Nguyễn Công Phương

Engineering Electromagnetics

Transmission Lines

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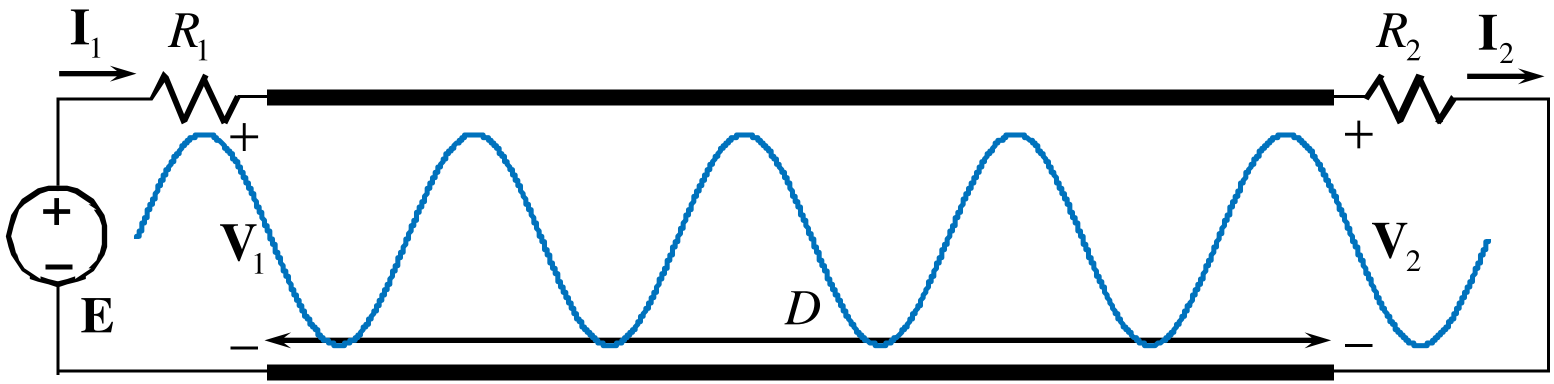
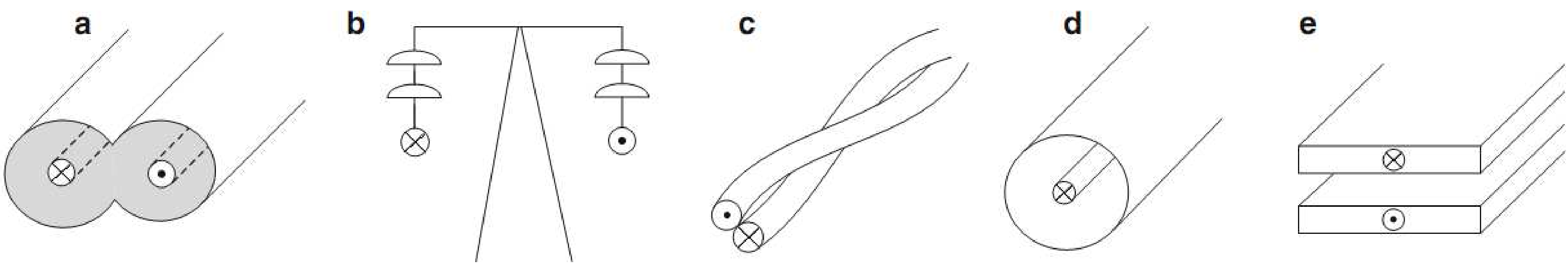
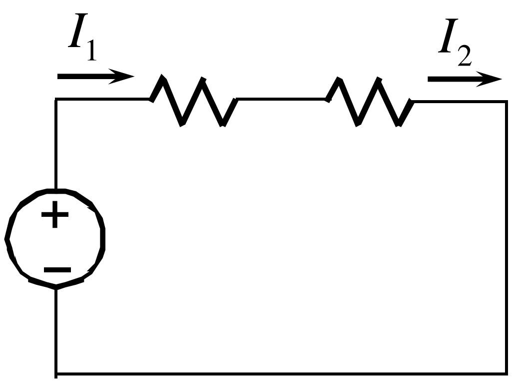
Introduction

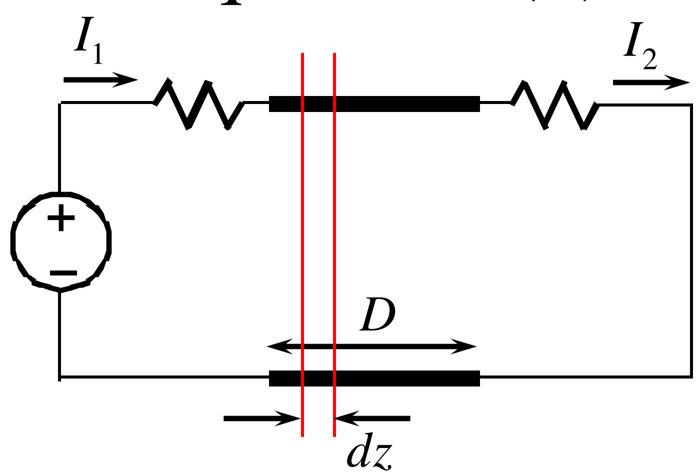


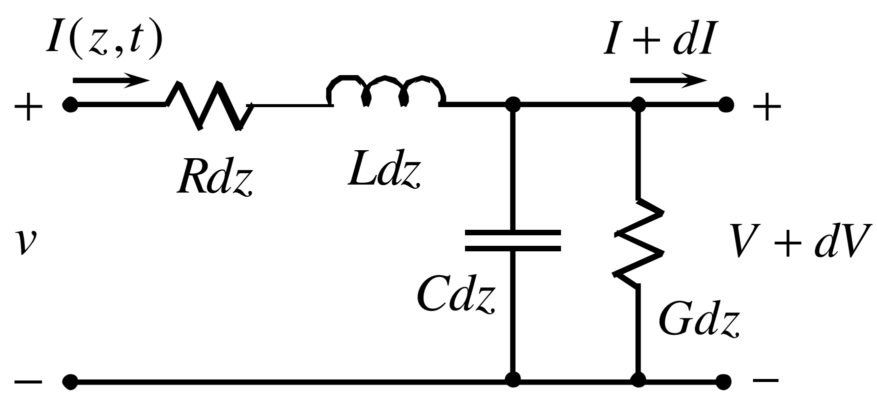


N. Ida. *Engineering Electromagnetics*. Springer 2015

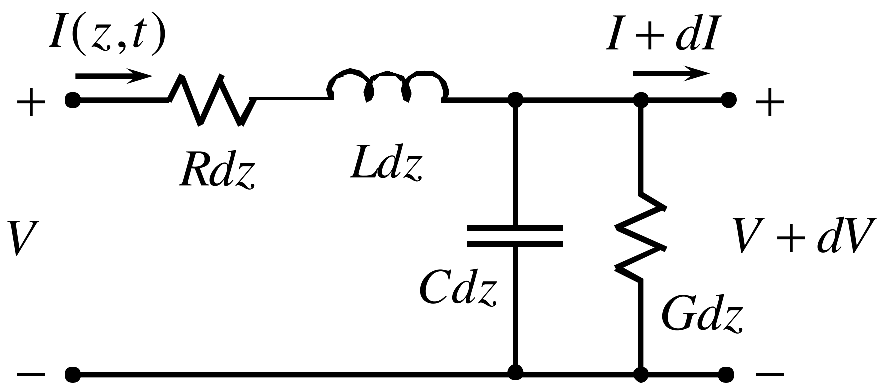
The Transmission Line Equations (1)







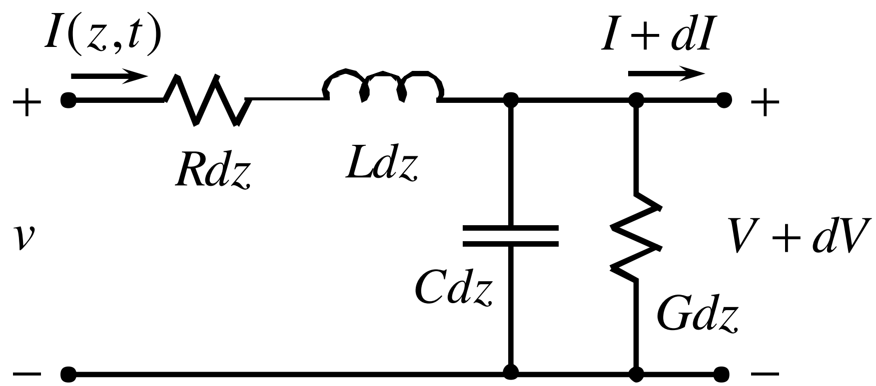
The Transmission Line Equations (2)



$$\begin{cases} I-(I+dI)-(Gdz)(V+dV)-(Cdz)(V+dV)'=0 \\ -V+(Rdz)I+(Ldz)I'+V+dV=0 \end{cases}$$

$$\rightarrow \begin{cases} dV+(Rdz)i+(Ldz)\dfrac{dI}{dt}=0 \\ dI+(Gdz)v+(Cdz)\dfrac{dV}{dt}=0 \end{cases} \quad \rightarrow \begin{cases} -\dfrac{\partial V}{\partial z}=RI+L\dfrac{\partial I}{\partial t} \\ -\dfrac{\partial I}{\partial z}=GV+C\dfrac{\partial V}{\partial t} \end{cases}$$

The Transmission Line Equations (3)



$$\begin{cases} -\dfrac{\partial V}{\partial z} = RI + L\dfrac{\partial I}{\partial t} \\ -\dfrac{\partial I}{\partial z} = GV + C\dfrac{\partial V}{\partial t} \end{cases}$$

$$\rightarrow \begin{cases} \dfrac{\partial^2 V}{\partial z^2} = LC\dfrac{\partial^2 V}{\partial t^2} + (LG + RC)\dfrac{\partial V}{\partial t} + RGV \\ \dfrac{\partial^2 I}{\partial z^2} = LC\dfrac{\partial^2 I}{\partial t^2} + (LG + RC)\dfrac{\partial V}{\partial t} + RGI \end{cases}$$

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Lossless Propagation (1)

$$\begin{cases} -\dfrac{\partial V}{\partial z} = RI + L\dfrac{\partial I}{\partial t} \\ -\dfrac{\partial I}{\partial z} = GV + C\dfrac{\partial V}{\partial t} \end{cases}, \quad \begin{cases} \dfrac{\partial^2 V}{\partial z^2} = LC\dfrac{\partial^2 V}{\partial t^2} + (LG+RC)\dfrac{\partial V}{\partial t} + RGV \\ \dfrac{\partial^2 I}{\partial z^2} = LC\dfrac{\partial^2 I}{\partial t^2} + (LG+RC)\dfrac{\partial V}{\partial t} + RGI \end{cases}$$

$$R=0, G=0 \rightarrow \begin{cases} -\dfrac{\partial V}{\partial z} = L\dfrac{\partial I}{\partial t} \\ -\dfrac{\partial I}{\partial z} = C\dfrac{\partial V}{\partial t} \end{cases}, \quad \begin{cases} \dfrac{\partial^2 V}{\partial z^2} = LC\dfrac{\partial^2 V}{\partial t^2} \\ \dfrac{\partial^2 I}{\partial z^2} = LC\dfrac{\partial^2 I}{\partial t^2} \end{cases}$$

$$\rightarrow V(z,t) = f_1\left(t - \frac{z}{v}\right) + f_2\left(t + \frac{z}{v}\right) = V^+ + V^-$$

Lossless Propagation (2)

$$V(z,t) = f_1\left(t - \frac{z}{v}\right) + f_2\left(t + \frac{z}{v}\right) = V^+ + V^-$$

$$\frac{\partial f_1}{\partial z} = \frac{\partial f_1}{\partial (t - z/v)} \frac{\partial (t - z/v)}{dz} = -\frac{1}{v} f_1'$$

$$\frac{\partial f_1}{\partial t} = \frac{\partial f_1}{\partial (t - z/v)} \frac{\partial (t - z/v)}{\partial t} = f_1'$$

$$\left.\begin{array}{l} \dfrac{\partial^2 f_1}{\partial t^2} = \dfrac{1}{v^2} f_1''; \quad \dfrac{\partial^2 f_1}{\partial t^2} = f_1'' \\[2ex] \dfrac{\partial^2 V}{\partial z} = LC \dfrac{\partial^2 V}{\partial t^2} \end{array}\right\} \rightarrow \boxed{v = \frac{1}{\sqrt{LC}}}$$

