Fundamentals of Electric Circuits

DC Circuits

Chapter 6. Capacitors and Inductors

6.1. Introduction

+ In this chapter → introduce two new and important passive linear circuit elements: capacitor and inductor

+ Capacitors and inductors → *not dissipate but store energy* which can be retrieved at a later time → be called *storage elements*

+ Introduction of capacitors and inductors: → able to analyze more important and practical circuits

+ Applications: Capacitors can be combined with op amps to form *integrators, differentiators, and analog computers*

6.2. Capacitors

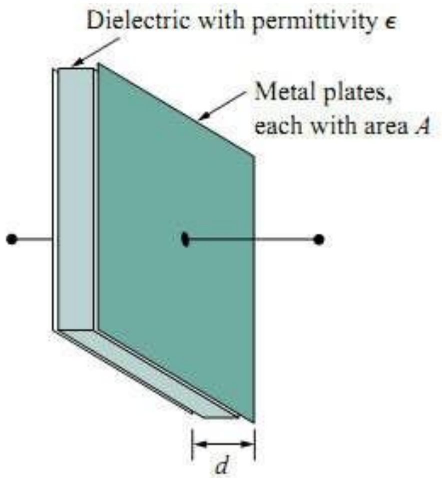
+ Capacitor → a passive element designed to store energy in its electric field

+ Capacitor → two conducting plates separated by an insulator (or dielectric): *air, ceramic, paper, mica,…*

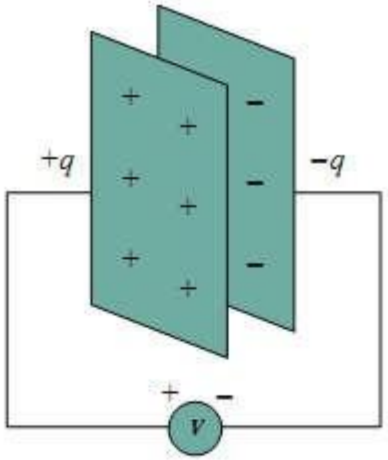
+ When a voltage v is connected:

→ Positive charge q on one plate

→ Negative charge $-q$ on another one



A typical capacitor



A capacitor with applied voltage v

+ Capacitance: → the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates, measured in farads

$$\boxed{C = \frac{q}{V}}$$

$$\boxed{1F = \frac{1C}{1V}}$$

6.2. Capacitors

+ Capacitance C: depend on capacitor physical dimension (not *q* and *V*)

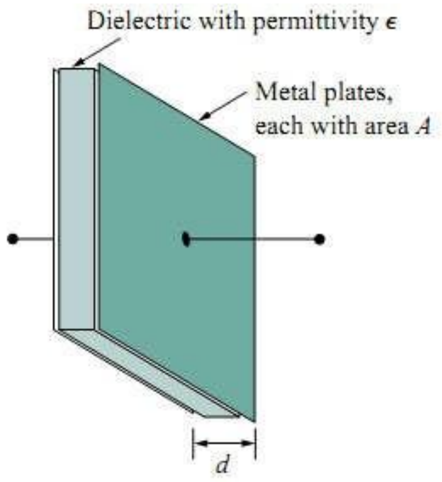
For parallel plate capacitors

$$C = \frac{\varepsilon A}{d}$$

A: surface area of each plate

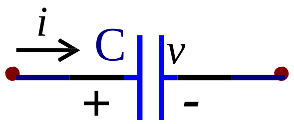
d: distance between the plates

Ɛ: permittivity of the dielectric material between the plates

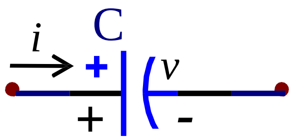


A typical capacitor

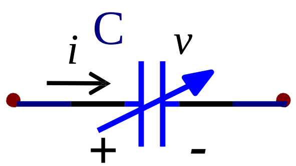
+ Typically, capacitance have values in the *pF* to μF



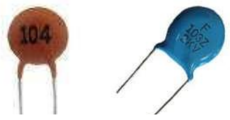
Fix, un-polarized capacitor



Fix, polarized capacitor



Variable capacitor



Ceramic capacitor

Electrolytic capacitor

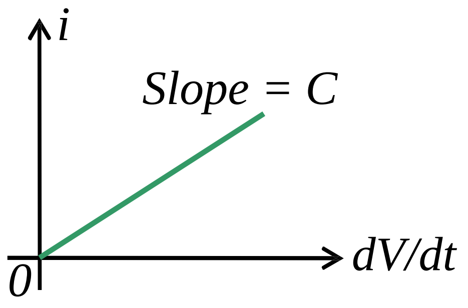
Trimmer capacitor

6.2. Capacitors

+ Capacitors → used to block DC, pass AC, shift phase, store energy, start motors, suppress noise,...

