Cardinals and Cardinal Arithmetics

Cardinals

Cardinals, or cardinal numbers, represent the size or "cardinality" of sets in set theory. The arithmetic of cardinal numbers, known as cardinal arithmetic, studies the operations of addition, multiplication, and exponentiation on cardinals. Here's a brief overview:

1. **Cardinals**:
    - Finite cardinals are just the natural numbers: `0, 1, 2, 3, ...`.
    - The size of the set of natural numbers is the first infinite cardinal, denoted as $\aleph_0$.
    - There are larger infinite cardinals, like $\aleph_0$, $\aleph_1$,....
2. **Cardinal Arithmetic**:
    - **Addition**:
        - For finite cardinals, addition is the usual arithmetic addition.
        - For infinite cardinals, if κ is any infinite cardinal, then $\kappa + \kappa = \kappa$. For example, $\aleph_0 + \aleph_0 = \aleph_0$.
        - For any cardinals κ and λ with $\kappa \leq \lambda$, $\kappa + \lambda = \lambda$.
        - **Abelian monoid**: associativity, closure, identity, commutativity.
    - **Multiplication**:
        - For finite cardinals, multiplication is the usual arithmetic multiplication.
        - For infinite cardinals, if κ is any infinite cardinal, then $\kappa \times \kappa = \kappa$. For instance, $\aleph_0 \times \aleph_0 = \aleph_0$.
        - For any cardinals κ and λ with $\kappa \leq \lambda$, $\kappa \times \lambda = \lambda$.
        - **Abelian monoid**: associativity, closure, identity, commutativity.
    - **Exponentiation**:
        - For finite cardinals, exponentiation is the usual arithmetic exponentiation.
        - "Continuum Hypothesis"

Isabelle Definition

The cardinality is established by building a bijection with the smallest ordinal number, which is based on the definition of ordinal numbers.

```
definition vcard :: "V⇒V"
  where "vcard a ≡ (LEAST i. Ord i ∧ elts i ≈ elts a)"
```

```
definition eqpoll :: "'a set ⇒ 'b set ⇒ bool" (infixl "≈" 50)
  where "eqpoll A B ≡ ∃f. bij_betw f A B"
```

```
definition bij_betw :: "('a ⇒ 'b) ⇒ 'a set ⇒ 'b set ⇒ bool"  — ‹bijective›
  where "bij_betw f A B ⟷ inj_on f A ∧ f ` A = B"
```

Addition is defined by the cardinality of disjoint union.

```
definition cadd :: "[V,V]⇒V"       (infixl ‹⊕› 65)
  where "κ ⊕ μ ≡ vcard (κ ⊎ μ)"
```

```
definition vsum :: "V ⇒ V ⇒ V" (infixl "⊎" 65) where
 "A ⊎ B ≡ (VSigma (set {0}) (λx. A)) ⊔ (VSigma (set {1}) (λx. B))"
```

```
definition VSigma :: "V ⇒ (V ⇒ V) ⇒ V"
  where "VSigma A B ≡ set(⋃x ∈ elts A. ⋃y ∈ elts (B x). {⟨x,y⟩})"
```

Multiplication is defined by the cardinality of Cartesian product.

```
definition cmult :: "[V,V]⇒V"       (infixl ‹⊗› 70)
  where "κ ⊗ μ ≡ vcard (VSigma κ (λz. μ))"
```