MEN SIUD

SET OF NUMBERS

1.0 INTRODUCTION

Although, the theory of sets is very general, important sets, which we meet in elementary mathematics, are sets of numbers. Of particular importance, especially in analysis, is the set of ***real numbers,*** which we denote by $\mathcal{R}$

In fact, we assume in this unit, unless otherwise stated, that the set of real numbers $\mathcal{R}$ is out universal set. We first review some elementary properties of real numbers before applying our elementary principles of set theory to sets of numbers. The set of real numbers and its properties is called the ***real number system.***

2.0 OBJECTIVES

After studying this unit, you should be able to do the following:

- Represent the set of numbers on the real line
- Perform the basic set operations on intervals

3.0 MAIN BODY

3.1 REAL NUMBERS, $\mathcal{R}$

One of the most important properties of the real numbers is that points on a straight line that can represent them. As in Fig 3.1, we choose a point, called the origin, to represent 0 and another point, usually to the right, to represent 1. Then there is a natural way to pair off the points on the line and the real numbers, that is, each point will represent a unique real number and each real number will be represented by a unique point. We refer to this line as the ***real line.*** Accordingly, we can use the words point and number interchangeably.

Those numbers to the right of 0, i.e. on the same side as 1, are called the *positive numbers* and those numbers to the left of 0 are called the *negative numbers*. The number 0 itself is neither positive nor negative.

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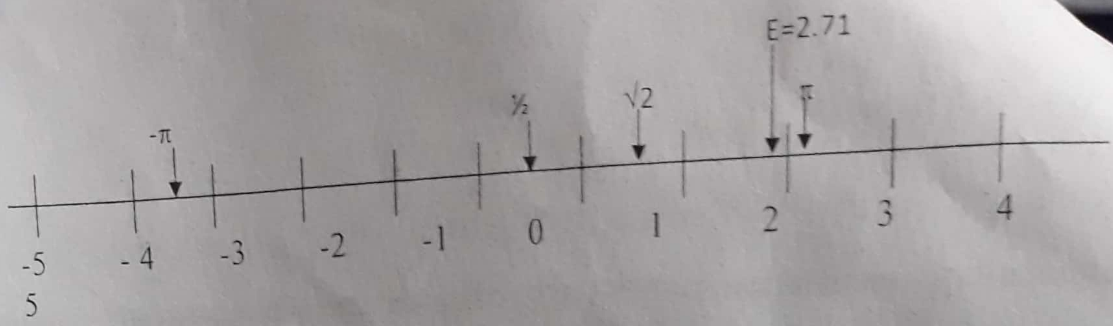

Fig 3.1

3.1.2 Integers, Z

The integers are those real numbers

…, -3, -2, -1, 0, 1, 2, 3,…

We denote the integers by Z; hence we can write

Z = { …, -2, - 1, 0, 1, 2,…}

The integers are also referred to as the "whole" numbers.

One important property of the integers is that they are "closed" under the operations of addition, multiplication and subtraction; that is, the sum, product and difference of two integers is again in integer. Notice that the quotient of two integers, e.g. 3 and 7, need not be an integer; hence the integers are not closed under the operation of division.

3.1.3 Rational Numbers, Q

The *rational numbers* are those real numbers, which can be expressed as the ratio of two integers. We denote the set of rational numbers by Q. Accordingly,

$Q = \{x \mid x = p|q \text{ where } p \in z, q \in z\}$

Notice that each integer is also a rational number since, for example, 5 = 5/1; hence Z is a subset of Q.

The rational numbers are closed not only under the operations of addition, multiplication and subtraction but also under the operation of division (except by 0). In other words, the sum, product, difference and quotient (except by 0) of two rational numbers is again a rational number.

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3.1.4 Natural Numbers, N

The *natural numbers* are the positive integers. We denote the set of natural numbers by N; hence N = {1,2,3.....}

The natural numbers were the first number system developed and were used primarily, at one time, for counting. Notice the following relationship between the above numbers systems:

$N \subset Z \subset Q \subset R$

The natural numbers are closed only under the operation of addition and multiplication. The difference and quotient of two natural numbers needed not be a natural number.

The *prime numbers* are those natural numbers p, excluding 1, which are only divisible 1 and p itself. We list the first few prime numbers: 2,3,5,7,11,13,17,19...

