The cardinality of $\mathbb{Z}$ is infinite. We also call $|A|$ the **size** of the set A.

Note: A set is called finite if its cardinality is an integer. Otherwise, if it is infinite.

Equality of Sets: Two sets are equal if they have exactly the same elements.

Example: Suppose $E = \{x \mid x \in \mathbb{Z}, 2 | x\}$.

$F = \{z \mid z \in \mathbb{Z}, z = a+b, \ a, b \text{ are odd}\}$.

Is $E = F$? **Solution:** (i) For every $x \in E \Rightarrow x \in F$;

(ii) and for $z \in F, \Rightarrow z \in E$.

Hence, $E = F$.

(i) Since $x \in E \Rightarrow x = 2y$ for some $y \in \mathbb{Z}$.

now, $x = (2y+1) + (-1)$ where $2y+1$ is odd and -1 is odd.

$\Rightarrow x \in F$.

(ii) If $z \in F$ then $z = a+b$ where a, b are odd.

$\Rightarrow z = 2c$ for some $c \in \mathbb{Z}$.

$\Rightarrow 2 | z \Rightarrow z \in E$. ∎

Subsets: Suppose A and B are sets. We say that A is a subset of B provided every element of A is also an element of B. The notation $A \subseteq B$ means A is a subset of B.

Example: Suppose $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$

Then, $A \subseteq B$.

Note:

(i) For any set A, we have $A \subseteq A$ because every element of A is of course in A.

(ii) For any set A, $\phi \subseteq A$.

(iii) $A \subset B$ means A is a "proper subset" of B. This means A is contained in B but not equal to B. That is,
$A \subseteq B$, $A \neq B$.

Becareful!!!

Distinction between "$\in$" and "$\subseteq$".

$x \in A$ means x is an element (or member) of A.

$A \subseteq B$ means every element of A is also an element of B.

Example, $\phi \subseteq \{1, 2, 3\}$ ✓

$\phi \in \{1, 2, 3\}$ ✗

Notation:

The symbols $\in$ and $\subseteq$ may be written backward; $\ni$ and $\supseteq$.

$A \ni x$ means "$x \in A$"..

"$B \supseteq A$" means exactly the same thing as "$A \subseteq B$". We say B is a "superset" of A.

Counting Subsets (Power Sets)

The family of all the subsets of any set S is called the power set of S. We denote the power set of S by:
$2^{|S|}$.

Example: How many subsets does $A = \{1, 2, 3\}$ have?

No. of elements	Subsets	Number
0	ϕ	1
1	$\{1\}, \{2\}, \{3\}$	3
2	$\{1, 2\}, \{1, 3\}, \{2, 3\}$	3
3	$\{1, 2, 3\}$	1

Total: 8

Disjoint Sets.

Two sets are disjoint if they have no elements in common.

Example: Let $A = \{1,3,7,8\}$ and $B = \{2,4,7,9\}$.

Then, A and B are not disjoint since they have element 7 in common.

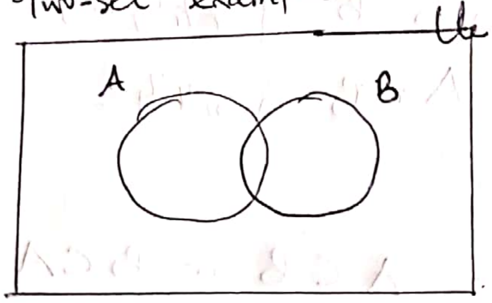
Example: $A = \{x \mid x \in \mathbb{Z}^-\}$ & $B = \{x \mid x \in \mathbb{Z}^+\}$.

Here A and B are disjoint since A contains only negative integers and B contains only positive integers.

VENN-EULER DIAGRAM:

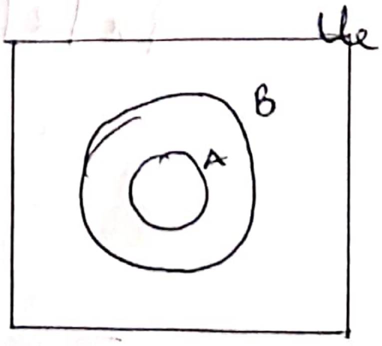
Is a pictoral representation of the relationships between sets.

