INEQUALITIES

Linear Inequalities in One Variable

Recall the following:

i) $6 > 4$ implies 6 is greater than 4;

ii) $4 < 6$ implies 4 is less than 6.

If $a > b$ (or $b < a$), it means $a - b > 0$ or $b - a < 0$.

Recall also that $\geqslant$ implies "is greater than or equal to" and $\leqslant$ implies "is less than or equal to".

Likewise, $\geqslant$ also implies "is not less than" and $\leqslant$ ~~impl~~ implies "is not greater than". For instance, if x is not less than 5, then $x \geqslant 5$.

Consider the following:

1) If $a > b$ then $a + x > b + x$ and $a - x > b - x$.

For example, given $5 > -3$, then

$$5 + 3 > -3 + 3$$

That is, $8 > 0$

2) If $a > b$, then $ax > bx$ and $\frac{a}{x} > \frac{b}{x}$, provided x is positive. For example,

$12 > -6$, then

$12(2) > -6(2)$, that is, $24 > -12$.

Also, $12 \div 2 > -6 \div 2$

That is, $6 > -3$

3) If $a > b$, then $ax < bx$ and $\frac{a}{x} < \frac{b}{x}$ if x is negative. For example,

$$12 > -6 \text{ then}$$

$$12(-2) < -6(-2)$$

That is, $-24 < 12$

Also, $12 \div (-2) < -6 \div (-2)$

That is, $-6 < 3$

Example 1: Solve $3x - 5 > -11$

$$3x - 5 > -11$$

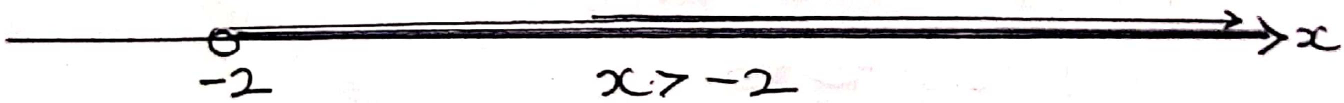
$$3x > -11 + 5$$

$$3x > -6$$

$$x > \frac{-6}{3}$$

$$x > -2$$

The solution on a real number line is given as:



Note that o at -2 in the figure above implies that -2 is NOT included.

Example 2: Solve $-3 < 5 - 3x \leq 11$

By solving each inequality separately,

$-3 < 5 - 3x$ and $5 - 3x \leq 11$

$3x < 5 + 3$ $\quad$ $-3x \leq 11 - 5$

$3x < 8$

$x < \frac{8}{3}$

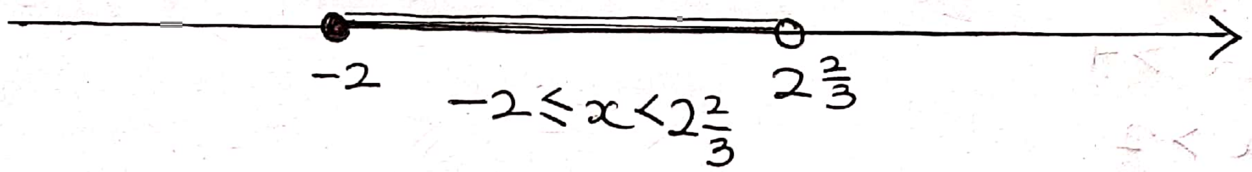
$x < 2\frac{2}{3}$

$-3x \leqslant 6$

That is, $x \geqslant -2$

The combined solutions are given as $-2 \leqslant x < 2\frac{2}{3}$.

This is shown on a number line as:

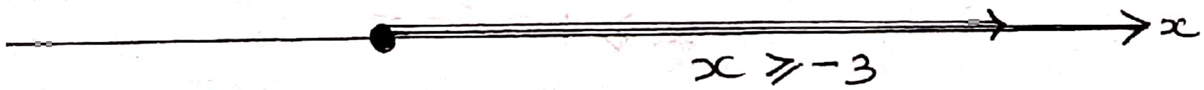


The ● at -2 means that -2 is included. x lies in an interval which is closed at -2 (●) but open at $2\frac{2}{3}$ (o).

