# SEQUENCES AND SERIES:

**Sequence:** A function which is defined in the set of natural numbers is called a sequence.

For example: The list of squares of a natural number.

$1^2, 2^2, 3^2, 4^2, \ldots$

$1, 4, 9, 16, \ldots$

This can be written as a function with n-th term given by $f(n) = n^2$.

Here, in sequence, instead of using a function, we denote each term of the sequence by $a_n$. For example, the first term is given by $a_1$, second term by $a_2$, and so on.

So, for the above example,

$a_1 = 1,\ a_2 = 4,\ a_3 = 9,\ a_4 = 16,\ a_5 = 25,\ a_6 = 36, \ldots ; a_n = n^2$.

Here, $a_1$ is the first term,
$a_2$ is the second term,
$a_3$ is the third term.
⋮
$a_n$ is the nth term, or the general term.

Since, this is just a matter of notation, we can use another letter instead of the letter "a". For example, we can also use $b_n, c_n, d_n$, e.t.c as the name for the general term of a sequence.

Different notations used for a sequence are:

1. $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$
2. $a_n, \quad n = 1, 2, 3, \ldots$
3. $\{a_n\}_{n=1}^{\infty}$
4. $(a_n)_{n=1}^{\infty}$

For example.



$(a_n) = (1, 4, 9, 16, \dots, n^2, \dots)$. when we want to list the terms or
$= 1, 4, 9, 16, \dots, n^2, \dots$.

Note: Sequence is defined for only natural numbers, hence,
$a_{2.5}$, $a_0$, $a_{-1}$ are all undefined and any other term of similar nature.

Ex: Write the first five terms of the sequence with general form (term) $a_n = \frac{1}{n}$.

Soln:
$n = 1, 2, 3, 4, 5$, which gives, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$.

Ex. Given the sequence with general term
$a_n = \frac{4n-5}{2n}$, find $a_5$, $a_{-2}$, $a_{100}$.

Soln
$$a_5 = \frac{4(5)-5}{2(5)} = \frac{20-5}{10} = \frac{15}{10} = 1.5 = \frac{3}{2}$$

$a_{-2}$ is undefined since $n$ can't be a negative number.

$$a_{100} = \frac{4(100)-5}{2(100)} = \frac{400-5}{200} = \frac{395}{200} = \frac{79}{40}$$

Ex: Find a suitable general term $b_n$ for the sequence whose first four terms are $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$.

Soln: We find a pattern and notice that the ~~denominator~~ ~~of~~ numerator is equal to the term position and denominator is one more than the term position, so we can write

$$b_n = \frac{n}{n+1}.$$

Ex: Given ~~that~~ the sequence $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$
Find the n-th term.

Soln:
Notice that
$$a_1 = 1 = \frac{1}{1^2}$$
$$a_2 = \frac{1}{4} = \frac{1}{2^2}$$
$$a_3 = \frac{1}{9} = \frac{1}{3^2}$$
$$\vdots$$
Hence $a_n = \frac{1}{n^2}$.

Ex: (1) Write the first five terms of the sequence whose general term is $C_n = (-1)^n$.

(2) Find a suitable general term $a_n$ for the sequence whose first four terms are $2, 4, 6, 8$.

(3) Given the sequence with general term $b_n = 2n+3$, find $b_5$, $b_0$ and $b_{43}$.

Soln
(1) $a_1 = (-1)^1 = (-1)$
$a_2 = (-1)^2 = 1$
$a_3 = (-1)^3 = -1$, $a_4 = (-1)^4 = 1$, $a_5 = (-1)^5 = -1$
$\Rightarrow -1, 1, -1, 1, -1$.

(2) note that
$$a_1 = 2 = 1 \times 2$$
$$a_2 = 4 = 2 \times 2$$
$$a_3 = 6 = 3 \times 2$$
$$a_4 = 8 = 4 \times 2$$
$$\vdots$$
Hence $a_n = n \times 2 = 2n$

(3) $b_5 = 2(5)+3 = 15$, $a_0$ is undefined since $0$ is not a natural number.
$b_{43} = 2(43)+3 = 86+3 = 89$

Ex: Find the general term for the sequence below: ④

1, 2, 4, 8, 16, 32, - - - .

