Fundamentals of Electric Circuits

DC Circuits

Chapter 3. Methods of Analysis

Methods of Analysis

3.1. Introduction

+ In chapter 2:
    - Geometric configuration of electric circuits (branch, node, loop/mesh)
    - Basic laws: Ohm's law and Kirchhoff's laws → circuit analysis
    - Some circuit transformation rules

+ In this chapter: → develop 2 powerful techniques for circuit analysis based on KCL and KVL
    - Nodal analysis → based on KCL
    - Mesh analysis → based on KVL

+ With the 2 techniques
    - Solve a set of equations to obtain the required values of current or voltage
    - Almost types of electric circuit can be analyzed

3.2. Nodal analysis

+ Using *node voltage as the circuit variables* for analyzing circuits *(node – voltage method)*

+ **Objective**: *reduces the number of equations*

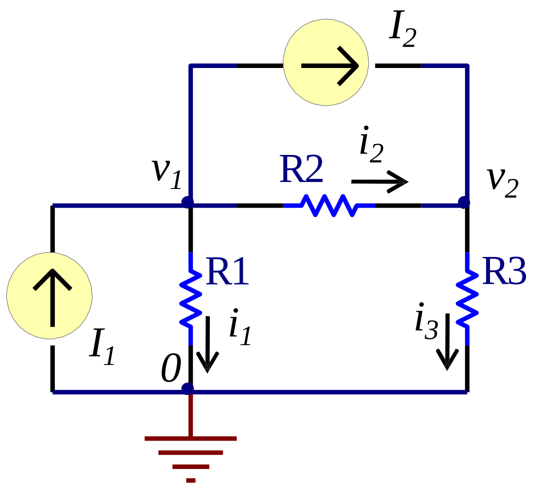
3.2.1. Nodal analysis without voltage source

Assuming that circuits with *n* nodes do not contain voltage source

- ***Select*** *a node as the reference node (ground, v = 0)*
- **Assign** voltages v_1, v_2, … v_{n-1} to the remaining *n-1* nodes → all voltages are referenced to the reference node
- **Apply** *KCL to each of the n-1 non-reference nodes.* Use Ohm's law to *express the branch currents in terms of node voltages*
- **Solve** the resulting simultaneous equations to obtain the unknown node voltages

3.2. Nodal analysis

3.2.1. Nodal analysis without voltage source



Example 1: Find the currents in this circuit

- Choose node 0 as a reference node ($v_0 = 0$), assign voltage of node *1* and node *2* with v_1 and v_2 , i_1, i_2, and i_3 as the currents on R_1, R_2, R_3
- Apply KCL for node 1 and 2: $\begin{cases} I_1 - I_2 = i_1 + i_2 \\ I_2 = i_3 - i_2 \end{cases}$
- Apply Ohm's law to express the currents in term of node voltages

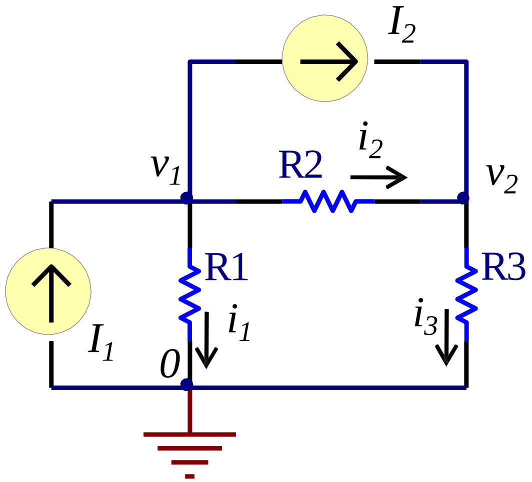
$$i_1 = \frac{v_1 - v_0}{R_1} = G_1 v_1 \qquad i_2 = \frac{v_1 - v_2}{R_2} = G_2(v_1 - v_2) \qquad i_3 = \frac{v_2 - v_0}{R_3} = G_3 v_2$$

$$\rightarrow \begin{cases} I_1 = I_2 + G_1 v_1 + G_2 (v_1 - v_2) \\ I_2 + G_2(v_1 - v_2) = G_3 v_2 \end{cases} \rightarrow \begin{cases} (G_1 + G_2) v_1 - G_2 v_2 = I_1 - I_2 \\ -G_2 v_1 + (G_2 + G_3) v_2 = I_2 \end{cases}$$

3.2. Nodal analysis

3.2.1. Nodal analysis without voltage source

Example 1: Find the currents in this circuit



- Solve this set of equations to obtain the node voltages v_1, v_2

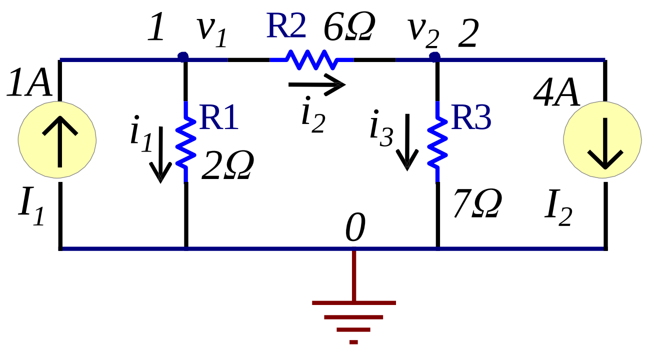
$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$

- Finally, calculate the currents in circuit

$$i_1 = \frac{v_1}{R_1} \qquad i_2 = \frac{v_1 - v_2}{R_2} \qquad i_3 = \frac{v_2}{R_3}$$

3.2. Nodal analysis

3.2.1. Nodal analysis without voltage source



Example 2: Find the currents in this circuit

Choose node 0 as reference node

Apply KCL to each non-reference node (node 1 and 2):

$$I_1 = i_1 + i_2$$
$$i_2 = i_3 + I_2$$

Apply Ohm law to branches

$$i_1 = \frac{v_1}{R_1} = G_1 v_1 \qquad i_2 = \frac{v_1 - v_2}{R_2} = G_2(v_1 - v_2) \qquad i_3 = \frac{v_2}{R_3} = G_3 v_2$$

Obtain set of equations

$$\begin{cases} (G_1 - G_2)v_1 - G_2 v_2 = I_1 \\ -G_2 v_1 + (G_2 + G_3)v_2 = -I_2 \end{cases} \rightarrow \begin{cases} 0{,}667 v_1 - 0{,}167 v_2 = 1 \\ -0{,}167 v_1 + 0{,}31 v_2 = -4 \end{cases} \rightarrow \begin{cases} v_1 = -2V \\ v_2 = -14V \end{cases}$$

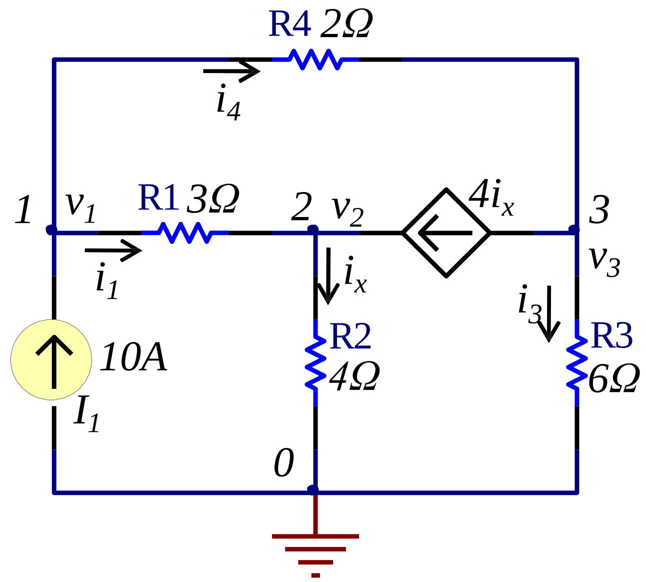
Calculate currents through resistors in circuit

$$i_1 = \frac{v_1}{R_1} = -1A \qquad i_2 = \frac{v_1 - v_2}{R_2} = 2A \qquad i_3 = \frac{v_2}{R_3} = -2A$$

3.2. Nodal analysis

3.2.1. Nodal analysis without voltage source

Example 3: Find the voltages at the three non-reference nodes in this circuit



- Choose node 0 ~ reference node, node *1* ~ v_1, *node 2* ~ v_2, node *3* ~ v_3
- Set of KCL equations for node 1, 2 and 3

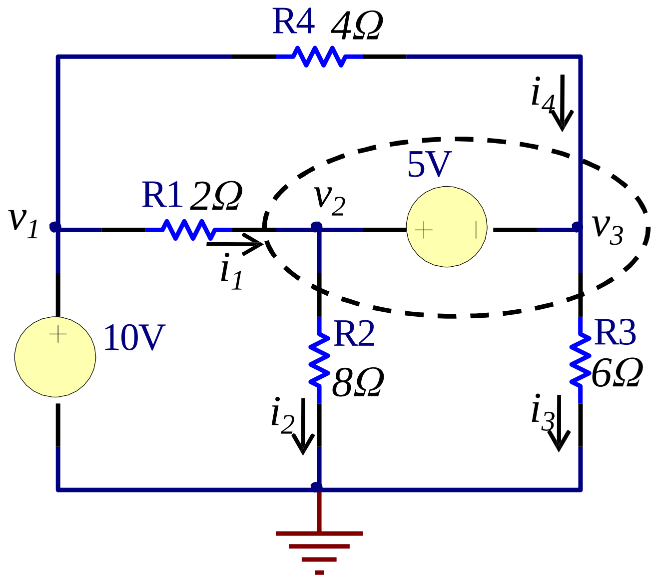
$$\begin{cases} i_1 + i_4 = I_1 \\ i_1 + 4i_x = i_x \\ i_4 = 4i_x + i_3 \end{cases}$$

