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Applied Quantum Mechanics

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Adiabatic Theorem and Quantum Annealing

A Mathematically Rigorous Treatment

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What is the Adiabatic Theorem?

Physical Statement

If a quantum system starts in an eigenstate of a time-dependent Hamiltonian, and if the Hamiltonian changes **sufficiently slowly**, the system will remain in the corresponding instantaneous eigenstate throughout the evolution.

Slow Change

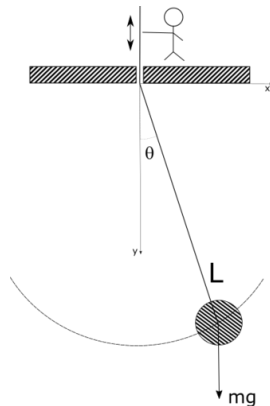
- Key word: “**sufficiently slowly**”
- The system acquires a **dynamical phase** and a **geometric phase** (Berry phase)
- Fundamental to quantum computing (adiabatic quantum computation)
- Applies to quantum chemistry, biology / condensed matter physics

Abrupt Change

- First formulated by Max Born and Vladimir Fock (1928)
- Further developed by Kato (1950)
- Geometric phase discovered by Berry (1984)
- Modern applications in quantum computing and quantum annealing

Classical Analogy

A pendulum whose length changes slowly will maintain its oscillation mode, but if changed rapidly, it will exhibit chaotic behavior.



Time-Dependent Schrödinger Equation

Consider a time-dependent Hamiltonian $\hat{H}(t)$:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle \quad (1)$$

Instantaneous Eigenstates

At each time t , we can solve the eigenvalue problem:

$$\hat{H}(t) |n(t)\rangle = E_n(t) |n(t)\rangle \quad (2)$$

where $|n(t)\rangle$ are the **instantaneous eigenstates** and $E_n(t)$ are the **instantaneous eigenvalues**.

Note: These are *not* solutions to the time-dependent Schrödinger equation, but rather eigenstates at fixed time t .

Substituting into Schrödinger Equation

Substitute the expansion $|\psi(t)\rangle = \sum_n c_n(t) e^{i\theta_n(t)} |n(t)\rangle$ into the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle$$
$$i\hbar \sum_n \left[\dot{c}_n e^{i\theta_n} |n\rangle + c_n i\dot{\theta}_n e^{i\theta_n} |n\rangle + c_n e^{i\theta_n} |\dot{n}\rangle \right] = \sum_n c_n e^{i\theta_n} E_n(t) |n\rangle \quad (3)$$

where dots denote time derivatives: $\dot{c}_n = \frac{dc_n}{dt}$, $|\dot{n}\rangle = \frac{d}{dt} |n(t)\rangle$.

Dividing by $i\hbar$:

$$\sum_n \left[\frac{\dot{c}_n}{i\hbar} e^{i\theta_n} |n\rangle - c_n \dot{\theta}_n e^{i\theta_n} |n\rangle + \frac{c_n}{i\hbar} e^{i\theta_n} |\dot{n}\rangle \right] = \sum_n c_n e^{i\theta_n} \frac{E_n}{\hbar} |n\rangle \quad (4)$$

Take the inner product with $\langle m(t)|$:

$$\frac{\dot{c}_m}{i\hbar} e^{i\theta_m} - c_m \dot{\theta}_m e^{i\theta_m} + \frac{c_m}{i\hbar} e^{i\theta_m} \langle m|\dot{n}\rangle = \sum_{n \neq m} \frac{c_n}{i\hbar} e^{i\theta_n} \langle m|\dot{n}\rangle \quad (5)$$

We used orthonormality: $\langle m|n\rangle = \delta_{mn}$ and $\sum_{n \neq m} \langle m|n\rangle = 0$.