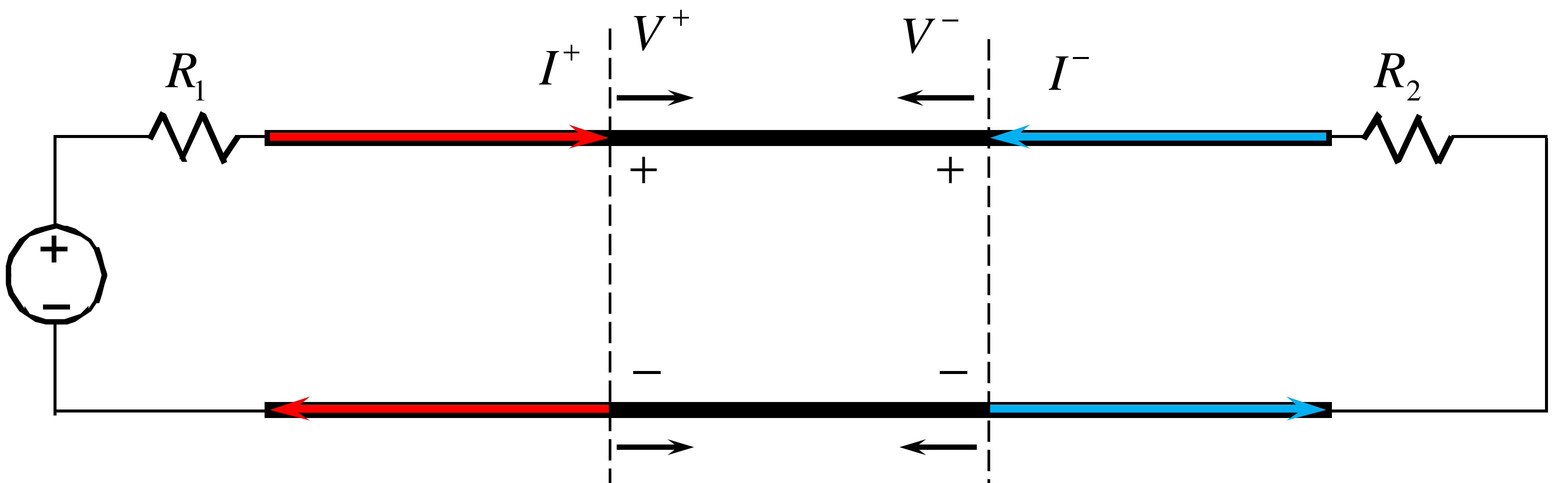
Lossless Propagation (3)

$$V(z,t) = f_1\left(t - \frac{z}{v}\right) + f_2\left(t + \frac{z}{v}\right) = V^+ + V^-$$

$$I(z,t) = \frac{1}{Lv}\left[f_1\left(t - \frac{z}{v}\right) - f_2\left(t + \frac{z}{v}\right)\right] = I^+ - I^-$$

$$Z_0 = Lv = \sqrt{L/C}$$

$$V^+ = Z_0 I^+, \quad V^- = -Z_0 I^-$$



Lossless Propagation (4)

$$V(z,t) = |V_0|\cos(\omega t - \beta z + \phi) + |V_0|\cos(\omega t + \beta z + \phi) = V_{forward} + V_{backward}$$

$$|V_0| = V(z=0, t=0)$$

$$\beta = \frac{\omega}{v}$$

$$e^{jx} = \cos(x) + j\sin(x) \rightarrow \cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\rightarrow \begin{cases} |V_0|\cos(\omega t - \beta z + \phi) = |V_0|\dfrac{e^{j(\omega t - \beta z + \phi)} + e^{-j(\omega t - \beta z + \phi)}}{2} = \dfrac{|V_0|e^{j\phi}}{2}e^{j(\omega t - \beta z)} + \dfrac{|V_0|e^{-j\phi}}{2}e^{-j(\omega t - \beta z)} \\ |V_0|\cos(\omega t + \beta z + \phi) = |V_0|\dfrac{e^{j(\omega t + \beta z + \phi)} + e^{-j(\omega t + \beta z + \phi)}}{2} = \dfrac{|V_0|e^{j\phi}}{2}e^{j(\omega t + \beta z)} + \dfrac{|V_0|e^{-j\phi}}{2}e^{-j(\omega t + \beta z)} \end{cases}$$

Lossless Propagation (5)

$$V(z,t) = |V_0|\cos(\omega t - \beta z + \phi) + |V_0|\cos(\omega t + \beta z + \phi) = V_{forward} + V_{backward}$$

$$\begin{cases} |V_0|\cos(\omega t - \beta z + \phi) = \dfrac{|V_0|e^{j\phi}}{2} e^{j(\omega t - \beta z)} + \dfrac{|V_0|e^{-j\phi}}{2} e^{-j(\omega t - \beta z)} \\ |V_0|\cos(\omega t + \beta z + \phi) = \dfrac{|V_0|e^{j\phi}}{2} e^{j(\omega t + \beta z)} + \dfrac{|V_0|e^{-j\phi}}{2} e^{-j(\omega t + \beta z)} \end{cases}$$

$$V_0 = |V_0|e^{j\phi}, \qquad V_c(z,t) = V_0 e^{\pm j\beta z} e^{j\omega t}, \qquad V_s(z) = V_0 e^{\pm j\beta z}$$

$$\rightarrow V_{f/b}(z,t) = |V_0|\cos(\omega t \pm \beta z + \phi) = \mathrm{Re}\left\{V_s(z)e^{j\omega t}\right\} = \frac{V_s(z)e^{j\omega t} + V_s^*(z)e^{-j\omega t}}{2}$$

Transmission Lines

Transmission Line Equations & Their Solutions in Phasor Form (1)

$$\left.\begin{aligned} \frac{\partial^2 V}{\partial z^2} &= LC\frac{\partial^2 V}{\partial t^2} + (LG+RC)\frac{\partial V}{\partial t} + RGV \\ V &= V_s(z)e^{j\omega t} \end{aligned}\right\}$$

$$\rightarrow \frac{d^2 V_s}{dz^2}e^{j\omega t} = (j\omega)^2 LCV_s e^{j\omega t} + j\omega(LG+RC)V_s e^{j\omega t} + RGV_s e^{j\omega t}$$

$$\rightarrow \frac{d^2 V_s}{dz^2} = -\omega^2 LCV_s + j\omega(LG+RC)V_s + RGV_s$$

$$= \underbrace{(R+j\omega L)}_{Z}\underbrace{(G+j\omega C)}_{Y}V_s = \gamma^2 V_s$$

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{ZY} = \alpha + j\beta$$

$$\begin{cases} V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \\ I_s(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \end{cases}$$

Transmission Line Equations & Their Solutions in Phasor Form (2)

$$\left.\begin{matrix}\begin{cases}-\dfrac{\partial V}{\partial z} = RI + L\dfrac{\partial I}{\partial t} \\ -\dfrac{\partial I}{\partial z} = GV + C\dfrac{\partial V}{\partial t}\end{cases} \\ V = V_s(z)e^{j\omega t}, \quad I = I_s(z)e^{j\omega t}\end{matrix}\right\}$$

$$\rightarrow \left.\begin{matrix}\begin{cases}-\dfrac{dV_s}{dz} = (R + j\omega L)I_s = ZI_s \\ -\dfrac{dI_s}{dz} = (R + j\omega L)V_s = YV_s\end{cases} \\ \begin{cases}V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \\ I_s(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}\end{cases}\end{matrix}\right\}$$

$$\rightarrow \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{Z}{\gamma} = \frac{Z}{\sqrt{ZY}} = \sqrt{\frac{Z}{Y}} = \boxed{\sqrt{\frac{R + j\omega L}{G + j\omega C}} = Z_0 = |Z_0|e^{j\theta}}$$

Transmission Line Equations & Their Solutions in Phasor Form (3)

Ex.

A lossless transmission line is 100 cm long & operates at a frequency of 600 MHz. The line parameters are $L = 0.25$ μH/m & $C = 100$ pF/m. Find the characteristic impedance, the phase constant, & the phase velocity?

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.25\times10^{-6}}{100\times10^{-12}}} = 50\,\Omega$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = j\omega\sqrt{LC} = \alpha + j\beta$$

$$\rightarrow \beta = \omega\sqrt{LC} = 2\pi\times600\times10^6\sqrt{(0.25\times10^{-6})(100\times10^{-12})} = 18.85 \text{ rad/m}$$

$$v_p = \frac{\omega}{\beta} = \frac{2\pi\times600\times10^6}{18.85} = 2\times10^8 \text{ m/s}$$

Transmission Line Equations & Their Solutions in Phasor Form (4)

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{ZY} = \alpha + j\beta$$

$$\left.\begin{aligned} &= j\omega\sqrt{LC}\sqrt{1+\frac{R}{j\omega L}}\sqrt{1+\frac{G}{j\omega C}} \\ &\sqrt{1+x} \approx 1+\frac{x}{2}-\frac{x^2}{8} \quad (x \ll 1) \end{aligned}\right\}$$

$$\rightarrow \gamma \approx j\omega\sqrt{LC}\left(1+\frac{R}{j2\omega L}+\frac{R^2}{8\omega^2 L^2}\right)\left(1+\frac{G}{j2\omega C}+\frac{G^2}{8\omega^2 C^2}\right) \quad (R \ll \omega L, G \ll \omega C)$$

$$\approx j\omega\sqrt{LC}\left[1+\frac{1}{j2\omega}\left(\frac{R}{L}+\frac{G}{C}\right)+\frac{1}{8\omega^2}\left(\frac{R^2}{L^2}-\frac{2RG}{LC}+\frac{G^2}{C^2}\right)\right]$$

$$\rightarrow \begin{cases} \alpha \approx \dfrac{1}{2}\left(R\sqrt{\dfrac{C}{L}}+G\sqrt{\dfrac{L}{C}}\right) \\ \beta \approx \omega\sqrt{LC}\left[1+\dfrac{1}{8}\left(\dfrac{G}{\omega C}-\dfrac{R}{\omega L}\right)^2\right] \end{cases}$$

Transmission Line Equations & Their Solutions in Phasor Form (5)

$$R \ll \omega L, G \ll \omega C \rightarrow \begin{cases} \gamma \approx j\omega\sqrt{LC}\left[1+\dfrac{1}{j2\omega}\left(\dfrac{R}{L}+\dfrac{G}{C}\right)+\dfrac{1}{8\omega^2}\left(\dfrac{R^2}{L^2}-\dfrac{2RG}{LC}+\dfrac{G^2}{C^2}\right)\right] \\ \alpha \approx \dfrac{1}{2}\left(R\sqrt{\dfrac{C}{L}}+G\sqrt{\dfrac{L}{C}}\right) \\ \beta \approx \omega\sqrt{LC}\left[1+\dfrac{1}{8}\left(\dfrac{G}{\omega C}-\dfrac{R}{\omega L}\right)^2\right] \\ Z_0 \approx \sqrt{\dfrac{L}{C}}\left\{1+\dfrac{1}{2\omega^2}\left[\dfrac{1}{4}\right]\right\} \end{cases}$$

Transmission Line Equations & Their Solutions in Phasor Form (6)

$$V_s(z) = V_0 e^{-\gamma z} = V_0 e^{-\alpha z} e^{-j\beta z}$$

$$I_s(z) = I_0 e^{-\gamma z} = I_0 e^{-\alpha z} e^{-j\beta z} = \frac{V_0}{Z_0} e^{-\alpha z} e^{-j\beta z}$$

$$P = \frac{1}{2}\mathrm{Re}\left\{V_s I_s^*\right\} = \frac{1}{2}\mathrm{Re}\left\{\left(V_0 e^{-\alpha z} e^{-j\beta z}\right)\left(\frac{V_0^*}{Z_0^*} e^{-\alpha z} e^{j\beta z}\right)\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\left(V_0 e^{-\alpha z} e^{-j\beta z}\right)\left(\frac{V_0^*}{|Z_0| e^{-j\theta}} e^{-\alpha z} e^{j\beta z}\right)\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\frac{V_0 V_0^*}{|Z_0|} e^{-2\alpha z} e^{j\theta}\right\} = \boxed{\frac{1}{2}\frac{|V_0|^2}{|Z_0|} e^{-2\alpha z}\cos\theta}$$

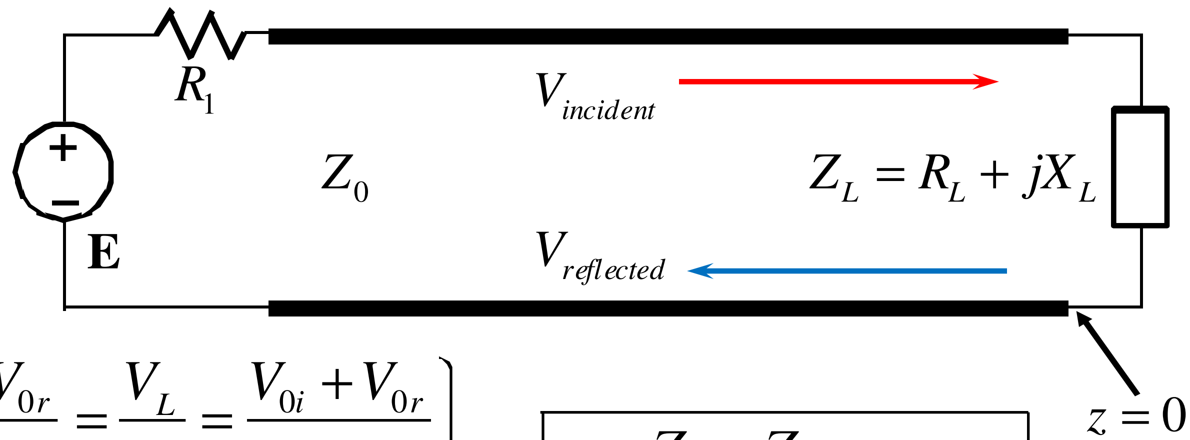
Transmission Lines

Wave Reflection at Discontinuities (1)

$$V_i(z) = V_{0i} e^{-\alpha z} e^{-j\beta z}$$

$$V_r(z) = V_{0r} e^{\alpha z} e^{j\beta z}$$

$$V_L = V_{0i} + V_{0r}$$



$$\left.\begin{array}{l} I_L = I_{0i} + I_{0r} = \dfrac{V_{0i} - V_{0r}}{Z_0} = \dfrac{V_L}{Z_L} = \dfrac{V_{0i} + V_{0r}}{Z_L} \\ \Gamma = \dfrac{V_{0r}}{V_{0i}} \end{array}\right\} \rightarrow \boxed{\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\phi_r}}$$

$$\left.\begin{array}{l} \rightarrow V_L = (1 + \Gamma) V_{0i} \\ \tau = \dfrac{V_L}{V_{0i}} \end{array}\right\}$$

