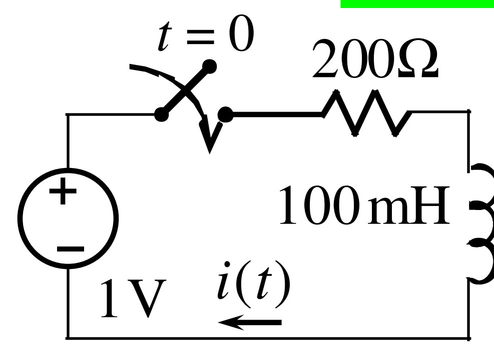
Inverse Transform

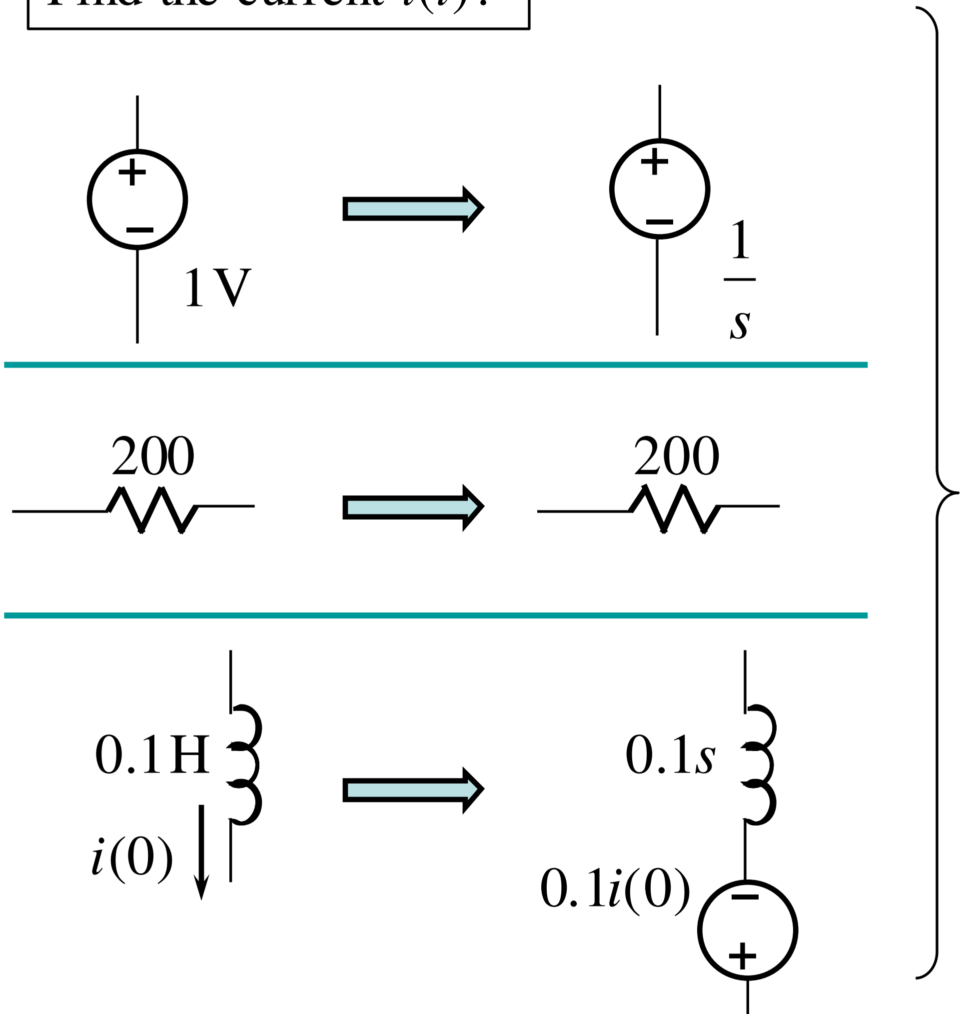
i(*t*), *v*(*t*), …

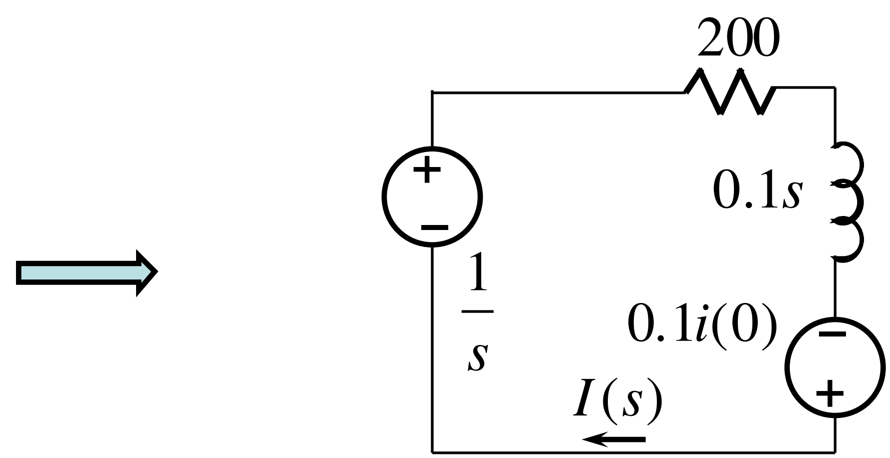
Ex. 1

Analysis Techniques (2)

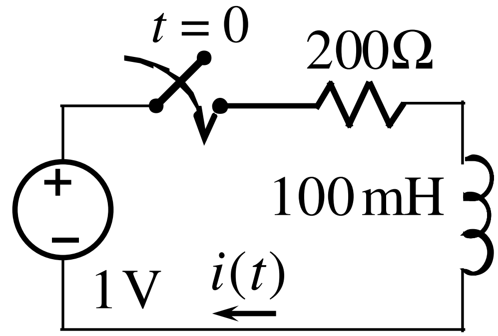
Find the current $i(t)$?







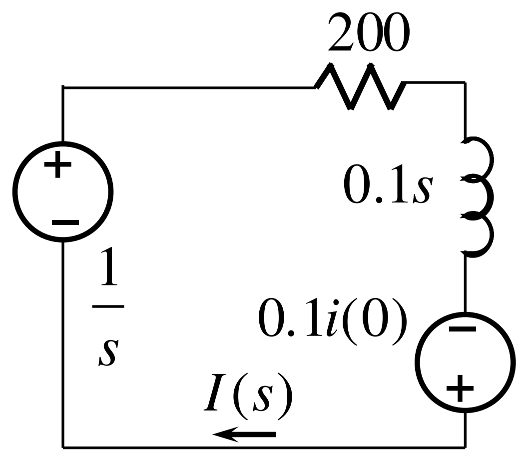
Ex. 1 Analysis Techniques (3)



Find the current $i(t)$?

$i(0) = 0$

$$200I(s) + 0.1sI(s) - 0.1i(0) = \frac{1}{s} = 200I(s) + 0.1sI(s)$$



$$\rightarrow I(s) = \frac{1}{s(0.1s + 200)} = \frac{10}{s(s+2000)} = \frac{K_1}{s} + \frac{K_2}{s+2000}$$

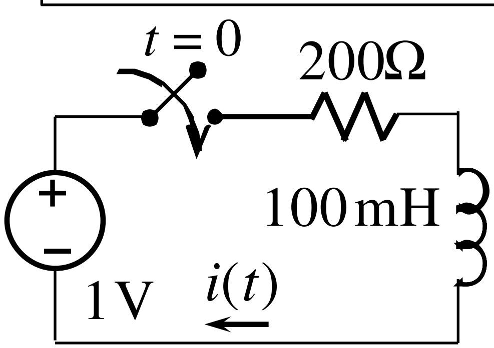
$$K_1 = \left.\frac{10}{s+2000}\right|_{s=0} = 0.005$$

$$K_2 = \left.\frac{10}{s}\right|_{s=-2000} = -0.005$$

$$\rightarrow I(s) = \frac{0.005}{s} - \frac{0.005}{s+2000} \rightarrow \boxed{i(t) = 0.005(1 - e^{-2000t})\text{ A}}$$

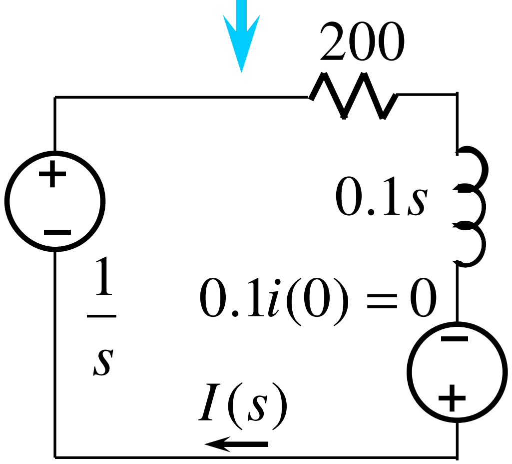
Ex. 1 Analysis Techniques (4)

Find the current $i(t)$?



$i(0)=0$

Circuit Element Models



1.	Solve for initial capacitor voltages & inductor currents
2.	Draw an s-domain circuit
3.	Use one of DC circuit analysis techniques to solve for voltages or/and currents in s-domain
4.	Find the inverse Laplace transform to convert them back to the time domain

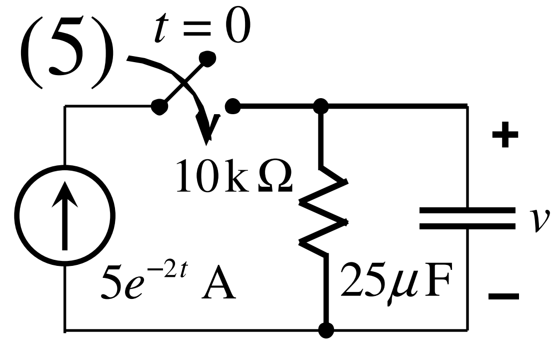
$$200I(s)+0.1sI(s)=\frac{1}{s}$$

$$I(s)=\frac{10}{s(s+2000)}$$

Inverse Transform

$$i(t)=0.005(1-e^{-2000t})\text{ A}$$

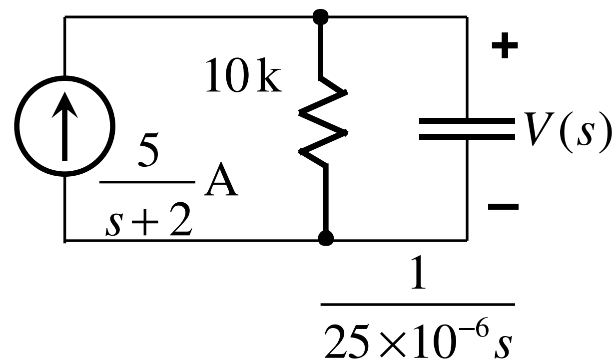
Ex. 2 Analysis Techniques (5)



Find the voltage $v(t)$?

$v(0) = 0$

$$V(s) = \left[R // \frac{1}{sC}\right] J(s) = \frac{10^4 \dfrac{1}{25\times10^{-6}s}}{10^4 + \dfrac{1}{25\times10^{-6}s}} \times \frac{5}{s+2}$$



$$= \frac{4\times10^4}{(s+2)(s+4)} = \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

$$K_1 = \left.\frac{4\times10^4}{s+4}\right|_{s=-2} = 2\times10^4$$

$$K_2 = \left.\frac{4\times10^4}{s+2}\right|_{s=-4} = -2\times10^4$$

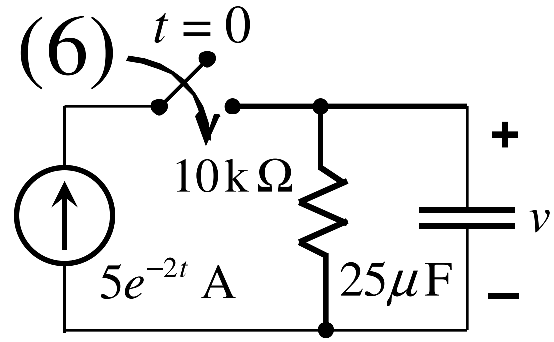
$$\rightarrow \boxed{v(t) = 2\times10^4(e^{-2t} - e^{-4t})\,\mathrm{V}}$$

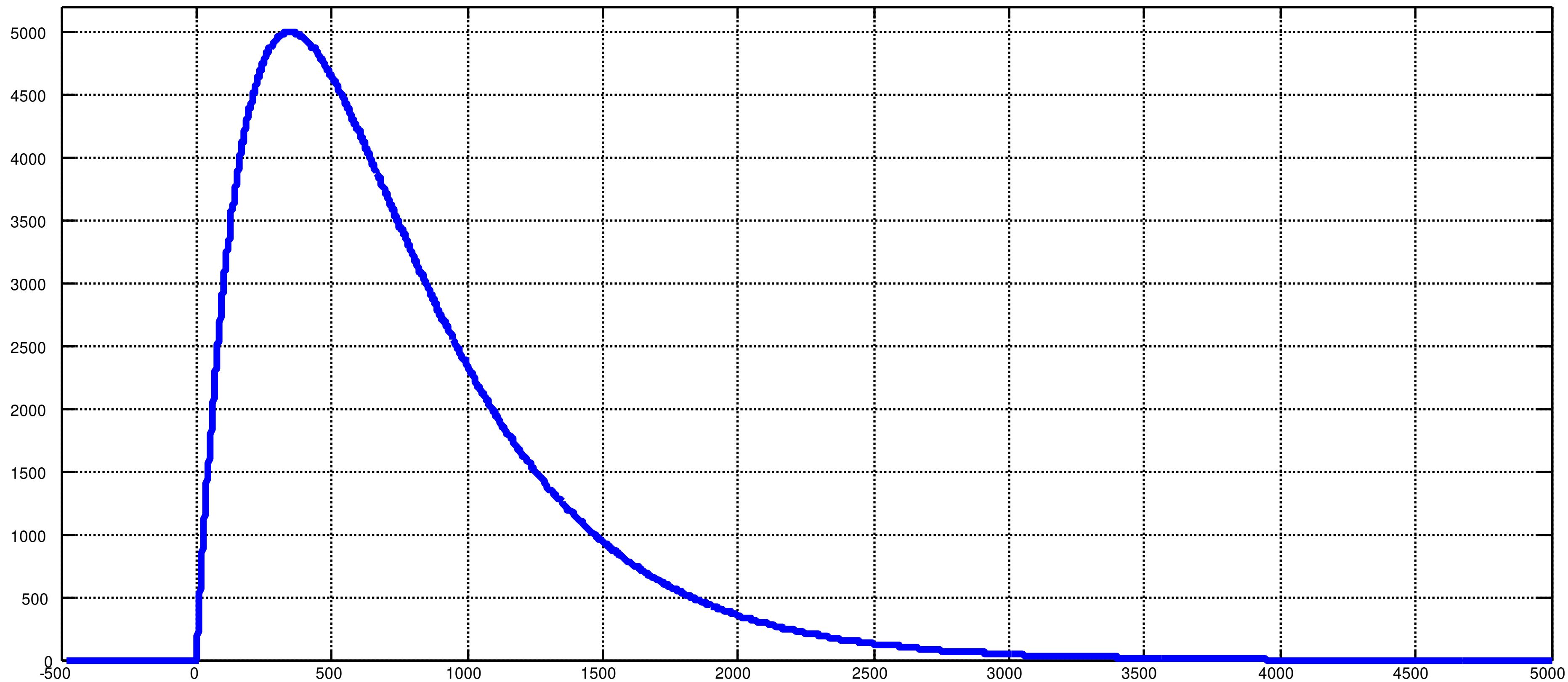
1. ✓ Solve for initial capacitor voltages & inductor currents
2. ✓ Draw an s-domain circuit
3. ✓ Use one of DC circuit analysis techniques to solve for voltages or/and currents in s-domain
4. ✓ Find the inverse Laplace transform to convert them back to the time domain

Ex. 2 Analysis Techniques (6)

Find the voltage $v(t)$?

$$v(t) = 2\times 10^4 (e^{-2t} - e^{-4t})\,\mathrm{V}$$



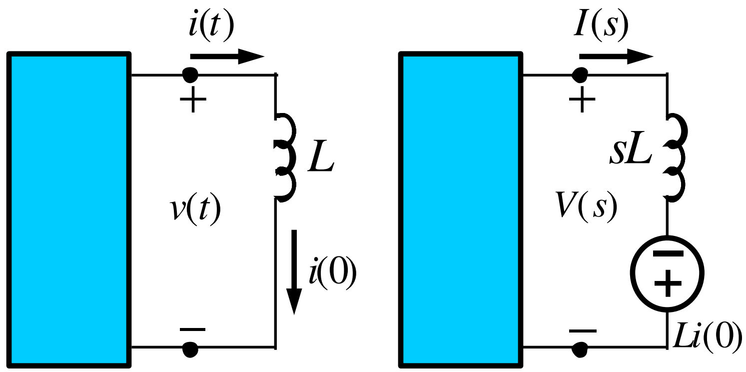


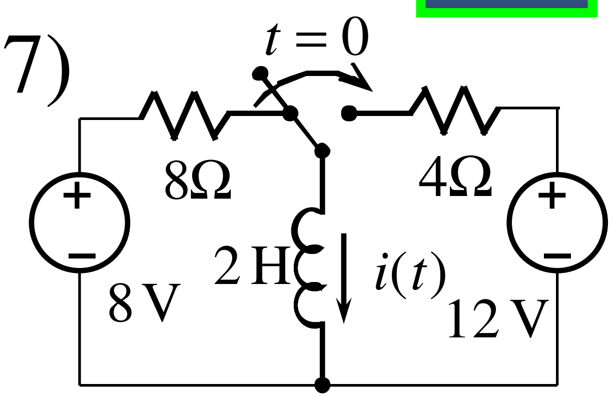
Ex. 3

Analysis Techniques (7)

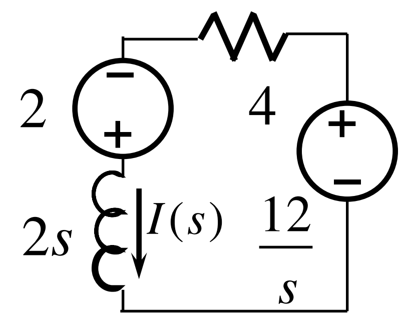
Find the current $i(t)$?