3.1.5 Irrational Numbers, Q'

The irrational numbers are those real numbers which are not rational, that is, the set of irrational numbers is the complement of the set of rational numbers Q in the real numbers R; hence Q' denote the irrational numbers. Examples of irrational numbers are $\sqrt{3}$, π, $\sqrt{2}$, etc.

3.1.6 Line Diagram of the Number Systems

Fig 3.2 below is a line diagram of the various sets of number, which we have investigated. (For completeness, the diagram include the sets of complex numbers, number of the form a + bi where a and b are real. Notice that the set of complex numbers is superset of the set of real numbers.)

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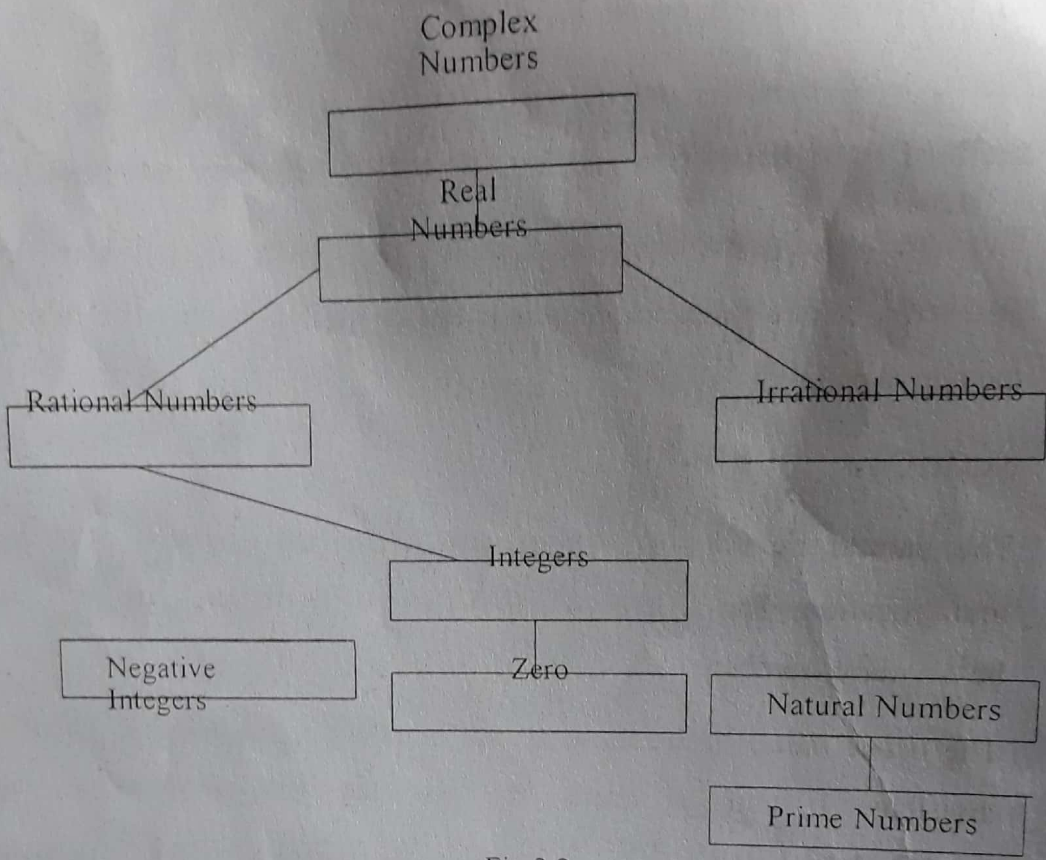

Fig 3.2

3.2 DECIMALS AND REAL NUMBERS

Every real number can be represented by a "non-terminating decimal". The decimal representation of a rational number p/q can be found by "dividing the denominator q into the numerator p". If the indicated division terminates, as for

3/8 = .375

We write 3/8 = .375000

Or 3/8 = .374999…

If we indicated division of q into p does not terminate, then it is known that a block of digits will continually be repeated; for example, 2/11 = .181818…

We now state the basic fact connecting decimals and real numbers. The rational numbers correspond precisely to those decimals in which a block of

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digits is continually repeated, and the irrational numbers correspond to the other non-terminating decimals.