Rearranging:

$$\dot{c}_m e^{i\theta_m} = c_m e^{i\theta_m} \left[\dot{\theta}_m - \frac{E_m}{i\hbar} - \langle m|\dot{n}\rangle \right] - \sum_{n \neq m} c_n e^{i\theta_n} \langle m|\dot{n}\rangle \quad (6)$$

Simplifying Terms / Choice of Dynamical Phase

Dynamical Phase

$$\theta_m(t) = -\frac{1}{\hbar} \int_0^t E_m(t') dt' + \gamma_m(t) \quad (7)$$

where $\gamma_m(t)$ is the **geometric phase** (Berry phase):

Geometric Phase

$$\gamma_m(t) = i \int_0^t \langle m(t') | \dot{m}(t') \rangle dt' \quad (8)$$

Then: $\dot{\theta}_m = -\frac{E_m}{\hbar} + i \langle m | \dot{m} \rangle$

And our equation becomes:

$$\dot{c}_m e^{i\theta_m} = - \sum_{n \neq m} c_n e^{i\theta_n} \langle m | \dot{n} \rangle \quad (9)$$

From the eigenvalue equation $\hat{H}(t) |n(t)\rangle = E_n(t) |n(t)\rangle$, differentiate with respect to t :

$$\dot{\hat{H}} |n\rangle + \hat{H} |\dot{n}\rangle = \dot{E}_n |n\rangle + E_n |\dot{n}\rangle \quad (10)$$

Take inner product with $\langle m|$ (where $m \neq n$):

$$\langle m|\dot{\hat{H}}|n\rangle + \langle m|\hat{H}|\dot{n}\rangle = \dot{E}_n \langle m|n\rangle + E_n \langle m|\dot{n}\rangle \quad (11)$$

Since $\langle m|\hat{H} = E_m \langle m|$ and $\langle m|n\rangle = 0$ for $m \neq n$:

$$\langle m|\dot{\hat{H}}|n\rangle + E_m \langle m|\dot{n}\rangle = E_n \langle m|\dot{n}\rangle \quad (12)$$

Therefore:

Matrix Element

$$\langle m|\dot{n}\rangle = \frac{\langle m|\dot{\hat{H}}|n\rangle}{E_n - E_m} \quad (m \neq n) \quad (13)$$

The Differential Equation for Coefficients

Substituting back:

$$\dot{c}_m = - \sum_{n \neq m} c_n e^{i(\theta_n - \theta_m)} \frac{\langle m | \dot{\hat{H}} | n \rangle}{E_n - E_m} \quad (14)$$

The phase difference is:

$$\theta_n - \theta_m = -\frac{1}{\hbar} \int_0^t [E_n(t') - E_m(t')] dt' + [\gamma_n(t) - \gamma_m(t)] \quad (15)$$

Key Observation

The right-hand side contains:

- › $\dot{\hat{H}}$ - rate of change of Hamiltonian
- › $E_n - E_m$ - energy gap (in denominator)
- › Rapidly oscillating phase factor

For the system to remain in state $|m\rangle$, we need $\dot{c}_m \approx 0$ for m equal to the initial state.

Adiabatic Condition

The Hamiltonian changes slowly enough that:

$$\left| \frac{\langle m | \dot{H} | n \rangle}{E_n - E_m} \right| \ll \frac{|E_n - E_m|}{\hbar} \quad (16)$$

for all $n \neq m$.

This can be rewritten as:

$$\left| \langle m | \dot{H} | n \rangle \right| \ll \frac{(E_n - E_m)^2}{\hbar} \quad (17)$$

Physical Interpretation: The time scale of Hamiltonian variation must be much larger than $\hbar/\Delta E$, where ΔE is the relevant energy gap.

Theorem (Quantum Adiabatic Theorem)

Let $\hat{H}(t)$ be a time-dependent Hamiltonian with instantaneous eigenstates $|n(t)\rangle$ and eigenvalues $E_n(t)$. Suppose:

- 1 The system starts in eigenstate $|n(0)\rangle$: $|\psi(0)\rangle = |n(0)\rangle$
- 2 The eigenvalues are non-degenerate
- 3 The adiabatic condition is satisfied for all $t \in [0, T]$

Then the state at time t is:

$$|\psi(t)\rangle = e^{i\theta_n(t)} |n(t)\rangle + \mathcal{O}(\epsilon) \quad (18)$$

where ϵ characterizes the adiabatic parameter and

$$\theta_n(t) = -\frac{1}{\hbar} \int_0^t E_n(t') dt' + i \int_0^t \langle n(t') | \partial_{t'} n(t') \rangle dt' \quad (19)$$

Shane's slides 5 minutes allocated here.