$$i = \frac{dq}{dt} = \frac{\partial q}{\partial v}\cdot\frac{dv}{dt} = C\frac{dv}{dt}$$

$$v = \frac{1}{C}\int_{-\infty}^{t} i\,dt = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$$



Current-voltage relationship

+ Instantaneous power:

$$p = vi = Cv\frac{dv}{dt}$$

+ Energy:

$$w = \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t} Cv\frac{dv}{dt}dt = \frac{1}{2}Cv^2$$

$$w = \frac{1}{2}Cv^2 = \frac{q^2}{2C}$$

+ Classification:

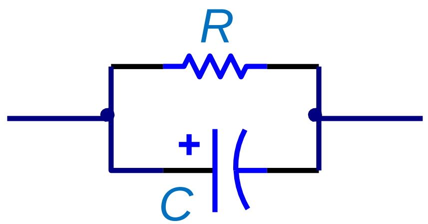
Linear capacitor

Nonlinear capacitor

6.2. Capacitors

+ Properties of a capacitor:

- A capacitor is an *open circuit to DC*
- The voltage on a capacitor *cannot change abruptly:* An discontinuous change in voltage requires an infinitive current
- The ideal capacitor does *not dissipate energy*: It takes power from the circuit when storing energy in its field and returns previously stored energy when delivering power to the circuit
- A *non-ideal capacitor* has a parallel-model leakage resistance (~100MΩ) → the leakage resistance may be neglected for most practical application)

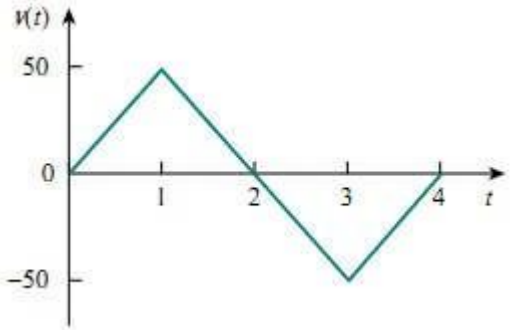


Circuit model of a non-ideal capacitor

6.2. Capacitors

+ Example 1: Determine the current through a 200μF capacitor whose voltage is

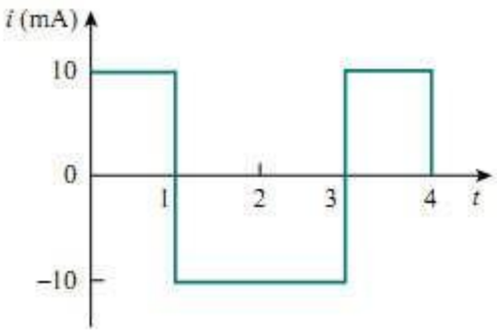
$$v(t)=\begin{cases} 50t\,V & 0<t<1 \\ 100-50t\,V & 1<t<3 \\ -200+50t\,V & 3<t<4 \\ 0V & otherwise \end{cases}$$



The current through a capacitor: $i=C\dfrac{dv}{dt}$

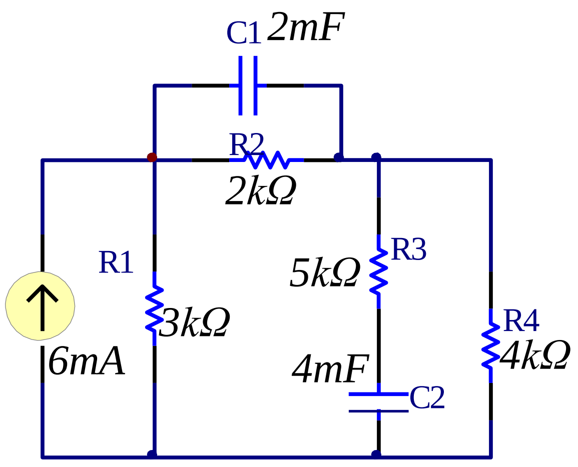
So we have:

$$i(t)=200.10^{-6}.\begin{cases} 50A & 0<t<1 \\ -50A & 1<t<3 \\ 50A & 3<t<4 \\ 0A & otherwise \end{cases}=\begin{cases} 10mA & 0<t<1 \\ -10mA & 1<t<3 \\ 10mA & 3<t<4 \\ 0mA & otherwise \end{cases}$$



6.2. Capacitors

+ Example 2: Obtain the energy stored in each capacitor in the given circuit under DC condition



Under DC conditions, each capacitor is an open circuit

The current through R1, R2 and R4 is:

$$i = \frac{jR_1}{R_1 + R_2 + R_4} = \frac{6.10^{-3}.3}{3+2+4} = 2mA$$

The voltages across C_1 and C_2 are:

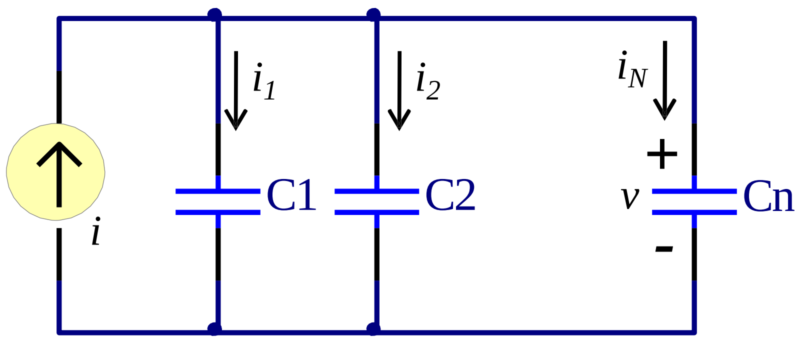
$$v_1 = R_2 i = 2.10^3.2.10^{-3} = 4V \qquad v_2 = R_4 i = 4.10^3.2.10^{-3} = 8V$$

So energies stored in C_1, C_2 are:

$$w_1 = \frac{1}{2}C_1 v_1^2 = 16mJ \qquad w_2 = \frac{1}{2}C_2 v_2^2 = 128mJ$$

6.3. Series and Parallel Capacitors

+ Equivalent capacitance C_{eq} of ***N* capacitor in parallel**:

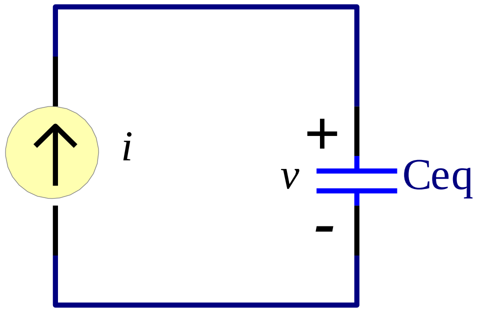


Apply KCL:

$$i = i_1 + i_2 + \ldots + i_N \rightarrow i = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + \ldots + C_N \frac{dv}{dt}$$

$$\rightarrow i = \left(\sum_{k=1}^{N} C_k \right) \frac{dv}{dt} = C_{eq} \frac{dv}{dt}$$

$$\boxed{C_{eq} = C_1 + C_2 + \ldots + C_N}$$



+ Equivalent capacitance of N parallel-connected capacitors → the sum of the individual capacitances

6.3. Series and Parallel Capacitors

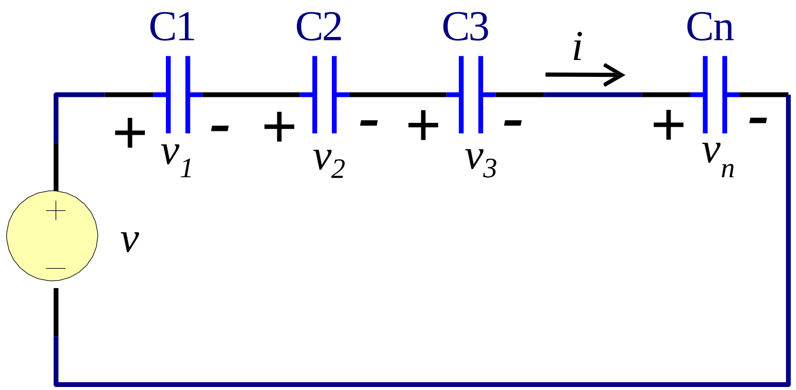
+ Equivalent capacitance C_{eq} of ***N* capacitor in series**:

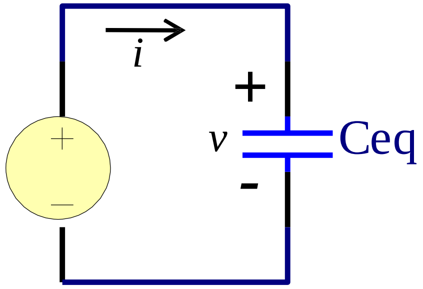
KVL to the loop:

$$v = v_1 + v_2 + \ldots + v_N$$

$$\rightarrow v = \frac{1}{C_1}\int_{t_0}^{t} idt + v_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} idt + v_2(t_0) + \ldots + \frac{1}{C_N}\int_{t_0}^{t} idt + v_N(t_0)$$

$$\rightarrow v = \left(\frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}\right)\int_{t_0}^{t} idt + v_1(t_0) + \ldots + v_N(t_0) = \frac{1}{C_{eq}}\int_{t_0}^{t} idt + v(t_0)$$



