POLYNOMIALS

Definition: An algebraic expression which comprises a single real number or the product of a real number and one or more several variables raised to whole number powers is called a "monomial".

E.g. $6, -2x^3, 5a^2b^3, -\frac{7}{2}$ and $3x^4yz^5$ are all monomials.

Each number preceding the variable(s) in a monomial is called a coefficient.

In the examples above $6, -2, 5, -7/2$ and 3 are the coefficients.

A <u>polynomial</u> is the sum or difference of a set of monomials.

<u>Ex</u>: $4x^3 + 5x - 3$, $5xy^2 + 4x + 3y$, and $-7a^3b^3 + 4ab^2$ are all polynomials.

The term of a polynomial that does not contain a variable is called the "constant term".

The coefficient of the term containing the variable raised to the highest power is called the "leading coefficient".

Ex: Consider $9x^6 - 5x^4 + 3x^3 - 8x + 2$.

$9x^6, -5x^4, 3x^3, -8x$ and 2 are the terms of polynomial.

$9, -5, -8$, and 2 are the coefficients.

2 is the constant term.

9 is the leading coefficient.

A polynomial is said to be in <u>standard form</u> if the terms are written in descending order of degree,

$x^4 - 4x^2 + 6$ is in standard form, $-4x^2 + 6 + x^4$ is not in standard form.

Ex: Determine whether each algebraic expression is a polynomial.

a. -8 b. $5x - \frac{3}{4}$ c. $4x^2 - 3x^{-2}$ d. $\frac{6}{x} + \frac{5}{x^2}$

e. $x^{56} - x^{45} + 3x^2 - 3$

Soln

a. -8 is a polynomial containing only a constant term.

b. 5 and $-\frac{3}{4}$ are real numbers, so this is a polynomial with two terms

c. $4x^2 - 3x^{-2}$ is not a polynomial because -2 is not a whole number power.

d. $\frac{6}{x} + \frac{5}{x^2} = 6x^{-1} + 5x^{-2}$. So, this is not a polynomial,

e. The expression is a polynomial.

Degree of a Polynomial: Is the highest degree of the terms in the polynomial.

e.g $-6x^5 + 4x^3 - 7$ has degree 5.

$9z^7 - 8z^6 + 7z^3 + 3$ has degree 7.

$-12x^3yz + 5x^4yz^3 - 6y^4z^3$ has degree 8

$8 = (4 + 1 + 3)$.

Ex: Find the degree of the polynomial

a) $4x^5 - 4x^2 + 3$

(b) $-3x^2y^5 + 6xyz^4 + 12x^3y^2z^4$

(c) $12x^2 - 10x^7 + 4x^2 + 2x + 1$

Polynomial function.

Defin: Every function defined from $\mathbb{R}$ to $\mathbb{R}$ of the form
$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ is called a polynomial function.

Example: Let $P(x) = 2x^2 - 3x + 4$. Find $P(1)$ and $P(-2)$

solution:
$$P(1) = 2(1)^2 - 3(1) + 4$$
$$= 2 - 3 + 4 = 3.$$
$$P(-2) = 2(-2)^2 - 3(-2) + 4$$
$$= (2 \cdot 4) + 6 + 4 = 18$$
$$= 8 + 6 + 4 = \underline{18}$$

Example: Given that $P(x) = 3x^2 - 5x + 4$
Find $P(-1)$ and $P(3)$.

Rule: In order to find the constant term of a polynomial, compute $P(0)$.

Ex: Given $P(x) = 3x^2 - 5x + 4$. What is the constant term?

Soln
$$P(0) = 3(0^2) - 5(0) + 4$$
$$= 0 - 0 + 4 = \underline{4}$$

Rule: To add the sum of the coefficients of $P(x)$, compute $P(1)$;

Ex: Find the sum of the coefficients of $P(x) = 3x^2 - 5x + 4$

Soln
$$P(1) = 3(1^2) - 5(1) + 4 = 3 - 5 + 4 = \underline{2}$$

OPERATIONS ON POLYNOMIALS :

Defn: Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$
and $Q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_2 x^2 + b_1 x + b_0$
be two polynomials such that
$\deg|Q(x)| \geq \deg|P(x)|$. Then, the sum of the polynomials is defined as

$$P(x) + Q(x) = b_m x^m + \cdots + (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_2 + b_2)x^2 + (a_1 + b_1)x + a_0 + b_0.$$

Ex: Let $P(x) = -6x^4 + 5x^3 - 2x + 5$ and
$Q(x) = 4x^4 + 5x^3 + 12$.

