FUNCTION

A function is a relationship between two variables such that, to each value of the independent variable, there is exactly one corresponding value of the dependent variable.

For example. $A = \pi l^2$. A is a function of l.
$A = \pi r^2 h$; A is a function of r and h.

Unless we are dealing with a particular example we shall generally use the symbols y to represent the dependent variable and x to denote the independent variable.

The statement y is a function of x is expressed mathematically as

$$y = f(x).$$

The "f" is used to indicate dependence on the bracketed quantity.

Deciding whether relation are functions

Which of the equation below defines y as a function of x.

Note: To have a functional relationship every value of x should have exactly one value of y that ~~produces it~~ it is assigned to.

In otherwords, every value of x should be mapped to exactly one value of y.

(a) $x + y = 1$ (b) $x^2 + y^2 = 1$ (c) $x^2 + y = 1$, (d) $x + y^2 = 1$

Solution:

To decide if an equation defines a function, it is helpful to isolate the dependent variable on the left.

(a) $y = 1 - x$ Yes, each value of x gives exactly one value of y.

(b) $x^2 + y^2 = 1$, $\Rightarrow y = \pm\sqrt{1-x^2}$ No, some values of x give two values of y.

(c) $x^2 + y = 1$, $y = 1 - x^2$ Yes.

(d) $x + y^2 = 1$, $y = \pm\sqrt{1-x}$ No.

Example: If $f(x) = x^2 - 3x$ evaluate $f(2), f(3), f(-5), f(a)$

Solution

$f(2) = 2^2 - 3(2) = -2$

$f(3) = 3^2 - 3(3) = 0$

$f(-5) = (-5)^2 - 3(-5) = 40$

$f(a) = a^2 - 3a$.

A function $y = f(x)$ is said to be defined for a certain value a, of x $(x = a)$, if a definite value of $y = f(a)$ exist within the range of f.

Example: For what values of x are the following functions defined? Take the range of f to be the real line.

(i) $f(x) = 2x - 5$ (ii) $f(x) = \dfrac{1}{x-2}$ (iii) $f(x) = +\sqrt{4 - x^2}$.

Solution:

(i) $y = f(x) = 2x - 5$ is defined for all values of $x \in \mathbb{R}$.
$D = \mathbb{R}$.

(ii) $f(x) = \dfrac{1}{x-2}$. is defined for every value of $x \in \mathbb{R}$ except at $x = 2$

Hence the domain of f is $\mathbb{R} \setminus \{2\}$ or

$D = (-\infty, 2) \cup (2, \infty)$.

(iii) $f(x) = +\sqrt{4-x^2}$: is only defined for values of x that satisfy $4 - x^2 \geq 0$

$$(2-x)(2+x) \geq 0$$

$$\Rightarrow (2-x) \geq 0 \ \& \ (2+x) \geq 0$$

or

$$(2+x) \leq 0 \ \& \ (2-x) \leq 0$$

If
$$\left.\begin{aligned} 2-x \geq 0 &\Rightarrow x \leq 2 \\ 2+x \geq 0 &\Rightarrow x \geq -2 \end{aligned}\right\} \boxed{-2 \leq x \leq 2}.$$

If
$$\left.\begin{aligned} 2+x \leq 0 &\Rightarrow x \leq -2 \\ 2-x \leq 0 &\Rightarrow x \geq 2 \end{aligned}\right\} \text{Impossible}$$

Hence. $D = [-2, 2]$

Explicit Function:

If the functional relationship between y and x is expressed by a formula giving y in terms of x, we say that y is an explicit function of x.

Example, $y = x^2 - 3x$, $y = 2x - 5$, $y = \dfrac{1}{x-5}$ are all

$y = \sqrt[3]{x^3 - 3x + 1}$ cases where y is an explicit function of x.

Implicit Function:

If the relationship between the quantities y and x is expressed by means of an equation of the types for example. $3y + 4x - 5 = 0$ or

$x^3 + y^3 = 27$, it is called implicit function.

It is infact possible to express y as an explicit function in both cases i.e.,

$y = \frac{1}{3}(5 - 4x)$ and $y = \sqrt[3]{27 - x^3}$

This will not always be the case.

Exercise

1. $\phi(x) = x^2 - 5x + 6$. Evaluate $\phi(0)$, $\phi(1)$. For what values of x is $\phi(x) = 0$?

2. $F(\theta) = \cos\theta - \sin\theta$. Evaluate $F(0)$, $F(\frac{\pi}{2})$. For what values of θ is $F(\theta) = 0$?

3. For what values of x is the function $\phi(x) = \dfrac{2x}{(x-1)(x-2)}$ defined?

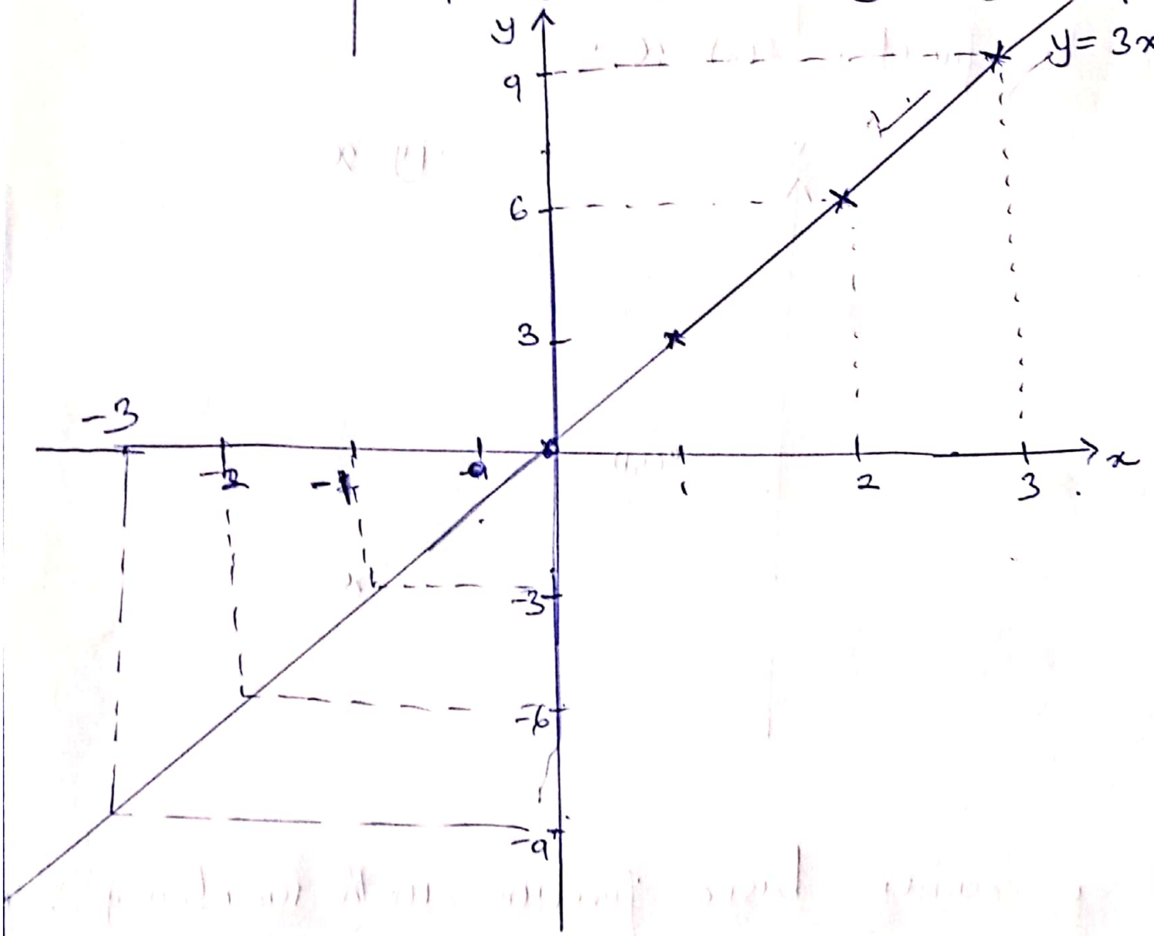
4. Define y as an explicit function of x (if possible) when (i) $xy + 4y = x^3$ (ii) $x + y + y^2 = x^2$
   (iii) $x^5 + y^5 + xy = 3$

5. If y is defined as an implicit function of x by the relations (i) $xy + y^2 = 2$
   (ii) $xy^2 + y^4 + 1 = 0$,
   evalute y when $x = 1$ and when $x = 2$ (if possible) in each case.

