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# Adiabatic Theorem and Quantum Annealing

*A Mathematically Rigorous Treatment*

- 1 Introduction
- 2 Mathematical Framework
- 3 Derivation of the Adiabatic Theorem
- 4 The Adiabatic Theorem
- 5 Examples and Applications
- 6 Geometric Phase (Berry Phase)
- 7 Advanced Topics
- 8 Quantum Annealing: QUBO and Ising Models
- 9 Conclusion

## Know it alls

Einstein 1911 at Solvay Conference had to deal with  
cheeky Lorentz questioning the quantum hypothesis  
 $E = nh\nu$  for atomic oscillators.

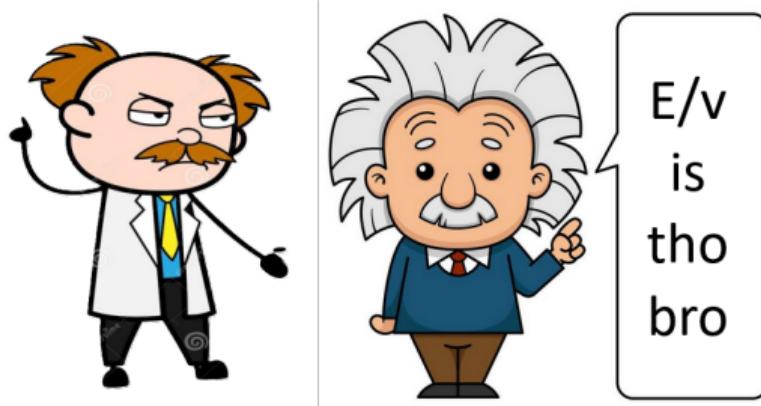


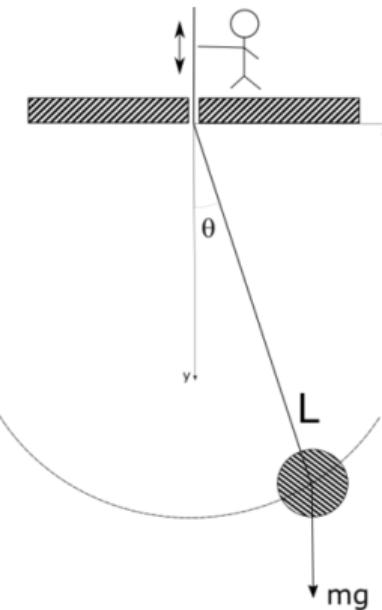
Figure: 1911 Solvay Conference  
Wikimedia Commons

Figure: 1911 Solvay Conference Windows Paint and Bill Gates

## Classical Analogy

A pendulum whose length changes slowly will maintain its oscillation mode, but if changed rapidly, it will exhibit chaotic behavior.

Also called adiabatic invariant, written by Ehrenfest in 1916, and extended to quantum mechanics by Born and Fock in 1928.



## Physical Statement

If a quantum system starts in an eigenstate of a time-dependent Hamiltonian, and if the Hamiltonian changes **sufficiently slowly**, the system will remain in the corresponding instantaneous eigenstate throughout the evolution.

Slow Change

- Key word: “**sufficiently slowly**”
- The system acquires a **dynamical phase** and a **geometric phase** (Berry phase)
- Fundamental to quantum computing (adiabatic quantum computation)
- Applies to quantum chemistry, biology / condensed matter physics

Abrupt Change

# Time-Dependent Schrödinger Equation

Consider a time-dependent Hamiltonian  $\hat{H}(t)$ :

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle \quad (1)$$

## Instantaneous Eigenstates

At each time  $t$ , we can solve the eigenvalue problem:

$$\hat{H}(t) |n(t)\rangle = E_n(t) |n(t)\rangle \quad (2)$$

where  $|n(t)\rangle$  are the **instantaneous eigenstates** and  $E_n(t)$  are the **instantaneous eigenvalues**.

**Note:** These are *not* solutions to the time-dependent Schrödinger equation, but rather eigenstates at fixed time  $t$ .

