Animation & Simulation

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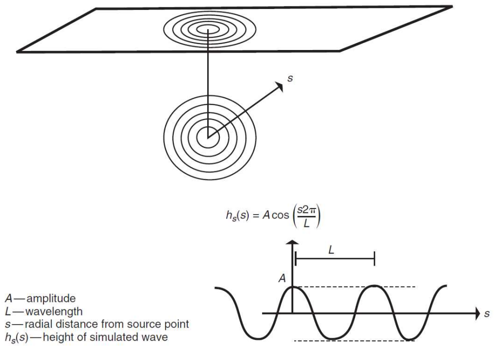
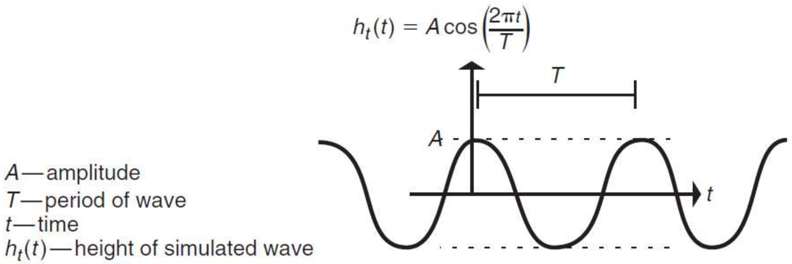
Fluids: Liquids and Gases

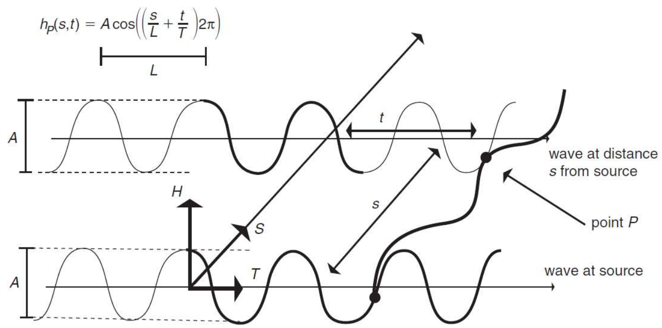
- Universally seen (clouds, water, fire, etc.)
- Not defined by static, rigid, topologically simple structure
- Math description: Computational Fluid Dynamics (CFD)
- Cutting-edge research:
  - https://www.youtube.com/watch?v=QJePco0_os8
  - https://vimeo.com/163601169
  - https://www.youtube.com/watch?v=3pojNrRfJFM

Fluids: Liquids and Gases

- Specific Fluid Models
  - Models of water
    - Still water and small-amplitude waves
      - Assign blue below a depth
      - Normal perturbation (bump mapping)
        - Sinusoidal functions to create small waves







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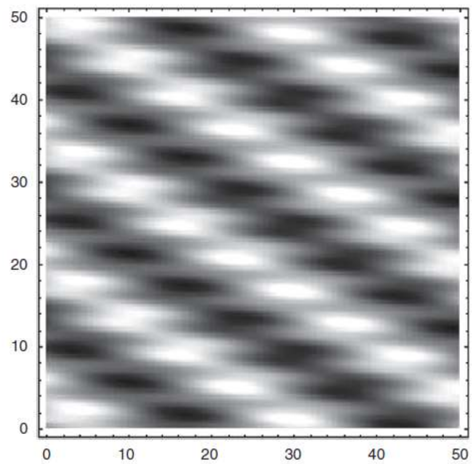
        - Superimposing multiple functions with different amplitude (different sources and directions)



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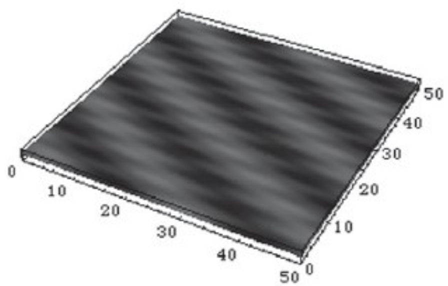
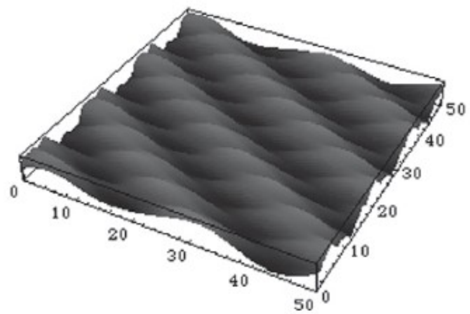
        - Sinusoidal functions to create small waves
        - Superimposing multiple functions with different amplitude (different sources and directions)
        - Heights can be computed in a similar way