Note that $a_1 = 1 = 2^0$

$a_2 = 2^1$

$a_3 = 4 = 2^2$

$a_4 = 8 = 2^3$

$a_5 = 16 = 2^4$

Hence $\underline{\underline{a_n = 2^{n-1}}}$.

Criteria for the existence of a Sequence.

If there is a natural number which makes the general term of a sequence undefined, then there is no such sequence.

Ex: Is $a_n = \dfrac{2n+1}{n-2}$ a general term of a sequence? Why?

Soln. No, because we can't find a proper value for $n = 2$.

Ex: Is $a_n = \sqrt{\dfrac{4-n}{2n+1}}$ a general term of a sequence? Why.

Soln

Note that $a_n$ is only meaningful only when

$\dfrac{4-n}{2n+1} \geq 0 \Rightarrow 4-n \geq 0$ and $2n+1 \geq 0$

$\Rightarrow n \leq 4 \quad n \geq -\frac{1}{2}$

Since ~~a~~ $\dfrac{4-n}{2n+1}$ is undefined at $n = -\frac{1}{2}$

then we have the solution $n \leq 4$ and $n > -\frac{1}{2}$

$\Rightarrow n \in (-\frac{1}{2}, 4]$.

This means only $a_1, a_2, a_3, a_4$ are defined. So, $a_n$ is not a general term of a sequence.

Ex: Is $a_n = \frac{n+1}{2n-1}$ a general term of a sequence? If yes, find $a_1 + a_2 + a_3$.

Soln

$\cdot \frac{n+1}{2n-1}$ is not meaningful when $n = \frac{1}{2} \notin \mathbb{N}$. Since $a_n$ is defined for any natural number, it is a general term of a sequence. Choosing $n = 1, 2, 3$ we get

$a_1 = 2,\ a_2 = 1,\ a_3 = 0.8$. So, $a_1 + a_2 + a_3 = \underline{3.8}$

Ex (1) Is $a_n = \frac{3n+1}{n+2}$ a general term of a sequence? Why?

(2) Which term of the sequence with general term $a_n = \frac{3n-1}{5n+7}$ is $\frac{7}{12}$?

## Types of Sequences

1. Finite and Infinite Sequences.

A sequence may contain a finite or infinite number of terms.

For example, the sequence

$(a_n) = (1, 4, 9, \ldots, n^2)$ contains $n$ terms, which is finite number of terms.

$(b_n) = (1, 4, 9, \ldots, n^2, \ldots)$ contains infinitely many terms.

If a sequence contains a countable number of terms, then we say it is a finite sequence.

If a sequence contains infinitely many terms, then we say it is an infinite sequence.

Ex: State whether the following sequences are finite or infinite.

a. The sequence of all odd numbers

(b) $(a_n) = (-10, -5, 0, 5, 10, 15, \ldots, 150)$

c. $1, 1, 2, 3, 5, 8, \ldots$

Soln

a. The sequence of all odd numbers is 1,3,5,7,- - -, ⑥
=> It is infinite.

b. Sequence has finite number of terms since the last term is 150 is given.

c. Sequence is infinite, since we have the continuation symbol "..."

## Monotone Sequences

• If each term of a sequence is greater than the previous term, then the sequence is called an increasing sequence.

Symbollically $(a_n)$ is an increasing sequence if $a_{n+1} > a_n$.

If $a_{n+1} \geq a_n$, then $(a_n)$ is a nondecreasing sequence

• If each term of a sequence is less than the previous term, then that sequence is called a decreasing sequence.

Symbollically, $(a_n)$ is decreasing sequence if $a_{n+1} < a_n$.

If $a_{n+1} \leq a_n$, then $(a_n)$ is a nonincreasing sequence.

• In general any increasing, nondecreasing, decreasing, or nonincreasing sequence is called a monotone sequence.

Ex: 10, 8, 6, 4, - - - is a decreasing sequence since each consecutive term is less than the previous one. Thus, it is a monotone sequence.

Ex: 1, 1, 2, 3, 5, . . . . is a nondecreasing sequence, because the first two terms are equal.

Ex: Consider 4, 1, 0, 1, 4, . . . We can't put this sequence into any of the categories of sequence defined above. Therefore, it is not monotone.

Ex. Prove that the sequence $(a_n)$ with general term $a_n = 2n$ is an increasing sequence.