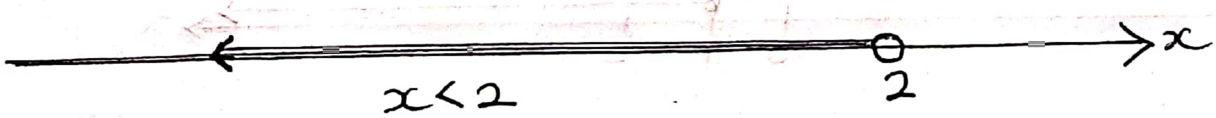
TYPES OF INTERVAL

Some of the types of intervals in which solutions of an inequality can lie are shown below:

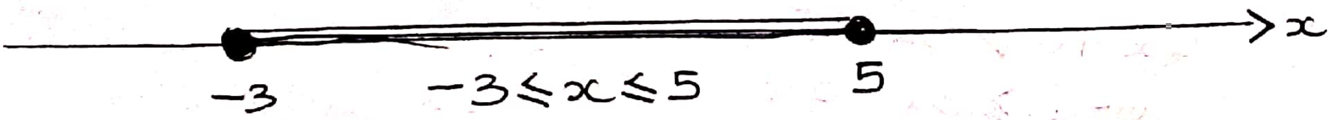
1) Infinite interval: Closed on the left hand

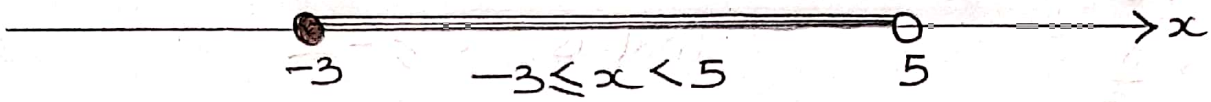


2) Infinite interval: open on the right hand



3) Finite intervals





$-3 \leq x < 5$

FURTHER EXAMPLES

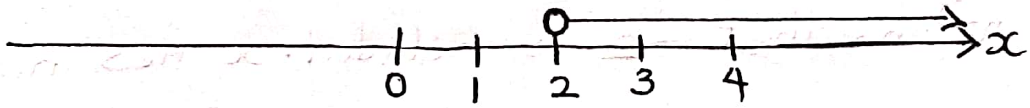
1) Find the values of x which satisfy:

$$4x + 5 > 2x + 9$$

$$4x - 2x > 9 - 5$$

$$2x > 4$$

$$\Rightarrow x > 2$$



2) Find the range of values of x for which $\frac{x+2}{4x-3} \leq 3$?

By cross multiplication, $x + 2 \leq 3(4x - 3)$

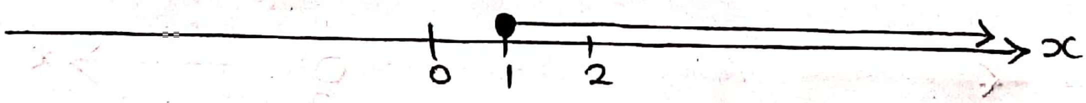
$$x + 2 \leq 12x - 9$$

$$2 + 9 \leq 12x - x$$

$$11 \leq 11x$$

$$1 \leq x$$

$$x \geq 1$$



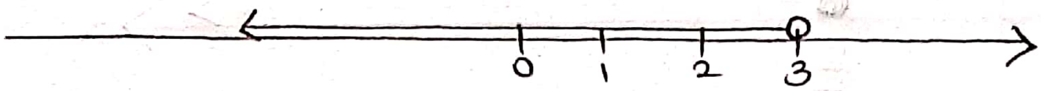
3) Find the values of x for which $2(x+3) > 3(x-1) + 6$?

$$2x + 6 > 3x - 3 + 6$$

$$2x + 6 > 3x + 3$$

$$6 - 3 > 3x - 2x$$

$$3 > x \Rightarrow x < 3$$



4) Given the inequalities (i) $2(3-x) > 9$ and

(ii) $\dfrac{x+1}{2} + \dfrac{2x-1}{3} \geq 1$

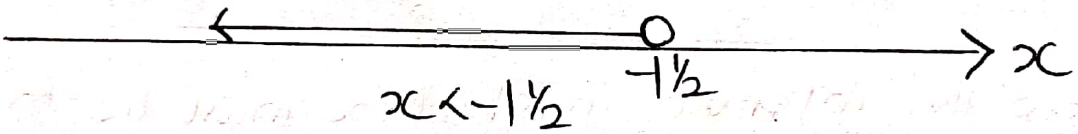
Show the intervals in which x can lie if it satisfies (i) OR (ii), and if it satisfies (i) AND (ii).

