QUADRATIC EQUATIONS AND FUNCTIONS

Roots of a quadratic equation

Recall the standard quadratic equation given as:

$ax^2+bx+c=0$ (provided $a \neq 0$)

The roots of this equation is given by:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{D}}{2a} \quad \text{where } D = b^2-4ac$$

Sum and Product of the Roots

Given the equation, $ax^2+bx+c=0$ has roots α and β.
It is equivalent to the equation;

$$(x-\alpha)(x-\beta)=0$$

This gives $x=\alpha$, $x=\beta$. By expanding $(x-\alpha)(x-\beta)$, gives

$$x^2-(\alpha+\beta)x+\alpha\beta.$$

Compare the following equations from above:

$$x^2+\frac{b}{a}x+\frac{c}{a}=0 \text{ (Dividing through by a)} \cdots (1)$$

$$x^2-(\alpha+\beta)x+\alpha\beta=0$$

Hence, $\alpha+\beta = -b/a$ and $\alpha\beta = \frac{c}{a}$

For any quadratic equation $ax^2+bx+c=0$ with roots α and β gives the following:

1) Sum of roots: $\alpha+\beta = -\frac{b}{a}$

2) Product of roots: $\alpha\beta = \frac{c}{a}$

The sum and the product of the roots can also be derived from the formula for the roots.

Let $\alpha = \frac{-b+\sqrt{D}}{2a}$ and $\beta = \frac{-b-\sqrt{D}}{2a}$

Then, $\alpha+\beta = \frac{-b+\sqrt{D}}{2a} + \frac{-b-\sqrt{D}}{2a}$

$$= \frac{-b+\sqrt{D}-b-\sqrt{D}}{2a}$$

$$= \frac{-2b}{2a}$$

$$= -\frac{b}{a}$$

And

$$\alpha\beta = \left(\frac{-b+\sqrt{D}}{2a}\right)\left(\frac{-b-\sqrt{D}}{2a}\right)$$

$$= \frac{b^2-D}{4a^2}$$

$$= \frac{b^2-(b^2-4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$$\alpha\beta = \frac{c}{a}$$

Example 1:- If the roots of $3x^2-4x-1=0$ are α and β. Find $\alpha+\beta$ and $\alpha\beta$.

Solution

By comparing the given equation with the standard form $ax^2+bx+c=0$, $a=3$, $b=-4$ and $c=-1$.

$$\text{Hence, } \alpha+\beta = -\frac{b}{a} = -\frac{(-4)}{3}$$
$$= \frac{4}{3}$$

$$\alpha\beta = \frac{-1}{3} = \frac{c}{a}$$

Note: The equation, $ax^2+bx+c=0$ is equivalent to $x^2+\frac{b}{a}x+\frac{c}{a}=0$. That is, $x^2-(\alpha+\beta)x+\alpha\beta=0$ where α and β are the roots. Then, any quadratic equation can be written as:

$$x^2 - \begin{pmatrix}\text{sum of} \\ \text{the roots} \\ \alpha+\beta\end{pmatrix}x + \begin{pmatrix}\text{product} \\ \text{of the} \\ \text{roots} \\ \alpha\beta\end{pmatrix} = 0$$

Example 2: Construct an equation with roots $\sqrt{2}+1, \sqrt{2}-1$

Sum of roots, $\alpha+\beta = \sqrt{2}+1+\sqrt{2}-1$
$$= 2\sqrt{2}$$

Product of roots, $\alpha\beta = (\sqrt{2}+1)(\sqrt{2}-1)$
$$= 1$$

From $x^2-(\alpha+\beta)x+\alpha\beta=0$, the equation is:
$$x^2-2\sqrt{2}x+1=0$$

Exerase

1) If α, β are roots of the following equations, state the values for each of $\alpha+\beta$ and $\alpha\beta$.

