Niall, Fintan, Shane

Applied Quantum Mechanics

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Adiabatic Theorem and Quantum Annealing

A Mathematically Rigorous Treatment

Outline

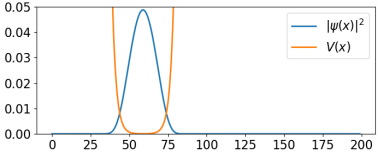
What is the Adiabatic Theorem?

Physical Statement

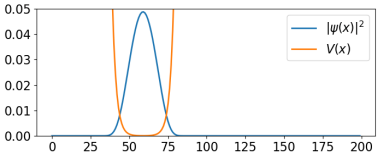
If a quantum system starts in an eigenstate of a time-dependent Hamiltonian, and if the Hamiltonian changes **sufficiently slowly**, the system will remain in the corresponding instantaneous eigenstate throughout the evolution.

- Key word: **"sufficiently slowly"**
- The system acquires a **dynamical phase** and a **geometric phase** (Berry phase)
- Fundamental to quantum computing (adiabatic quantum computation)
- Applies to quantum chemistry, biology / condensed matter physics

Slow Change

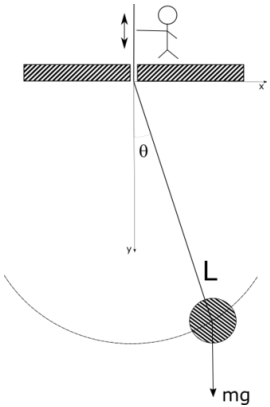


Abrupt Change



Historical Context

- First formulated by Max Born and Vladimir Fock (1928)
- Further developed by Kato (1950)
- Geometric phase discovered by Berry (1984)
- Modern applications in quantum computing and quantum annealing



Classical Analogy

A pendulum whose length changes slowly will maintain its oscillation mode, but if changed rapidly, it will exhibit chaotic behavior.

Time-Dependent Schrödinger Equation

Consider a time-dependent Hamiltonian $\hat{H}(t)$:

$$i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = \hat{H}(t)\left|\psi(t)\right\rangle \tag{1}$$

Instantaneous Eigenstates

At each time t, we can solve the eigenvalue problem:

$$\hat{H}(t)\left|n(t)\right\rangle = E_n(t)\left|n(t)\right\rangle \tag{2}$$

where $\left|n(t)\right\rangle$ are the **instantaneous eigenstates** and $E_n(t)$ are the **instantaneous eigenvalues**.

Note: These are *not* solutions to the time-dependent Schrödinger equation, but rather eigenstates at fixed time t.

Substituting into Schrödinger Equation

Substitute the expansion $|\psi(t)\rangle = \sum_n c_n(t) e^{i\theta_n(t)} |n(t)\rangle$ into the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle$$

$$i\hbar \sum_n \left[\dot{c}_n e^{i\theta_n} |n\rangle + c_n i\dot{\theta}_n e^{i\theta_n} |n\rangle + c_n e^{i\theta_n} |\dot{n}\rangle \right] = \sum_n c_n e^{i\theta_n} E_n(t) |n\rangle \quad (3)$$

where dots denote time derivatives: $\dot{c}_n = \frac{dc_n}{dt}$, $|\dot{n}\rangle = \frac{d}{dt} |n(t)\rangle$.

Dividing by $i\hbar$:

$$\sum_n \left[\frac{\dot{c}_n}{i\hbar} e^{i\theta_n} |n\rangle - c_n \dot{\theta}_n e^{i\theta_n} |n\rangle + \frac{c_n}{i\hbar} e^{i\theta_n} |\dot{n}\rangle \right] = \sum_n c_n e^{i\theta_n} \frac{E_n}{\hbar} |n\rangle \quad (4)$$

Projecting onto Eigenstates

Take the inner product with $\langle m(t)|$:

$$\frac{\dot{c}_m}{i\hbar}e^{i\theta_m} - c_m\dot{\theta}_m e^{i\theta_m} + \frac{c_m}{i\hbar}e^{i\theta_m}\langle m|\dot{m}\rangle = \sum_{n\neq m}\frac{c_n}{i\hbar}e^{i\theta_n}\langle m|\dot{n}\rangle \tag{5}$$

We used orthonormality: $\langle m|n\rangle = \delta_{mn}$ and $\sum_{n\neq m}\langle m|n\rangle = 0$.