By solving (i), $6 - 2x > 9$

$$-2x > 3$$

$$x < -\frac{3}{2}$$

$$x < -1\tfrac{1}{2}$$



By solving (ii), $\cancel{3(x+1) + 2(2x-1) \geq 6}$

$$\frac{x+1}{2} + \frac{2x-1}{3} \geq 1$$

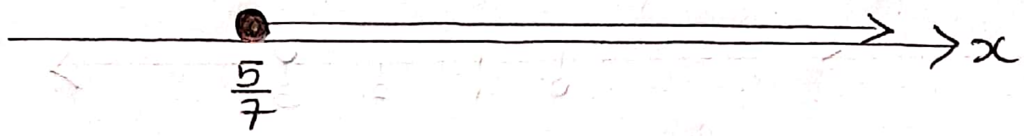
$$\frac{3(x+1) + 2(2x-1)}{6} \geq 1$$

$$3x + 3 + 4x - 2 \geq 6$$

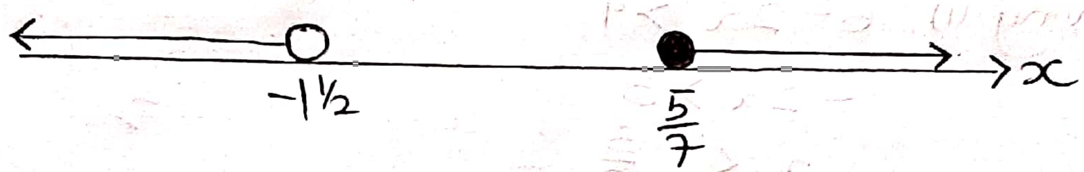
$$7x \geq 6 - 1$$

$$7x \geq 5$$

$$x \geq \frac{5}{7}$$



Hence, if x is to satisfy (i) OR (ii), then both intervals are required so that $x < -1\frac{1}{2}$ or $x \geq \frac{5}{7}$. ~~This represented~~ This is represented on the number line as:



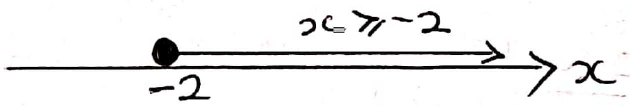
If x is to satisfy (i) AND (ii) then, no value of x is possible. There is no solution.

5) Show the interval in which x must lie to satisfy the inequalities: $2x+5 \geq 1$ and $3 - \frac{1-x}{2} < 4$.

i. $2x+5 \geq 1$

$2x \geq -4$

$x \geq -2$

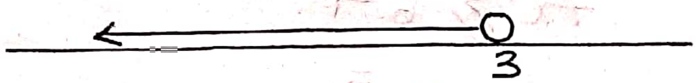


ii. $3 - \frac{1-x}{2} < 4$

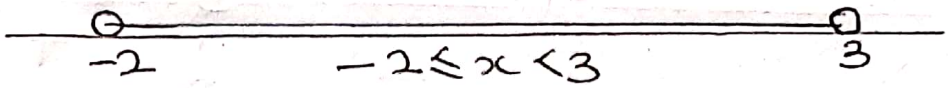
$\frac{6-(1-x)}{2} < 4$

$6-1+x < 8$

$x < 3$



For both inequalities to hold, x must lie in the common interval, that is, $-2 \leq x < 3$. This is shown on the real number line as:



$-2 \leq x < 3$

Exercise

1) Solve the following inequalities, showing the interval(s) obtained on a number line.