$$i(0) = \frac{8}{8} = 1\text{ A}$$





$$I(s) = \frac{2 + \frac{12}{s}}{2s + 4}$$



$$= \frac{s+6}{s(s+2)} = \frac{K_1}{s} + \frac{K_2}{s+2}$$

$$K_1 = \left.\frac{s+6}{s+2}\right|_{s=0} = 3$$

$$K_2 = \left.\frac{s+6}{s}\right|_{s=-2} = -2$$

$$\rightarrow \boxed{i(t) = 3 - 2e^{-2t}\text{ A}}$$

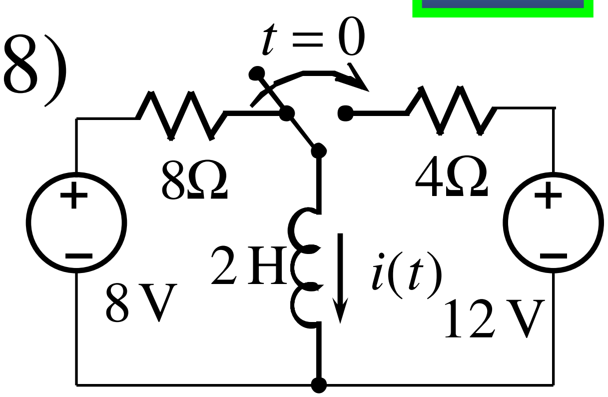
1. ✓ Solve for initial capacitor voltages & inductor currents
2. ✓ Draw an s-domain circuit
3. ✓ Use one of DC circuit analysis techniques to solve for voltages or/and currents in s-domain
4. ✓ Find the inverse Laplace transform to convert them back to the time domain

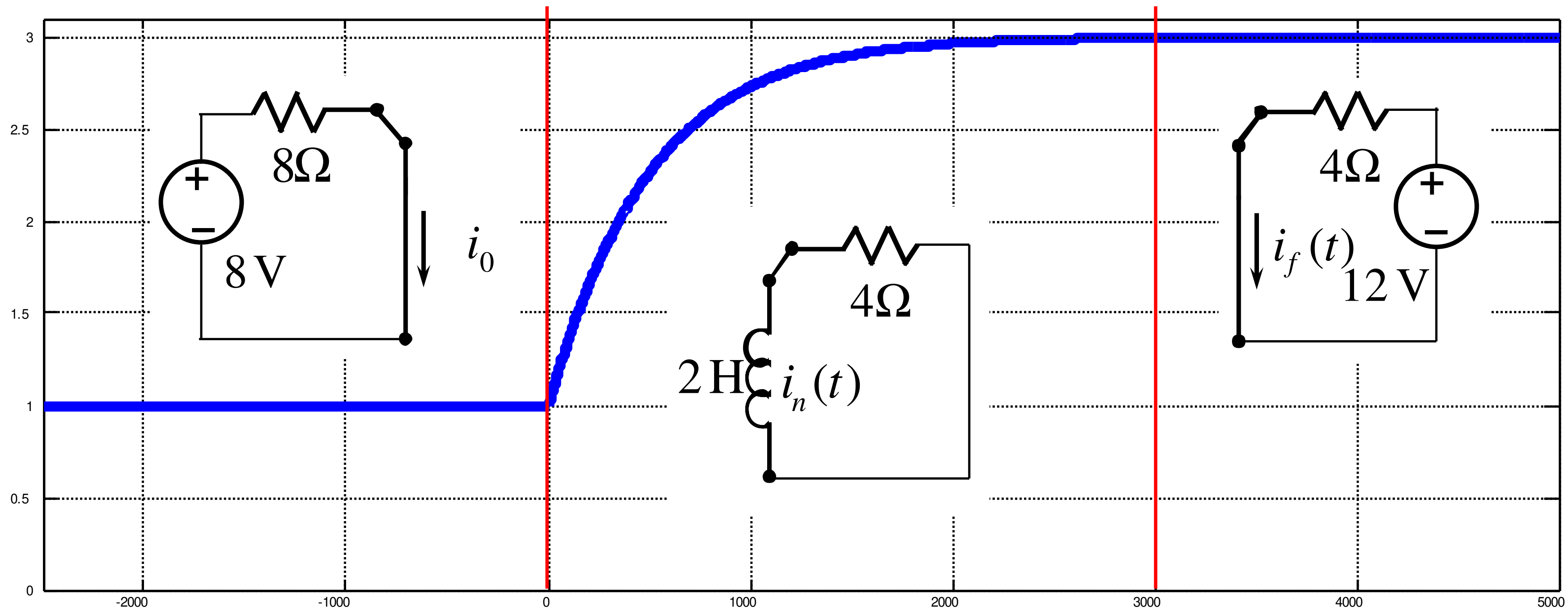
Ex. 3

Analysis Techniques (8)

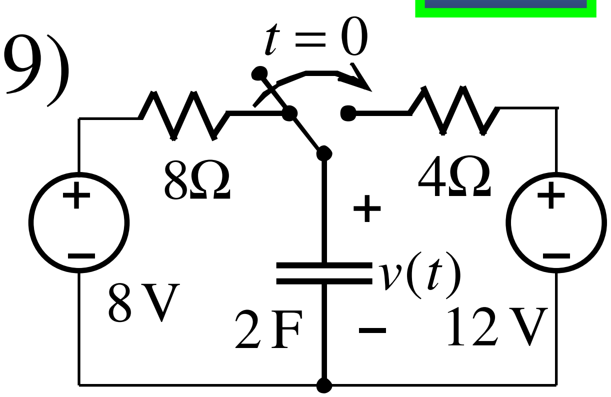
Find the current $i(t)$?

$$i(t) = 3 - 2e^{-2t}\ \text{A}$$



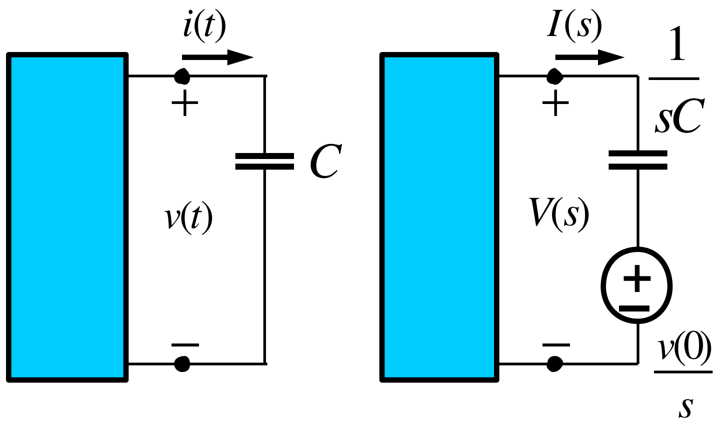


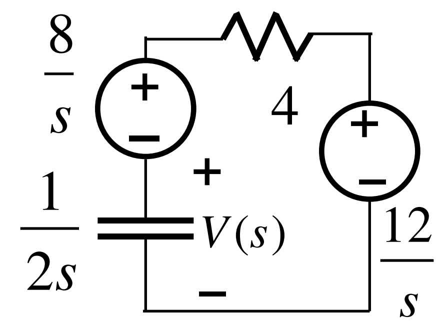
Ex. 4 Analysis Techniques (9)



Find the voltage $v(t)$?

$v(0) = 8\text{ V}$





$$V(s) = \frac{1}{2s} \times \frac{\frac{12}{s} - \frac{8}{s}}{4 + \frac{1}{2s}} + \frac{8}{s}$$

$$= \frac{8s + 1.5}{s(s + 0.125)} = \frac{K_1}{s} + \frac{K_2}{s + 0.125}$$

$$K_1 = \left.\frac{8s + 1.5}{s + 0.125}\right|_{s=0} = 12$$

$$K_2 = \left.\frac{0.5}{s}\right|_{s=-0.125} = -4$$

$$\rightarrow \boxed{v(t) = 12 - 4e^{-0.125t}\ \text{V}}$$

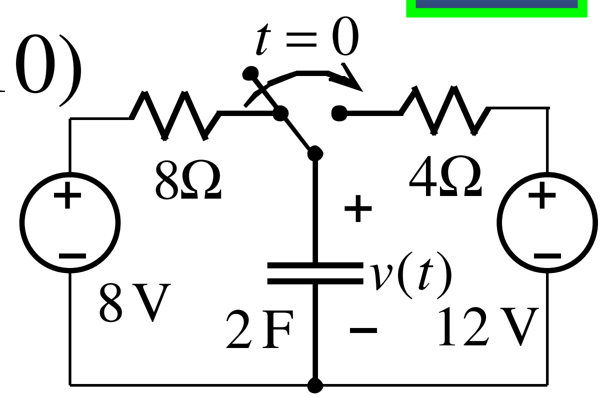
1. ✓ Solve for initial capacitor voltages & inductor currents
2. ✓ Draw an s-domain circuit
3. ✓ Use one of DC circuit analysis techniques to solve for voltages or/and currents in s-domain
4. ✓ Find the inverse Laplace transform to convert them back to the time domain

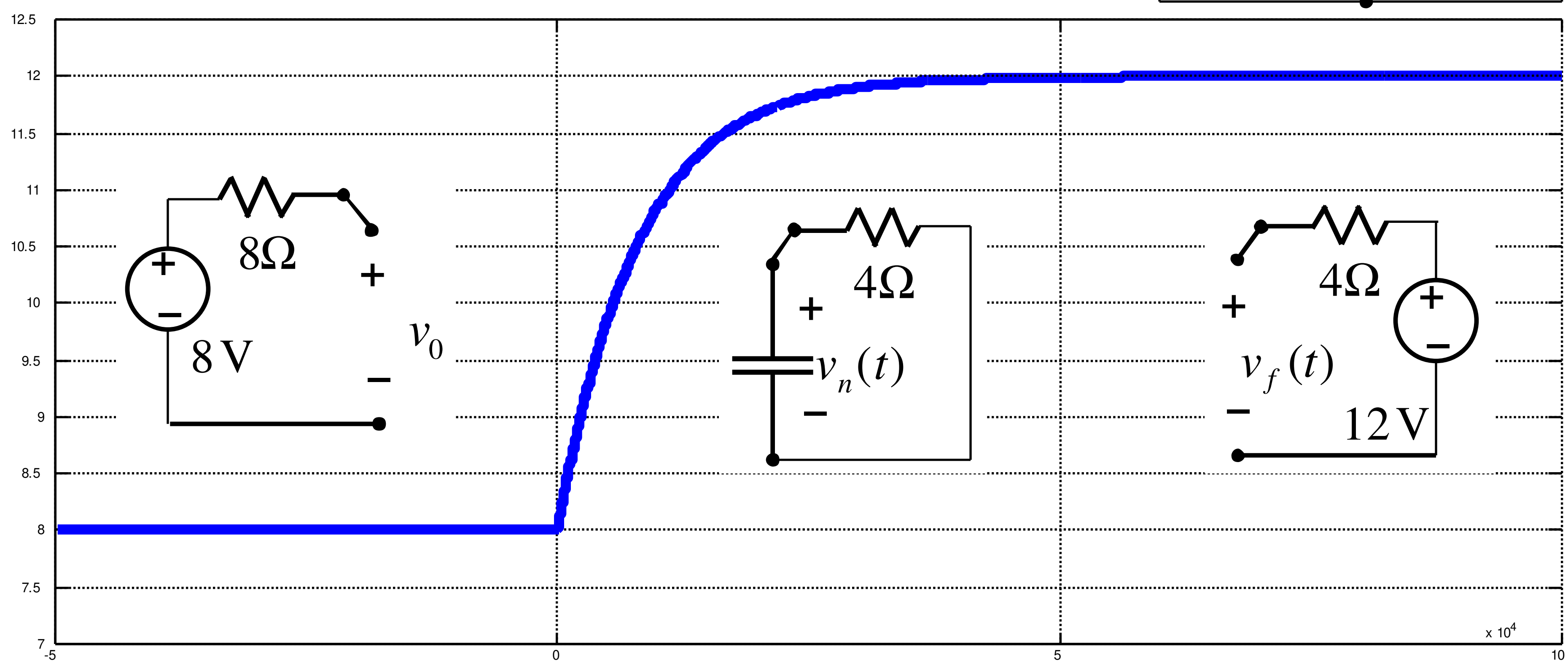
Ex. 4

Analysis Techniques (10)

Find the voltage $v(t)$?

$$v(t) = 12 - 4e^{-0.125t} \text{ V}$$

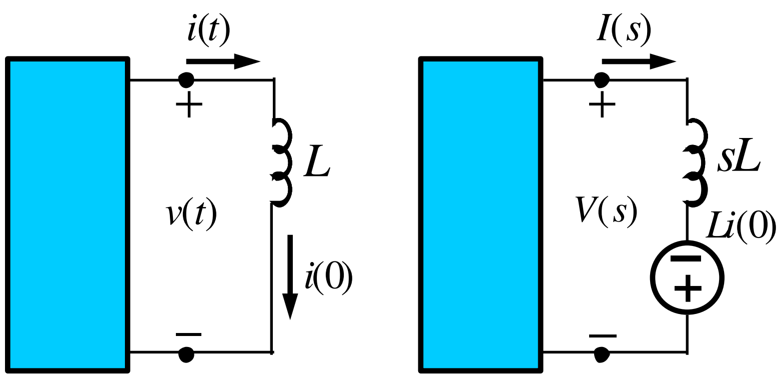


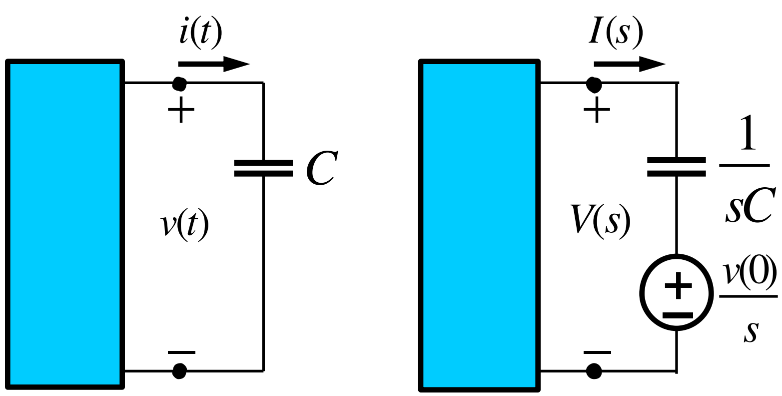


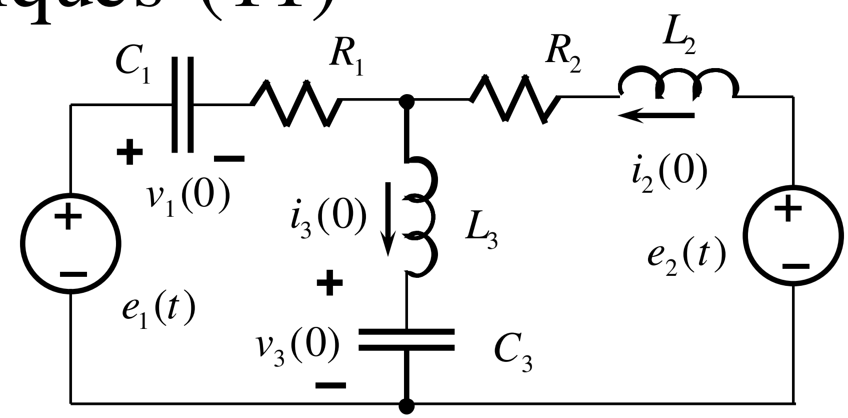
Ex. 5

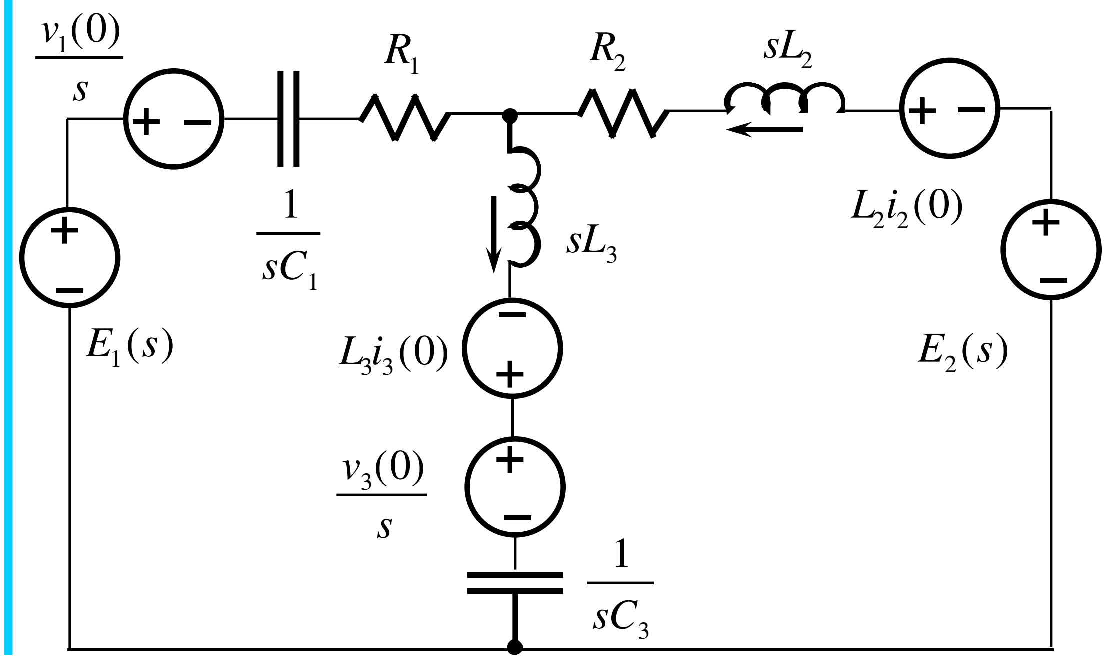
Analysis Techniques (11)

Write the mesh equations in the s-domain?