Then, $P(x) + Q(x) = (-6+4)x^4 + (5+5)x^3 - 2x + (12+5)$
$= -2x^4 + 10x^3 - 2x + 17$.

Ex: Let $P(x) = -6x^4 + 5x^3 - 2x + 5$
$Q(x) = 2x^5 + x^4 - x^2$.

Find $P(x) + Q(x)$.

Soln:

$P(x) + Q(x) = 2x^5 + (-6 + 1)x^4 + 5x^3 - x^2 - 2x + 5$
$= 2x^5 - 5x^4 + 5x^3 - x^2 - 2x + 5$

Subtracting Polynomials :

The difference of two polynomials $P(x)$ and $Q(x)$ is defined as $P(x) - Q(x) = P(x) + [-Q(x)]$.

In otherwords, we subtract the terms.

Ex: Let $P(x) = 7x^3 - 4x^2 + 5$ and $Q(x) = 4x^3 + 5x - 2$.

Then, $P(x) - Q(x) = (7x^3 - 4x^2 + 5) - (4x^3 + 5x - 2)$

$= (7-4)x^3 - 4x^2 - 5x + (5-(-2))$

$= 3x^3 - 4x^2 - 5x + 7$.

Ex: Let $P(x) = 8x^4 - 3x^3 + 5x - 4$

$Q(x) = 6x^3 + 2x^2 - 10x + 6$.

Calculate

a. $P(x) - Q(x)$ b. $Q(x) - P(x)$.

Soln

a. $P(x) - Q(x) = (8x^4 - 3x^3 + 5x - 4) - (6x^3 + 2x^2 - 10x + 6)$

$= 8x^4 + (-3-6)x^3 - 2x^2 + (5-(-10))x (-4-6)$

$= 8x^4 - 9x^3 - 2x^2 + 15x - 10$.

b. $Q(x) - P(x) = (6x^3 + 2x^2 - 10x + 6) - (8x^4 - 3x^3 + 5x - 4)$

$= -8x^4 + (6-3)x^3 + 2x^2 + (-10-5)x + (6-(-4))$

$= -8x^4 - 3x^3 + 2x^2 - 15x + 10$.

Ex: $P(x) = 9x^5 - 6x^3 + 2x^2$,

$Q(x) = 7x^4 - 3x^3 - x^2 + 5$.

$R(x) = 2x^5 + 5x^4 - x + 6$.

Perform the Calculations. a. $P(x) + Q(x)$ b. $Q(x) - R(x)$

c. $P(x) - R(x)$ d. $R(x) - Q(x)$.

Multiplying Polynomials:

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and

$Q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$.

Then, the product of $P(x)$ and $Q(x)$ is defined as

$$P(x) \cdot Q(x) = (a_n x^n + \cdots + a_1 x + a_0)(b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0).$$

Ex: Let $P(x) = 2x^3 + x - 3$ and $Q(x) = x^2 - 3x + 2$.

$$\begin{aligned}
P(x) \cdot Q(x) &= (2x^3 + x - 3) \cdot (x^2 - 3x + 2) \\
&= 2x^3(x^2 - 3x + 2) + x(x^2 - 3x + 2) - 3(x^2 - 3x + 2) \\
&= 2x^5 - 6x^4 + 4x^3 + x^3 - 3x^2 + 2x - 3x^2 + 9x - 6 \\
&= 2x^5 - 6x^4 + (4+1)x^3 + (-3-3)x^2 + (2+9)x - 6 \\
&= 2x^5 - 6x^4 + 5x^3 - 6x^2 + 11x - 6.
\end{aligned}$$

Ex: Let $P(x) = 2x + 1$ $Q(x) = x^2 - 2x + 1$

$$\begin{aligned}
Q(x) \cdot P(x) &= (x^2 - 2x + 1)(2x + 1) \\
&= x^2(2x + 1) - 2x(2x + 1) + 1(2x + 1) \\
&= 2x^3 + x^2 - 4x^2 - 2x + 2x + 1 \\
&= 2x^3 + (1-4)x^2 + (-2+2)x + 1 \\
&= \underline{2x^3 - 3x^2 + 1}
\end{aligned}$$

Ex: Let $P(x) = x^3 + 2x$, $Q(x) = 2x^2 - x + 1$, and $R(x) = -x^2 + 5$. Find each product.

a. $P(x) \cdot Q(x)$. b. $P(x) \cdot R(x)$. c. $Q(x) \cdot R(x)$.