## Substituting into Schrödinger Equation

Substitute the expansion  $|\psi(t)\rangle = \sum_n c_n(t) e^{i\theta_n(t)} |n(t)\rangle$  into the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle$$
$$i\hbar \sum_n \left[ \dot{c}_n e^{i\theta_n} |n\rangle + c_n i\dot{\theta}_n e^{i\theta_n} |n\rangle + c_n e^{i\theta_n} |\dot{n}\rangle \right] = \sum_n c_n e^{i\theta_n} E_n(t) |n\rangle \quad (3)$$

Dividing by  $i\hbar$ :

$$\sum_n \left[ \frac{\dot{c}_n}{i\hbar} e^{i\theta_n} |n\rangle - \frac{c_n \dot{\theta}_n}{\hbar} e^{i\theta_n} |n\rangle + \frac{c_n}{i\hbar} e^{i\theta_n} |\dot{n}\rangle \right] = \sum_n \frac{c_n E_n}{i\hbar} e^{i\theta_n} |n\rangle \quad (4)$$

# Projecting onto Eigenstates

Take the inner product with  $\langle m(t)|$ :

$$\frac{\dot{c}_m}{i\hbar}e^{i\theta_m} - \frac{c_m\dot{\theta}_m}{\hbar}e^{i\theta_m} + \frac{c_m}{i\hbar}e^{i\theta_m}\langle m|\dot{m}\rangle = \frac{c_mE_m}{i\hbar}e^{i\theta_m} \quad (5)$$

We used orthonormality:  $\langle m|n\rangle = \delta_{mn}$  so that  $\langle m|\hat{H}|n\rangle = E_m\delta_{mn}$ .

Multiplying by  $i\hbar$  and rearranging:

$$\dot{c}_m e^{i\theta_m} = c_m e^{i\theta_m} \left[ i\hbar\dot{\theta}_m - E_m + i\hbar\langle m|\dot{m}\rangle \right] - \sum_{n \neq m} c_n e^{i\theta_n} \langle m|\dot{n}\rangle \quad (6)$$

## Dynamical Phase

$$\theta_m(t) = -\frac{1}{\hbar} \int_0^t E_m(t') dt' + \gamma_m(t) \quad (7)$$

where  $\gamma_m(t)$  is the **geometric phase** (Berry phase):

## Geometric Phase

$$\gamma_m(t) = i \int_0^t \langle m(t') | \dot{m}(t') \rangle dt' \quad (8)$$

Then:  $\dot{\theta}_m = -\frac{E_m}{\hbar} + i \langle m | \dot{m} \rangle$

Substituting back:  $i\hbar \dot{\theta}_m - E_m + i\hbar \langle m | \dot{m} \rangle = 0$ , so:

$$\dot{c}_m e^{i\theta_m} = - \sum_{n \neq m} c_n e^{i\theta_n} \langle m | \dot{n} \rangle \quad (9)$$

How long did it take?

Where have you been bro?



Figure: Michael Berry (The man himself) *Wikimedia Commons*

From the eigenvalue equation  $\hat{H}(t)|n(t)\rangle = E_n(t)|n(t)\rangle$ , differentiate with respect to  $t$ :

$$\dot{\hat{H}}|n\rangle + \hat{H}|\dot{n}\rangle = \dot{E}_n|n\rangle + E_n|\dot{n}\rangle \quad (10)$$

Take inner product with  $\langle m|$  (where  $m \neq n$ ):

$$\langle m| \dot{\hat{H}}|n\rangle + \langle m| \hat{H}|\dot{n}\rangle = \dot{E}_n \langle m|n\rangle + E_n \langle m|\dot{n}\rangle \quad (11)$$

Since  $\langle m| \hat{H} = E_m \langle m|$  and  $\langle m|n\rangle = 0$  for  $m \neq n$ :

$$\langle m| \dot{\hat{H}}|n\rangle + E_m \langle m|\dot{n}\rangle = E_n \langle m|\dot{n}\rangle \quad (12)$$

## Matrix Element

$$\langle m|\dot{n}\rangle = \frac{\langle m| \dot{\hat{H}}|n\rangle}{E_n - E_m} \quad (m \neq n) \quad (13)$$

# The Differential Equation for Coefficients

Substituting back:

$$\dot{c}_m = - \sum_{n \neq m} c_n e^{i(\theta_n - \theta_m)} \frac{\langle m | \dot{\hat{H}} | n \rangle}{E_n - E_m} \quad (14)$$

The phase difference is:

$$\theta_n - \theta_m = -\frac{1}{\hbar} \int_0^t [E_n(t') - E_m(t')] dt' + [\gamma_n(t) - \gamma_m(t)] \quad (15)$$

## Key Observation

The right-hand side contains:

- $\dot{\hat{H}}$  - rate of change of Hamiltonian
- $E_n - E_m$  - energy gap (in denominator)
- Rapidly oscillating phase factor

For the system to remain in state  $|m\rangle$ , we need  $\dot{c}_m \approx 0$  for  $m$  equal to the initial state.