- Apply Ohm law, we have $i_1 = G_1(v_1 - v_2), i_x = G_2 v_2, i_3 = G_3 v_3, i_4 = G_4(v_1 - v_3)$
- Substitute to set of KCL equations and solve it to obtain v_1, v_2, v_3

$$\begin{cases} (G_1 + G_4)v_1 - G_1 v_2 - G_4 v_3 = I_1 \\ -G_1 v_1 + (G_1 - 3G_2)v_2 = 0 \\ -G_4 v_1 + 4G_2 v_2 + (G_3 + G_4)v_3 = 0 \end{cases} \rightarrow \begin{cases} \frac{5}{6}v_1 - \frac{1}{3}v_2 - \frac{1}{2}v_3 = 10 \\ -\frac{1}{3}v_1 - \frac{5}{12}v_2 = 0 \\ -\frac{1}{2}v_1 + v_2 + \frac{2}{3}v_3 = 0 \end{cases} \rightarrow \begin{cases} v_1 = 80V \\ v_2 = -64V \\ v_3 = 156V \end{cases}$$

3.2. Nodal analysis

3.2.2. Nodal analysis with voltage sources



+ Voltage source connects between reference node and non-reference node:

Voltage of non-reference node = voltage source

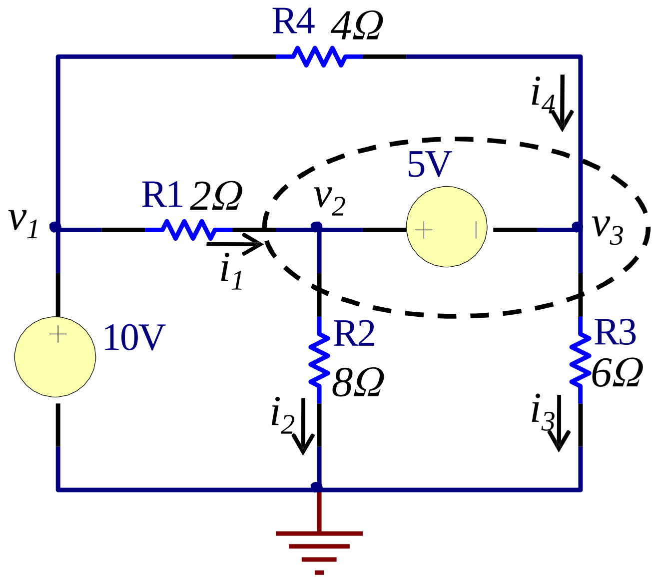
+ Voltage source connects between 2 non-reference nodes
→ form a *super-node*

Super-node *is formed by enclosing a (dependent or independent) voltage source connected between two non-reference nodes and any elements connected in parallel with it*

+ To analyze circuit → applying the same three steps presented in 3.2.1, except the super-node

3.2. Nodal analysis

3.2.2. Nodal analysis with voltage sources



+ For example:

KCL at super-node: $i_1 + i_4 = i_2 + i_3 \rightarrow \dfrac{v_1 - v_2}{R_1} + \dfrac{v_1 - v_3}{R_4} = \dfrac{v_2}{R_2} + \dfrac{v_3}{R_3}$

and: $v_2 - v_3 = 5V$

We have a set of equations and its solution : $\begin{cases} v_1 = 10V \\ v_2 - v_3 = 5V \\ \dfrac{5}{8}v_2 + \dfrac{5}{12}v_3 = 7.5 \end{cases} \rightarrow \begin{cases} v_1 = 10V \\ v_2 = 9.2V \\ v_3 = 4.2V \end{cases}$

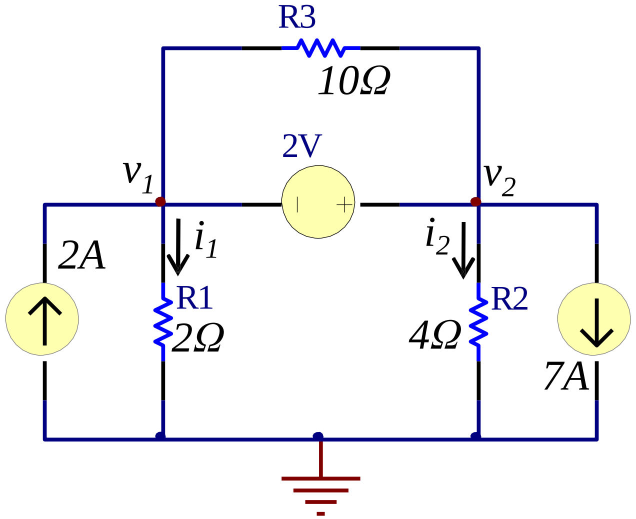
+ Note:

- The voltage source inside the super-node provides a constraint equation needed to solve for the node voltages
- A super-node has no voltage of its own
- A super-node requires the application of both KCL and KVL

3.2. Nodal analysis

3.2.2. Nodal analysis with voltage sources

+ Example 1: find the voltage node in this circuit using nodal analysis



Super-node includes the *2V* source node 1, node *2* and R_3

Apply KCL to the super-node:

$$2 = i_1 + i_2 + 7 \rightarrow 2 = \frac{v_1}{R_1} + \frac{v_2}{R_2} + 7 \leftrightarrow 2v_1 + v_2 = -20$$

Apply KVL to the super-node: $v_2 = v_1 + 2$

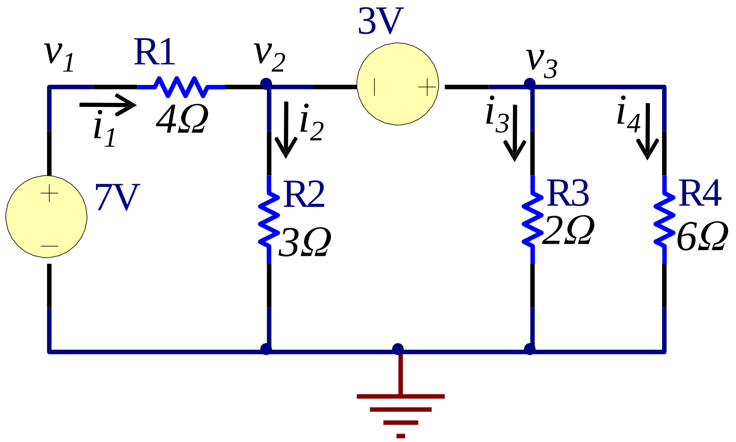
Solve the set of 2 equations:

$$\begin{cases} v_1 = -7.33V \\ v_2 = -5.33V \end{cases}$$

3.2. Nodal analysis

3.2.2. Nodal analysis with voltage sources

+ Example 2: find the voltage nodes and the currents in this circuit using nodal analysis



Super-node includes the *3V* source, node *2,* node *3*

Apply the KCL and KVL to the super-node

$$\begin{cases} \dfrac{v_1 - v_2}{R_1} = \dfrac{v_2}{R_2} + \dfrac{v_3}{R_3} + \dfrac{v_3}{R_4} \\ v_1 = 7 \\ v_3 = v_2 + 3 \end{cases} \rightarrow \begin{cases} v_1 = 7V \\ v_2 = -0.2V \\ v_3 = 2.8V \end{cases}$$

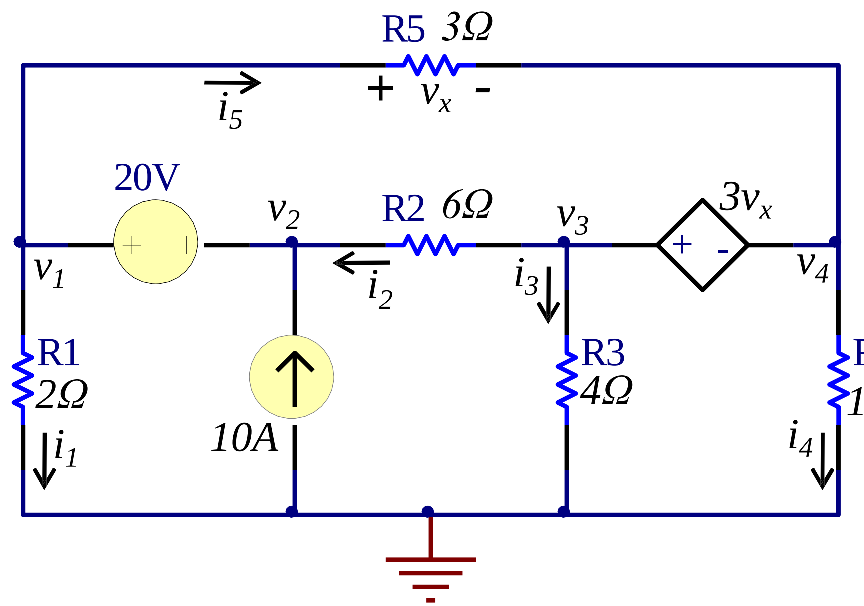
Apply Ohm's law to get the currents $i_1 = \dfrac{v_1 - v_2}{R_1} = 1.8A$ $\quad i_2 = G_2 v_2 = -0.067A$ $\quad i_3 = G_3 v_3 = 1.4A$

$$i_4 = G_4 v_3 = 0.467A$$

3.2. Nodal analysis

3.2.2. Nodal analysis with voltage sources

+ Example 3: find the node voltages in this circuit using nodal analysis



Super-node 1: Node 1 + node 2

$$i_1 + i_5 = i_2 + 10$$

$$\frac{v_1}{2} + \frac{v_1 - v_4}{3} = \frac{v_3 - v_2}{6} + 10 \rightarrow \boxed{\begin{aligned} &5v_1 + v_2 - v_3 - 2v_4 = 60 \\ &v_1 = 20 + v_2 \end{aligned}}$$

$$\rightarrow 6v_1 - v_3 - 2v_4 = 80$$

Super-node 2: Node 3 + node 4

$$i_5 = i_2 + i_3 + i_4 \rightarrow \frac{v_1 - v_4}{3} = \frac{v_3 - v_2}{6} + \frac{v_3}{4} + v_4 \rightarrow 4v_1 + 2v_2 - 5v_3 - 16v_4 = 0$$