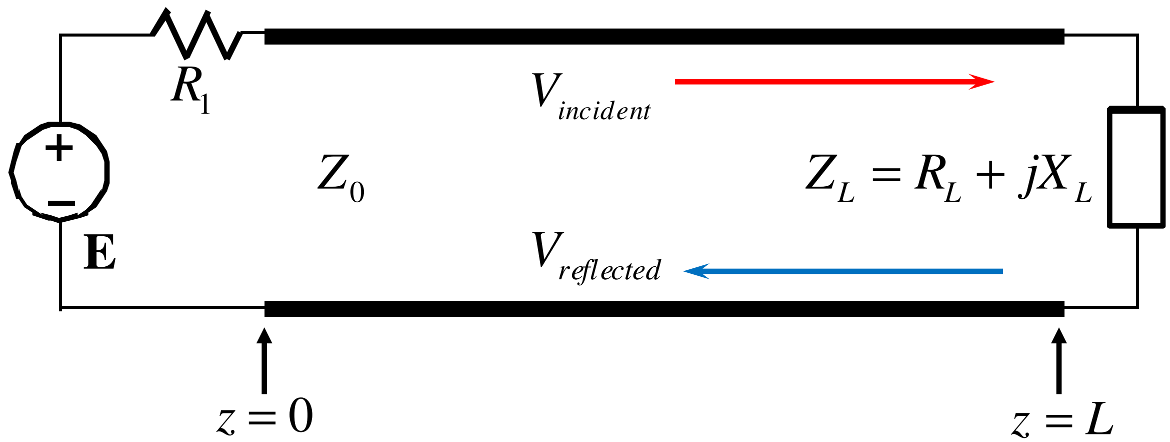
$$\rightarrow \boxed{\tau = 1 + \Gamma = \dfrac{2Z_L}{Z_L + Z_0} = |\tau| e^{j\phi_t}}$$

Wave Reflection at Discontinuities (2)

$$\Gamma = \frac{V_{0r}}{V_{0i}} = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\phi_r}$$

$$P = \frac{1}{2}\mathrm{Re}\left\{\frac{V_0 V_0^*}{|Z_0|} e^{-2\alpha z} e^{j\theta}\right\}$$

$$= \frac{1}{2}\frac{|V_0|^2}{|Z_0|} e^{-2\alpha z}\cos\theta$$



$$P_i = P\big|_{z=L} = \frac{1}{2}\mathrm{Re}\left\{\frac{V_0 V_0^*}{|Z_0|} e^{-2\alpha L} e^{j\theta}\right\} = \frac{1}{2}\frac{|V_0|^2}{|Z_0|} e^{-2\alpha L}\cos\theta$$

$$P_r = P\big|_{z=L} = \frac{1}{2}\mathrm{Re}\left\{\frac{V_0 V_0^*}{|Z_0|} e^{-2\alpha L} e^{j\theta}\right\} = \frac{1}{2}\frac{|V_0|^2}{|Z_0|} e^{-2\alpha L}\cos\theta$$

$$= \frac{1}{2}\mathrm{Re}\left\{\frac{(\Gamma V_0)(\Gamma^* V_0^*)}{|Z_0|} e^{-2\alpha L} e^{j\theta}\right\} = \frac{1}{2}\frac{|\Gamma|^2 |V_0|^2}{|Z_0|} e^{-2\alpha L}\cos\theta$$

$$\rightarrow \boxed{\frac{P_r}{P_i} = \Gamma\Gamma^* = |\Gamma|^2}$$

Wave Reflection at Discontinuities (3)

$$\Gamma = \frac{V_{0r}}{V_{0i}} = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\phi_r}$$

$$V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$V_{sT}(z) = V_0 e^{-j\beta z} + \Gamma V_0 e^{j\beta z}$$

$$= V_0 \left[e^{-j\beta z} + |\Gamma| e^{j(\beta z + \phi)} \right]$$

$$= V_0 e^{j\phi/2} \left[e^{-j\beta z} e^{-j\phi/2} + |\Gamma| e^{j\beta z} e^{j\phi/2} \right]$$

$$= V_0 \left(1 - |\Gamma|\right) e^{-j\beta z} + V_0 |\Gamma| e^{j\phi/2} \left(e^{-j\beta z} e^{-j\phi/2} + e^{j\beta z} e^{j\phi/2} \right)$$

$$= V_0 \left(1 - |\Gamma|\right) e^{-j\beta z} + V_0 |\Gamma| e^{j\phi/2} \cos(\beta z + \phi/2)$$

$$\rightarrow V(z,t) = \mathrm{Re}\left\{ V_{sT}(z) e^{j\omega t} \right\} = \underbrace{V_0 \left(1 - |\Gamma|\right) \cos(\omega t - \beta z)}_{traveling\ wave} + \underbrace{2|\Gamma| V_0 \cos(\beta z + \phi/2) \cos(\omega t + \phi/2)}_{standing\ wave}$$

Wave Reflection at Discontinuities (4)

$$\Gamma = \frac{V_{0r}}{V_{0i}} = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\phi_r}$$

$$V_{sT}(z) = V_0 \left[e^{-j\beta z} + |\Gamma| e^{j(\beta z + \phi)} \right]$$

$$\left.\begin{aligned} & -\beta z - (\beta z + \phi) = (2m+1)\pi \\ & \rightarrow \boxed{z_{\min} = -\frac{\phi + (2m+1)\pi}{2\beta}} \qquad (m = 0, 1, 2, \ldots) \\ & V(z,t) = V_0 \left(1 - |\Gamma|\right) \cos(\omega t - \beta z) + 2|\Gamma| V_0 \cos(\beta z + \phi/2) \cos(\omega t + \phi/2) \end{aligned}\right\}$$

$$\rightarrow V(z,t) = V_0 \left(1 - |\Gamma|\right) \cos\left(\omega t + \beta \frac{\phi + \pi}{2\beta}\right) + 2|\Gamma| V_0 \cos\left(-\beta \frac{\phi + \pi}{2\beta} + \phi/2\right) \cos(\omega t + \phi/2)$$

$$= V_0 \left(1 - |\Gamma|\right) \cos\left(\omega t + \frac{\phi + \pi}{2}\right) + 2|\Gamma| V_0 \cos\left(-\frac{\phi + \pi}{2} + \phi/2\right) \cos(\omega t + \phi/2)$$

$$= -V_0 \left(1 - |\Gamma|\right) \sin(\omega t + \phi/2) + 2|\Gamma| V_0 \cos(\pi/2) \cos(\omega t + \phi/2)$$

$$= -V_0 \left(1 - |\Gamma|\right) \sin(\omega t + \phi/2) \rightarrow \boxed{V(z_{\min}, t) = \pm V_0 \left(1 - |\Gamma|\right) \sin(\omega t + \phi/2)}$$

Wave Reflection at Discontinuities (5)

$$\Gamma = \frac{V_{0r}}{V_{0i}} = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\phi_r}$$

$$V_{sT}(z) = V_0 \left[e^{-j\beta z} + |\Gamma| e^{j(\beta z + \phi)} \right]$$

$$-\beta z - (\beta z + \phi) = 2m\pi$$

$$\rightarrow \boxed{z_{\max} = -\frac{\phi + 2m\pi}{2\beta}} \qquad (m = 0, 1, 2, \ldots)$$

$$V(z,t) = V_0 \left(1 - |\Gamma|\right) \cos(\omega t - \beta z) + 2|\Gamma| V_0 \cos(\beta z + \phi/2) \cos(\omega t + \phi/2)$$

$$\rightarrow V(z,t) = V_0 \left(1 - |\Gamma|\right) \cos\left(\omega t + \beta \frac{\phi}{2\beta} \right) + 2|\Gamma| V_0 \cos\left(-\beta \frac{\phi}{2\beta} + \phi/2 \right) \cos(\omega t + \phi/2)$$

$$= V_0 \left(1 - |\Gamma|\right) \cos\left(\omega t + \frac{\phi}{2} \right) + 2|\Gamma| V_0 \cos\left(-\frac{\phi}{2} + \phi/2 \right) \cos(\omega t + \phi/2)$$

$$= V_0 \left(1 + |\Gamma|\right) \cos(\omega t + \phi/2)$$

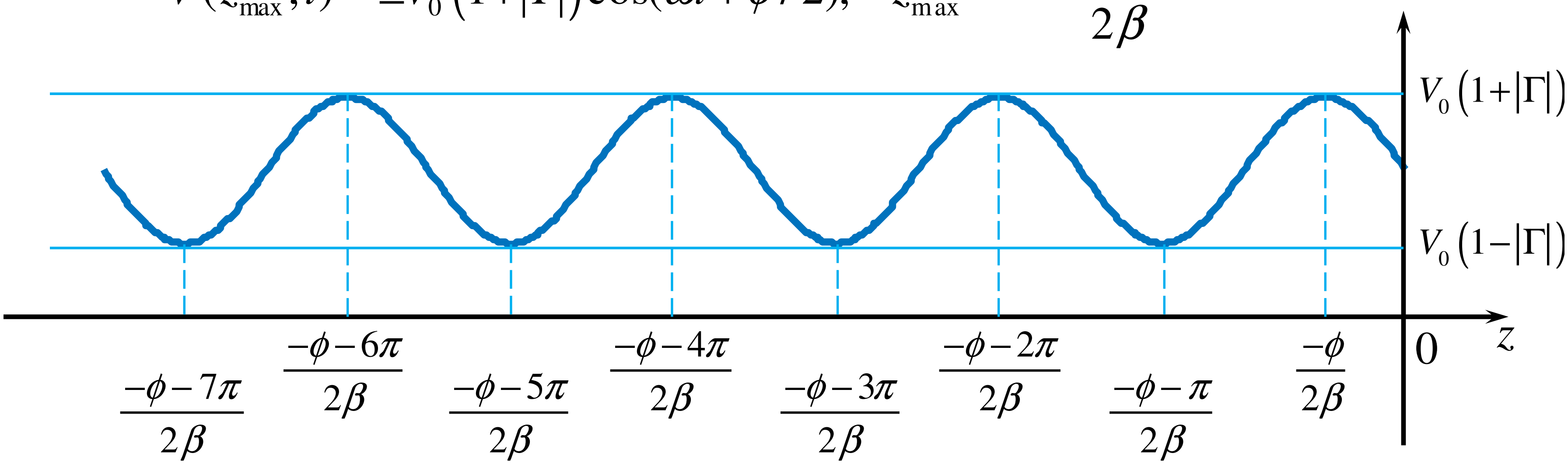
$$\rightarrow \boxed{V(z_{\max}, t) = \pm V_0 \left(1 + |\Gamma|\right) \cos(\omega t + \phi/2)}$$

Wave Reflection at Discontinuities (6)

$$\Gamma = \frac{V_{0r}}{V_{0i}} = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\phi_r}$$

$$V(z_{\min}, t) = \pm V_0 \left(1 - |\Gamma|\right) \sin(\omega t + \phi/2), \quad z_{\min} = -\frac{\phi + (2m+1)\pi}{2\beta}$$

$$V(z_{\max}, t) = \pm V_0 \left(1 + |\Gamma|\right) \cos(\omega t + \phi/2), \quad z_{\max} = -\frac{\phi + 2m\pi}{2\beta}$$



Voltage standing wave ratio (VSWR): $\boxed{s = \frac{V_{sT}(z_{\max})}{V_{sT}(z_{\min})} = \frac{1+|\Gamma|}{1-|\Gamma|}}$