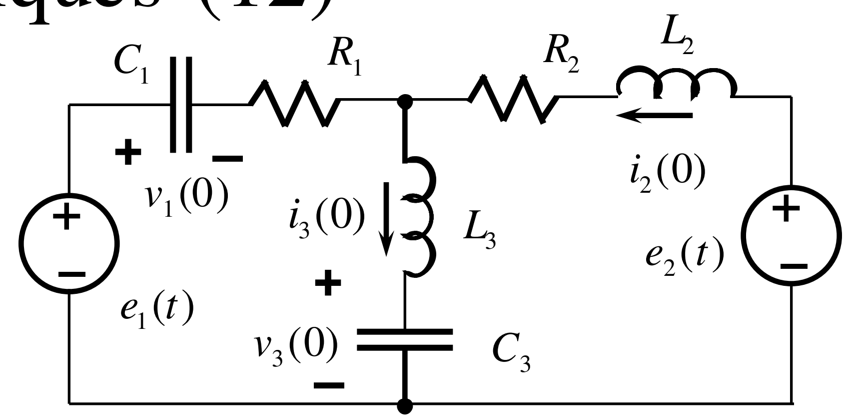




Ex. 5

Analysis Techniques (12)

Write the mesh equations in the *s*-domain?



$$A:\ \left(R_1+\frac{1}{sC_1}\right)I_A(s)+$$

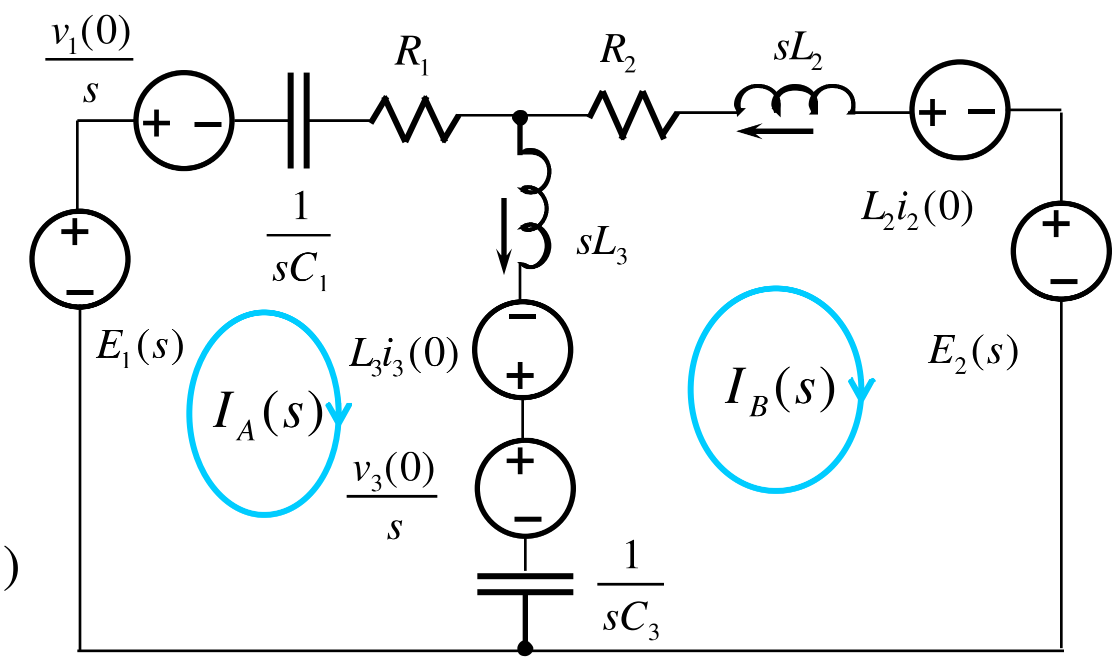
$$+\left(sL_3+\frac{1}{sC_3}\right)[I_A(s)-I_B(s)]=$$

$$=E_1(s)-\frac{v_1(0)}{s}+L_3i_3(0)-\frac{v_3(0)}{s}$$

$$B:\ (R_2+sL_2)\,I_B(s)+$$

$$+\left(sL_3+\frac{1}{sC_3}\right)[I_B(s)-I_A(s)]=$$

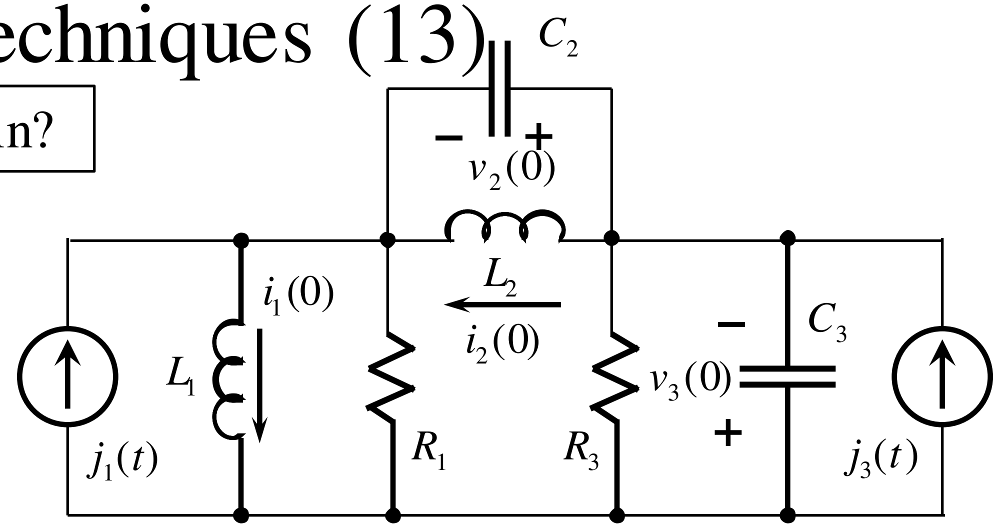
$$=\frac{v_3(0)}{s}-L_3i_3(0)-L_2i_2(0)-E_2(s)$$



Ex. 6

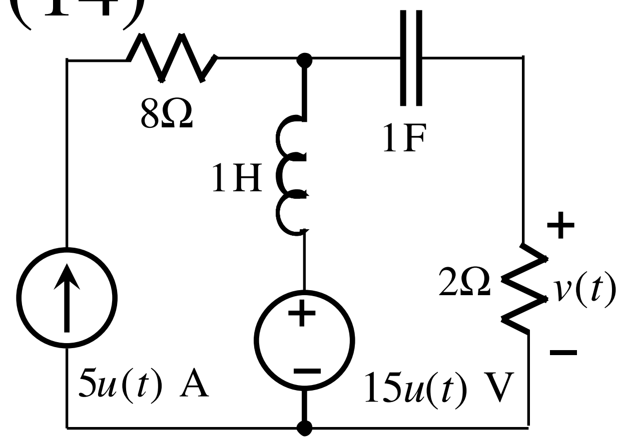
Analysis Techniques (13)

Write the node equations in the *s*-domain?



Ex. 7

Analysis Techniques (14)



Solve for $v(t)$?

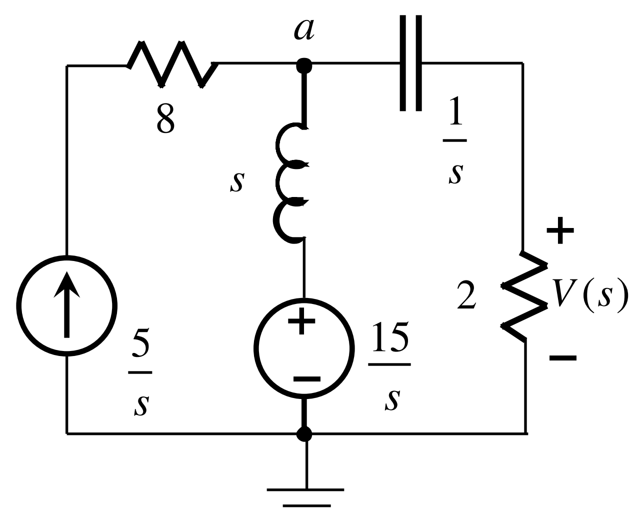
$i_L(0) = 0;\ v_C(0) = 0;$

$$a: \quad \frac{5}{s} + \frac{\frac{15}{s} - V_a(s)}{s} - \frac{V_a(s)}{2 + \frac{1}{s}} = 0$$

$$\to V_a(s) = \frac{10s^2 + 35s + 15}{s^3 + 2s^2 + s}$$

$$\to V(s) = 2\frac{V_a(s)}{2 + \frac{1}{s}} = 2\frac{10s^2 + 35s + 15}{s^3 + 2s^2 + s} \times \frac{s}{2s + 1}$$

$$= \frac{10(s+3)}{(s+1)^2} = \frac{K_{11}}{s+1} + \frac{K_{12}}{(s+1)^2}$$

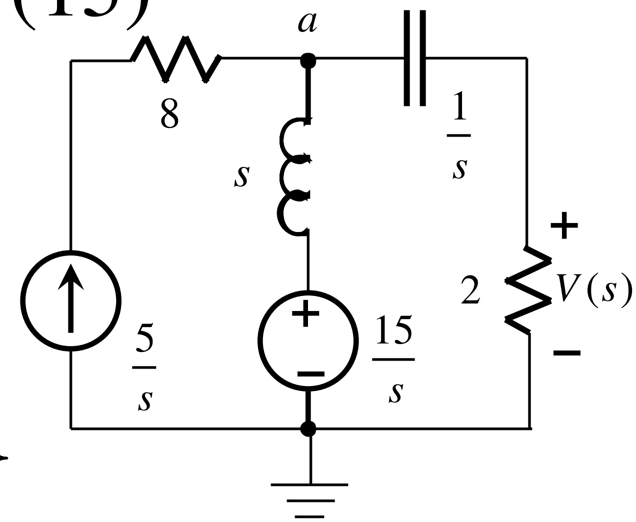


Ex. 7

Analysis Techniques (15)

Solve for $v(t)$?

Method 1



$$V(s) = \frac{10(s+3)}{(s+1)^2} = \frac{K_{11}}{s+1} + \frac{K_{12}}{(s+1)^2}$$

$$K_{12} = (s+1)^2 \left.\frac{10(s+3)}{(s+1)^2}\right|_{s=-1} = 10(s+3)\big|_{s=-1} = 20$$

$$K_{11} = \frac{d}{ds}\left[(s+1)^2 \frac{10(s+3)}{(s+1)^2}\right]\bigg|_{s=-1} = \frac{d}{ds}10(s+3)\bigg|_{s=-1} = 10$$

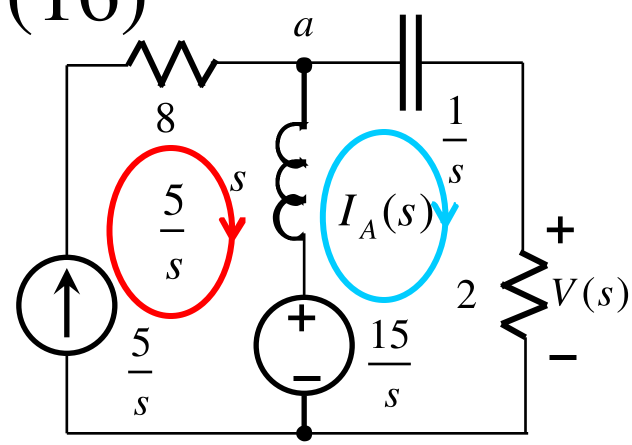
$$\rightarrow V(s) = \frac{10}{s+1} + \frac{20}{(s+1)^2} \rightarrow v(t) = \boxed{10(2t+1)e^{-t}\ \text{V}}$$

Ex. 7

Analysis Techniques (16)

Solve for $v(t)$?

Method 2



$$s\left[I_A(s)-\frac{5}{s}\right]+\left(2+\frac{1}{s}\right)I_A(s)=\frac{15}{s}$$

$$\rightarrow I_A(s)=5\frac{s+3}{(s+1)^2}$$

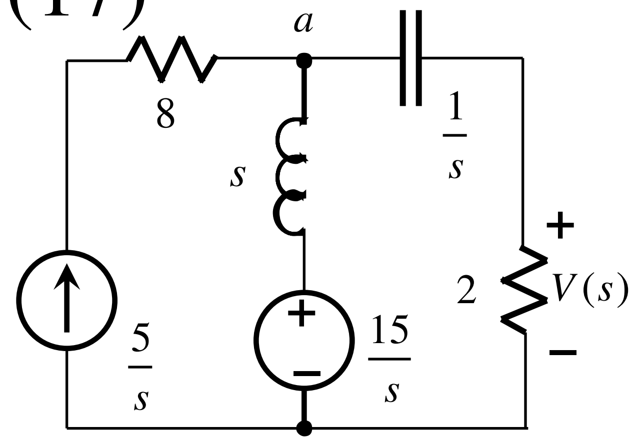
$$\rightarrow \boxed{V(s)=10\frac{s+3}{(s+1)^2}}$$

Ex. 7

Analysis Techniques (17)

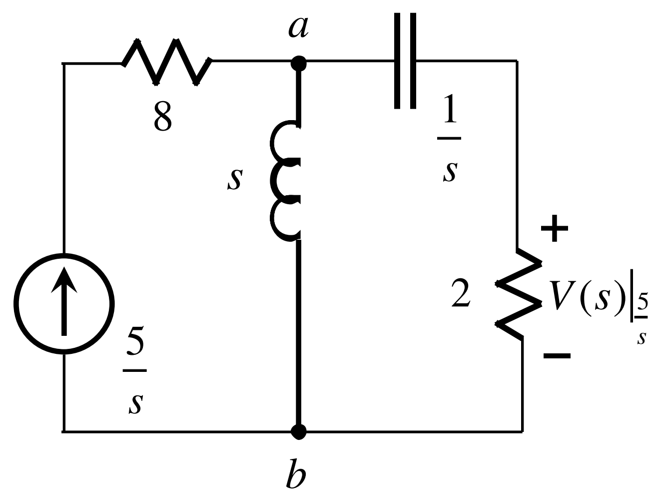
Solve for $v(t)$?

Method 3



$$V_{ab}(s)\Big|_{\frac{5}{s}} = \frac{s\left(2+\frac{1}{s}\right)}{s+2+\frac{1}{s}} \times \frac{5}{s} = \frac{10(s+0.5)}{(s+1)^2}$$

$$\rightarrow V(s)\Big|_{\frac{5}{s}} = \frac{10(s+0.5)}{(s+1)^2} \times \frac{2}{2+\frac{1}{s}} = \frac{10s}{(s+1)^2}$$

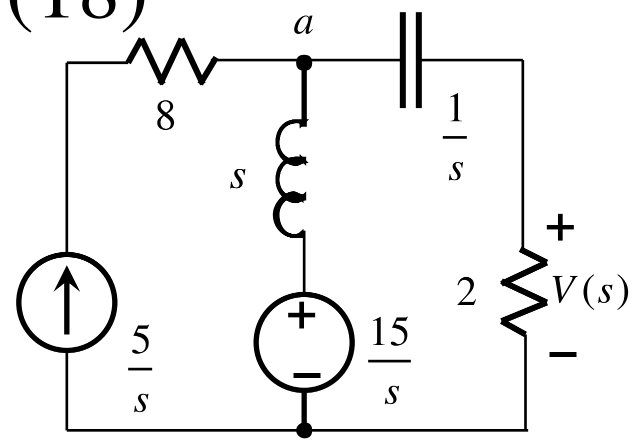


Ex. 7

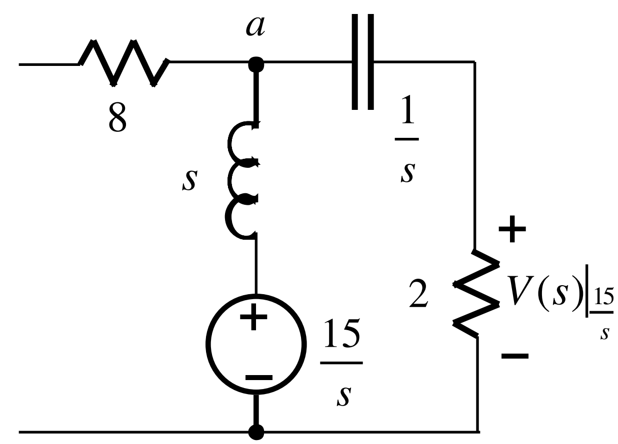
Analysis Techniques (18)

Solve for $v(t)$?

Method 3



$$I_C(s)\Big|_{\frac{15}{s}} = \frac{\frac{15}{s}}{s+\frac{1}{s}+2} = \frac{15}{(s+1)^2}$$



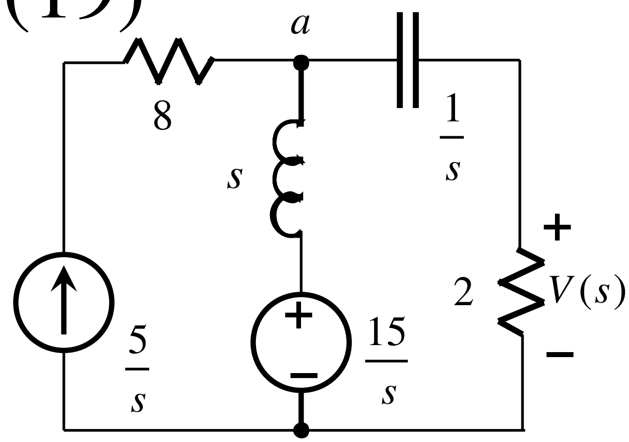
$$\rightarrow V(s)\Big|_{\frac{15}{s}} = 2\frac{15}{(s+1)^2}$$

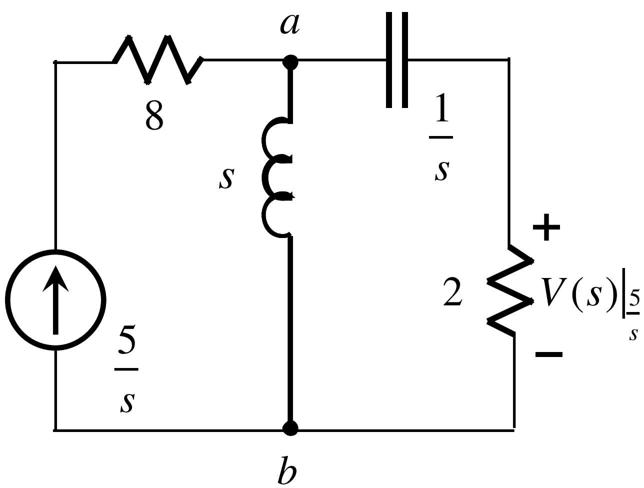
Ex. 7

Analysis Techniques (19)

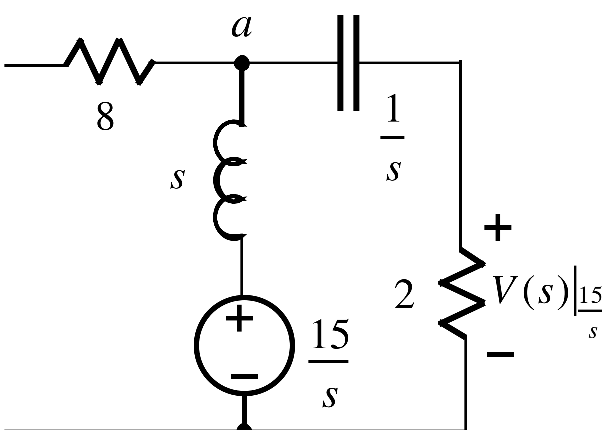
Solve for $v(t)$?

Method 3





$$\rightarrow V(s)\big|_{\frac{5}{s}} = \frac{10s}{(s+1)^2}$$



$$\rightarrow V(s)\big|_{\frac{15}{s}} = 2\frac{15}{(s+1)^2}$$

$$\rightarrow V(s) = V(s)\big|_{\frac{15}{s}} + V(s)\big|_{\frac{15}{s}}$$

$$= \frac{10s}{(s+1)^2} + \frac{30}{(s+1)^2}$$

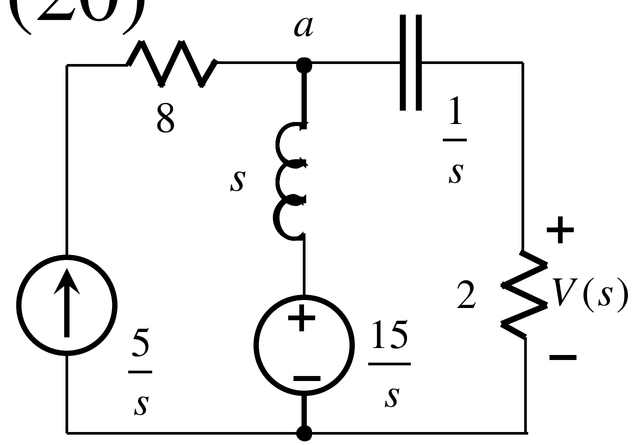
$$= \boxed{\frac{10(s+3)}{(s+1)^2}}$$

Ex. 7

Analysis Techniques (20)

Solve for $v(t)$?

