The process of calculating the derivative or differential coefficient is called differentiation. More specifically, the process of calculating the derivative $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ is called differentiation from "first principle".

1) Differentiation of polynomial function $y = ax^n$.

$f(x) = ax^n$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{a(x+h)^n - ax^n}{h}$$

$$= \lim_{h\to 0} \frac{a\left((x+h)^n - x^n\right)}{h} = \lim_{h\to 0} \frac{a\left(x^n + \binom{n}{1}x^{n-1}h + \cdots + \binom{n}{n-1}xh^{n-1} + h^n - x^n\right)}{h}$$

$$= \lim_{h\to 0} \frac{a\left(x^n + {}^nC_1 x^{n-1}h + \cdots + {}^nC_{n-1} x h^{n-1} + h^n - x^n\right)}{h}$$

$$= \lim_{h\to 0} \frac{a\left({}^nC_1 x^{n-1}h + \cdots + {}^nC_{n-1} x h^{n-1} + h^n\right)}{h}$$

$$= \lim_{h\to 0} a\left({}^nC_1 x^{n-1} + {}^nC_2 x^{n-2}h + {}^nC_3 x^{n-3}h^2 + \cdots + h^{n-1}\right)$$

$$= a\left({}^nC_1 x^{n-1}\right) \text{ as the other terms become } 0.$$

$$= anx^{n-1} \qquad \left({}^nC_1 = \frac{n!}{(n-1)!\,1!} = \frac{n(n-1)!}{(n-1)!} = n.\right)$$

Thus $\frac{dy}{dx} = f'(x) = anx^{n-1}$.

(2) Derivative of $\sin x$

$f(x) = y = \sin x$.

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}.$$

Recall: The following Trigonometric identities.

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Also; the limit of the function

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

$$\Rightarrow f'(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}.$$

$$= \lim_{h\to 0}\frac{\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h/2}$$

$$= \cancel{\lim \cos}$$

$$= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\underbrace{\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{h/2}}_{1}$$

$$= \underline{\underline{\cos x}}$$

(3) Derivative of $\cos x$:

If $f(x) = y = \cos x$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}.$$

$$= \lim_{h\to 0}\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}.$$

$$= \lim_{h\to 0}-1\sin\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{h/2}$$

$$= \lim_{h\to 0}-\left(\sin\left(x+\frac{h}{2}\right)\right)\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{h/2}$$

$$= -\sin x\cdot 1 = \underline{\underline{-\sin x}}$$

Further examples on differentiation from first principles.

Example: $f(x) = y = x^2 + 3x$.

Then, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{(x+h)^2 + 3(x+h) - (x^2+3x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x}{h}$

$= \lim\limits_{h \to 0} \dfrac{2xh + h^2 + 3h}{h}$

$= \lim\limits_{h \to 0} 2x + h + 3$

$\Rightarrow = \underline{\underline{2x + 3}}$

Example: $y = f(x) = \sin 3x$.

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{\sin 3(x+h) - \sin 3x}{h}$

$= \lim\limits_{h \to 0} \dfrac{2 \cos\left(\frac{6x+3h}{2}\right) \sin\left(\frac{3h}{2}\right)}{h}$

$= \lim\limits_{h \to 0} \cos\left(3x + \frac{3}{2}h\right) \dfrac{\sin\left(\frac{3h}{2}\right)}{\frac{h}{2}}$

$= \lim\limits_{h \to 0} \dfrac{3 \cos\left(3x + \frac{3h}{2}\right) \sin\left(\frac{3h}{2}\right)}{3 \times \frac{h}{2}}$

$\Rightarrow = \lim\limits_{h \to 0} 3 \cos\left(3x + \frac{3h}{2}\right) \underbrace{\lim\limits_{h \to 0} \dfrac{\sin\left(\frac{3h}{2}\right)}{\frac{3h}{2}}}_{1}$

$= \cancel{\lim} \ \underline{\underline{3 \cos(3x)}}$.

Example: $f(x) = x^3$.

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{(x+h)^3 - x^3}{h}$

$= \lim\limits_{h \to 0} \dfrac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \lim\limits_{h \to 0} \dfrac{3x^2h + 3xh^2 + h^3}{h} = \lim\limits_{h \to 0} 3x^2 + 3xh + h^2$

$= \underline{\underline{3x^2}}$

Exercises :

Differentiat the following functions using first principles.