Solution : If $a_n = 2n$, then $a_{n+1} = 2(n+1) = 2n+2$

So, $a_{n+1} - a_n = 2n+2 - 2n = 2$

Since $2 > 0$, $(a_n)$ is an increasing sequence

Ex: State whether the following sequences are finite or infinite.

a. $3, 6, 9, \ldots, 54$

b. $3, 6, 9, \ldots$

c. $-1, 1, 1, 4, 3, 1.$

d. $2, 3, 4, 4, 7, 9, 11, \ldots$

Piecewise Sequences

If the general term of a sequence is defined by more than one formula, then it is called a piecewise sequence.

For example, the sequence with general term

$$a_n = \begin{cases} \frac{1}{n}, & n \text{ is even} \\ \frac{2}{n+1}, & n \text{ is odd} \end{cases}$$

is a piecewise sequence.

$a_1 = \frac{2}{1+1} = 1$

$a_2 = \frac{1}{2}$

$a_3 = \frac{2}{3+1} = \frac{1}{2}$, etc.

Ex: Given the piecewise with general term $a_n = \begin{cases} n^2-5n, & n<10 \\ n-8, & n\geq 10 \end{cases}$

a. Find $a_{20}$.
b. find $a_1$
c. Find the term which is equal to 0.

Soln

a. $n=20$, $a_{20} = 20-8 = 12$.

b. $n=1$, $a_1 = 1^2-5(1) = 1-5 = -4$

c. $n^2-5n=0$, $n(n-5)=0 \Rightarrow \underset{\times}{n=0}$ or $\boxed{n=5}$ since n can't be 0.

$n-8=0$, $\Rightarrow n=8$ X since $n\geq 10$,

## Recursively Defined Sequences

Sometimes the terms in a sequence may depend on the other terms. Such a sequence is called a recursively defined sequence.

Ex: $a_{n+1} = a_n + 2$, and first term $a_1 = 4$ is a recursively defined sequence.

a. find $a_2$.
(b) find $a_4$
(c) find the general term.

Soln

a. $a_2 = a_1 + 2 = 4+2 = 6$

(c) $a_4 = a_3 + 2$ ; $a_3 = a_2 + 2 = 6+2 = 8$.
$= 8+2 = 10$

(d) $a_1 = 4$

$a_2 = a_1 + 2$

$a_3 = a_2 + 2 = a_1 + 2 + 2 = \cancel{a_1 + 2(3-1)} = a_1 + 2(3-1)$

$a_4 = a_3 + 2 = a_1 + 2 + 2 + 2 = a_1 + 2(4-1)$

⋮

$a_n = a_1 + 2(n-1) = 4 + 2n - 2 = \underline{\underline{2n+2}}$.

# ARITHMETIC & SERIES

A series is a succession of numbers of which each number is formed according to a definite law which is the same through out the series.
Here, we will be considering basically the following series AP and GP.

## ARITHMETIC PROGRESSION

If a sequence $(a_n)$ has the same difference d between its consecutive terms, then it is called an arithmetic sequence.

In otherwords, if $(a_n)$ is arithmetic then

$$a_{n+1} = a_n + d \text{ such that } n \in \mathbb{N},\ d \in \mathbb{R}.$$

d is called the common difference of the arithmetic sequence.

Note: If "d" is positive, we say the AP is increasing, if "d" is negative, we say the AP is decreasing.
- If $d = 0$, then AP is non-increasing and non-decreasing sequence.

Ex: State whether the following sequences are Arithmetic Progression or not.

a. 7, 10, 13, 16,  b. 3, −2, −7, −12, . . .
c. 1, 4, 9, 16, . . . ,  d. 6, 6, 6, 6, . . .

Soln
a. AP, $d = 3$,  b. AP, $d = -5$,  c. Not AP  d. AP, $d = 0$

Ex: State whether the sequences with the following general terms are arithmetic or not. If a sequence is arithmetic, find the common difference.

a. $a_n = 4n - 3$,  b. $a_n = 2^n$  c. $a_n = n^2 - n$

Soln

a. $a_{n+1} = 4(n+1) - 3 = 4n+1$

So, $a_{n+1} - a_n = (4n+1) - (4n-3) = 4$, which is constant.

$\Rightarrow (a_n)$ is AP; $d = 4$.

b. $a_{n+1} = 2^{n+1}$, $a_n = 2^n$.