Graphs of Functions

The graph of a function is the graph of its ordered pairs. For example, the graph of $f(x) = 3x$ is the set of points (x, y) in the rectangular coordinate system satisfying $y = 3x$. That is.

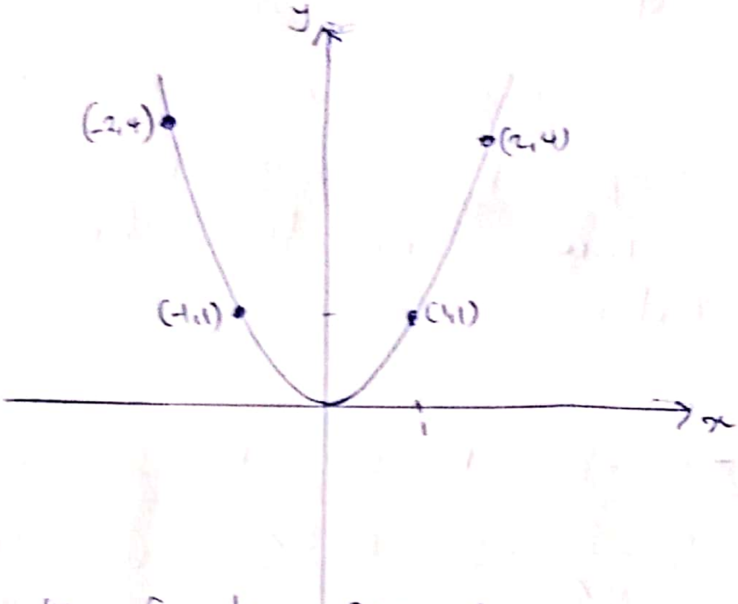
x	-3	-2	-1	0	1	2	3
y	-9	-6	-3	0	3	6	9



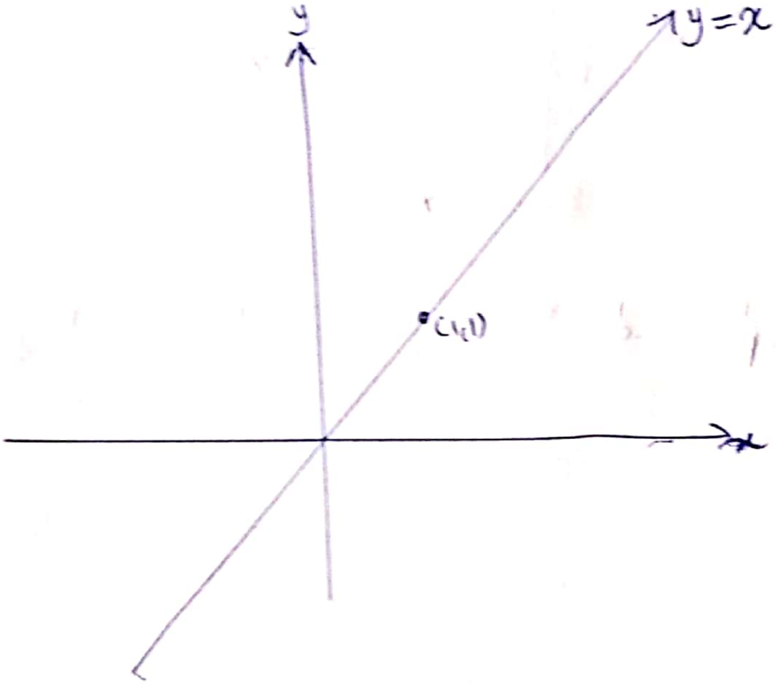
Example: Graph the function $f(x) = x^2$.

Solution: Make a table of (x, y) pairs that satisfy $y = x^2$.

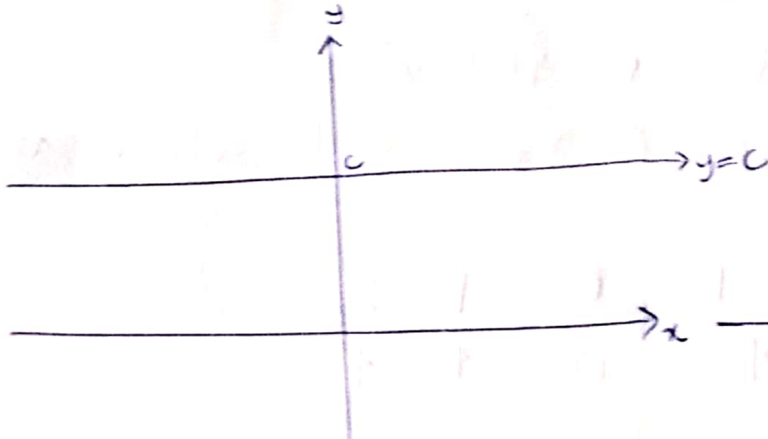
x	-2	-1	0	1	2
y	4	1	0	1	4



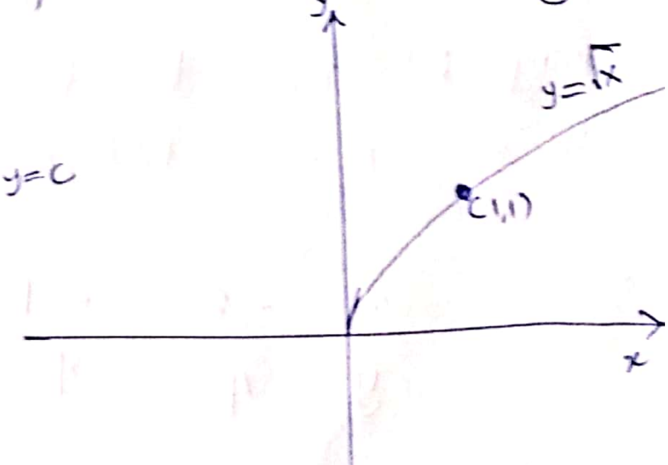
Example: Graph the function $f(x) = x$.



Some graphs of common basic functions worth remembering.



Graph of a constant function $f(x) = c$



Graph of $f(x) = \sqrt{x}$.