Example: Two-set example.



Example 1: Suppose $A \subset B$ and say $A \neq B$, this means A is contained in the set B. This can be represented in a Venn-diagram as.

Solution.



Set Operations:

Union and Intersection:

Let A and B be sets.

The "Union" of A and B is the set of all elements that are in A or B. The union of A and B is denoted "$A \cup B$".

The "intersection" of A and B is the set of all elements that are in both A and B. The intersection of A and B is denoted. "$A \cap B$".

In symbols, we can write this as:

$$A \cup B = \{x \mid x \in B \text{ or } x \in B\},$$ and

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Example: Suppose $A = \{1,2,3,4\}$ and $B = \{3,4,5,6\}$. Then $A \cup B = \{1,2,3,4,5,6\}$ and $A \cap B = \{3,4\}$.

Comparability:

Two sets A&B are comparable if: $A \subset B$ or $B \subset A$.

Example:

$A = \{2,4,6,8\}$

$B = \{x \mid x \text{ is an even natural no.}\} = \{x \mid 2 \mid x,\ x \in \mathbb{N}\}$. or

$= \{x \mid 2 \mid x,\ x \in \mathbb{Z}^+\}$

Theorem Let A, B and C denote sets. The following are true:

(i) $A \cup B = B \cup A$ and $A \cap B = B \cap A$. (Commutative properties).

(ii) $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$. (Associative properties).

(iii) $A \cup \phi = A$ and $A \cap \phi = \phi$.

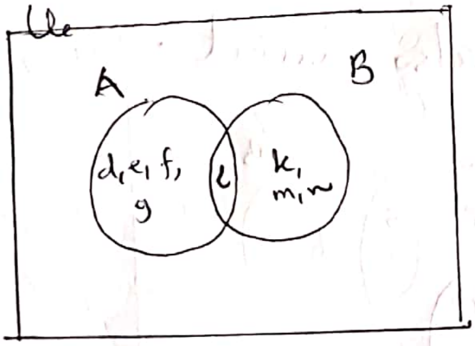
(iv) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive properties).

Example.

Let $A = \{d, e, f, g, l\}$ and $B = \{k, l, m, n\}$.

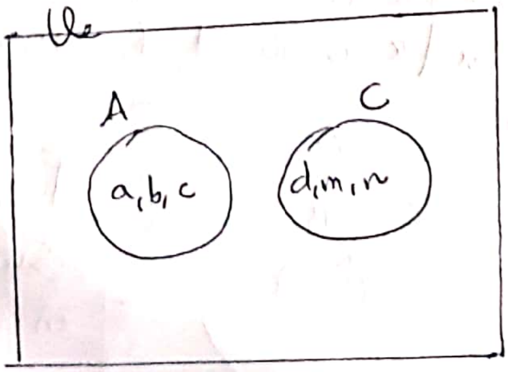
Represent these sets in a Venn-diagram.



Example.

Let $A = \{a, b, c\}$ and $C = \{d, m, n\}$.

Show this on a venn-diagram.



Sets A and C are not comparable.

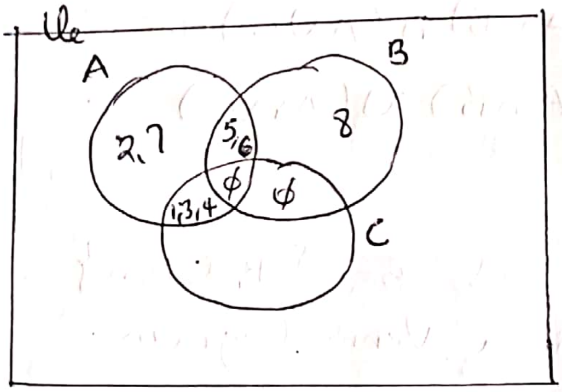
Example.

Let $A = \{x \mid x \text{ is an integer from } 1 \text{ to } 7\} = \{1, 2, 3, 4, 5, 6, 7\}$.