3.3 INEQUALITIES

The concept of "order" is introduced in the real number system by the

Definition: The real number a is less than the real number b,

written $a < b$

If $b - a$ is a positive number.

The following properties of the relation $a < b$ can be proven. Let a, b and c be real numbers; then:

P_1: Either $a < b$, $a = b$ or $b < a$.

P_2: If $a < b$ and $b < c$, then $a < c$.

P_3: If $a < b$, then $a + c < b + c$

P_4: If $a < b$ and c is positive, then $ac < bc$

P_5: If $a < b$ and c is negative, then $bc < ac$.

4

3<4

4<5

3<5.

Geometrically, if $a < b$ then the point a on the real line lies to the left of the point b.

We also denote $a < b$ by $b > a$

Which reads "b is *greater then* a". Furthermore, we write

$a < b$ or $b > a$

if $a < b$ or $a = b$, that is, if a is not greater than b.

Example 1.1: $2 < 5$; $-6 < -3$ and $-4 < +4$; $5 > -8$

Example 1.2: The notation $x < 5$ means that x is a real number which is less than 5; hence x lies to the left of 5 on the real line.

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The notation $2 < x < 7$; means $2 < x$ and also $x < 7$; hence x will lie between 2 and 7 on the real line.

Remark 3.1: Notice that the concept of order, i.e. the relation $a < b$, is defined in terms of the concept of positive numbers. The fundamental property of the positive numbers which is used to prove properties of the relation $a < b$ is that the positive numbers are closed under the operations of addition and multiplication. Moreover, this fact is intimately connected with the fact that the natural numbers are also closed under the operations of addition and multiplication.

Remark 3.2: The following statements are true when a, b, c are any real numbers:

1. $a < a$

2. if $a < b$ and $b < a$ then $a = b$.

3. if $a < b$ and $b < c$ then $a < c$.

3.4 ABSOLUTE VALUE

The absolute value of a real number x, denoted by $|x|$ is defined by the formula

$$|x| = \begin{cases} x \text{ if } x \geqslant 0 \\ -x \text{ if } x < 0 \end{cases}$$

that is, if x is positive or zero then $|x|$ equals x, and if x is negative then $|x|$ equals $-x$. Consequently, the absolute value of any number is always non-negative, i.e. $|x| > 0$ for every $x \in R$.

Geometrically speaking, the absolute value of x is the distance between the point x on the real line and the origin, i.e. the point 0. Moreover, the distance between any two points, i.e. real numbers, a and b is $|a - b| = |b - a|$.

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Example 2.1: $|-2| = 2$, $|7| = 7$. $|-p| = p$

Example 2.2: The statement $|x| < 5$ can be interpreted to mean that the distance between x and the origin is less than 5, i.e. x must lies between -5 and 5 on the real line. In other words,

$|x| < 5$ and $-5 < x < 5$

have identical meaning. Similarly,

$|x| < 5$ and $-5 < x < 5$

have identical meaning.

3.5 INTERVALS

Consider the following set of numbers;

$A_1 = \{x| \ 2 < x < 5\}$

$A_2 = \{x| \ 2 \leq x \leq 5\}$

$A_3 = \{x| \ 2 < x \leq 5\}$

$A_4 = \{x| 2 \leq x < 5\}$

Notice, that the four sets contain only the points that lie between 2 and 5 with the possible exceptions of 2 and/or 5. We call these sets intervals, the numbers 2 and 5 being the endpoints of each interval. Moreover, A_1 is an ***open interval*** as it does not contain either end point: A_2 is a ***closed interval*** as it contains bother endpoints; A_3 and A_4 are ***open-closed*** and ***closed-open*** respectively.

