Example

Find $\frac{dy}{dx}$ if (i) $y = (3x^2-4)^4$ (ii) $y = \left(\frac{x-1}{x+1}\right)^2$

Solution

(i) $y = (3x^2-4)^4$. Let $v = 3x^2-4$ then $y = v^4$.

$\frac{dy}{dv} = 4v^3$, $\frac{dv}{dx} = 6x$

then $\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{dx} = 4v^3 \times 6x = 24x(3x^2-4)^3$.

(2) $y = \left(\frac{x-1}{x+1}\right)^2$, let $v = \frac{x-1}{x+1}$, then $y = v^2$.

$\frac{dy}{dv} = 2v$; $\frac{dv}{dx} = \frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}$

$= \frac{x+1-x+1}{(x+1)^2} = \frac{2}{(x+1)^2}$

Hence $\frac{dy}{dx} = 2v \times \frac{2}{(x+1)^2} = \frac{4(x-1)}{(x+1)} \times \frac{1}{(x+1)^2}$

$= \frac{4(x-1)}{(x+1)^3}$,

Example:

(3). Evaluate $\frac{dy}{dx}$ if (i) $y = \sin(4x^2+3x)$ (ii) $y = \sec^2 4\theta$.

Soln

(1) $y = \sin(4x^2+3x)$ let $v = 4x^2+3x \Rightarrow y = \sin v$.

$\frac{dv}{dx} = 8x+3$ $\frac{dy}{dv} = \cos v$

$\Rightarrow \frac{dy}{dx} = (8x+3)\cos v = (8x+3)\cos(4x^2+3x)$

(2) $y = \sec^2 4\theta$ let $v = \sec 4\theta \Rightarrow y = v^2$.

$$\frac{dy}{dv} = 2v \, ; \quad \cancel{\frac{dv}{d\theta} = \cos 4\theta \times 4 = 4\cos 4\theta}$$

$$\Rightarrow \frac{dv}{d\theta} = \sec 4\theta \tan 4\theta \times 4$$

$$= 4\sec 4\theta \tan 4\theta$$

Note that if $v = \sec 4\theta$, take $u = 4\theta$

$\Rightarrow v = \sec u$

$$\frac{dv}{d\theta} = \frac{dv}{du} \times \frac{du}{d\theta}$$

$$\Rightarrow \frac{dv}{d\theta} = \sec u \tan u \times 4$$

$$= 4\sec 4\theta \tan 4\theta$$

$$\Rightarrow \frac{dy}{dx} = 2v \times 4\sec 4\theta \tan 4\theta$$

$$= 8\sec 4\theta \sec 4\theta \tan 4\theta$$

$$= \underline{\underline{8\sec^2 4\theta \tan 4\theta}}$$

Exercises.

Evaluate the first derivative of the following functions.

1. $y = (x^2 + 2x + 1)^4$
2. $y = x^2 \cos 3x$
3. $y = (2x^2 - 3x)^5$
4. $y = x \sin 4x$
5. $y = \sec 3x$

DERIVATIVE OF x^n where n is NEGATIVE or A FRACTION.

1). If n is a negative integer, let $n = -m$.

$$y = x^{-m} = \frac{1}{x^m},$$

$$\frac{dy}{dx} = -m x^{-m-1}.$$

2). If n is a fraction, let $n = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ (not necessarily positive). Then.

$$y = x^{p/q} = (x^{1/q})^p$$

$$\frac{dy}{dx} = \frac{p}{q} x^{p/q - 1}.$$

Example: Find $\frac{dy}{dx}$ of (1) $y = \sqrt{x}$, (2) $y = \frac{1}{x^4}$.

Solution.

1) $y = \sqrt{x} = x^{1/2}$. $\quad \frac{dy}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

(2) $y = \frac{1}{x^4} = x^{-4}$; $\quad \frac{dy}{dx} = -4x^{-5} = \frac{-4}{x^5}$

Example Evaluate $\frac{dy}{dx}$ if (i) $y = \left(x^2 - \frac{2}{x^2}\right)^2$

(ii) $y = \sqrt{\left(\frac{x}{1+x}\right)}$

Solution

1) $y = \left(x^2 - \frac{2}{x^2}\right)^2$. Let $v = x^2 - \frac{2}{x^2}$, then $y = v^2$.

$$\frac{dy}{dv} = 2v \qquad \frac{dv}{dx} = 2x - 2(-2)x^{-3} = 2x + \frac{4}{x^3}.$$

Hence, $\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{dx}$

$$= 2\left(x^2 - \frac{2}{x^2}\right)\left(2x + \frac{4}{x^3}\right)$$

$$= 4\left(x^2 - \frac{2}{x^2}\right)\left(x + \frac{2}{x^3}\right).$$

(2) $y = \sqrt{\frac{x}{1+x}} = \sqrt{v}$ where $v = \frac{x}{1+x}$.

Then, $\frac{dy}{dv} = \frac{1}{2\sqrt{v}}$, $\frac{dv}{dx} = \frac{(1+x)(1) - x(1)}{(1+x)^2}$

$$= \frac{1+x-x}{(1+x)^2} = \frac{1}{(1+x)^2}.$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\frac{x}{1+x}}} \times \frac{1}{(1+x)^2}.$$

SOME BASIC FUNCTIONS & THEIR DERIVATIVES.