Method 4

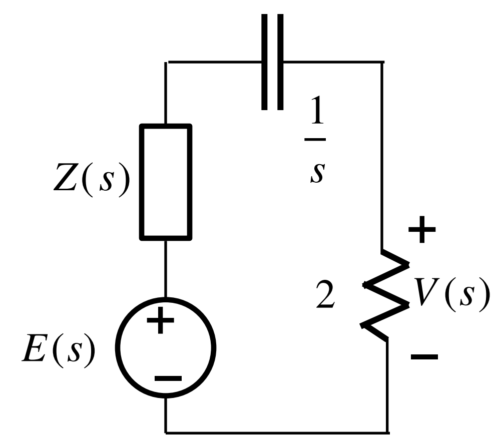


$$Z(s) = s$$

$$E(s) = \left(\frac{5}{s} + \frac{15}{s^2}\right)s = 5\frac{s+3}{s}$$

$$I(s) = \frac{5\dfrac{s+3}{s}}{s + \dfrac{1}{s} + 2} = 5\frac{s+3}{(s+1)^2}$$

$$V(s) = 2 \times 5\frac{s+3}{(s+1)^2} = \boxed{10\frac{s+3}{(s+1)^2}}$$

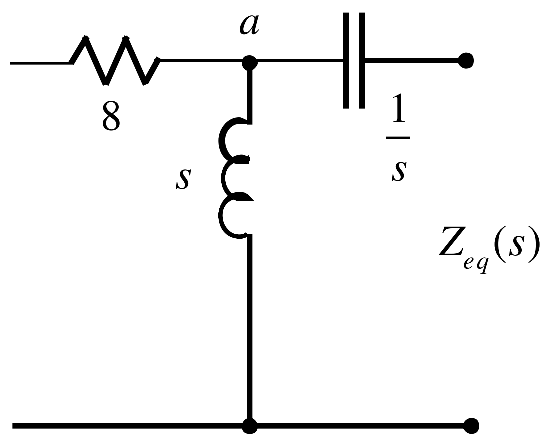


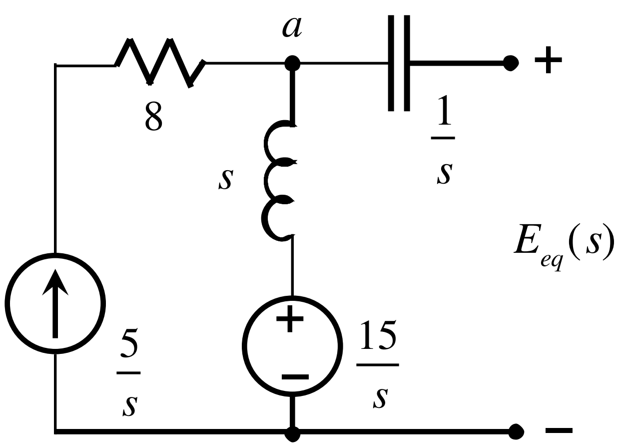
Ex. 7

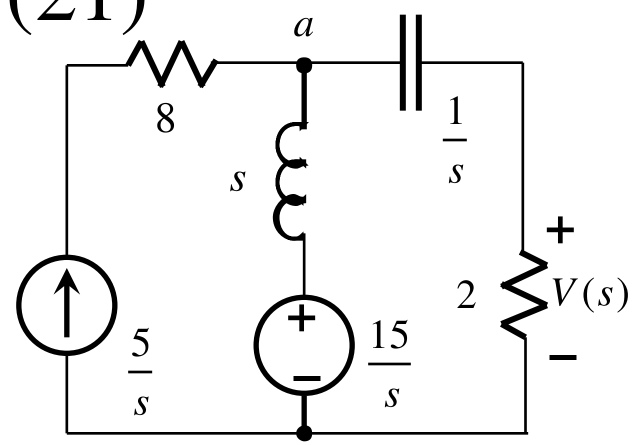
Analysis Techniques (21)

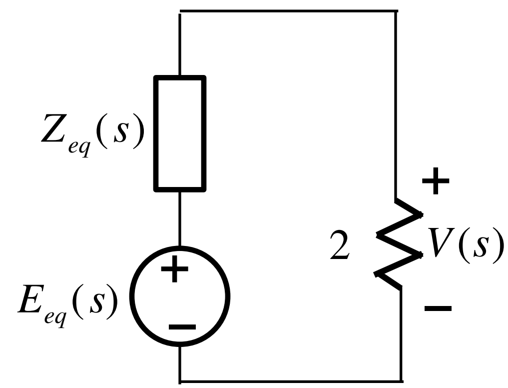
Solve for $v(t)$?

Method 5









$$Z_{eq}(s) = s + \frac{1}{s}$$

$$E_{eq}(s) - s\frac{5}{s} = \frac{15}{s}$$

$$\rightarrow E_{eq}(s) = 5\frac{s+3}{s}$$

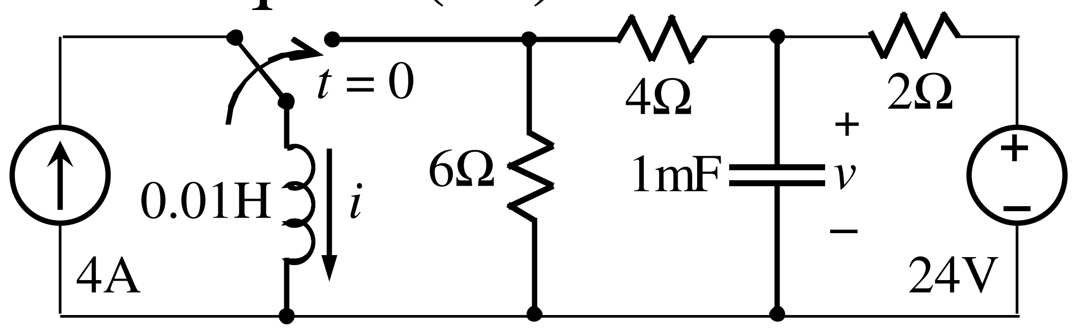
$$\rightarrow V(s) = 2\frac{E_{eq}(s)}{2 + Z_{eq}(s)}$$

$$= \boxed{10\frac{s+3}{(s+1)^2}}$$

Ex. 8

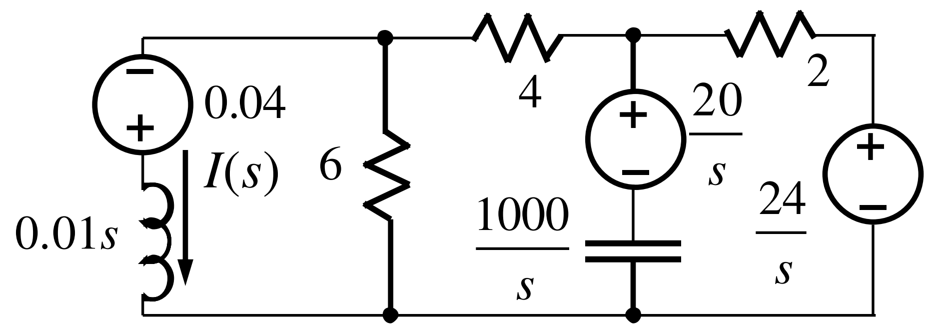
Analysis Techniques (22)

Solve for $i(t)$?

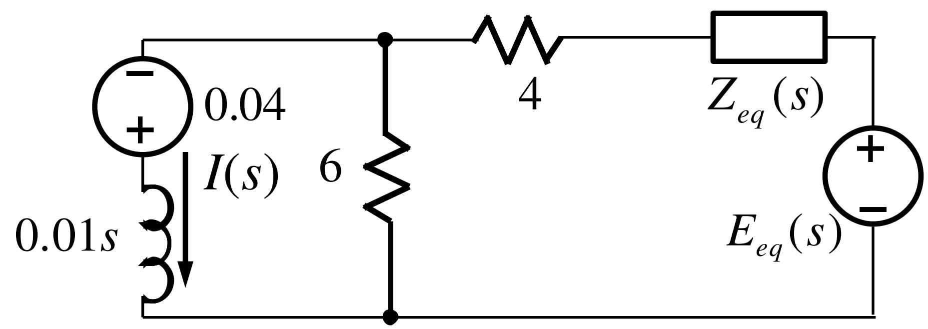


$$i(0) = 4\text{A}$$

$$v(0) = \frac{24}{6+4+2}(6+4) = 20\text{V}$$



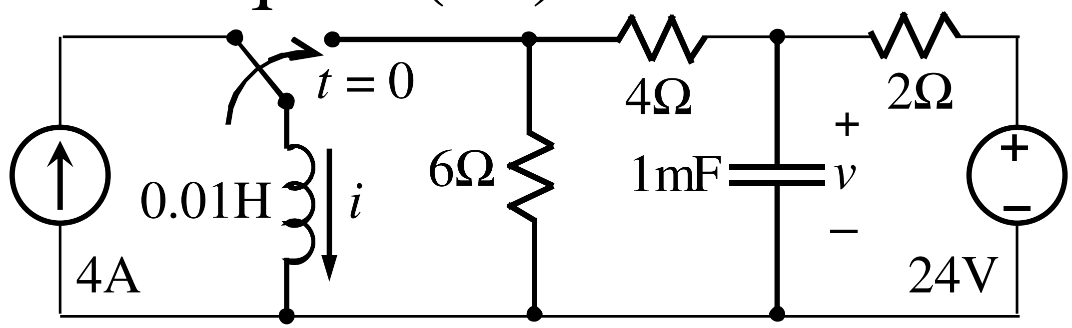
$$Z_{eq}(s) = \frac{2\dfrac{1000}{s}}{2+\dfrac{1000}{s}} = \frac{1000}{s+500}\,\Omega$$



$$E_{eq}(s) = \frac{\dfrac{20/s}{1000/s} + \dfrac{24/s}{2}}{\dfrac{1}{1000/s} + \dfrac{1}{2}} = \frac{20s + 12000}{s(s+500)}\ \text{V}$$

Ex. 8 Analysis Techniques (23)

Solve for $i(t)$?

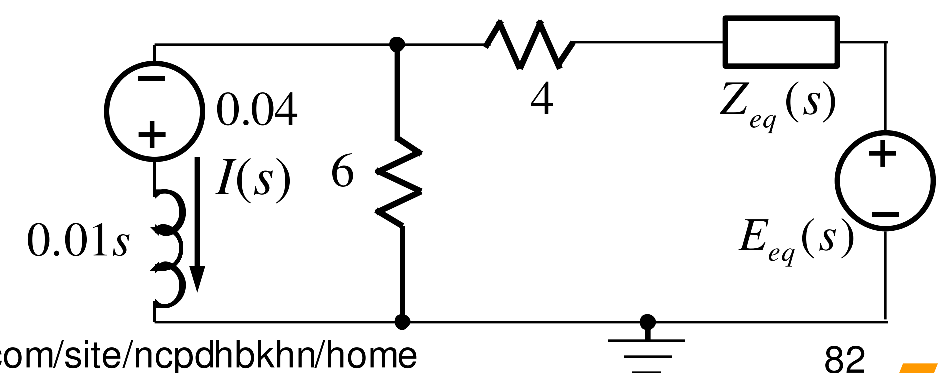


$$V(s)=\frac{-\dfrac{0.04}{0.01s}+\dfrac{E_{eq}(s)}{4+Z_{eq}(s)}}{\dfrac{1}{0.01s}+\dfrac{1}{6}+\dfrac{1}{4+Z_{eq}(s)}}$$

$$=\frac{12s}{5s^2+4200s+9\times10^5}\ \text{V}$$

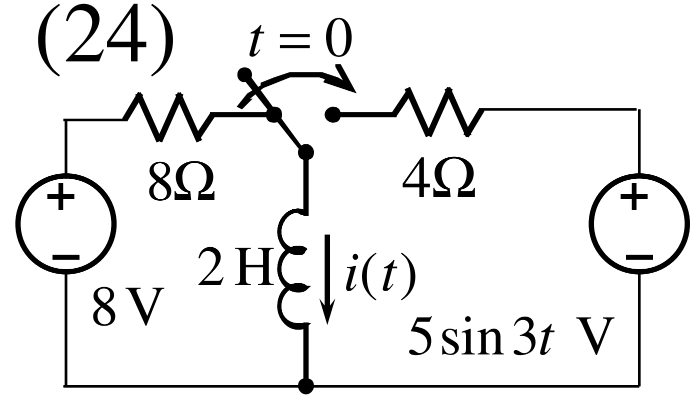
$$I(s)=\frac{0.04+V(s)}{0.01s}=\frac{4s^2+3600s+720000}{s(s^2+840s+180000)}\ \text{A}$$

$$\boxed{i(t)=4+4e^{-420t}\sin(60t)\ \text{A}}$$



Ex. 9

Analysis Techniques (24)



Find the current $i(t)$?

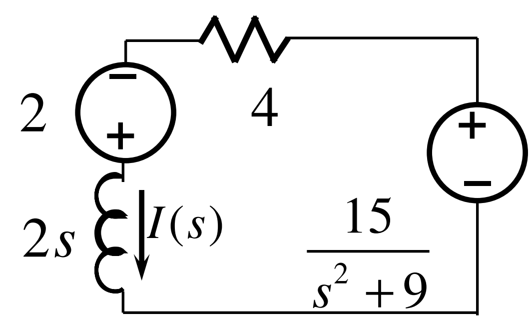
$$i(0) = \frac{8}{8} = 1\text{ A}$$

$$I(s) = \frac{2 + \dfrac{15}{s^2+9}}{2s+4} = \frac{s^2+16.5}{(s+2)(s^2+9)}$$

$$= \frac{K_1}{s+2} + \frac{K_2}{s-j3} + \frac{K_2^*}{s+j3}$$

$$K_1 = \left.\frac{s^2+16.5}{s^2+9}\right|_{s=-2} = 1.58$$

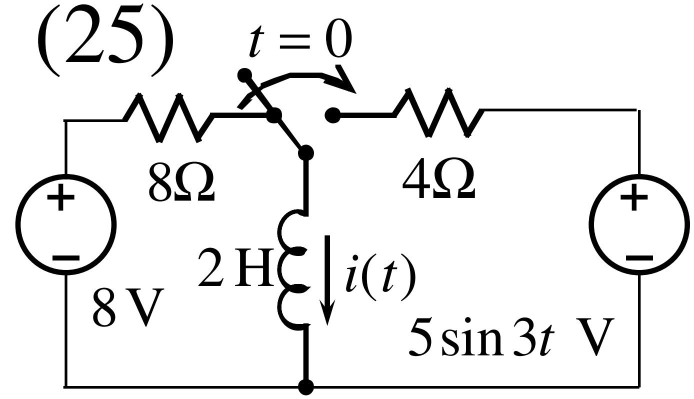
$$K_2 = \left.\frac{s^2+16.5}{(s+2)(s+j3)}\right|_{s=j3} = 0.35\underline{/-146.3^\circ}$$



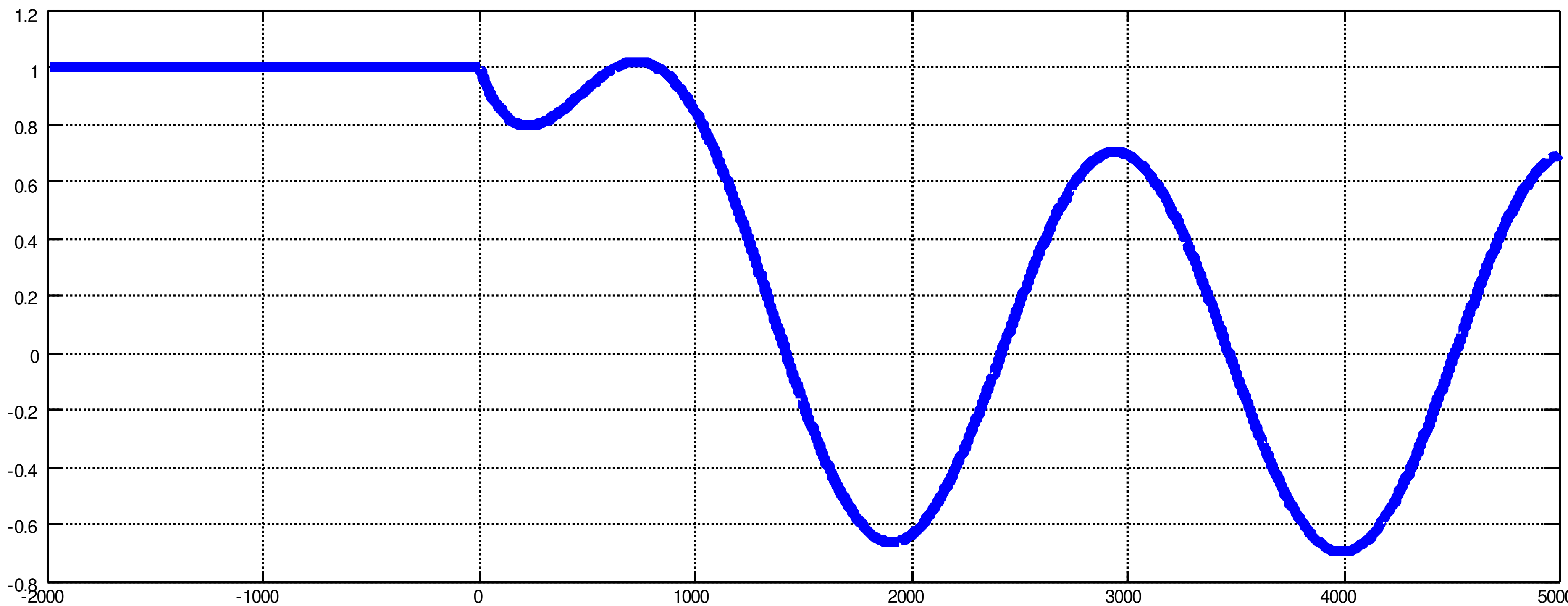
$$\rightarrow i(t) = \boxed{1.58e^{-2t} + 0.70\cos(3t - 146.3^\circ)\text{ A}}$$

Ex. 9 Analysis Techniques (25)

Find the current $i(t)$?