a) $x^2-x+1=0$

b) $2x^2+x-3=0$

c) $x^2+\sqrt{3}x+1=0$

d) $px^2=q$

e) $pt^2-qt-r=0$

f) $2x^2+x-3=0$

g) $2y^2-(a+3)y+a^2=0$

2) Construct and simplify equations whose roots will be:

a) $-3, 1$ (b) $\frac{1}{2}, 2$ (c) $-5, -6$ (d) $\sqrt{3}-2, \sqrt{3}+2$

e) $\frac{1}{1+\sqrt{2}}, \frac{1}{1-\sqrt{2}}$ (f) $\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}$

Example 3: If α, β are the roots of $2x^2-x-2=0$, find the values of

(a) $\alpha^2+\beta^2$ (b) $\alpha-\beta$ (c) $\alpha^2-\beta^2$ (d) $\frac{1}{\alpha}+\frac{1}{\beta}$ (e) $\alpha^3+\beta^3$

(f) $\alpha^3-\beta^3$

Solution

Compare $ax^2+bx+c=0$ to $2x^2-x-2=0$.

$a=2$, $b=-1$, $c=-2$. Therefore,

$$\alpha+\beta = \frac{-b}{a} = -\left(\frac{-1}{2}\right) = \frac{1}{2} \quad \text{and} \quad \alpha\beta = \frac{c}{a} = \frac{-2}{2} = -1$$

a) $\alpha^2+\beta^2$

$=(\alpha+\beta)^2-2\alpha\beta$

$=\left(\frac{1}{2}\right)^2-2(-1)$

$=\frac{1}{4}+2$

$=\frac{9}{4}$

b) $\alpha-\beta$ can be obtained from $\cancel{(\alpha+\beta)^2-4\alpha\beta}$

$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$

$=\left(\frac{1}{2}\right)^2-4(-1)$

$=\frac{1}{4}+4$

$(\alpha-\beta)^2=\frac{17}{4}$

Hence, $\alpha-\beta=\pm\sqrt{\frac{17}{4}}=\pm\frac{\sqrt{17}}{2}$ depending on whethere $\alpha>\beta$ OR $\alpha<\beta$.

c) $\alpha^2-\beta^2=(\alpha+\beta)(\alpha-\beta)$

$=\left(\frac{1}{2}\right)\left(\frac{\sqrt{17}}{2}\right)$ (taking $\alpha>\beta$)

$=\frac{\sqrt{17}}{4}$

d) $\frac{1}{\alpha}+\frac{1}{\beta}$

$=\frac{\beta+\alpha}{\alpha\beta}$

$=\frac{\frac{1}{2}}{-1}$

$=-\frac{1}{2}$

e) $\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$

$=(\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)$

$$= (\alpha+\beta)\left[(\alpha+\beta)^2 - 3\alpha\beta\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}\right)^2 - 3(-1)\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + 3\right]$$

$$= \frac{13}{8}$$

f) $\alpha^3 - \beta^3 = (\alpha-\beta)(\alpha^2 + \alpha\beta + \beta^2)$

$$= (\alpha-\beta)\left[\alpha^2 + \beta^2 + \alpha\beta\right]$$

$$= (\alpha-\beta)\left[(\alpha+\beta)^2 - 2\alpha\beta + \alpha\beta\right]$$

$$= (\alpha-\beta)\left[(\alpha+\beta)^2 - \alpha\beta\right]$$

$$= \frac{\sqrt{17}}{2}\left[\left(\frac{1}{2}\right)^2 - (-1)\right]$$

$$= \frac{\sqrt{17}}{2}\left[\frac{1}{4} + 1\right]$$

$$= \frac{5\sqrt{17}}{8}$$

<u>Example 4</u>: If α, β are roots of $3x^2 + 5x - 1 = 0$. Construct equations whose roots are:

a) $5\alpha, 5\beta$ (b) α^2, β^2 (c) $\frac{1}{\alpha}, \frac{1}{\beta}$ (d) $\alpha + \frac{1}{\beta}, \beta + \frac{1}{\alpha}$

