BINOMIAL THEOREM. ①

A Binomial expression is the sum or difference of two terms:
For example;

$x+1$, $3x+2y$, $a-b$.

You are already familiar with expanding the square of such expression. That is, $(x+1)^2 = (x+1)(x+1) = x^2+2x+1$

If we want to raise a binomial expression to a power higher than 2 for example, if you want to find $(x+1)^4$, it is cumbersome to do this by repeatedly multiplying by $(x+1)$ by itself.

In this topic, you will learn how a triangular pattern of numbers known as Pascal's Triangle can be used to obtain the required results very quickly.

Pascal's Triangle.

Start with the one raised to power of zero.

$$
\begin{array}{ccccccccccccccl}
 & & & & & & 1 & & & & & & & \rightarrow & a^0 \\
 & & & & & 1 & & 1 & & & & & & \rightarrow & a^1 \\
 & & & & 1 & & 2 & & 1 & & & & & \rightarrow & a^2 \\
 & & & 1 & & 3 & & 3 & & 1 & & & & \rightarrow & a^3 \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & & \rightarrow & a^4 \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \rightarrow & a^5 \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \rightarrow & a^6
\end{array}
$$

The above Pascal's triangle is used to expand $(a+b)^n$.

Ex: $(a+b)^2$ when $n=2$; we use the second row (of Pascal's triangle), the numbers represent the coefficients in $(a+b)^n$ of the expansion.
The powers of the first term is decreasing with each term in the expansion while the second term (in this case b) will be increasing. That is,

$$(a+b)^2 = 1a^2 + 2a^1 \cdot b^1 + 1 \cdot b^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = 1 \cdot a^3 + 3a^2 \cdot b + 3a \cdot b^2 + 1 \cdot b^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Ex: $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Hence, we can use this for different forms of binomial expansion.

Ex: $(2x+y)^3 = (2x)^3 + 3(2x)^2(y) + 3(2x)(y^2) + y^3$.

$$= 8x^3 + 3(4x^2)y + 6xy^2 + y^3$$

$$= 8x^3 + 12x^2y + 6xy^2 + y^3.$$

<u>Example</u>: Find $(1+p)^4$

$$= 1^4 + 4(1^3)p + 6(1^2)p^2 + 4(1)p^3 + p^4$$

$$= 1 + 4p + 6p^2 + 4p^3 + p^4.$$

<u>Ex:</u> $(3a-2b)^5$.

$$= (3a)^5 + 5(3a)^4(-2b) + 10(3a)^3(-2b)^2 + 10(3a)^2(-2b)^3 + 5(3a)(-2b)^4 + (-2b)^5$$

$$= 3^5a^5 + 5(3^4a^4)(-2b) + 10(3^3a^3)((-2)^2b^2) + 10(3^2a^2)(-2)^3b^3 + 5(3a)(-2)^4(b^4) + (-2)^5b^5$$

$$= 243a^5 - 810a^4b + 1080a^3b^2 - 720a^2b^3 + 240ab^4 - 32b^5.$$

Ex: $\left(1+\frac{2}{x}\right)^3 = 1^3 + 3(1^2)\left(\frac{2}{x}\right) + 3(1)\left(\frac{2}{x}\right)^2 + \left(\frac{2}{x}\right)^3$

$= 1 + \frac{6}{x} + \frac{12}{x^2} + \frac{8}{x^3}$

Exercise:

Expand the following.

(i) $(1-x)^3$

(ii) $(x+6)^3$

(3) $\left(x-\frac{1}{x}\right)^6$

(4) $(2x-3y)^5$.

Binomial Theorem.

As the power of binomial expression increases, the use of Pascal's triangle becomes difficult, hence there is need for an alternative method. Binomial theorem enables us to expand $(a+b)^n$ in increasing powers of b and decreasing powers of a.

That is,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n.$$

$$= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

Note that $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$

For example; $\binom{3}{2} = \frac{3!}{(3-2)!\,2!} = \frac{3!}{1!\,2!} = \frac{3\cdot2\cdot1}{1\times(2\cdot1)} = \underline{\underline{3}}$

$\binom{5}{2} = \frac{5!}{(5-2)!\,2!} = \frac{5!}{3!\,2!} = \frac{5\cdot4\cdot3!}{3!\,2!} = \frac{5\cdot4}{2!} = \frac{5\cdot4}{2\cdot1} = \underline{\underline{10}}$

$$\binom{6}{3} = \frac{6!}{3!\,3!} = \frac{6\cdot4\cdot3!}{3!\,3!} = \frac{6\cdot4}{3!} = \frac{6\times4}{3\times2\times1} = 4$$

$$\binom{10}{6} = \frac{10!}{4!\,6!} = \frac{10\cdot9\cdot8\cdot7\cdot6!}{4!\cdot6!} = \frac{10\cdot9\cdot8\cdot7}{4!}$$

$$= \frac{10\times9\times8\times7}{4\times3\times2\times1} = 10\times3\times7 = 210$$

Note:

$1! = 1$, $0! = 1$.