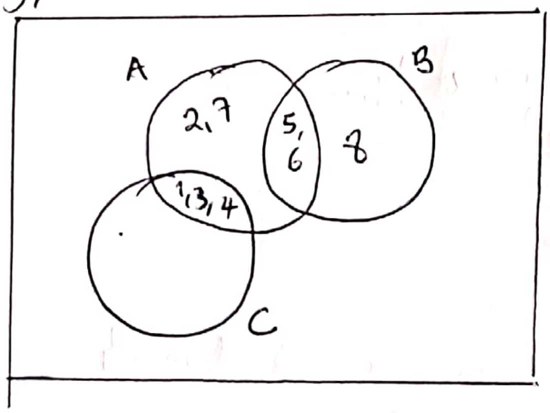
$B = \{5, 6, 8\}$

$C = \{1, 3, 4\}$

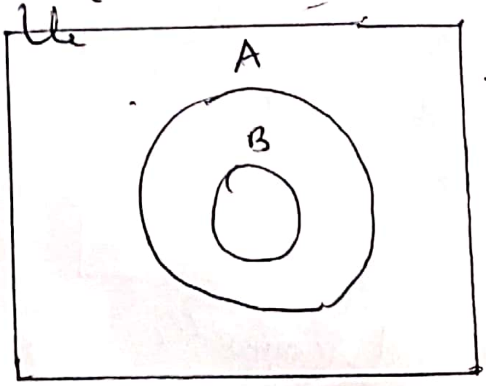
Illustrate this on a venn-diagram.



Alternatively, this is also correct.



Example: Is the following comparable or not. Illustrate on a Venn diagram. $A = \{x \mid x \text{ is the set of chairs in the class}\}$.
$B = \{x \mid x \text{ is the set of chairs in the middle column}\}$.



Yes, it is comparable since set B is contained in set A; i.e

$\boxed{B \subset A}$.

Example

Let $A = \{4, 5, 7, 8, 9, 1\}$

$B = \{7, 4, 17\}$

$C = \{7, 9, 8\}$.

Show that the theorem above are true through verification.

Solution.

(i) $A \cup B = \{1, 4, 5, 7, 8, 9, 17\}$

$B \cup A = \{1, 4, 5, 7, 8, 9, 17\}$. $\Rightarrow A \cup B = B \cup A$.

(iv) $A \cup (B \cap C)$:- $B \cap C = \{7\}$.

$A \cup (B \cap C) = \{4, 5, 7, 8, 9, 1\} \cup \{7\}$

$= \{1, 4, 5, 7, 8, 9\}$

$(A \cup B) \cap (A \cup C)$:- $A \cup B = \{1, 4, 5, 7, 8, 9, 17\}$.

$A \cup C = \{4, 5, 7, 8, 9, 1\}$.

$\therefore (A \cup B) \cap (A \cup C) = \{1, 4, 5, 7, 8, 9\}$.

True

Proposition: Suppose A and B are finite sets. There is a relationship between the quantities $|A|$, $|B|$, $|A \cup B|$ and $|A \cap B|$. $\therefore$ $\boxed{|A| + |B| = |A \cup B| + |A \cap B|}$.

Example: How many integers in the range of 1 to 1000 (inclusive) are divisible by 2 or by 5?

Soln :

Let $A = \{x \mid x \in \mathbb{Z}, 1 \le x \le 1000 \text{ and } 2 \mid x\}$ and

$B = \{x \mid x \in \mathbb{Z}, 1 \le x \le 1000 \text{ and } 5 \mid x\}$.

The problem asks for $|A \cup B|$.