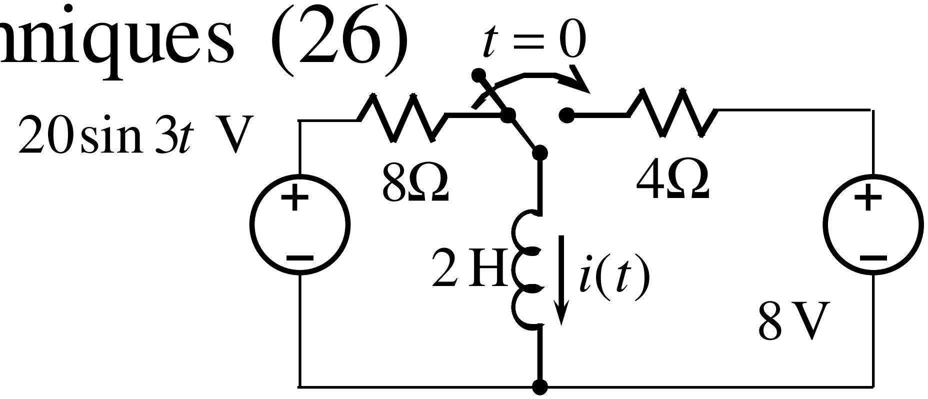


$$i(t) = 1.58e^{-2t} + 0.70\cos(3t - 146.3^{\circ})\ \text{A}$$



Ex. 10

Analysis Techniques (26)

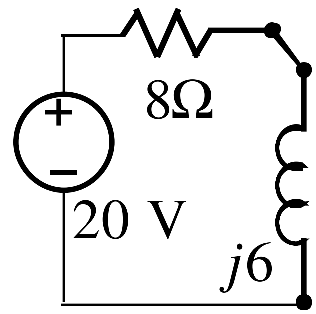


Find the current $i(t)$?

$$\mathbf{I}_0 = \frac{20}{8+j6} = 2\angle -36,9^\circ \text{ A}$$

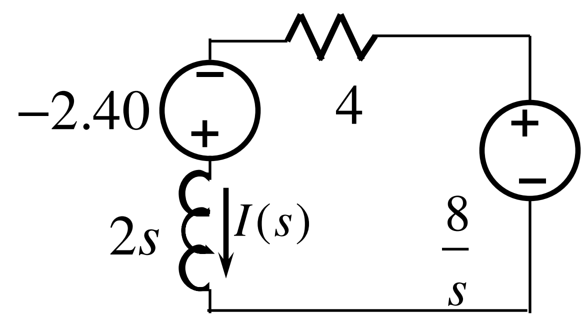
$$\rightarrow i_0(t) = 2\sin(3t - 36,9^\circ)\text{ A}$$

$$\rightarrow i(0) = 2\sin(-36,9^\circ) = -1.20\text{ A}$$



$$I(s) = \frac{-2.40 + \frac{8}{s}}{2s+4} = \frac{-1.2s+4}{s(s+2)}\text{ A} = \frac{K_1}{s} + \frac{K_2}{s+2}$$

$$K_1 = \left.\frac{-1.2s+4}{s+2}\right|_{s=0} = 2; \quad K_2 = \left.\frac{-1.2s+4}{s}\right|_{s=-2} = -3.2$$

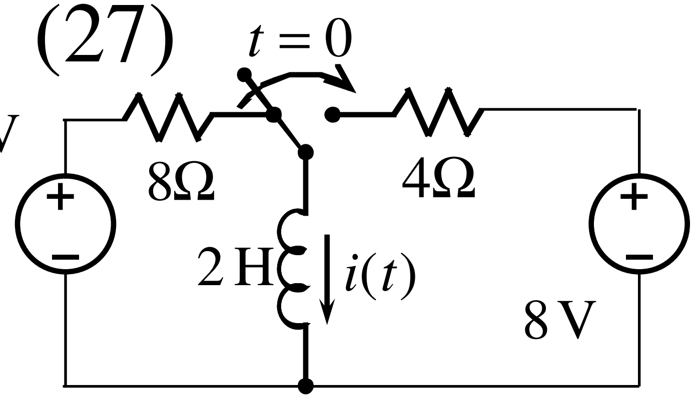


$$\rightarrow \boxed{i(t) = 2 - 3.2e^{-2t}\text{ A}}$$

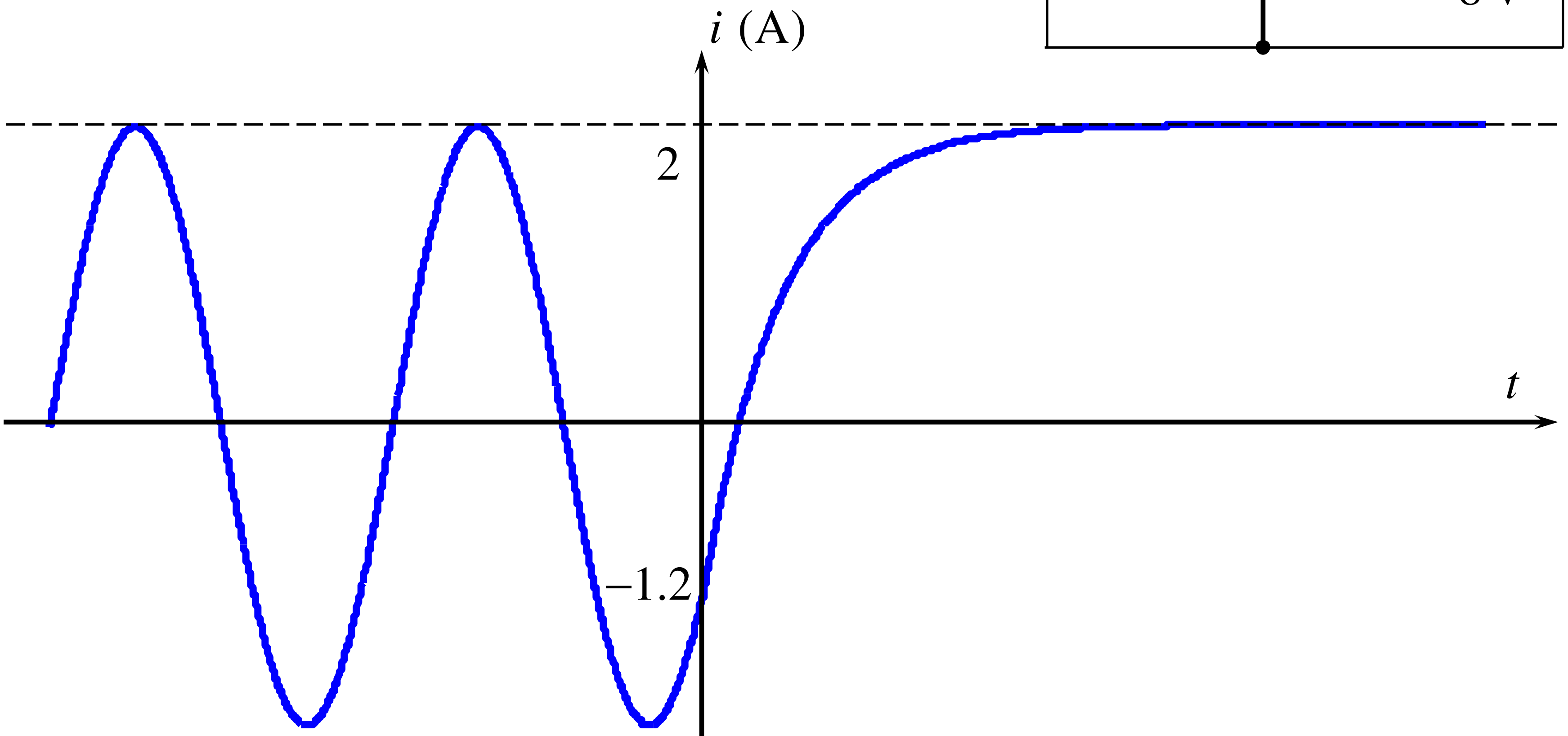
Ex. 10

Analysis Techniques (27)

Find the current $i(t)$?

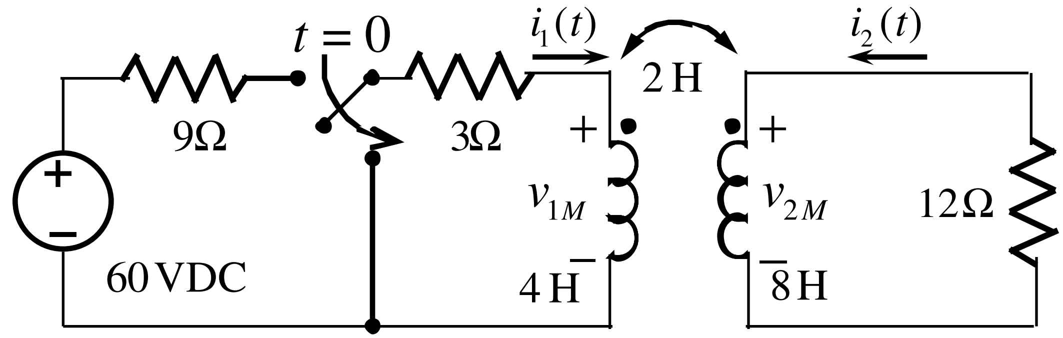


$$i(t) = 2 - 3.2e^{-2t} \text{ A}$$



Ex. 11 Analysis Techniques (28)

Find the current $i_2(t)$?



$$i_1(0)=\frac{60}{12}=5\text{A};\qquad i_2(0)=0$$

$$v_{1M}=2i_2';\qquad v_{2M}=2i_1'$$

$$\left.\begin{array}{l}\begin{cases}3i_1+4i_1'+2i_2'=0\\12i_2+8i_2'+2i_1'=0\end{cases}\\ kx(t)\to kX(s)\\ x'(t)\to sX(s)-x(-0)\end{array}\right\}\to\begin{cases}3I_1(s)+4[sI_1(s)-i_1(0)]+2[sI_2(s)-i_2(0)]=0\\2[sI_1(s)-i_1(0)]+12I_2(s)+8[sI_2(s)-i_2(0)]=0\end{cases}$$

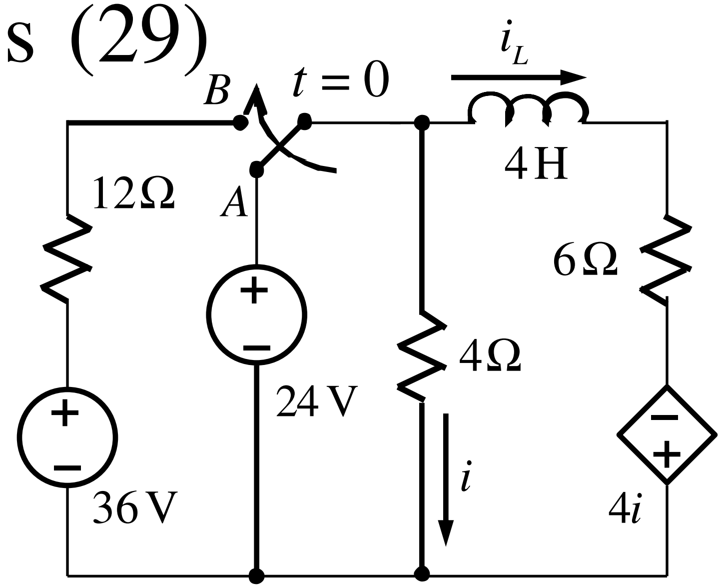
$$\to\begin{cases}3I_1(s)+4[sI_1(s)-5]+2sI_2(s)=0\\2[sI_1(s)-5]+12I_2(s)+8sI_2(s)=0\end{cases}$$

$$\to I_2(s)=\frac{15}{2(7s^2+18s+9)}\ \text{A}$$

$$\to\boxed{i_2(t)=0.8838(e^{-0.6796t}-e^{-1.8918t})\ \text{A}}$$

Ex. 12 Analysis Techniques (29)

The switch has been at A for a long time, and it moves to B at $t = 0$; find i_L for $t \geq 0$?



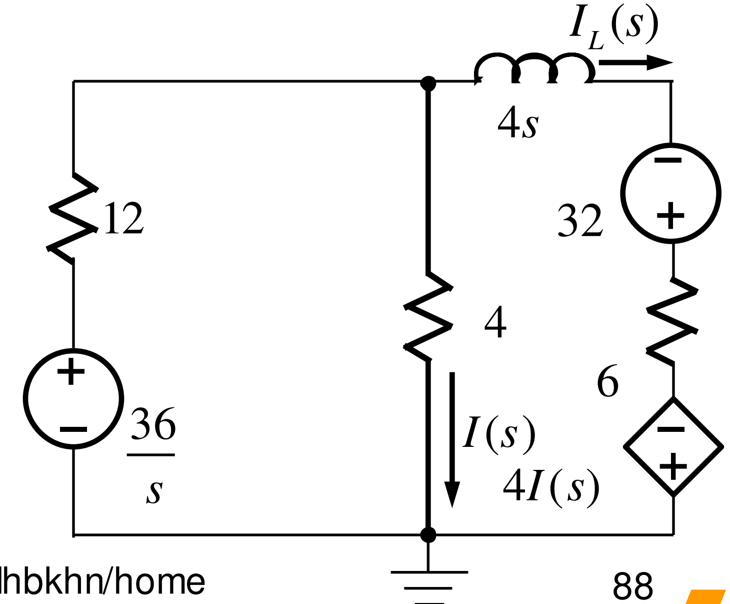
$$i = \frac{24}{4} = 6\,\text{A}$$

$$6i_L(0) = 4i + 24$$

$$\rightarrow i_L(0) = \frac{4i + 24}{6} = \frac{4 \times 6 + 24}{6} = 8\,\text{A}$$

$$\left.\begin{aligned} \left(\frac{1}{12} + \frac{1}{4} + \frac{1}{4s+6}\right)V(s) &= \frac{36/s}{12} - \frac{4I(s) + 32}{4s+6} \\ V(s) &= 4I(s) \end{aligned}\right\}$$

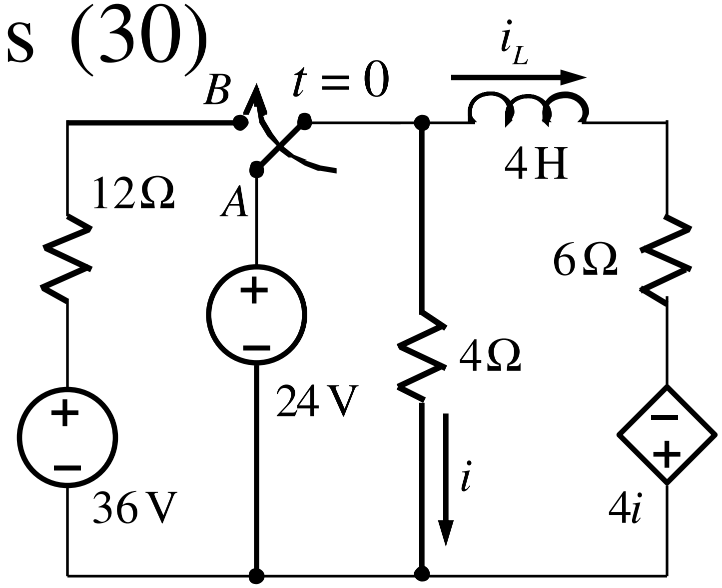
$$\rightarrow V(s) = -\frac{30s - 27}{2s(s+3)}\ \text{V}$$



Ex. 12

Analysis Techniques (30)

The switch has been at A for a long time, and it moves to B at $t = 0$; find i_L for $t \geq 0$?

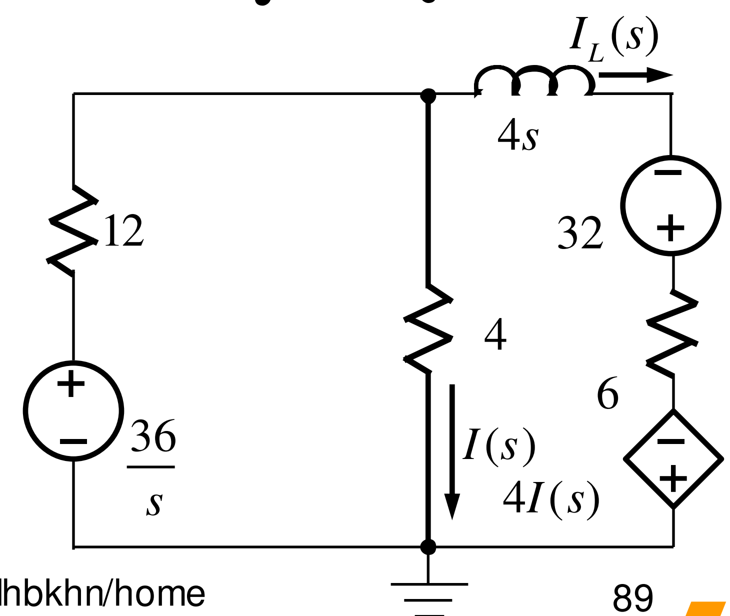


$$V(s) = -\frac{30s - 27}{2s(s+3)}\ \text{V}$$

$$\left.\begin{aligned} &\to I_L(s) = \frac{32 + 4I(s) + V(s)}{4s+6} \\ &V(s) = 4I(s) \end{aligned}\right\}$$

$$\to I_L(s) = \frac{32 + 2V(s)}{4s+6}$$

$$= \frac{16s+9}{2s(s+3)}\ \text{A}$$



Ex. 12

Analysis Techniques (31)

The switch has been at A for a long time, and it moves to B at $t = 0$; find i_L for $t \geq 0$?