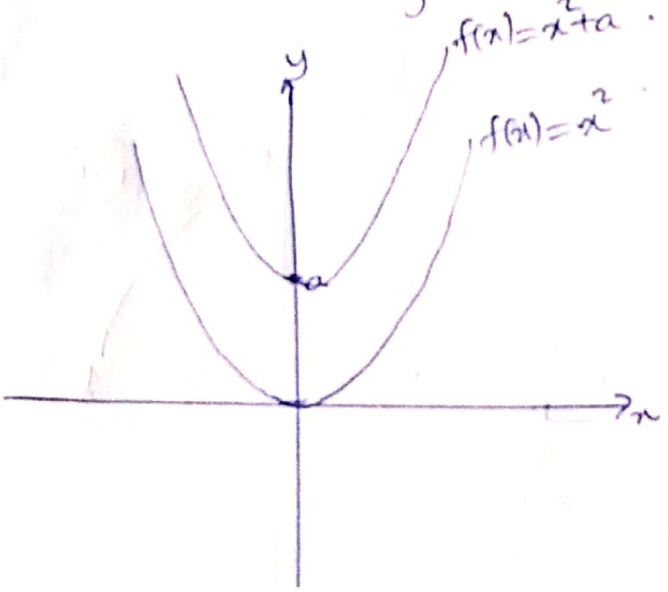
Shifting of Graphs of functions.

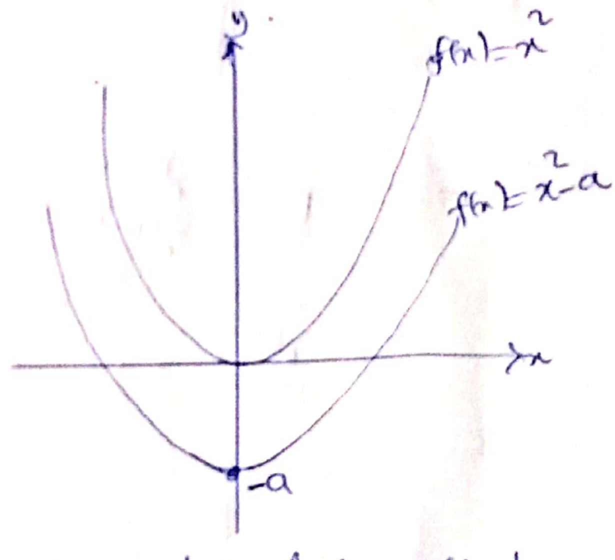
Sketch the graphs of $f(x) = x^2 + a$, $f(x) = (x-a)^2$, $f(x) = (x-a)^2 + b$, $f(x) = x^2 - a$, $f(x) = (x+a)^2$, $f(x) = -x^2$, $f(x) = -x^2 + a$, $f(x) = -(x-a)^2$.

Solution.

$f(x) = x^2 + a$ (a) When adding a constant term to a basic function like x^2, we shift the function in positive y-direction by a-units. (shift upwards).

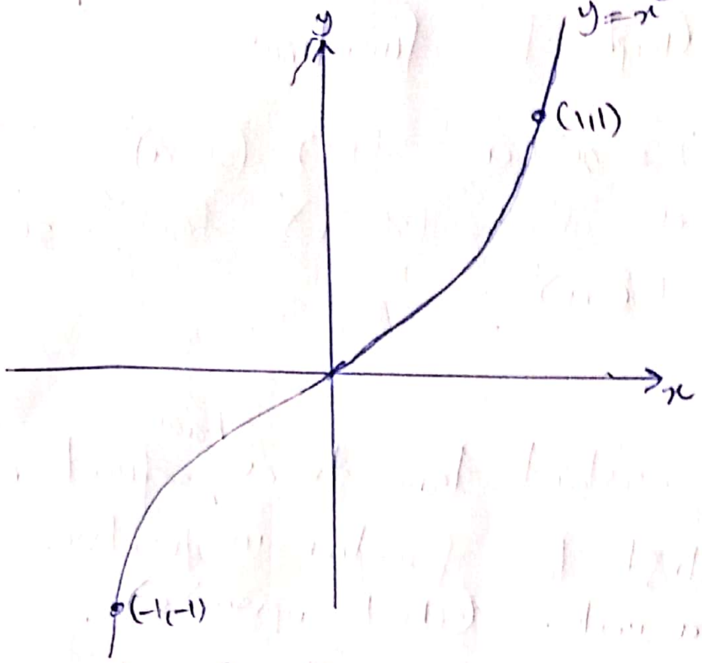
$f(x) = x^2 - a$ (b) We shift the basic function downwards (in negative y-direction) by a-units.



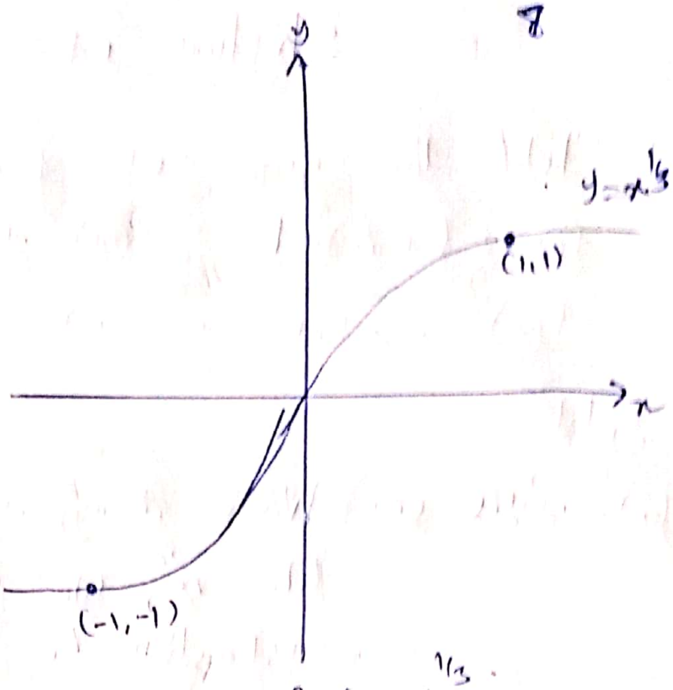


$f(x) = (x+a)^2$ (c) when the basic function is modified by subtracting a from x, we shift the function to the right by a-units. (in +ve x-direction) from the centre of the function.

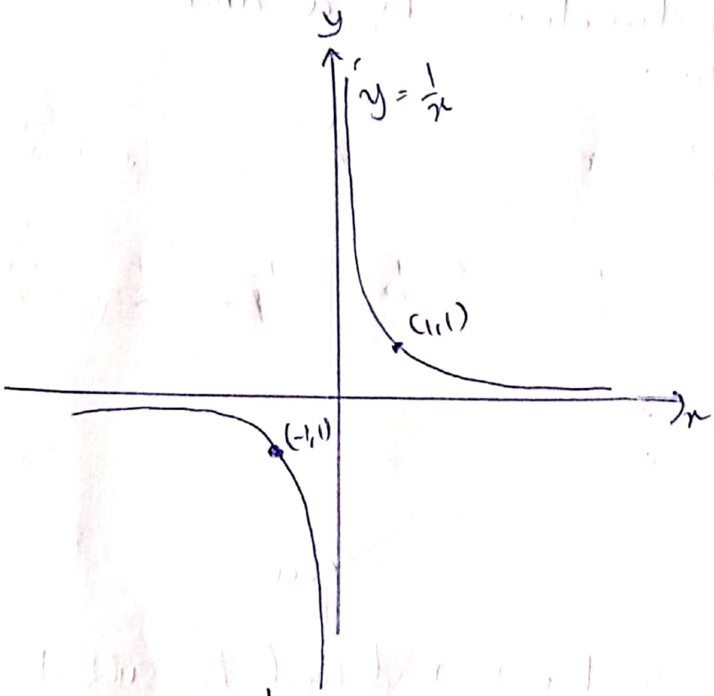
(d) Similarly, we shift the function to the left if the basic function is modified by adding a constant term to x.