$a_{n+1} - a_n = 2^{n+1} - 2^n = 2^n(2-1) = 2^n$ is not constant.

$\Rightarrow (a_n)$ is not AP.

c. $a_{n+1} = (n+1)^2 - (n+1)$, $= n^2 + 2n + 1 - (n+1) = n^2 + n$

$a_n = (n^2 - n)$ $\Rightarrow a_{n+1} - a_n = (n^2+n) - (n^2-n) = 2n$ is not constant

$\Rightarrow (a_n)$ is not AP.

## General Term of A.P.

If $(a_n)$ is arithmetic, $a_{n+1} = a_n + d$

$a_1 = a_1$

$a_2 = a_1 + d$

$a_3 = a_2 + d = (a_1+d) + d = a_1 + 2d$

$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d$

$a_5 = a_1 + 4d$

$\vdots$

$a_n = a_1 + (n-1)d$. This is the general term of an arithmetic sequence.

Ex: $-3, 2, 7$. are the first three term of an arithmetic sequence $(a_n)$. Find the twentieth term.

Soln: We know $a_1 = -3$, $d = a_3 - a_2 = a_2 - a_1 = 5$

$\Rightarrow a_n = a_1 + (n-1)d$

$a_{20} = -3 + (20-1)5 = -3 + 19(5) = 92$

Ex: $(a_n)$ is an arithmetic sequence with $a_1 = 4$, $a_8 = 25$.
Find the common difference and $a_{101}$.

Soln



$a_n = a_1 + (n-1)d$

$a_8 = a_1 + 7d \qquad a_1 = 4.$

$a_8 = 25 = 4 + 7d \Rightarrow 7d = 21 \Rightarrow d = \frac{21}{7} = \underline{3}$

$a_{101} = a_1 + 100d = 4 + 100(3) = \underline{304}$

Ex: $(a_n)$ is an arithmetic sequence with $a_1 = 3$ and $d = 4$. Is 59 a term of this sequence?

Soln

$a_n = a_1 + (n-1)d$

$59 = 3 + (n-1)4$

$\Rightarrow 56 = (n-1)4$

$\Rightarrow n-1 = 14 \Rightarrow \underline{n = 15}$.

Ex. Find the number of terms in the arithmetic sequence $1, 4, 7, \ldots, 91$.

Soln

$a_n = a_1 + (n-1)d$

$91 = 1 + (n-1)3$

$90 = (n-1)3 \Rightarrow 30 = n-1 \Rightarrow \underline{n = 31}$

Ex: 1. Is the sequence with general term $a_n = 5n + 9$ an arithmetic sequence? Why?

2. $6, 2, -2$ are the first three terms of an AP $(a_n)$. Find the 30th term.

3. $(a_n)$ is an AP with $a_1 = 7$, $a_{10} = 70$. Find $d$ and $a_{101}$.

4. $(a_n)$ is AP with $a_1 = -1$ and $d = 9$. What term ~~is~~ of the sequence is 89?

Ex: $(a_n)$ is an AP with $a_{11}=34$ and $d=3$. Find $a_3$

$a_n = a_1 + (n-1)d$

$a_{11} = a_1 + (11-1)3$

$34 = a_1 + 30$

$\Rightarrow a_1 = 4,$

$a_3 = a_1 + 2d = 4+6 = \underline{10}$

Ex: $(a_n)$ is an arithmetic sequence with $a_5 = 14$ and $a_{10} = 34$. Find the common difference and first term.

Soln

$a_5 = 14 = a_1 + 4d \quad (1)$

$a_{10} = 34 = a_1 + 9d. \quad (2)$

subtract (1) from (2)

$20 = 9d - 4d = 5d \Rightarrow \underline{\underline{d=4}}$

$\Rightarrow a_1 = 14 - 4d = 14 - 4(4) = 14 - 16 = \underline{\underline{-2}}$

Ex: $(a_n)$ is an AP with $a_9 - a_2 = 42$. Find $a_{10} - a_7$.

Soln.

