



Niall, Fintan, Shane

*Applied Quantum Mechanics*

November 7, 2025

# Adiabatic Theorem and Quantum Annealing

*A Mathematically Rigorous Treatment*

- 1 Introduction
- 2 Mathematical Framework
- 3 Derivation of the Adiabatic Theorem
- 4 The Adiabatic Theorem
- 5 Examples and Applications
- 6 Geometric Phase (Berry Phase)
- 7 Advanced Topics
- 8 Quantum Annealing: QUBO and Ising Models
- 9 Conclusion

## Physical Statement

If a quantum system starts in an eigenstate of a time-dependent Hamiltonian, and if the Hamiltonian changes **sufficiently slowly**, the system will remain in the corresponding instantaneous eigenstate throughout the evolution.

Slow Change

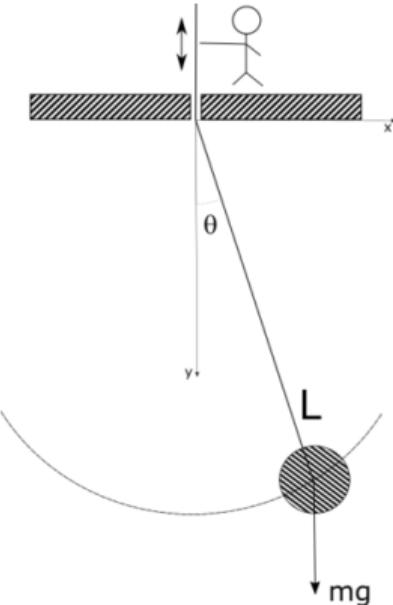
- Key word: “**sufficiently slowly**”
- The system acquires a **dynamical phase** and a **geometric phase** (Berry phase)
- Fundamental to quantum computing (adiabatic quantum computation)
- Applies to quantum chemistry, biology / condensed matter physics

Abrupt Change

- First formulated by Max Born and Vladimir Fock (1928)
- Further developed by Kato (1950)
- Geometric phase discovered by Berry (1984)
- Modern applications in quantum computing and quantum annealing

### Classical Analogy

A pendulum whose length changes slowly will maintain its oscillation mode, but if changed rapidly, it will exhibit chaotic behavior.



# Time-Dependent Schrödinger Equation

Consider a time-dependent Hamiltonian  $\hat{H}(t)$ :

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle \quad (1)$$

## Instantaneous Eigenstates

At each time  $t$ , we can solve the eigenvalue problem:

$$\hat{H}(t) |n(t)\rangle = E_n(t) |n(t)\rangle \quad (2)$$

where  $|n(t)\rangle$  are the **instantaneous eigenstates** and  $E_n(t)$  are the **instantaneous eigenvalues**.

**Note:** These are *not* solutions to the time-dependent Schrödinger equation, but rather eigenstates at fixed time  $t$ .

## Substituting into Schrödinger Equation

Substitute the expansion  $|\psi(t)\rangle = \sum_n c_n(t) e^{i\theta_n(t)} |n(t)\rangle$  into the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle$$
$$i\hbar \sum_n \left[ \dot{c}_n e^{i\theta_n} |n\rangle + c_n i\dot{\theta}_n e^{i\theta_n} |n\rangle + c_n e^{i\theta_n} |\dot{n}\rangle \right] = \sum_n c_n e^{i\theta_n} E_n(t) |n\rangle \quad (3)$$

where dots denote time derivatives:  $\dot{c}_n = \frac{dc_n}{dt}$ ,  $|\dot{n}\rangle = \frac{d}{dt} |n(t)\rangle$ .

Dividing by  $i\hbar$ :

$$\sum_n \left[ \frac{\dot{c}_n}{i\hbar} e^{i\theta_n} |n\rangle - c_n \dot{\theta}_n e^{i\theta_n} |n\rangle + \frac{c_n}{i\hbar} e^{i\theta_n} |\dot{n}\rangle \right] = \sum_n c_n e^{i\theta_n} \frac{E_n}{\hbar} |n\rangle \quad (4)$$

# Projecting onto Eigenstates

Take the inner product with  $\langle m(t) |$ :

$$\frac{\dot{c}_m}{i\hbar}e^{i\theta_m} - c_m \dot{\theta}_m e^{i\theta_m} + \frac{c_m}{i\hbar}e^{i\theta_m} \langle m | \dot{m} \rangle = \sum_{n \neq m} \frac{c_n}{i\hbar}e^{i\theta_n} \langle m | \dot{n} \rangle \quad (5)$$

We used orthonormality:  $\langle m | n \rangle = \delta_{mn}$  and  $\sum_{n \neq m} \langle m | n \rangle = 0$ .