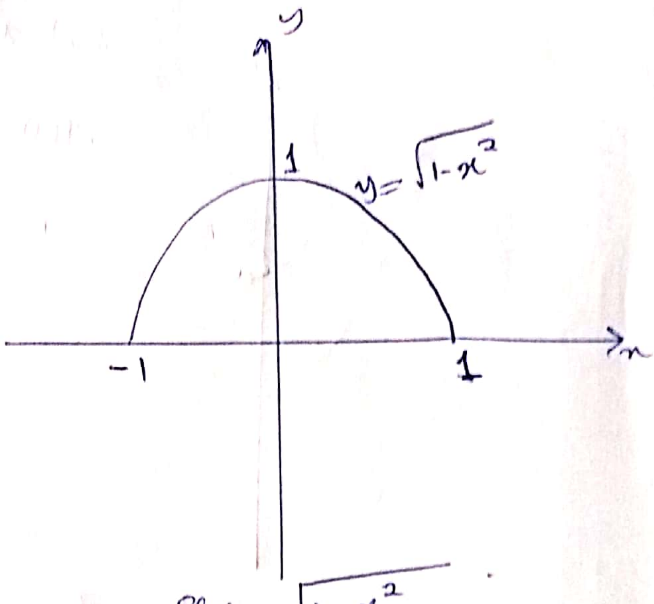
$f(x)=x^3$.



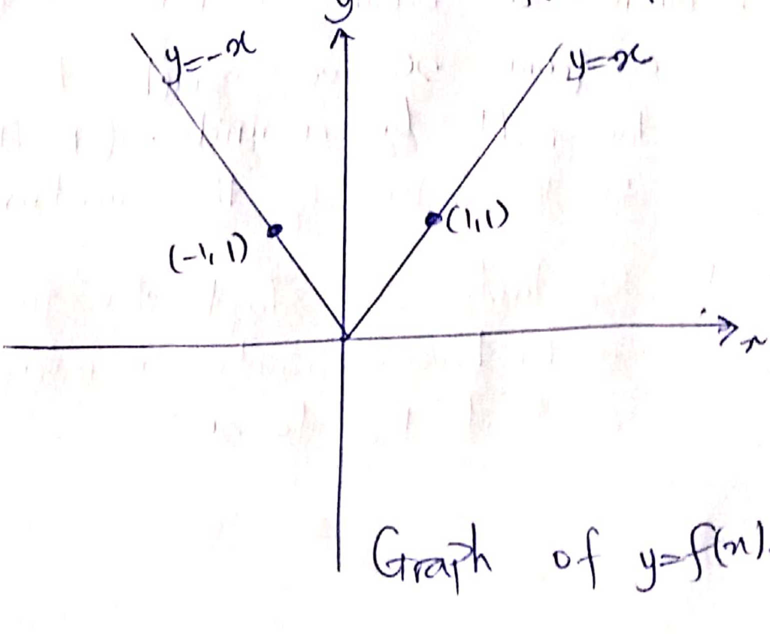
$f(x)=x^{1/3}$.



$f(x)=\frac{1}{x}$

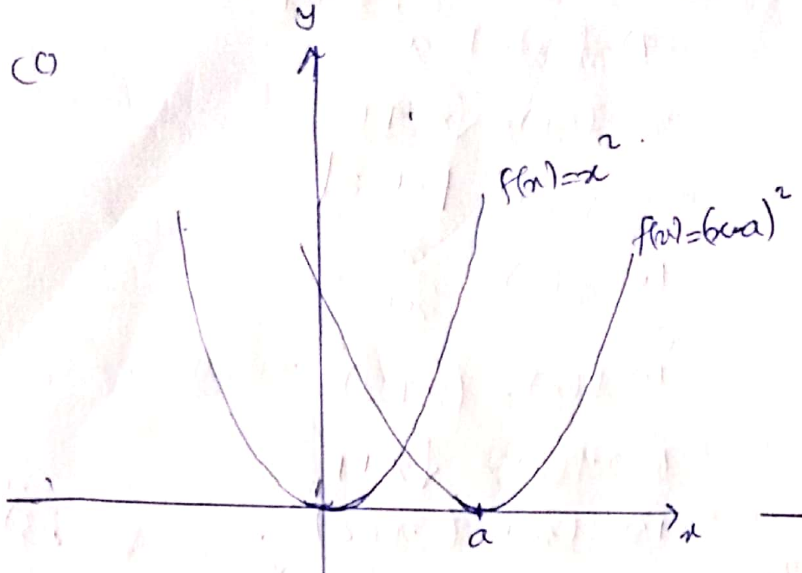


$f(x)=\sqrt{1-x^2}$

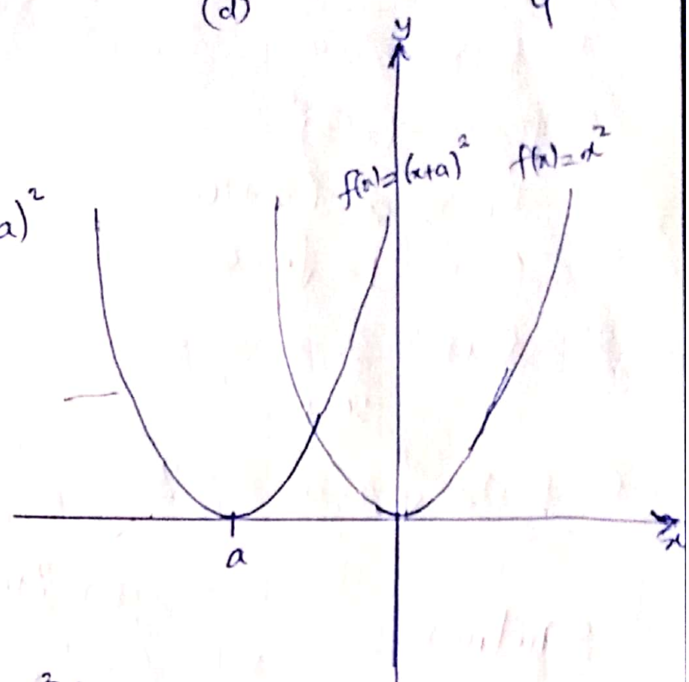


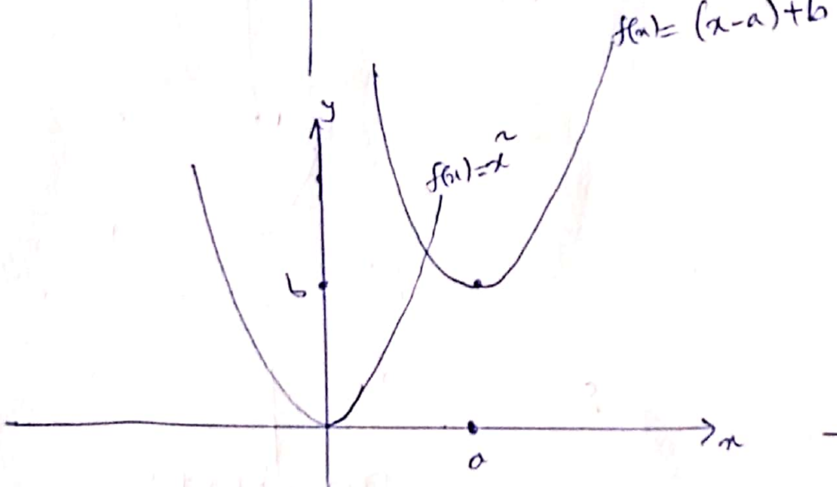
Graph of $y=f(x)=|x|$

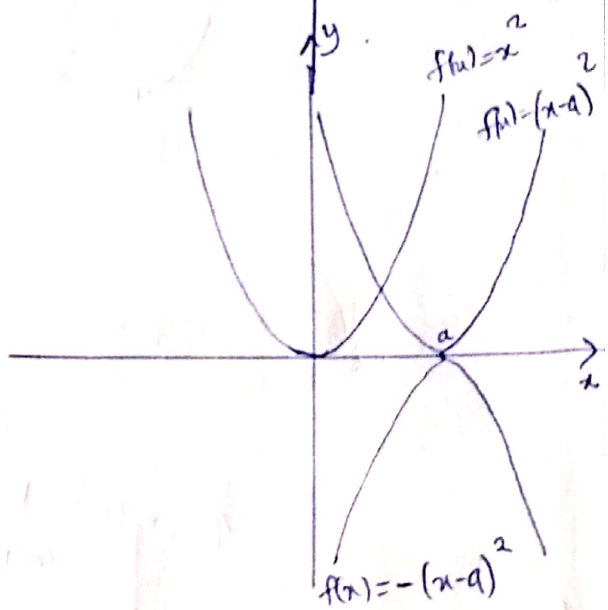
(c)