1. $y=7x^2$
2. $y=\sin 2x$
3. $y=\cos 3x$
4. $y=\frac{1}{x^2}$
5. $y=x^2+x+1$
6. $y=\tan x$
7. $y=\sin^2 x$ (Hint: express $\sin^2 x$ in terms of $\cos 2x$).
8. Show that if $y=\frac{1}{\sqrt{x}}$ then $\frac{dy}{dx}=\frac{-1}{2x^{3/2}}$.

DIFFERENTIATION TECHNIQUES

SOME TECHNIQUES OF DIFFERENTIATION:

In the previous lesson we have shown how to calculate the derivative of a function by evaluating the limiting value of the rate of change of the function ($\frac{\Delta y}{\Delta x}$) as h approaches zero.

This method was quite easy for the simple functions considered. It could however, be hard if we happened to be dealing with a rather complicated function.

Differentiation of a Constant.

If $y = c$, a constant, whatever the value of x, then $f(x+h) = c$, and so dy is identically zero.

i.e $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = 0$

Differentiation of the sum or difference of functions

If $y(x) = u(x) + v(x)$, where u and v are both functions of x, then if x is increased by h to $x+h$, u and v will change to $u(x+h)$ and $v(x+h)$ respectively.

Thus; $y(x+h) = u(x+h) + v(x+h)$.

Thus
$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$
$$= \lim_{h \to 0} \frac{u(x+h) + v(x+h) - (u(x) + v(x))}{h}$$
$$= \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \to 0} \frac{v(x+h) - v(x)}{h}$$
$$= \frac{du}{dx} + \frac{dv}{dx}.$$

Thus, we have shown that the limits of the sum is the sum of the limits.

Same thing applies to difference

$y(x) = u(x) - v(x)$

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}.$$

Hence if $y = u \pm v$

$$\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

Examples:

1. Find the differential coefficient of $x^6 - 7x^3 - 6x + 4$.

Recall: The differential coefficient of ax^n is nax^{n-1}.

$$\Rightarrow \frac{d}{dx}(x^6 - 7x^3 - 6x + 4) = \frac{d}{dx}(x^6) - \frac{d}{dx}(7x^3) - \frac{d}{dx}(6x) + \frac{d}{dx}(4)$$

$$= 6x^5 - 21x^2 - 6.$$

2. Find the derivative of $\cos x - \sin x$

$$= \frac{d}{dx}(\cos x - \sin x) = \frac{d}{dx}(\cos x) - \frac{d}{dx}(\sin x)$$

$$= -\sin x - \cos x.$$

3. If $y = x^2 + 3x$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 3x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(3x)$$

$$= 2x + 3$$

4. $y = x^3 + 5x^2 - 4x + 2$

$$\frac{dy}{dx} = 3x^2 + 10x - 4 + 0 = 3x^2 + 10x - 4.$$

5. If $y = x^4 + 6x^3 - 4x^2 + 7x - 2$. find $f'(x)$ and value of $f'(x)$ at $x = 2$.

$$\frac{dy}{dx} = 4x^{4-1} + 6(3)x^{3-1} - 4(2)x^{2-1} + 7(1)x^{1-1} - 0.$$

$$= 4x^3 + 18x^2 - 8x + 7$$

$$f'(2) = 4(2)^3 + 18(2)^2 - 8(2) + 7 = \underline{\underline{95}}$$

Exercise 5.

Find the derivative of the following.

(1) $f(x) = 5x^2 + 2$

(2) $F(x) = 6x^2 - 1$

(3) $y = 4x^3$

(4) $y = 6x^3 + 4x^2 - 7x + 2$

(5) $y = 15x^3 - 6x^2 + 10$, Evaluate $f'(3), f'(-2)$.

DIFFERENTIATION OF A PRODUCT.

If $y(x) = u(x)v(x)$, where $u(x)$ and $v(x)$ are both functions of x, then If x given the increment dx, u, v and y will have the increment du, dv, dy respectively.