Example 1: Spin-1/2 in Rotating Magnetic Field

Consider a spin-1/2 particle in a magnetic field that rotates slowly:

$$\vec{B}(t) = B_0(\sin \theta \cos \omega t, \sin \theta \sin \omega t, \cos \theta) \quad (20)$$

The Hamiltonian is:

$$\hat{H}(t) = -\gamma \vec{B}(t) \cdot \vec{\sigma} = -\gamma B_0 \begin{pmatrix} \cos \theta & \sin \theta e^{-i\omega t} \\ \sin \theta e^{i\omega t} & -\cos \theta \end{pmatrix} \quad (21)$$

Instantaneous eigenvalues: $E_{\pm} = \mp \gamma B_0$

If the spin starts aligned with $\vec{B}(0)$ and ω is small enough:

- The spin remains aligned with $\vec{B}(t)$ (adiabatic following)
- After one full rotation ($t = 2\pi/\omega$), acquires Berry phase: $\gamma = \pi(1 - \cos \theta)$

Example 2: Two-Level System (Landau-Zener Model)

Consider the Landau-Zener Hamiltonian:

$$\hat{H}(t) = -\Delta\sigma_x - \epsilon(t)\sigma_z \quad (22)$$

where $\epsilon(t)$ is swept linearly from $\epsilon_i < 0$ to $\epsilon_f > 0$.

Energy eigenvalues:

$$E_{\pm}(t) = \pm\sqrt{\Delta^2 + \epsilon^2(t)} \quad (23)$$

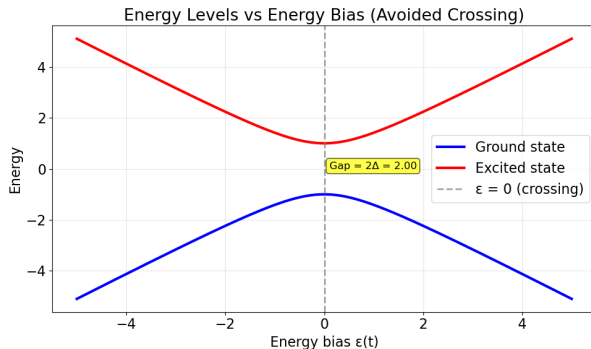
Energy gap at crossing ($\epsilon = 0$): $\Delta E_{\min} = 2\Delta$

Adiabatic Condition

$$T \gg \frac{\hbar}{\Delta E_{\min}} = \frac{\hbar}{2\Delta} \quad \text{and} \quad |\dot{\epsilon}| \ll \frac{4\Delta^2}{\hbar} \quad (24)$$

This is the canonical model for avoided crossings and adiabatic transitions!

Landau-Zener Model: Avoided Crossing Visualization



- The energy levels exhibit an **avoided crossing** at $\epsilon(t) = 0$
- Minimum gap: $\Delta E_{\min} = 2\Delta$ determines adiabatic time scale

Discussion: Avoided Crossing Physics

Key Questions

- 1 What happens at the avoided crossing point?
- 2 Why is this the bottleneck for adiabatic evolution?
- 3 How does the gap size affect the required evolution time?

Physical Insights:

- › At $\epsilon = 0$: The eigenstates are equal superpositions of diabatic states
- › The gap 2Δ prevents level crossing (quantum mechanical effect)
- › Smaller gap \Rightarrow easier to violate adiabaticity
- › Without the $-\Delta\sigma_x$ term, levels would cross diabatically

Critical Insight

The adiabatic condition is hardest to satisfy near the **minimum gap**. This is where non-adiabatic transitions (Landau-Zener tunneling) are most likely to occur.

For a cyclic adiabatic evolution, where the Hamiltonian returns to its initial form:

$$\hat{H}(R(T)) = \hat{H}(R(0)) \quad (25)$$

The total phase acquired is:

$$e^{i\phi} = e^{i\theta_{\text{dyn}}} e^{i\gamma_n} \quad (26)$$

Berry Phase (1984)

$$\gamma_n = i \oint_{\mathcal{C}} \langle n(R) | \nabla_R n(R) \rangle \cdot dR \quad (27)$$

where \mathcal{C} is the closed path in parameter space.