a) $2x - 1 \geq 5$

(b) $3 - x > 7$

c) $-5 \leq \frac{x}{3} - 1 \leq 1$

(d) $1 \leq \frac{3x+1}{4} < 2$

e) $\frac{x+1}{3} - \frac{x-1}{4} \geq \frac{1}{2}$

(f) $-\frac{1}{2} \leq \frac{3-2x}{5} \leq 1$

2) Find the intervals in which x can lie if:

a) $3x - 1 > 5$ or $1 - 2x \geq 4$

b) $3x - 1 > 5$ and $1 - 2x \geq 4$

c) $\frac{x-1}{2} > \frac{2}{3}$ and $\frac{2x-1}{3} > \frac{1}{2}$

d) $5 - (4 - x) \leq 3$ and $5 - \frac{x+6}{3} < 4$

Quadratic Inequalities

Quadratic inequalities in one variable, such as $x^2 - 4x - 5 \geq 0$ can be solved using the factors of $x^2 - 4x - 5$.

Example 1: Solve $x^2 - 4x - 5 \geq 0$

This can be written as $(x-5)(x+1) \geq 0$. Consider how the signs of each linear factor vary as x moves along the number line. $(x-5)$ changes sign at $x = 5$, and $(x+1)$ at $x = -1$.

x		-1		5		
$(x-5)$	$-$		$-$	0	$+$	$(x-5)\geq 0$ for $x>5$ but <0 for $x<5$
$(x+1)$	$-$	0	$+$		$+$	$(x+1)<0$ for $x<-1$, but >0 for $x>-1$
Product: $(x-5)(x+1)$	$+$	0	$-$	0	$+$	

Considering the sign of the product $P=(x-5)(x+1)$:

for $x<-1$	$P>0$	$(-\times-)$
at $x=-1$	$P=0$	
$-1<x<5$	$P<0$	$(-\times+)$
at $x=5$	$P=0$	
$x>5$	$P>0$	$(+\times+)$

Therefore, $x^2-4x-5=P$, will be ≥ 0 when $x\leq -1$ or $x\geq 5$.

Example 2: Find the interval which x can lie if $2x^2+3x<2$?

This can be expressed as: $2x^2+3x-2<0$

$$(2x-1)(x+2)<0$$

The values of x are: $x=-2$ and $x=\frac{1}{2}$.

x		-2		$\frac{1}{2}$	
$(2x-1)$	$-$		$-$	0	$+$
$(x+2)$	$-$	0	$+$		$+$
Product: $(2x-1)(x+2)$	$+$	0	$-$	0	$+$

Hence, $2x^2+3x-2=(2x-1)(x+2)<0$ when $-2<x<\frac{1}{2}$.

Example 3: Solve the inequality $x^2-x\leq 3$?

Rewrite the equation as: $x^2-x-3\leq 0$

Using the quadratic formula: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x=\frac{1\pm\sqrt{1-4(1)(-3)}}{2}$$

$x\approx -1.30$ or 2.30 (to 2 decimal places).

Then, $x^2-x-3\approx(x+1.30)(x-2.30)$

The changes in signs of the factors are

x		-1.30		2.30	
$(x+1.30)$	$-$	0	$+$		$+$
$(x-2.30)$	$-$		$-$	0	$+$
Product	$+$	0	$-$	0	$+$

So $x^2-x-3=(x+1.30)(x-2.30)\leq 0$ when $-1.30\leq x\leq 2.30$

Example 4: Find the range of values of x for which

$0\leq \log_{10}(x^2+2x-2)\leq 1$?

$0\leq \log_{10}(x^2+2x-2)\leq 1$ can be written as

$10^0\leq x^2+2x-2\leq 10^1$

$\Rightarrow 1\leq x^2+2x-2\leq 10$

Take these separately,

$1 \leqslant x^2+2x-2$ implies $x^2+2x-3 \geqslant 0$

$$(x+3)(x-1) \geqslant 0$$

x		-3		1	
$(x+3)$	$-$	0	$+$		$+$
$(x-1)$	$-$		$-$	0	$+$
Product: $(x+3)(x-1)$	$+$	0	$-$	0	$+$

So $x^2+2x-3 \geqslant 0$ if $x \leqslant -3$ or $x \geqslant 1$ (i)