It is not hard to see that $|A| = \frac{1000}{2} = 500$ and

$$|B| = \frac{1000}{5} = 200.$$

Now, an integer is divisible by both 2 and 5 if and only if it is divisible by 10. So,

$$A \cap B = \{x \mid x \in \mathbb{Z},\ 1 \le x \le 1000 \text{ and } 10 \mid x\}.$$

$$\therefore |A \cap B| = \frac{1000}{10} = 100.$$

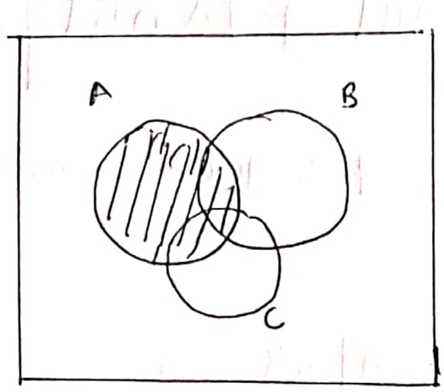
$$\Rightarrow |A \cup B| = |A| + |B| - |A \cap B| = 500 + 200 - 100$$
$$= \underline{\underline{600}}$$

There are 600 integers in the range 1 to 1000 that are divisible by 2 or by 5.

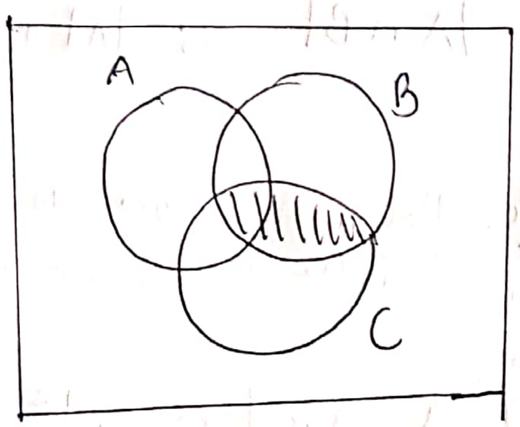
<u>Corollary</u>: If sets A and B are disjoint then $|A \cup B| = |A| + |B|$; since $|A \cap B| = 0$.

<u>MORE ON VENN DIAGRAM</u>.

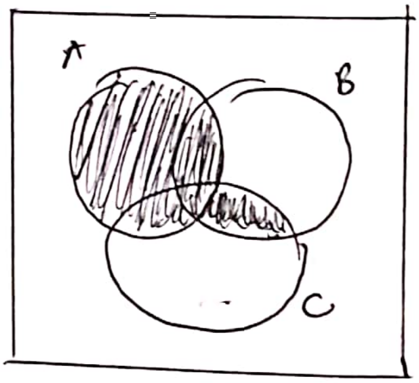
Suppose one has three sets A, B, C. We can represent the relationship between these sets in a Venn-diagram and characterize each segment.



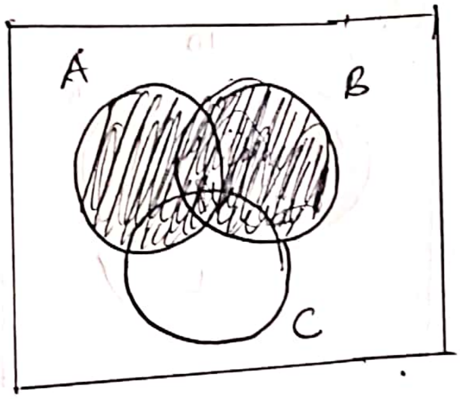
A.



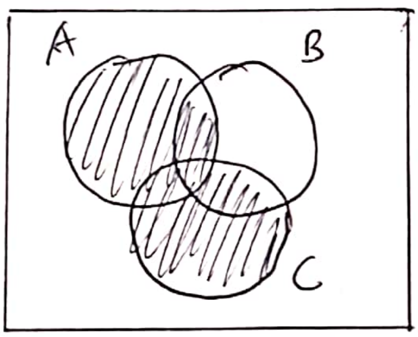
$B \cap C$



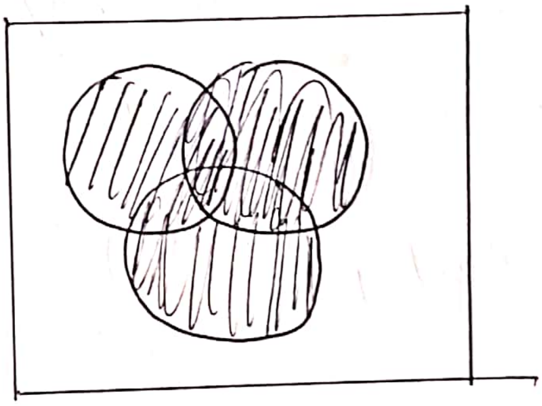
$A \cup (B \cap C)$



$A \cup B$



$A \cup C$



$A \cup B \cup C$.