Rearranging:

$$\dot{c}_m e^{i\theta_m} = c_m e^{i\theta_m} \left[ \dot{\theta}_m - \frac{E_m}{i\hbar} - \langle m | \dot{m} \rangle \right] - \sum_{n \neq m} c_n e^{i\theta_n} \langle m | \dot{n} \rangle \quad (6)$$

# Simplifying Terms / Choice of Dynamical Phase

## Dynamical Phase

$$\theta_m(t) = -\frac{1}{\hbar} \int_0^t E_m(t') dt' + \gamma_m(t) \quad (7)$$

where  $\gamma_m(t)$  is the **geometric phase** (Berry phase):

## Geometric Phase

$$\gamma_m(t) = i \int_0^t \langle m(t') | \dot{m}(t') \rangle dt' \quad (8)$$

Then:  $\dot{\theta}_m = -\frac{E_m}{\hbar} + i \langle m | \dot{m} \rangle$

And our equation becomes:

$$\dot{c}_m e^{i\theta_m} = - \sum_{n \neq m} c_n e^{i\theta_n} \langle m | \dot{n} \rangle \quad (9)$$

From the eigenvalue equation  $\hat{H}(t)|n(t)\rangle = E_n(t)|n(t)\rangle$ , differentiate with respect to  $t$ :

$$\dot{\hat{H}}|n\rangle + \hat{H}|\dot{n}\rangle = \dot{E}_n|n\rangle + E_n|\dot{n}\rangle \quad (10)$$

Take inner product with  $\langle m|$  (where  $m \neq n$ ):

$$\langle m| \dot{\hat{H}}|n\rangle + \langle m| \hat{H}|\dot{n}\rangle = \dot{E}_n \langle m|n\rangle + E_n \langle m|\dot{n}\rangle \quad (11)$$

Since  $\langle m| \hat{H} = E_m \langle m|$  and  $\langle m|n\rangle = 0$  for  $m \neq n$ :

$$\langle m| \dot{\hat{H}}|n\rangle + E_m \langle m|\dot{n}\rangle = E_n \langle m|\dot{n}\rangle \quad (12)$$

Therefore:

## Matrix Element

$$\langle m|\dot{n}\rangle = \frac{\langle m| \dot{\hat{H}}|n\rangle}{E_n - E_m} \quad (m \neq n) \quad (13)$$

# The Differential Equation for Coefficients

Substituting back:

$$\dot{c}_m = - \sum_{n \neq m} c_n e^{i(\theta_n - \theta_m)} \frac{\langle m | \dot{\hat{H}} | n \rangle}{E_n - E_m} \quad (14)$$

The phase difference is:

$$\theta_n - \theta_m = -\frac{1}{\hbar} \int_0^t [E_n(t') - E_m(t')] dt' + [\gamma_n(t) - \gamma_m(t)] \quad (15)$$

## Key Observation

The right-hand side contains:

- $\dot{\hat{H}}$  - rate of change of Hamiltonian
- $E_n - E_m$  - energy gap (in denominator)
- Rapidly oscillating phase factor

For the system to remain in state  $|m\rangle$ , we need  $\dot{c}_m \approx 0$  for  $m$  equal to the initial state.

## Adiabatic Condition

The Hamiltonian changes slowly enough that:

$$\left| \frac{\langle m | \dot{H} | n \rangle}{E_n - E_m} \right| \ll \frac{|E_n - E_m|}{\hbar} \quad (16)$$

for all  $n \neq m$ .

This can be rewritten as:

$$\left| \langle m | \dot{H} | n \rangle \right| \ll \frac{(E_n - E_m)^2}{\hbar} \quad (17)$$

**Physical Interpretation:** The time scale of Hamiltonian variation must be much larger than  $\hbar/\Delta E$ , where  $\Delta E$  is the relevant energy gap.

## Theorem (Quantum Adiabatic Theorem)

Let  $\hat{H}(t)$  be a time-dependent Hamiltonian with instantaneous eigenstates  $|n(t)\rangle$  and eigenvalues  $E_n(t)$ . Suppose:

- 1 The system starts in eigenstate  $|n(0)\rangle$ :  $|\psi(0)\rangle = |n(0)\rangle$
- 2 The eigenvalues are non-degenerate
- 3 The adiabatic condition is satisfied for all  $t \in [0, T]$

Then the state at time  $t$  is:

$$|\psi(t)\rangle = e^{i\theta_n(t)} |n(t)\rangle + \mathcal{O}(\epsilon) \quad (18)$$

where  $\epsilon$  characterizes the adiabatic parameter and

$$\theta_n(t) = -\frac{1}{\hbar} \int_0^t E_n(t') dt' + i \int_0^t \langle n(t') | \partial_{t'} n(t') \rangle dt' \quad (19)$$

Shane's slides 5 minutes allocated here.