<u>Solution</u>

From the given equation, $\alpha + \beta = -b/a = -5/3$

$\alpha\beta = \frac{c}{a} = -1/3$

a) Sum of roots $= 5\alpha + 5\beta$

$= 5(\alpha+\beta)$

$= 5(-5/3)$

$= -25/3$

Product of roots $= (5\alpha)(5\beta)$

$= 25\alpha\beta$

$= 25\left(-\frac{1}{3}\right)$

$= -\frac{25}{3}$

Therefore, the required equation is $x^2 - (\alpha+\beta)x + \alpha\beta = 0$. This e is

$x^2 - \left(-\frac{25}{3}\right)x + \left(-\frac{25}{3}\right) = 0$

$x^2 + \frac{25}{3}x - \frac{25}{3} = 0$

$3x^2 + 25x - 25 = 0$

b) Sum of roots $= \alpha^2 + \beta^2$

$= (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(\frac{-5}{3}\right)^2 - 2\left(-\frac{1}{3}\right)$

$= \frac{25}{9} + \frac{2}{3}$

$= 31/9$

Product of roots $= (\alpha^2)(\beta^2)$

$= (\alpha\beta)^2$

$= \left(\frac{-1}{3}\right)^2$

$= \frac{1}{9}$

Therefore, the equation is given as $x^2 - (31/9)x + \frac{1}{9} = 0$

$= 9x^2 - 31x + 1 = 0$

c) Sum of roots $= \frac{1}{\alpha} + \frac{1}{\beta}$

$$= \frac{\beta + \alpha}{\alpha\beta}$$

$$= \frac{-5/3}{-1/3}$$

$$= 5$$

Product of roots $= \frac{1}{\alpha}\left(\frac{1}{\beta}\right)$

$$= \frac{1}{\alpha\beta} = -3$$

The equation is given as $x^2 - 5x + (-3) = 0$

$$x^2 - 5x - 3 = 0$$

d) Sum of roots $= \alpha + \frac{1}{\beta} + \beta + \frac{1}{\alpha}$

$$= \alpha + \beta + \frac{\beta + \alpha}{\alpha\beta}$$

$$= -5/3 + 5$$

$$= 10/3$$

Product of roots $= \left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$

$$= \alpha\beta + 1 + 1 + \frac{1}{\alpha\beta}$$

$$= -\frac{1}{3} + 2 - 3$$

$$= -4/3$$

The equation is $x^2 - 10/3\, x + (-4/3) = 0$

$$3x^2 - 10x - 4 = 0$$

Example 5: One root of the equation $27x^2+bx+8=0$ is Known to be the square of the other. Find b?

Solution

Let the roots be α and α^2

Then, sum of roots, $\alpha+\alpha^2=\frac{-b}{a}=\frac{-b}{27}$;

Product of roots $\alpha(\alpha^2)=\frac{c}{a}=\frac{8}{27}$

$$\alpha^3=\frac{8}{27}$$

$$\alpha=\frac{2}{3} \quad \text{-----}(*)$$

Substitute (*) into the sum of roots,

$$\frac{2}{3}+\left(\frac{2}{3}\right)^2=\frac{-b}{27}$$

$$\frac{2}{3}+\frac{4}{9}=\frac{-b}{27}$$

Therefore, $b=-30$

Exercise

1) If α,β are the roots of $x^2+3x=5$. Calculate the values of the following:

a) $\alpha+\beta$ (b) $\alpha\beta$ (c) $\alpha-\beta$ (d) $\alpha^2+\beta^2$ (e) $\alpha^3+\beta^3$

(f) $\frac{1}{\alpha}+\frac{1}{\beta}$

2) If α,β are the roots of the given equations, construct new equations whose roots are:

a) $2\alpha, 2\beta$ (b) α^2, β^2 (c) $\frac{1}{\alpha}, \frac{1}{\beta}$ (d) α^3, β^3 (e) $\alpha+1, \beta+1$