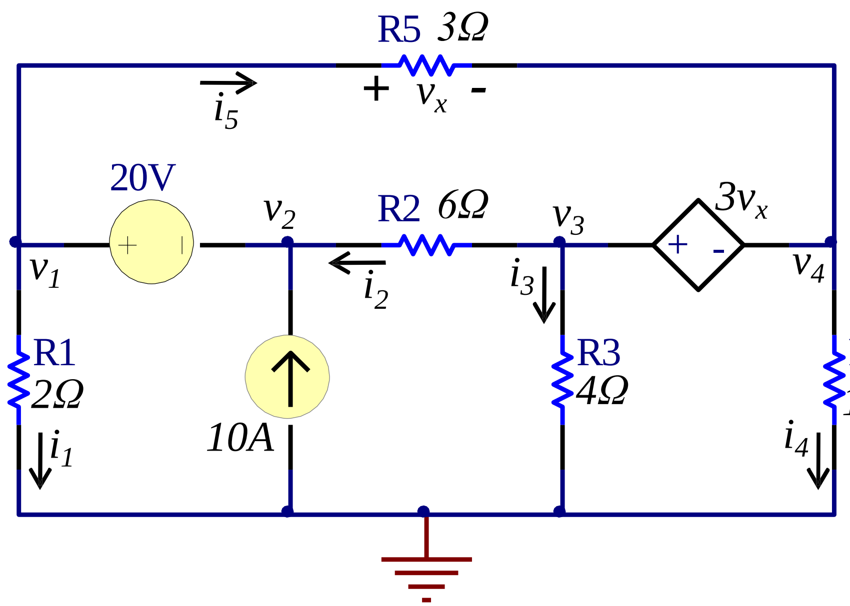
$$v_3 = 3v_x + v_4 = 3(v_1 - v_4) + v_4 \rightarrow 3v_1 - v_3 - 2v_4 = 0$$

$$\boxed{\begin{aligned} &4v_1 + 2v_2 - 5v_3 - 16v_4 = 0 \\ &3v_1 - v_3 - 2v_4 = 0 \end{aligned}} \rightarrow 6v_1 - 5v_3 - 16v_4 = 40$$

3.2. Nodal analysis

3.2.2. Nodal analysis with voltage sources

+ Example 3: find the node voltages in this circuit using nodal analysis



We have a set of equations

$$\begin{cases} 3v_1 - v_3 - 2v_4 = 0 \\ 6v_1 - v_3 - 2v_4 = 80 \\ 6v_1 - 5v_3 - 16v_4 = 40 \end{cases}$$

Using Cramer's rule to calculate node voltages

$$\Delta = \begin{vmatrix} 3 & -1 & -2 \\ 6 & -1 & -2 \\ 6 & -5 & -16 \end{vmatrix} = -18 \qquad \Delta_1 = \begin{vmatrix} 0 & -1 & -2 \\ 80 & -1 & -2 \\ 40 & -5 & -16 \end{vmatrix} = -480 \qquad \Delta_3 = \begin{vmatrix} 3 & 0 & -2 \\ 6 & 80 & -2 \\ 6 & 40 & -16 \end{vmatrix} = -3120$$

$$\Delta_4 = \begin{vmatrix} 3 & -1 & 0 \\ 6 & -1 & 80 \\ 6 & -5 & 40 \end{vmatrix} = 840$$

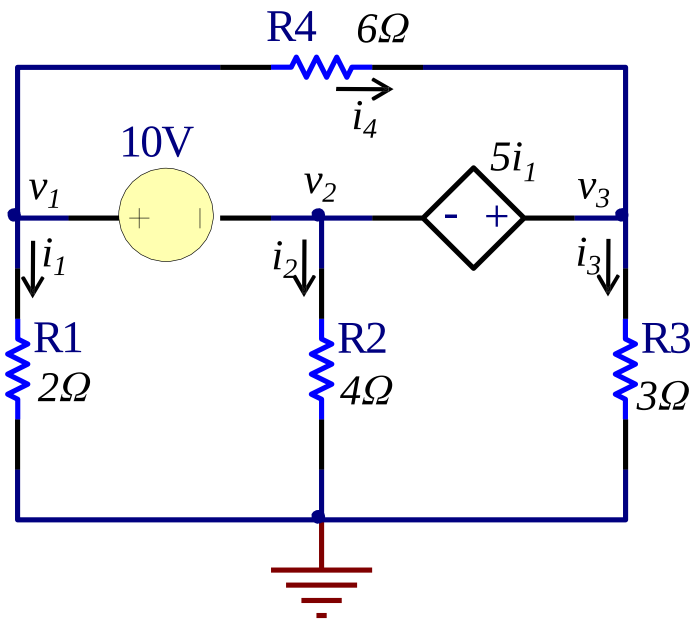
$$v_1 = \frac{\Delta_1}{\Delta} = 26.67V \qquad v_4 = \frac{\Delta_4}{\Delta} = -46.67V$$

$$v_3 = \frac{\Delta_3}{\Delta} = 173.33V \qquad v_2 = v_1 - 20 = 6.67V$$

3.2. Nodal analysis

3.2.2. Nodal analysis with voltage sources

+ Example 4: find the node voltages and the branch currents in this circuit using nodal analysis



Super-node consists of 10V source, *$5i_1$ dependent source, and R4*

We have $$i_1 + i_2 + i_3 = 0 \rightarrow \frac{v_1}{2} + \frac{v_2}{4} + \frac{v_3}{3} = 0$$

$$v_1 - v_2 = 10$$

$$v_3 = 5i_1 + v_2 \quad \rightarrow 5v_1 + 2v_2 - 2v_3 = 0$$

Solve the set of KCL and KVL equations at super node to get node voltages:

$$\begin{cases} v_1 = 3.043V \\ v_2 = -6.956V \\ v_3 = 0.652V \end{cases}$$

And branch currents: $$i_1 = \frac{v_1}{R_1} = 1.522A; i_2 = \frac{v_2}{R_2} = -1.739A; i_3 = \frac{v_3}{R_3} = 0.217A; i_2 = \frac{v_1 - v_3}{R_4} = 0.399A$$

3.2. Nodal analysis

3.2.3. Nodal analysis by inspection

In case:

If a circuit with only ***independent current sources*** has N non-reference nodes → the node-voltage equations can be written in terms of the conductance as

$$\begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1N} \\ G_{21} & G_{22} & \cdots & G_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ G_{N1} & G_{N2} & \cdots & G_{NN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} \quad \leftrightarrow \quad \mathbf{Gv} = \mathbf{i}$$

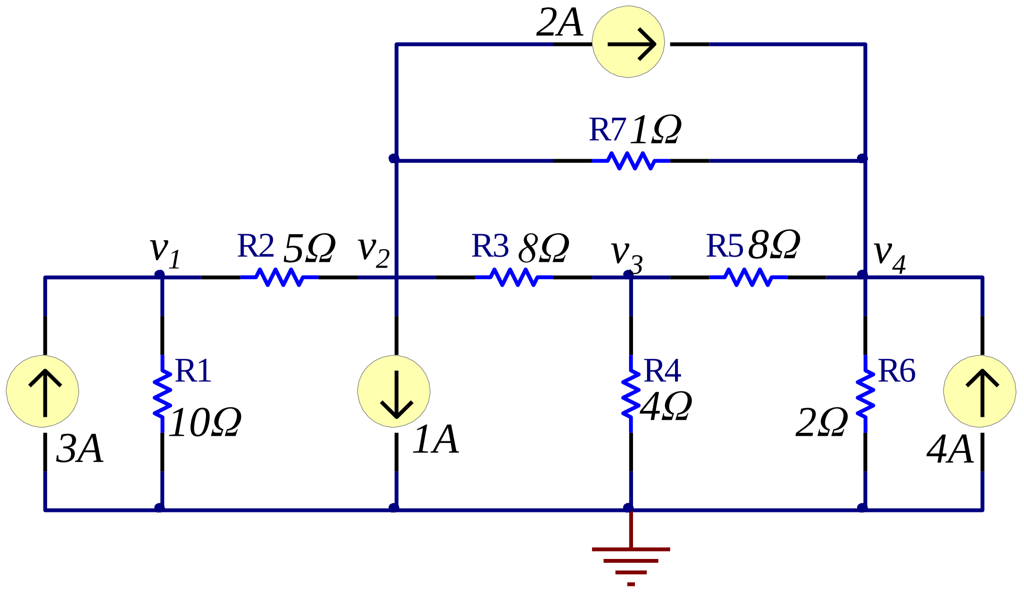
where:

- G_{kk} : Sum of the conductances connected to node k
- $G_{kj} = G_{jk}$: **Negative** of the sum of the conductances directly connecting nodes k and j, $k \neq j$.
- v_k : Unknown voltage at node k.
- i_k : Sum of all independent current sources directly connected to node k, with currents entering the node treated as positive.