Ex: Let $P(x) = 5x^7 + 4x^3 - 3x$, $Q(x) = 6x^6 + 8x^5$.

Find (a) $4 \cdot P(x)$. b. $5\,Q(x)$. c. $\deg |P(x) \cdot Q(x)|$.

Dividing Two Polynomials

Quotient of two polynomials:

Let $P(x)$ and $D(x)$ be two polynomials such that $\deg |P(x)| \geq \deg |D(x)| \geq 1$. If there exist $Q(x)$ and $R(x)$ such that $P(x) = D(x) \cdot Q(x) + R(x)$ where $\deg |R(x)| < \deg |D(x)|$, then

$P(x)$ is called the dividend

$D(x)$ is called the divisor

$Q(x)$ is called the quotient, and

$R(x)$ is called the remainder.

$$\begin{array}{r|l} & Q(x) \\ \hline D(x) & P(x) \\ & - \\ \hline & R(x). \end{array}$$

i.e $P(x) = D(x)\,Q(x) + R(x)$.

<u>Note</u>: If $R(x) = 0$, then we say that $P(x)$ is divisible by $D(x)$ and write $P(x) = D(x) \cdot Q(x)$

Ex. Divide $2x^2 - x - 6$ by $x - 2$.

$$
\begin{array}{r|l}
 & 2x + 3 \\
\hline
x - 2 & 2x^2 - x - 6 \\
 & -\ 2x^2 - 4x \\
\hline
 & (-1 - (-4))x - 6 \\
 & = 3x - 6 \\
 & -\ (3x - 6) \\
\hline
 & 0 + 0
\end{array}
$$

$(-1 + 4)x - 6$
$= 3x - 6$

Steps.

$$
\begin{array}{r|l}
 & 2x \\
\hline
x - 2 & 2x^2 - x - 6
\end{array}
$$

Step 1: Divide the first term of $2x^2 - x - 6$ by the first term of $x - 2$ and write the result:

$$\frac{2x^2}{x} = 2x$$

$$
\begin{array}{r|l}
 & 2x \\
\hline
x - 2 & 2x^2 - x - 6 \\
 & -(2x^2 - 4) \\
\hline
 & 3x - 6
\end{array}
$$

Step 2: Multiply $(x - 2)$ by $2x$ and subtract the result from $2x^2 - x - 6$.

$$
\begin{array}{r|l}
 & 2x + 3 \\
\hline
x - 2 & 2x^2 - x - 6 \\
 & -(2x^2 - 4) \\
\hline
 & 3x - 6
\end{array}
$$

Step 3: Divide $3x - 6$ by x. The result is the second term of the quotient: $\frac{3x}{x} = 3$.

$$
\begin{array}{r|l}
 & 2x + 3 \\
\hline
x - 2 & 2x^2 - x - 6 \\
 & -(2x^2 - 4) \\
\hline
 & 3x - 6 \\
 & -\ 3x - 6 \\
\hline
 & 0
\end{array}
$$

Step 4: Multiply $(x - 2)$ by 3 and subtract from $(3x - 6)$.

Here, $(2x + 3)$ is the quotient, and $2x^2 - x - 6 = (x - 2)(2x + 3)$. The remainder is zero, so $2x^2 - x - 6$ is divisible by $x - 2$.

Ex. Divide $6x^3-9x^2+12x-7$ by $2x-3$.

$$
\begin{array}{r|l}
 & 3x^2+6 \\
\hline
2x-3 & 6x^3-9x^2+12x-7 \\
 & -(6x^3-9x^2) \\
\hline
 & 0+0+12x-7 \\
 & -(12x-18) \\
\hline
 & -7+18 \\
 & =11
\end{array}
$$

step 1: $\frac{6x^3}{2x} = 3x^2$

step 2: $3x^2(2x-3) = 6x^3-9x^2$

step 3: $\frac{12x}{2x} = 6$

step 4: $6(2x-3) = 12x-18$.

So,

$$\underbrace{6x^3-9x^2+12x-7}_{\text{dividend}} = \underbrace{(2x-3)}_{\text{divisor}}\underbrace{(3x^2+6)}_{\text{quotient}} + \underbrace{11}_{\text{remainder}}.$$

Ex: Divide $2x^2+4x^4-3+5x$ by $2x^2+x$

Soln

$2x^2+x \,\big|\, 2x^2+4x^4+5x-3 = 4x^4+2x^2+5x-3$. (rearrange in decreasing order of power).