## Example 1: Spin-1/2 in Rotating Magnetic Field

Consider a spin-1/2 particle in a magnetic field that rotates slowly:

$$\vec{B}(t) = B_0(\sin \theta \cos \omega t, \sin \theta \sin \omega t, \cos \theta) \quad (20)$$

The Hamiltonian is:

$$\hat{H}(t) = -\gamma \vec{B}(t) \cdot \vec{\sigma} = -\gamma B_0 \begin{pmatrix} \cos \theta & \sin \theta e^{-i\omega t} \\ \sin \theta e^{i\omega t} & -\cos \theta \end{pmatrix} \quad (21)$$

Instantaneous eigenvalues:  $E_{\pm} = \mp \gamma B_0$

If the spin starts aligned with  $\vec{B}(0)$  and  $\omega$  is small enough:

- The spin remains aligned with  $\vec{B}(t)$  (adiabatic following)
- After one full rotation ( $t = 2\pi/\omega$ ), acquires Berry phase:

$$\gamma = \pi(1 - \cos \theta)$$

## Example 2: Two-Level System (Landau-Zener Model)

Consider the Landau-Zener Hamiltonian:

$$\hat{H}(t) = -\Delta\sigma_x - \epsilon(t)\sigma_z \quad (22)$$

where  $\epsilon(t)$  is swept linearly from  $\epsilon_i < 0$  to  $\epsilon_f > 0$ .

Energy eigenvalues:

$$E_{\pm}(t) = \pm\sqrt{\Delta^2 + \epsilon^2(t)} \quad (23)$$

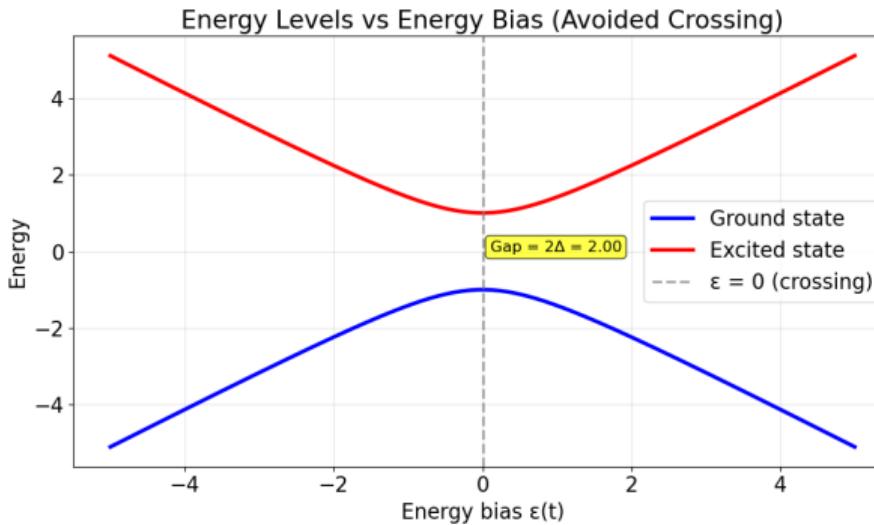
Energy gap at crossing ( $\epsilon = 0$ ):  $\Delta E_{\min} = 2\Delta$

### Adiabatic Condition

$$T \gg \frac{\hbar}{\Delta E_{\min}} = \frac{\hbar}{2\Delta} \quad \text{and} \quad |\dot{\epsilon}| \ll \frac{4\Delta^2}{\hbar} \quad (24)$$

This is the canonical model for avoided crossings and adiabatic transitions!

# Landau-Zener Model: Avoided Crossing Visualization



- The energy levels exhibit an **avoided crossing** at  $\epsilon(t) = 0$
- Minimum gap:  $\Delta E_{\min} = 2\Delta$  determines adiabatic time scale

## Key Questions

- 1 What happens at the avoided crossing point?
- 2 Why is this the bottleneck for adiabatic evolution?
- 3 How does the gap size affect the required evolution time?

## Physical Insights:

- At  $\epsilon = 0$ : The eigenstates are equal superpositions of diabatic states
- The gap  $2\Delta$  prevents level crossing (quantum mechanical effect)
- Smaller gap  $\Rightarrow$  easier to violate adiabaticity
- Without the  $-\Delta\sigma_x$  term, levels would cross diabatically

## Critical Insight

The adiabatic condition is hardest to satisfy near the **minimum gap**. This is where non-adiabatic transitions (Landau-Zener tunneling) are most likely to occur.