Example.

Given that $U = \{1, 2, 3, \ldots, 10\}$

$A = \{2, 3, 4, 5, 6\}$.

$B = \{2, 7, 8, 9, 6\}$

$C = \{2, 1, 3, 5, 7\}$.

Find the following sets and shade their regions on a venn-diagram:

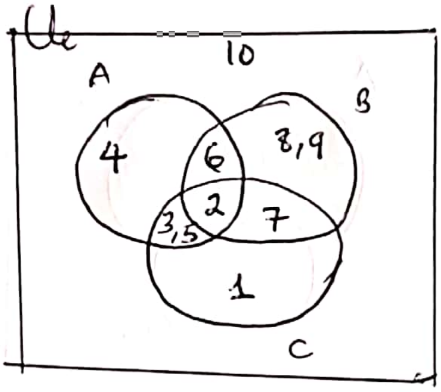
(i) $A \cup B \cup C$ (ii) $(A \cup B) \cap C$ (iii) $A \cap C$ (iv) $(A \cap C) \cap B$
(v) $(A \cup B) \cap (B \cup C)$. (vi) $(A \cap C) \cup (B \cap C)$.

Solution:

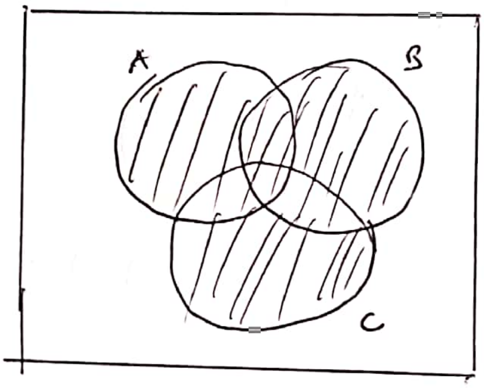
First let's represent the information in a venn-diagram.

$A \cap B = \{6, 2\}$ $B \cap C = \{7, 2\}$ $A \cap C = \{2, 3, 5\}$.

$A \cap B \cap C = \{2\}$



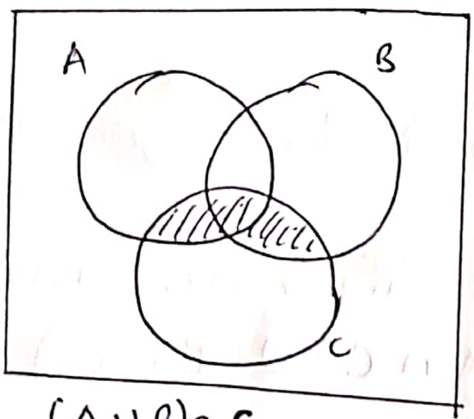
(i) $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.



(ii) $(A \cup B) \cap C \Rightarrow$ $A \cup B = \{3, 4, 5, 6, 7, 8, 9, 2\}$.

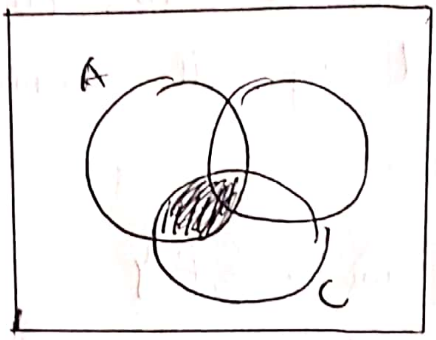
Then; $A \cup B \cap C = \{2, 3, 4, 5, 6, 7, 8, 9\} \cap \{2, 1, 3, 5, 7\}$

$= \{2, 3, 5, 7\}$.



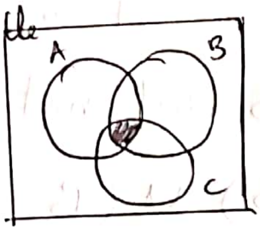
$(A \cup B) \cap C$

(iii) $A \cap C = \{2,3,5\}$.