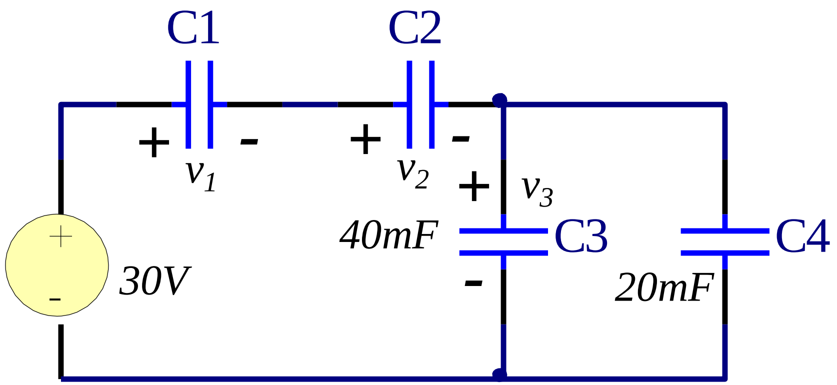


+ Equivalent capacitance of series-connected capacitors: → the reciprocal of the sum of the reciprocals of the individual capacitances

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}$$

6.3. Series and Parallel Capacitors

+ Example 3: Find the voltage across each capacitor



C_3 and C_4 can be combined to get: $C_{34} = C_3 + C_4 = 60mF$

C_{34} is in series with C_1 and C_2 : $C_{eq} = \dfrac{1}{\dfrac{1}{C_{34}} + \dfrac{1}{C_1} + \dfrac{1}{C_2}} = 10mF$

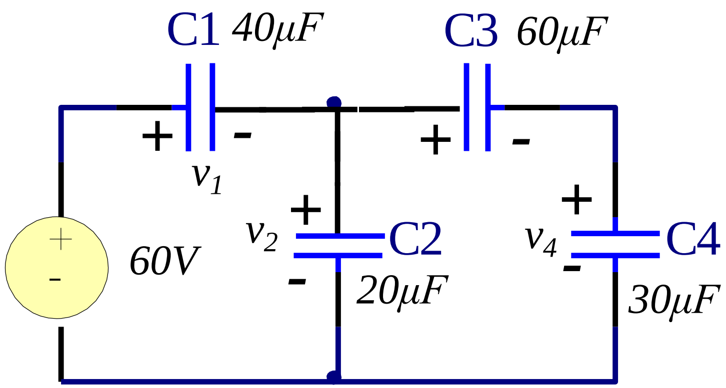
Total charge : $q = C_{eq}v = 10.10^{-3}.30 = 0.3C$

$$v_1 = \frac{q}{C_1} = \frac{0.3}{20.10^{-3}} = 15V \qquad v_2 = \frac{q}{C_2} = \frac{0.3}{30.10^{-3}} = 10V \qquad v_3 = 30 - (v_1 + v_2) = 5V$$

or : $v_3 = \dfrac{q}{C_{34}} = \dfrac{0.3}{60.10^{-3}} = 5V$

6.3. Series and Parallel Capacitors

+ Example 4: Find the voltage across each capacitor



$$C_{34} = \frac{1}{\frac{1}{C_3}+\frac{1}{C_4}} = 20\mu F \qquad C_{234} = C_2 + C_{34} = 40\mu F$$

$$C_{eq} = \frac{1}{\frac{1}{C_1}+\frac{1}{C_{234}}} = 20\mu F \qquad \rightarrow q = 60.C_{eq} = 1200\mu C$$

$$\rightarrow v_1 = \frac{q}{C_1} = \frac{1200}{40} = 30V \qquad \rightarrow v_2 = v_{34} = \frac{q}{C_{234}} = 60 - v_1 = 30V \qquad \rightarrow q_{34} = C_{34}v_{34} = 20.30 = 600\mu C$$

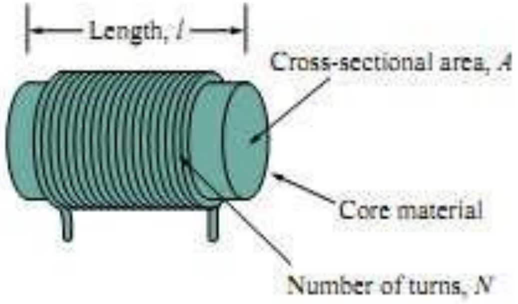
$$\rightarrow v_3 = \frac{q_{34}}{C_3} = 10V \qquad \rightarrow v_4 = \frac{q_{34}}{C_4} = v_{34} - v_3 = 20V$$

6.4. Inductors

+ An inductor:

→ a passive element designed to store energy in its magnetic field

→ consists of a coil of conducting wire

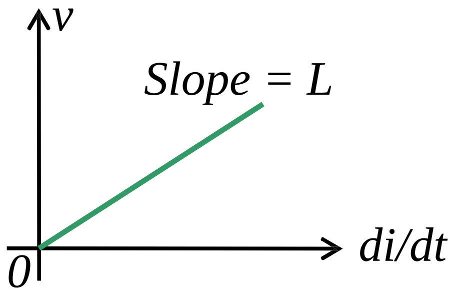


Typical form of an inductor

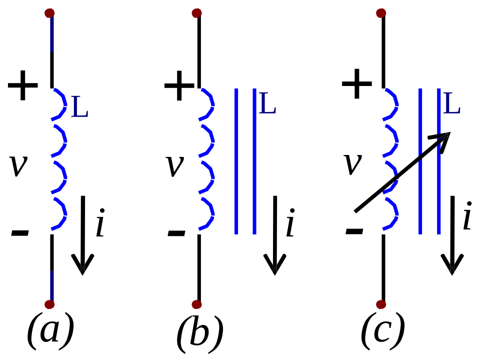
+ If current is allowed to pass through an inductor → the voltage across the inductor is directly proportional to the time rate of change of the current

$$v = L\frac{di}{dt} \leftrightarrow i = \frac{1}{L}\int_{t_0}^{t} v\,dt + i(t_0)$$

L: inductance of the inductor [H]



Voltage-current relationship



Circuit symbols for inductors: (a) air-core, (b) iron-core, (c) variable iron-core

6.4. Inductors

+ Inductance:

→ property of an inductor by which a charge in current through it induces an electromotive force in the conductor

→ depends on physical dimension and construction of inductor

$$\boxed{L = \frac{N^2 \mu A}{l}}$$

N: number of turns

l: length of coil

A: cross-section area

μ: permeability of the core


Toroidal inductor


Solenoidal wound inductor


Inductor

+ Inductor classification:

→ Fixed – Variable inductance

→ Linear – Non-linear inductors

→ Core material: iron, plastic, air,...

6.4. Inductors

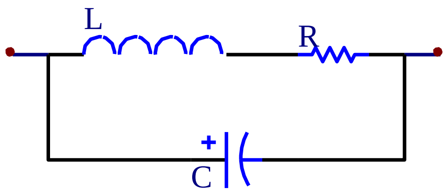
+ Power: $$p = vi = \left(L\frac{di}{dt}\right)i$$

+ Energy: $$w = \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t}\left(L\frac{di}{dt}\right)i\,dt \rightarrow w = \frac{1}{2}Li^2$$

+ Note:

- inductor acts like a *short circuit to DC*
- current through an inductor *cannot change instantaneously*
- ideal inductor does *not dissipate energy*
- Non-ideal inductor:

→ very small winding resistance
→ very small winding capacitance, except at high frequencies



Circuit model for a practical inductor

6.4. Inductors

+ Example 5: Find the current through a 5H inductor and the energy stored within 0<t< 5s, if its voltage is:

$$v(t)=\begin{cases}30t^2 & t>0\\ 0 & t\le 0\end{cases}$$

The current through the inductor $$i=\frac{1}{L}\int_{t_0}^{t} v(t)dt+i(t_0)=\frac{1}{5}\int_0^t 30t^2dt+0=2t^3A$$

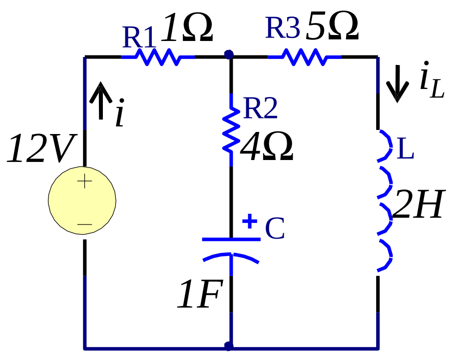
The power delivered to the inductor: $$p=vi=60t^5W$$

The energy stored in the inductor: $$w=\int_0^t pdt=\int_0^5 60t^5dt=60\frac{t^6}{6}\Big|_0^5=156.25kJ$$

Or it could be calculated by using equation: $$w=\frac{1}{2}Li^2=\frac{1}{2}5\left(2.5^3\right)^2=156.25kJ$$

6.4. Inductors

+ Example 6: Find i, v_C, i_L, energy stored in the capacitor and inductor under DC condition



Under DC source: capacitor → open circuit, inductor → short circuit

$$i = i_L = \frac{12}{R_1 + R_3} = \frac{12}{1+5} = 2A \rightarrow v_C = R_3 i = 10V$$

Energy stored in the capacitor:

$$w_C = \frac{1}{2}Cv_C^2 = \frac{1}{2}.1.10^2 = 50J$$

Energy stored in the inductor:

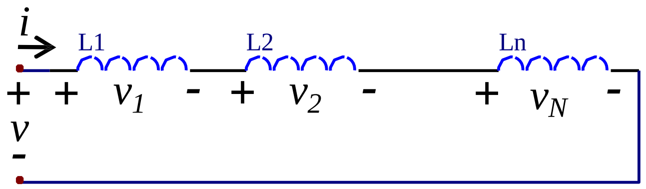
$$w_L = \frac{1}{2}Li_L^2 = \frac{1}{2}.2.2^2 = 4J$$