For a cyclic adiabatic evolution, where the Hamiltonian returns to its initial form:

$$\hat{H}(R(T)) = \hat{H}(R(0)) \quad (25)$$

The total phase acquired is:

$$e^{i\phi} = e^{i\theta_{\text{dyn}}} e^{i\gamma_n} \quad (26)$$

## Berry Phase (1984)

$$\gamma_n = i \oint_{\mathcal{C}} \langle n(R) | \nabla_R n(R) \rangle \cdot dR \quad (27)$$

where  $\mathcal{C}$  is the closed path in parameter space.

### Properties:

- Gauge invariant (physical observable)
- Geometric: depends only on path, not speed
- Can be written as flux of Berry curvature

## Berry Connection (Vector Potential)

$$\vec{A}_n(R) = i \langle n(R) | \nabla_R n(R) \rangle \quad (28)$$

## Berry Curvature (Field Strength)

$$\vec{F}_n(R) = \nabla_R \times \vec{A}_n(R) \quad (29)$$

By Stokes' theorem:

$$\gamma_n = \oint_C \vec{A}_n \cdot dR = \iint_S \vec{F}_n \cdot d\vec{S} \quad (30)$$

**Analogy:** Berry phase is like the Aharonov-Bohm phase, but in parameter space rather than real space!

When adiabaticity breaks down, transitions occur between energy levels.

## Landau-Zener Formula

For a linear crossing:  $E_1(t) = \alpha t$ ,  $E_2(t) = -\alpha t + \Delta$

Transition probability:

$$P_{1 \rightarrow 2} = \exp\left(-\frac{\pi\Delta^2}{2\hbar\alpha}\right) \quad (31)$$

- › Small gap  $\Delta \Rightarrow$  high transition probability
- › Fast sweep (large  $\alpha$ )  $\Rightarrow$  high transition probability
- › Slow sweep (small  $\alpha$ )  $\Rightarrow$  adiabatic regime,  $P \rightarrow 0$

We can systematically expand in the adiabatic parameter  $\epsilon = \tau_{\text{dyn}}/T$ :

$$|\psi(t)\rangle = e^{i\theta_n(t)} \left[ |n(t)\rangle + \epsilon |\psi_n^{(1)}(t)\rangle + \epsilon^2 |\psi_n^{(2)}(t)\rangle + \dots \right] \quad (32)$$

First-order correction:

$$|\psi_n^{(1)}(t)\rangle = \sum_{m \neq n} \frac{\langle m | \dot{\hat{H}} | n \rangle}{(E_n - E_m)^2} |m\rangle \quad (33)$$

This shows:

- › Transition amplitude  $\propto 1/(E_n - E_m)^2$
- › Small gaps make adiabatic evolution difficult
- › Can estimate fidelity:  $|\langle n(t) | \psi(t) \rangle|^2 \approx 1 - \mathcal{O}(\epsilon^2)$

When eigenvalues are degenerate, the theorem must be modified:

- › System can transition within degenerate subspace
- › Need to consider the entire degenerate subspace
- › Berry phase becomes a **matrix** (non-Abelian Berry connection)

## Wilczek-Zee Connection

For degenerate subspace with basis  $\{|n_a(R)\rangle\}$ :

$$(A_i)_{ab} = i \langle n_a | \partial_i n_b \rangle \quad (34)$$

This is a matrix-valued gauge potential.

Leads to non-Abelian geometric phases (important in topological quantum computation).

# Discussion: Diabatic vs Adiabatic Evolution

## Diabatic Evolution:

- Fast changes in  $\hat{H}(t)$
- System doesn't track instantaneous eigenstates
- Transitions between levels
- Example: Sudden approximation

## When applicable:

$$T \ll \frac{\hbar}{\Delta E} \quad (35)$$

## Adiabatic Evolution:

- Slow changes in  $\hat{H}(t)$
- System follows instantaneous eigenstate
- No transitions (to first order)
- Example: Born-Oppenheimer

## When applicable:

$$T \gg \frac{\hbar}{\Delta E} \quad (36)$$

## Discussion Question

In between these limits, how do we calculate the transition probability?

**Answer:** Landau-Zener formula, perturbation theory, or numerical methods

Fintan's slides 5 minutes allocated here.

# Application: Adiabatic Quantum Computation

## Basic Idea

- 1 Encode problem in Hamiltonian  $\hat{H}_{\text{problem}}$  whose ground state is the solution
- 2 Start with simple Hamiltonian  $\hat{H}_{\text{initial}}$  with known ground state
- 3 Slowly interpolate:  $\hat{H}(t) = (1 - s(t))\hat{H}_{\text{initial}} + s(t)\hat{H}_{\text{problem}}$
- 4 If evolution is adiabatic, system remains in ground state
- 5 At  $t = T$ : measure to obtain solution