Fluids: Liquids and Gases

- Specific Fluid Models
  - Models of water
    - The anatomy of waves

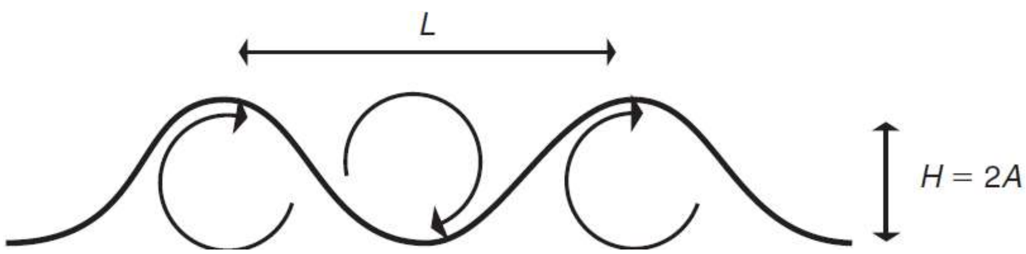
$$f(s,t) = \cos^{-1}\left(\frac{2\pi(s - Ct)}{L}\right)$$

period of the wave, T

C is the propagation speed, and L is the wavelength $\quad C = L/T.$

Fluids: Liquids and Gases

- Specific Fluid Models
  - Models of water
    - The anatomy of waves (assuming no water transport)
      - Water particles travel in a nearly circular orbit



$$Q_{\text{ave}} = \frac{\pi H}{T} = \frac{\pi H C}{L} = \pi S C$$

Avg. orbit speed

Steepness

H is defined as twice the amplitude

When Q at the wave top is bigger than C, waves break.

Fluids: Liquids and Gases

- Specific Fluid Models
  - Models of water
    - The anatomy of waves
      - Airy model (a simplified CFD model)

$$C = \sqrt{\frac{g}{\kappa}\tanh(\kappa d)} = \sqrt{\frac{gL}{2\pi} \times \boxed{\tanh\left(\frac{2\pi d}{L}\right)}}$$

$$L = CT$$

When d increases, tanh(kd) approx. 1
When d decreases, tanh(kd) approx. kd

the depth of the water, d, the propagation speed, C, and the wavelength of the wave, L

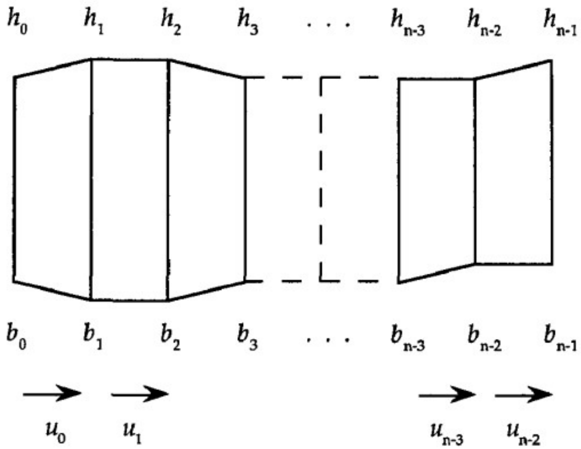
g is the acceleration of a body due to gravity at sea level, 9.81 m/sec^2, and $k = 2\pi/L$ is the spatial equivalent of wave frequency.

Fluids: Liquids and Gases

- Specific Fluid Models
  - Models of water
    - Finding its way downhill
      - Non-transported water -> transported water
      - Simplified Navier-Stokes equation (the book is wrong, refer to the paper Kass and Miller Siggraph 1990)

$h(x)$ be the height of the water

$b(x)$ be the height of the ground



$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial d}{\partial t} + \frac{\partial}{\partial x}(ud) = 0$$

The height of the water is $d(x) = h(x) - b(x)$

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The book is wrong here (page 260)!

F = ma
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

Volume conservation
$$\frac{\partial d}{\partial t} + \frac{\partial}{\partial x}(ud) = 0$$

Small velocity

Slowly varying depth

$$\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} = 0$$

derivative on x

$$\frac{\partial h}{\partial t} + d\frac{\partial u}{\partial x} = 0$$

derivative on t

Solve by finite difference
$$\frac{\partial^2 h}{\partial t^2} = gd\frac{\partial^2 h}{\partial x^2}$$