3.2. Nodal analysis

3.2.3. Nodal analysis by inspection

+ Example 5: write the node voltage matrix equations for this circuit



$$\begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}$$

where

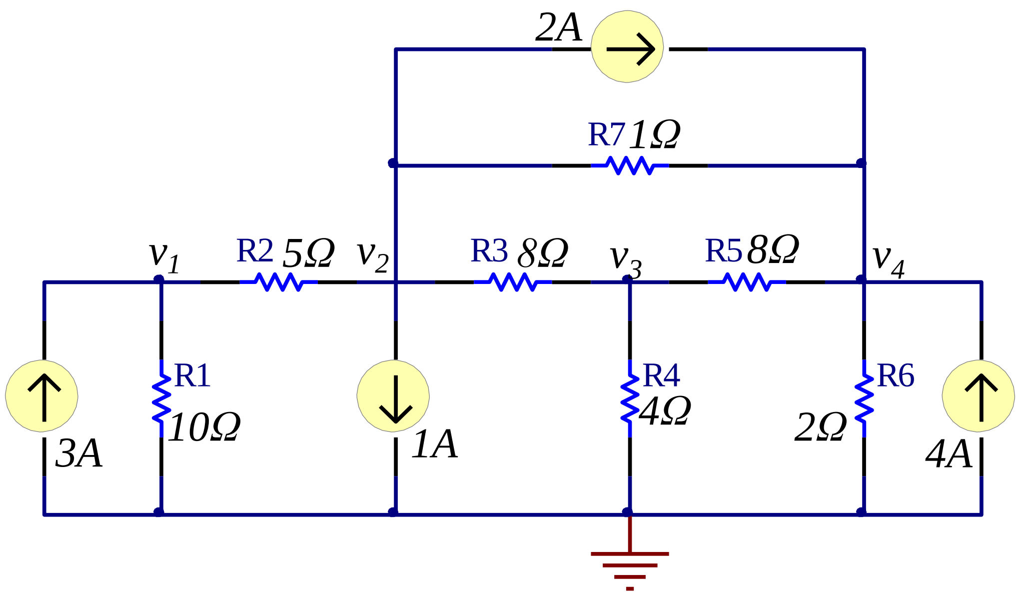
$$G_{11} = \frac{1}{R_1} + \frac{1}{R_2} = 0.3S \qquad G_{22} = \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_7} = 1.325S \qquad G_{13} = G_{31} = 0$$

$$G_{33} = \frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5} = 0.5S \qquad G_{44} = \frac{1}{R_5} + \frac{1}{R_6} + \frac{1}{R_7} = 1.625S \qquad G_{14} = G_{41} = 0$$

3.2. Nodal analysis

3.2.3. Nodal analysis by inspection

+ Example 5: write the node voltage matrix equations for this circuit



$$\begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}$$

$$G_{12} = G_{21} = -\frac{1}{R_2} = -0.2S$$

$$G_{32} = G_{23} = -\frac{1}{R_3} = -0.125S$$

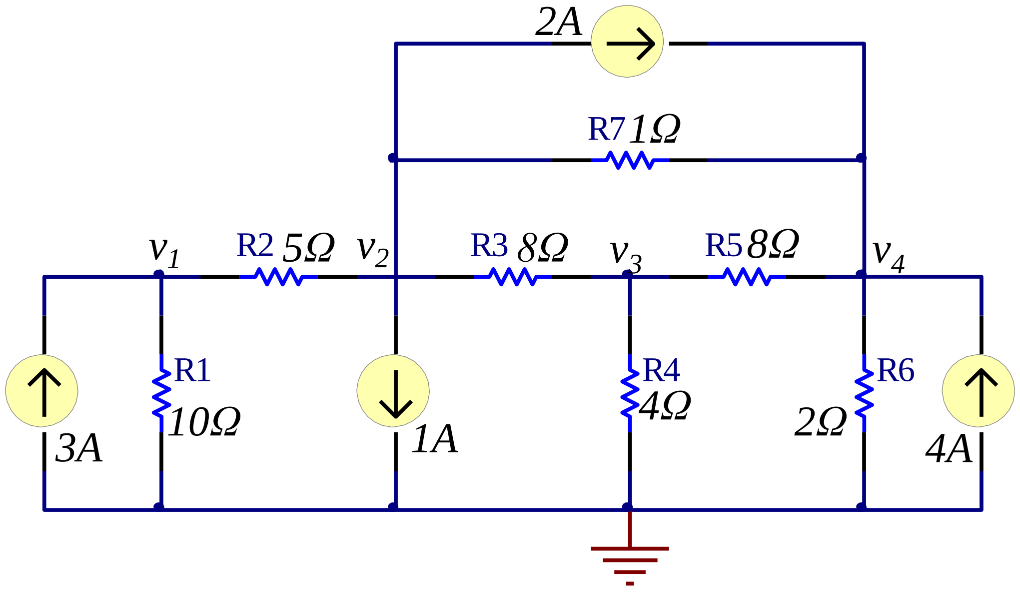
$$G_{34} = G_{43} = -\frac{1}{R_5} = -0.125S$$

$$G_{42} = G_{24} = -\frac{1}{R_7} = -1S$$

3.2. Nodal analysis

3.2.3. Nodal analysis by inspection

+ Example 5: write the node voltage matrix equations for this circuit



$$\begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}$$

Sum of current sources at node 1, 2, 3 and 4: $i_1 = 3A$ $\quad i_2 = -1 - 2 = -3A$ $\quad i_3 = 0A$ $\quad i_4 = 2 + 4 = 6A$

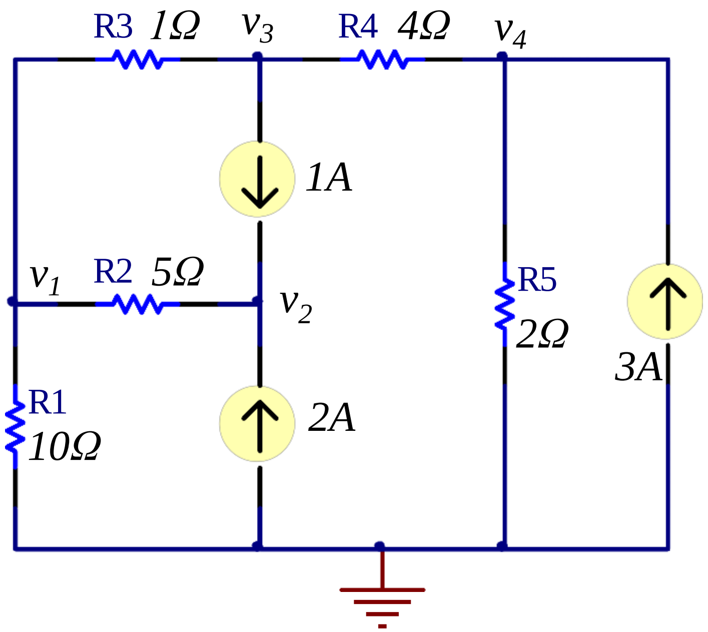
So we have node voltage matrix equations:

$$\begin{bmatrix} 0.3 & -0.2 & 0 & 0 \\ -0.2 & 1.325 & -0.125 & -1 \\ 0 & -0.125 & 0.5 & -0.125 \\ 0 & -1 & -0.125 & 1.625 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \\ 6 \end{bmatrix}$$

3.2. Nodal analysis

3.2.3. Nodal analysis by inspection

+ Example 6: write the node voltage matrix equations for this circuit



From 4 non-reference nodes, we can write

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}$$

where

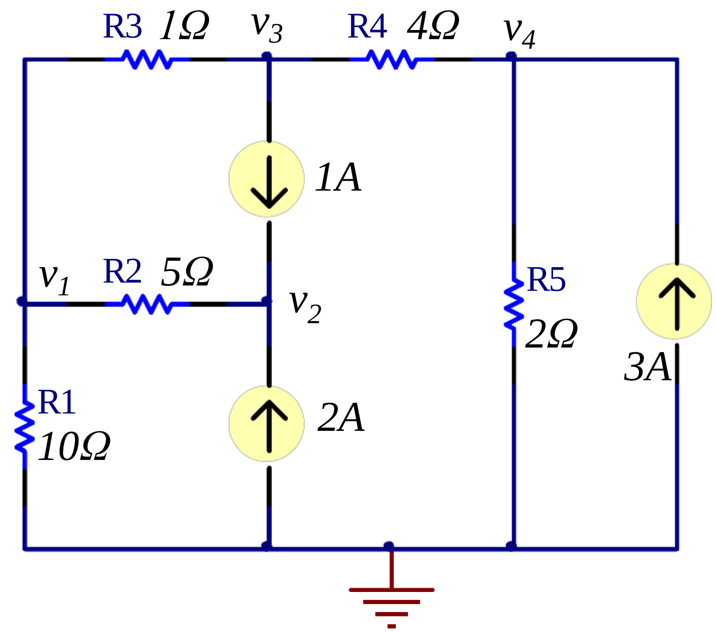
$$G_{11} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = 1.3S \qquad G_{22} = \frac{1}{R_2} = 0.2S \qquad G_{23} = G_{32} = 0 \qquad G_{14} = G_{41} = 0$$

$$G_{33} = \frac{1}{R_3} + \frac{1}{R_4} = 1.25S \qquad G_{44} = \frac{1}{R_4} + \frac{1}{R_5} = 0.75S \qquad G_{24} = G_{42} = 0$$