**Runtime:**  $T \sim \mathcal{O}(\hbar/\Delta_{\min}^2)$  where  $\Delta_{\min}$  is minimum gap

**Challenge:** Gap can be exponentially small for some problems!

**Goal:** Solve optimization problems using quantum adiabatic evolution

## Standard Approach

- 1 Encode optimization problem in a Hamiltonian
- 2 Prepare system in ground state of simple initial Hamiltonian
- 3 Adiabatically evolve to problem Hamiltonian
- 4 Measure final state to read out solution

## Two Main Formulations:

- **Ising Model:** Spin variables  $s_i \in \{-1, +1\}$
- **QUBO:** Binary variables  $x_i \in \{0, 1\}$
- These are equivalent and interconvertible

**Examples:** Graph coloring, traveling salesman, portfolio optimization, protein folding

## Classical Ising Hamiltonian

$$H_{\text{Ising}} = \sum_{i < j} J_{ij} s_i s_j + \sum_i h_i s_i \quad (37)$$

where  $s_i \in \{-1, +1\}$  are spin variables (Pauli-z eigenvalues)

**Parameters:**

- $J_{ij}$ : Coupling between spins  $i$  and  $j$ 
  - »  $J_{ij} > 0$ : Ferromagnetic (spins want to align)
  - »  $J_{ij} < 0$ : Antiferromagnetic (spins want to anti-align)
- $h_i$ : Local field on spin  $i$  (bias)

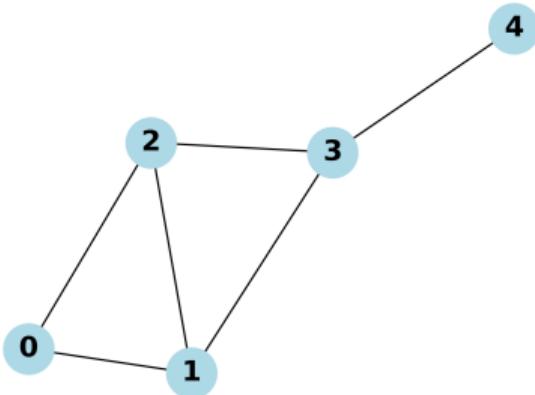
**Quantum Version:**

$$\hat{H}_{\text{Ising}} = \sum_{i < j} J_{ij} \sigma_i^z \sigma_j^z + \sum_i h_i \sigma_i^z \quad (38)$$

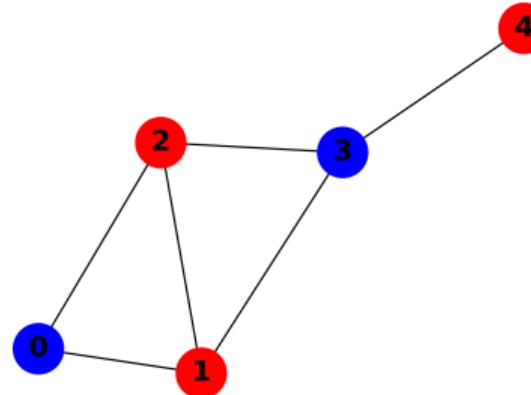
This is the **problem Hamiltonian** in quantum annealing!

# Visualizing the Ising Model: Max-Cut Example

Example Graph for Max-Cut

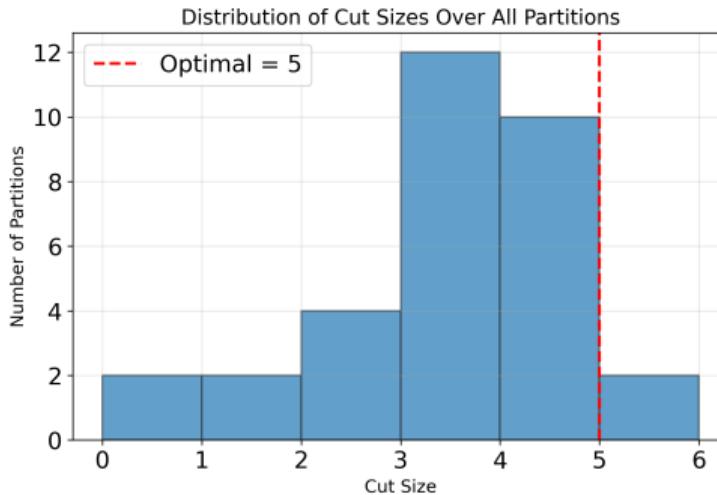


Optimal Max-Cut (size = 5)



- **Left:** Original graph with 5 vertices and 6 edges
- **Right:** Optimal partition (red vs blue) achieves cut size = 5
- Only 1 edge (out of 6) remains within a partition
- Ising formulation: Antiferromagnetic couplings ( $J_{ij} = 1$ ) favor opposite spins

# Energy Landscape: Solution Distribution



- Out of  $2^5 = 32$  possible partitions, only 2 achieve the optimal cut size
- Most partitions achieve sub-optimal cuts (sizes 3-4)
- **Challenge for quantum annealing:** Find the needle in the haystack!
- Energy landscape roughness determines difficulty

