Infinite Sum of Geometric Sequence.

When the common ratio is between -1 and 1; i.e $-1<q<1$, each successive term in the sequence gets closer to zero.

Ex: when $q=\frac{1}{2}$, $(b_n)=\left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right)$.

when $q=-\frac{1}{30}$ $(b_n)=\left(3, \frac{-1}{10}, \frac{1}{300}, -\frac{1}{9000}, \dots\right)$.

In both examples the terms get closer to zero as n increases.

<u>Theorem</u>: The infinite sum of a geometric sequence (b_n) with common ratio $|q|<1$ is denoted by S, and is given by the formula $S=\frac{b_1}{1-q}$.

Ex: Find $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$

Soln

$q=\frac{\frac{1}{2}}{1}=\frac{1}{2}$. then $S=\frac{b_1}{1-q}=\frac{1}{1-\frac{1}{2}}=\frac{1}{\frac{1}{2}}=\underline{\underline{2}}$

Ex. Find $100+50+25+\dots$

Soln

$q=\frac{50}{100}=\frac{1}{2} \Rightarrow S=\frac{b_1}{1-q}=\frac{100}{1-\frac{1}{2}}=\frac{100}{\frac{1}{2}}=\underline{\underline{200}}$

Ex: Find $-5+10-20+\dots$

Soln: $q=\frac{10}{-5}=-2$. Therefore, there is no infinite sum since $-2<-1$.

MATHEMATICAL INDUCTION

This is a powerful tool when we are asked to prove something is true for integers.

Suppose we are asked to prove that a given assertion is true for all positive integers. First, we show that it is true for 1 (or some other base case, often 0). Second, we show that it is true for some integer k, then we finally show it must be true for the number k+1 (inductive step).

Having prove this we argue that, since it is true for 1, it must be true for 1+1=2. Since it is true for 2, it is true for 2+1=3, and so on. Thus, the assertion is true for all positive integers.

Ex: Show that $1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

Soln

Step 1: Show it is true for $n=1$.

This is obvious as $1 = \frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$.

step 2: Assume the assertion is true for k, i.e

$$1+2+3+\cdots+k = \frac{k(k+1)}{2}.$$

Step 3 (inductive step). Show it is true for k+1.

$$\begin{aligned}
1+2+3+\cdots+k+(k+1) &= (1+2+3+\cdots+k)+(k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= (k+1)\frac{(k+2)}{2} \\
&= \frac{(k+1)((k+1)+1)}{2}.
\end{aligned}$$

Thus, we have shown that if

$1+2+3+\cdots+n = \frac{n(n+1)}{2}$ is true for $n=k$, then it is true for $n=k+1$. Since it is true for $n=1$, it is therefore true for $2, 3, 4, \ldots$; that is, all integers.

Ex: Using induction, show that the sum of n-terms of an arithmetic sequence (a_n) is given by

$S_n = \frac{n}{2}\left(2a_1 + (n-1)d\right)$ where a_1 and d are the first term and common difference respectively.

Soln:

Step 1: Show it is true for $n=1$:

$$S_1 = \frac{1}{2}\left(2a_1 + (1-1)d\right) = \frac{1}{2}(2a_1) = a_1 \checkmark.$$

Step 2: Assume, it is true for $n=k$; i.e

$$S_k = \frac{k}{2}\left(2a_1 + (k-1)d\right) = a_1 + a_2 + \cdots + a_k$$

Step 3: Show it is true for $n=k+1$.

$$S_{k+1} = a_1 + a_2 + \cdots + a_k + a_{k+1}$$

$$= S_k + a_{k+1} = \frac{k}{2}\left(2a_1 + (k-1)d\right) + \left(a_1 + (k+1-1)d\right).$$

$$= \frac{k}{2}\left(2a_1 + (k-1)d\right) + \left(a_1 + kd\right).$$

$$= \frac{k(2a_1 + (k-1)d) + 2a_1 + 2kd}{2}$$

$$= \frac{2a_1(k+1) + d\left(k(k-1) + 2k\right)}{2}$$

$$= \frac{2a_1(k+1) + d(k)(k+1)}{2}$$

$$= \frac{k+1}{2}\left(2a_1 + kd\right) = \frac{k+1}{2}\left(2a_1 + (k+1-1)d\right).$$

Thus, it's true for $n=k+1$. Hence, it is true for all integers n.

Ex: Show that the sum of n-terms of a Geometric sequence (b_n) is given by $S_n = \frac{b_1(1-q^n)}{1-q}$ $q \neq 1$.

Soln

for $n=1$; $S_1 = \frac{b_1(1-q^1)}{1-q} = b_1$ which is true.

assume, it is true for $n=k$. That is,

$$S_k = \frac{b_1(1-q^k)}{1-q}. \quad q \neq 1, \quad S_k = b_1 + b_2 + \cdots + b_k.$$

Then, let's show that it is true for $n=k+1$.

$$S_{k+1} = b_1 + b_2 + b_3 + \cdots + b_k + b_{k+1}$$
$$= (b_1 + b_2 + \cdots + b_k) + b_{k+1} = S_k + b_{k+1}$$
$$= \frac{b_1(1-q^k)}{1-q} + b_1 q^k \quad ; \text{ note } b_{k+1} = b_1 q^k.$$
$$= \frac{b_1(1-q^k) + b_1 q^k(1-q)}{1-q} = \frac{b_1 - b_1 q^k + b_1 q^k - b_1 q^{k+1}}{1-q}$$
$$= \frac{b_1 - b_1 q^{k+1}}{1-q} = \frac{b_1(1-q^{k+1})}{1-q}$$

Hence; it is true for $n = k+1$, therefore, it is true for all n.

Ex: Show that for all positive integers n,

$$7 + 6\cdot7 + 6\cdot7^2 + \cdots + 6\cdot7^n = 7^{n+1}.$$

Soln

for $n=1$, $7 + 6\cdot7^1 = 7(1+6) = 7\cdot7 = 7^2 = 7^{1+1} = 7^2$ ✓.

assume it is true for $n=k$; that is,

$$7 + 6\cdot7 + 6\cdot7^2 + \cdots + 6\cdot7^k = 7^{k+1}.$$

Show it is true for $n=k+1$

$$7+6\cdot7+6\cdot7^2+6\cdot7^3+\cdots+6\cdot7^k+6\cdot7^{k+1}$$

$$= \left(7+6\cdot7+6\cdot7^2+\cdots+6\cdot7^k\right)+6\cdot7^{k+1}$$

$$= 7^{k+1}+6\cdot7^{k+1} = 7^{k+1}(1+6)$$

$$= 7^{k+1}\cdot7$$

$$= \underline{\underline{7^{k+2}}}.$$

Hence, it is true for $n=k+1$, $\Rightarrow$ it is true for all n.