## Adiabatic Condition

The Hamiltonian changes slowly enough that:

$$\left| \frac{\langle m | \dot{H} | n \rangle}{E_n - E_m} \right| \ll \frac{|E_n - E_m|}{\hbar} \quad (16)$$

for all  $n \neq m$ .

This can be rewritten as:

$$\left| \langle m | \dot{H} | n \rangle \right| \ll \frac{(E_n - E_m)^2}{\hbar} \quad (17)$$

**Physical Interpretation:** The time scale of Hamiltonian variation must be much larger than  $\hbar/\Delta E$ , where  $\Delta E$  is the relevant energy gap.

## Theorem (Quantum Adiabatic Theorem)

Let  $\hat{H}(t)$  be a time-dependent Hamiltonian with instantaneous eigenstates  $|n(t)\rangle$  and eigenvalues  $E_n(t)$ . Suppose:

- 1 The system starts in eigenstate  $|n(0)\rangle$ :  $|\psi(0)\rangle = |n(0)\rangle$
- 2 The eigenvalues are non-degenerate
- 3 The adiabatic condition is satisfied for all  $t \in [0, T]$

Then the state at time  $t$  is:

$$|\psi(t)\rangle = e^{i\theta_n(t)} |n(t)\rangle + \mathcal{O}(\epsilon) \quad (18)$$

where  $\epsilon$  characterizes the adiabatic parameter and

$$\theta_n(t) = -\frac{1}{\hbar} \int_0^t E_n(t') dt' + i \int_0^t \langle n(t') | \partial_{t'} n(t') \rangle dt' \quad (19)$$

Shane's slides 5 minutes allocated here.

## Example 1: Spin-1/2 in Rotating Magnetic Field

Consider a spin-1/2 particle in a magnetic field that rotates slowly:

$$\vec{B}(t) = B_0(\sin \theta \cos \omega t, \sin \theta \sin \omega t, \cos \theta) \quad (20)$$

The Hamiltonian is:

$$\hat{H}(t) = -\gamma \vec{B}(t) \cdot \vec{\sigma} = -\gamma B_0 \begin{pmatrix} \cos \theta & \sin \theta e^{-i\omega t} \\ \sin \theta e^{i\omega t} & -\cos \theta \end{pmatrix} \quad (21)$$

Instantaneous eigenvalues:  $E_{\pm} = \mp \gamma B_0$

If the spin starts aligned with  $\vec{B}(0)$  and  $\omega$  is small enough:

- The spin remains aligned with  $\vec{B}(t)$  (adiabatic following)
- After one full rotation ( $t = 2\pi/\omega$ ), acquires Berry phase:  
$$\gamma = \pi(1 - \cos \theta)$$

## Example 2: Two-Level System (Landau-Zener Model)

Consider the Landau-Zener Hamiltonian:

$$\hat{H}(t) = -\Delta\sigma_x - \epsilon(t)\sigma_z \quad (22)$$

where  $\epsilon(t)$  is swept linearly from  $\epsilon_i < 0$  to  $\epsilon_f > 0$ .