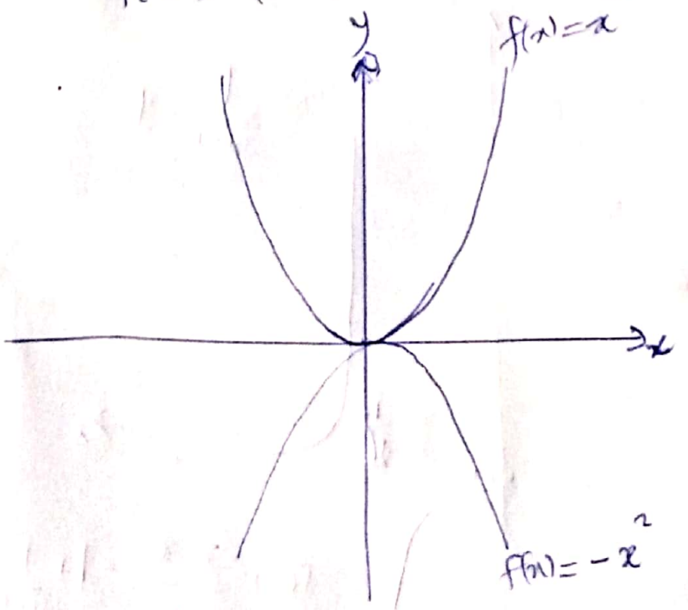
(d)







$f(x) = (x-a)^2 + b$.



When negative values is added to front of the basic function, we reflect the function across x-axis from the centre of the function.

Example: Sketch the graphs of the following functions.

(1) $f(x) = x^2+4$

(2) $f(x) = (x-3)^2$

(3) $f(x) = x^2-6x+10$

(4) $f(x) = 1+\sqrt{x-4}$

(5) $f(x) = \dfrac{2-x}{x-1}$

(6) $f(x) = \sqrt{2-x} - 3$

(7) $f(x) = 1-x^3$

(8) $f(x) = \sqrt{x} + 1$

(9) $f(x) = |x-2|$

(10) $f(x) = \dfrac{x}{x+1}$

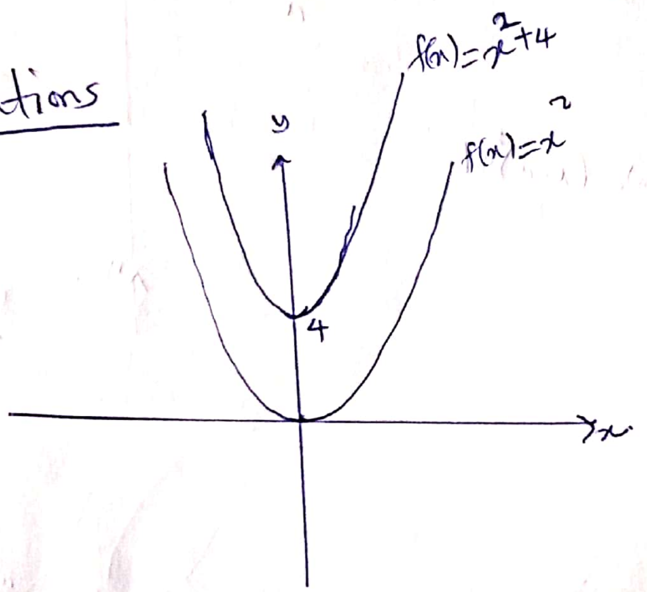
(11) $f(x) = (x+2)^3$

(12) $f(x) = \sqrt{x+1}$

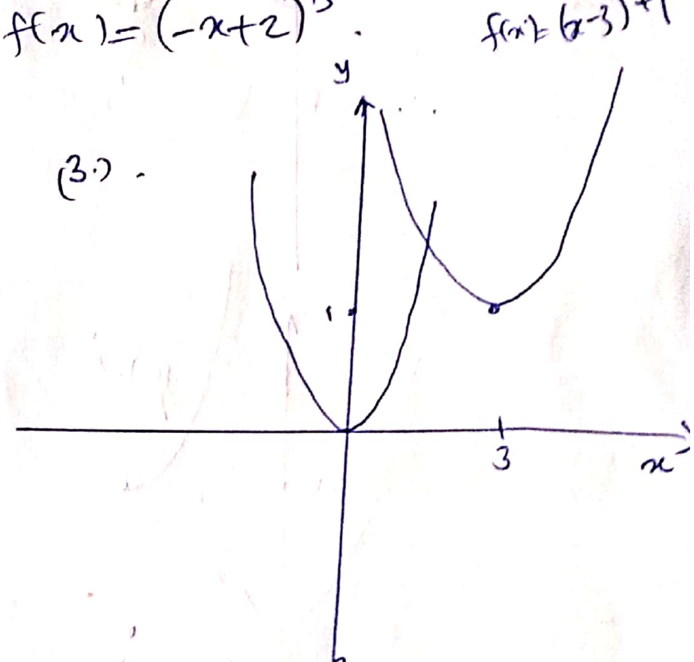
(13) $f(x) = (-x+2)^{1/3}$

Solutions

(1)