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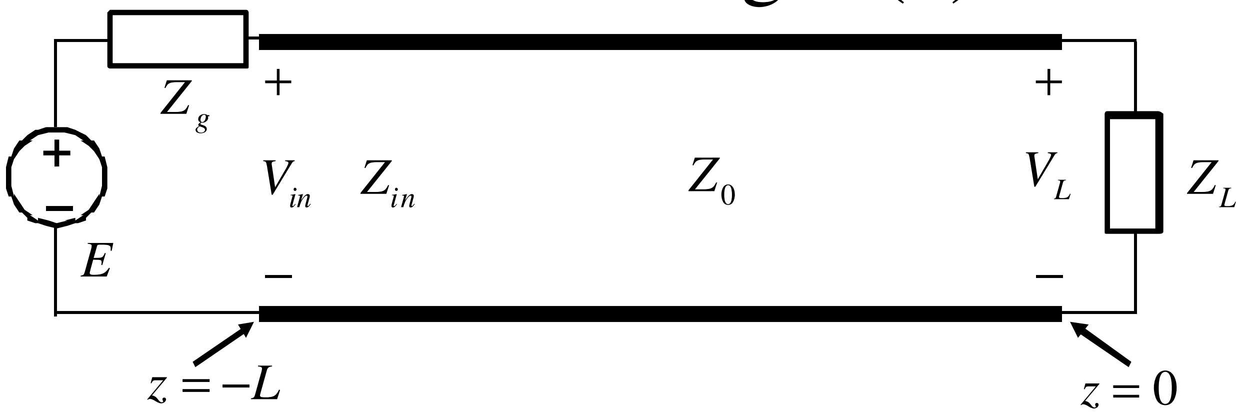
Transmission Lines of Finite Length (1)

$$V_{sT}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$I_{sT}(z) = I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z}$$

$$Z_w(z) = \frac{V_{sT}(z)}{I_{sT}(z)}$$



$$= \frac{V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}}{I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z}}$$

$$V_0^- = \Gamma V_0^+,\ I_0^+ = V_0^+ / Z_0,\ I_0^- = -V_0^- / Z_0$$

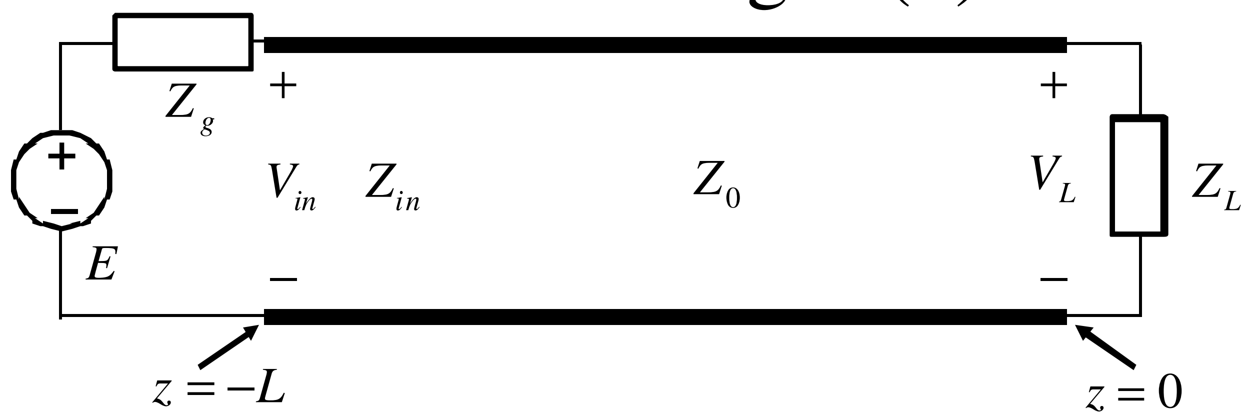
$$\rightarrow Z_w(z) = Z_0 \frac{e^{-j\beta z} + \Gamma e^{j\beta z}}{e^{-j\beta z} - \Gamma e^{j\beta z}}$$

$$\Gamma = \frac{V_{0r}}{V_{0i}} = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\phi_r}$$

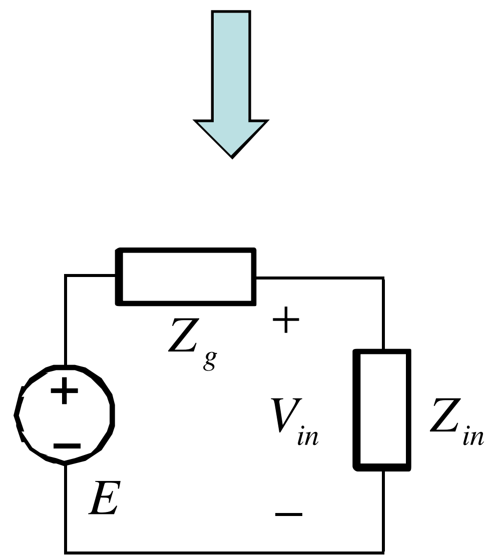
$$e^{j\phi} = \cos\phi + j\sin\phi$$

$$\rightarrow Z_w(z) = Z_0 \frac{Z_L \cos(\beta z) - jZ_0 \sin(\beta z)}{Z_0 \cos(\beta z) - jZ_L \sin(\beta z)}$$

Transmission Lines of Finite Length (2)



$$Z_w(z) = Z_0 \frac{Z_L \cos(\beta z) - jZ_0 \sin(\beta z)}{Z_0 \cos(\beta z) - jZ_L \sin(\beta z)}$$

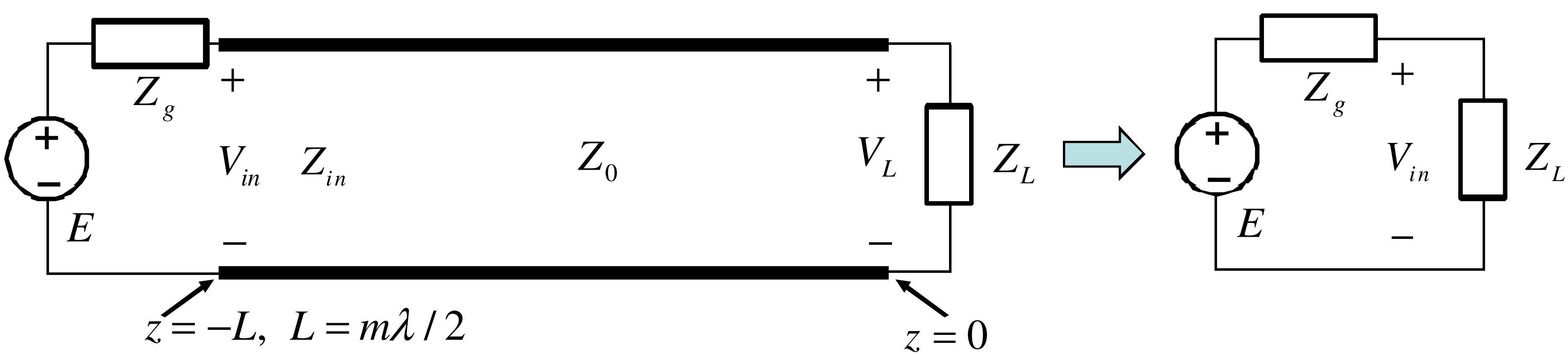


$$\rightarrow Z_{in} = Z_w(z = -L) = \boxed{Z_0 \frac{Z_L \cos(\beta L) + jZ_0 \sin(\beta L)}{Z_0 \cos(\beta L) + jZ_L \sin(\beta L)}}$$

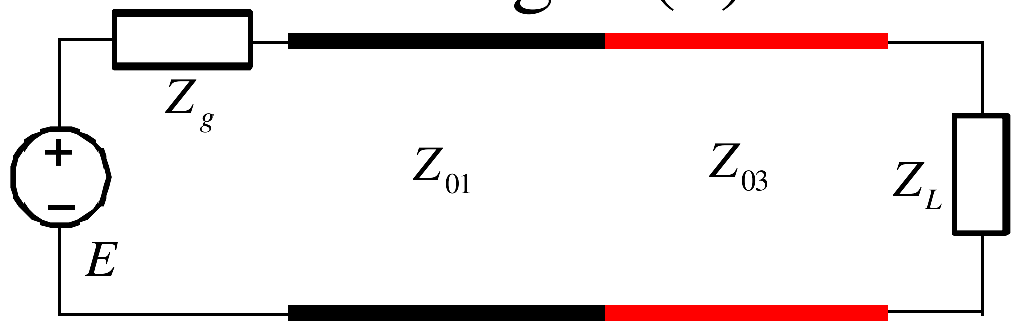
Transmission Lines of Finite Length (3)

$$\left.\begin{array}{l} Z_{in} = Z_0 \dfrac{Z_L\cos(\beta L) + jZ_0\sin(\beta L)}{Z_0\cos(\beta L) + jZ_L\sin(\beta L)} \\[2ex] \left.\begin{array}{l} \beta = \dfrac{2\pi}{\lambda} \\ \boxed{L = \dfrac{m\lambda}{2}} \quad (m = 0, 1, 2, \ldots) \end{array}\right\} \to \beta L = m\pi \end{array}\right\} \to \boxed{Z_{in}(L = m\lambda/2) = Z_L}$$

$$\boxed{Z_{in}(L = \lambda/4) = \frac{Z_0^2}{Z_L}}$$



Transmission Lines of Finite Length (4)

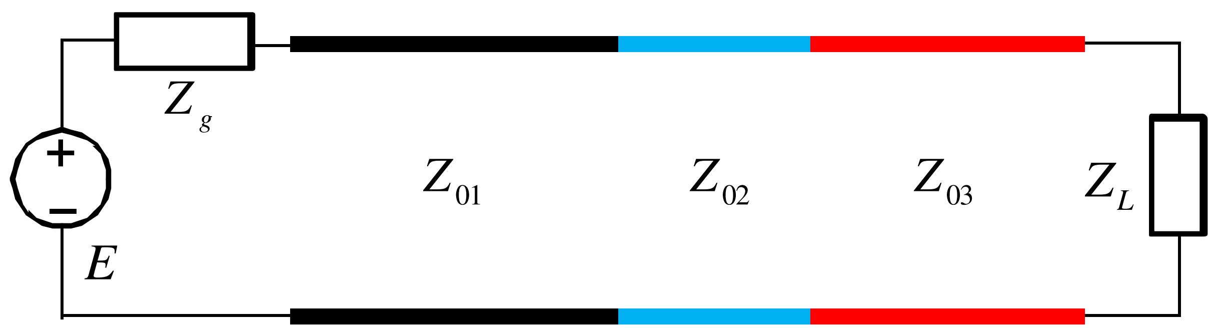


$$\Gamma_{01-03} = \frac{Z_{03} - Z_{01}}{Z_{03} + Z_{01}}$$

$$\left.\begin{aligned} Z_{in02} &= Z_{02}\frac{Z_{03}\cos(\beta_2 L_2) + jZ_{02}\sin(\beta_2 L_2)}{Z_{02}\cos(\beta_2 L_2) + jZ_{03}\sin(\beta_2 L_2)} \\ L_2 &= \lambda/4 \end{aligned}\right\} \rightarrow Z_{in02} = \frac{Z_{02}^2}{Z_{03}}$$

$$\Gamma_{01-02} = \frac{Z_{in02} - Z_{01}}{Z_{in02} + Z_{01}} = 0 \quad \rightarrow Z_{in02} - Z_{01} = 0 \quad \rightarrow Z_{in02} = Z_{01}$$

$$\rightarrow \boxed{Z_{02} = \sqrt{Z_{01}Z_{03}}}$$



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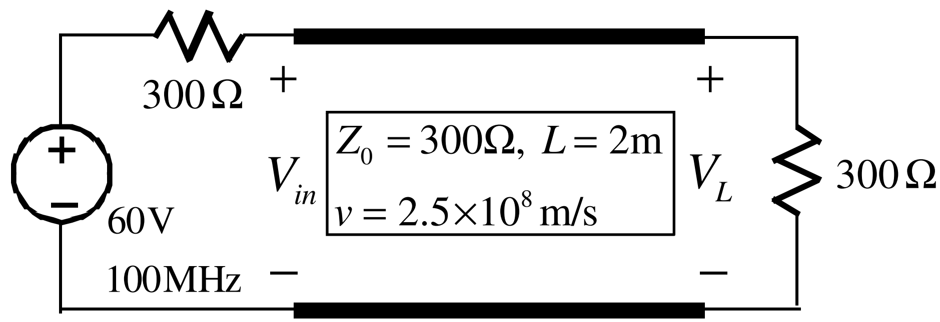
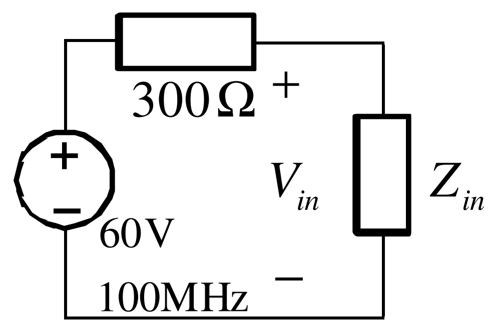
9. Graphical Method
10. Transients Analysis

Some Transmission Line Examples (1)

Ex. 1





$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{300-300}{300+300} = 0$$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0}{1-0} = 1$$

$$\beta = \frac{\omega}{v} = \frac{2\pi\times10^8}{2.5\times10^8} = 0.8\pi \text{ rad/m}$$

$$Z_{in} = Z_0\frac{Z_L\cos(\beta L) + jZ_0\sin(\beta L)}{Z_0\cos(\beta L) + jZ_L\sin(\beta L)} = Z_0 = 300\,\Omega$$

$$V_{in} = \frac{E}{300+300}300 = 30\cos(2\pi10^8 t)\text{ V}$$

$$V_L = 30\cos(2\pi10^8 t - \beta L) = 30\cos(2\pi10^8 t - 1.6\pi)\text{ V}$$

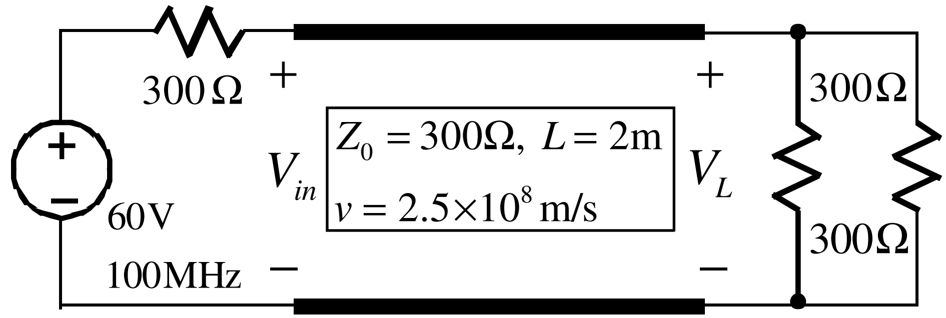
$$I_{in} = \frac{V_{in}}{300} = \frac{30\cos(2\pi10^8 t)}{300} = 0.1\cos(2\pi10^8 t)\text{ A}$$

$$I_L = 0.1\cos(2\pi10^8 t - 1.6\pi)\text{ A}$$

$$P_{in} = P_L = \frac{1}{2}30\times0.1 = 1.5\text{ W}$$

Some Transmission Line Examples (2)