Rearranging:

$$\dot{c}_m e^{i\theta_m} = c_m e^{i\theta_m}\left[\dot{\theta}_m - \frac{E_m}{i\hbar} - \langle m|\dot{m}\rangle\right] - \sum_{n\neq m} c_n e^{i\theta_n}\langle m|\dot{n}\rangle \tag{6}$$

Simplifying Terms / Choice of Dynamical Phase

Dynamical Phase

$$\theta_m(t) = -\frac{1}{\hbar}\int_0^t E_m(t')dt' + \gamma_m(t) \tag{7}$$

where $\gamma_m(t)$ is the **geometric phase** (Berry phase):

Geometric Phase

$$\gamma_m(t) = i\int_0^t \left\langle m(t')\middle|\dot{m}(t')\right\rangle dt' \tag{8}$$

Then: $\dot{\theta}_m = -\frac{E_m}{\hbar} + i\left\langle m|\dot{m}\right\rangle$

And our equation becomes:

$$\dot{c}_m e^{i\theta_m} = -\sum_{n\neq m} c_n e^{i\theta_n}\left\langle m|\dot{n}\right\rangle \tag{9}$$

What is $\langle m|\dot{n}\rangle$

From the eigenvalue equation $\hat{H}(t)\,|n(t)\rangle = E_n(t)\,|n(t)\rangle$, differentiate with respect to t:

$$\dot{\hat{H}}\,|n\rangle + \hat{H}\,|\dot{n}\rangle = \dot{E}_n\,|n\rangle + E_n\,|\dot{n}\rangle \tag{10}$$

Take inner product with $\langle m|$ (where $m \neq n$):

$$\langle m|\,\dot{\hat{H}}\,|n\rangle + \langle m|\,\hat{H}\,|\dot{n}\rangle = \dot{E}_n\,\langle m|n\rangle + E_n\,\langle m|\dot{n}\rangle \tag{11}$$

Since $\langle m|\,\hat{H} = E_m\,\langle m|$ and $\langle m|n\rangle = 0$ for $m \neq n$:

$$\langle m|\,\dot{\hat{H}}\,|n\rangle + E_m\,\langle m|\dot{n}\rangle = E_n\,\langle m|\dot{n}\rangle \tag{12}$$

Therefore:

Matrix Element

$$\langle m|\dot{n}\rangle = \frac{\langle m|\,\dot{\hat{H}}\,|n\rangle}{E_n - E_m} \quad (m \neq n) \tag{13}$$

The Differential Equation for Coefficients

Substituting back:

$$\dot{c}_m = -\sum_{n \neq m} c_n e^{i(\theta_n - \theta_m)} \frac{\langle m| \dot{\hat{H}} |n\rangle}{E_n - E_m} \tag{14}$$

The phase difference is:

$$\theta_n - \theta_m = -\frac{1}{\hbar} \int_0^t [E_n(t') - E_m(t')]dt' + [\gamma_n(t) - \gamma_m(t)] \tag{15}$$

Key Observation

The right-hand side contains:

- $\dot{\hat{H}}$ - rate of change of Hamiltonian
- $E_n - E_m$ - energy gap (in denominator)
- Rapidly oscillating phase factor

The Adiabatic Condition

For the system to remain in state $|m\rangle$, we need $\dot{c}_m \approx 0$ for m equal to the initial state.

Adiabatic Condition

The Hamiltonian changes slowly enough that:

$$\left| \frac{\langle m| \dot{\hat{H}} |n\rangle}{E_n - E_m} \right| \ll \frac{|E_n - E_m|}{\hbar} \tag{16}$$

for all $n \neq m$.

This can be rewritten as:

$$\left| \langle m| \dot{\hat{H}} |n\rangle \right| \ll \frac{(E_n - E_m)^2}{\hbar} \tag{17}$$

Physical Interpretation: The time scale of Hamiltonian variation must be much larger than $\hbar/\Delta E$, where ΔE is the relevant energy gap.