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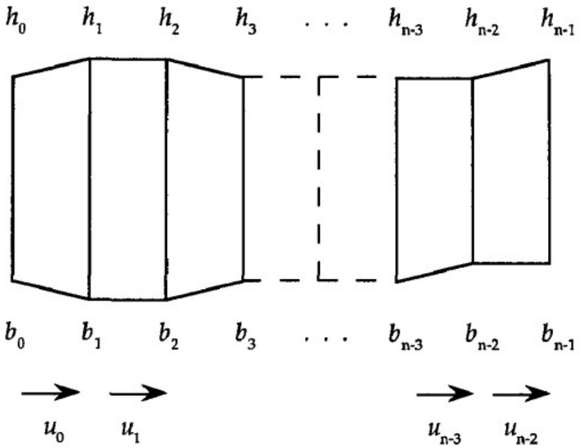

$$\frac{\partial h_i}{\partial t} = \left(\frac{d_{i-1} + d_i}{2\Delta x}\right)u_{i-1} - \left(\frac{d_i + d_{i+1}}{2\Delta x}\right)u_i$$

$$\frac{\partial u_i}{\partial t} = \frac{-g(h_{i+1} - h_i)}{\Delta x}$$

$$\frac{\partial^2 h_i}{\partial t^2} = -g\left(\frac{d_{i-1} + d_i}{2(\Delta x)^2}\right)(h_i - h_{i-1}) + g\left(\frac{d_i + d_{i+1}}{2(\Delta x)^2}\right)(h_{i+1} - h_i)$$

Update height for next time step
However, diverge fast

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Solve by finite difference $\frac{\partial^2 h}{\partial t^2} = gd\,\frac{\partial^2 h}{\partial x^2}$

First-order implicit method is stable enough!

$$\frac{h(n) - h(n-1)}{\Delta t} = \dot{h}(n)$$

$$\frac{\dot{h}(n) - \dot{h}(n-1)}{\Delta t} = \ddot{h}(n)$$

$$\Rightarrow$$

$$h(n) = h(n-1) + \Delta t\, \dot{h}(n-1) + (\Delta t)^2 \ddot{h}(n)$$

$$h(n) = 2h(n-1) - h(n-2) + (\Delta t)^2 \ddot{h}(n)$$

$$\Rightarrow$$

$$h_i(n) = 2h_i(n-1) - h_i(n-2) - g(\Delta t)^2\left(\frac{d_{i-1}+d_i}{2(\Delta x)^2}\right)(h_i(n) - h_{i-1}(n)) + g(\Delta t)^2\left(\frac{d_i+d_{i+1}}{2(\Delta x)^2}\right)(h_{i+1}(n) - h_i(n))$$

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Solve by finite difference $$\frac{\partial^2 h}{\partial t^2} = gd\frac{\partial^2 h}{\partial x^2}$$

d still depends on h, assuming d is constant within a step

$$\begin{aligned} h_i(n) &= 2h_i(n-1) - h_i(n-2) \\ &- g(\Delta t)^2\left(\frac{d_{i-1}+d_i}{2(\Delta x)^2}\right)\left(h_i(n) - h_{i-1}(n)\right) \\ &+ g(\Delta t)^2\left(\frac{d_i+d_{i+1}}{2(\Delta x)^2}\right)\left(h_{i+1}(n) - h_i(n)\right) \end{aligned}$$

$$\Longrightarrow \quad \mathbf{A}h_i(n) = 2h_i(n-1) - h_i(n-2)$$

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Solve by finite difference $$\frac{\partial^2 h}{\partial t^2} = gd\,\frac{\partial^2 h}{\partial x^2}$$

$$\mathbf{A}h_i(n) = 2h_i(n-1) - h_i(n-2) \Rightarrow$$

$$\mathbf{A} = \begin{pmatrix} e_0 & f_0 & & & & & \\ f_0 & e_1 & f_1 & & & & \\ & f_1 & e_2 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & e_{n-3} & f_{n-3} & \\ & & & & f_{n-3} & e_{n-2} & f_{n-2} \\ & & & & & f_{n-2} & e_{n-1} \end{pmatrix}$$

$$e_0 = 1 + g(\Delta t)^2\left(\frac{d_0 + d_1}{2(\Delta x)^2}\right)$$

$$e_i = 1 + g(\Delta t)^2\left(\frac{d_{i-1} + 2d_i + d_{i+1}}{2(\Delta x)^2}\right) \quad (0 < i < n-1)$$

$$e_{n-1} = 1 + g(\Delta t)^2\left(\frac{d_{n-2} + d_{n-1}}{2(\Delta x)^2}\right)$$

$$f_i = -g(\Delta t)^2\left(\frac{d_i + d_{i+1}}{2(\Delta x)^2}\right)$$

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The right-hand side can be seen as extrapolation

$$\mathbf{A}h_i(n) = 2h_i(n-1) - h_i(n-2) \quad \Longrightarrow \quad \mathbf{A}h_i(n) = h_i(n-1) + (1-\tau)(h_i(n-1) - h_i(n-2))$$