Properties:

- Gauge invariant (physical observable)
- Geometric: depends only on path, not speed
- Can be written as flux of Berry curvature

Berry Connection (Vector Potential)

$$\vec{A}_n(R) = i \langle n(R) | \nabla_R n(R) \rangle \quad (28)$$

Berry Curvature (Field Strength)

$$\vec{F}_n(R) = \nabla_R \times \vec{A}_n(R) \quad (29)$$

By Stokes' theorem:

$$\gamma_n = \oint_{\mathcal{C}} \vec{A}_n \cdot dR = \iint_S \vec{F}_n \cdot d\vec{S} \quad (30)$$

Analogy: Berry phase is like the Aharonov-Bohm phase, but in parameter space rather than real space!

When adiabaticity breaks down, transitions occur between energy levels.

Landau-Zener Formula

For a linear crossing: $E_1(t) = \alpha t$, $E_2(t) = -\alpha t + \Delta$

Transition probability:

$$P_{1 \rightarrow 2} = \exp\left(-\frac{\pi \Delta^2}{2\hbar\alpha}\right) \quad (31)$$

- › Small gap $\Delta \Rightarrow$ high transition probability
- › Fast sweep (large α) \Rightarrow high transition probability
- › Slow sweep (small α) \Rightarrow adiabatic regime, $P \rightarrow 0$

We can systematically expand in the adiabatic parameter $\epsilon = \tau_{\text{dyn}}/T$:

$$|\psi(t)\rangle = e^{i\theta_n(t)} \left[|n(t)\rangle + \epsilon |\psi_n^{(1)}(t)\rangle + \epsilon^2 |\psi_n^{(2)}(t)\rangle + \dots \right] \quad (32)$$

First-order correction:

$$|\psi_n^{(1)}(t)\rangle = \sum_{m \neq n} \frac{\langle m | \dot{H} | n \rangle}{(E_n - E_m)^2} |m\rangle \quad (33)$$

This shows:

- Transition amplitude $\propto 1/(E_n - E_m)^2$
- Small gaps make adiabatic evolution difficult
- Can estimate fidelity: $|\langle n(t) | \psi(t) \rangle|^2 \approx 1 - \mathcal{O}(\epsilon^2)$

When eigenvalues are degenerate, the theorem must be modified:

- › System can transition within degenerate subspace
- › Need to consider the entire degenerate subspace
- › Berry phase becomes a **matrix** (non-Abelian Berry connection)

Wilczek-Zee Connection

For degenerate subspace with basis $\{|n_a(R)\rangle\}$:

$$(A_i)_{ab} = i \langle n_a | \partial_i n_b \rangle \quad (34)$$

This is a matrix-valued gauge potential.

Leads to non-Abelian geometric phases (important in topological quantum computation).

Discussion: Diabatic vs Adiabatic Evolution

Diabatic Evolution:

- Fast changes in $\hat{H}(t)$
- System doesn't track instantaneous eigenstates
- Transitions between levels
- Example: Sudden approximation

When applicable:

$$T \ll \frac{\hbar}{\Delta E} \quad (35)$$

Adiabatic Evolution:

- Slow changes in $\hat{H}(t)$
- System follows instantaneous eigenstate
- No transitions (to first order)
- Example: Born-Oppenheimer

When applicable:

$$T \gg \frac{\hbar}{\Delta E} \quad (36)$$

Discussion Question

In between these limits, how do we calculate the transition probability?

Answer: Landau-Zener formula, perturbation theory, or numerical methods

Fintan's slides 5 minutes allocated here.

Application: Adiabatic Quantum Computation

Basic Idea

- 1 Encode problem in Hamiltonian \hat{H}_{problem} whose ground state is the solution
- 2 Start with simple Hamiltonian \hat{H}_{initial} with known ground state
- 3 Slowly interpolate: $\hat{H}(t) = (1 - s(t))\hat{H}_{\text{initial}} + s(t)\hat{H}_{\text{problem}}$
- 4 If evolution is adiabatic, system remains in ground state
- 5 At $t = T$: measure to obtain solution

Runtime: $T \sim \mathcal{O}(\hbar/\Delta_{\min}^2)$ where Δ_{\min} is minimum gap

Challenge: Gap can be exponentially small for some problems!