Theorem (Quantum Adiabatic Theorem)

Let $\hat{H}(t)$ be a time-dependent Hamiltonian with instantaneous eigenstates $|n(t)\rangle$ and eigenvalues $E_n(t)$. Suppose:

1. *The system starts in eigenstate $|n(0)\rangle$: $|\psi(0)\rangle = |n(0)\rangle$*
2. *The eigenvalues are non-degenerate*
3. *The adiabatic condition is satisfied for all $t \in [0, T]$*

Then the state at time t is:

$$|\psi(t)\rangle = e^{i\theta_n(t)} |n(t)\rangle + \mathcal{O}(\epsilon) \tag{18}$$

where ϵ characterizes the adiabatic parameter and

$$\theta_n(t) = -\frac{1}{\hbar} \int_0^t E_n(t')dt' + i \int_0^t \langle n(t') | \partial_{t'} n(t') \rangle \, dt' \tag{19}$$

Shane's Slides Placeholder

Shane's slides 5 minutes allocated here.

Example 1: Spin-1/2 in Rotating Magnetic Field

Consider a spin-1/2 particle in a magnetic field that rotates slowly:

$$\vec{B}(t) = B_0(\sin\theta\cos\omega t, \sin\theta\sin\omega t, \cos\theta) \tag{20}$$

The Hamiltonian is:

$$\hat{H}(t) = -\gamma\vec{B}(t)\cdot\vec{\sigma} = -\gamma B_0 \begin{pmatrix} \cos\theta & \sin\theta e^{-i\omega t} \\ \sin\theta e^{i\omega t} & -\cos\theta \end{pmatrix} \tag{21}$$

Instantaneous eigenvalues: $E_\pm = \mp\gamma B_0$

If the spin starts aligned with $\vec{B}(0)$ and ω is small enough:

- The spin remains aligned with $\vec{B}(t)$ (adiabatic following)
- After one full rotation ($t = 2\pi/\omega$), acquires Berry phase: $\gamma = \pi(1 - \cos\theta)$

Example 2: Two-Level System (Landau-Zener Model)

Consider the Landau-Zener Hamiltonian:

$$\hat{H}(t) = -\Delta\sigma_x - \epsilon(t)\sigma_z \tag{22}$$

where $\epsilon(t)$ is swept linearly from $\epsilon_i < 0$ to $\epsilon_f > 0$.

Energy eigenvalues:

$$E_{\pm}(t) = \pm\sqrt{\Delta^2 + \epsilon^2(t)} \tag{23}$$

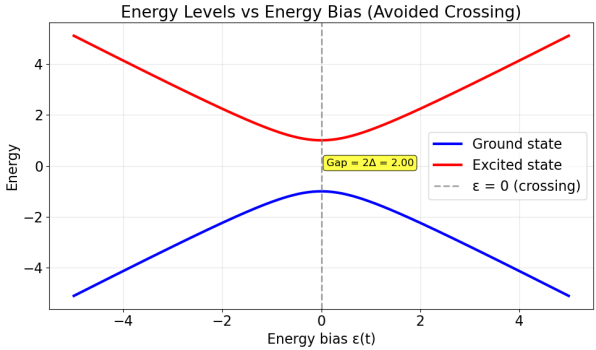
Energy gap at crossing ($\epsilon = 0$): $\Delta E_{\min} = 2\Delta$

Adiabatic Condition

$$T \gg \frac{\hbar}{\Delta E_{\min}} = \frac{\hbar}{2\Delta} \quad \text{and} \quad |\dot{\epsilon}| \ll \frac{4\Delta^2}{\hbar} \tag{24}$$

This is the canonical model for avoided crossings and adiabatic transitions!

Landau-Zener Model: Avoided Crossing Visualization



- The energy levels exhibit an **avoided crossing** at $\epsilon(t) = 0$
- Minimum gap: $\Delta E_{\min} = 2\Delta$ determines adiabatic time scale

Discussion: Avoided Crossing Physics

Key Questions

1. What happens at the avoided crossing point?
2. Why is this the bottleneck for adiabatic evolution?
3. How does the gap size affect the required evolution time?

Physical Insights:

- At $\epsilon = 0$: The eigenstates are equal superpositions of diabatic states
- The gap 2Δ prevents level crossing (quantum mechanical effect)
- Smaller gap $\Rightarrow$ easier to violate adiabaticity
- Without the $-\Delta\sigma_x$ term, levels would cross diabatically

Critical Insight

The adiabatic condition is hardest to satisfy near the **minimum gap**. This is where non-adiabatic transitions (Landau-Zener tunneling) are most likely to occur.