Ex. 2



$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{150-300}{150+300} = -0.33$$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.33}{1-0.33} = 2$$

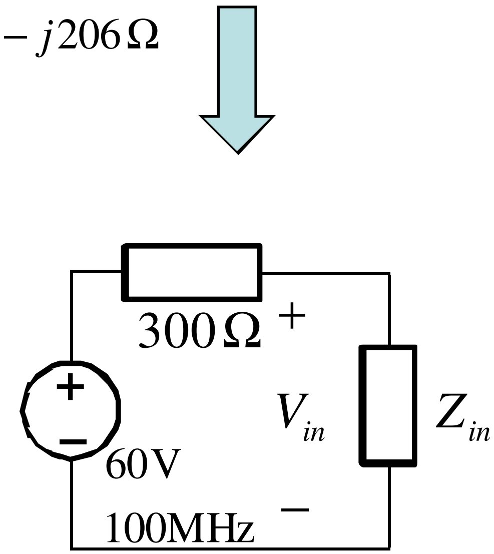
$$Z_{in} = Z_0 \frac{Z_L \cos(\beta L) + jZ_0 \sin(\beta L)}{Z_0 \cos(\beta L) + jZ_L \sin(\beta L)} = 300\frac{150\cos(1.6\pi) + j300\sin(1.6\pi)}{300\cos(1.6\pi) + j150\sin(1.6\pi)} = 466 - j206\,\Omega$$

$$I_{s,in} = \frac{60}{300 + (466 - j206)} = 0.0756\angle 15^\circ \text{ A}$$

$$V_{s,in} = Z_{in} I_{s,in} = 38.54\angle -8.8^\circ \text{ V}$$

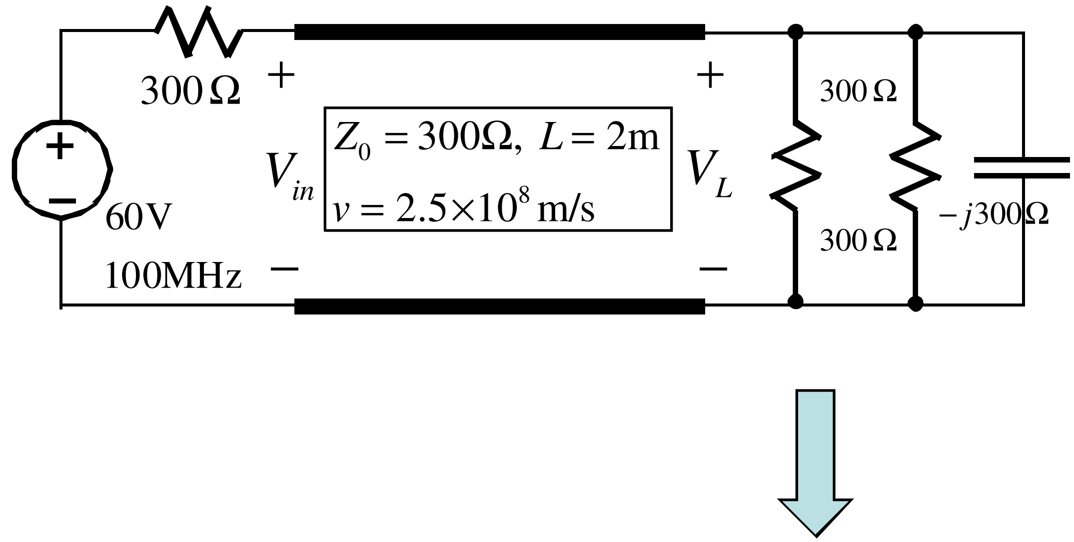
$$P_{in} = \frac{1}{2} R_{in} |I_{s,in}|^2 = \frac{1}{2}\times 466 \times 0.0756^2 = 1.333\text{W} = P_L$$

$$P_L = \frac{1}{2}\frac{|V_{s,L}|^2}{Z_L} \to |V_{s,L}| = \sqrt{2P_L Z_L} = \sqrt{2\times1.333\times150} = 20\,\text{V}$$



Some Transmission Line Examples (3)

Ex. 3



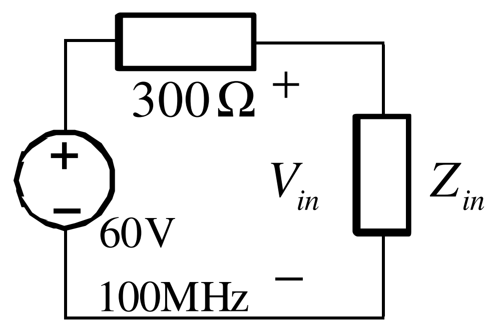
$$Z_L = \frac{150(-j300)}{150 - j300} = 120 - j60\,\Omega$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{120 - j60 - 300}{120 - j60 + 300}$$

$$= 0.447\angle -153.4^\circ$$

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.447}{1 - 0.447} = 2.62$$

$$Z_{in} = 300\frac{(120 - j60)\cos(1.6\pi) + j300\sin(1.6\pi)}{300\cos(1.6\pi) + j(120 - j60)\sin(1.6\pi)} = 775 - j138.5\,\Omega$$

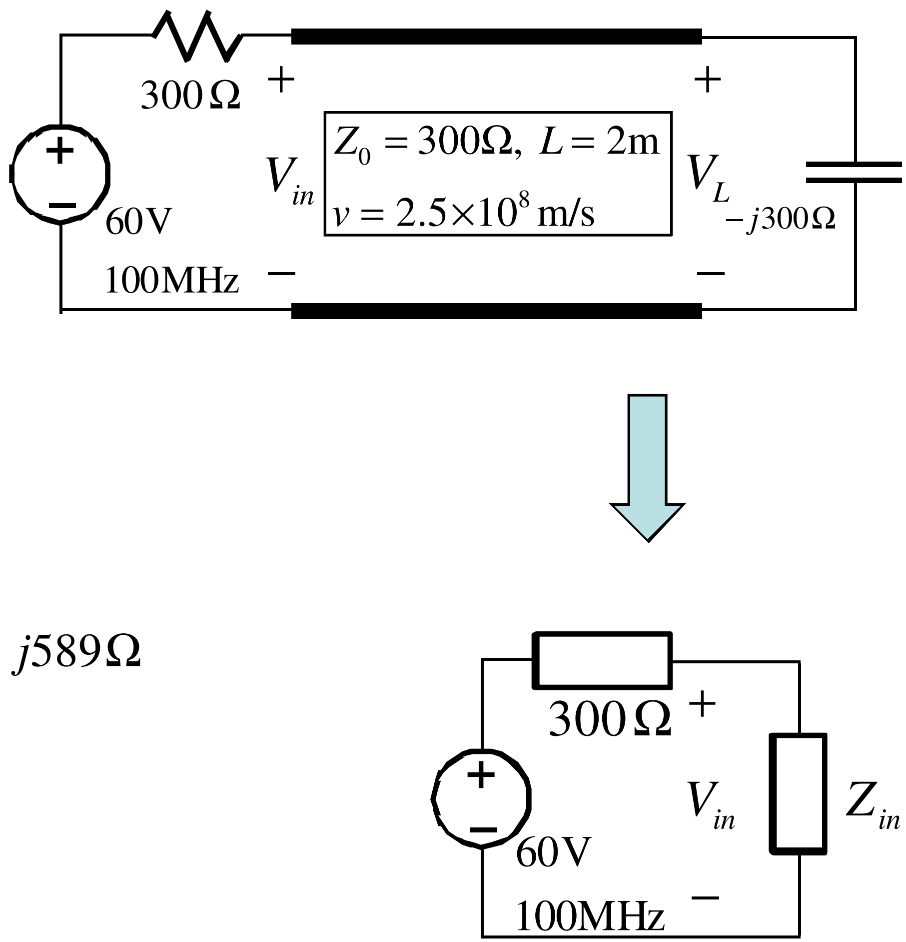


$$I_{s,in} = \frac{60}{300 + 755 - j138.5} = 0.0564\angle 7.47^\circ \text{ A}$$

$$P_{in} = \frac{1}{2}R_{in}\left|I_{s,in}\right|^2 = \frac{1}{2}\times 755\times 0.0564^2 = 1.2\,\text{W} = P_L$$

Some Transmission Line Examples (4)

Ex. 4



$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{-j300-300}{-j300+300} = 1\angle -90^{\circ}$$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+1}{1-1} = \infty$$

$$Z_{in} = 300\frac{-j300\cos(1.6\pi)+j300\sin(1.6\pi)}{300\cos(1.6\pi)+j(-j300)\sin(1.6\pi)} = j589\,\Omega$$

Transmission Lines

Graphical Method (1)

$$\left.\begin{aligned}
&\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \;\rightarrow Z_L = Z_0 \frac{1+\Gamma}{1-\Gamma} \\
&\frac{Z_L}{Z_0} = z_L \quad \text{(normalized load impedance)}
\end{aligned}\right\} \rightarrow z_L = \frac{1+\Gamma}{1-\Gamma}$$

$$\rightarrow \mathrm{Re}\{z_L\} + j\,\mathrm{Im}\{z_L\} = \frac{1 + \left[\mathrm{Re}\{\Gamma\} + j\,\mathrm{Im}\{\Gamma\}\right]}{1 - \left[\mathrm{Re}\{\Gamma\} - j\,\mathrm{Im}\{\Gamma\}\right]}$$

$$= \frac{1 - \mathrm{Re}^2\{\Gamma\} - \mathrm{Im}^2\{\Gamma\} + j2\,\mathrm{Im}\{\Gamma\}}{\left[1 - \mathrm{Re}\{\Gamma\}\right]^2 + \mathrm{Im}^2\{\Gamma\}}$$

Graphical Method (2)

$$\mathrm{Re}\{z_L\} + j\,\mathrm{Im}\{z_L\} = \frac{1-\mathrm{Re}^2\{\Gamma\}-\mathrm{Im}^2\{\Gamma\}+j2\,\mathrm{Im}\{\Gamma\}}{[1-\mathrm{Re}\{\Gamma\}]^2+\mathrm{Im}^2\{\Gamma\}}$$

$$\mathrm{Re}\{z_L\} = \frac{1-\mathrm{Re}^2\{\Gamma\}-\mathrm{Im}^2\{\Gamma\}}{[1-\mathrm{Re}\{\Gamma\}]^2+\mathrm{Im}^2\{\Gamma\}}$$

$$\rightarrow \mathrm{Re}\{z_L\}[\mathrm{Re}\{\Gamma\}-1]^2 + \left[\mathrm{Re}^2(\{\Gamma\}-1\right] +$$

(= 0)

$$+\mathrm{Re}\{\Gamma\}\,\mathrm{Im}^2\{\Gamma\}+\mathrm{Im}^2\{\Gamma\} + \boxed{\frac{1}{1+\mathrm{Re}\{z_L\}} - \frac{1}{1+\mathrm{Re}\{z_L\}}} = 0$$

$$\rightarrow \left(\mathrm{Re}\{\Gamma\} - \frac{\mathrm{Re}\{z_L\}}{1+\mathrm{Re}\{z_L\}}\right)^2 + \mathrm{Im}^2\{\Gamma\} = \left(\frac{\mathrm{Re}\{z_L\}}{1+\mathrm{Re}\{z_L\}}\right)^2$$

Graphical Method (3)

$$\mathrm{Re}\{z_L\} + j\,\mathrm{Im}\{z_L\} = \frac{1 - \mathrm{Re}^2\{\Gamma\} - \mathrm{Im}^2\{\Gamma\} + j2\,\mathrm{Im}\{\Gamma\}}{[1 - \mathrm{Re}\{\Gamma\}]^2 + \mathrm{Im}^2\{\Gamma\}}$$