$a_9 - a_2 = (a_1 + 8d) - (a_1 + d) = 7d = 42$

$\Rightarrow \underline{\underline{d=6}}$

$a_{10} - a_7 = (a_1 + 9d) - (a_1 + 6d)$

$= 3d = 3(6) = \underline{18}$

Ex: The 10th term of an AP is $-15$. and the 31st term is $-57$. Find the 15th term

Soln

$a_{10} = -15 = a_1 + 9d \quad (1)$

$a_{31} = -57 = a_1 + 30d \quad (2)$

Subtract (1) from (2).

$-57-(-15) = 0 + 30d - 9d$

$-57+15 = 21d$

$-42 = 21d$

$d = -\frac{42}{21} = \underline{\underline{-2}}$

$a_1 = -15 - 9d = -15 - 9(-2) = -15 + 18 = \underline{\underline{3}}$.

$a_{15} = a_1 + 14d = 3 + 14(-2) = 3 - 28 = \underline{\underline{-25}}$

## Sum of the first n Terms.

Consider the AP whose first few terms are 3, 7, 11, 15, 19.

$S_n = a_1 + a_2 + \cdots + a_n$

$S_1 = 3$

$S_2 = 3+7 = 10$

$S_3 = 3+7+11 = 21$

$S_4 = 3+7+11+15 = 36$

$S_5 = 3+7+11+15+19 = 55$.

**EX:** Given the AP with general term $a_n = 3n+1$, find the sum of first three terms.

Soln

$S_3 = a_1 + a_2 + a_3 = 4+7+10 = 21$.

**Theorem:** The sum of the first n terms of an arithmetic sequence $(a_n)$ is $S_n = \left(\frac{a_1+a_n}{2}\right)n$.

Proof:

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$ or

$S_n = a_n + a_{n-1} + \cdots + a_2 + a_1$.

Adding these equations side by side.

~~57~~ ~~57~~

$$2S_n = (a_1+a_n)+(a_2+a_{n-1})+\cdots+(a_{n-1}+a_2)+(a_n+a_1)$$

$$\cancel{2S_n = (a_1+a_n) + (a_1+d + a_n}$$

$$2S_n = (a_1+a_n) + \big(a_1+d+a_1+(n-2)d\big)+\cdots+\big(a_1+(n-2)d+a_1+d\big) + (a_n+a_1)$$

$$= (a_1+a_n) + \big(a_1+\underbrace{a_1+(n-1)d}_{a_n}\big)+\cdots+\big(a_1+\underbrace{a_1+(n-1)d}_{a_n}\big) + (a_n+a_1)$$

$$= (a_1+a_n)+(a_1+a_n)+\cdots+(a_1+a_n)+(a_n+a_1)$$

$$= \underbrace{\qquad\qquad\qquad\qquad}_{n\text{-times}}$$

$$2S_n = (a_1+a_n)\cdot n$$

$$\Rightarrow S_n = \frac{(a_1+a_n)}{2}\cdot n = \frac{n(a_1+a_1+(n-1)d)}{2} = \frac{n}{2}\big(2a_1+(n-1)d\big)$$

Ex: Given the Arithmetic sequence with $a_1 = 2$ and $a_6 = 17$. Find $S_6$.

Soln

$$S_6 = \left(\frac{a_1+a_6}{2}\right)6 = (2+17)3 = 57.$$

Ex: The first term of an AP is 7. The last term is 70 and the sum is 385. Find the no of terms in the sequence and the common difference.

Soln

$$S_n = \frac{n}{2}(a_1+a_n).$$

$$\Rightarrow 385 = \frac{n}{2}(7+70) = \frac{n}{2}(77)$$

$$n = \frac{385\times 2}{77} = \underline{\underline{10}}$$

$\cancel{S_n} \Longrightarrow a_{10} = 70 = a_1 + 9d = 7 + 9d$

$\Rightarrow 63 = 9d \Rightarrow \underline{\underline{d = 7}}$.

or

$\cancel{S_n}$ :

Ex: Given the AP with $a_1 = -14$ and $d = 5$, find $S_{27}$.

Soln

$S_{27} = \left(\frac{a_1 + a_{27}}{2}\right)27$ . $a_{27} = a_1 + 26d = -14 + 26(5)$
$= 116$.

$= \left(\frac{-14 + 116}{2}\right)27 = \underline{\underline{1377}}$

Ex: Find the number of terms in an AP whose first term is 5, $d = 3$ and sum is 55.