So;

$$
\begin{array}{r|l}
 & 2x^2-x+3/2 \\
\hline
2x^2+x & 4x^4+2x^2+5x-3 \\
 & -(4x^4+2x^3) \\
\hline
 & -2x^3+2x^2+5x-3 \\
 & -(-2x^3-x^2) \\
\hline
 & 3x^2+5x-3 \\
 & -(3x^2+3/2x) \\
\hline
 & 7/2x-3.
\end{array}
$$

Hence, $(2x^2-x+\frac{3}{2})(2x^2+x) + \frac{7}{2}x-3 = 4x^4+2x^2+5x-3$

Synthetic Division

We can use this method to speed up the division procedure by a binomial of the form $(x-c)$ or $(ax+b)$ where $a, b, c \in \mathbb{R}$. In synthetic division we consider only the coefficients of polynomials. To understand the process let's look at some examples.

a. Dividing by $x-c$.

Divide $2x^4 - x^3 + 5x^2 - 3$ by $x-3$.

$x-3=0$

$x=3$

$$
\begin{array}{c|ccccc}
 & 2x^4 & -x^3 & +5x^2 & +0\cdot x & -3 \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & 2 & -1 & 5 & 0 & -3 \\
 & & \underbrace{6}_{\text{add}} & & & \\
\hline
\text{multiply} & 2 & & & &
\end{array}
$$

continuing in the fashion gives the remainder. That is,

$$
\begin{array}{c|ccccc}
 & x^4 & x^3 & x^2 & x^1 & 1 \\
3 & 2 & -1 & 5 & 0 & -3 \\
 & \downarrow & 6 & 15 & 60 & 180 \\
\hline
 & 2 & 5 & 20 & 60 & \| \ 177 \\
 & x^3 & x^2 & x & 1 & \text{remainder}
\end{array}
$$

So, the quotient is $2x^3 + 5x^2 + 20x + 60$

and remainder is 177.

b. Dividing by $ax+b$.

Ex: Divide $P(x) = 6x^3 + 13x^2 - x + 1$ by $2x+1$ using synthetic division.

$2x+1=0 \Rightarrow x = \frac{-1}{2}$.

$$\begin{array}{c|cccc}
 & x^3 & x^2 & x & 1 \\
-\frac{1}{2} & 6 & 13 & -1 & 1 \\
 & & -3 & -5 & 3 \\
\hline
 & 6 & 10 & -6 & \| \; 4 \\
 & x^2 & x & 1 &
\end{array}$$

So; $6x^3 + 13x^2 - x + 1 = (2x+1)(3x^2+5x-3) + 4$.

Exercise: Divide $p(x) = 8x^5 - 6x^4 + 4x^2 + 6$ by each divisor a. $x+2$ b. $2x-1$.

The Remainder Theorem

When $P(x)$ is divided by $x-a$, the remainder is $P(a)$.

Ex: Find the remainder when $P(x) = x^4 - 2x^3 + 3x$ is divided by $Q(x) = x-2$.

$P(x) = (x-2)Q(x) + R$

$P(2) = R$

$R = 2^4 - 2(2^3) + 3(2)$

$= 16 - 16 + 6 = \underline{\underline{6}}$

$$\begin{array}{r|l}
 & x^3 + 3 \\
\hline
x-2 & x^4 - 2x^3 + 3x \\
 & -(x^4 - 2x^3) \\
\hline
 & 3x \\
 & -(3x - 6) \\
\hline
 & \boxed{6}
\end{array}$$

Factor Theorem

Let $P(x)$ be a polynomial. Then the following statements are true:

1) If $P(x)$ has a factor $(x-a)$, then $P(a)=0$.
2) If $P(a)=0$ then $(x-a)$ is a factor of $P(x)$.

Example: Let $P(x) = (x-2)(2x+3)$

$$= x(2x+3) - 2(2x+3)$$
$$= 2x^2 + 3x - 4x - 6 = 2x^2 - x - 6,$$

Then $x-2$ is a ~~divisor~~ factor of $2x^2-x-6$

Since $P(2) = 2(2^2) - 2 - 6 = 8-2-6 = \underline{\underline{0}}$.