Animation and Simulation

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Collisions and Body Dynamics

- Rigid Body Dynamics
  - Physics System
    - Classic Newtonian Mechanics
    - Rigid Bodies
    - Constraints (collisions, etc.)
    - Can be computed from equations

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Foundations
    - Metric system (meters, kilograms, seconds, etc.)
    - Linear vs angular dynamics
      - A rigid body can translate and rotate (6 degrees of freedom)
        - Translate (3 degrees of freedom)
        - Rotate(3 degrees of freedom)
    - Centre of mass
      - Mass concentrated on one point, a simplified model

$$\mathbf{r}_{CM} = \frac{\sum_{\forall i} m_i \, \mathbf{r}_i}{\sum_{\forall i} m_i} = \frac{\sum_{\forall i} m_i \, \mathbf{r}_i}{m},$$

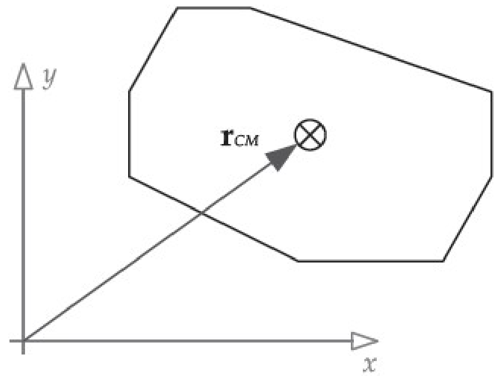
Collisions and Body Dynamics

- Rigid Body Dynamics
  - Linear Dynamics
    - Velocity and acceleration

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \dot{\mathbf{r}}(t),$$

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \dot{\mathbf{v}}(t)$$

$$= \frac{d^2\mathbf{r}(t)}{dt^2} = \ddot{\mathbf{r}}(t).$$

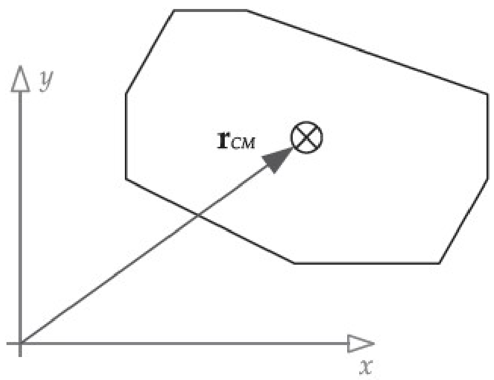


Collisions and Body Dynamics

- Rigid Body Dynamics
  - Linear Dynamics
    - Force and momentum

$$\mathbf{F}_{\text{net}} = \sum_{i=1}^{N} \mathbf{F}_i . \qquad \mathbf{F}(t) = m\,\mathbf{a}(t) = m\,\ddot{\mathbf{r}}(t) \qquad \mathbf{p}(t) = m\,\mathbf{v}(t)$$

$$\mathbf{F}(t) = \frac{d\mathbf{p}(t)}{dt} = \frac{d(m\,\mathbf{v}(t))}{dt}.$$



Collisions and Body Dynamics

- Rigid Body Dynamics
  - Solving Motion Equations
    - Force

$$\mathbf{F}\big(t,\ \mathbf{r}(t),\ \mathbf{v}(t),\ \ldots\big) = m\,\mathbf{a}(t).$$

$$\mathbf{F}\big(t,\ \mathbf{r}(t),\ \dot{\mathbf{r}}(t),\ \ldots\big) = m\,\ddot{\mathbf{r}}(t).$$

Spring

$$F\big(t,\ x(t)\big) = -k\ x(t),$$

Damping

$$F\big(t,\ v(t)\big) = -b\ v(t),$$

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Solving Motion Equations
    - Ordinary Differential Equations (ODEs)

$$\frac{d^n x}{dt^n} = f\left(t\,,\; x(t),\; \frac{dx(t)}{dt},\; \frac{d^2 x(t)}{dt^2},\; \ldots,\; \frac{d^{n-1} x(t)}{dt^{n-1}}\right).$$

$$\mathbf{F}\left(t,\; \mathbf{r}(t),\; \mathbf{v}(t),\; \ldots\right) = m\,\mathbf{a}(t). \longrightarrow \ddot{\mathbf{r}}(t) = \frac{1}{m}\mathbf{F}\left(t\,,\; \mathbf{r}(t),\; \dot{\mathbf{r}}(t)\right).$$

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Solving Motion Equations
    - Analytical Solutions (a simple, close-form function can be used)

$$\ddot{y}(t) = g \qquad \dot{y}(t) = gt + v_0 \qquad y(t) = \tfrac{1}{2}gt^2 + v_0 t + y_0$$

    - Almost never happen in game engines

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Solving Motion Equations
    - Numerical Integration (compute the system step by step)