Berry's Geometric Phase

For a cyclic adiabatic evolution, where the Hamiltonian returns to its initial form:

$$\hat{H}(R(T)) = \hat{H}(R(0)) \tag{25}$$

The total phase acquired is:

$$e^{i\phi} = e^{i\theta_{\text{dyn}}} e^{i\gamma_n} \tag{26}$$

Berry Phase (1984)

$$\gamma_n = i \oint_{\mathcal{C}} \langle n(R) | \nabla_R n(R) \rangle \cdot dR \tag{27}$$

where $\mathcal{C}$ is the closed path in parameter space.

Properties:

- Gauge invariant (physical observable)
- Geometric: depends only on path, not speed
- Can be written as flux of Berry curvature

Berry Connection and Curvature

Berry Connection (Vector Potential)

$$\vec{A}_n(R) = i \langle n(R) | \nabla_R n(R) \rangle \tag{28}$$

Berry Curvature (Field Strength)

$$\vec{F}_n(R) = \nabla_R \times \vec{A}_n(R) \tag{29}$$

By Stokes' theorem:

$$\gamma_n = \oint_{\mathcal{C}} \vec{A}_n \cdot dR = \iint_{\mathcal{S}} \vec{F}_n \cdot d\vec{S} \tag{30}$$

Analogy: Berry phase is like the Aharonov-Bohm phase, but in parameter space rather than real space!

Non-Adiabatic Transitions

When adiabaticity breaks down, transitions occur between energy levels.

Landau-Zener Formula

For a linear crossing: $E_1(t) = \alpha t$, $E_2(t) = -\alpha t + \Delta$
Transition probability:

$$P_{1\to 2} = \exp\left(-\frac{\pi\Delta^2}{2\hbar\alpha}\right) \tag{31}$$

- Small gap $\Delta \Rightarrow$ high transition probability
- Fast sweep (large α) $\Rightarrow$ high transition probability
- Slow sweep (small α) $\Rightarrow$ adiabatic regime, $P \to 0$

Adiabatic Perturbation Theory

We can systematically expand in the adiabatic parameter $\epsilon = \tau_{\text{dyn}}/T$:

$$|\psi(t)\rangle = e^{i\theta_n(t)} \left[|n(t)\rangle + \epsilon \left|\psi_n^{(1)}(t)\right\rangle + \epsilon^2 \left|\psi_n^{(2)}(t)\right\rangle + \cdots \right] \tag{32}$$

First-order correction:

$$\left|\psi_n^{(1)}(t)\right\rangle = \sum_{m \neq n} \frac{\langle m| \dot{\hat{H}} |n\rangle}{(E_n - E_m)^2} |m\rangle \tag{33}$$

This shows:

- Transition amplitude $\propto 1/(E_n - E_m)^2$
- Small gaps make adiabatic evolution difficult
- Can estimate fidelity: $|\langle n(t)|\psi(t)\rangle|^2 \approx 1 - \mathcal{O}(\epsilon^2)$

Generalization to Degenerate Levels

When eigenvalues are degenerate, the theorem must be modified:

- System can transition within degenerate subspace
- Need to consider the entire degenerate subspace
- Berry phase becomes a **matrix** (non-Abelian Berry connection)

Wilczek-Zee Connection

For degenerate subspace with basis $\{|n_a(R)\rangle\}$:

$$(A_i)_{ab} = i \langle n_a | \partial_i n_b \rangle \tag{34}$$

This is a matrix-valued gauge potential.

Leads to non-Abelian geometric phases (important in topological quantum computation).

Discussion: Diabatic vs Adiabatic Evolution

Diabatic Evolution:

- Fast changes in $\hat{H}(t)$
- System doesn't track instantaneous eigenstates
- Transitions between levels
- Example: Sudden approximation

When applicable:

$$T \ll \frac{\hbar}{\Delta E} \tag{35}$$

Adiabatic Evolution:

- Slow changes in $\hat{H}(t)$
- System follows instantaneous eigenstate
- No transitions (to first order)
- Example: Born-Oppenheimer

When applicable:

$$T \gg \frac{\hbar}{\Delta E} \tag{36}$$

Discussion Question

In between these limits, how do we calculate the transition probability?

Answer: Landau-Zener formula, perturbation theory, or numerical methods

Fintan's Slides Placeholder

Fintan's slides 5 minutes allocated here.