Simplify the results as much as possible

i) $x^2+x+1=0$ (ii) $x^2-x+2=0$

iii) $2x^2+x-4=0$ (iv) $3x^2-2x=1$

v) $ax^2+bx+c=0$

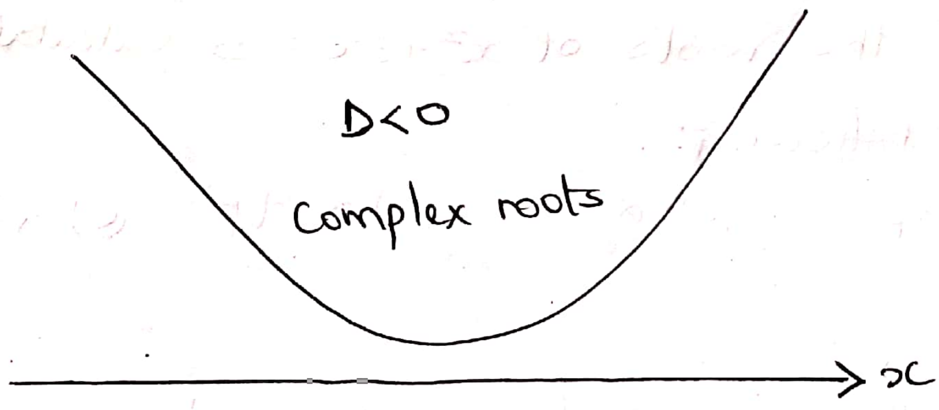
3) If α, β are the roots of $ax^2+bx+c=0$. Form equations whose roots are:

(a) $-\alpha, -\beta$ (b) $\alpha-2, \beta-2$

4) Given that α, β are the roots of the equation $3x^2-x-5=0$, form the equation whose roots are $2\alpha-\frac{1}{\beta}, 2\beta-\frac{1}{\alpha}$.

Types of Roots of a Quadratic Equation

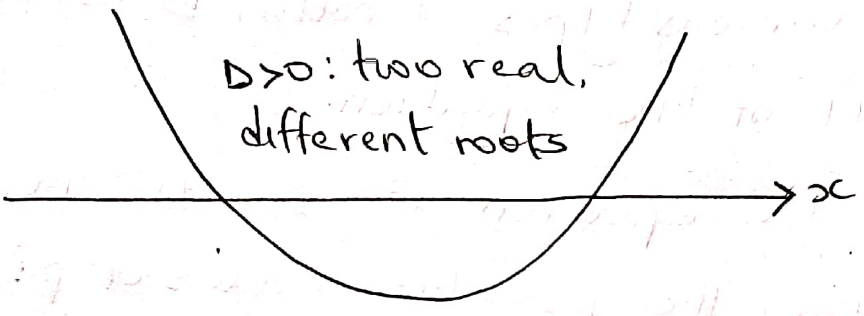
I) If D is negative ($D<0$ or $b^2<4ac$), then there is no real value for $\sqrt{D}$. The function has no zeros and the curve does not cut the x-axis. ~~The equation~~



The equation has COMPLEX roots. For example, $2x^2-x+3=0$ has no real roots. That is,

$(-1)^2<4(2\times3)$

II) If D is positive ($D > 0$ or $b^2 > 4ac$), then there are two values for $\sqrt{D}$ and the equation will have two different real roots. The curve cuts the x-axis twice, as shown below:



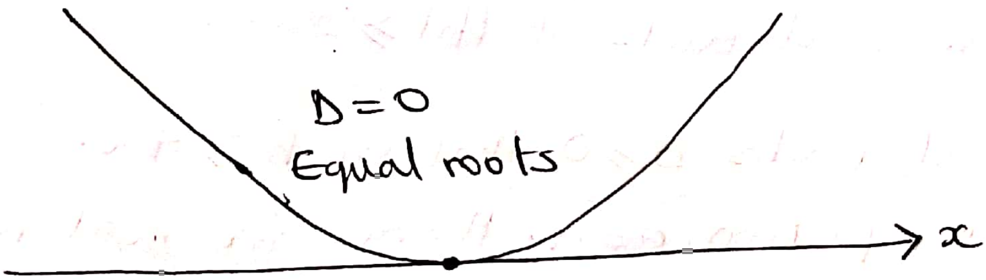
For example, $2x^2 - 3x - 1 = 0$ gives $x = \frac{3 \pm \sqrt{17}}{4}$, two different irrational roots.

· If D is a perfect square, the roots will be rational numbers. For example, $2x^2 + 3x - 5 = 0$ gives

$$x = \frac{-3 \pm \sqrt{49}}{4} = \frac{-3 \pm 7}{4}$$

That is, $x = 1$ or $-5/2$

III) If $D = 0$ ($b^2 = 4ac$) then $x = -b/2a$ and the roots are equal. The curve touches the x-axis in this case with the two roots merging into one.



For example, $4x^2-12x+9=0$ has equal roots as $(-12)^2=4(4)(9)$

Hence, for the equation $ax^2+bx+c=0$ to have real roots, $D \geq 0$, that is, $b^2 \geq 4ac$. Since the value of D discriminates between the various types of roots, D is called the DISCRIMINANT of the equation.

Example: If the equation $x^2-3x+1=p(x-3)$ has equal roots, find the possible values of p?

Rewriting the equation in standard form

$$x^2-x(3+p)+(1+3p)=0$$

then, $a=1$, $b=-(3+p)$, $c=(1+3p)$

For equal root, $D=0$ that is, $b^2=4ac$

that is, $(3+p)^2=4(1)(1+3p)$

$$9+6p+p^2=4+12p$$

$$p^2-6p+5=0$$

By factorisation, we have, $(p-5)(p-1)=0$

$$p=5 \text{ or } 1$$

Example: Show that the equation $ax^2+bx+a=0$ $(a>0)$ can only have real roots if $|b| \geq 2a$.

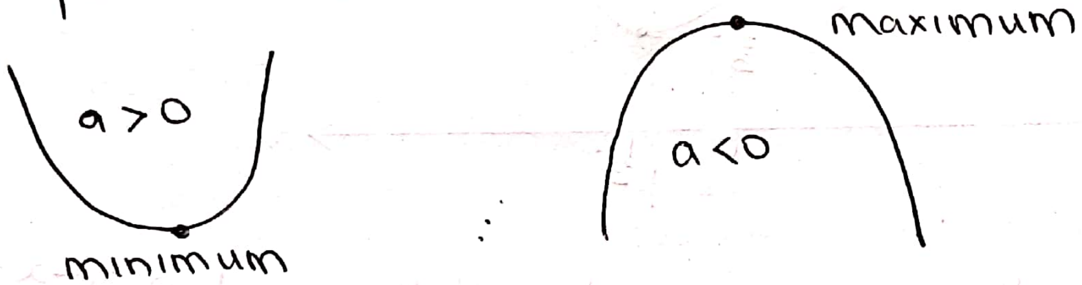
To have real roots, $D \geq 0$, that is, $b^2 \geq 4ac$.

In the given equation, $c=a$. Hence, for real roots $b^2 \geq 4a^2$, that is, $|b| \geq 2a$.

Maximum and Minimum Values of a Quadratic Function

The turning points on the graph, which give the maximum and minimum values of a quadratic function, can be found by completing the square. An alternative method, using calculus, shall be discussed later.

For a graph of a function, $y = ax^2 + bx + c$ where $a \neq 0$, is a parabola as shown below:



If $a > 0$, e.g. $y = 2x^2 + 3x - 7$, the function has a MINIMUM value at the bottom of the curve.