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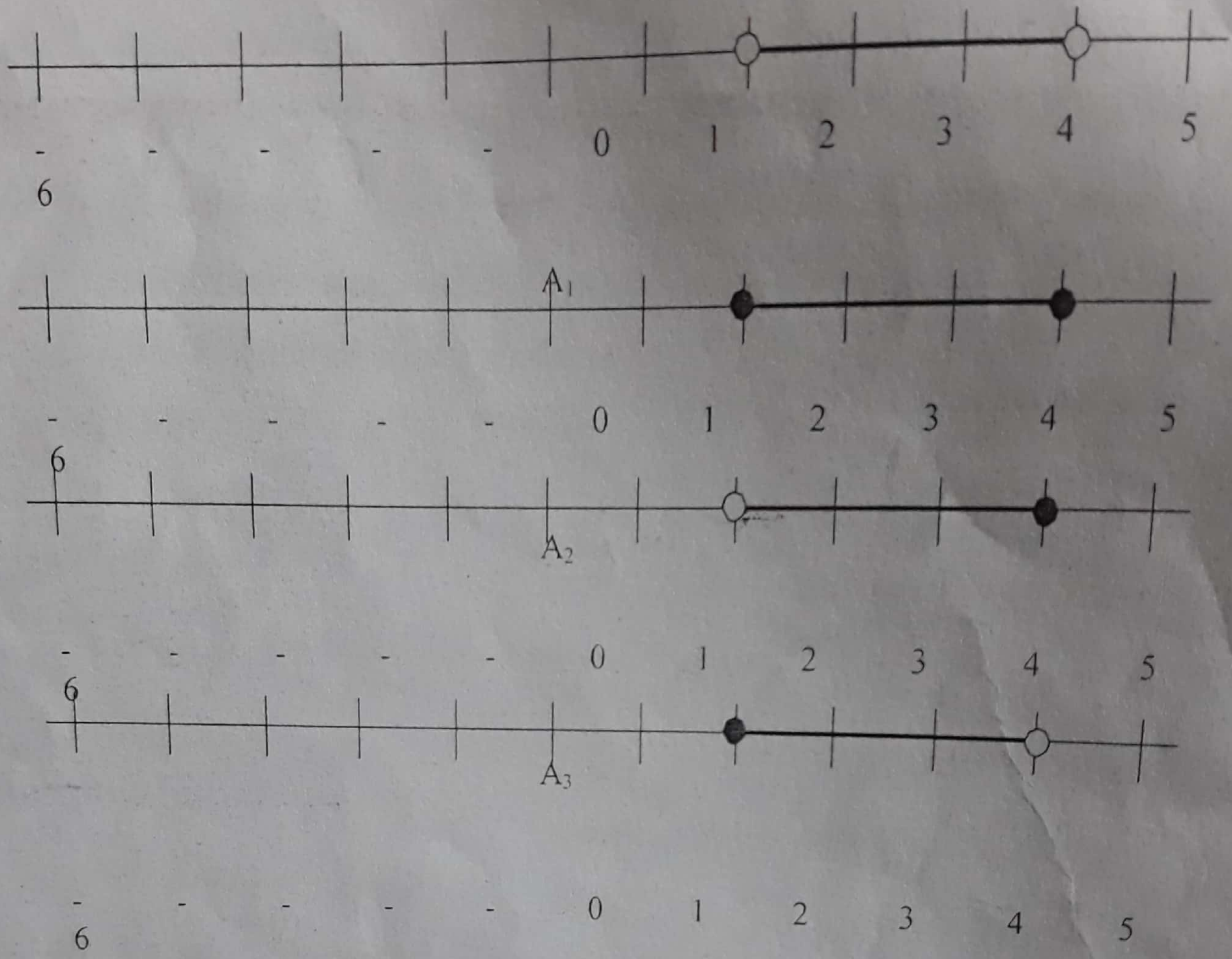
We display, i.e. graph, these sets on the real line as follows.



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Notice that in each diagram we circle the endpoints 2 and 5 and thicken (or shade) the line segment between the points. If an interval includes an endpoint, then this is denoted by shading the circle about the endpoint.

Since intervals appear very often in mathematics, a shorter notation is frequently used to designated intervals, Specifically, the above intervals are sometimes denoted by;

$A_1 = (2, 5)$

$A_2 = [2, 5]$

$A_3 = (2, 5]$

$A_4 = [2, 5)$

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Notice that a parenthesis is used to designate an open endpoint, i.e. an endpoint that is not in the interval, and a bracket is used to designate a closed endpoint.