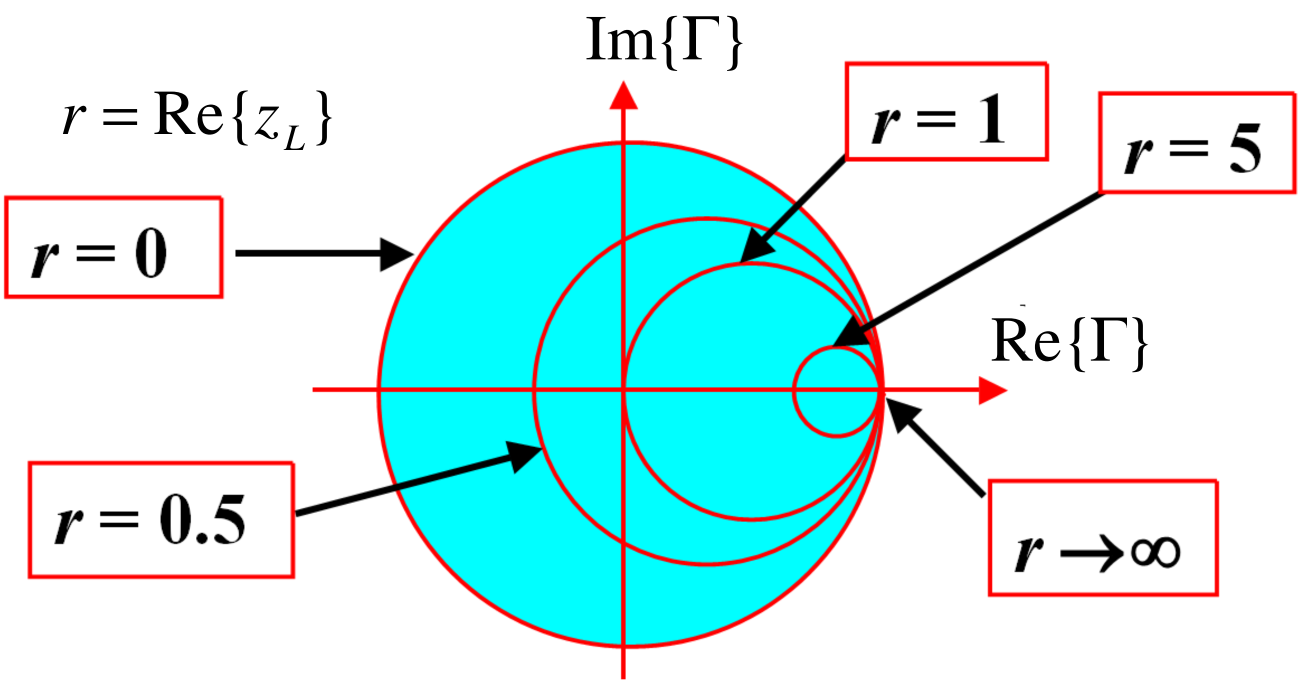
$$\boxed{\left(\mathrm{Re}\{\Gamma\} - \frac{\mathrm{Re}\{z_L\}}{1 + \mathrm{Re}\{z_L\}}\right)^2 + \mathrm{Im}^2\{\Gamma\} = \left(\frac{\mathrm{Re}\{z_L\}}{1 + \mathrm{Re}\{z_L\}}\right)^2}$$

$$\boxed{(\mathrm{Re}\{\Gamma\} - 1)^2 + \left(\mathrm{Im}\{\Gamma\} - \frac{1}{\mathrm{Im}\{z_L\}}\right)^2 = \frac{1}{\mathrm{Im}^2\{z_L\}}}$$

Graphical Method (4)

$$\left(\mathrm{Re}\{\Gamma\} - \frac{\mathrm{Re}\{z_L\}}{1+\mathrm{Re}\{z_L\}}\right)^2 + \mathrm{Im}^2\{\Gamma\} = \left(\frac{\mathrm{Re}\{z_L\}}{1+\mathrm{Re}\{z_L\}}\right)^2$$

Equation of a circle, centered at $\left\{\frac{1}{1+\mathrm{Re}\{z_L\}}, 0\right\}$ & a radius of $\frac{1}{1+\mathrm{Re}\{z_L\}}$

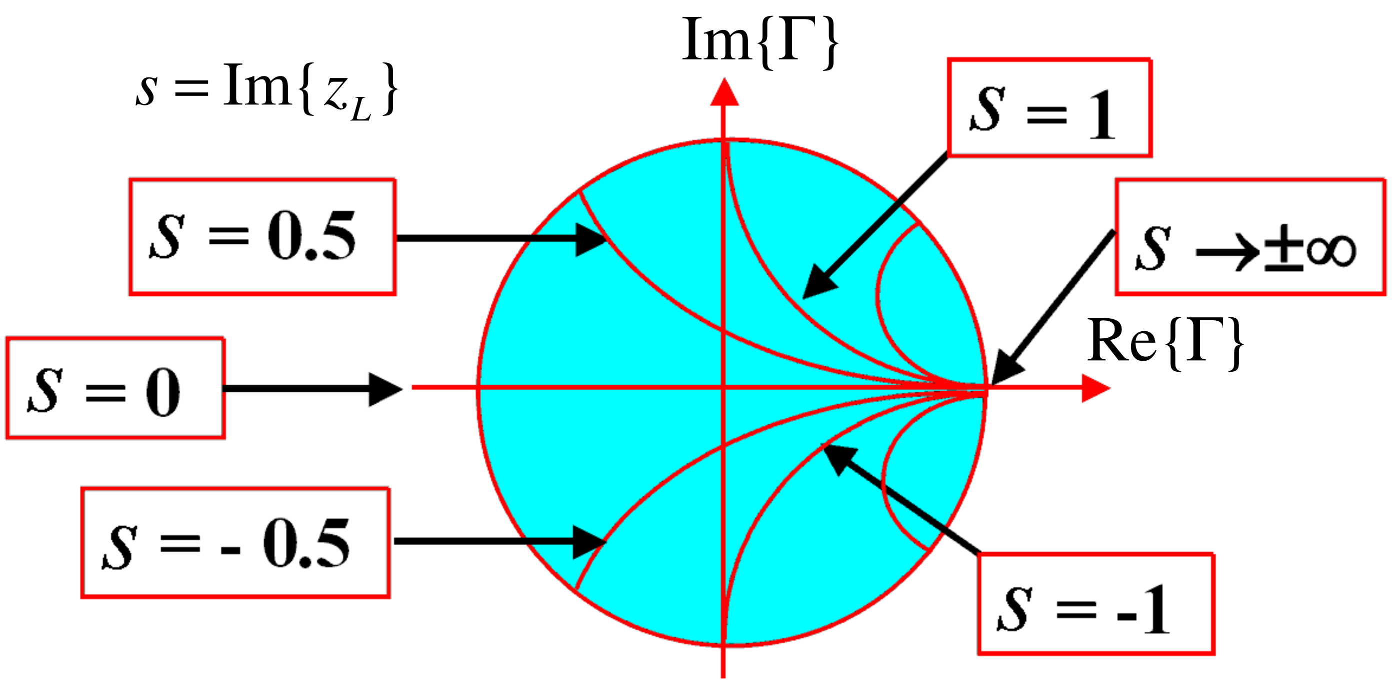


Graphical Method (5)

$$\boxed{\left(\mathrm{Re}\{\Gamma\}-1\right)^2+\left(\mathrm{Im}\{\Gamma\}-\frac{1}{\mathrm{Im}\{z_L\}}\right)^2=\frac{1}{\mathrm{Im}^2\{z_L\}}}$$

Equation of a circle, centered at $\left\{1,\dfrac{1}{\mathrm{Im}\{z_L\}}\right\}$ & a radius of $\dfrac{1}{\mathrm{Im}\{z_L\}}$

$s=\mathrm{Im}\{z_L\}$



Graphical Method (6)

1. Find the normalized load impedance

$$z_L = \frac{Z_L}{Z_0} = \mathrm{Re}\{z_L\} + j\,\mathrm{Im}\{z_L\}$$

2. Find the circle corresponding to $\mathrm{Re}\{z_L\}$
3. Find the arc corresponding to $\mathrm{Im}\{z_L\}$
4. The intersection of the circle & the arc is Γ.

Graphical Method (7)

Ex.: $Z_L = 25 + j100\ \Omega$, $Z_0 = 50\ \Omega$; $\Gamma = ?$

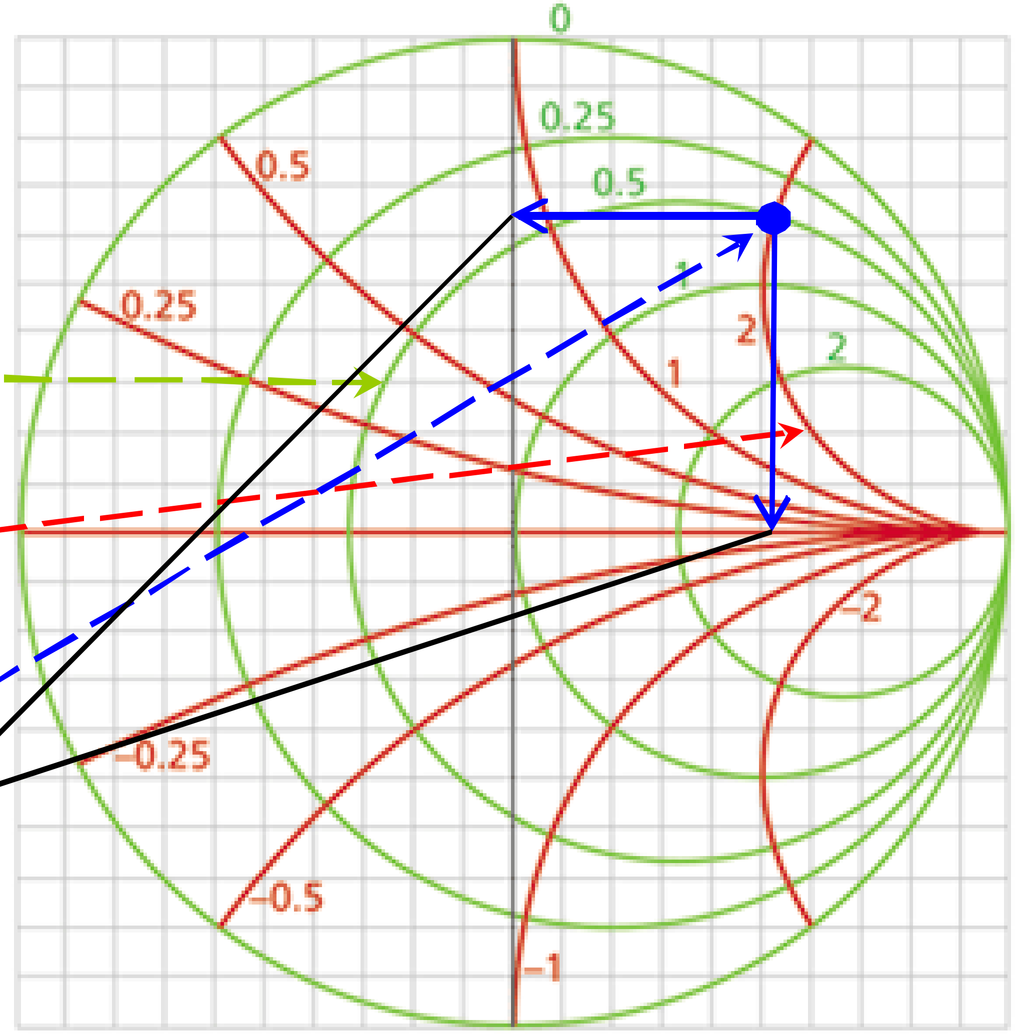
1. Normalization:

$z_L = (25 + j100)/50 = 0.5 + j2$

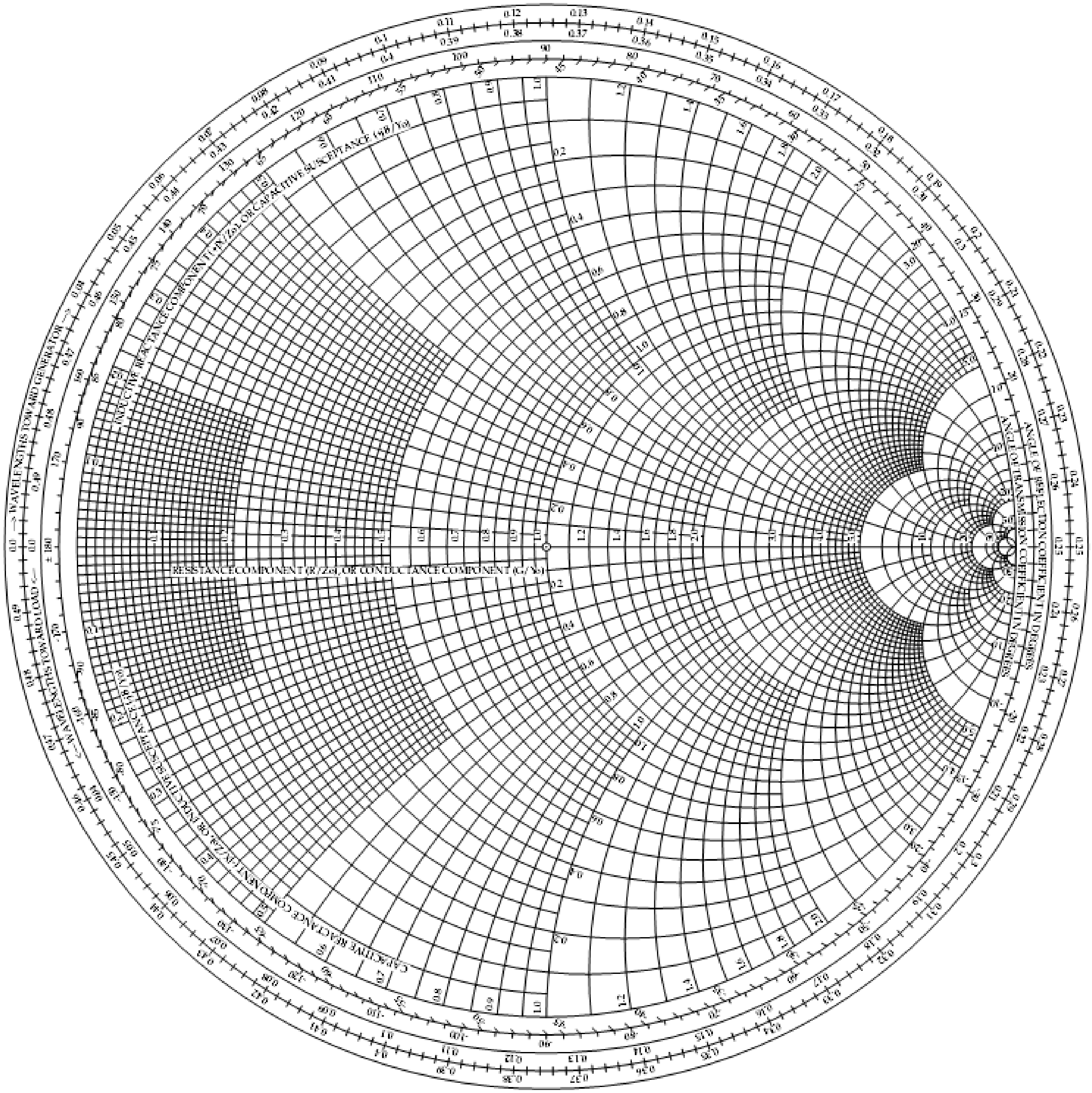
2. The circle corresponds to 0.5

3. The arc corresponds to 2

4. Γ is the intersection of the circle & the arc.