If $a < 0$, e.g. $5 - 3x - x^2$, the function has a maximum value at the ~~curve~~ top of the curve.

Example

What is the minimum value of $3x^2 - 2x + 1$, and for what value of x does it occur?

$$y = 3x^2 - 2x + 1$$

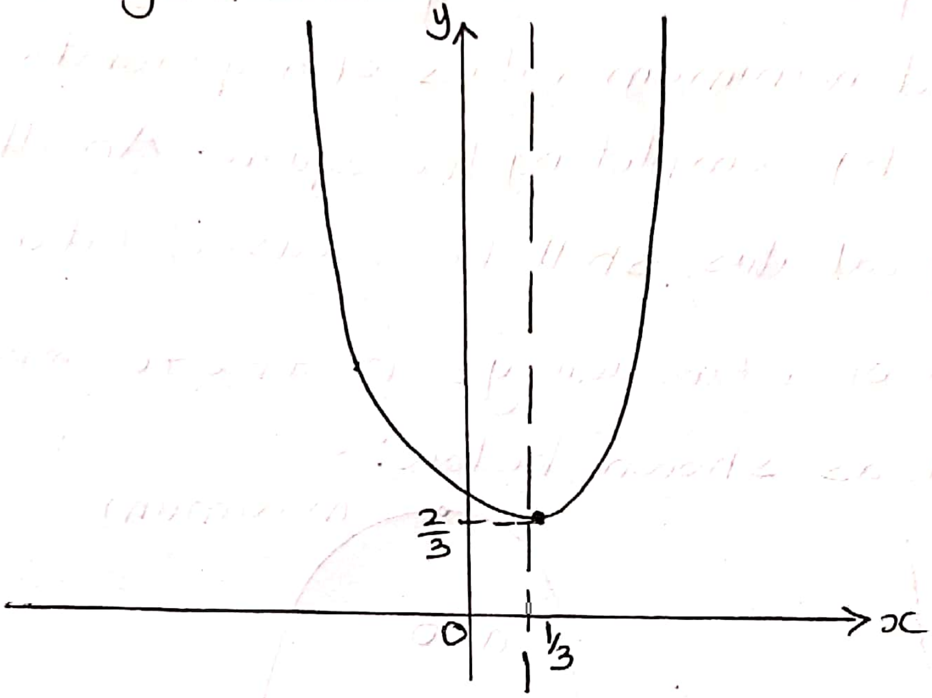
By completing the square, it becomes

$$y = 3\left(x - \tfrac{1}{3}\right)^2 + \tfrac{2}{3}$$

The least value of $\left(x - \tfrac{1}{3}\right)^2$ is 0, when $x = \tfrac{1}{3}$.

The least value of y is $\frac{2}{3}$, when $x = \frac{1}{3}$.

The line $x = \frac{1}{3}$ is called the axis of the parabola and the curve is symmetrical about this axis.