(iv) $(A \cap C) \cap B = A \cap C \cap B$.

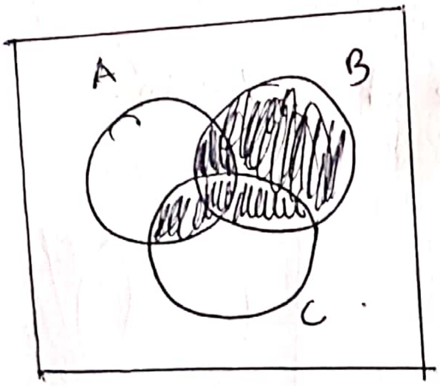
$= \{2\}$.



(v) $(A \cup B) \cap (B \cup C)$ ∴ $A \cup B = \{2,3,4,5,6,7,8,9\}$.

$B \cup C = \{1,2,3,5,6,7,8,9\}$.

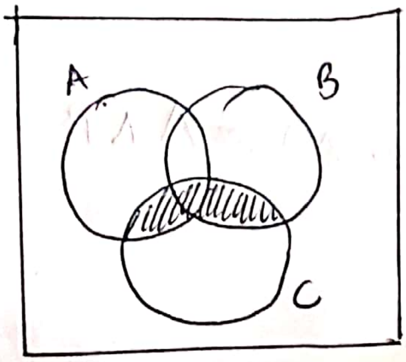
$(A \cup B) \cap (B \cup C) = \{2,3,5,6,7,8,9\}$.



(vi) $(A \cap C) \cup (B \cap C)$ ∴ $A \cap C = \{2,3,5\}$

$B \cap C = \{2,7\}$.

$(A \cap C) \cup (B \cap C) = \{2,3,5,7\}$



Difference and Symmetric Difference:

Set Difference: Let A and B be sets. The set difference, $A-B$, is the set of all elements of A that are not in B:

$$A-B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Symmetric Difference: The symmetric difference of A and B, denoted $A \Delta B$, is the set of all elements in A but not B or in B but not A. That is,

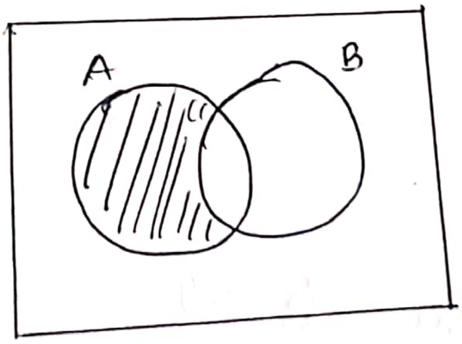
$$A \Delta B = (A-B) \cup (B-A).$$

Example: Suppose $A = \{1,2,3,4\}$ and $B = \{3,4,5,6\}$.
Then, $A-B = \{1,2\}$, $B-A = \{5,6\}$.

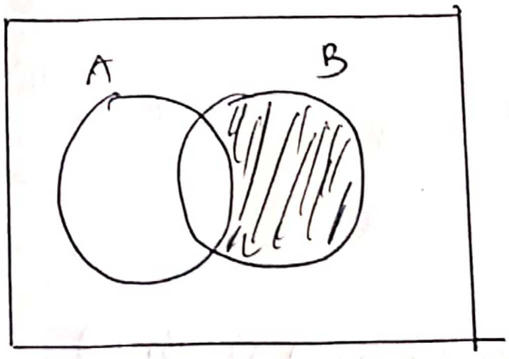
$$A \Delta B = \{1,2\} \cup \{5,6\} = \{1,2,5,6\}.$$

Remark:

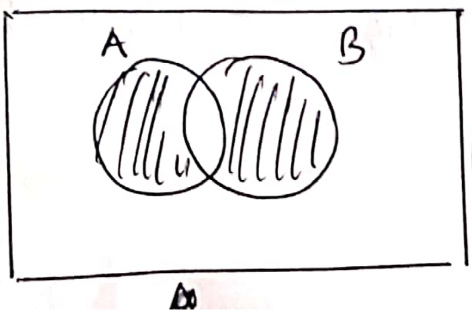
1) $A-B$, $B-A$ on a venn-diagram are shown below.