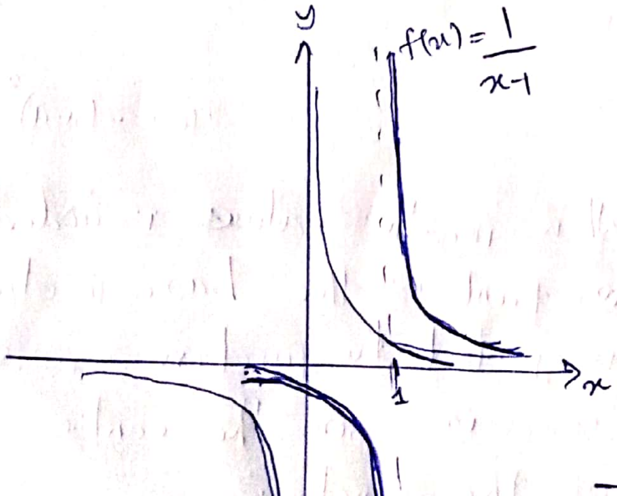


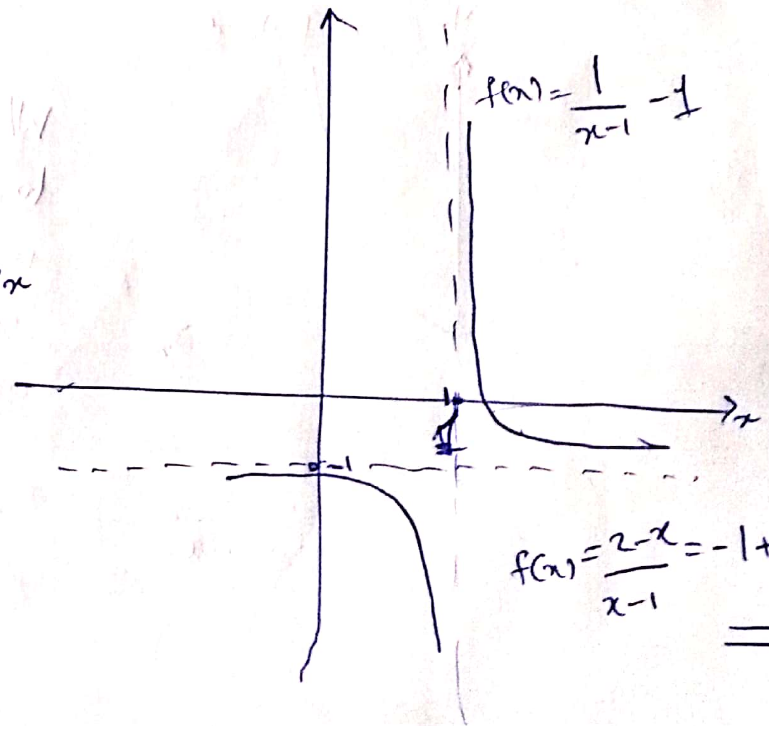
(3)



$f(x) = x^2-6x+10$
$= (x-3)^2+1$
(complete the square)

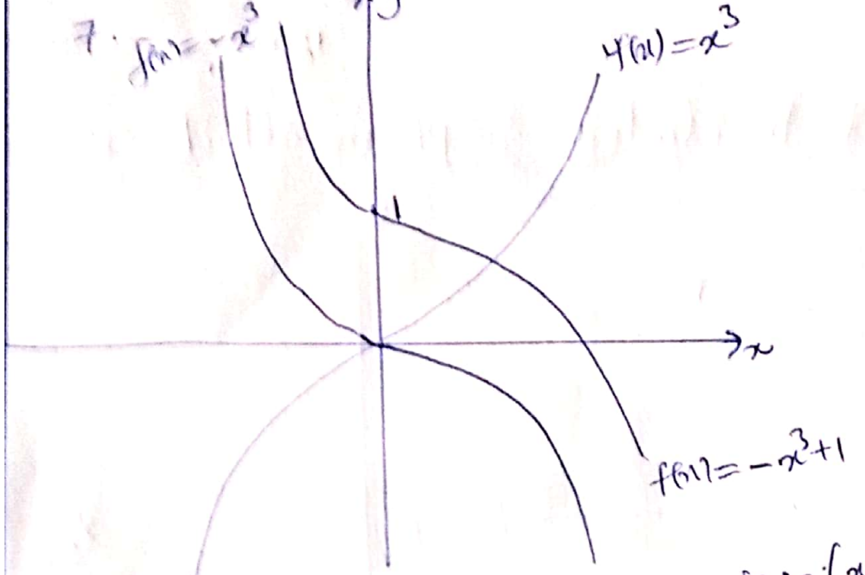
(5)



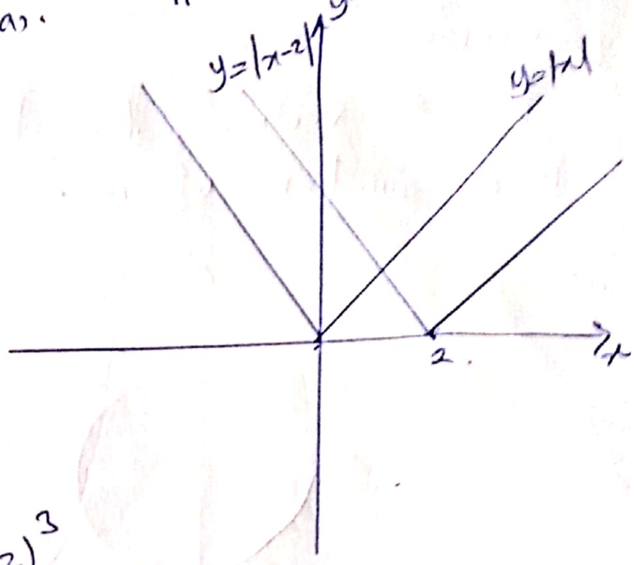


$f(x) = \dfrac{2-x}{x-1} = -1 + \dfrac{1}{x-1}$

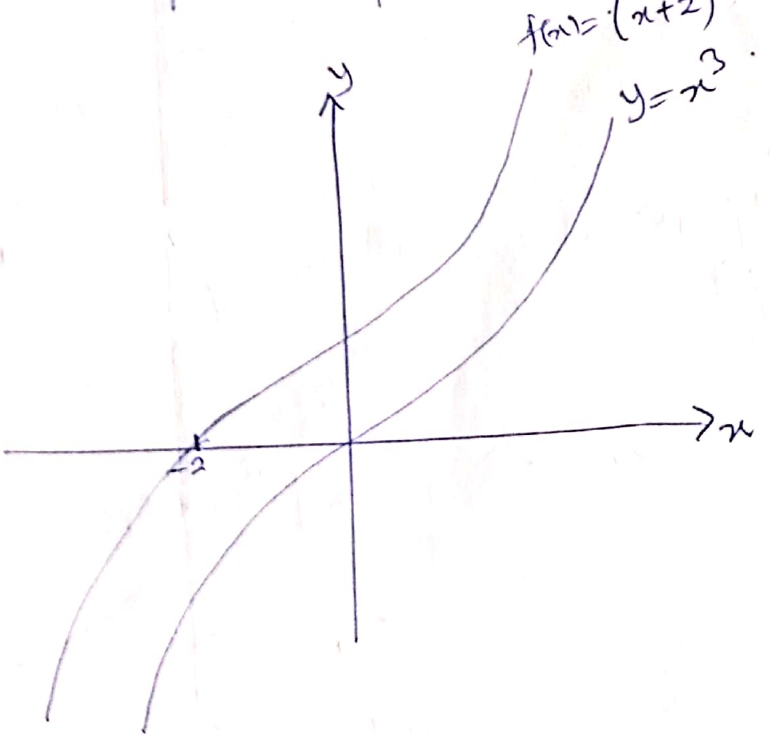
7.



(a). 11



11.



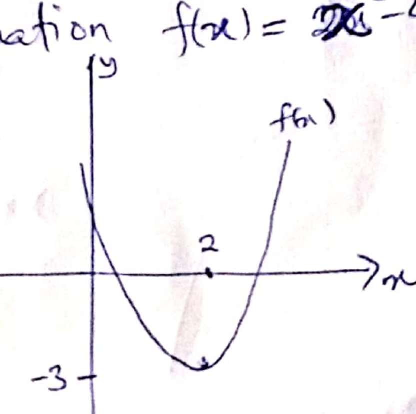
Vertical Line Test

This test states that if every vertical line intersects the graph of an equation at most once, then the equation defines y as a function of x.

Example: Check if the equation $f(x) = \cancel{2x}x^2-4x+1$ is a function.

Since

$f(x) = (x-2)^2 - 3$
$= x^2 - 4x + 1$

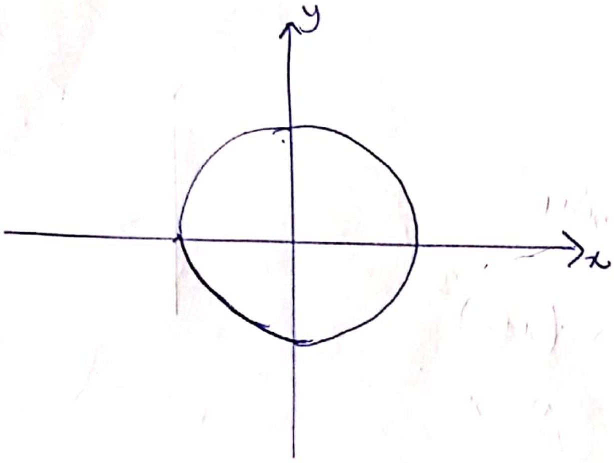


From the graph, it shows that $f(x)$ passes vertical line test, hence a function.