Now;
$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$

$$= \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

$$= \lim_{h \to 0} \frac{(u(x) + du)(v(x) + dv) - u(x)v(x)}{h}.$$

Note that
$u(x+h) = u(x) + du$
$v(x+h) = v(x) + dv$

$$= \lim_{h \to 0} \frac{u(x)v(x) + u(x)dv + v(x)du + du\,dv - u(x)v(x)}{h}$$

$$= \lim_{h \to 0} \frac{u(x)dv + v(x)du + du\,dv}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{u(x)\,dv}{h} + \lim_{h\to 0} \frac{v(x)\,du}{h} + \lim_{h\to 0} \frac{du\,dv}{h}.$$

$$= \lim_{h\to 0} u(x)\left(\frac{v(x+h)-v(x)}{h}\right) + \lim_{h\to 0} v(x)\left(\frac{u(x+h)-u(x)}{h}\right) + \lim_{h\to 0}\frac{du\,dv}{h}$$

$$= u(x)v'(x) + v(x)u'(x) + 0.$$

$$= u(x)v'(x) + v(x)u'(x) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Note that. $\lim_{h\to 0} \frac{du\,dv}{h} = \lim_{h\to 0} \left(\frac{u(x+h)-u(x)}{h}\right)\left(v(x+h)-v(x)\right)$

$= u'(x)\cdot 0$ as $v(x+h) \to v(x)$ as $h \to 0$.

$= 0.$

Remark: (1) If u is a constant, i.e $y(x) = c\,v(x)$.

then, $\frac{dy}{dx} = c\frac{dv}{dx}$ $\left[\text{Since } \frac{du}{dx} = 0 \text{ when } u \text{ is constant}\right]$.

(2) If $a_1, a_2, a_3, \dots, a_n$ are constants and $u_1, u_2, \dots, u_n$ are functions of x and

$y = a_1u_1 + a_2u_2 + \dots + a_nu_n$; Then,

$$\frac{dy}{dx} = a_1\frac{du_1}{dx} + a_2\frac{du_2}{dx} + \dots + a_n\frac{du_n}{dx}.$$

(3). If $y = u(x)v(x)w(x)$ we may consider y as the product of two functions (uv) and (w). Thus, we have

$$\frac{dy}{dx} = uv\frac{dw}{dx} + w\frac{d(uv)}{dx}$$

$$= uv\frac{dw}{dx} + w\left(u\frac{dv}{dx} + v\frac{du}{dx}\right)$$

$$= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}.$$

Example.

Find the differential coefficient (derivate) of the following functions.

1) $y = 6\sin x$

(ii) $y = 8\cos x + 3\sin x$.

Solution.

1) $y = 6\sin x$ $\quad \frac{dy}{dx} = \frac{d}{dx}(6\sin x) = 6\frac{d}{dx}(\sin x)$

$= \underline{6\cos x}$.

2) $y = 8\cos x + 3\sin x$ $\quad \frac{dy}{dx} = \frac{d}{dx}(8\cos x + 3\sin x)$

$= 8\frac{d}{dx}(\cos x) + 3\frac{d}{dx}(\sin x)$.

$= \underline{-8\sin x + 3\cos x}$.

Example:

Find $f'(x)$ if (i) $y = x^6\cos x$ (ii) $y = \sin^2 x$.

Solution:

1. $y = x^6\cos x$, then $\frac{dy}{dx} = x^6\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x^6)$.

$= -x^6\sin x + 6x^5\cos x$

$= \underline{x^5(6\cos x - x\sin x)}$.

(2) $y = \sin^2 x = \sin x \sin x$

$\Rightarrow \frac{dy}{dx} = \sin x\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(\sin x)$

$= \sin x\cos x + \sin x\cos x$.

$= 2\sin x\cos x$.

Example: Find $\frac{dy}{dx}$ if $y = 6x^2\sin x\cos x$.

Solution. $\frac{dy}{dx} = 6x^2\sin x\frac{d}{dx}(\cos x) + \cos x\, 6x^2\frac{d}{dx}(\sin x)$

$+ \sin x\cos x\frac{d}{dx}(6x^2)$.

$$\frac{dy}{dx} = -6x^2\sin^2 x + 6x^2\cos^2 x + 12x\sin x\cos x$$
$$= 6x^2(\cos^2 x - \sin^2 x) + 12x\sin x\cos x$$
$$= 6x^2(\cos 2x) + 6x\sin 2x$$
$$= 6x(\sin 2x + x\cos 2x)$$

Exercise :

Differentiate the following functions with respect to x.