$A-B$



$B-A$.



$(A-B) \cup (B-A) = A \Delta B$

2). The sets $(A-B)$, $(A\cap B)$, and $(B-A)$ are mutually disjoints, i.e intersection of any two is the null set.

Proposition: (De Morgan's Laws).

Let A, B, and C be sets. Then,

$$\boxed{A-(B\cup C) = (A-B)\cap(A-C)}$$ and

$$A-(B\cap C) = (A-B)\cup(A-C).$$

Example:

Suppose $A=\{2,3,4,5,6\}$

$B=\{2,7,8,9,6\}$

$C=\{1,2,3,5,7\}$.

Compute the following:

(i) $A-B$ (ii) $B-C$ (iii) $A-(B\cup C)$ (iv) $A-(B\cap C)$

(v) $B-(C\cap A)$.

Solution:

(i) $A-B=\{3,4,5\}$ (ii) $B-C=\{8,9,6\}$.

(iii) $A-(B\cup C) = (A-B)\cap(A-C)$ Using De Morgan's Law.

$= \{3,4,5\}\cap(A-C)$.

$A-C=\{4,6\}$ $\quad = \{3,4,5\}\cap\{4,6\}$

$= \{4\}$.

(iv) $A-(B\cap C) = (A-B)\cup(A-C)$. Using De Morgan's Laws

$= \{3,4,5\}\cup\{4,6\} = \{3,4,5,6\}$

(v). $B-(C\cap A) = (B-C)\cup(B-A)$

$= \{8,9,6\} \cup \{7,8,9\}$

$B-A = \{7,8,9\}$

$= \underline{\underline{\{6,7,8,9\}}}.$

COMPLEMENT:

The complement of a set A is the set of elements that do not belong to A. That is, the difference of the universal set $\mathcal{U}$ and A. We denote the complement by A' or A^c.

Example:

Let $\mathcal{U} = \{2, 9, 10, 14, 20\}$.

$A = \{9, 14\}$

$B = \{2, 20\}$

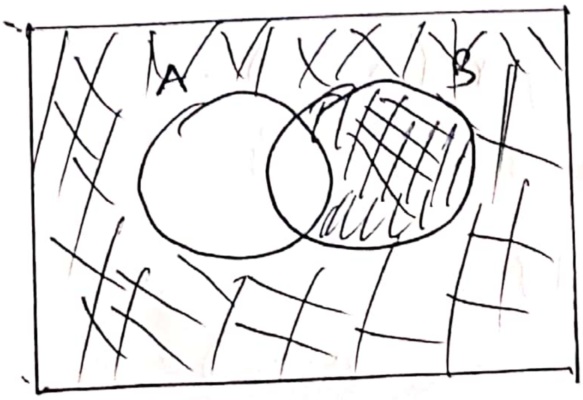
Then $A' = \underline{\underline{\{2, 10, 20\}}}.$

$B' = \underline{\underline{\{9, 10, 14\}}}.$

$(A\cup B)' = \{2, 9, 14, 20\}' = \underline{\underline{\{10\}}}$

$(A\cap B)' = \{\emptyset\}' = \mathcal{U} = \{2, 9, 10, 14, 20\}.$

On a Venn-diagram for 2-set case.



A' or A^c.

Remark.

(1) The union of any set A and its complement A' is the universal set i.e, $A \cup A' = \mathcal{U}$.

(2) The complement of the universal set $\mathcal{U}$ is the null set ϕ and vice versa.
i.e $\mathcal{U}' = \phi$; $\phi' = \mathcal{U}$.

(3) Set A and its complement are disjoint, i.e
$$A \cap A' = \phi.$$

(4) The complement of the complement of set A is the set A itself $\therefore (A')' = A$.

(5). $A - B = A \cap B'$, i.e

$$A - B = \{x \mid x \in A, x \notin B\}$$
$$= \{x \mid x \in A, x \in B'\}.$$
$$= A \cap B'.$$