3.2. Nodal analysis

3.2.3. Nodal analysis by inspection

+ Example 6: write the node voltage matrix equations for this circuit



From 4 non-reference nodes, we can write

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}$$

where

$$G_{12} = G_{21} = -\frac{1}{R_2} = -0.2S \qquad G_{13} = G_{31} = -\frac{1}{R_3} = -1S$$

$$G_{34} = G_{43} = -\frac{1}{R_4} = -0.25S$$

$$i_1 = 0$$

$$i_2 = 1 + 2 = 3A$$

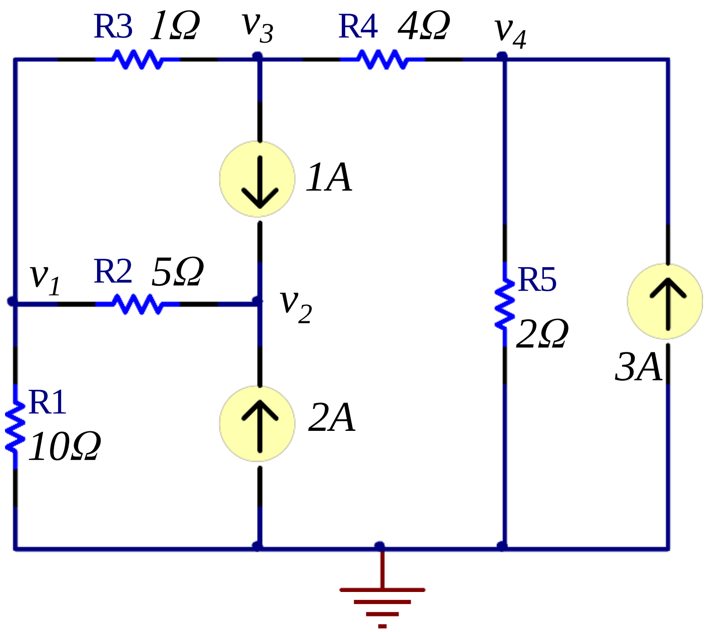
$$i_3 = -1A$$

$$i_4 = 3A$$

3.2. Nodal analysis

3.2.3. Nodal analysis by inspection

+ Example 6: write the node voltage matrix equations for this circuit



From 4 non-reference nodes, we can write

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}$$

Finally, the matrix equations is

$$\begin{bmatrix} 1.3 & -0,2 & -1 & 0 \\ -0.2 & 0.2 & 0 & 0 \\ -1 & 0 & 1.25 & -0.25 \\ 0 & 0 & -0.25 & 0.75 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 3 \end{bmatrix}$$

3.3. Mesh analysis

+ **Mesh analysis** provides another general procedure for analyzing circuits → using *mesh current as the circuit variables* (loop analysis or mesh current method)

+ **Objective:** reduces the number of equations

+ **Mesh:** loop that does not contain any other loop within

+ **Apply KVL** to find unknown currents

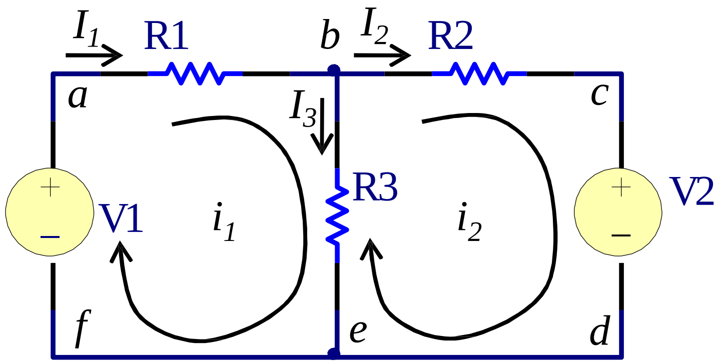
+ **Only applicable** to a planar circuit

3.3. Mesh analysis

+ An example to introduce mesh and mesh current

Meshes: *abefa*, and *bcdeb* (*abcdefa* is not a mesh)

Mesh current → Current through a mesh (i_1 and i_2)

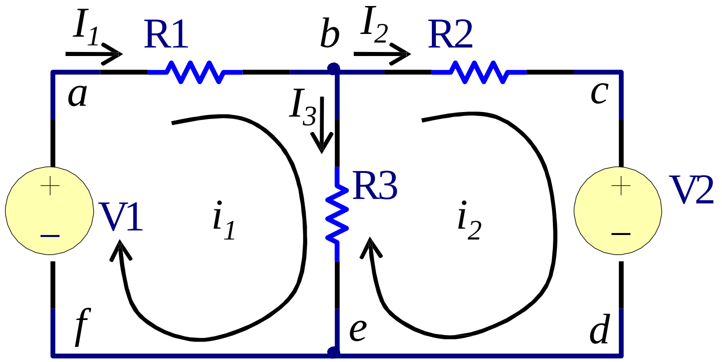


+ Step to determine mesh currents and element currents

- Assign mesh current $i_1, i_2, \ldots, i_n$ to the *n* meshes in a given circuit
- Apply KVL to each of the n meshes (using Ohm's law to express the voltages in terms of the mesh currents)
- Solve the resulting n equations to get the mesh currents
- Calculate current through each element: sum of the mesh currents through it, (including current sources)

3.3. Mesh analysis

3.3.1. Mesh analysis without current sources



+ For example: find branch currents in the given circuit using mesh current method

+ Apply KVL to mesh I, II:

$$\begin{cases} (R_1+R_3)i_1 - R_3 i_2 = V_1 \\ -R_3 i_1 + (R_2+R_3)i_2 = -V_2 \end{cases} \qquad \begin{bmatrix} R_1+R_3 & -R_3 \\ -R_3 & R_2+R_3 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}$$

+ Calculate the current through circuit elements $I_1 = i_1, I_2 = i_2, I_3 = i_1 - i_2$

+ *Note:*

- A circuit having *n* nodes, *b* branches, and *l* independent loops (mesh)

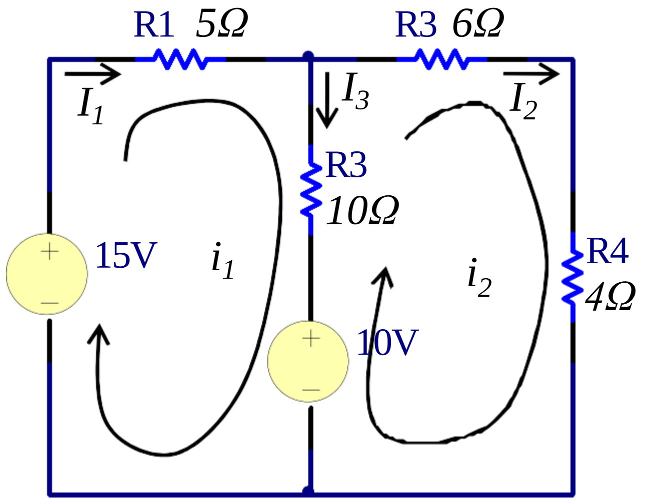
$$\boxed{l = b - n + 1}$$

- Branch currents are different from the mesh currents unless the mesh is isolated

3.3. Mesh analysis

3.3.1. Mesh analysis without current sources

+ Example 1: find the branch current I_1, I_2, I_3 in the given circuit using mesh current method



Apply KVL to 2 meshes:

$$-15+5i_1+10(i_1-i_2)+10=0$$

$$6i_2+4i_2+10(i_2-i_1)-10=0$$

Calculate mesh currents:

$$\begin{cases} 3i_1-2i_2=1 \\ i_1-2i_2=-1 \end{cases} \rightarrow \begin{cases} i_1=1A \\ i_2=1A \end{cases}$$

Calculate branch (element) currents:

$$I_1=i_1=1A$$

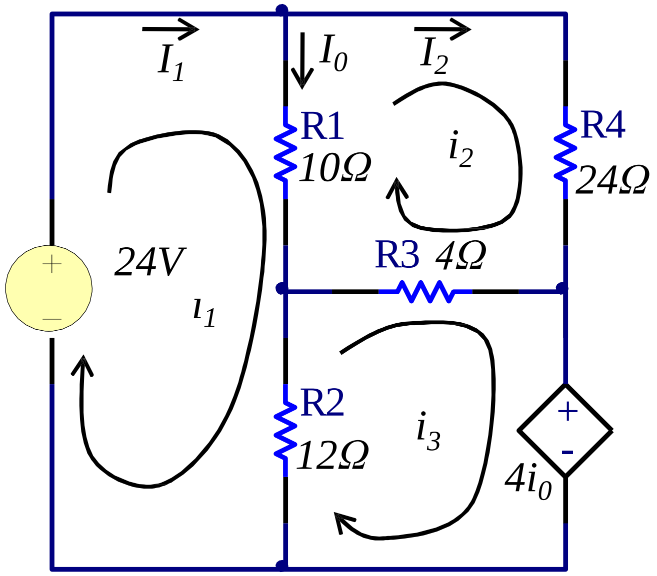
$$I_2=i_2=1A$$

$$I_3=i_1-i_2=0A$$

3.3. Mesh analysis

3.3.1. Mesh analysis without current sources

+ Example 2: find the current I_0, in the given circuit using mesh current method



Apply KVL to 3 meshes:

$$-24+R_1(i_1-i_2)+R_2(i_1-i_3)=0 \rightarrow 11i_1-5i_2-6i_3=12$$

$$R_4i_2+R_3(i_2-i_3)+R_1(i_2-i_1)=0 \rightarrow -5i_1+19i_2-2i_3=0$$

$$4I_0+R_2(i_3-i_1)+R_3(i_3-i_2)=0 \rightarrow -i_1-i_2+2i_3=0$$

$$I_0=i_1-i_2$$

Solve the set of mesh equations to calculate I_0:

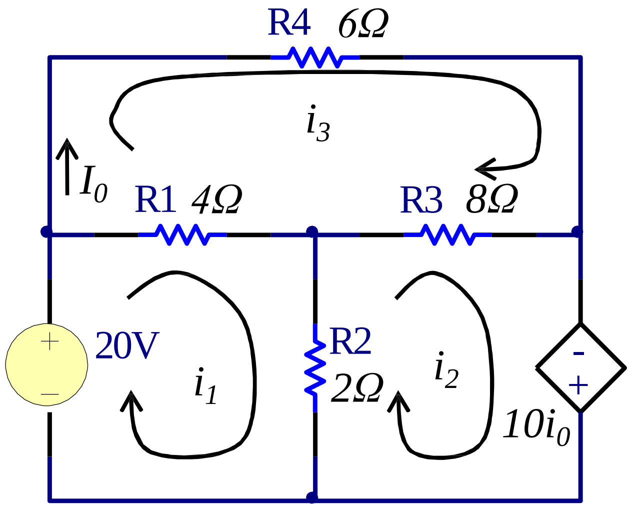
$$\begin{cases} 11i_1-5i_2-6i_3=12 \\ -5i_1+19i_2-2i_3=0 \\ -i_1-i_2+2i_3=0 \end{cases} \rightarrow \begin{cases} i_1=2.25A \\ i_2=0.75A \\ i_3=1.5A \end{cases}$$