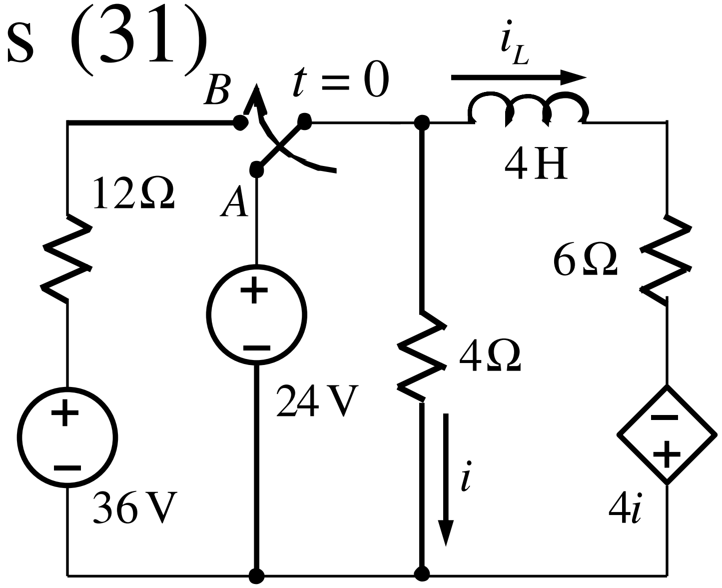
$$I_L(s) = \frac{16s+9}{2s(s+3)} = \frac{8s+4.5}{s(s+3)} \text{ A}$$

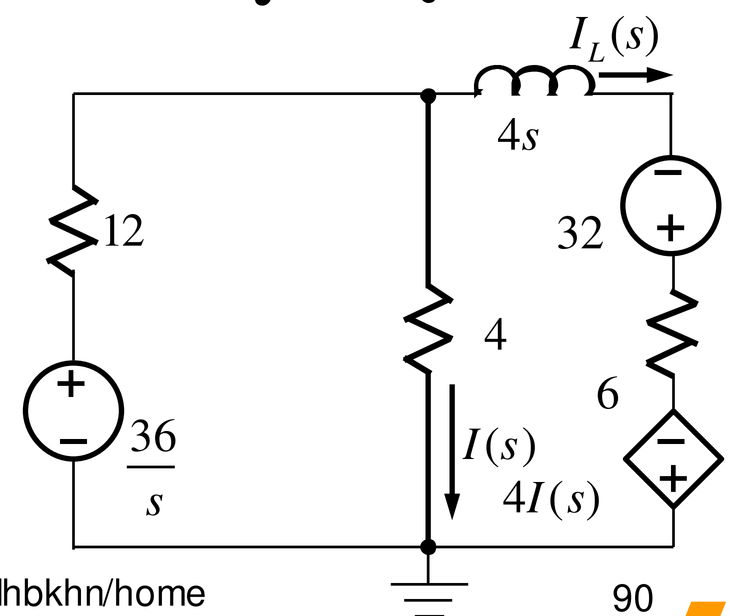
$$= \frac{K_1}{s} + \frac{K_2}{s+3}$$

$$K_1 = \left.\frac{8s+4.5}{s+3}\right|_{s=0} = 1.5$$

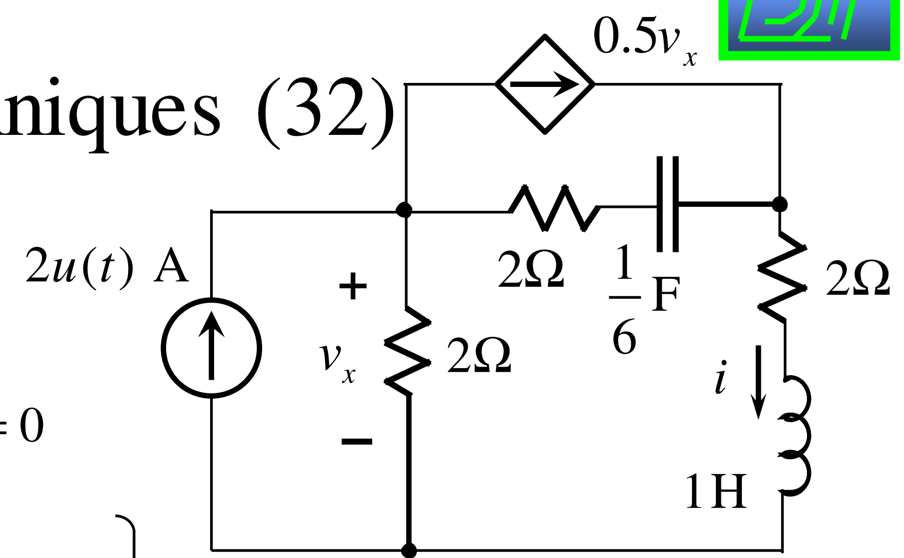
$$K_2 = \left.\frac{8s+4.5}{s}\right|_{s=-3} = 6.5$$

$$\rightarrow \boxed{i_L(t) = 1.5 + 6.5e^{-3t} \text{ A}}$$





Ex. 13 Analysis Techniques (32)



Solve for $i(t)$?

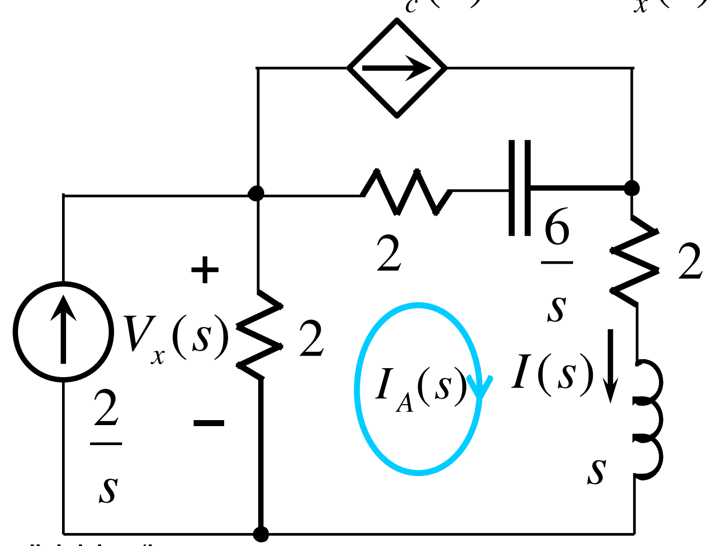
$v_C(0)=0;\ \ i_L(0)=0$

$$-V_x(s)+\left(2+\frac{6}{s}\right)[I_A(s)-I_c(s)]+(s+2)I_A(s)=0$$

$$\rightarrow -V_x(s)+\left(2+\frac{6}{s}\right)[I_A(s)-0.5V_x(s)]+(s+2)I_A(s)=0$$

$$V_x(s)=2\left[\frac{2}{s}-I_A(s)\right]=\frac{4}{s}-2I_A(s)$$

$I_c(s)=0.5V_x(s)$



$$\rightarrow -\left[\frac{4}{s}-2I_A(s)\right]+\left(2+\frac{6}{s}\right)\left\{I_A(s)-0.5\left[\frac{4}{s}-2I_A(s)\right]\right\}+$$

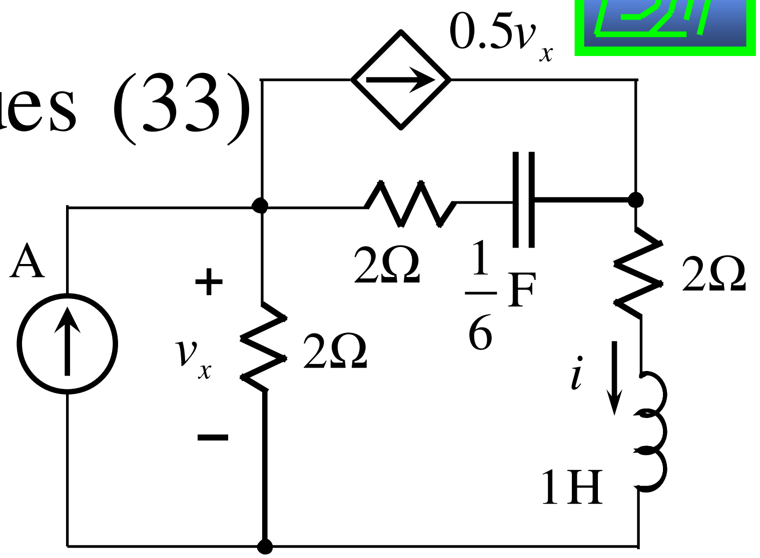
$$+(s+2)I_A(s)=0$$

$$\rightarrow I_A(s)=\frac{8s+12}{s(s+2)(s+6)}=I(s)$$

Ex. 13

Analysis Techniques (33)

Solve for $i(t)$?



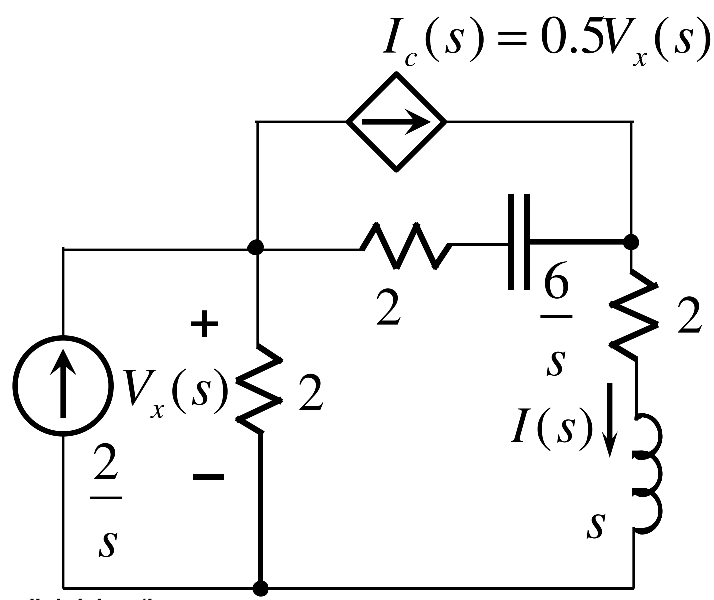
$$I(s)=\frac{8s+12}{s(s+2)(s+6)}=\frac{K_1}{s}+\frac{K_2}{s+2}+\frac{K_3}{s+6}$$

$$K_1=\left.\frac{8s+12}{(s+2)(s+6)}\right|_{s=0}=1$$

$$K_2=\left.\frac{8s+12}{s(s+6)}\right|_{s=-2}=0.5$$

$$K_3=\left.\frac{8s+12}{s(s+2)}\right|_{s=-6}=-1.5$$

$$\rightarrow i(t)=\boxed{1+0.5e^{-2t}-1.5e^{-6t}\ \text{A}}$$



Analysis Techniques (34)

Circuit

Circuit Element Models

Circuit in s-domain

DC circuit analysis techniques (KVL, KCL, nodal analysis, mesh analysis, source transformation, superposition, Thevenin/Norton equivalent, …)

$I(s)$, $V(s)$, …

Inverse Transform

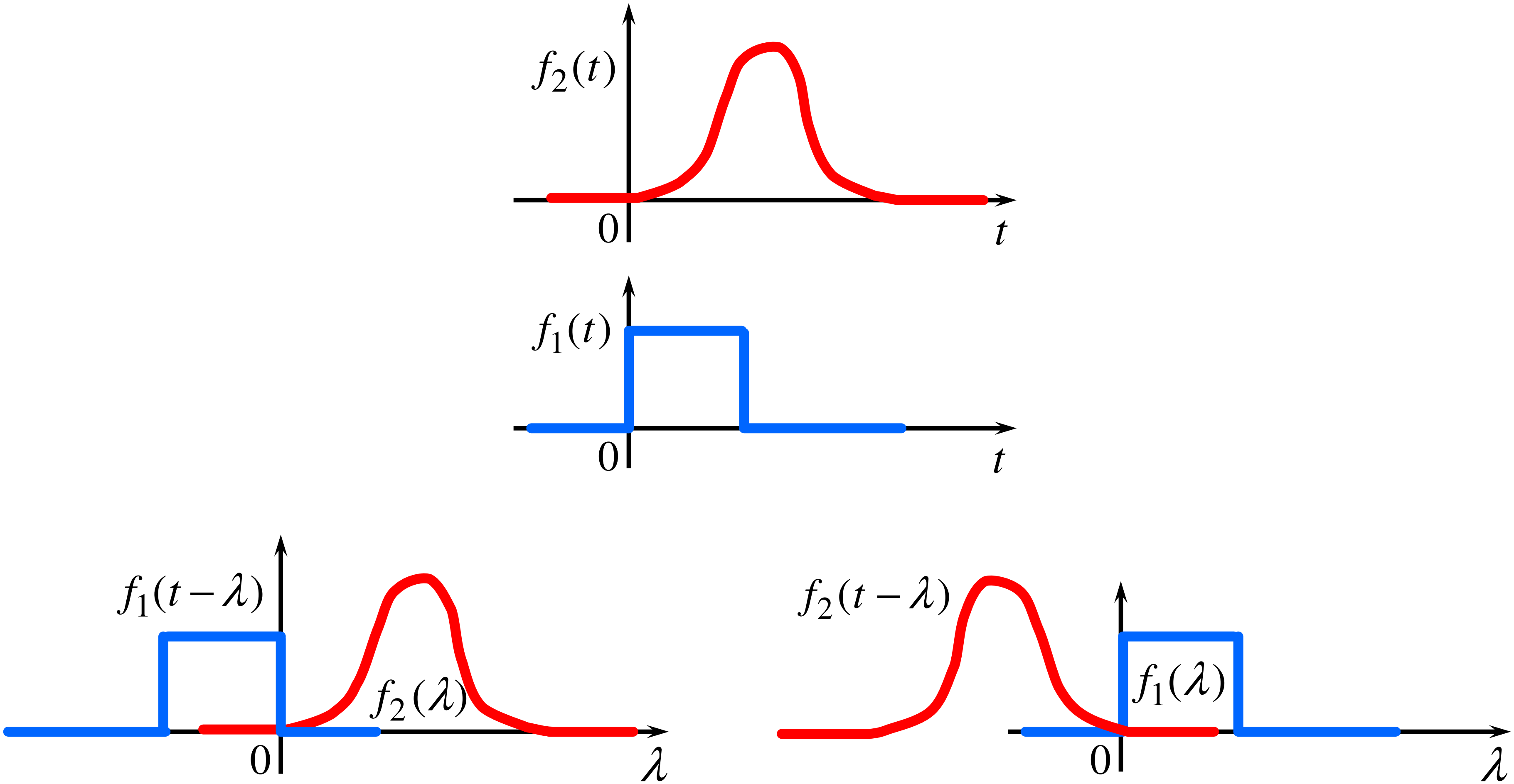
$i(t)$, $v(t)$, …

The Laplace Transform

1. Definition
2. Two Important Singularity Functions
3. Transform Pairs
4. Properties of the Transform
5. Inverse Transform
6. Initial-Value & Final-Value Theorems
7. Laplace Circuit Solutions
8. Circuit Element Models
9. Analysis Techniques
10. **Convolution Integral**
11. Transfer Function

Convolution Integral (1)

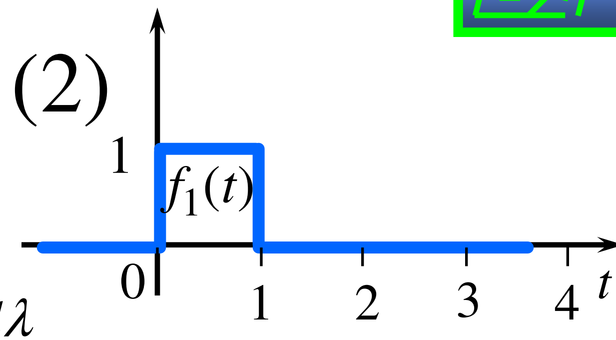
$$f(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\lambda) f_2(\lambda) d\lambda = \int_0^t f_1(\lambda) f_2(t-\lambda) d\lambda$$

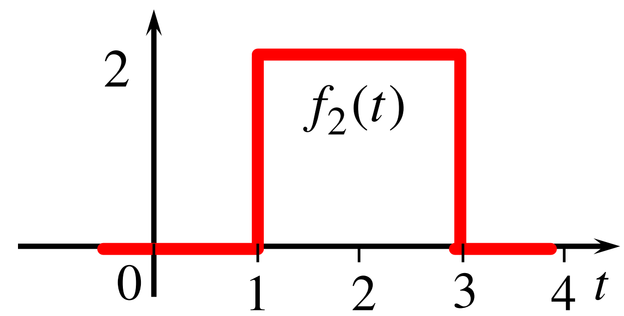


Convolution Integral (2)

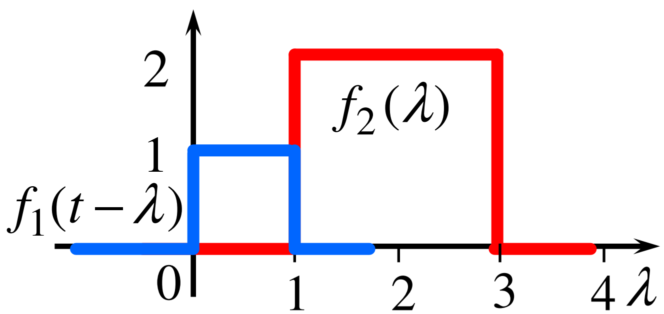
Ex. 1

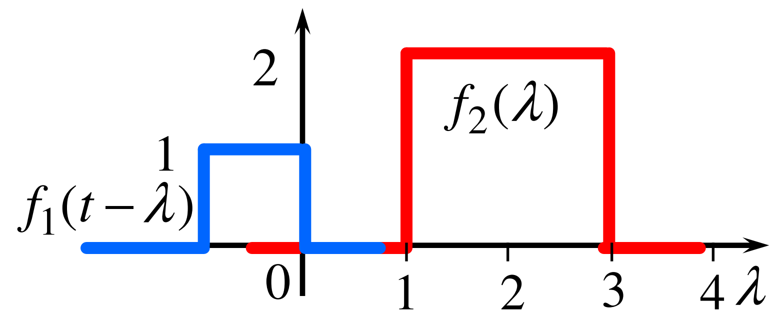
Find the convolution of the two signals?





$$f(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\lambda) f_2(\lambda) d\lambda = \int_0^t f_1(\lambda) f_2(t-\lambda) d\lambda$$





$$0 < t < 1: f_1 = 1;\ f_2 = 0$$

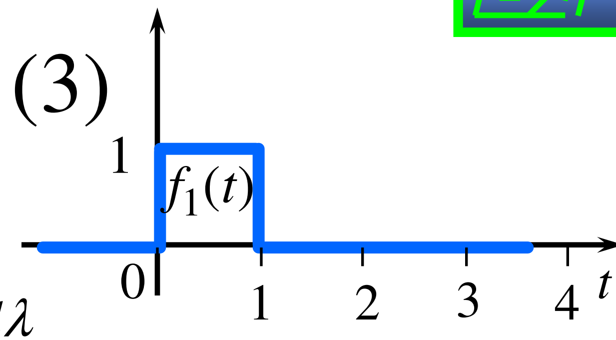
$$f_1(t) * f_2(t) = 0$$

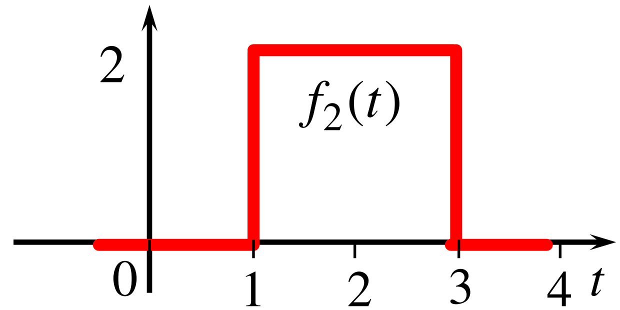
Convolution Integral (3)

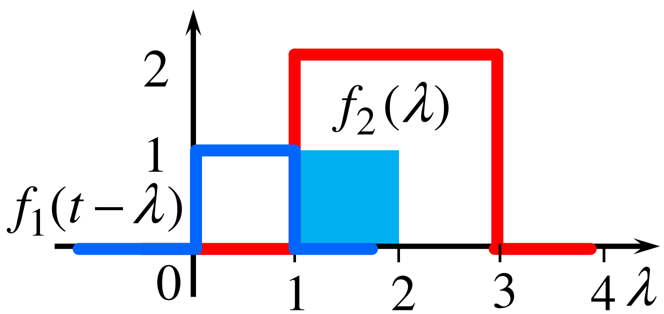
Ex. 1

Find the convolution of the two signals?

$$f(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\lambda) f_2(\lambda) d\lambda = \int_0^t f_1(\lambda) f_2(t-\lambda) d\lambda$$