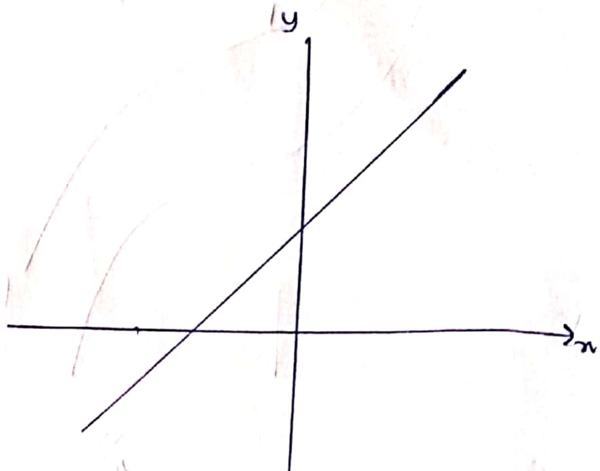
Exercise:

Use the vertical line test to identify graph in which y is a function of x.

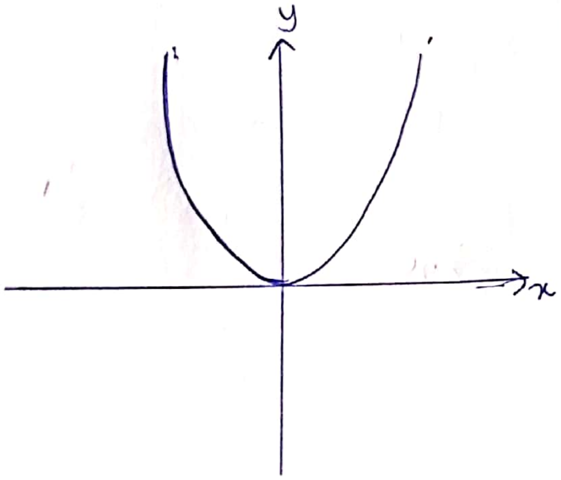
a).



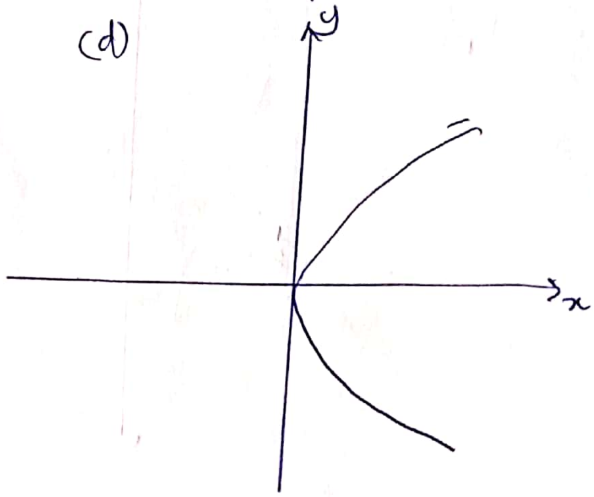
b.



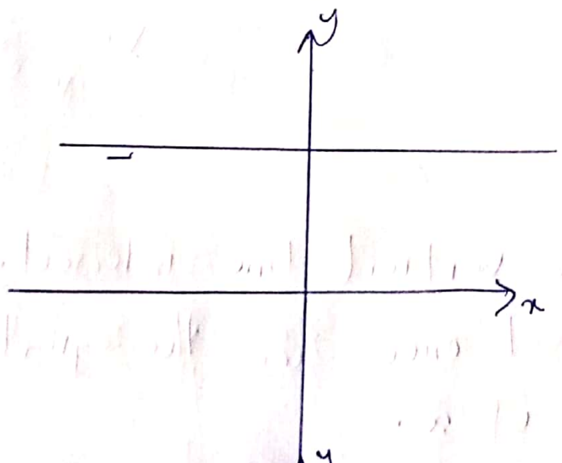
(c).



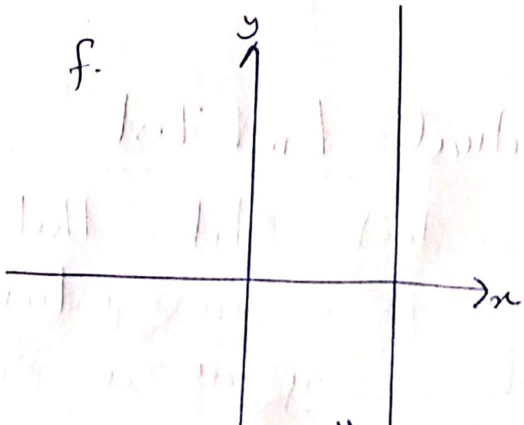
(d)



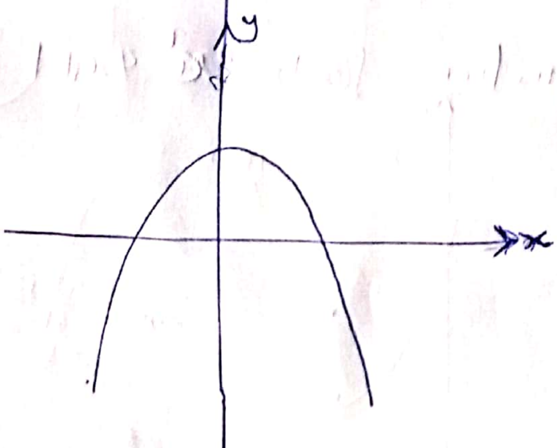
(e).



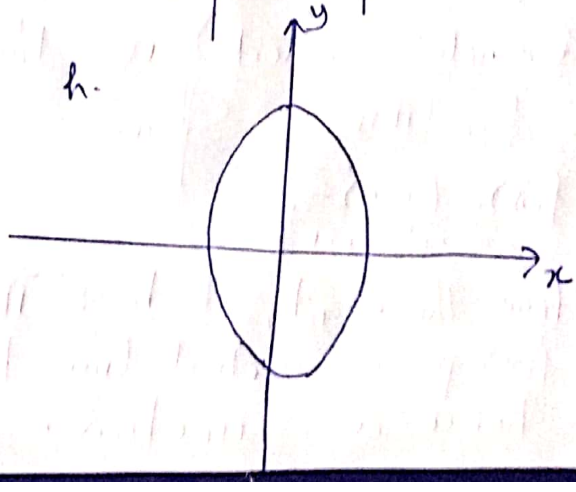
f.



g.



h.