Example:

$$(1+y)^{10} = \binom{10}{0}1^{10} + \binom{10}{1}1^{9}y + \binom{10}{2}1^{8}y^{2} + \binom{10}{3}1^{7}y^{3} + \binom{10}{4}1^{6}y^{4}$$

$$+\binom{10}{5}1^{5}y^{5} + \binom{10}{6}1^{4}y^{6} + \binom{10}{7}1^{3}y^{7} + \binom{10}{8}1^{2}y^{8} + \binom{10}{9}1\,y^{9} + \binom{10}{10}y^{10}$$

$$= \cancel{y^5+40y}$$

$$= 1 + 10y + 45y^2 + 120y^3 + 210y^4 + 252y^5 + 210y^6$$

$$+ 120y^7 + 45y^8 + 10y^9 + y^{10}.$$

$\binom{10}{0} = \frac{10!}{10!\,0!} = 1$

$\binom{10}{1} = \frac{10!}{9!\,1!} = 10$

$\binom{10}{2} = \frac{10!}{8!\,2!} = \frac{10\times9}{2} = 45$ $\quad \binom{10}{3} = \frac{10!}{3!\,7!} = \frac{10\times9\times8}{3\times2\times1} = 120$

$\binom{10}{4} = \frac{10!}{6!\,4!} = \frac{10\times9\times8\times7}{4\times3\times2\times1} = 210$, $\quad \binom{10}{5} = \frac{10!}{5!\,5!} = \frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1} = 6\times7\times6 = 252.$

$\binom{10}{6} = \frac{10!}{6!\,4!} = 210$, $\quad \binom{10}{7} = \frac{10!}{7!\,3!} = 120$ $\quad \binom{10}{8} = \frac{10!}{8!\,2!} = 45$

$\binom{10}{10} = \frac{10!}{10!\,0!} = 1$ $\quad \binom{10}{9} = \frac{10!}{9!\,1!} = 10$

Ex Find the first three terms in the expansion $(3-5x)^{14}$ using Binomial expansion.

$$(3-5x)^{14} = \binom{14}{0} 3^{14} + \binom{14}{1} 3^{13}(-5x) + \binom{14}{2} 3^{12}(-5x)^2 + \cdots$$

$$= 3^{14} + \binom{14}{1}(3^{13})(-5)x + \binom{14}{2} 3^{12}(25)x^2 + \cdots$$

Example: Find the coefficient of x^2 in the expansion. $(2x+1)^4$.

Since, $\binom{n}{r}\cdot(2x)^{n-r}(1)^r$ as the general form of each term where $r = 0, 1, 2, \ldots, n$.

we get x^2 when $n-r=2 \Rightarrow 4-r=2 \Rightarrow \underline{\underline{r=2}}$.

$$\binom{4}{2}(2x)^2 1^2 = \frac{4!}{2!\,2!} 4x^2 = \frac{4\times3}{2}\cdot 4x^2$$

$$= 24x^2$$

Coefficient of x^2 is $\underline{\underline{24}}$.

Ex: Find the coefficient of x^3y^3 in the expansion $(2x+5y)^6$.

$\underline{\underline{\text{Soln}}}$

$$\binom{n}{r}(2x)^{n-r}(5y)^r = \binom{6}{r}(2x)^{6-r}(5y)^r \quad n=6.$$

we get y^3x^3 when $r=3$.

$$\Rightarrow \binom{6}{3}(2x)^{6-3}(5y)^3 = \binom{6}{3}(2x)^3(5y)^3$$

$$= \frac{6!}{3!\,3!}\ 8x^3 \times 75y^3$$

$$= \frac{6\times5\times4}{6} \times 8x^3 \times 75y^3$$

$$= 20 \times 8x^3 \times 75y^3$$

$$= 12\,000x^3y^3$$

Hence coefficient of x^3y^3 is $\underline{\underline{12,000}}$.

Ex: a. Compute the following using Binomial expansion.

(1) $(1-3x)^4$ (2) $(1+2x)^3$

b. Find the coefficient of x^4 in $(1-\frac{x}{2})^8$.

c. Find the coefficient of x^5 in the expansion $(1+4x)^9$.

d. Find the first four terms in the expansion of $(2+\frac{x}{3})^{12}$.