# QUBO: Quadratic Unconstrained Binary Optimization

## QUBO Formulation

Minimize:

$$f(x) = \sum_i Q_{ii}x_i + \sum_{i < j} Q_{ij}x_i x_j = x^T Q x \quad (39)$$

where  $x_i \in \{0, 1\}$  and  $Q$  is the QUBO matrix

## Why QUBO?

- Many combinatorial optimization problems naturally expressed as QUBO
- NP-hard in general (finding global minimum)
- Can encode constraints via penalty terms

## Examples:

- Max-Cut:  $\max \sum_{(i,j) \in E} (1 - x_i x_j)$
- Number Partitioning:  $\min (\sum_i a_i x_i)^2$
- Graph Coloring: Use penalty for adjacent same-colored vertices

The mapping is straightforward:

## QUBO → Ising

Variable transformation:  $x_i = \frac{1-s_i}{2}$  where  $s_i \in \{-1, +1\}$

Then:  $x_i x_j = \frac{(1-s_i)(1-s_j)}{4} = \frac{1-s_i-s_j+s_i s_j}{4}$

## Ising → QUBO

Inverse transformation:  $s_i = 1 - 2x_i$  where  $x_i \in \{0, 1\}$

Then:  $s_i s_j = (1 - 2x_i)(1 - 2x_j) = 1 - 2x_i - 2x_j + 4x_i x_j$

**Key Point:** The two formulations are equivalent - choose based on:

- Natural problem representation
- Hardware constraints (quantum annealers often use Ising)
- Software/solver availability

## Standard Quantum Annealing Hamiltonian:

$$\hat{H}(t) = A(t)\hat{H}_{\text{driver}} + B(t)\hat{H}_{\text{problem}} \quad (40)$$

where:

- $\hat{H}_{\text{driver}} = - \sum_i \sigma_i^x$  (transverse field - creates superposition)
- $\hat{H}_{\text{problem}} = \sum_{i < j} J_{ij} \sigma_i^z \sigma_j^z + \sum_i h_i \sigma_i^z$  (Ising model)
- $A(t), B(t)$  are annealing schedules with  $A(0) \gg B(0)$  and  $A(T) \ll B(T)$

## Standard Linear Schedule:

$$A(t) = A_0 \left(1 - \frac{t}{T}\right) \quad (41)$$

$$B(t) = B_0 \frac{t}{T} \quad (42)$$

At  $t = 0$ : Easy ground state  $|+\rangle^{\otimes N}$  (all spins in  $+x$  direction)

At  $t = T$ : Ground state encodes solution to optimization problem

## Example: Max-Cut Problem

**Problem:** Given graph  $G = (V, E)$ , partition vertices into two sets to maximize edges between sets.

### QUBO Formulation:

Maximize:

$$f(x) = \sum_{(i,j) \in E} (x_i - x_j)^2 \quad (43)$$

Expanding:

$$= \sum_{(i,j) \in E} (x_i + x_j - 2x_i x_j) \quad (44)$$

For minimization (quantum annealer):

$$H_{\text{QUBO}} = - \sum_{(i,j) \in E} (x_i + x_j - 2x_i x_j) \quad (45)$$

### Ising Formulation:

Using  $s_i = 1 - 2x_i$ :

$$H_{\text{Ising}} = - \sum_{(i,j) \in E} (1 - s_i s_j) \quad (46)$$

Simplifies to:

$$H_{\text{Ising}} = \sum_{(i,j) \in E} s_i s_j + \text{const} \quad (47)$$

### Interpretation:

- $s_i = +1$ : vertex in set A
- $s_i = -1$ : vertex in set B
- Minimize  $\Rightarrow$  maximize cut

# Adiabatic Theorem in Quantum Annealing

Success condition based on adiabatic theorem:

## Required Annealing Time

$$T \gg \frac{\hbar}{\Delta_{\min}^2} \quad (48)$$

where  $\Delta_{\min}$  is the minimum energy gap during evolution

## Problem-Dependent Gap:

- Different problems  $\Rightarrow$  different gap structures
- Gap often smallest near  $t \approx T/2$  (mid-anneal)
- For many NP-hard problems:  $\Delta_{\min} \sim e^{-\alpha N}$  (exponentially small!)