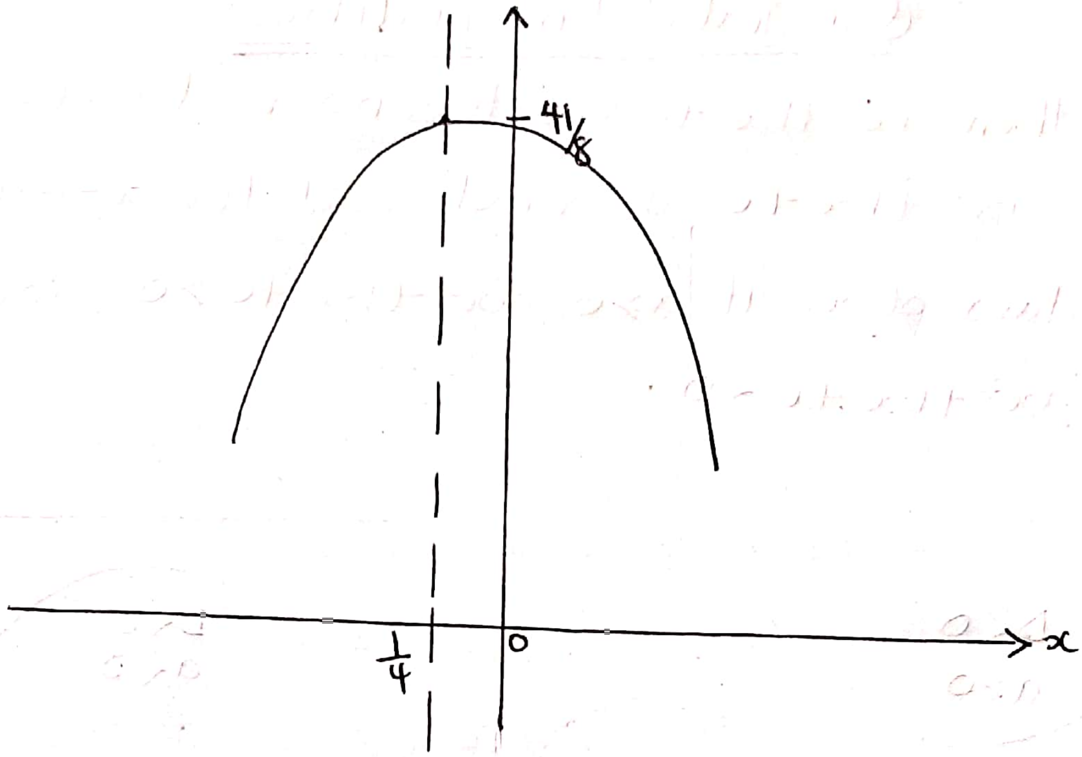
Example: Find the maximum value of the function $5 - x - 2x^2$, and the equation of the axis of its curve.

$$y = -2x^2 - x + 5$$
$$= -2\left(x^2 + \frac{x}{2} - \frac{5}{2}\right)$$
$$= -2\left[\left(x + \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{5}{2}\right]$$
$$= -2\left(x + \frac{1}{4}\right)^2 + \frac{41}{8}$$

Hence, the greatest value of y is $\frac{41}{8}$ when $x = -\frac{1}{4}$.

The axis is therefore the line $x = -\frac{1}{4}$.

Below is the diagram for the function.



Exercise

1) Describe the nature of the roots of the following equations.

a) $2x^2 - x - 1 = 0$

(b) $x^2 + 5x - 1 = 0$

c) $3x^2 - 2x + 4 = 0$

(d) $4x^2 - 28x + 49 = 0$

e) $x^2 = x - 5$

(f) $ax^2 + bx - a = 0$

2) If the equation $x^2 + px + 9 = 0$ has equal roots, find the possible values of p?

3) Find the maximum or minimum values (as appropriate) of the following functions, and the values of x at which they occur:

a) $2x^2 - x - 1$

(b) $2 - 4x - x^2$

c) $3x^2 - x - 6$

(d) $1 - x - x^2$

e) $x^2 - 20x$

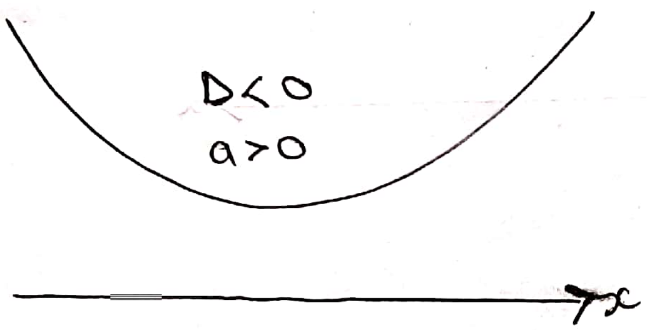
(f) $(x+3)(x-1)$

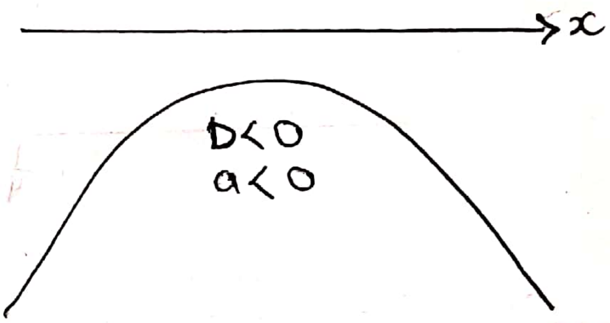
(g) $(1-x)(2+x)$

(h) $x^2 + bx + c$

Quadratic Inequalities

I) If $D<0$, then $ax^2+bx+c=0$ has no real roots and the curve $y=ax^2+bx+c$ does not meet the x-axis. Hence, for all values of x, if $a>0$, $ax^2+bx+c>0$ and if $a<0$, $ax^2+bx+c<0$.



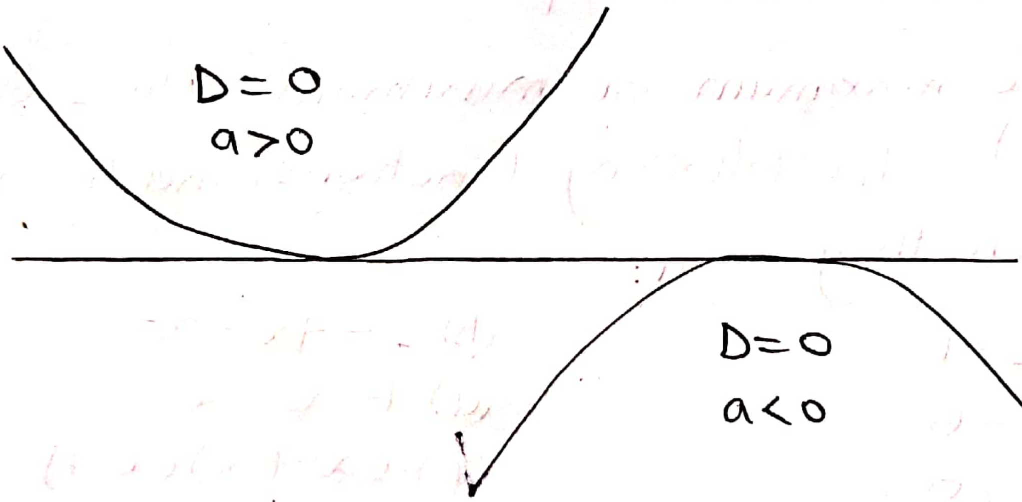


For example, $3x^2-x+1>0$ for all values of x, as $D=-11$;

and

$x-x^2-2<0$ for all values of x as $D=-8$.

II) If $D=0$, $ax^2+bx+c=0$ has equal roots and the curve will touch the x-axis where $x=\frac{-b}{2a}$ as shown below:



Example: For what range of values of x is $2x^2+x \leq 1$?

Rewrite as $2x^2+x-1 \leq 0$.

$a > 0$ and $D = 1+8 = 9 > 0$

Hence, $\alpha \leq x \leq \beta$ where α, β are the roots of $2x^2+x-1=0$.

$\alpha = -1, \beta = \frac{1}{2}$.

Hence, $-1 \leq x \leq \frac{1}{2}$

Example: If the equation $x^2-(p+1)x+p^2=5$ has real roots. Find the range of possible values of p?

Rewrite as $x^2-(p+1)x+(p^2-5)=0$

For real roots, $D \geq 0$, that is, $(p+1)^2 \geq 4(p^2-5)$

$$3p^2-2p-21 \leq 0$$

$$(3p+7)(p-3) \leq 0$$

Hence, $-\frac{7}{3} \leq p \leq 3$.