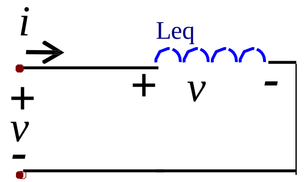
6.5. Series and Parallel Inductors

+ N inductors connected in series:

$$v = v_1 + v_2 + \ldots + v_N$$

$$\rightarrow v = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \ldots + L_N \frac{di}{dt} = \left(\sum_{k=1}^{N} L_k \right) \frac{di}{dt} = L_{eq} \frac{di}{dt}$$





$$\boxed{L_{eq} = L_1 + L_2 + \ldots + L_N}$$

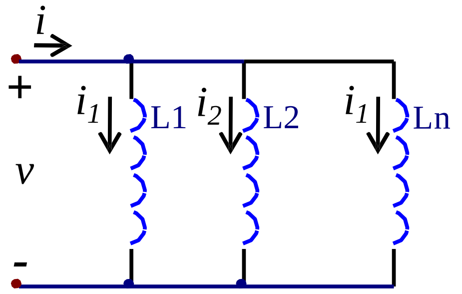
+ Equivalent inductance of series-connected inductors: sum of the individual inductances

6.5. Series and Parallel Inductors

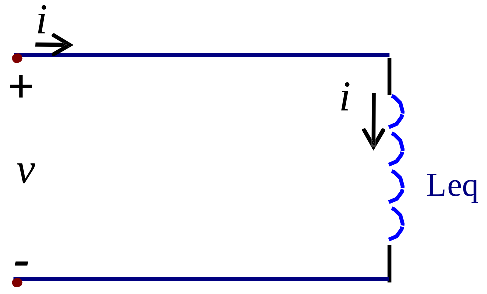
+ N inductors connected in parallel:

$$i = i_1 + i_2 + \ldots + i_N$$

$$\rightarrow i = \frac{1}{L_1}\int_{t_0}^{t} v dt + i_1(t_0) + \ldots + \frac{1}{L_N}\int_{t_0}^{t} v dt + i_N(t_0) = \left(\sum_{k=1}^{N}\frac{1}{L_k}\right)\int_{t_0}^{t} v dt + \sum_{k=1}^{N} i_k(t_0)$$



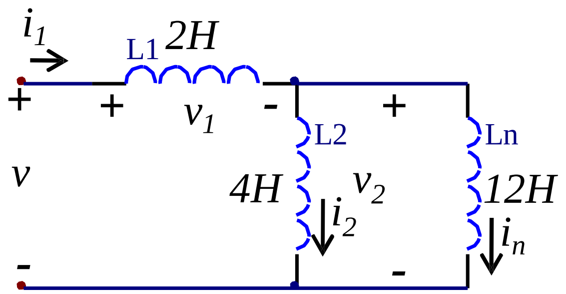
$$\boxed{L_{eq} = \frac{1}{\frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N}}}$$



+ Equivalent inductance of parallel-connected inductors: reciprocal of the sum of the reciprocals of the individual inductances

6.5. Series and Parallel Inductors

+ Example 7: Find $i_2(0)$, $v(t)$, $v_1(t)$, $v_2(t)$, $i_2(t)$, $i_n(t)$ if $i_1(t) = 4.(2 - e^{-10t})\ mA$ and $i_n(0) = -1mA$



Equivalent inductance is: $$L_{eq} = L_1 + \frac{L_2.L_N}{L_2 + L_N} = 2 + \frac{4.12}{4+12} = 5H$$

So: $$\boxed{v(t) = L_{eq}\frac{di_1}{dt} = 5.(-4)(-10)e^{-10t} = 200e^{-10t}V}$$

From: $$i_1(t) = 4\left(2 - e^{-10t}\right)mA \rightarrow i_1(0) = 4mA$$

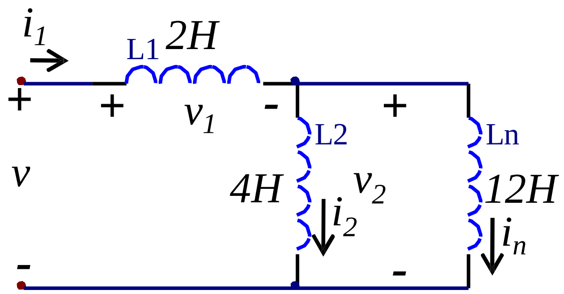
$$i_1 = i_2 + i_N \rightarrow \boxed{i_2(0) = i_1(0) - i_N(0) = 4 - (-1) = 5mA}$$

$$\boxed{v_1(t) = L_1\frac{di_1}{dt} = 2.(-4).\left(-10e^{-10t}\right) = 80e^{-10t}mV}$$

$$\boxed{v_2(t) = v(t) - v_1(t) = 200e^{-10t} - 80e^{-10t} = 120e^{-10t}mV}$$

6.5. Series and Parallel Inductors

+ Example 7: Find $i_2(0)$, $v(t)$, $v_1(t)$, $v_2(t)$, $i_2(t)$, $i_n(t)$ if $i_1(t) = 4.(2 - e^{-10t})\ mA$ and $i_n(0) = -1mA$