## Practical Implications:

- + Easy problems: Fast annealing possible (microseconds)
- Hard problems: May need exponentially long annealing time
- ± Thermal effects can help/hinder (quantum vs simulated annealing)

# Discussion: Quantum Annealing vs Gate Model

## Quantum Annealing:

- + Analog, continuous evolution
- + Natural for optimization
- + Potentially more robust to some errors
- + Easier to scale (1000s of qubits)
- Limited to specific problem types
- No quantum error correction
- Success depends on gap structure

## Gate-Based Quantum Computing:

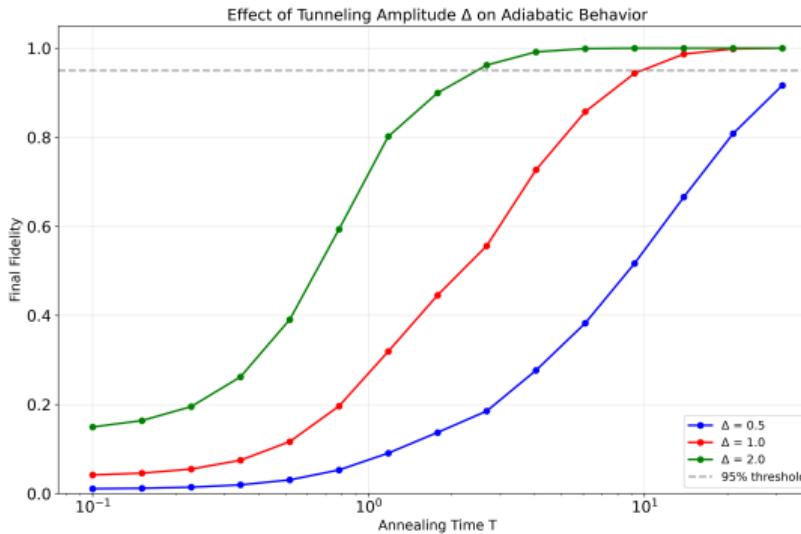
- + Universal quantum computation
- + Quantum error correction possible
- + Well-defined circuit depth
- + Can implement any algorithm
- Currently limited qubits (<1000)
- High gate error rates
- Requires precise control

## Open Question

Does quantum annealing provide speedup over classical algorithms?

Evidence mixed - depends on problem structure, noise, and implementation.

# Parameter Dependence: Effect of Tunneling Amplitude $\Delta$



- **Larger  $\Delta$ :** Larger gap  $\Rightarrow$  longer time needed for 95% fidelity
- **Smaller  $\Delta$ :** Smaller gap  $\Rightarrow$  easier to satisfy adiabatic condition
- Critical time scale:  $T_{\text{critical}} \propto 1/\Delta^2$

# Discussion: Scaling and the Adiabatic Theorem

## Discussion Points

- 1 Why does larger  $\Delta$  require *longer* evolution time?
- 2 How does this relate to the adiabatic condition  $T \gg \hbar/\Delta E$ ?
- 3 What is the trade-off between gap size and computation time?

## Counterintuitive Result:

- Larger gap  $\Delta$  = more separation between energy levels
- BUT: Also means stronger coupling between diabatic states
- Stronger coupling  $\Rightarrow$  harder to maintain adiabatic evolution
- Mathematical:  $|\langle m|\dot{n} \rangle| \propto \Delta$  but gap  $\propto \Delta$

## Key Scaling

Landau-Zener transition probability:  $P_{\text{LZ}} = \exp\left(-\frac{\pi\Delta^2}{2\hbar v}\right)$

where  $v = |\dot{\epsilon}|$  is the sweep rate. Adiabatic limit:  $v \rightarrow 0 \Rightarrow P_{\text{LZ}} \rightarrow 0$

# Experimental Realization and Fidelity

## Experimental Platforms:

- Superconducting qubits
- Trapped ions
- Cold atoms
- Nitrogen-vacancy centers
- Quantum dots

## Typical Parameters:

- $\Delta/h \sim 0.1\text{-}10 \text{ GHz}$
- $T \sim 1\text{-}1000 \text{ ns}$
- Fidelity  $> 99\%$  achievable

## Challenges:

- Decoherence ( $T_1, T_2$ )
- Control noise
- Calibration errors
- Finite temperature

## Fidelity Metric

$$F = |\langle \psi_{\text{target}} | \psi_{\text{final}} \rangle|^2 \quad (49)$$

Measures overlap with desired state

**Discussion:** How do environmental effects modify the adiabatic theorem?

Gap can be exponentially small for some problems!

# Open Questions and Research Frontiers

1 **Many-body systems:** How does the adiabatic theorem scale with system size?

- » Ground state degeneracy
- » Exponentially small gaps
- » Quantum phase transitions

2 **Decoherence:** How does noise affect adiabatic evolution?

- » Open quantum systems
- » Markovian vs non-Markovian dynamics
- » Error correction strategies

3 **Shortcuts to adiabaticity:** Can we achieve adiabatic-like results faster?

- » Counter-diabatic driving
- » Optimal control theory
- » Fast-forward protocols

4 **Quantum annealing:** What problems can quantum annealers solve?