THE RATE OF CHANGE OF A FUNCTION:

If y is a function of x, as x changes y will generally change. We relate change in y to the corresponding change in x by defining the average rate of change of the function to be the function divided by the corresponding change in x.

i.e $$\frac{y_2-y_1}{x_2-x_1} = \frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

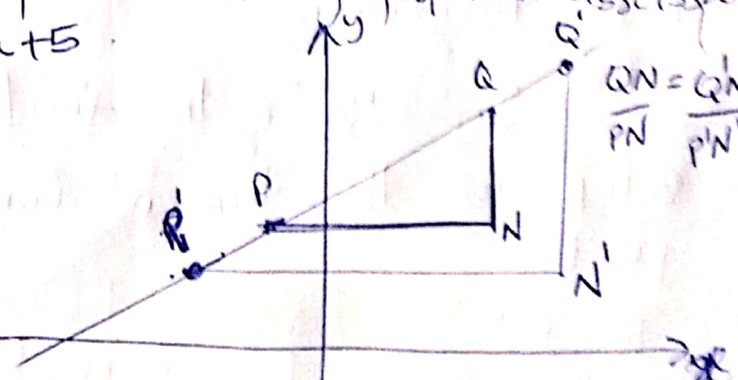
<u>Example:</u> Find an expression for the average rate of change of the functions (i) $y=2x+5$ (2) $y=x^2$ (3) $y=(3x-1)^2$ in the interval x_1 to x_2.

<u>Soln.</u>

(i) $y=2x+5$.
$$\frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$
$$= \frac{(2x_2+5)-(2x_1+5)}{x_2-x_1} = \frac{2(x_2-x_1)}{x_2-x_1} = \underline{\underline{2}}$$

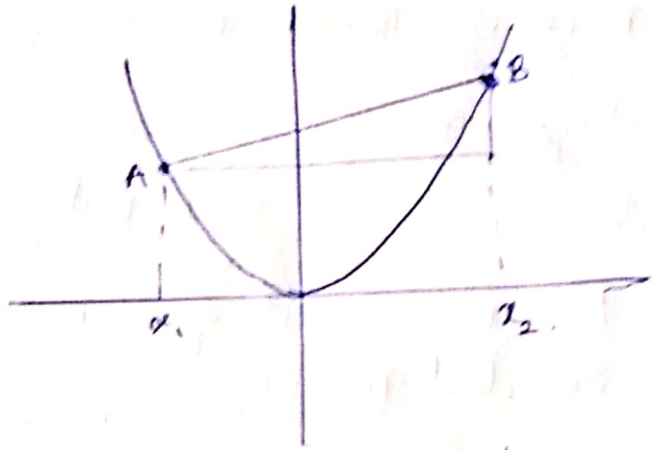
(ii) $y = x^2$.
$$\frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1} = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{(x_2^2)-(x_1^2)}{x_2-x_1}$$
$$= \frac{(x_2-x_1)(x_2+x_1)}{x_2-x_1} = x_1+x_2$$

If we represent the function graphically the average rate of change of the function in the interval x_1 to x_2 may be interpreted as the chord joining the points on the graph with abscissae x_1 and x_2. E.g $y=2x+5$.



$$\frac{QN}{PN} = \frac{Q'N'}{P'N'} = 2$$

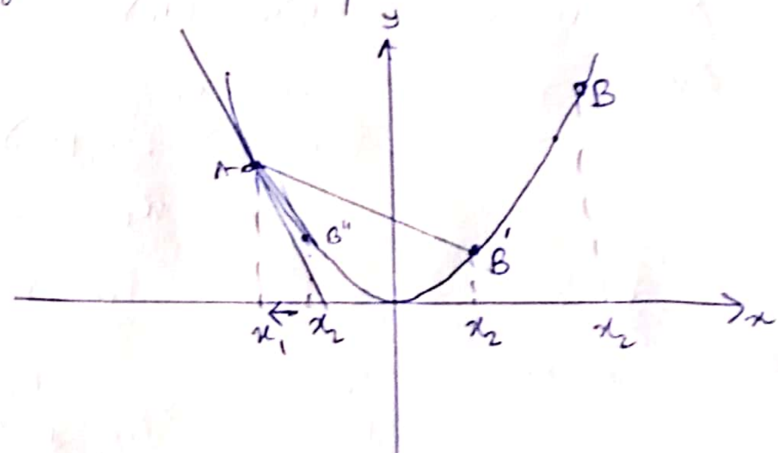
Ex. $y = x^2$



The rate of change of the function between x_1 & x_2 is given by $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{x_2^2 - x_1^2}{x_2 - x_1} = x_2 + x_1$

This refers to the slope of the ~~tangent~~ chord joining $|AB|$ the two points on the curve $y = x^2$.

Consider, that point x_2 moves closer to x_1. How does the chord change & its slope?



Notice that as $x_2 \to x_1$ (tends to x_1), the ~~t~~chord becomes the tangent line at x_2 to the curve.

Hence, the slope of the chord becomes the slope of the tangent.

$\Rightarrow$ Rate of change of the function at a point $\Rightarrow$ slope of the tangent line to the curve at that point.

Example: Find the gradient of the chord joining the points with abscissae 2 and x_2 on the curves (1) $y=\frac{1}{x}$ and (2) $y=\frac{3}{x^2}$. (3) $y=3x+1$

What is the gradient of the tangent at the point with $x=2$ on these curves?

Solution.

(1) $y=\frac{1}{x}$

$$\frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1} = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{\frac{1}{x_2}-\frac{1}{2}}{x_2-2}$$

$$= \frac{2-x_2}{2x_2(x_2-2)} = \frac{-1}{2x_2}.$$

As x_2 approaches the value 2, i.e

$x_2 \to 2 \Rightarrow \frac{\Delta y}{\Delta x} \to \frac{-1}{2(2)} = \frac{-1}{4}$ (which is the gradient/slope of the tangent as required).

(2) $$\frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{\frac{3}{x_2^2}-\frac{3}{2^2}}{x_2-2} = \frac{3(4-x_2^2)}{4x_2^2(x_2-2)}$$

$$= \frac{3(2-x_2)(2+x_2)}{4x_2^2(x_2-2)}$$

$$= \frac{-3(2+x_2)}{4x_2^2}$$

As $x_2 \to 2$ $\frac{\Delta y}{\Delta x} \to \frac{-3(2+2)}{4\times 2^2} = -\frac{3}{4}$ (gradient of the tangent at $x=2$).

(3) Exercise.

Exercises.

(1) Find the gradient of the chord joining the points with $x=3$ and x_2 on the curve $y=x^2+5x$. What is the gradient of the tangent to the curve at the point with $x=3$?

(2) Show that the gradient of the chord joining the points with abscissae x_1 and x_2 on the curve $y=\frac{1}{x}$ is $\frac{-1}{x_1 x_2}$

Deduce the gradient of the tangent at the point with abscissae (i) 1 (ii) x

LIMIT AND LIMIT NOTATION

The value approached by $\frac{y_2-y_1}{x_2-x_1}$ is called the limiting value or the limit as x_2 tends to x_1, we abbreviate this as;

$$\lim_{x_2\to x_1} \frac{y_2-y_1}{x_2-x_1} = \lim_{x_2\to x_1} \frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

Now, let's take an arbitrary point x.

Then, the ~~tangent~~ slope of the tangent line to the curve at x is given by

$$\lim_{x_2\to x} \frac{f(x_2)-f(x)}{x_2-x}.$$

Now, take $x_2-x=h \Rightarrow x_2=x+h$.

$\Rightarrow$ as $x_2\to x \Rightarrow h\to 0$.

$$\Rightarrow \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

This limiting value is called the derivative of the function with respect to x, given by

~~Hence~~ $f'(x)=\frac{dy}{dx}$.

This method of finding the derivative of a function is called the 1st principle.

Thus $\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Example: Find the derivative of the function $y = 3x^2$ and the gradient of the tangent to the curve $y = 3x^2$ at the point with $x = 3$.

Soln

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h} = \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{6xh + h^2}{h} = \lim_{h \to 0} 6x + h.$$

$$= \underline{\underline{6x}}.$$

Thus, the gradient of the tangent at the point with $x = 3$ is $f'(3) = 6 \times 3 = 18$.