Current through L_2 is:

$$i_2(t) = \frac{1}{L_2}\int_0^t v_2 dt + i_2(0) = \frac{1}{4}\int_0^t 120e^{-10t} dt + 5 = 8 - 3e^{-10t} mA$$

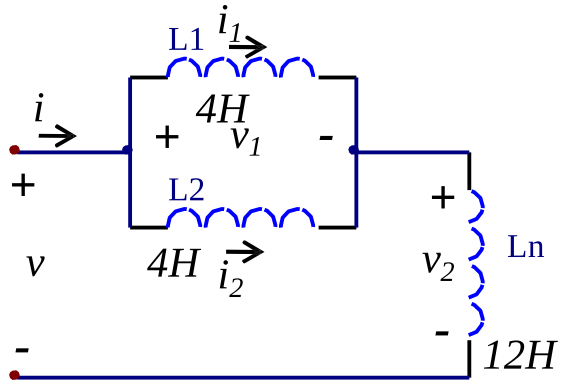
Current through L_N is:

$$i_N(t) = \frac{1}{L_N}\int_0^t v_2 dt + i_N(0) = \frac{1}{12}\int_0^t 120e^{-10t} dt - 1 = -e^{-10t} mA$$

Or by applying the KCL we otain the same result:

$$i_N(t) = i_1(t) - i_2(t) = -e^{-10t} mA$$

6.5. Series and Parallel Inductors



+ Example 8: Find $i_2(0)$, $i_2(t)$, $i(t)$, $v(t)$, $v_1(t)$, $v_2(t)$ if $i_1(t) = 0.6.e^{-2t}$ and $i(0) = 1.4A$

$$i = i_1 + i_2 \rightarrow i(0) = i_1(0) + i_2(0) \rightarrow i_2(0) = i(0) - i_1(0) = 1.4 - 0.6 = 0.8A$$

$$v_1(t) = L_1 \frac{di_1}{dt} = -4.8e^{-2t}V = v_2(t)$$

$$\rightarrow i_2(t) = \frac{1}{L_2}\int_0^t v_2 dt + i_2(0) = \frac{1}{4}\int_0^t (-4.8)e^{-2t}dt + 0.8 = 0.6e^{-2t} + 0.2 \qquad \rightarrow i(t) = i_1(t) + i_2(t) = 1.2e^{-2t} + 0.2A$$

$$\rightarrow v_2(t) = L_n \frac{di}{dt} = 12.1.2.(-2)e^{-2t} = -28.8e^{-2t}V$$

$$\rightarrow v(t) = v_1(t) + v_2(t) = -33.6e^{-2t}V$$

6.6. Applications

+ Circuit elements (R, C) are available in either discrete form or integrated circuit (IC) form, but inductance are difficult to produce on IC substrates

+ Inductor are used in applications:

→ Telephone circuits, radio, TV receivers

→ Power supplies, electric motors, microphones, loudspeakers

→ Relays, delays, sensing devices

+ Capacitors and inductors possess 03 special properties:

→ Useful for generating a current or voltage in short period of time (DC circuit)

→ Useful for suppression and converting pulsating DC voltage into relatively smooth DC voltage (DC circuit)

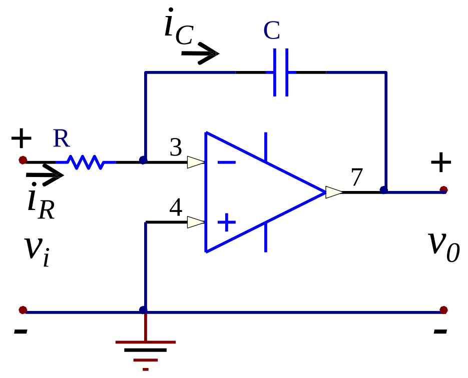
→ Useful for frequency discrimination (AC circuit)