Soln

$S_n = \frac{n}{2}(2a_1 + (n-1)d) \Rightarrow 55 = \frac{n}{2}(2(5) + (n-1)3)$

$= 110 = n(10 + 3n - 3)$

$110 = n(7 + 3n)$

$\Rightarrow 3n^2 + 7n - 110 = 0$.

$n = \frac{-7 \pm \sqrt{7^2 - 4(3)(-110)}}{2(3)} = \frac{-7 \pm \sqrt{49 + 1320}}{6}$

$n = \frac{-7 \pm \sqrt{1369}}{6} = \frac{-7 \pm 37}{6} \Rightarrow -\frac{42}{6}$ or $\frac{30}{6}$

$= -7$ or $5$.

$\Rightarrow \underline{\underline{n = 5}}$ since $n$ is a positive integer.

Ex: (i) Given an AP with $a_1 = 4$ and $a_{10} = 15$. find $S_{10}$.

(ii) " " " $a_{13} = 26$ and $d = -2$, find $S_{13}$.

(iii) " " " " $a_1 = 9$, and $S_8 = 121$, find $d$.

# GEOMETRIC PROGRESSION (SEQUENCE)

If a sequence $(b_n)$ has the same ratio $q$ between its consecutive terms, then it is called a <u>geometric sequence</u>.

In otherwords, $(b_n)$ is geometric if $b_{n+1} = b_n \cdot q$ such that $n \in \mathbb{N}, q \in \mathbb{R}$.
$q$ is called <u>common ratio</u> of the sequence.

- If $q > 1$, then the geometric sequence is <u>increasing</u> when $b_1 > 0$ and <u>decreasing</u> when $b_1 < 0$
- If $0 < q < 1$, G.P is <u>increasing</u> when $b_1 < 0$ and <u>decreasing</u> when $b_1 > 0$.
- If $q < 0$, then the sequence is not monotone.
- If $q = 1$, then the sequence is non-increasing or non-decreasing.
- If $q = 0$, then " " " " " "

Ex: $2, 4, 8, 10, \dots$ $\qquad q = +2$.
$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ $\qquad q = +\frac{1}{2}$.
$\frac{1}{3}, 1, -3, 9, \dots$ $\qquad q = -3$.

Ex: State whether the following sequence is geometric or not. If it is a GP, find the common ratio?

a. $1, 2, 4, 8, \dots$ b. $3, 3, 3, 3, \dots$ c. $1, 4, 9, 16, \dots$
d. $5, -1, \frac{1}{5}, -\frac{1}{25}$.

<u>Soln</u>

a. GP, $q = 2$, b. GP, $q = 1$ c. Not GP.
d. GP; $q = -\frac{1}{5}$.

Ex: State whether the following is a GP or not.

a. $b_n = 3^n$  b. $b_n = n^2+3$  c. $3\cdot 2^{n+3}$.

Soln

a. $b_{n+1} = 3^{n+1}$, $b_n = 3^n$ $\Rightarrow$ $\frac{b_{n+1}}{b_n} = \frac{3^{n+1}}{3^n} = \underline{\underline{3}}$ is constant $\Rightarrow$ Geometric sequence

b. $b_{n+1} = (n+1)^2+3$, $b_n = n^2+3$ $\Rightarrow$ $\frac{b_{n+1}}{b_n} = \frac{(n+1)^2+3}{n^2+3} = \frac{n^2+2n+4}{n^2+3}$ which is not constant.

So, $(b_n)$ is not a Geometric sequence.

c. $b_{n+1} = 3\cdot 2^{(n+1)+3} = 3\cdot 2^{n+4}$, $b_n = 3\cdot 2^{n+3}$ $\Bigg\}$ $\frac{b_{n+1}}{b_n} = \frac{3\cdot 2^{n+4}}{3\cdot 2^{n+3}} = \underline{\underline{2}}$ is constant

$\Rightarrow$ Sequence is Geometric.

## General Term.

If $(b_n)$ is geometric, then we only know that $b_{n+1} = b_n \cdot q$.

So,

$b_1 = b_1$

$b_2 = b_1 \cdot q$

$b_3 = b_2 \cdot q = b_1 q \cdot q = b_1 \cdot q^2$

$b_4 = b_3 \cdot q = b_1 q^2 \cdot q = b_1 q^3$

$\vdots$

$b_n = b_1 \cdot q^{n-1}$ . (General Term of a Geometric sequence).