$\rightarrow$ $\boldsymbol{I_0=i_1-i_2 = 1.5A}$

3.3. Mesh analysis

3.3.1. Mesh analysis without current sources

+ Example 3: find the current I_0, in the given circuit using mesh current method



Apply KVL to 3 meshes:

$$-20 + R_1(i_1 - i_3) + R_2(i_1 - i_2) = 0 \rightarrow 6i_1 - 2i_2 - 4i_3 = 20$$

$$R_2(i_2 - i_1) + R_3(i_2 - i_3) - 10i_0 = 0 \rightarrow -2i_1 + 10i_2 - 18i_3 = 0$$

$$I_0 = i_3$$

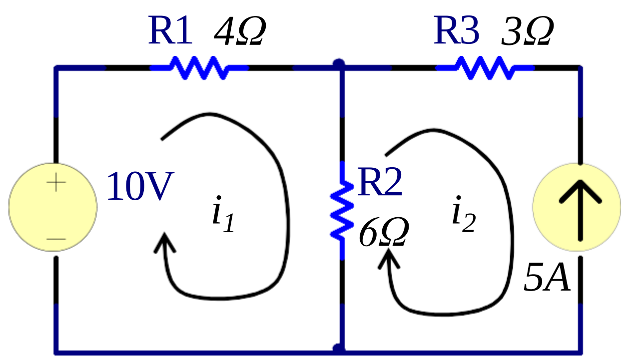
$$R_1(i_3 - i_1) + R_3(i_3 - i_2) + R_4 i_3 = 0 \rightarrow -4i_1 - 8i_2 + 18i_3 = 0$$

Solve the set of mesh equations to calculate I_0:

$$\begin{cases} 6i_1 - 2i_2 - 4i_3 = 20 \\ -2i_1 + 10i_2 - 18i_3 = 0 \\ -4i_1 - 8i_2 + 18i_3 = 0 \end{cases} \rightarrow \begin{cases} i_1 = -3.21A \\ i_2 = -9.64A \\ i_3 = -5A \end{cases} \rightarrow I_0 = -5A$$

3.3. Mesh analysis

3.3.2. Mesh analysis with current sources



+ The presence of the current sources → reduces the number of equations in the mesh analysis

+ Two cases:

- Current source exists only in one mesh → mesh current = current source
- Current source exists between two meshes → create a ***super-mesh*** by **excluding** the current source and any elements connected in series with it

> ***A super-mesh*** results when two meshes have a (dependent or independent) current source in common

+ For the given circuit: Current source exists only in one mesh

$$\begin{cases} 4i_1 + 6(i_1 - i_2) = 10 \\ i_2 = -5A \end{cases} \rightarrow \begin{cases} i_1 = -2A \\ i_2 = -5A \end{cases}$$

(one equation)

3.3. Mesh analysis

3.3.2. Mesh analysis with current sources

+ Example 1: find branch currents using mesh current method

A current source *6A* between two mesh → super mesh

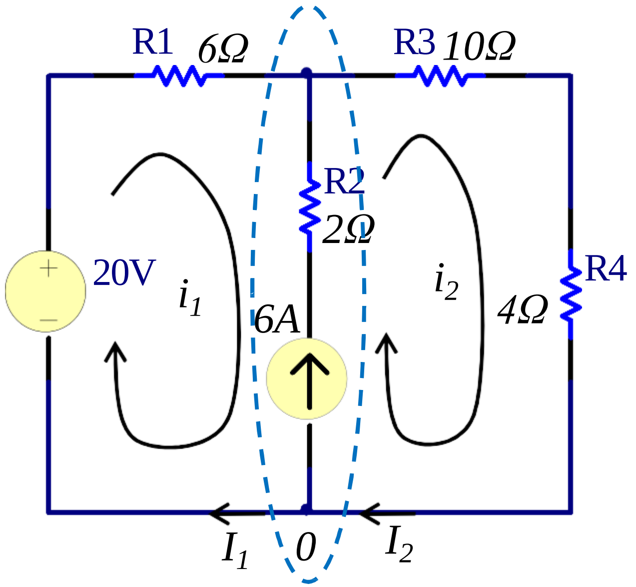
Apply KVL to the super-mesh:

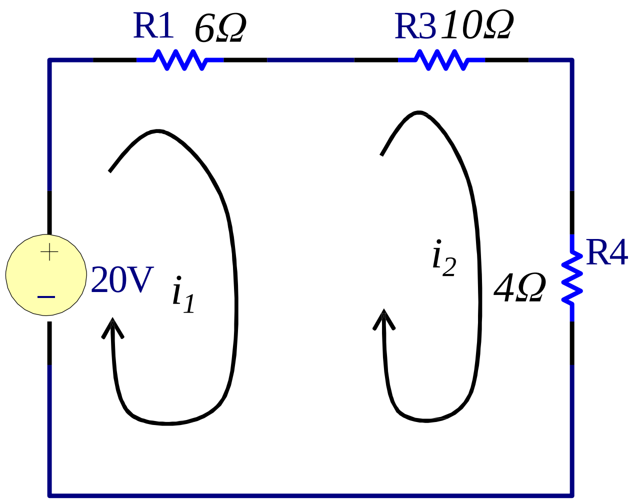
$$-20 + R_1 i_1 + R_3 i_2 + R_4 i_2 = 0 \rightarrow \boxed{6i_1 + 14i_2 = 20}$$

Apply KCL to node 0: $I_2 = I_1 + 6 \quad i_1 = I_1 \quad i_2 = I_2 \rightarrow \boxed{i_2 = i_1 + 6}$

Note:

- Current source in the super-mesh provides the constraint equation to solve for the mesh currents
- Super mesh has no current of its own
- Super mesh requires the using of both KVL and KCL

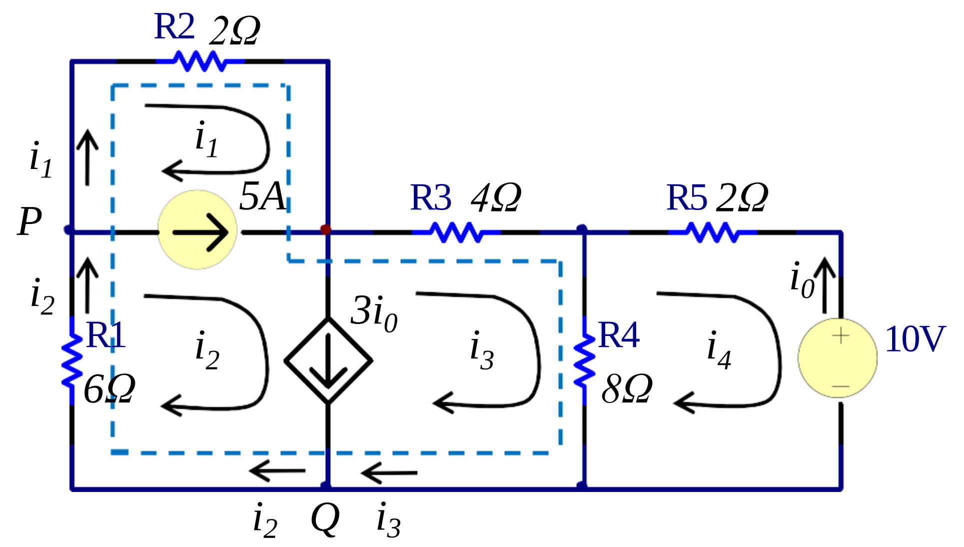




3.3. Mesh analysis

3.3.2. Mesh analysis with current sources

+ Example 2: find the current i_1, i_4 in the given circuit using mesh current method



Apply KVL to the super mesh in blue dash-line:

$$R_2 i_1 + R_3 i_3 + R_4(i_3 - i_4) + R_1 i_2 = 0 \rightarrow i_1 + 3i_2 + 6i_3 - 4i_4 = 0$$

Apply KCL to node P: $i_2 = i_1 + 5$

Apply KCL to node Q: $i_2 = i_3 + 3i_0 \quad i_0 = -i_4 \rightarrow \quad i_2 = i_3 - 3i_4$

Apply KVL to mesh IV: $R_5 i_4 + R_4(i_4 - i_3) + 10 = 0 \rightarrow 5i_4 - 4i_3 = -5$