Application: Adiabatic Quantum Computation

Basic Idea

1. Encode problem in Hamiltonian $\hat{H}_{\text{problem}}$ whose ground state is the solution
2. Start with simple Hamiltonian $\hat{H}_{\text{initial}}$ with known ground state
3. Slowly interpolate: $\hat{H}(t) = (1 - s(t))\hat{H}_{\text{initial}} + s(t)\hat{H}_{\text{problem}}$
4. If evolution is adiabatic, system remains in ground state
5. At $t = T$: measure to obtain solution

Runtime: $T \sim \mathcal{O}(\hbar/\Delta_{\min}^2)$ where $\Delta_{\min}$ is minimum gap

Challenge: Gap can be exponentially small for some problems!

Quantum Annealing Framework

Goal: Solve optimization problems using quantum adiabatic evolution

Standard Approach

1. Encode optimization problem in a Hamiltonian
2. Prepare system in ground state of simple initial Hamiltonian
3. Adiabatically evolve to problem Hamiltonian
4. Measure final state to read out solution

Two Main Formulations:

- **Ising Model:** Spin variables $s_i \in \{-1, +1\}$
- **QUBO:** Binary variables $x_i \in \{0, 1\}$
- These are equivalent and interconvertible

Examples: Graph coloring, traveling salesman, portfolio optimization, protein folding

Classical Ising Hamiltonian

$$H_{\text{Ising}} = \sum_{i<j} J_{ij} s_i s_j + \sum_i h_i s_i \tag{37}$$

where $s_i \in \{-1, +1\}$ are spin variables (Pauli-z eigenvalues)

Parameters:

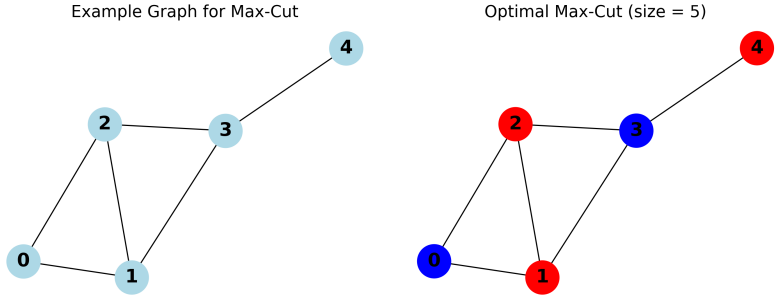
- J_{ij}: Coupling between spins i and j
  - $J_{ij} > 0$: Ferromagnetic (spins want to align)
  - $J_{ij} < 0$: Antiferromagnetic (spins want to anti-align)
- h_i: Local field on spin i (bias)

Quantum Version:

$$\hat{H}_{\text{Ising}} = \sum_{i<j} J_{ij} \sigma_i^z \sigma_j^z + \sum_i h_i \sigma_i^z \tag{38}$$

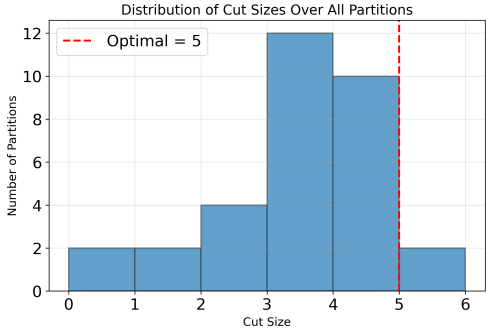
This is the **problem Hamiltonian** in quantum annealing!

Visualizing the Ising Model: Max-Cut Example



- **Left:** Original graph with 5 vertices and 6 edges
- **Right:** Optimal partition (red vs blue) achieves cut size $= 5$
- Only 1 edge (out of 6) remains within a partition
- Ising formulation: Antiferromagnetic couplings ($J_{ij} = 1$) favor opposite spins

Energy Landscape: Solution Distribution



- Out of $2^5 = 32$ possible partitions, only 2 achieve the optimal cut size
- Most partitions achieve sub-optimal cuts (sizes 3-4)
- **Challenge for quantum annealing:** Find the needle in the haystack!
- Energy landscape roughness determines difficulty

QUBO: Quadratic Unconstrained Binary Optimization

QUBO Formulation

Minimize:

$$f(x) = \sum_i Q_{ii} x_i + \sum_{i<j} Q_{ij} x_i x_j = x^T Q x \tag{39}$$

where $x_i \in \{0, 1\}$ and Q is the QUBO matrix

Why QUBO?