Quantum Annealing Framework

Goal: Solve optimization problems using quantum adiabatic evolution

Standard Approach

- 1 Encode optimization problem in a Hamiltonian
- 2 Prepare system in ground state of simple initial Hamiltonian
- 3 Adiabatically evolve to problem Hamiltonian
- 4 Measure final state to read out solution

Two Main Formulations:

- › **Ising Model:** Spin variables $s_i \in \{-1, +1\}$
- › **QUBO:** Binary variables $x_i \in \{0, 1\}$
- › These are equivalent and interconvertible

Examples: Graph coloring, traveling salesman, portfolio optimization, protein folding

Classical Ising Hamiltonian

$$H_{\text{Ising}} = \sum_{i < j} J_{ij} s_i s_j + \sum_i h_i s_i \quad (37)$$

where $s_i \in \{-1, +1\}$ are spin variables (Pauli-z eigenvalues)

Parameters:

- J_{ij} : Coupling between spins i and j
 - $J_{ij} > 0$: Ferromagnetic (spins want to align)
 - $J_{ij} < 0$: Antiferromagnetic (spins want to anti-align)
- h_i : Local field on spin i (bias)

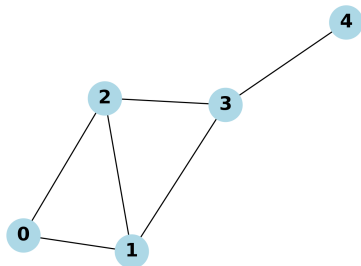
Quantum Version:

$$\hat{H}_{\text{Ising}} = \sum_{i < j} J_{ij} \sigma_i^z \sigma_j^z + \sum_i h_i \sigma_i^z \quad (38)$$

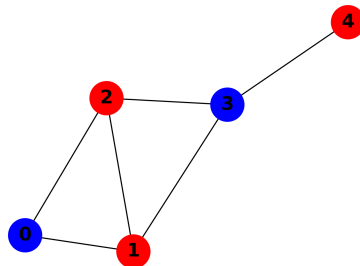
This is the **problem Hamiltonian** in quantum annealing!

Visualizing the Ising Model: Max-Cut Example

Example Graph for Max-Cut

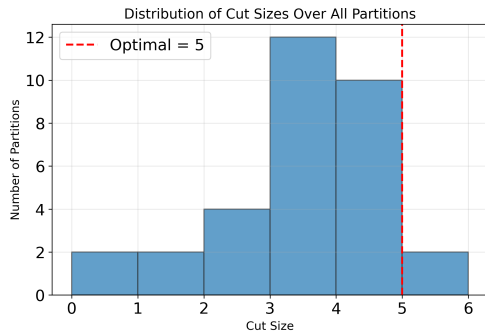


Optimal Max-Cut (size = 5)



- **Left:** Original graph with 5 vertices and 6 edges
- **Right:** Optimal partition (red vs blue) achieves cut size = 5
- Only 1 edge (out of 6) remains within a partition
- Ising formulation: Antiferromagnetic couplings ($J_{ij} = 1$) favor opposite spins

Energy Landscape: Solution Distribution



- › Out of $2^5 = 32$ possible partitions, only 2 achieve the optimal cut size
- › Most partitions achieve sub-optimal cuts (sizes 3-4)
- › **Challenge for quantum annealing:** Find the needle in the haystack!
- › Energy landscape roughness determines difficulty

QUBO: Quadratic Unconstrained Binary Optimization

QUBO Formulation

Minimize:

$$f(x) = \sum_i Q_{ii}x_i + \sum_{i<j} Q_{ij}x_ix_j = x^T Q x \quad (39)$$

where $x_i \in \{0, 1\}$ and Q is the QUBO matrix

Why QUBO?