1. $y = (x^3+1)(x^4+1)$
2. $y = x(3x + 4\cos x)$
3. $y = 8x^2(1+\sin x)(1+\cos x)$
4. $y = (x^2+1)^2$
5. $y = 3x\sin x\cos x$
6. $y = (x^2+1)^2(2x+1)$
7. $y = x\cos x + 3(x+1)(x-1)$

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DIFFERENTIATION OF A QUOTIENT.

If $y = \frac{u(x)}{v(x)}$ where $u(x)$ and $v(x)$ are functions of x, then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Remark: In case where $u = 1$, i.e $y = \frac{1}{v}$ is the reciprocal of the function v; $\frac{du}{dx} = 0$. Thus

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{v}\right) = \frac{v(0) - 1\cdot\frac{dv}{dx}}{v^2} = \frac{-\frac{dv}{dx}}{v^2}$$

Example: Find the derivative of

(i) $y = \frac{x}{x+1}$ (ii) $y = \frac{\sin x}{x^2 + \cos x}$ (iii) $y = \frac{1}{x^2+4}$

Solution.

1. $y = \frac{x}{x+1}$ $\Rightarrow$ $\frac{dy}{dx} = \frac{(x+1)\frac{d}{dx}(x) - x\frac{d}{dx}(x+1)}{(x+1)^2}$

$$= \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$$

2. $y = \frac{\sin x}{x^2 + \cos x}$ $\Rightarrow$ $\frac{dy}{dx} = \frac{(x^2+\cos x)\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(x^2+\cos x)}{(x^2+\cos x)^2}$

$$= \frac{(x^2+\cos x)\cos x - \sin x\,(2x - \sin x)}{(x^2+\cos x)^2}$$

$$= \frac{x^2\cos x + \cos^2 x - 2x\sin x + \sin^2 x}{(x^2+\cos x)^2}$$

$$= \frac{1 + x(x\cos x - 2\sin x)}{(x^2+\cos x)^2}$$

Note: $\sin^2 x + \cos^2 x = 1$

(iii) $y = \frac{1}{x^2+4} \Rightarrow \frac{dy}{dx} = -\frac{\frac{dv}{dx}}{v^2} = -\frac{\frac{d}{dx}(x^2+4)}{(x^2+4)^2}$

Since $u(x) = 1$ (constant)

$= -\frac{2x}{(x^2+4)^2}$.

Example:

Find $\frac{du}{d\theta}$ if (i) $u = \frac{\theta\cos\theta}{\theta+3}$. (ii) $u = \frac{\theta\cos\theta}{(\theta+1)\sin\theta}$.

Solution.

1. $u = \frac{\theta\cos\theta}{\theta+3}$. $\Rightarrow \frac{du}{d\theta} = \frac{(\theta+3)\frac{d}{d\theta}(\theta\cos\theta) \cancel{+} \theta\cos\theta\frac{d}{d\theta}(\theta+3)}{(\theta+3)^2}$

$$= \frac{(\theta+3)(\cos\theta - \theta\sin\theta) - \theta\cos\theta\,(1)}{(\theta+3)^2}$$

$$= \frac{\theta\cos\theta + \theta^2\sin\theta + 3\cos\theta - 3\theta\sin\theta - \theta\cos\theta}{(\theta+3)^2}$$

$$= \frac{3\cos\theta - (\theta+3)\theta\sin\theta}{(\theta+3)^2}.$$

(ii) $u = \frac{\theta\cos\theta}{(\theta+1)\sin\theta} \Rightarrow \frac{du}{d\theta} = \frac{(\theta+1)\sin\theta\frac{d}{d\theta}(\theta\cos\theta) - \theta\cos\theta\frac{d}{d\theta}((\theta+1)\sin\theta)}{[(\theta+1)\sin\theta]^2}$

$$= \frac{(\theta+1)\sin\theta(\cos\theta - \theta\sin\theta) - \theta\cos\theta[(\theta+1)\cos\theta + \sin\theta]}{(\theta+1)^2\sin^2\theta}.$$