+ Capacitors + Op amp: → Integrator and Differentiator

6.6. Applications

6.6.1. Integrator

+ Integrator : an op amp circuit whose output is proportional to the integral of the input signal



$$i_R = i_C \rightarrow \frac{v_i}{R} = -C\frac{dv_0}{dt} \rightarrow dv_0 = -\frac{1}{RC}v_i dt$$

$$\rightarrow \boxed{v_0(t) = -\frac{1}{RC}\int_0^t v_i(t)dt}$$

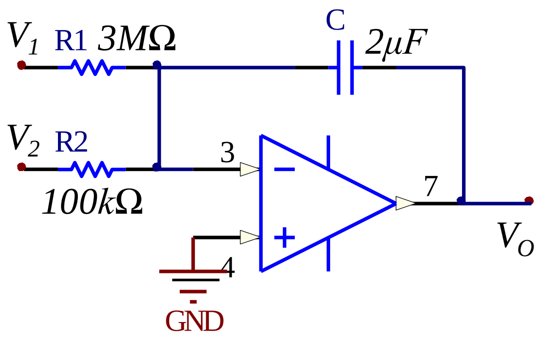
+ In practice, note that:

- Op amp integrator → requires a *feedback resistor to reduce DC gain and prevent saturation*
- Op amp integrator operates within the *linear range* → does not saturate

6.6. Applications

6.6.1. Integrator

+ Example 9: Find v_O in the op amp circuit if $v_1 = 10.\cos 2t\ (mV)$ and $v_2 = 0.5t\ (mV)$



We have a summing integrator:

$$v_0 = -\frac{1}{R_1C}\int V_1 dt - \frac{1}{R_2C}\int V_2 dt$$

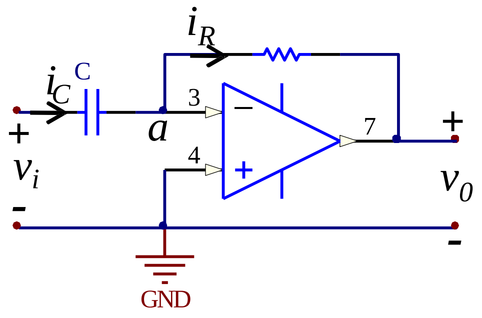
$$v_0 = -\frac{1}{3.10^6.2.10^{-6}}\int_0^t 10\cos 2t dt - \frac{1}{100.10^3.2.10^{-6}}\int_0^t 0.5t dt$$

$$v_0 = -\frac{10}{6.2}\sin 2t - \frac{1}{0.2\times 2}0.5t^2 = -0.833\sin 2t - 1.25t^2 mV$$

6.6. Applications

6.6.2. Differentiator

+ Differentiator: an op amp circuit whose output is proportional to the rate of change of the inputs signal



$$i_C = i_R \rightarrow C\frac{dv_i}{dt} = -\frac{v_0}{R}$$

$$\boxed{v_0(t) = -RC\frac{dv_i}{dt}}$$

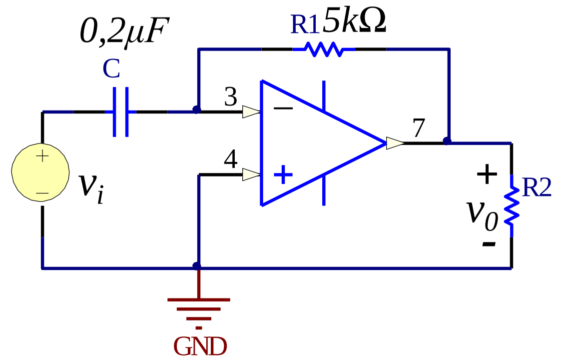
+ Note that:

- Differentiator circuits → *electronically unstable* because any electrical noise within the circuit is exaggerated by the differentiator
- Differentiator circuits → not as useful and popular as integrators

6.6. Applications

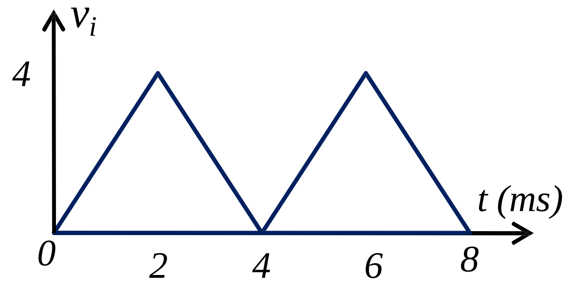
6.6.2. Differentiator

+ Example 10: Find the output voltage with the given input. Take $v_0 = 0$ at $t = 0$



For 0 < t < 4ms or 4 < t < 8ms, the input voltage is:

$$v_i = \begin{cases} 2t & 0 < t < 2ms \quad 4 < t < 6ms \\ 8 - 2t & 2 < t < 4ms \quad 6 < t < 8ms \end{cases}$$



This is a differentiator with $RC = 5\times10^3 \times 0.2\times10^{-6} = 10^{-3}$

$$v_0 = -RC\frac{dv_i}{dt} = \begin{cases} -2mV & 0 < t < 2ms \quad 4 < t < 6ms \\ 2mV & 2 < t < 4ms \quad 6 < t < 8ms \end{cases}$$