Energy eigenvalues:

$$E_{\pm}(t) = \pm\sqrt{\Delta^2 + \epsilon^2(t)} \quad (23)$$

Energy gap at crossing ( $\epsilon = 0$ ):

$$\Delta E_{\min} = 2\Delta$$

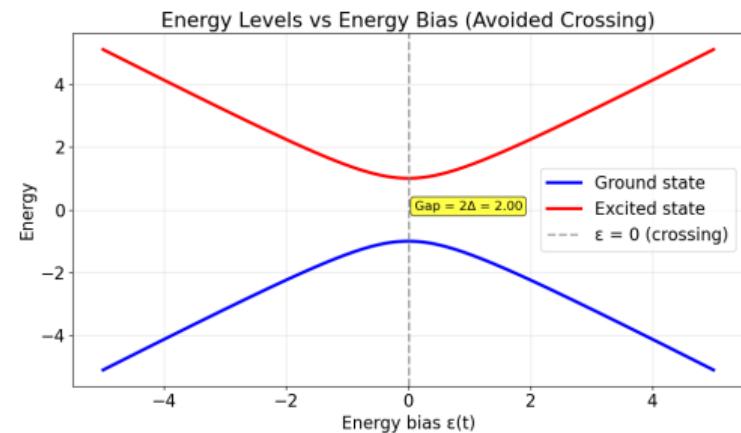
# Landau-Zener Model: Avoided Crossing Visualization

- The energy levels exhibit an **avoided crossing** at  $\epsilon(t) = 0$
- Minimum gap:  $\Delta E_{\min} = 2\Delta$  determines adiabatic time scale

## Adiabatic Condition

$$T \gg \frac{\hbar}{\Delta E_{\min}} = \frac{\hbar}{2\Delta} \quad \text{and} \quad |\dot{\epsilon}| \ll \frac{4\Delta^2}{\hbar} \quad (24)$$

This is the canonical model for avoided crossings and adiabatic transitions!



## Key Questions

- 1 What happens at the avoided crossing point?
- 2 Why is this the bottleneck for adiabatic evolution?
- 3 How does the gap size affect the required evolution time?

## Physical Insights:

- At  $\epsilon = 0$ : The eigenstates are equal superpositions of diabatic states
- The gap  $2\Delta$  prevents level crossing (quantum mechanical effect)
- Smaller gap  $\Rightarrow$  easier to violate adiabaticity
- Without the  $-\Delta\sigma_x$  term, levels would cross diabatically

## Critical Insight

The adiabatic condition is hardest to satisfy near the **minimum gap**. This is where non-adiabatic transitions (Landau-Zener tunneling) are most likely to occur.

For a cyclic adiabatic evolution, where the Hamiltonian returns to its initial form:

$$\hat{H}(R(T)) = \hat{H}(R(0)) \quad (25)$$

The total phase acquired is:

$$e^{i\phi} = e^{i\theta_{\text{dyn}}} e^{i\gamma_n} \quad (26)$$

## Berry Phase (1984)

$$\gamma_n = i \oint_{\mathcal{C}} \langle n(R) | \nabla_R n(R) \rangle \cdot dR \quad (27)$$

where  $\mathcal{C}$  is the closed path in parameter space.

### Properties:

- Gauge invariant (physical observable)
- Geometric: depends only on path, not speed
- Can be written as flux of Berry curvature

## Berry Connection (Vector Potential)

$$\vec{A}_n(R) = i \langle n(R) | \nabla_R n(R) \rangle \quad (28)$$

## Berry Curvature (Field Strength)

$$\vec{F}_n(R) = \nabla_R \times \vec{A}_n(R) \quad (29)$$

By Stokes' theorem:

$$\gamma_n = \oint_C \vec{A}_n \cdot dR = \iint_S \vec{F}_n \cdot d\vec{S} \quad (30)$$

**Analogy:** Berry phase is like the Aharonov-Bohm phase, but in parameter space rather than real space!

When adiabaticity breaks down, transitions occur between energy levels.

## Landau-Zener Formula

For a linear crossing:  $E_1(t) = \alpha t$ ,  $E_2(t) = -\alpha t + \Delta$

Transition probability:

$$P_{1 \rightarrow 2} = \exp\left(-\frac{\pi\Delta^2}{2\hbar\alpha}\right) \quad (31)$$

- Small gap  $\Delta \Rightarrow$  high transition probability
- Fast sweep (large  $\alpha$ )  $\Rightarrow$  high transition probability
- Slow sweep (small  $\alpha$ )  $\Rightarrow$  adiabatic regime,  $P \rightarrow 0$

We can systematically expand in the adiabatic parameter  $\epsilon = \tau_{\text{dyn}}/T$ :

$$|\psi(t)\rangle = e^{i\theta_n(t)} \left[ |n(t)\rangle + \epsilon |\psi_n^{(1)}(t)\rangle + \epsilon^2 |\psi_n^{(2)}(t)\rangle + \dots \right] \quad (32)$$