- » Quantum advantage vs classical algorithms
- » Role of thermal fluctuations
- » Embedding optimization problems

## Mathematical Rigor

The adiabatic theorem is not just a hand-waving argument:

- Precise conditions on energy gaps and rates of change
- Systematic perturbation theory for corrections
- Geometric interpretation via Berry connection and curvature

## Physical Consequences

- Quantum systems can "remember" their path through parameter space
- Topological effects in quantum mechanics
- Foundation for quantum computation paradigm
- Explains robustness of certain quantum states

# Numerical Verification: What We Observed

From our Landau-Zener simulations:

- 1 **Critical Time Scale:** For  $\Delta = 1.0$ , achieve 95% fidelity at  $T \approx 10\hbar$ 
  - » Consistent with  $T \gg \hbar/(2\Delta) = 0.5\hbar$
- 2 **Scaling with  $\Delta$ :**
  - »  $\Delta = 0.5$ :  $T_{95\%} \approx 3\hbar$  (easier)
  - »  $\Delta = 1.0$ :  $T_{95\%} \approx 10\hbar$  (baseline)
  - »  $\Delta = 2.0$ :  $T_{95\%} \approx 30\hbar$  (harder)
  - » Approximately:  $T_{95\%} \propto \Delta^2$
- 3 **Non-adiabatic regime:** For  $T < 1\hbar$ , fidelity drops dramatically
  - » Landau-Zener transitions dominate
  - » Exponential suppression with  $T$

## Key Takeaway

The adiabatic theorem is not just qualitative - it provides quantitative predictions that match numerical simulations!

## Questions for Consideration

- 1 For quantum computing, is longer evolution time always acceptable?
- 2 How do we balance fidelity vs speed in real applications?
- 3 What role does the adiabatic theorem play in current quantum technologies?

## Real-World Trade-offs:

### Slower Evolution:

- + Higher fidelity
- + Better adiabaticity
- More decoherence
- Longer computation time

### Faster Evolution:

- + Less decoherence
- + Faster results
- Lower fidelity
- Non-adiabatic errors

**Optimal Strategy:** Find the "sweet spot" where  $T$  is long enough for adiabaticity but short enough to avoid decoherence. This is an active area of research!

# Thank You!

Questions?

## Appendix: Proof of $\langle n|\dot{n} \rangle$ is Purely Imaginary

---

From normalization:  $\langle n|n \rangle = 1$

Differentiate with respect to time:

$$\frac{d}{dt} \langle n|n \rangle = \langle \dot{n}|n \rangle + \langle n|\dot{n} \rangle = 0 \quad (50)$$

Therefore:

$$\langle n|\dot{n} \rangle = -\langle \dot{n}|n \rangle = -\langle n|\dot{n} \rangle^* \quad (51)$$

This implies  $\langle n|\dot{n} \rangle$  is purely imaginary.

We can write:  $\langle n|\dot{n} \rangle = i\gamma_n(t)$  where  $\gamma_n(t)$  is real.

## Appendix: Derivation of Berry Curvature

---

Starting from Berry connection:

$$A_i = i \langle n | \partial_i n \rangle \quad (52)$$

The Berry curvature is:

$$F_{ij} = \partial_i A_j - \partial_j A_i \quad (53)$$

Expanding:

$$F_{ij} = i [\partial_i \langle n | \partial_j n \rangle - \partial_j \langle n | \partial_i n \rangle] \quad (54)$$

$$= i [\langle \partial_i n | \partial_j n \rangle + \langle n | \partial_i \partial_j n \rangle - \langle \partial_j n | \partial_i n \rangle - \langle n | \partial_j \partial_i n \rangle] \quad (55)$$

Since  $\partial_i \partial_j = \partial_j \partial_i$ :

$$F_{ij} = i [\langle \partial_i n | \partial_j n \rangle - \langle \partial_j n | \partial_i n \rangle] \quad (56)$$

## Appendix: Completeness Relation

Using completeness:  $\sum_m |m\rangle \langle m| = \mathbb{I}$

We can write:

$$|\partial_i n\rangle = \sum_m |m\rangle \langle m| \partial_i n \rangle \quad (57)$$

Therefore:

$$\langle \partial_i n | \partial_j n \rangle = \sum_{m,k} \langle \partial_i n | k \rangle \langle k | m \rangle \langle m | \partial_j n \rangle \quad (58)$$

$$= \sum_m \langle \partial_i n | m \rangle \langle m | \partial_j n \rangle \quad (59)$$

The Berry curvature becomes:

$$F_{ij} = i \sum_{m \neq n} [\langle \partial_i n | m \rangle \langle m | \partial_j n \rangle - \langle \partial_j n | m \rangle \langle m | \partial_i n \rangle] \quad (60)$$