- Many combinatorial optimization problems naturally expressed as QUBO
- NP-hard in general (finding global minimum)
- Can encode constraints via penalty terms

Examples:

- Max-Cut: $\max \sum_{(i,j) \in E} (1 - x_i x_j)$
- Number Partitioning: $\min (\sum_i a_i x_i)^2$
- Graph Coloring: Use penalty for adjacent same-colored vertices

The mapping is straightforward:

QUBO \rightarrow Ising

Variable transformation: $x_i = \frac{1-s_i}{2}$ where $s_i \in \{-1, +1\}$

$$\text{Then: } x_i x_j = \frac{(1-s_i)(1-s_j)}{4} = \frac{1-s_i-s_j+s_i s_j}{4}$$

Ising \rightarrow QUBO

Inverse transformation: $s_i = 1 - 2x_i$ where $x_i \in \{0, 1\}$

$$\text{Then: } s_i s_j = (1 - 2x_i)(1 - 2x_j) = 1 - 2x_i - 2x_j + 4x_i x_j$$

Key Point: The two formulations are equivalent - choose based on:

- › Natural problem representation
- › Hardware constraints (quantum annealers often use Ising)
- › Software/solver availability

Standard Quantum Annealing Hamiltonian:

$$\hat{H}(t) = A(t)\hat{H}_{\text{driver}} + B(t)\hat{H}_{\text{problem}} \quad (40)$$

where:

- $\hat{H}_{\text{driver}} = -\sum_i \sigma_i^x$ (transverse field - creates superposition)
- $\hat{H}_{\text{problem}} = \sum_{i<j} J_{ij}\sigma_i^z\sigma_j^z + \sum_i h_i\sigma_i^z$ (Ising model)
- $A(t), B(t)$ are annealing schedules with $A(0) \gg B(0)$ and $A(T) \ll B(T)$

Standard Linear Schedule:

$$A(t) = A_0 \left(1 - \frac{t}{T}\right) \quad (41)$$

$$B(t) = B_0 \frac{t}{T} \quad (42)$$

At $t = 0$: Easy ground state $|+\rangle^{\otimes N}$ (all spins in $+x$ direction)

At $t = T$: Ground state encodes solution to optimization problem

Example: Max-Cut Problem

Problem: Given graph $G = (V, E)$, partition vertices into two sets to maximize edges between sets.

QUBO Formulation:

Maximize:

$$f(x) = \sum_{(i,j) \in E} (x_i - x_j)^2 \quad (43)$$

Expanding:

$$= \sum_{(i,j) \in E} (x_i + x_j - 2x_i x_j) \quad (44)$$

For minimization (quantum annealer):

$$H_{\text{QUBO}} = - \sum_{(i,j) \in E} (x_i + x_j - 2x_i x_j) \quad (45)$$

Ising Formulation:

Using $s_i = 1 - 2x_i$:

$$H_{\text{Ising}} = - \sum_{(i,j) \in E} (1 - s_i s_j) \quad (46)$$

Simplifies to:

$$H_{\text{Ising}} = \sum_{(i,j) \in E} s_i s_j + \text{const} \quad (47)$$

Interpretation:

- $s_i = +1$: vertex in set A
- $s_i = -1$: vertex in set B
- Minimize \Rightarrow maximize cut

Adiabatic Theorem in Quantum Annealing

Success condition based on adiabatic theorem:

Required Annealing Time

$$T \gg \frac{\hbar}{\Delta_{\min}^2} \quad (48)$$

where Δ_{\min} is the minimum energy gap during evolution

Problem-Dependent Gap:

- Different problems \Rightarrow different gap structures
- Gap often smallest near $t \approx T/2$ (mid-anneal)
- For many NP-hard problems: $\Delta_{\min} \sim e^{-\alpha N}$ (exponentially small!)