First-order correction:

$$|\psi_n^{(1)}(t)\rangle = \sum_{m \neq n} \frac{\langle m | \dot{\hat{H}} | n \rangle}{(E_n - E_m)^2} |m\rangle \quad (33)$$

This shows:

- › Transition amplitude  $\propto 1/(E_n - E_m)^2$
- › Small gaps make adiabatic evolution difficult
- › Can estimate fidelity:  $|\langle n(t) | \psi(t) \rangle|^2 \approx 1 - \mathcal{O}(\epsilon^2)$

When eigenvalues are degenerate, the theorem must be modified:

- › System can transition within degenerate subspace
- › Need to consider the entire degenerate subspace
- › Berry phase becomes a **matrix** (non-Abelian Berry connection)

## Wilczek-Zee Connection

For degenerate subspace with basis  $\{|n_a(R)\rangle\}$ :

$$(A_i)_{ab} = i \langle n_a | \partial_i n_b \rangle \quad (34)$$

This is a matrix-valued gauge potential.

Leads to non-Abelian geometric phases (important in topological quantum computation).

# Discussion: Diabatic vs Adiabatic Evolution

## Diabatic Evolution:

- Fast changes in  $\hat{H}(t)$
- System doesn't track instantaneous eigenstates
- Transitions between levels
- Example: Sudden approximation

## When applicable:

$$T \ll \frac{\hbar}{\Delta E} \quad (35)$$

## Adiabatic Evolution:

- Slow changes in  $\hat{H}(t)$
- System follows instantaneous eigenstate
- No transitions (to first order)
- Example: Born-Oppenheimer

## When applicable:

$$T \gg \frac{\hbar}{\Delta E} \quad (36)$$

## Discussion Question

In between these limits, how do we calculate the transition probability?

**Answer:** Landau-Zener formula, perturbation theory, or numerical methods

Fintan's slides 5 minutes allocated here.

# Application: Adiabatic Quantum Computation

## Basic Idea

- 1 Encode problem in Hamiltonian  $\hat{H}_{\text{problem}}$  whose ground state is the solution
- 2 Start with simple Hamiltonian  $\hat{H}_{\text{initial}}$  with known ground state
- 3 Slowly interpolate:  $\hat{H}(t) = (1 - s(t))\hat{H}_{\text{initial}} + s(t)\hat{H}_{\text{problem}}$
- 4 If evolution is adiabatic, system remains in ground state
- 5 At  $t = T$ : measure to obtain solution

**Runtime:**  $T \sim \mathcal{O}(\hbar/\Delta_{\min}^2)$  where  $\Delta_{\min}$  is minimum gap

**Challenge:** Gap can be exponentially small for some problems!

# Adiabatic Theorem in Quantum Annealing

Success condition based on adiabatic theorem:

## Required Annealing Time

$$T \gg \frac{\hbar}{\Delta_{\min}^2} \quad (37)$$

where  $\Delta_{\min}$  is the minimum energy gap during evolution

## Problem-Dependent Gap:

- Different problems  $\Rightarrow$  different gap structures
- Gap often smallest near  $t \approx T/2$  (mid-anneal)
- For many NP-hard problems:  $\Delta_{\min} \sim e^{-\alpha N}$  (exponentially small!)

## Practical Implications:

- + Easy problems: Fast annealing possible (microseconds)
- Hard problems: May need exponentially long annealing time
- ± Thermal effects can help/hinder (quantum vs simulated annealing)

**Goal:** Solve optimization problems using quantum adiabatic evolution

## Standard Approach

- 1 Encode optimization problem in a Hamiltonian
- 2 Prepare system in ground state of simple initial Hamiltonian
- 3 Adiabatically evolve to problem Hamiltonian
- 4 Measure final state to read out solution

## Two Main Formulations:

- **Ising Model:** Spin variables  $s_i \in \{-1, +1\}$
- **QUBO:** Binary variables  $x_i \in \{0, 1\}$
- These are equivalent and interconvertible