$0 < t < 1: \quad f_1(t) * f_2(t) = 0$

$1 < t < 2: f_1 = 1;\ f_2 = 2$

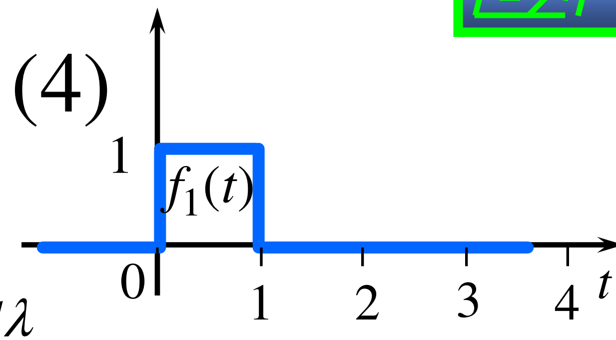
$$f_1(t) * f_2(t) = \int_1^t f_1(t-\lambda) f_2(\lambda) d\lambda = \int_1^t 1 \times 2 d\lambda = 2\lambda\Big|_{\lambda=1}^{t} = 2(t-1)$$

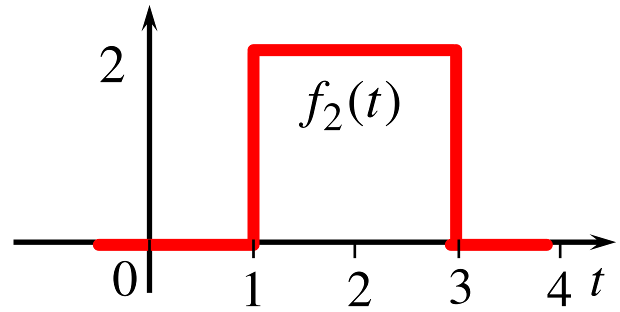
Convolution Integral (4)

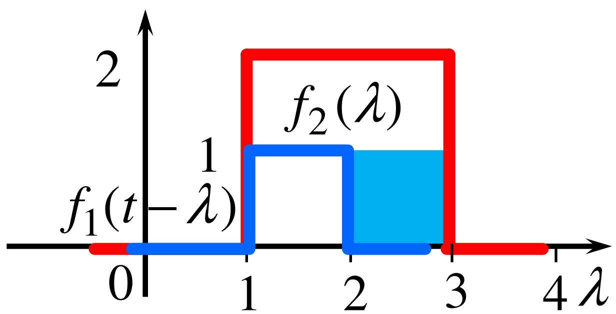
Ex. 1

Find the convolution of the two signals?

$$f(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\lambda) f_2(\lambda) d\lambda = \int_0^t f_1(\lambda) f_2(t-\lambda) d\lambda$$







$0 < t < 1: \quad f_1(t) * f_2(t) = 0$

$1 < t < 2: \quad f_1(t) * f_2(t) = 2(t-1)$

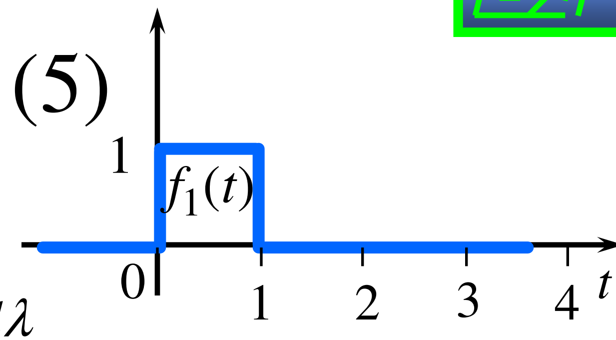
$2 < t < 3: f_1 = 1; \; f_2 = 2$

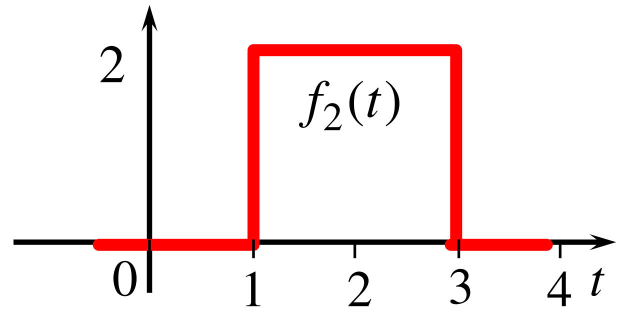
$$f_1(t) * f_2(t) = \int_{t-1}^t f_1(t-\lambda) f_2(\lambda) d\lambda = \int_{t-1}^t 1 \times 2 d\lambda = 2\lambda\Big|_{\lambda=t-1}^t = 2$$

Convolution Integral (5)

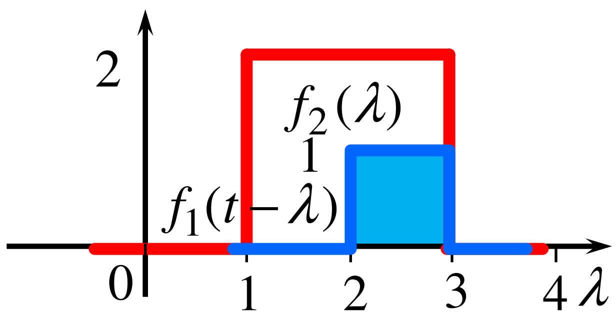
Ex. 1

Find the convolution of the two signals?





$$f(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\lambda) f_2(\lambda) d\lambda = \int_0^t f_1(\lambda) f_2(t-\lambda) d\lambda$$



$$0 < t < 1: \quad f_1(t) * f_2(t) = 0$$

$$1 < t < 2: \quad f_1(t) * f_2(t) = 2(t-1)$$

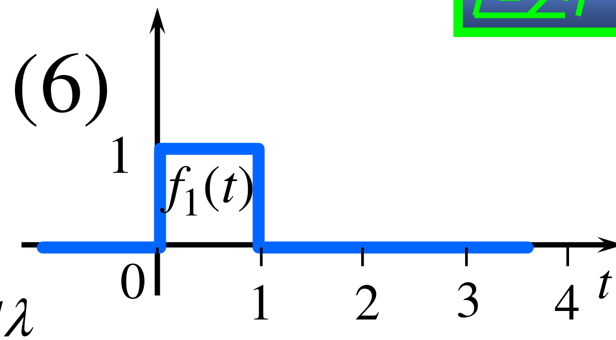
$$2 < t < 3: \quad f_1(t) * f_2(t) = 2$$

$$3 < t < 4: f_1 = 1;\ f_2 = 2$$

$$f_1(t) * f_2(t) = \int_{t-1}^{3} f_1(t-\lambda) f_2(\lambda) d\lambda = \int_{t-1}^{3} 1 \times 2 d\lambda = 2\lambda\Big|_{\lambda=t-1}^{3} = 8 - 2t$$

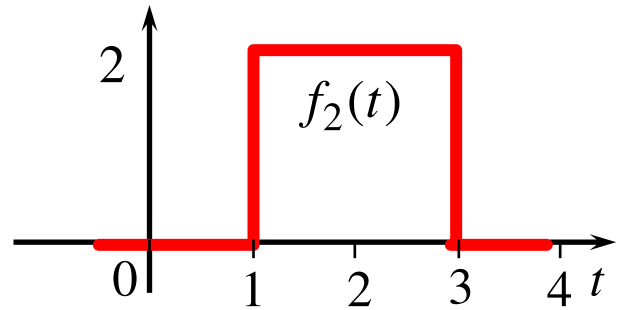
Ex. 1

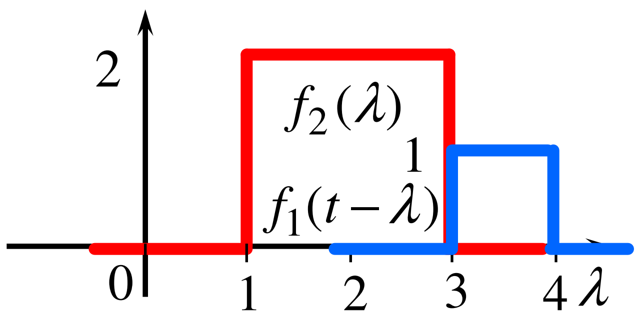
Convolution Integral (6)



Find the convolution of the two signals?

$$f(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\lambda) f_2(\lambda) d\lambda = \int_0^t f_1(\lambda) f_2(t-\lambda) d\lambda$$





$0 < t < 1: \quad f_1(t) * f_2(t) = 0$

$1 < t < 2: \quad f_1(t) * f_2(t) = 2(t-1)$

$2 < t < 3: \quad f_1(t) * f_2(t) = 2$

$3 < t < 4: \quad f_1(t) * f_2(t) = 8 - 2t$

$t > 4: f_1 = 1; \; f_2 = 0$

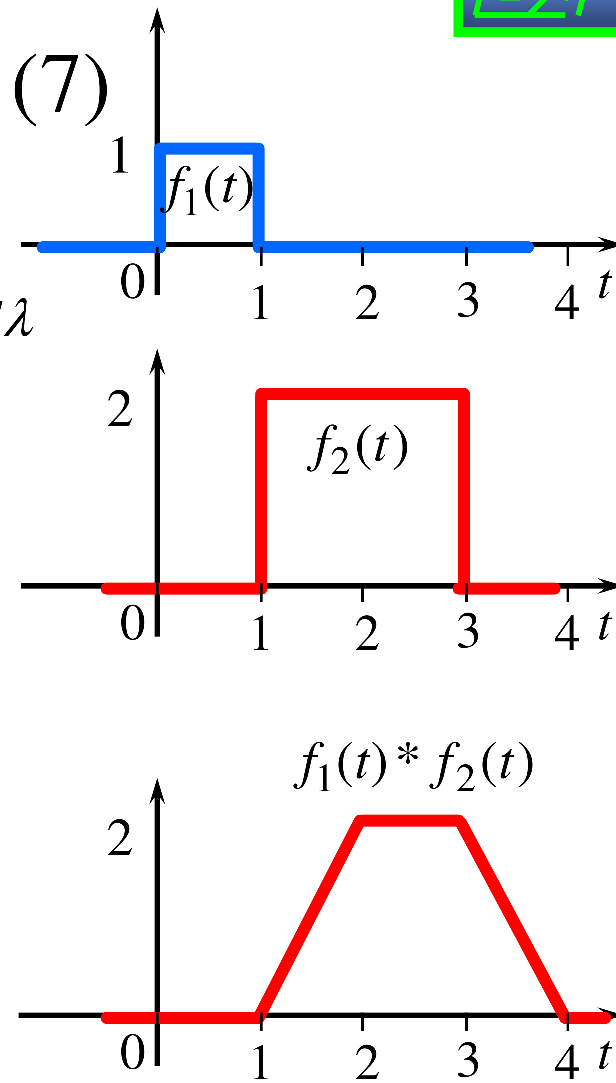
$f_1(t) * f_2(t) = 0$

Convolution Integral (7)

Ex. 1

Find the convolution of the two signals?

$$f(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\lambda) f_2(\lambda) d\lambda = \int_0^t f_1(\lambda) f_2(t-\lambda) d\lambda$$



$0 < t < 1: \quad f_1(t) * f_2(t) = 0$

$1 < t < 2: \quad f_1(t) * f_2(t) = 2(t-1)$

$2 < t < 3: \quad f_1(t) * f_2(t) = 2$

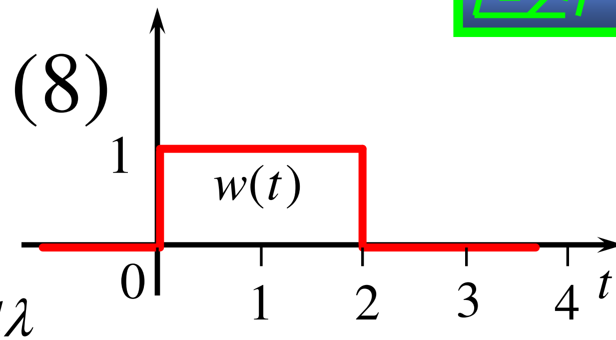
$3 < t < 4: \quad f_1(t) * f_2(t) = 8 - 2t$

$t > 4: \quad f_1(t) * f_2(t) = 0$

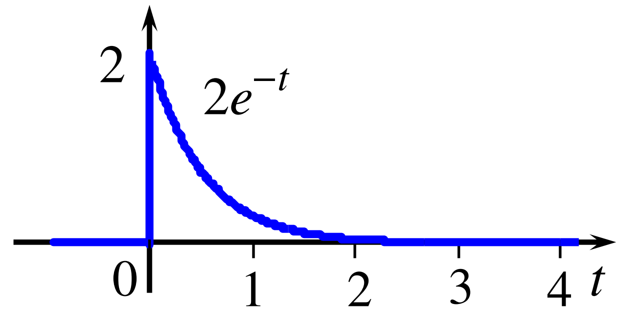
Convolution Integral (8)

Ex. 2

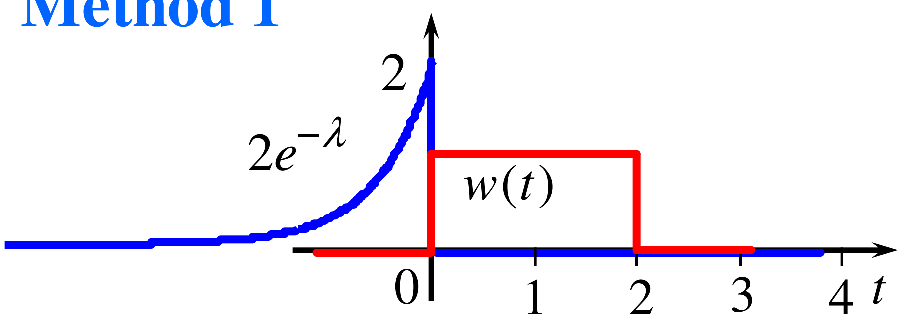
Find the convolution of the two signals?

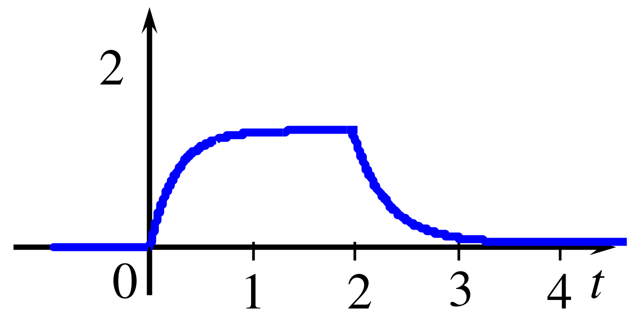


$$f(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\lambda) f_2(\lambda) d\lambda = \int_0^t f_1(\lambda) f_2(t-\lambda) d\lambda$$



Method 1





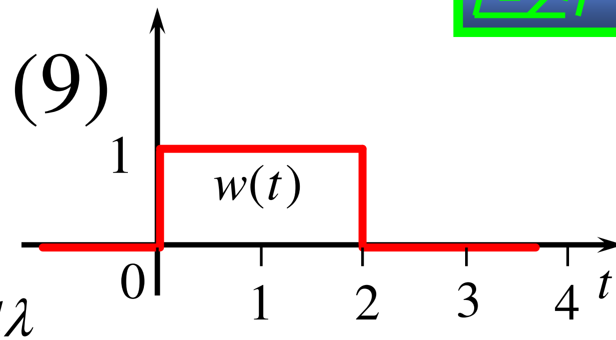
$$0 < t < 2: \; f(t) = \int_0^t 2e^{-(t-\lambda)} \times 1 d\lambda = 2e^{\lambda - t}\Big|_{\lambda=0}^{\lambda=t} = 2(1 - e^{-t})$$

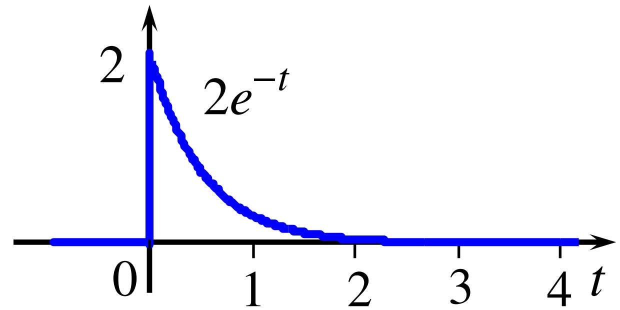
$$t > 2: \quad f(t) = \int_0^2 2e^{-(t-\lambda)} \times 1 d\lambda = 2e^{\lambda - t}\Big|_{\lambda=0}^{\lambda=2} = 2(e^2 - 1)e^{-t}$$

Convolution Integral (9)

Ex. 2

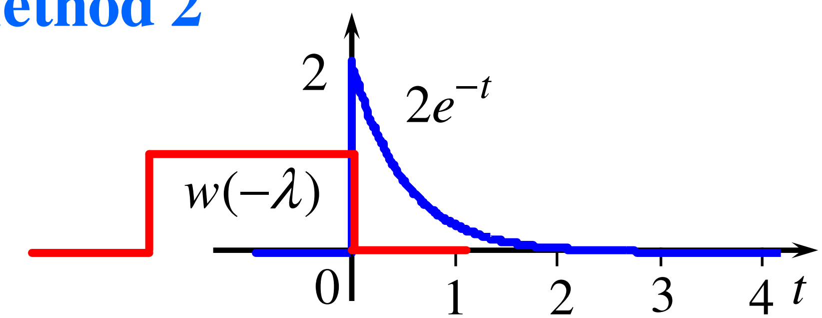
Find the convolution of the two signals?