For the second part: $x^2+2x-2 \leqslant 10$

$$x^2+2x-12 \leqslant 0$$

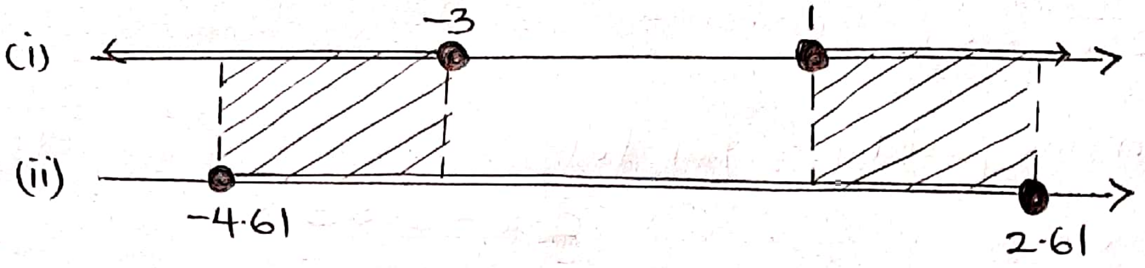
The roots of $x^2+2x-12=0$ are $x \approx 2.61$ or -4.61

That is, $x^2+2x-12 \approx (x-2.61)(x+4.61)$

x		-4.61		2.61	
$(x-2.61)$	$-$		$-$	0	$+$
$(x+4.61)$	$-$	0	$+$		$+$
Product $(x-2.61)(x+4.61)$	$+$	0	$-$	0	$+$

So $x^2+2x-12 \leqslant 0$ if $-4.61 \leqslant x \leqslant 2.61$ (ii)

By combining (i) and (ii), x must lie in the common intervals of (i) and ii, that is, $-4.61 \leqslant x \leqslant -3$ OR $1 \leqslant x \leqslant 2.61$. This is shown in the diagram below:



ABSOLUTE VALUES, $|x|$

The notation $|x|$ means the modulus or absolute value of x. For example,

$|5| = 5$, Also $|-5| = 5$

$\cos|-240^\circ| = \cos 240^\circ = -\frac{1}{2}$; $|\cos 240^\circ| = |-\frac{1}{2}| = \frac{1}{2}$

We define $|x| \begin{cases} = x & \text{if } x \geq 0 \\ = -x & \text{if } x < 0 \end{cases}$

Note: $|2x-3| \Rightarrow \begin{cases} 2x-3 \text{ if } 2x-3 \geq 0 \text{ ie. if } x \geq 1\frac{1}{2} \\ -(2x-3) \text{ if } 2x-3 < 0 \Rightarrow 3x-2 \text{ if } x < 1\frac{1}{2} \end{cases}$

Also, $|x| < K$ means $x < K$ or $-x < K$ ie $x > -K$

Then, $|x| < K$ implies $-K < x < K$

Also, $|x| > K$ implies $x > K$ or $x < -K$

Therefore, we can write

$|ax + b| \leq K$ as $-K \leq ax + b \leq K$

But

$|ax+b| \geq$ is $ax + b \leq -K$ or $ax + b \geq K$

Example 1: Solve (a) $|2x-3| \leq 5$ (b) $|2x-3| > 5$

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a) $-5 \leq 2x-3 \leq 5$

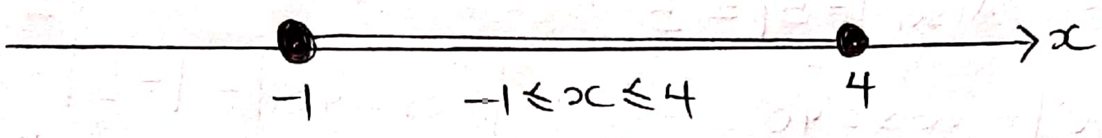
Taking each inequality separately,

$-5 \leq 2x-3$ $\quad\quad$ $2x-3 \leq 5$

$-2 \leq 2x$ $\quad\quad$ $2x \leq 8$

$-1 \leq x$ $\quad\quad$ $x \leq 4$

Therefore, the interval is $-1 \leq x \leq 4$.



b) $|2x-3| > 5$

$2x-3 > 5$ and $-(2x-3) > 5$

$2x-3 > 5$ and $2x-3 < -5$

$2x > 8$ and $2x < -2$

$x > 4$ and $x < -1$