$f(x)$	$f'(x)$
x^n	$n x^{n-1}$
$\ln x$	$1/x$ $\quad x > 0$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \tan x$
$\sec x$	$\sec x \tan x$
c	0
a^x	$a^x \ln a$
e^x	e^x

$f(x)$	$f'(x)$	
$\log_a x$	$\frac{1}{x}\log_a e$	$x>0$
$\sin^{-1}(x)$	$1/\sqrt{1-x^2}$	for $\lvert x\rvert<1$
$\cos^{-1}(x)$	$\frac{-1}{\sqrt{1-x^2}}$	for $\lvert x\rvert<1$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$	
$\ln g(x)$	$\frac{1}{g(x)}\, g'(x)$	
$e^{g(x)}$	$g'(x)\, e^{g(x)}$	

DIFFERENTIATION OF INVERSE FUNCTIONS.

In general if $y=f(x)$ then the value of x will depend on the value of y and so x is a function of y
$x=g(y)$.

Example: if $y=x^2$, then $x=\sqrt{y}$
if $y=\sin x$ then $x=\sin^{-1}y$.

Thus, if $y=f(x)$, and $x=g(y)$, then the derivative of $\frac{dx}{dy}$ of $g(y)$ in terms of the derivative $\frac{dy}{dx}$ of $f(x)$ is given by

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\frac{dx}{dy}} \quad \text{or} \quad \frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}$$

Examples :

1. If $y = \tan^{-1} x \Rightarrow x = \tan y$.

$$\frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y$$
$$= 1 + x^2$$

Hence, $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{1+x^2}$

(Note that once you get $\frac{dx}{dy}$, you have to write your answer in terms of x and not y).

2. If $y = \sin^{-1} x \Rightarrow x = \sin y$

$$\frac{dx}{dy} = \cos y = \sqrt{1 - \sin^2 y}$$
$$= \sqrt{1 - x^2}$$

Note that $\sin^2 y + \cos^2 y = 1$
$\Rightarrow \cos^2 y = 1 - \sin^2 y$
$\cos y = \sqrt{1 - \sin^2 y}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

3) If $y = \cos^{-1} x \Rightarrow x = \cos y$

$$\Rightarrow \frac{dx}{dy} = -\sin y = -\sqrt{1 - \cos^2 y}$$
$$= -\sqrt{1 - x^2}$$

Note that $\sin^2 y + \cos^2 y = 1$
$\Rightarrow \sin y = \sqrt{1 - \cos^2 y}$

Hence $\frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}}$

SECOND AND HIGHER DERIVATIVES.

If $y = f(x)$ is a function of x, then in general the derivative $\frac{dy}{dx} = f'(x)$ will be some other function of x. We might as well enquire what is the rate of change of the derivative with respect to x. The derivative of $\frac{dy}{dx}$ is called the second derivative or the second differential coefficient of y with respect to x and is written as $\frac{d^2y}{dx^2}$. The third derivative of y is denoted by $\frac{d^3y}{dx^3}$ and the nth differential coefficient of y with respect to x is $\frac{d^ny}{dx^n}$.

If the notation $f(x)$ is used, the first, second, third, ... nth derivatives are denoted by $f'(x), f''(x), f'''(x), \ldots, f^{(n)}(x)$.

Example

1. If $y = \sin x$, show that $\frac{d^2y}{dx^2} = -y$, $\frac{d^4y}{dx^4} = y$.

$$\frac{dy}{dx} = \cos x$$

$$\frac{d^2y}{dx^2} = -\sin x = -y$$

$$\frac{d^3y}{dx^3} = -\cos x$$

$$\frac{d^4y}{dx^4} = -(-\sin x) = \sin x = y.$$

2). Find the 2nd derivative of $y = f(\theta) = \frac{\sin\theta}{1+\cos\theta}$

<u>Soln</u>

$$\frac{dy}{d\theta} = \frac{(1+\cos\theta)\cos\theta - \sin\theta(-\sin\theta)}{(1+\cos\theta)^2}$$ using quotient rule

$$= \frac{\cos\theta + \cos^2\theta + \sin^2\theta}{(1+\cos\theta)^2} = \frac{\cos\theta + 1}{(1+\cos\theta)^2} = \frac{1}{1+\cos\theta}$$

$$\frac{d^2y}{d\theta^2} = \frac{-\frac{d}{d\theta}(1+\cos\theta)}{(1+\cos\theta)^2} = \frac{-(-\sin\theta)}{(1+\cos\theta)^2} = \frac{\sin\theta}{(1+\cos\theta)^2}$$

(3). If $y = \tan\theta$ Show that $\frac{d^2y}{d\theta^2} = 2y(1+y^2)$.

$$\frac{dy}{d\theta} = \sec^2\theta = \sec\theta \cdot \sec\theta .$$

$$\frac{d^2y}{d\theta^2} = \sec\theta(\sec\theta\tan\theta) + \sec\theta(\sec\theta\tan\theta).$$

$$= 2\sec^2\theta\tan\theta$$ (Since $\sec^2\theta = 1+\tan^2\theta$).

$$= 2\tan\theta(1+\tan^2\theta)$$

$$= 2y(1+y^2).$$

$$\Rightarrow \frac{d^2y}{d\theta^2} = 2y(1+y^2)$$

<u>Exercise</u>

1. Find $\frac{d^2y}{dx^2}$ if (i) $y = \cos^2 x$ (ii) $y = \frac{\cos x}{1-\sin x}$.
2. If $y = \theta^n$ where $n \in \mathbb{Z}^+$, show that (i) $\theta\frac{dy}{d\theta} = ny$ and (ii) $\theta^2\frac{d^2y}{d\theta^2} = n(n-1)y$
3. If $y = \sec\theta$ show that $\frac{d^2y}{d\theta^2} = y(2y^2-1)$