- Many combinatorial optimization problems naturally expressed as QUBO
- NP-hard in general (finding global minimum)
- Can encode constraints via penalty terms

Examples:

- Max-Cut: $\max \sum_{(i,j)\in E} (1 - x_i x_j)$
- Number Partitioning: $\min \left(\sum_i a_i x_i\right)^2$
- Graph Coloring: Use penalty for adjacent same-colored vertices

Converting Between QUBO and Ising

The mapping is straightforward:

QUBO → Ising

Variable transformation: $x_i = \frac{1-s_i}{2}$ where $s_i \in \{-1, +1\}$

Then: $x_i x_j = \frac{(1-s_i)(1-s_j)}{4} = \frac{1-s_i-s_j+s_i s_j}{4}$

Ising → QUBO

Inverse transformation: $s_i = 1 - 2x_i$ where $x_i \in \{0, 1\}$

Then: $s_i s_j = (1-2x_i)(1-2x_j) = 1 - 2x_i - 2x_j + 4x_i x_j$

Key Point: The two formulations are equivalent - choose based on:

- Natural problem representation
- Hardware constraints (quantum annealers often use Ising)
- Software/solver availability

Quantum Annealing Schedule

Standard Quantum Annealing Hamiltonian:

$$\hat{H}(t) = A(t)\hat{H}_{\text{driver}} + B(t)\hat{H}_{\text{problem}} \tag{40}$$

where:

- $\hat{H}_{\text{driver}} = -\sum_i \sigma_i^x$ (transverse field - creates superposition)
- $\hat{H}_{\text{problem}} = \sum_{i<j} J_{ij}\sigma_i^z\sigma_j^z + \sum_i h_i\sigma_i^z$ (Ising model)
- $A(t)$, $B(t)$ are annealing schedules with $A(0) \gg B(0)$ and $A(T) \ll B(T)$

Standard Linear Schedule:

$$A(t) = A_0\left(1 - \frac{t}{T}\right) \tag{41}$$

$$B(t) = B_0\frac{t}{T} \tag{42}$$

At $t = 0$: Easy ground state $|+\rangle^{\otimes N}$ (all spins in $+x$ direction)
At $t = T$: Ground state encodes solution to optimization problem

Example: Max-Cut Problem

Problem: Given graph $G = (V, E)$, partition vertices into two sets to maximize edges between sets.

QUBO Formulation:

Maximize:

$$f(x) = \sum_{(i,j)\in E} (x_i - x_j)^2 \tag{43}$$

Expanding:

$$= \sum_{(i,j)\in E} (x_i + x_j - 2x_i x_j) \tag{44}$$

For minimization (quantum annealer):

$$H_{\text{QUBO}} = -\sum_{(i,j)\in E} (x_i + x_j - 2x_i x_j) \tag{45}$$

Ising Formulation:

Using $s_i = 1 - 2x_i$:

$$H_{\text{Ising}} = -\sum_{(i,j)\in E} (1 - s_i s_j) \tag{46}$$

Simplifies to:

$$H_{\text{Ising}} = \sum_{(i,j)\in E} s_i s_j + \text{const} \tag{47}$$

Interpretation:

- $s_i = +1$: vertex in set A
- $s_i = -1$: vertex in set B
- Minimize $\Rightarrow$ maximize cut

Adiabatic Theorem in Quantum Annealing

Success condition based on adiabatic theorem:

Required Annealing Time

$$T \gg \frac{\hbar}{\Delta_{\min}^2} \tag{48}$$

where $\Delta_{\min}$ is the minimum energy gap during evolution

Problem-Dependent Gap:

- Different problems $\Rightarrow$ different gap structures
- Gap often smallest near $t \approx T/2$ (mid-anneal)
- For many NP-hard problems: $\Delta_{\min} \sim e^{-\alpha N}$ (exponentially small!)