Ex: If 100, 50, 25 are the first three terms of a geometric sequence $(b_n)$, find the sixth term.

Soln

$q = \frac{b_3}{b_2} = \frac{b_2}{b_1} = \frac{1}{2}$, So, $b_1 = 100$, $q = \frac{1}{2}$.

$b_6 = b_1 q^5 = 100\left(\frac{1}{2}\right)^5 = \frac{100}{32} = \underline{\underline{\frac{25}{8}}}$

Ex: $(b_n)$ is a GP with $b_1 = \frac{1}{3}$, $q = 3$. Find $b_4$.

Soln

$$b_4 = \frac{1}{3} \cdot 3^{4-1} = 9. \qquad (b_n = b_1 q^{n-1}).$$

Ex: $(b_n)$ is a GP with $b_1 = -15$, $q = \frac{1}{5}$. Find the general term.

Soln:

$$b_n = b_1 q^{n-1}$$

$$b_n = -15\left(\frac{1}{5}\right)^{n-1} = -15\left(\frac{1}{5}\right)^n \cdot \left(\frac{1}{5}\right)^{-1} = -15\left(\frac{1}{5}\right)^n \cdot 5 = -75\left(\frac{1}{5}\right)^n.$$

Ex: Consider the GP $(b_n)$ with $b_1 = \frac{1}{9}$ and $q = 3$.

Is 243 a term of this sequence?

Soln

$b_n = b_1 \cdot q^{n-1}$. So, $b_n = \left(\frac{1}{9}\right) \cdot 3^{n-1}$

$$243 = \left(\frac{1}{9}\right) \cdot 3^n \cdot 3^{-1} = \left(\frac{1}{9}\right) 3^n \cdot \left(\frac{1}{3}\right)$$

$$\Rightarrow \quad 3^n = 243 \times 9 \times 3 = 3^5 \times 3^2 \times 3^1 = 3^8$$

$$\underline{\underline{n = 8}}.$$

Since n is a natural no, 243 is the eight term of this sequence.

Ex: Given a ~~monotone~~ GP with $b_4 = 56$, $q = -1/2$. Find $b_9$.

Soln

$$b_4 = b_1 \cdot q^3 \Rightarrow 56 = b_1\left(-\frac{1}{2}\right)^3 \Rightarrow \text{So, } b_1 = (56)(-8) = \underline{\underline{-448}}$$

$$b_9 = b_1 q^8 = -448\left(-\frac{1}{2}\right)^8 = -7/4.$$

Ex: $(b_n)$ is a GP with $b_5 = \frac{1}{32}$, $b_8 = 4^{-4}$. Find the common ratio.

Soln

$$b_5 = \frac{1}{32} = 2^{-5} \text{ and } b_8 = 4^{-4} = 2^{-8}.$$

$$b_5 = b_1 q^4 = 2^{-5} \qquad \frac{b_8}{b_5} = \frac{b_1 q^7}{b_1 q^4} = \frac{2^{-8}}{2^{-5}} \Rightarrow q^3 = 2^{-8-(-5)} = 2^{-3}$$

$$b_8 = b_1 q^7 = 2^{-8} \qquad \Rightarrow q^3 = \left(\frac{1}{2}\right)^3$$

$$\boxed{q = \frac{1}{2}}$$

Ex: Given a monotone GP with $b_3 = 9$, $b_5 = 16$.
Find the common ratio.

Soln

$$\frac{b_5}{b_3} = \frac{b_1 q^4}{b_1 q^2} = q^2 = \frac{16}{9} \Rightarrow q = \sqrt{\frac{16}{9}} = \pm \frac{4}{3}.$$

$\underline{\underline{q = \frac{4}{3}}}$, since GP is monotone; that is each successive term in this case is greater than the previous.

If $q$ was $-\frac{4}{3}$, the sequence won't be a monotonic.

Ex. $(b_n)$ is a non-monotonic GP with $b_2 = 2$, $b_4 = \frac{8}{9}$,
Which term is $\frac{32}{81}$? What is the common ratio?