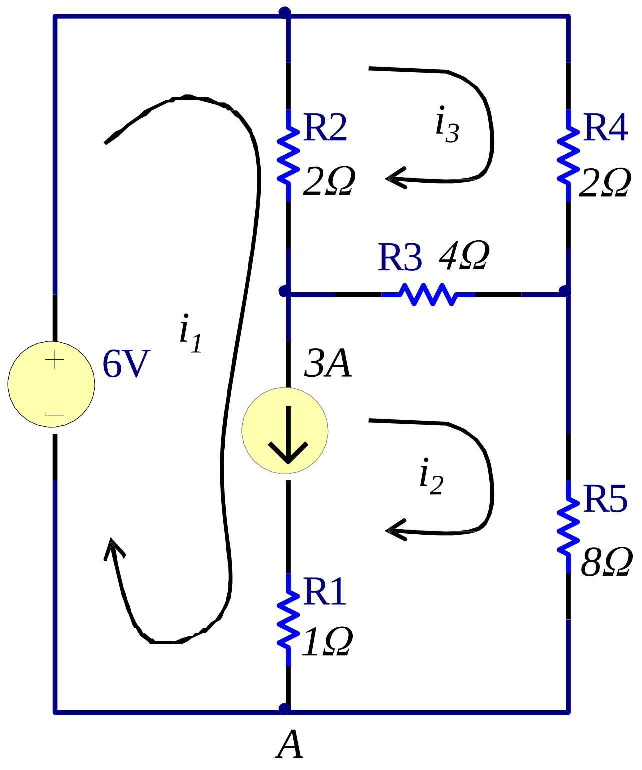
We have a set of 4 equations to calculate 4 mesh currents:

$$\begin{cases} i_1 + 3i_2 + 6i_3 - 4i_4 = 0 \\ -i_1 + i_2 = 5 \\ i_2 - i_3 + 3i_4 = 0 \\ -4i_3 + 5i_4 = -5 \end{cases} \rightarrow \begin{cases} i_1 = -7.5A \\ i_2 = -2.5A \\ i_3 = 3.93A \\ i_4 = 2.14A \end{cases}$$

3.3. Mesh analysis

3.3.2. Mesh analysis with current sources

+ Example 3: find the current i_1, i_2, i_3 in the given circuit using mesh current method



Apply KVL to super mesh:

$$-6+R_2(i_1-i_3)+R_3(i_2-i_3)+R_5 i_2=0 \rightarrow 2i_1+12i_2-6i_3=6$$

Apply KCL to node A: $i_1-i_2=3$

Apply KVL to mesh III: $R_2(i_3-i_1)+R_3(i_3-i_2)+R_4 i_3=0 \rightarrow -2i_1-4i_2+8i_3=0$

We have a set of 3 equations to calculate 3 mesh currents:

$$\begin{cases} 2i_1+12i_2-6i_3=6 \\ i_1-i_2=3 \\ -2i_1-4i_2+8i_3=0 \end{cases} \rightarrow \begin{cases} i_1=3.47A \\ i_2=0.47A \\ i_3=1.11A \end{cases}$$

Branch currents ?

3.3. Mesh analysis

3.3.3. Mesh analysis by inspection

+ If a circuit (with only *independent voltage sources*) has N meshes, the mesh current equations can be expressed in terms of the resistances as

$$\begin{bmatrix} R_{11} & R_{12} & \dots & R_{1N} \\ R_{21} & R_{22} & \dots & R_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ R_{N1} & R_{N2} & \dots & R_{NN} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} \quad \leftrightarrow \quad R\mathbf{i} = \mathbf{v}$$

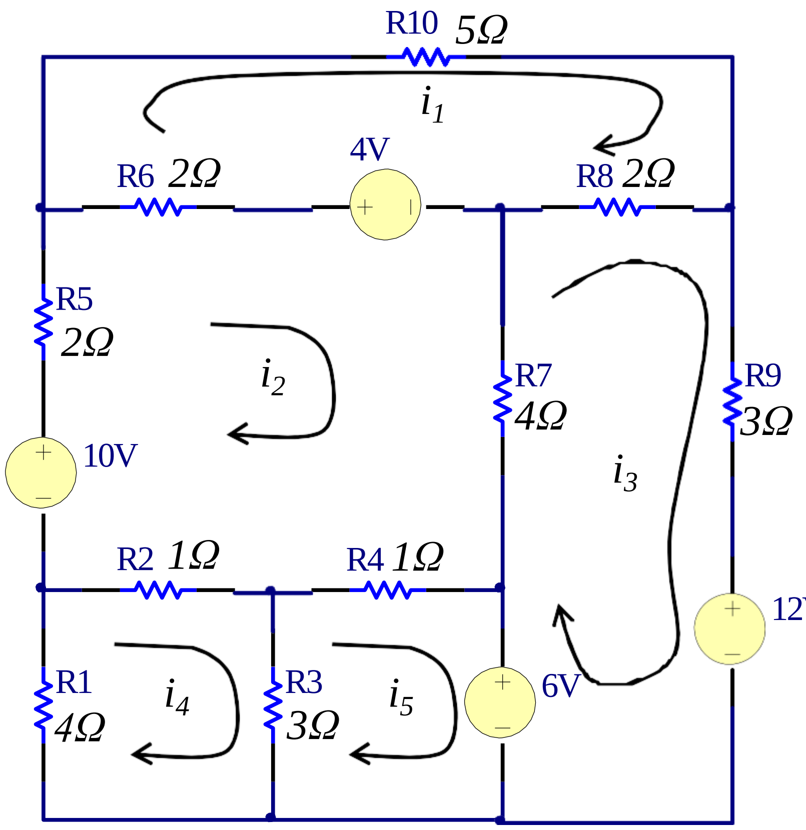
where:

- $\boldsymbol{R_{kk}}$: Sum of the resistances in mesh k.
- $\boldsymbol{R_{kj} = R_{jk}}$: Negative of the sum of the resistances in common with meshes k and j, $k \neq j$.
- $\boldsymbol{i_k}$: Unknown mesh current for mesh k in the clockwise direction.
- $\boldsymbol{v_k}$: Sum taken clockwise of all independent voltage sources in mesh k, with voltage rise treated as positive.

3.3. Mesh analysis

3.3.3. Mesh analysis by inspection

+ Example 4: write the mesh current equations



5 meshes in matrix form:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

where:

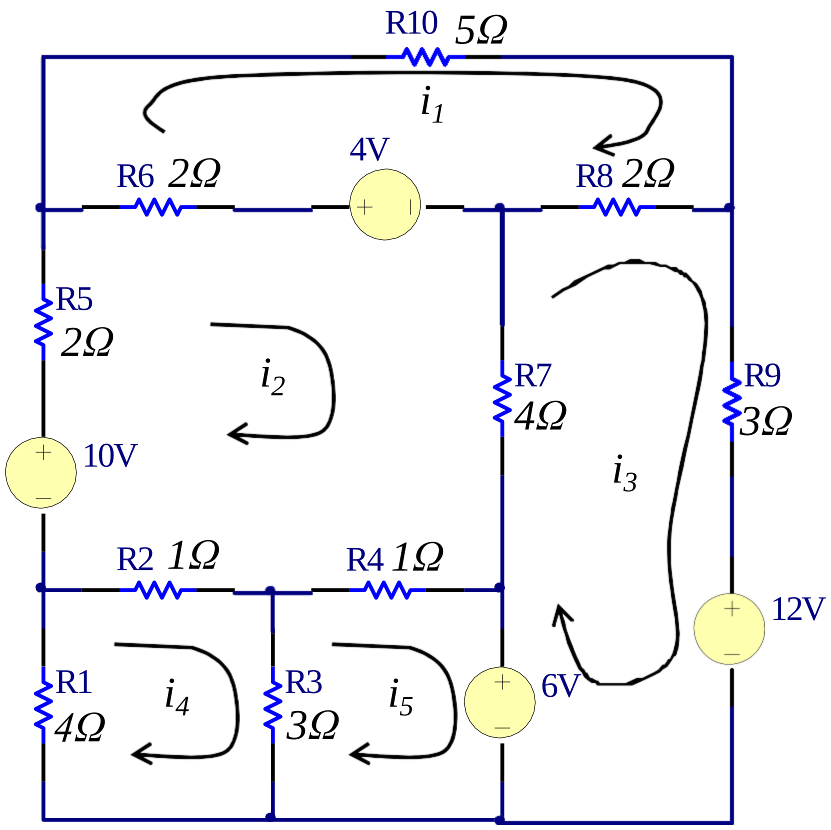
$$R_{11} = R_6 + R_8 + R_{10} = 9\Omega \qquad R_{22} = R_2 + R_4 + R_5 + R_6 + R_7 = 10\Omega$$

$$R_{33} = R_7 + R_8 + R_9 = 9\Omega \qquad R_{44} = R_1 + R_2 + R_3 = 8\Omega \qquad R_{55} = R_3 + R_4 = 4\Omega$$

3.3. Mesh analysis

3.3.3. Mesh analysis by inspection

+ Example 4: write the mesh current equations



5 meshes in matrix form:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

where:

$$R_{12} = R_{21} = -R_6 = -2\Omega \qquad R_{23} = R_{32} = -R_7 = -4\Omega$$

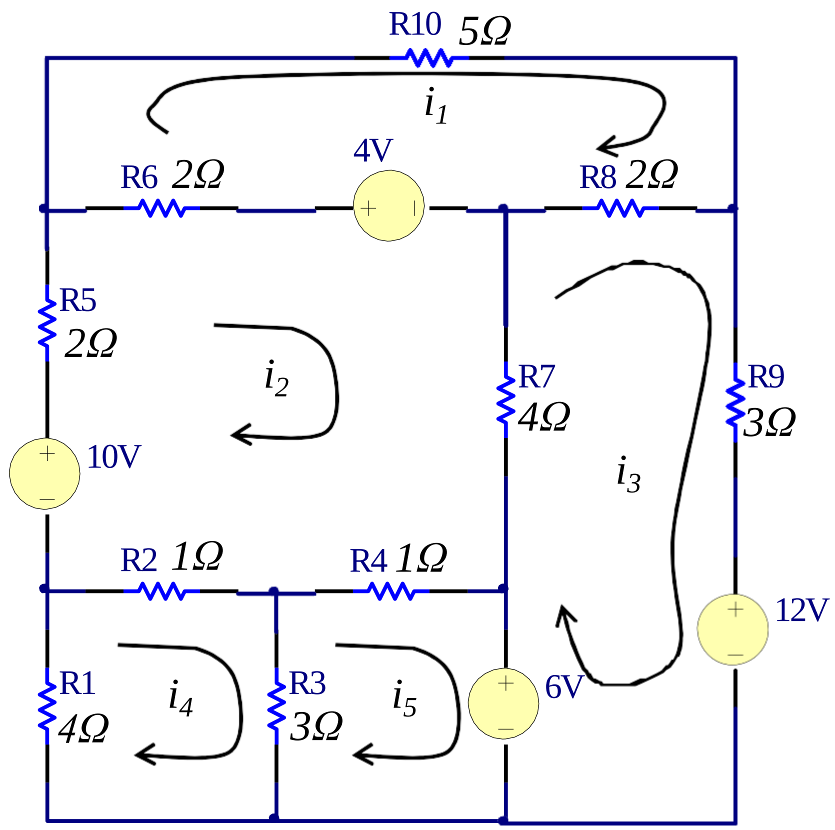
$$R_{13} = R_{31} = -R_8 = -2\Omega \qquad R_{24} = R_{42} = -R_2 = -1\Omega \qquad R_{34} = R_{43} = 0$$

$$R_{14} = R_{41} = 0 \qquad R_{25} = R_{52} = -R_4 = -1\Omega \qquad R_{35} = R_{53} = 0 \qquad R_{45} = R_{54} = -R_3 = -3\Omega$$