$\Gamma = 0.52 + j0.64$



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Transient Analysis (1)

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{R_L - Z_0}{R_L + Z_0}$$

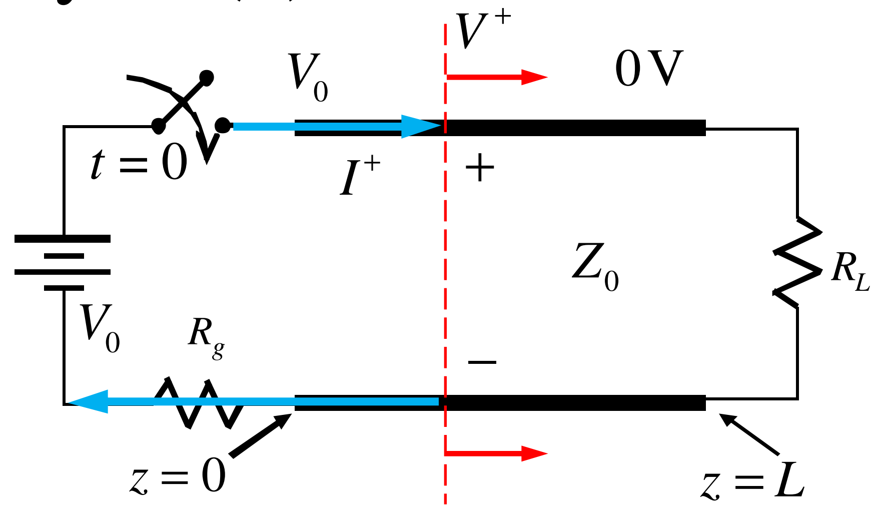
$$R_L = Z_0 \rightarrow \Gamma = 0$$

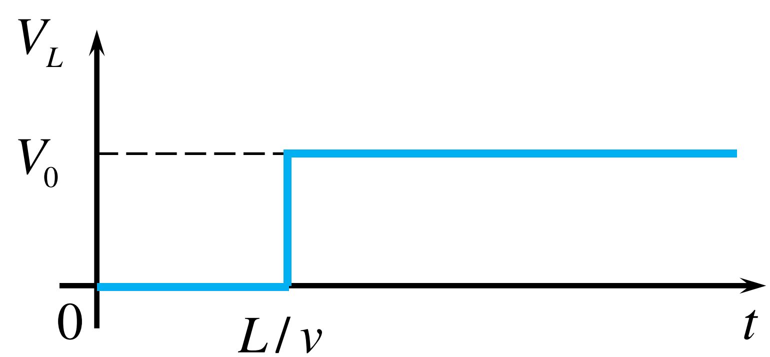
$$R_L = 0 \rightarrow \Gamma = -1$$

$$R_L = \infty \rightarrow \Gamma = 1$$

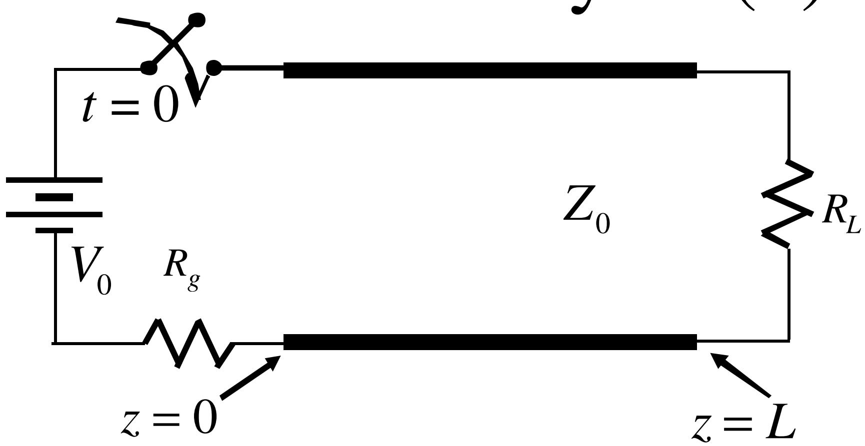
$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}$$

$$\Gamma_g = \frac{R_g - Z_0}{R_g + Z_0}$$





Transient Analysis (2)



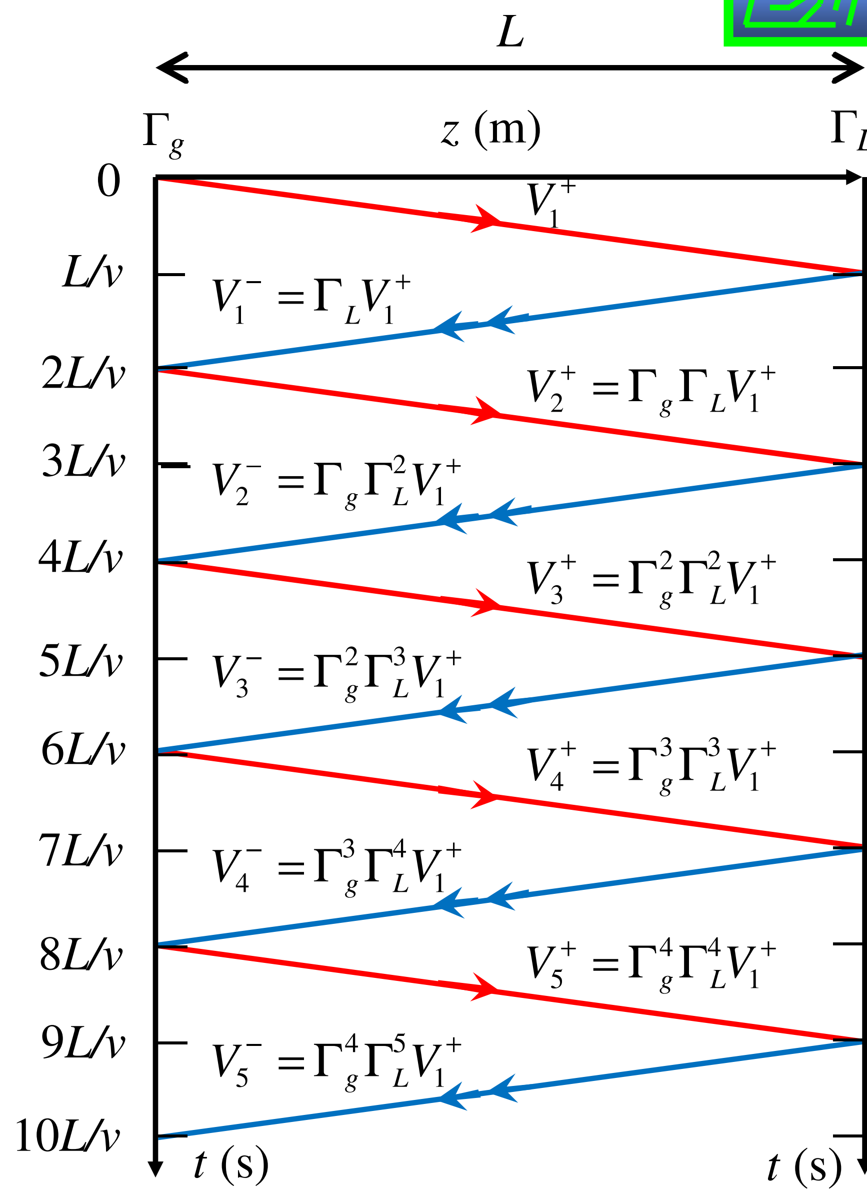
$$V_L = V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- + \ldots$$

$$= V_1^+(1+\Gamma_L + \Gamma_g\Gamma_L + \Gamma_g\Gamma_L^2 + \Gamma_g^2\Gamma_L^2 + \ldots)$$

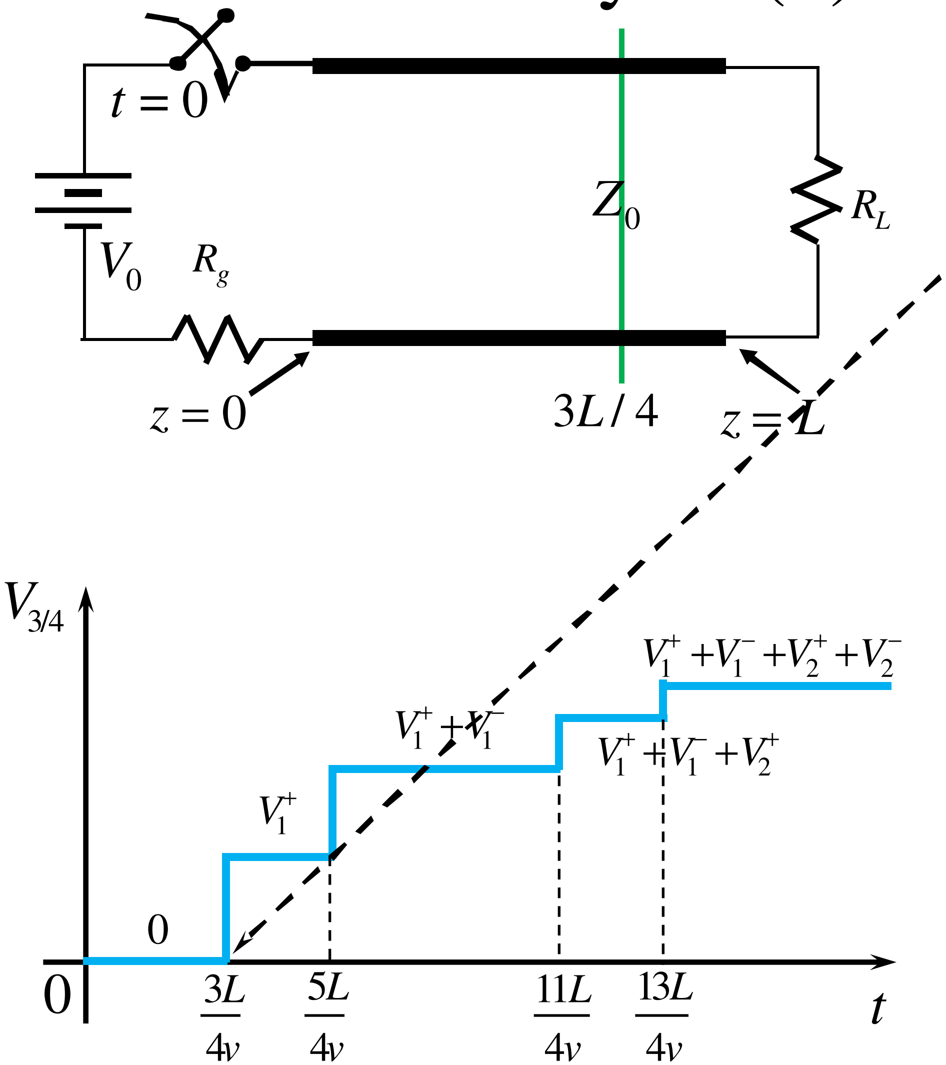
$$= V_1^+(1+\Gamma_L)(1+\Gamma_g\Gamma_L + \Gamma_g^2\Gamma_L^2 + \ldots)$$