Soln

$$\frac{b_4}{b_2} = \frac{b_1 q^3}{b_1 q} = q^2 = \frac{\frac{8}{9}}{2} = \frac{4}{9} \Rightarrow q = \sqrt{\frac{4}{9}} = \pm \frac{2}{3}.$$

$\Rightarrow \underline{\underline{q = -\frac{2}{3}}}$ for $(b_n)$ to be a non-monotone sequence.

$$\frac{32}{81} = b_n = b_1 q^{n-1} \Rightarrow \frac{32}{81} = \left(\frac{b_2}{q}\right)(q^{n-1}) = b_2 q^{n-2} = 2 \cdot \left(-\frac{2}{3}\right)^{n-2}.$$

$$= \frac{16}{81} = \left(-\frac{2}{3}\right)^{n-2} = \left(-\frac{2}{3}\right)^n \cdot \left(-\frac{2}{3}\right)^{-2} = \left(-\frac{2}{3}\right)^n \left(-\frac{3}{2}\right)^2$$

$$= \frac{16}{81} = \left(-\frac{2}{3}\right)^n \frac{9}{4} \Rightarrow \frac{64}{81 \times 9} = \left(-\frac{2}{3}\right)^n$$

$$= \frac{2^6}{3^6} = \left(-\frac{2}{3}\right)^n$$

$$\Rightarrow \left(\frac{2}{3}\right)^6 = \left(-\frac{2}{3}\right)^n$$

$\Rightarrow \underline{\underline{n = 6}}$ since $\left(-\frac{2}{3}\right)^6 = \left(\frac{2}{3}\right)^6$.

## Sum of the First n Terms:

Consider $(b_n)$ with first few terms $1, 2, 4, 8, 16$.

· The sum of the first n terms i.e $S_n = b_1 + b_2 + \cdots + b_n$.

Now;

$$S_1 = 1$$
$$S_2 = 1 + 2 = 3$$
$$S_3 = 1 + 2 + 4 = 7$$
$$S_4 = 1 + 2 + 4 + 8 = 15.$$

Ex: Given $(b_n)$, a GP with general term $b_n = 3(-2)^n$; find the first three terms.

Soln

$$S_3 = b_1 + b_2 + b_3 = -6 + 12 - 24 = \underline{-18}.$$

**Theorem:** The sum of the first n terms of a Geometric sequence $(b_n)$ is $S_n = b_1 \cdot \frac{1-q^n}{1-q}$, $q \neq 1$.

Proof:

$$S_n = b_1 + b_2 + b_3 + \cdots + b_{n-1} + b_n.$$
$$S_n = b_1 + b_1 q + b_1 q^2 + \cdots + b_1 q^{n-2} + b_1 q^{n-1} \quad (1)$$
$$q S_n = b_1 q + b_1 q^2 + b_1 q^3 + \cdots + b_1 q^{n-1} + b_1 q^n \quad (2)$$

Subtract (2) from (1).

$$S_n - q S_n = b_1 - b_1 q^n$$
$$S_n = \underline{\frac{b_1(1-q^n)}{1-q}} = \underline{\underline{\frac{b_1(q^n-1)}{q-1}}}$$

Ex: Given a geometric sequence with $b_1 = \frac{1}{81}$ and $q = 3$, find $S_6$.

Soln

$$S_n = b_1 \frac{1-q^n}{1-q}, \text{ so, } S_6 = \left(\frac{1}{81}\right) \cdot \frac{1-3^6}{1-3} = \frac{364}{81}.$$

Ex: Given a GP with $S_6 = 3640$ and $q = 3$, find $b_1$.

Soln

$S_6 = b_1 \frac{1-q^5}{1-q}$ , so $3640 = b_1 \cdot \frac{1-3^6}{1-3}$ , and so

$\underline{\underline{b_1 = 10}}$.

Ex: Given a GP with $q = \frac{1}{3}$ , $b_p = 5$ and $S_p = 1820$. find $b_1$

Soln

$S_p = b_1 \cdot \frac{1-q^p}{1-q} = \frac{b_1 - b_1 q^p}{1-q} = \frac{b_1 - b_{p+1}}{1-q} \Rightarrow$.

$1820 = \frac{b_1 - 5\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)} \Rightarrow \underline{\underline{b_1 = 1215}}$.

Ex: (i) Given a GS with $b_1 = 1$ and $q = -2$. find $S_7$.

(2) Given a GP with $S_9 = 513$, & $q = -2$ find $b_5$.