3.3. Mesh analysis

3.3.3. Mesh analysis by inspection

+ Example 4: write the mesh current equations



5 mesh voltages:

$$\begin{cases} v_1 = 4V \\ v_2 = 10 - 4 = 6V \\ v_3 = 6 - 12 = -6V \\ v_4 = 0V \\ v_5 = -6V \end{cases}$$

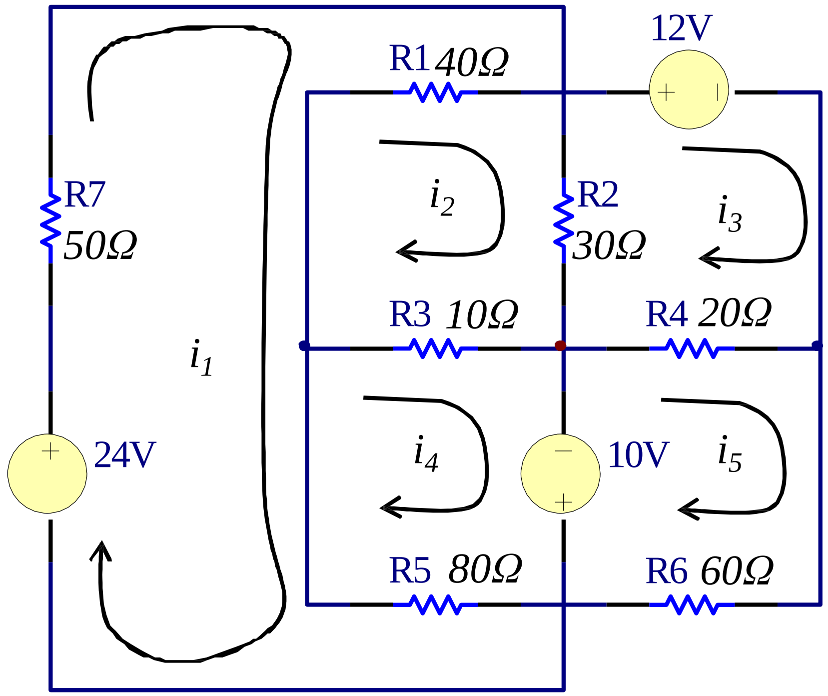
So we have the mesh current equations written in matrix form:

$$\begin{bmatrix} 9 & -2 & -2 & 0 & 0 \\ -2 & 10 & -4 & -1 & -1 \\ -2 & -4 & 9 & 0 & 0 \\ 0 & -1 & 0 & 8 & -3 \\ 0 & -1 & 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -6 \\ 0 \\ -6 \end{bmatrix}$$

3.3. Mesh analysis

3.3.3. Mesh analysis by inspection

+ Example 5: write the mesh current equations



5 meshes in matrix form:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

where:

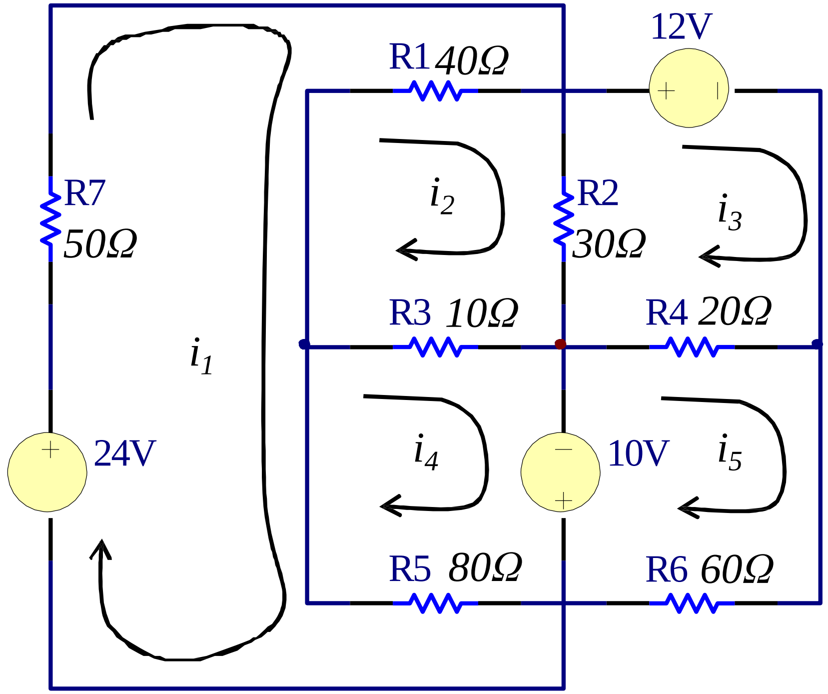
$$R_{11} = R_1 + R_5 + R_7 = 170\Omega \qquad R_{22} = R_1 + R_2 + R_3 = 80\Omega$$

$$R_{33} = R_2 + R_4 = 50\Omega \qquad R_{44} = R_3 + R_5 = 90\Omega \qquad R_{55} = R_4 + R_6 = 80\Omega$$

3.3. Mesh analysis

3.3.3. Mesh analysis by inspection

+ Example 5: write the mesh current equations



5 meshes in matrix form:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

where:

$$R_{12} = R_{21} = -R_1 = -40\Omega \qquad R_{23} = R_{32} = -R_2 = -30\Omega \qquad R_{34} = R_{43} = 0$$

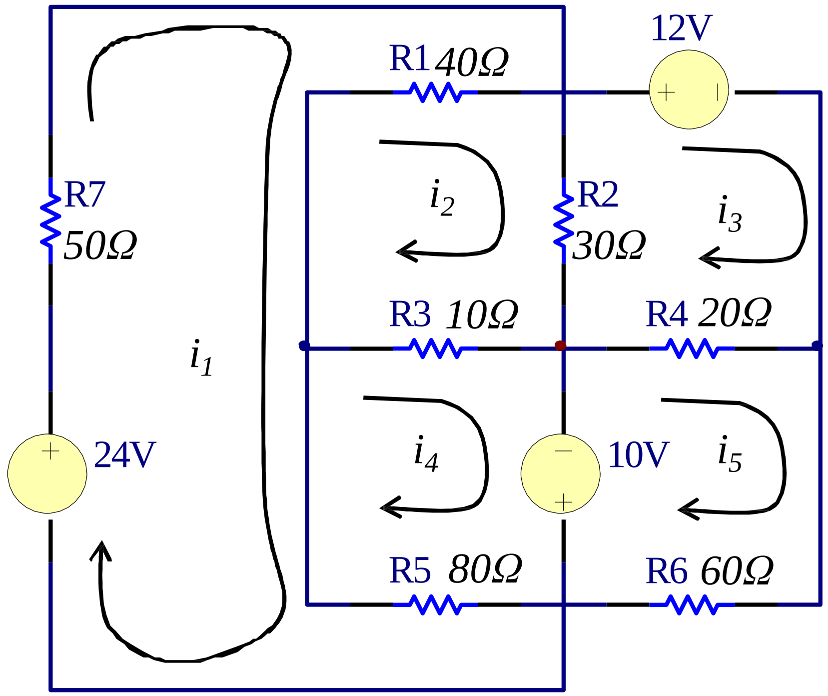
$$R_{13} = R_{31} = 0 \qquad R_{24} = R_{42} = -R_3 = -10\Omega \qquad R_{45} = R_{54} = 0$$

$$R_{14} = R_{41} = -R_5 = -80\Omega \qquad R_{25} = R_{52} = 0 \qquad R_{35} = R_{53} = -R_4 = -20\Omega$$

3.3. Mesh analysis

3.3.3. Mesh analysis by inspection

+ Example 5: write the mesh current equations



5 mesh voltages:

$$\begin{cases} v_1 = 24V \\ v_2 = 0V \\ v_3 = -12V \\ v_4 = 10V \\ v_5 = -10V \end{cases}$$

So we have the mesh current equations written in matrix form:

$$\begin{bmatrix} 170 & -40 & 0 & -80 & 0 \\ -40 & 80 & -30 & -10 & 0 \\ 0 & -30 & 50 & 0 & -20 \\ -80 & -10 & 0 & 90 & 0 \\ 0 & 0 & -20 & 0 & 80 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} 24 \\ 0 \\ -12 \\ 10 \\ -10 \end{bmatrix}$$

3.3. Mesh analysis

3.3.4. Nodal versus Mesh analysis

+ Nodal and Mesh analysis: → provide a systematic way of analyzing a complex circuit

+ *Mesh analysis* → many series-connected elements, voltage sources, or super-meshes

+ *Nodal analysis* → parallel-connected elements, current sources, or super-nodes

smaller number of equations