Practical Implications:

- + Easy problems: Fast annealing possible (microseconds)
- Hard problems: May need exponentially long annealing time
- ± Thermal effects can help/hinder (quantum vs simulated annealing)

Discussion: Quantum Annealing vs Gate Model

Quantum Annealing:

- + Analog, continuous evolution
- + Natural for optimization
- + Potentially more robust to some errors
- + Easier to scale (1000s of qubits)
- Limited to specific problem types
- No quantum error correction
- Success depends on gap structure

Gate-Based Quantum Computing:

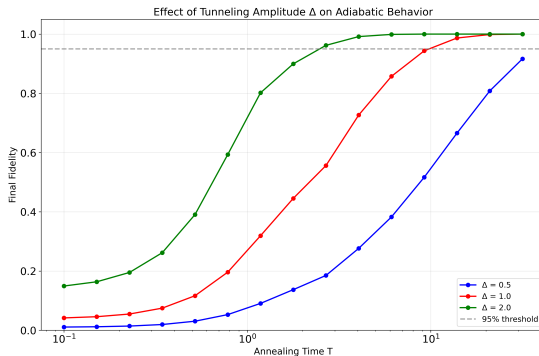
- + Universal quantum computation
- + Quantum error correction possible
- + Well-defined circuit depth
- + Can implement any algorithm
- Currently limited qubits (<1000)
- High gate error rates
- Requires precise control

Open Question

Does quantum annealing provide speedup over classical algorithms?

Evidence mixed - depends on problem structure, noise, and implementation.

Parameter Dependence: Effect of Tunneling Amplitude Δ



- **Larger Δ :** Larger gap \Rightarrow longer time needed for 95% fidelity
- **Smaller Δ :** Smaller gap \Rightarrow easier to satisfy adiabatic condition
- Critical time scale: $T_{\text{critical}} \propto 1/\Delta^2$

Discussion: Scaling and the Adiabatic Theorem

Discussion Points

- 1 Why does larger Δ require *longer* evolution time?
- 2 How does this relate to the adiabatic condition $T \gg \hbar/\Delta E$?
- 3 What is the trade-off between gap size and computation time?

Counterintuitive Result:

- Larger gap Δ = more separation between energy levels
- BUT: Also means stronger coupling between diabatic states
- Stronger coupling \Rightarrow harder to maintain adiabatic evolution
- Mathematical: $|\langle m|\dot{n}\rangle| \propto \Delta$ but gap $\propto \Delta$

Key Scaling

Landau-Zener transition probability: $P_{LZ} = \exp\left(-\frac{\pi\Delta^2}{2\hbar v}\right)$

where $v = |\dot{\epsilon}|$ is the sweep rate. Adiabatic limit: $v \rightarrow 0 \Rightarrow P_{LZ} \rightarrow 0$

Experimental Realization and Fidelity

Experimental Platforms:

- Superconducting qubits
- Trapped ions
- Cold atoms
- Nitrogen-vacancy centers
- Quantum dots

Typical Parameters:

- $\Delta/h \sim 0.1\text{-}10$ GHz
- $T \sim 1\text{-}1000$ ns
- Fidelity $> 99\%$ achievable

Challenges:

- Decoherence (T_1, T_2)
- Control noise
- Calibration errors
- Finite temperature

Fidelity Metric

$$F = |\langle \psi_{\text{target}} | \psi_{\text{final}} \rangle|^2 \quad (49)$$

Measures overlap with desired state

Discussion: How do environmental effects modify the adiabatic theorem?

Gap can be exponentially small for some problems!

- 1 **Many-body systems:** How does the adiabatic theorem scale with system size?
 - » Ground state degeneracy
 - » Exponentially small gaps
 - » Quantum phase transitions
- 2 **Decoherence:** How does noise affect adiabatic evolution?
 - » Open quantum systems
 - » Markovian vs non-Markovian dynamics
 - » Error correction strategies
- 3 **Shortcuts to adiabaticity:** Can we achieve adiabatic-like results faster?
 - » Counter-diabatic driving
 - » Optimal control theory
 - » Fast-forward protocols
- 4 **Quantum annealing:** What problems can quantum annealers solve?
 - » Quantum advantage vs classical algorithms
 - » Role of thermal fluctuations
 - » Embedding optimization problems

Mathematical Rigor

The adiabatic theorem is not just a hand-waving argument:

- › Precise conditions on energy gaps and rates of change
- › Systematic perturbation theory for corrections
- › Geometric interpretation via Berry connection and curvature

Physical Consequences

- › Quantum systems can "remember" their path through parameter space
- › Topological effects in quantum mechanics
- › Foundation for quantum computation paradigm
- › Explains robustness of certain quantum states