3.5.1 Properties of Intervals

Let $\Im$ be the family of all intervals on the real line. We include in $\Im$ the null set $\varnothing$ and single points a = [a, a]. Then the intervals have the following properties:

1. The intersection of two intervals is an interval, that is, $A \in \Im$, $B \in \Im$ implies $A \cap B \in \Im$

2. The union of two non-disjoint intervals is an interval, that is, $A \in \Im$, $B \in \Im$, $A \cap B \neq \varnothing$ ipmlies $A \cup B \in \Im$

3. The difference of two non-comparable intervals is an interval, that is, $A \in \Im$, $B \in \Im$, $A \not\subset B$, $B \not\subset A$ implies $A - B \in \Im$

Example 3.1: Let A = [2, 4), B = (3, 8). Then

$A \cap B = (3, 4)$, $A \cup B = [2, 8)$

$A - B = [2, 3]$, $B - A = [4, 8)$

3.5.2 Infinite Intervals

Sets of the form

$A = \{x \mid x > 1\}$

$B = \{x \mid x \geq 2$

$C = \{x \mid x < 3\}$

$D = \{x \mid x \leq 4\}$

$E = \{x \mid x \in \Re\}$

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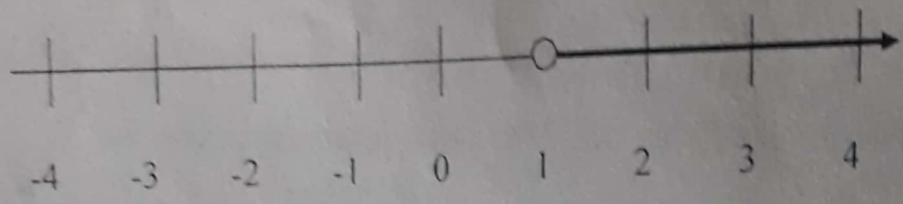
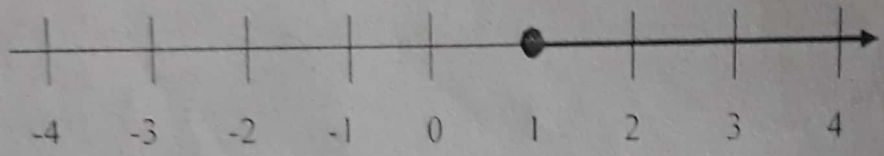
Are called infinite intervals and are also denoted by

$A = (1, \infty)$, $B = [2, \infty)$, $C = (-\infty, 3)$, $D = (-\infty, 4]$, $E = (-\infty, \infty)$

We plot these intervals on the real line as follows:



A is Shaded



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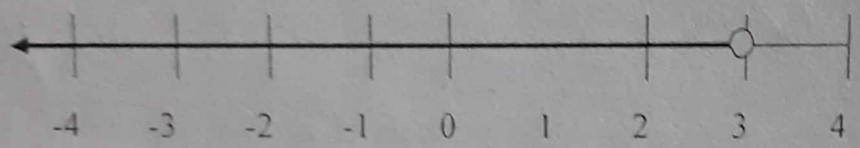
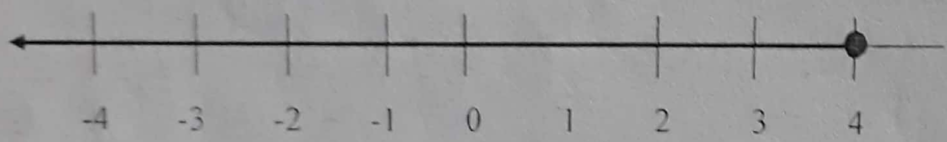
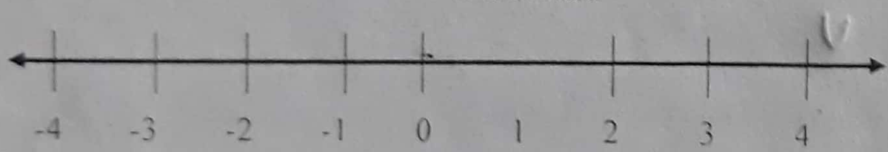


C is Shaded



D is Shaded



E is Shaded

3.6 BOUNDED AND UNBOUNDED SETS

Let A be a set of numbers, then A is called ***bounded*** set if A is the subset of a finite interval. An equivalent definition of boundedness is;

Definition 3.1: Set A is ***bounded*** if there exists a positive number M such that $|x| \leq M$.

for all $x \in A$. A set is called ***unbounded*** if it is not bounded

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