Infinite Integration by Parts

Wadood Rehman

Abstract

We explore formal manipulations of the divergent series

$$S_\infty = 1 - 1 + 1 - 1 + 1 - ...$$

using repeated tabular integration by parts. This symbolic computation suggests an assignment of zero to the series. While this does not constitute a rigorous summation in the classical sense, it is inspired by formal methods employed by Ramanujan for other divergent series, such as

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}.$$

This note highlights intriguing patterns in divergent series and invites further exploration of alternative summation techniques.

Introduction

Let $f(x)$ be a function of x, where $f^{(n)}(x)$ denotes the n-th derivative of $f(x)$. For the purposes of this document, let $F^{(n)}(x)$ denote the n-th integral (antiderivative) of $f(x)$. The tabular integration of $f(x)$ times $g(x)$ written as $\int f(x)g(x)\,dx$

$$\begin{array}{ccc} & D & I \\ + & f(x) & g(x) \\ - & f^{(1)}(x) & G^{(1)}(x) \\ + & f^{(2)}(x) & G^{(2)}(x) \\ - & f^{(3)}(x) & G^{(3)}(x) \\ \vdots & \vdots & \vdots \end{array}$$

The next step is to multiply them diagonally downwards from the left side to the right one, so $f(x)$ multiplies with $G^{(1)}(x)$, $f^{(1)}(x)$ multiplies with $G^{(2)}(x)$ and finally we could take $\int f^{(n)}(x)G^{(n)}(x)\,dx$ resulting in a series

$$f(x)G^{(1)}(x) - f^{(1)}(x)G^{(2)}(x) + f^{(2)}(x)G^{(3)}(x) - f^{(3)}(x)G^{(4)}(x) + \cdots + \int f^{(n)}(x)G^{(n)}(x)\,dx$$

Alternating Diverging Series

In the first example $\int e^x \sin x\,dx$ we will use the tabular integration but repeat it infinitely where $f(x) = \sin x$ and $g(x) = e^x$

$$\begin{aligned} I &= \int e^x \sin x\,dx \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x\,dx \\ &\Rightarrow 2I = e^x(\sin x - \cos x) \\ &\boxed{I = \frac{1}{2}e^x(\sin x - \cos x) + C} \end{aligned}$$

Now we use the tabular integration method to expand this integral into an infinite series

$$I = \int e^x \sin x \, dx = \sin x \cdot e^x - \cos x \cdot e^x - \sin x \cdot e^x + \cos x \cdot e^x + ...$$

Factoring e^x

$$\frac{1}{2} e^x (\sin x - \cos x) = e^x \big((\sin x - \cos x) - (\sin x - \cos x) + ... \big)$$

Factoring $(\sin x - \cos x)$

$$\frac{1}{2} e^x (\sin x - \cos x) = e^x (\sin x - \cos x)(1 - 1 + 1 - 1 + 1 - 1 + ...)$$

The result of this is

$$\boxed{S_\infty = 1 - 1 + 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}}$$

Which is the same result as the Cesàro summation of this series More generally, the integral $\int e^{ax} \sin x \, dx$ can be used to derive the non-standard sum

$$1 - r + r^2 - r^3 + r^4 - r^5 + \cdots = \frac{1}{1+r}$$

This follows from the appearance of terms proportional to $\sin x$ and $\cos x$ with coefficients depending rationally on r, mirroring the structure of the alternating geometric series.

Beyond Zeta Regularized Values

Non-standard values can be found for diverging series, other than Reimann Zeta Regularized Values For example by using

$$\int \frac{e^x}{x} \, dx = \mathrm{Ei}(x) + C = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n \, n!} + C,$$

Again we use recursive integration by parts to get

$$\begin{aligned} I = \int \frac{e^x}{x} \, dx &= \frac{e^x}{x} + \frac{e^x}{x^2} + \frac{2e^x}{x^3} + \frac{6e^x}{x^4} + ... \\ &= \frac{0! \cdot e^x}{x} + \frac{1! \cdot e^x}{x^2} + \frac{2! \cdot e^x}{x^3} + \frac{3! \cdot e^x}{x^4} + ... \end{aligned}$$

Writing in summation notation, this leads to

$$\mathrm{Ei}(x) = e^x \sum_{n=0}^{\infty} \frac{n!}{x^{n+1}}$$

Putting $x = 1$

$$\mathrm{Ei}(1) = e \sum_{n=0}^{\infty} n!$$

Dividing by e on both sides

$$\frac{\mathrm{Ei}(1)}{e} = \sum_{n=0}^{\infty} n!$$

We reach a result which is not possible by Zeta Regularized Function

$$\boxed{0! + 1! + 2! + 3! + 4! + \cdots = \frac{Ei(x)}{e} \approx 0.69717}$$

This actually gives non standard values for all values of x in $\mathrm{Ei}(x)$ except $x = 0$

It should be noted that the method works particularly well with functions involving e^x and may fail for other forms. It is a heuristic approach rather than a formal proof, and with that this paper is concluded.

References

[1] Youtube Video by BlackPenRedPen, *integration by parts, DI method, VERY EASY*
Youtube Video Link

[2] Hobart Pao, Tapas Mazumdar, and Mahindra Jain, *Tabular Integration* Brilliant.org
Webpage Link

[3] Wolfram *Exponential Integral*
Webpage Link

[4] Nicolas M Robles *Zeta Function Regularization* Department of Theoretical Physics Imperial College London, September 25th 2009
PDF Link

[5] Andoni Benito Gonz´alez *The Riemann Zeta Function and Zeta Regularization in Casimir Effect* Leioa, 17 June 2020
PDF Link