$$= \frac{(\theta+1)\sin\theta\cos\theta - \theta(\theta+1)\sin^2\theta - \theta(\theta+1)\cos^2\theta - \theta\sin\theta\cos\theta}{(\theta+1)^2\sin^2\theta}$$

$$= \frac{\sin\theta\cos\theta - \theta(\theta+1)}{(\theta+1)^2\sin^2\theta}$$

7. Find the gradient of the tangent line to the curve $y = \frac{\sin\theta}{\cos\theta + \sin\theta}$ at the point where $\theta = \frac{\pi}{3}$.

DIFFERENTIATION OF OTHER TRIGONOMETRIC FUNCTIONS.

The result of preceding section enable us to obtain the derivatives of the remaining four basic trigonometric functions: $\tan x$, $\cot x$, $\sec x$. and $\operatorname{cosec} x$.

1. $y = \tan x = \frac{\sin x}{\cos x}$

$$\frac{dy}{dx} = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos^2 x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

2. $y = \cot x = \frac{\cos x}{\sin x}$

$$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

Note: $\div\theta(\theta+1)[\sin^2\theta+\cos^2\theta] = -\theta(\theta+1)$.

Also, $\sin\theta\cos\theta\,[(\theta+1)-\theta] = \sin\theta\cos\theta$.

Example: Find the gradient of the tangent line to the curve $y = \frac{x^2}{x^2+1}$ at the point with abscissa 1 (ie $x=1$).

Solution.

$$y = \frac{x^2}{x^2+1} \Rightarrow \frac{dy}{dx} = \frac{(x^2+1)\frac{d}{dx}(x^2) - (x^2)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}.$$

Hence, $f'(1) = \frac{2(1)}{(1^2+1)^2} = \frac{2}{2^2} = \underline{\underline{\frac{1}{2}}}$.

The gradient of the tangent line is $1/2$

Exercise:

Differentiate with respect to x.

1. $y = \frac{x}{x+\sin x}$

2. $y = \frac{x+\sin x}{1+\cos x}$

3. $y = \frac{x^3+3x}{(x+1)(x-2)}$

4. $y = \frac{x^2\sin x}{(x+1)(x^2-1)}$

5. $y = \frac{3}{(x+1)^2}$.

6. Find the gradient of the tangent line to the curve $w = \frac{z^3}{z^2+1}$ at the point with abscissae 3.

3) $y = \sec x = \dfrac{1}{\cos x}$

$$\frac{dy}{dx} = -\frac{\frac{d}{dx}(\cos x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \underline{\sec x \tan x}.$$

4). $y = \operatorname{cosec} x = \dfrac{1}{\sin x}$.

$$\frac{dy}{dx} = -\frac{\frac{d}{dx}(\sin x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$$= -\cot x \operatorname{cosec} x$$

$$= \underline{-\operatorname{cosec} x \cot x}.$$

DIFFERENTIATION OF A FUNCTION OF A FUNCTION (CHAIN RULE).

The function $y = (2x+1)^3$ is a function of $2x+1$ which in turn is a function of x. Also, consider the function $\sin x^2$, y is the sine of the function x^2.
That is, y is a function of some quantity which in turn is a function of x.

Thus; $y = F(v)$ where $v = f(x)$.

If $y = (2x+1)^3$, then $y = v^3$ where $v = 2x+1$

If $y = \sin x^2$, then $y = \sin v$ where $v = x^2$.

The general rule for differentiating a function of a function is

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{dx}.$$

For example, let apply the rule to the examples above

1. $y = (2x+1)^3 = v^3$ where $v = 2x+1$.

$$\frac{dy}{dv} = 3v^2, \quad \frac{dv}{dx} = 2.$$

Then, $$\frac{dy}{dx} = 3v^2 \times 2 = \underline{6(2x+1)^2}.$$

2. $y = \sin x^2 = \sin v$ where $v = x^2$.

Then, $$\frac{dy}{dv} = \cos v, \quad \frac{dv}{dx} = 2x$$

Hence, $$\frac{dy}{dx} = \cos v \times 2x = \underline{2x \cos x^2}.$$