Practical Implications:

- $+$ Easy problems: Fast annealing possible (microseconds)
- $-$ Hard problems: May need exponentially long annealing time
- $\pm$ Thermal effects can help/hinder (quantum vs simulated annealing)

Discussion: Quantum Annealing vs Gate Model

Quantum Annealing:

+ Analog, continuous evolution
+ Natural for optimization
+ Potentially more robust to some errors
+ Easier to scale (1000s of qubits)
− Limited to specific problem types
− No quantum error correction
− Success depends on gap structure

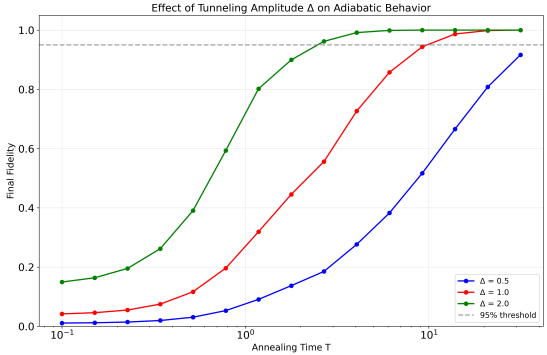
Gate-Based Quantum Computing:

+ Universal quantum computation
+ Quantum error correction possible
+ Well-defined circuit depth
+ Can implement any algorithm
− Currently limited qubits (<1000)
− High gate error rates
− Requires precise control

Open Question

Does quantum annealing provide speedup over classical algorithms?
Evidence mixed - depends on problem structure, noise, and implementation.

Parameter Dependence: Effect of Tunneling Amplitude Δ



- **Larger** Δ: Larger gap $\Rightarrow$ longer time needed for 95% fidelity
- **Smaller** Δ: Smaller gap $\Rightarrow$ easier to satisfy adiabatic condition
- Critical time scale: $T_{\text{critical}} \propto 1/\Delta^2$

Discussion: Scaling and the Adiabatic Theorem

Discussion Points

1. Why does larger Δ require *longer* evolution time?
2. How does this relate to the adiabatic condition $T \gg \hbar/\Delta E$?
3. What is the trade-off between gap size and computation time?

Counterintuitive Result:

- Larger gap Δ = more separation between energy levels
- BUT: Also means stronger coupling between diabatic states
- Stronger coupling $\Rightarrow$ harder to maintain adiabatic evolution
- Mathematical: $|\langle m|\dot{n}\rangle| \propto \Delta$ but gap $\propto \Delta$

Key Scaling

Landau-Zener transition probability: $P_{\text{LZ}} = \exp\left(-\frac{\pi\Delta^2}{2\hbar v}\right)$
where $v = |\dot{\epsilon}|$ is the sweep rate. Adiabatic limit: $v \to 0 \Rightarrow P_{\text{LZ}} \to 0$

Experimental Realization and Fidelity

Experimental Platforms:

- Superconducting qubits
- Trapped ions
- Cold atoms
- Nitrogen-vacancy centers
- Quantum dots

Typical Parameters:

- $\Delta/h \sim$ 0.1-10 GHz
- $T \sim$ 1-1000 ns
- Fidelity $>$ 99% achievable

Challenges:

- Decoherence (T_1, T_2)
- Control noise
- Calibration errors
- Finite temperature

Fidelity Metric

$$F = |\langle \psi_{\text{target}} | \psi_{\text{final}} \rangle|^2 \tag{49}$$

Measures overlap with desired state

Discussion: How do environmental effects modify the adiabatic theorem?

Challenge:

Gap can be exponentially small for some problems!

Open Questions and Research Frontiers

1. **Many-body systems:** How does the adiabatic theorem scale with system size?
   - Ground state degeneracy
   - Exponentially small gaps
   - Quantum phase transitions
2. **Decoherence:** How does noise affect adiabatic evolution?
   - Open quantum systems
   - Markovian vs non-Markovian dynamics
   - Error correction strategies
3. **Shortcuts to adiabaticity:** Can we achieve adiabatic-like results faster?
   - Counter-diabatic driving
   - Optimal control theory
   - Fast-forward protocols
4. **Quantum annealing:** What problems can quantum annealers solve?
   - Quantum advantage vs classical algorithms
   - Role of thermal fluctuations
   - Embedding optimization problems

Key Insights

Mathematical Rigor

The adiabatic theorem is not just a hand-waving argument:

- Precise conditions on energy gaps and rates of change
- Systematic perturbation theory for corrections
- Geometric interpretation via Berry connection and curvature