**Examples:** Graph coloring, traveling salesman, portfolio optimization, protein folding

## Classical Ising Hamiltonian

$$H_{\text{Ising}} = \sum_{i < j} J_{ij} s_i s_j + \sum_i h_i s_i \quad (38)$$

where  $s_i \in \{-1, +1\}$  are spin variables (Pauli-z eigenvalues)

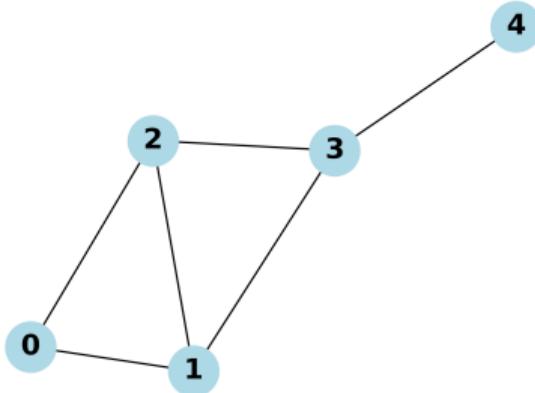
### Parameters:

- $J_{ij}$ : Coupling between spins  $i$  and  $j$ 
  - »  $J_{ij} > 0$ : Ferromagnetic (spins want to align)
  - »  $J_{ij} < 0$ : Antiferromagnetic (spins want to anti-align)
- $h_i$ : Local field on spin  $i$  (bias)

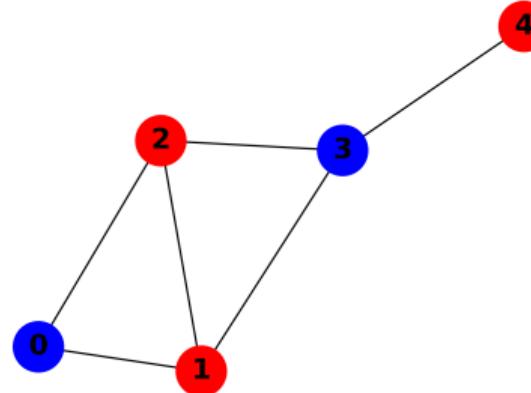
This is the **problem Hamiltonian** in quantum annealing!

# Visualizing the Ising Model: Max-Cut Example

Example Graph for Max-Cut



Optimal Max-Cut (size = 5)



- **Left:** Original graph with 5 vertices and 6 edges
- **Right:** Optimal partition (red vs blue) achieves cut size = 5
- Only 1 edge (out of 6) remains within a partition
- Ising formulation: Antiferromagnetic couplings ( $J_{ij} = 1$ ) favor opposite spins

# QUBO: Quadratic Unconstrained Binary Optimization

## QUBO Formulation

Minimize:

$$f(x) = \sum_i Q_{ii}x_i + \sum_{i < j} Q_{ij}x_i x_j = x^T Q x \quad (39)$$

where  $x_i \in \{0, 1\}$  and  $Q$  is the QUBO matrix

## Why QUBO?

- Many combinatorial optimization problems naturally expressed as QUBO
- NP-hard in general (finding global minimum)
- Can encode constraints via penalty terms

## Examples:

- Max-Cut:  $\max \sum_{(i,j) \in E} (1 - x_i x_j)$
- Number Partitioning:  $\min (\sum_i a_i x_i)^2$
- Graph Coloring: Use penalty for adjacent same-colored vertices

The mapping is straightforward:

## QUBO → Ising

Variable transformation:  $x_i = \frac{1-s_i}{2}$  where  $s_i \in \{-1, +1\}$

Then:  $x_i x_j = \frac{(1-s_i)(1-s_j)}{4} = \frac{1-s_i-s_j+s_i s_j}{4}$

## Ising → QUBO

Inverse transformation:  $s_i = 1 - 2x_i$  where  $x_i \in \{0, 1\}$

Then:  $s_i s_j = (1 - 2x_i)(1 - 2x_j) = 1 - 2x_i - 2x_j + 4x_i x_j$

**Key Point:** The two formulations are equivalent - choose based on:

- Natural problem representation
- Hardware constraints (quantum annealers often use Ising)
- Software/solver availability