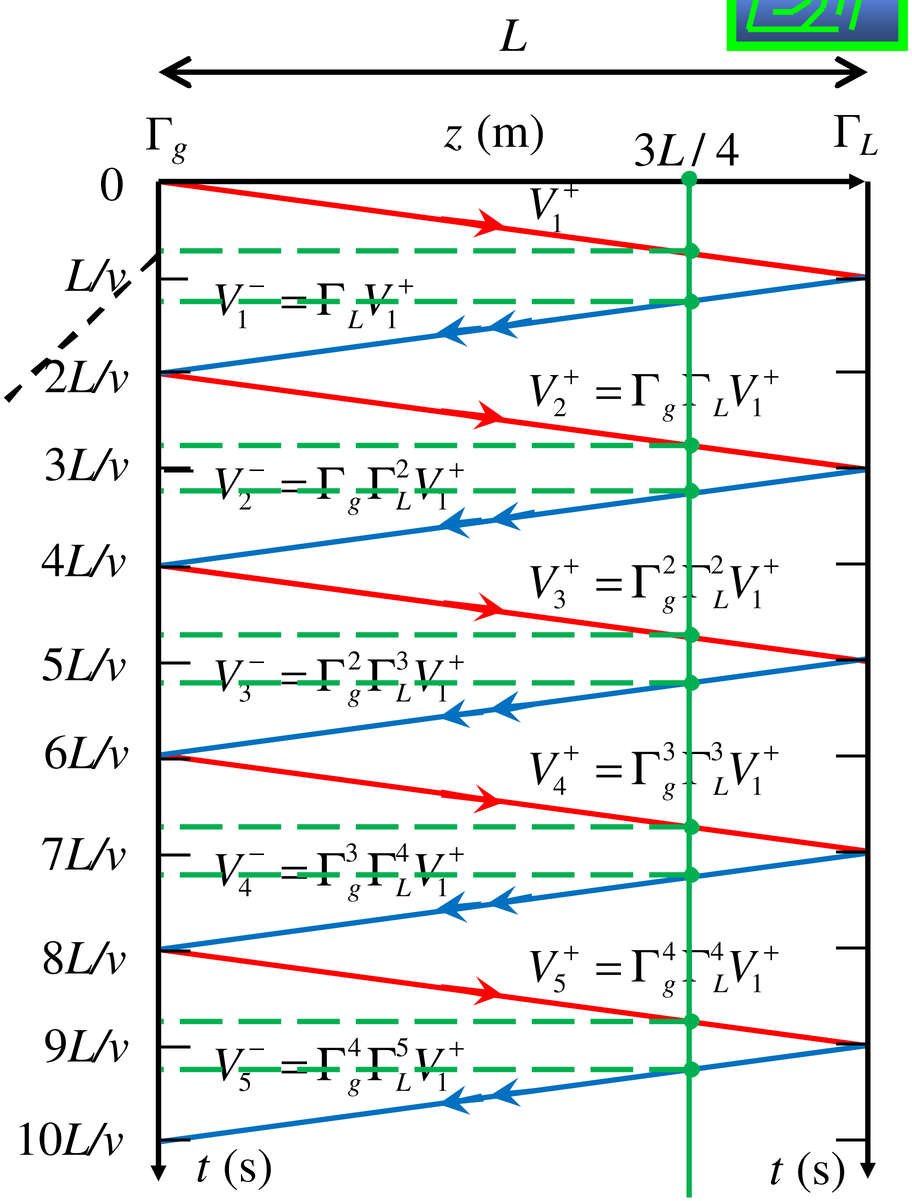
$$= V_1^+(1+\Gamma_L)\frac{1}{1-\Gamma_g\Gamma_L}$$

$$V_1^+ = \frac{V_0 Z_0}{R_g + Z_0}$$

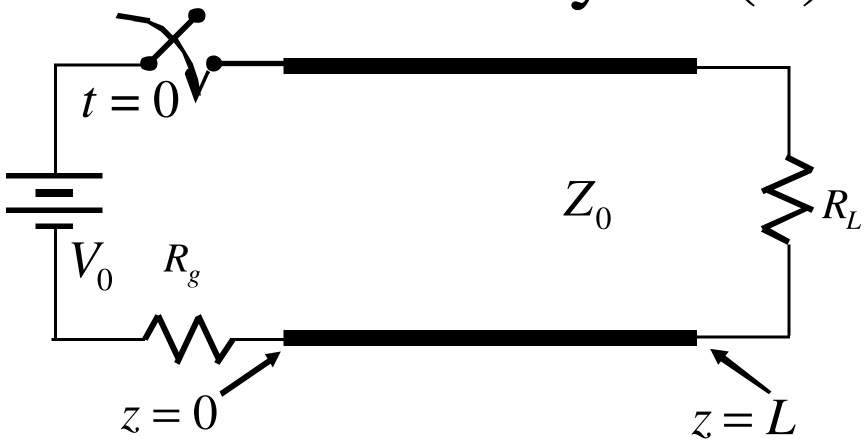


Transient Analysis (3)

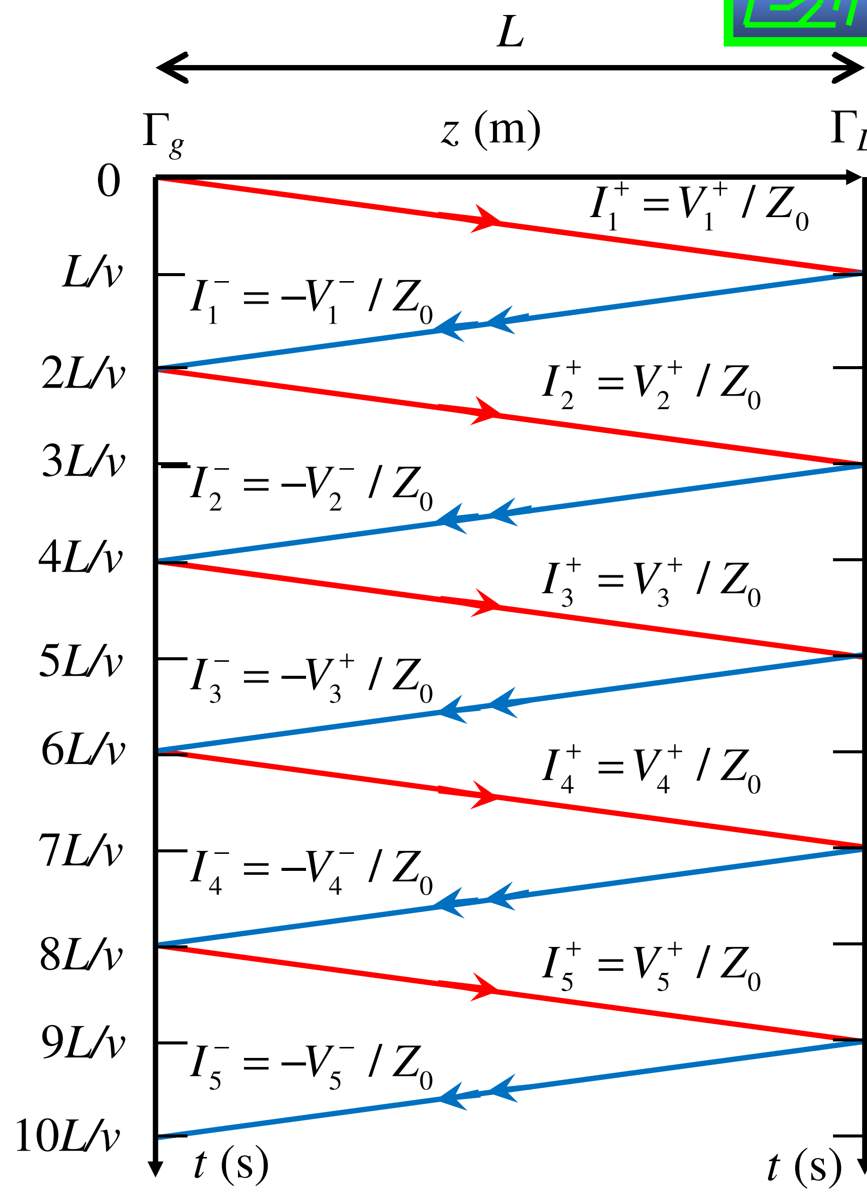




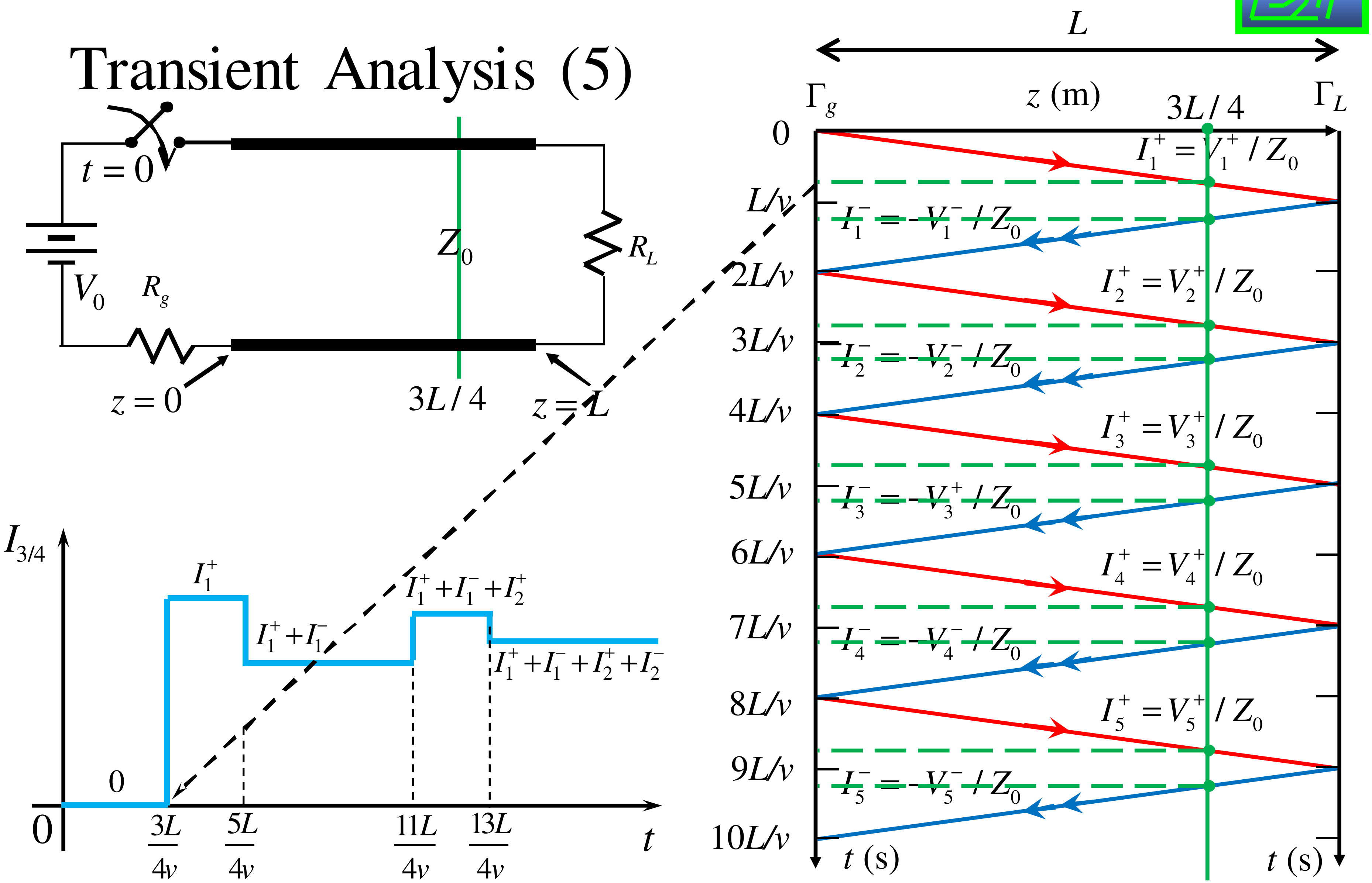
Transient Analysis (4)



$$I_L = I_1^+ + I_1^- + I_2^+ + I_2^- + I_3^+ + I_3^- + \dots$$



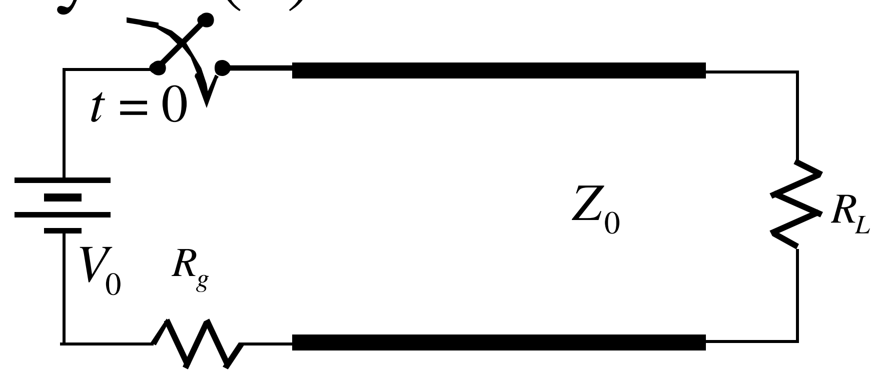
Transient Analysis (5)



Transient Analysis (6)

Ex. 1

$R_g = Z_0 = 50\ \Omega$, $R_L = 25\ \Omega$, $V_0 = 10$ V. The switch is closed at time $t = 0$. Determine the voltage at the load resistor and the current in the battery as functions of time.



$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{25-50}{25+50} = -0.33$$

$$\Gamma_g = \frac{R_g - Z_0}{R_g + Z_0} = \frac{50-50}{50+50} = 0$$

$$V_1^+ = \frac{V_0}{R_g + Z_0} Z_0 = \frac{10}{50+50} 50 = 5\text{V}$$

$$V_1^- = \Gamma_L V_1^+ = (-0.33)5 = -1.67\text{V}$$

$$I_1^+ = \frac{V_1^+}{Z_0} = \frac{5}{50} = 0.1\text{A}$$

$$I_1^- = -\frac{V_1^-}{Z_0} = -\frac{(-1.67)}{50} = 0.033\text{A}$$