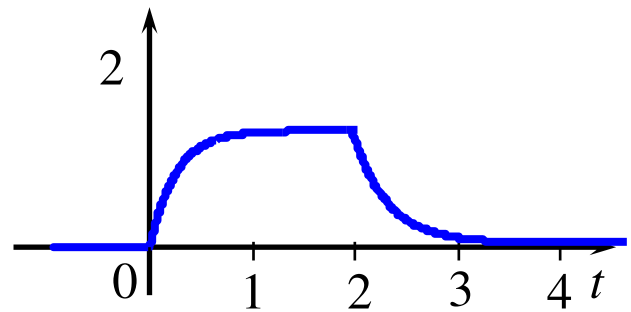




$$f(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\lambda) f_2(\lambda) d\lambda = \int_0^t f_1(\lambda) f_2(t-\lambda) d\lambda$$

Method 2





$$0 < t < 2: \ f(t) = \int_0^t 1 \times 2e^{-\lambda} d\lambda = -2e^{-\lambda}\Big|_{\lambda=0}^{\lambda=t} = 2(1 - e^{-t})$$

$$t > 2: \quad f(t) = \int_{t-2}^t 1 \times 2e^{-\lambda} d\lambda = -2e^{-\lambda}\Big|_{\lambda=t-2}^{\lambda=t} = 2(e^2 - 1)e^{-t}$$

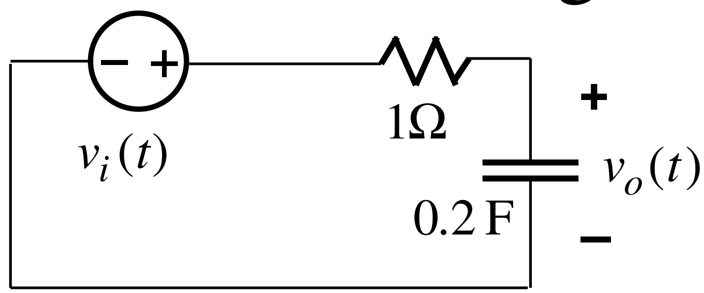
Convolution Integral (10)

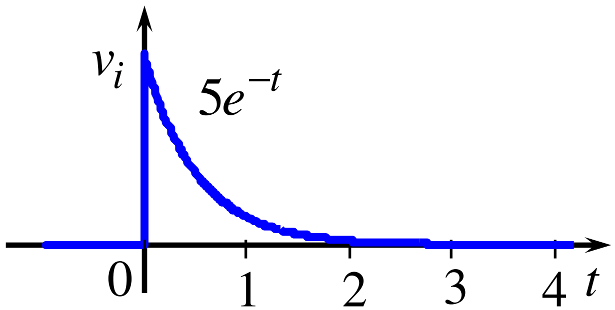
Property	$f(t)$	$F(s)$
1. Magnitude scaling	$Af(t)$	$AF(s)$
2. Addition/subtraction	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
3. Time scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
4. Time shifting	$f(t-a)u(t-a),\ a \geq 0$ $f(t)u(t-a),\ a \geq 0$	$e^{-as}F(s)$ $e^{-as}L[f(t+a)]$
5. Frequency shifting	$e^{-at}f(t)$	$F(s+a)$
6. Differentiation	$d^n f(t)/dt^n$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f^1(0) \ldots - s^0 f^{n-1}(0)$
7. Multiplication by t	$t^n f(t)$	$(-1)^n d^n F(s)/ds^n$
8. Division by t	$f(t)/t$	$\int_s^{\infty} F(\lambda)d\lambda$
9. Integration	$\int_0^t f(\lambda)d\lambda$	$F(s)/s$
10. Convolution	$f_1(t) * f_2(t) = \int_0^t f_1(\lambda) f_2(t-\lambda)d\lambda$	$F_1(s)F_2(s)$

Convolution Integral (11)

Ex. 3

Find $v_o(t)$?





$$V_o(s) = \frac{Z_C(s)}{R + Z_C(s)} V_i(s) = \frac{\frac{1}{0.2s}}{1 + \frac{1}{0.2s}} \times \frac{5}{s+1} = \frac{5}{s+5} \times \frac{5}{s+1}$$

Method 1: $V_o(s) \to v_o(t) = \boxed{6.25(e^{-t} - e^{-5t})\ \text{V}}$

Method 2: $V_o(s) = H(s)V_i(s) \to v_o(t) = h(t) * v_i(t)$

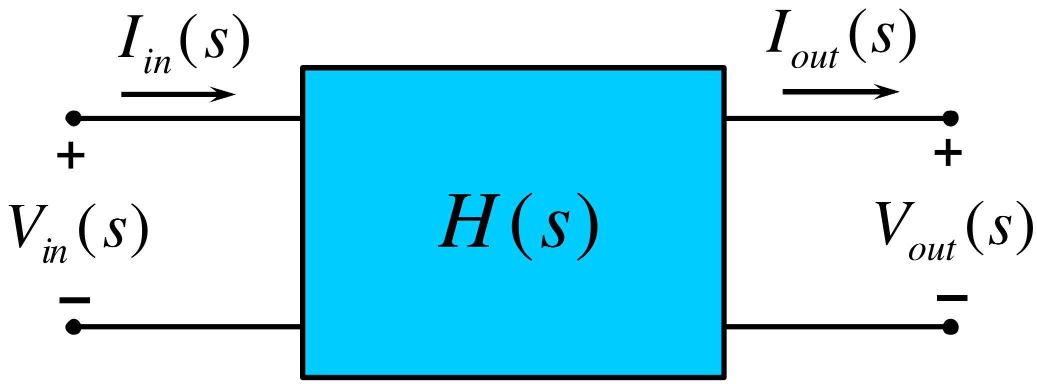
$$H(s) = \frac{5}{s+5} \to h(t) = 5e^{-5t}$$

$$v_o(t) = \int_0^t h(t-\lambda) v_i(\lambda) d\lambda = \int_0^t 5e^{-5(t-\lambda)} 5e^{-\lambda} d\lambda = 25\int_0^t e^{-5t+4\lambda} d\lambda = 6.25e^{-5t+4\lambda}\Big|_{\lambda=0}^{\lambda=t}$$

$$= \boxed{6.25(e^{-t} - e^{-5t})\ \text{V}}$$

The Laplace Transform

Transfer Function (1)



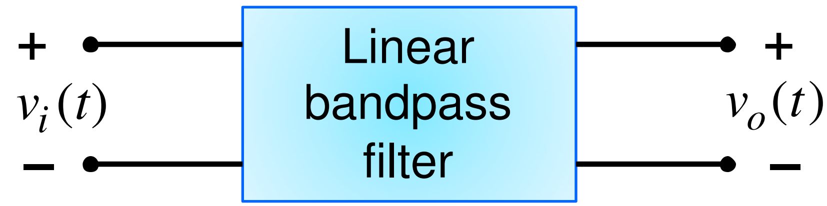
$$\boxed{H(s) = \frac{Out(s)}{In(s)}}$$

If $in(t) = \delta(t) \rightarrow In(s) = 1 \rightarrow H(s) = Out(s)$

Transfer Function (2)

Ex. 1

Find the transfer function $h(t)$ of the filter?



$$v_i(t) = 10u(t)$$

$$V_o(s) = H(s)V_i(s) = H(s)\frac{10}{s}$$

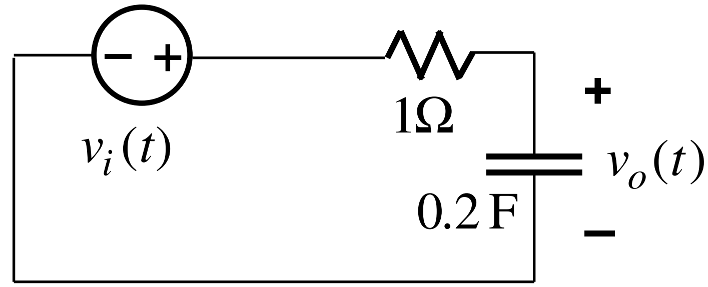
$$\rightarrow H(s) = \frac{1}{10} sV_o(s)$$

$$\rightarrow h(t) = \frac{1}{10}\frac{dv_o(t)}{dt}$$

Ex. 2

Transfer Function (3)

Find the transfer function $H(s)$?



$$V_o(s) = \frac{Z_C(s)}{R + Z_C(s)} V_i(s) = \frac{\frac{1}{0.2s}}{1 + \frac{1}{0.2s}} V_i(s) = \frac{5}{s+5} V_i(s) = H(s) V_i(s)$$

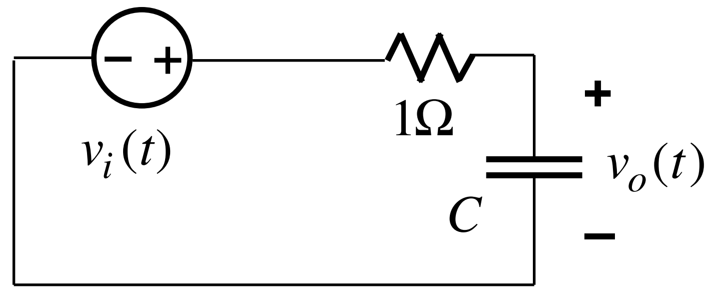
$$\rightarrow \boxed{H(s) = \frac{V_o(s)}{V_i(s)} = \frac{5}{s+5}}$$

Transfer Function (4)

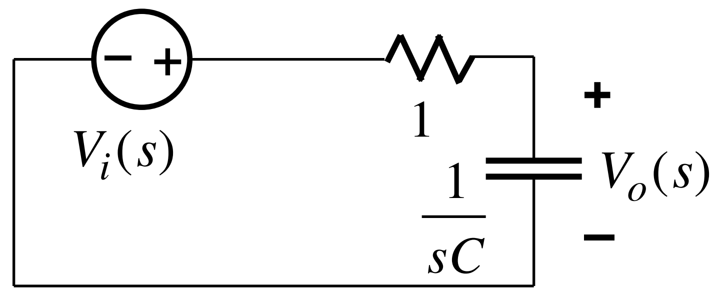
Ex. 3

Given the transfer function $H(s) = \dfrac{5}{s+5}$

Find C?



$$V_o(s) = \frac{Z_C(s)}{R + Z_C(s)} V_i(s) = \frac{\frac{1}{sC}}{1 + \frac{1}{sC}} V_i(s) = \frac{1}{Cs+1} V_i(s)$$



$$\rightarrow H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{Cs+1} = \frac{5}{5Cs+5}$$

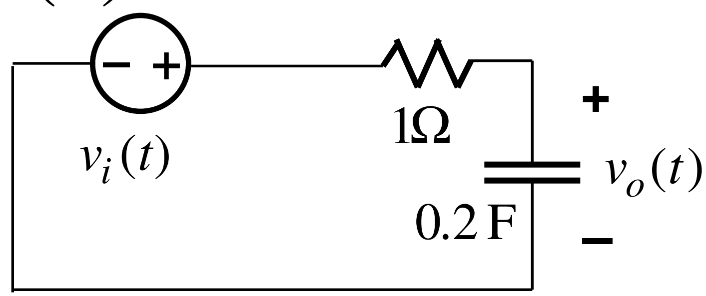
$$\rightarrow 5C = 1$$

$$\rightarrow \boxed{C = 0.2\,\text{F}}$$

Transfer Function (5)

Ex. 4

Find $v_o(t)$ in two cases: $v_i = 10\text{VDC}$; and $v_i(t) = 10\cos 2t$ V?



$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{5}{s+5}$$

$$V_{o1}(s) = V_{i1}(s)H(s) = \frac{10}{s}.\frac{5}{s+5}$$

$$\rightarrow v_{o1}(t) = 10(1 - e^{-5t}) \text{ V}$$

$$V_{o2}(s) = V_{i2}(s)H(s) = \frac{10s}{s^2+4}.\frac{5}{s+5}$$

$$\rightarrow v_{o2}(t) = -8.62e^{-5t} + 9.28\cos(2t + 21.8^\circ) \text{ V}$$

Ex. 5

Transfer Function (6)

Find the step responses of $H_1(s) = \dfrac{1}{s+1}$, $H_2(s) = \dfrac{4}{s^2+4}$, $H_3(s) = \dfrac{3}{s-3}$.

$$Y_1(s) = X(s)H_1(s) = \frac{1}{s}.\frac{1}{s+1} \qquad \rightarrow y_1(t) = 1 - e^{-t}$$

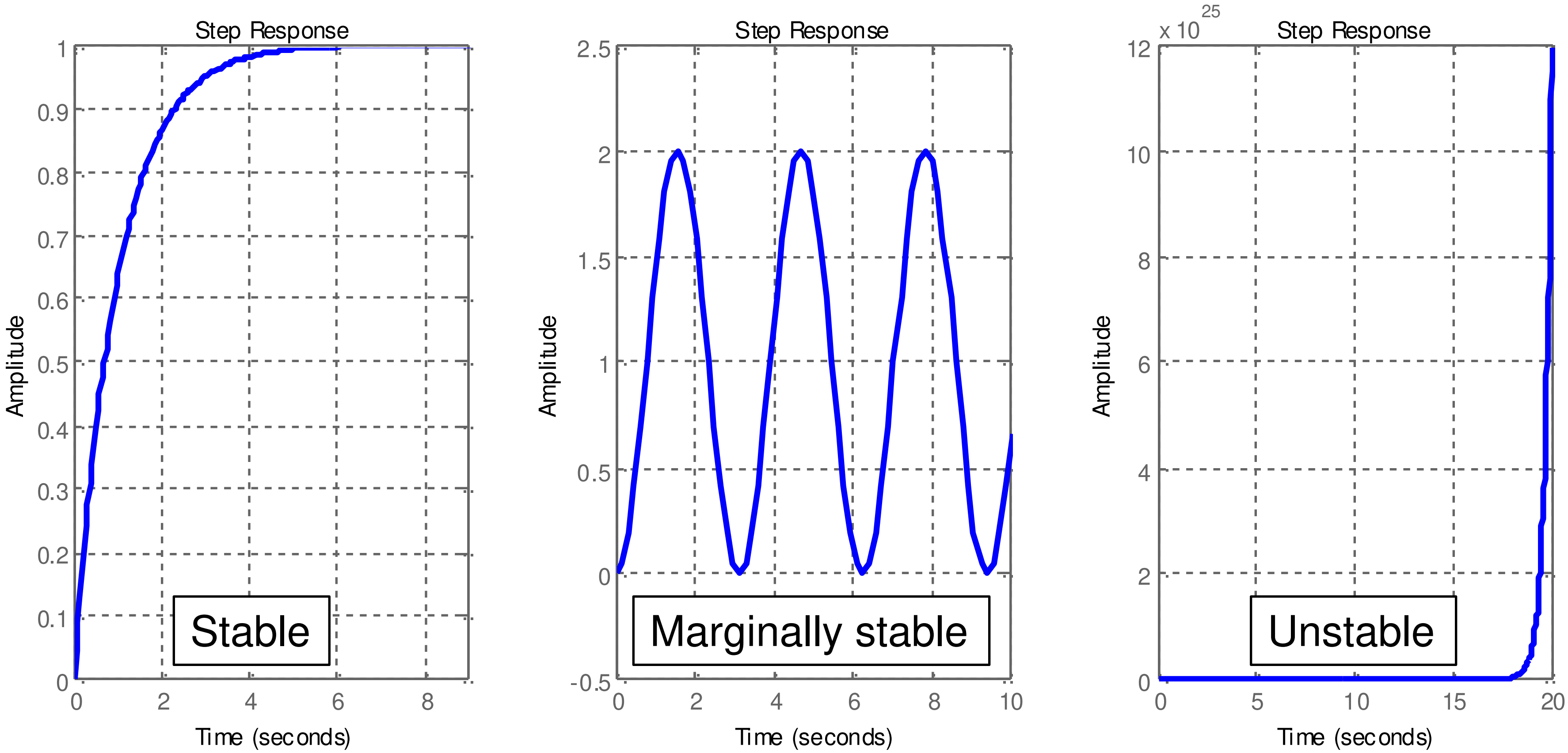
$$Y_2(s) = X(s)H_2(s) = \frac{1}{s}.\frac{4}{s^2+4} \qquad \rightarrow y_2(t) = 1 - \cos 2t$$

$$Y_3(s) = X(s)H_3(s) = \frac{1}{s}.\frac{3}{s-3} \qquad \rightarrow y_1(t) = -1 + e^{3t}$$

Transfer Function (7)

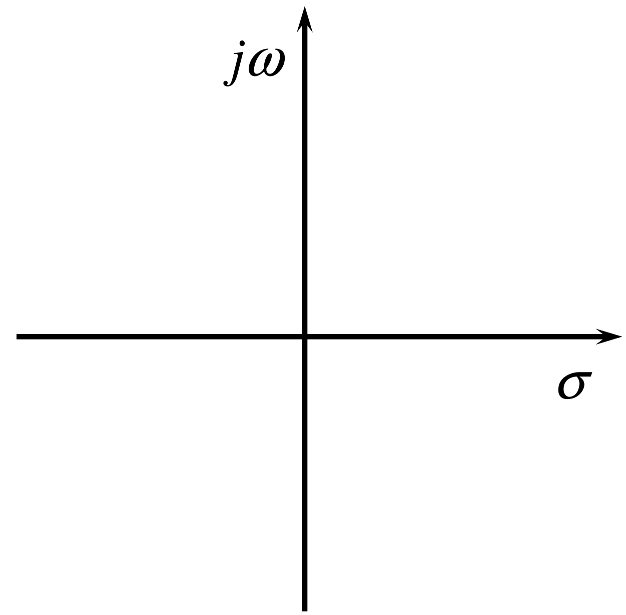
$$H(s)=\frac{1}{s+1} \qquad H(s)=\frac{4}{s^2+4} \qquad H(s)=\frac{3}{s-3}$$

$$y(t)=1-e^{-t} \qquad y(t)=1-\cos 2t \qquad y(t)=-1+e^{3t}$$



Transfer Function (8)

A circuit is stable if : $\lim_{t \to \infty} |h(t)| = \text{finite}$



$$H(s) = \frac{N(s)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$$\to h(t) = (k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \ldots + k_n e^{-p_n t})u(t)$$

A circuit is stable when all the poles of its transfer function $H(s)$ lie in the left half of the s-plane

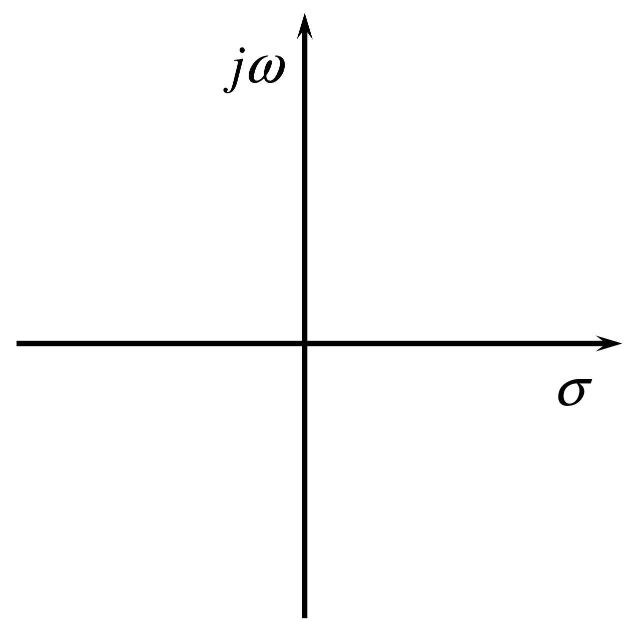
Ex. 6

Transfer Function (9)

An active filter has the transfer function $H(s) = \dfrac{k}{s^2 + (4-k)s + 1}$

For what values of k is the filter stable?

A circuit is stable when all the poles of its transfer function $H(s)$ lie in the left half of the s-plane



$$p_{1,2} = \frac{-(4-k) \pm \sqrt{(4-k)^2 - 4}}{2}$$

$$\rightarrow 4 - k > 0$$

$$\rightarrow \boxed{k < 4}$$

Ex. 7

Transfer Function (10)

Find the frequency responses of the system $H(s) = \dfrac{1}{s+1}$.

$$\mathbf{H}(j\omega) = H(s)\Big|_{s=j\omega} = \frac{1}{j\omega+1} = \frac{j\omega-1}{(j\omega)^2-1^2} = \frac{1}{1+\omega^2} - j\frac{\omega}{1+\omega^2}$$

$$|\mathbf{H}| = \sqrt{\left(\frac{1}{1+\omega^2}\right)^2 + \left(\frac{\omega}{1+\omega^2}\right)^2}$$

$$\angle\mathbf{H} = \operatorname{atan}\left(\frac{\dfrac{\omega}{1+\omega^2}}{\dfrac{1}{1+\omega^2}}\right)$$