## Example: Max-Cut Problem

**Problem:** Given graph  $G = (V, E)$ , partition vertices into two sets to maximize edges between sets.

### QUBO Formulation:

Maximize:

$$f(x) = \sum_{(i,j) \in E} (x_i - x_j)^2 \quad (40)$$

Expanding:

$$= \sum_{(i,j) \in E} (x_i + x_j - 2x_i x_j) \quad (41)$$

For minimization (quantum annealer):

$$H_{\text{QUBO}} = - \sum_{(i,j) \in E} (x_i + x_j - 2x_i x_j) \quad (42)$$

### Ising Formulation:

Using  $s_i = 1 - 2x_i$ :

$$H_{\text{Ising}} = - \sum_{(i,j) \in E} (1 - s_i s_j) \quad (43)$$

Simplifies to:

$$H_{\text{Ising}} = \sum_{(i,j) \in E} s_i s_j + \text{const} \quad (44)$$

### Interpretation:

- $s_i = +1$ : vertex in set A
- $s_i = -1$ : vertex in set B
- Minimize  $\Rightarrow$  maximize cut

# Born–Oppenheimer Approximation: Adiabatic Physics in Molecules

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$$\Psi(\mathbf{r}, \mathbf{R}) \approx \psi_e(\mathbf{r}; \mathbf{R}) \chi(\mathbf{R})$$

- **Separation of timescales:** Nuclei (heavy) move slowly, electrons adapt instantaneously.
- The electronic Hamiltonian  $H_e(\mathbf{R})$  depends *parametrically* on the nuclear coordinates.
- By the **adiabatic theorem**, the electronic state remains in the instantaneous ground state as nuclei evolve slowly.

## Analogy to Adiabatic Quantum Computing

Slow nuclear motion  $\leftrightarrow$  slow Hamiltonian evolution. Electrons follow adiabatically  $\leftrightarrow$  qubit states follow instantaneous ground state.

# Discussion: Quantum Annealing vs Gate Model

## Quantum Annealing:

- + Analog, continuous evolution
- + Natural for optimization
- + Potentially more robust to some errors
- + Easier to scale (1000s of qubits)
- Limited to specific problem types
- No quantum error correction
- Success depends on gap structure

## Gate-Based Quantum Computing:

- + Universal quantum computation
- + Quantum error correction possible
- + Well-defined circuit depth
- + Can implement any algorithm
- Currently limited qubits (<1000)
- High gate error rates
- Requires precise control

## Open Question

Does quantum annealing provide speedup over classical algorithms?

Evidence mixed - depends on problem structure, noise, and implementation.

# Experimental Realization and Fidelity

## Experimental Platforms:

- Superconducting qubits
- Trapped ions
- Cold atoms
- Nitrogen-vacancy centers
- Quantum dots

## Typical Parameters:

- $\Delta/h \sim 0.1\text{-}10 \text{ GHz}$
- $T \sim 1\text{-}1000 \text{ ns}$
- Fidelity  $> 99\%$  achievable

## Challenges:

- Decoherence ( $T_1, T_2$ )
- Control noise
- Calibration errors
- Finite temperature

## Fidelity Metric

$$F = |\langle \psi_{\text{target}} | \psi_{\text{final}} \rangle|^2 \quad (45)$$

Measures overlap with desired state

**Discussion:** How do environmental effects modify the adiabatic theorem?

# Open Questions and Research Frontiers

1 **Many-body systems:** How does the adiabatic theorem scale with system size?

- » Ground state degeneracy
- » Exponentially small gaps
- » Quantum phase transitions

2 **Decoherence:** How does noise affect adiabatic evolution?

- » Open quantum systems
- » Markovian vs non-Markovian dynamics
- » Error correction strategies

3 **Shortcuts to adiabaticity:** Can we achieve adiabatic-like results faster?

- » Counter-diabatic driving
- » Optimal control theory
- » Fast-forward protocols

4 **Quantum annealing:** What problems can quantum annealers solve?