Physical Consequences

- Quantum systems can "remember" their path through parameter space
- Topological effects in quantum mechanics
- Foundation for quantum computation paradigm
- Explains robustness of certain quantum states

Numerical Verification: What We Observed

From our Landau-Zener simulations:

1. **Critical Time Scale:** For $\Delta = 1.0$, achieve 95% fidelity at $T \approx 10\hbar$
   - Consistent with $T \gg \hbar/(2\Delta) = 0.5\hbar$
2. **Scaling with Δ:**
   - $\Delta = 0.5$: $T_{95\%} \approx 3\hbar$ (easier)
   - $\Delta = 1.0$: $T_{95\%} \approx 10\hbar$ (baseline)
   - $\Delta = 2.0$: $T_{95\%} \approx 30\hbar$ (harder)
   - Approximately: $T_{95\%} \propto \Delta^2$
3. **Non-adiabatic regime:** For $T < 1\hbar$, fidelity drops dramatically
   - Landau-Zener transitions dominate
   - Exponential suppression with T

Key Takeaway

The adiabatic theorem is not just qualitative - it provides quantitative predictions that match numerical simulations!

Discussion: Practical Implications

Questions for Consideration

1. For quantum computing, is longer evolution time always acceptable?
2. How do we balance fidelity vs speed in real applications?
3. What role does the adiabatic theorem play in current quantum technologies?

Real-World Trade-offs:

Slower Evolution:

+ Higher fidelity
+ Better adiabaticity
− More decoherence
− Longer computation time

Faster Evolution:

+ Less decoherence
+ Faster results
− Lower fidelity
− Non-adiabatic errors

Optimal Strategy: Find the "sweet spot" where T is long enough for adiabaticity but short enough to avoid decoherence. This is an active area of research!

Thank You!

Questions?

Appendix: Proof of $\langle n|\dot{n}\rangle$ is Purely Imaginary

From normalization: $\langle n|n\rangle = 1$
Differentiate with respect to time:

$$\frac{d}{dt}\langle n|n\rangle = \langle \dot{n}|n\rangle + \langle n|\dot{n}\rangle = 0 \tag{50}$$

Therefore:

$$\langle n|\dot{n}\rangle = -\langle \dot{n}|n\rangle = -\langle n|\dot{n}\rangle^* \tag{51}$$

This implies $\langle n|\dot{n}\rangle$ is purely imaginary.

We can write: $\langle n|\dot{n}\rangle = i\gamma_n(t)$ where $\gamma_n(t)$ is real.

Appendix: Derivation of Berry Curvature

Starting from Berry connection:

$$A_i = i \langle n | \partial_i n \rangle \tag{52}$$

The Berry curvature is:

$$F_{ij} = \partial_i A_j - \partial_j A_i \tag{53}$$

Expanding:

$$F_{ij} = i \left[\partial_i \langle n | \partial_j n \rangle - \partial_j \langle n | \partial_i n \rangle \right] \tag{54}$$

$$= i \left[\langle \partial_i n | \partial_j n \rangle + \langle n | \partial_i \partial_j n \rangle - \langle \partial_j n | \partial_i n \rangle - \langle n | \partial_j \partial_i n \rangle \right] \tag{55}$$

Since $\partial_i \partial_j = \partial_j \partial_i$:

$$F_{ij} = i \left[\langle \partial_i n | \partial_j n \rangle - \langle \partial_j n | \partial_i n \rangle \right] \tag{56}$$

Appendix: Completeness Relation

Using completeness: $\sum_m |m\rangle \langle m| = \mathbb{I}$
We can write:

$$|\partial_i n\rangle = \sum_m |m\rangle \langle m|\partial_i n\rangle \tag{57}$$

Therefore:

$$\langle \partial_i n|\partial_j n\rangle = \sum_{m,k} \langle \partial_i n|k\rangle \langle k|m\rangle \langle m|\partial_j n\rangle \tag{58}$$

$$= \sum_m \langle \partial_i n|m\rangle \langle m|\partial_j n\rangle \tag{59}$$

The Berry curvature becomes:

$$F_{ij} = i \sum_{m\neq n} \left[\langle \partial_i n|m\rangle \langle m|\partial_j n\rangle - \langle \partial_j n|m\rangle \langle m|\partial_i n\rangle\right] \tag{60}$$