Numerical Verification: What We Observed

From our Landau-Zener simulations:

- 1 **Critical Time Scale:** For $\Delta = 1.0$, achieve 95% fidelity at $T \approx 10\hbar$
 - » Consistent with $T \gg \hbar/(2\Delta) = 0.5\hbar$
- 2 **Scaling with Δ :**
 - » $\Delta = 0.5$: $T_{95\%} \approx 3\hbar$ (easier)
 - » $\Delta = 1.0$: $T_{95\%} \approx 10\hbar$ (baseline)
 - » $\Delta = 2.0$: $T_{95\%} \approx 30\hbar$ (harder)
 - » Approximately: $T_{95\%} \propto \Delta^2$
- 3 **Non-adiabatic regime:** For $T < 1\hbar$, fidelity drops dramatically
 - » Landau-Zener transitions dominate
 - » Exponential suppression with T

Key Takeaway

The adiabatic theorem is not just qualitative - it provides quantitative predictions that match numerical simulations!

Discussion: Practical Implications

Questions for Consideration

- 1 For quantum computing, is longer evolution time always acceptable?
- 2 How do we balance fidelity vs speed in real applications?
- 3 What role does the adiabatic theorem play in current quantum technologies?

Real-World Trade-offs: Slower Evolution:

- + Higher fidelity
- + Better adiabaticity
- More decoherence
- Longer computation time

Faster Evolution:

- + Less decoherence
- + Faster results
- Lower fidelity
- Non-adiabatic errors

Optimal Strategy: Find the "sweet spot" where T is long enough for adiabaticity but short enough to avoid decoherence. This is an active area of research!

Thank You!

Questions?

Appendix: Proof of $\langle n|\dot{n} \rangle$ is Purely Imaginary

From normalization: $\langle n|n \rangle = 1$

Differentiate with respect to time:

$$\frac{d}{dt} \langle n|n \rangle = \langle \dot{n}|n \rangle + \langle n|\dot{n} \rangle = 0 \quad (50)$$

Therefore:

$$\langle n|\dot{n} \rangle = -\langle \dot{n}|n \rangle = -\langle n|\dot{n} \rangle^* \quad (51)$$

This implies $\langle n|\dot{n} \rangle$ is purely imaginary.

We can write: $\langle n|\dot{n} \rangle = i\gamma_n(t)$ where $\gamma_n(t)$ is real.

Appendix: Derivation of Berry Curvature

Starting from Berry connection:

$$A_i = i \langle n | \partial_i n \rangle \quad (52)$$

The Berry curvature is:

$$F_{ij} = \partial_i A_j - \partial_j A_i \quad (53)$$

Expanding:

$$F_{ij} = i [\partial_i \langle n | \partial_j n \rangle - \partial_j \langle n | \partial_i n \rangle] \quad (54)$$

$$= i [\langle \partial_i n | \partial_j n \rangle + \langle n | \partial_i \partial_j n \rangle - \langle \partial_j n | \partial_i n \rangle - \langle n | \partial_j \partial_i n \rangle] \quad (55)$$

Since $\partial_i \partial_j = \partial_j \partial_i$:

$$F_{ij} = i [\langle \partial_i n | \partial_j n \rangle - \langle \partial_j n | \partial_i n \rangle] \quad (56)$$

Appendix: Completeness Relation

Using completeness: $\sum_m |m\rangle \langle m| = \mathbb{I}$

We can write:

$$|\partial_i n\rangle = \sum_m |m\rangle \langle m|\partial_i n\rangle \quad (57)$$

Therefore:

$$\langle \partial_i n | \partial_j n \rangle = \sum_{m,k} \langle \partial_i n | k \rangle \langle k | m \rangle \langle m | \partial_j n \rangle \quad (58)$$

$$= \sum_m \langle \partial_i n | m \rangle \langle m | \partial_j n \rangle \quad (59)$$

The Berry curvature becomes:

$$F_{ij} = i \sum_{m \neq n} [\langle \partial_i n | m \rangle \langle m | \partial_j n \rangle - \langle \partial_j n | m \rangle \langle m | \partial_i n \rangle] \quad (60)$$