- » Quantum advantage vs classical algorithms
- » Role of thermal fluctuations
- » Embedding optimization problems

## Mathematical Rigor

The adiabatic theorem is not just a hand-waving argument:

- Precise conditions on energy gaps and rates of change
- Systematic perturbation theory for corrections
- Geometric interpretation via Berry connection and curvature

## Physical Consequences

- Quantum systems can "remember" their path through parameter space
- Topological effects in quantum mechanics
- Foundation for quantum computation paradigm
- Explains robustness of certain quantum states

# Numerical Verification: What We Observed

From our Landau-Zener simulations:

- 1 **Critical Time Scale:** For  $\Delta = 1.0$ , achieve 95% fidelity at  $T \approx 10\hbar$ 
  - » Consistent with  $T \gg \hbar/(2\Delta) = 0.5\hbar$
- 2 **Scaling with  $\Delta$ :**
  - »  $\Delta = 0.5$ :  $T_{95\%} \approx 3\hbar$  (easier)
  - »  $\Delta = 1.0$ :  $T_{95\%} \approx 10\hbar$  (baseline)
  - »  $\Delta = 2.0$ :  $T_{95\%} \approx 30\hbar$  (harder)
  - » Approximately:  $T_{95\%} \propto \Delta^2$
- 3 **Non-adiabatic regime:** For  $T < 1\hbar$ , fidelity drops dramatically
  - » Landau-Zener transitions dominate
  - » Exponential suppression with  $T$

## Key Takeaway

The adiabatic theorem is not just qualitative - it provides quantitative predictions that match numerical simulations!

## Questions for Consideration

- 1 For quantum computing, is longer evolution time always acceptable?
- 2 How do we balance fidelity vs speed in real applications?
- 3 What role does the adiabatic theorem play in current quantum technologies?

## Real-World Trade-offs: Slower Evolution:

- + Higher fidelity
- + Better adiabaticity
- More decoherence
- Longer computation time

## Faster Evolution:

- + Less decoherence
- + Faster results
- Lower fidelity
- Non-adiabatic errors

**Optimal Strategy:** Find the "sweet spot" where  $T$  is long enough for adiabaticity but short enough to avoid decoherence. This is an active area of research!

# Thank You!

Questions?

## Appendix: Proof of $\langle n|\dot{n} \rangle$ is Purely Imaginary

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From normalization:  $\langle n|n \rangle = 1$

Differentiate with respect to time:

$$\frac{d}{dt} \langle n|n \rangle = \langle \dot{n}|n \rangle + \langle n|\dot{n} \rangle = 0 \quad (46)$$

Therefore:

$$\langle n|\dot{n} \rangle = -\langle \dot{n}|n \rangle = -\langle n|\dot{n} \rangle^* \quad (47)$$

This implies  $\langle n|\dot{n} \rangle$  is purely imaginary.

We can write:  $\langle n|\dot{n} \rangle = i\gamma_n(t)$  where  $\gamma_n(t)$  is real.

## Appendix: Derivation of Berry Curvature

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Starting from Berry connection:

$$A_i = i \langle n | \partial_i n \rangle \quad (48)$$

The Berry curvature is:

$$F_{ij} = \partial_i A_j - \partial_j A_i \quad (49)$$

Expanding:

$$F_{ij} = i [\partial_i \langle n | \partial_j n \rangle - \partial_j \langle n | \partial_i n \rangle] \quad (50)$$

$$= i [\langle \partial_i n | \partial_j n \rangle + \langle n | \partial_i \partial_j n \rangle - \langle \partial_j n | \partial_i n \rangle - \langle n | \partial_j \partial_i n \rangle] \quad (51)$$

Since  $\partial_i \partial_j = \partial_j \partial_i$ :

$$F_{ij} = i [\langle \partial_i n | \partial_j n \rangle - \langle \partial_j n | \partial_i n \rangle] \quad (52)$$

## Appendix: Completeness Relation

Using completeness:  $\sum_m |m\rangle \langle m| = \mathbb{I}$

We can write:

$$|\partial_i n\rangle = \sum_m |m\rangle \langle m| \partial_i n \rangle \quad (53)$$

Therefore:

$$\langle \partial_i n | \partial_j n \rangle = \sum_{m,k} \langle \partial_i n | k \rangle \langle k | m \rangle \langle m | \partial_j n \rangle \quad (54)$$

$$= \sum_m \langle \partial_i n | m \rangle \langle m | \partial_j n \rangle \quad (55)$$

The Berry curvature becomes:

$$F_{ij} = i \sum_{m \neq n} [\langle \partial_i n | m \rangle \langle m | \partial_j n \rangle - \langle \partial_j n | m \rangle \langle m | \partial_i n \rangle] \quad (56)$$