Collisions and Body Dynamics

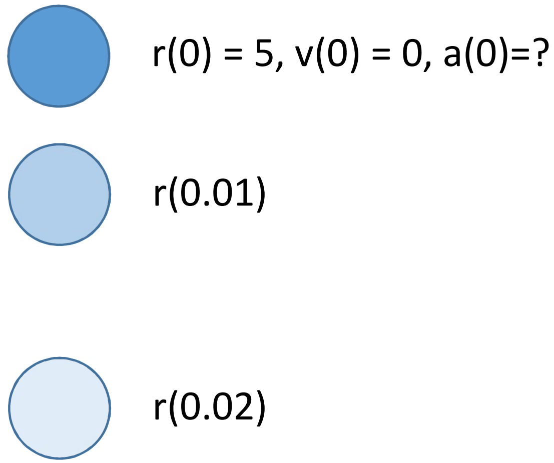
Initial height 5m, initial velocity 0m/s, time step 0.01s, g=-9.8 m/s^2, mass 1kg

- Rigid Body Dynamics
  - Solving Motion Equations
    - An example of rigid ball free-fall

$$\mathbf{r}(t_2) = \mathbf{r}(t_1) + \mathbf{v}(t_1)\Delta t.$$

$$\mathbf{a}(t) = \frac{\mathbf{F}_{\text{net}}(t)}{m} = \dot{\mathbf{v}}(t)$$

$$\mathbf{v}(t_2) = \mathbf{v}(t_1) + \frac{\mathbf{F}_{\text{net}}(t_1)}{m}\Delta t$$



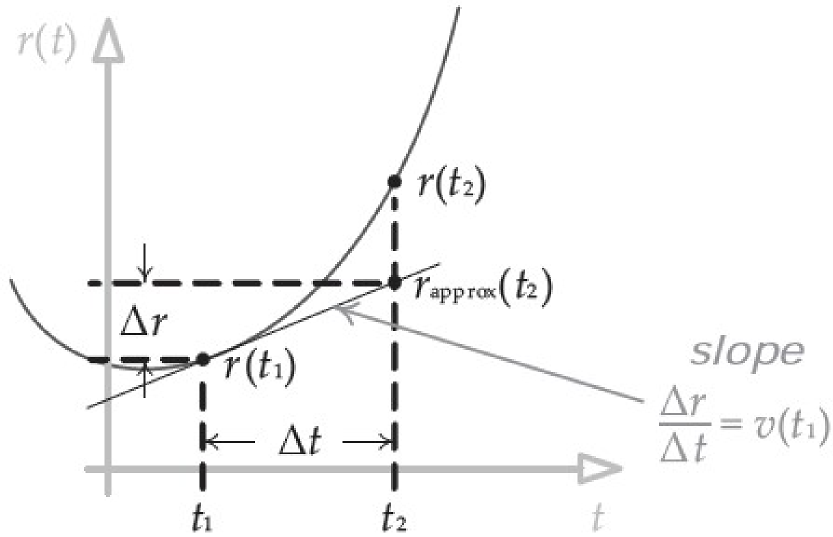
Collisions and Body Dynamics

- Rigid Body Dynamics
  - Solving Motion Equations
    - Numerical Integration (compute the system step by step)
      - Explicit Euler

$$\mathbf{r}(t_2) = \mathbf{r}(t_1) + \mathbf{v}(t_1)\Delta t.$$

$$\mathbf{a}(t) = \frac{\mathbf{F}_{\text{net}}(t)}{m} = \dot{\mathbf{v}}(t)$$

$$\mathbf{v}(t_2) = \mathbf{v}(t_1) + \frac{\mathbf{F}_{\text{net}}(t_1)}{m}\Delta t$$



Collisions and Body Dynamics

- Rigid Body Dynamics
  - Solving Motion Equations
    - Numerical Integration (compute the system step by step)
    - Properties
      - Convergence
      - Order

$$\mathbf{r}(t_2) = \mathbf{r}(t_1) + \dot{\mathbf{r}}(t_1)\,\Delta t. \qquad \mathbf{r}(t_2) = \mathbf{r}(t_1) + \dot{\mathbf{r}}(t_1)\,\Delta t + \tfrac{1}{2}\ddot{\mathbf{r}}(t_1)\,\Delta t^2 + \tfrac{1}{6}\dddot{\mathbf{r}}(t_1)\,\Delta t^3 + \ldots.$$

$$\mathbf{E} = \tfrac{1}{2}\ddot{\mathbf{r}}(t_1)\,\Delta t^2 + \tfrac{1}{6}\dddot{\mathbf{r}}(t_1)\,\Delta t^3 + \ldots \; = O(\Delta t^2) \qquad \mathbf{r}(t_2) = \mathbf{r}(t_1) + \dot{\mathbf{r}}(t_1)\,\Delta t + O(\Delta t^2)$$

      - Stability

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Solving Motion Equations
    - Numerical Integration (compute the system step by step)
    - Properties
      - Convergence
      - Order
      - Stability

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Solving Motion Equations
    - Numerical Integration (compute the system step by step)
    - Alternative to Explicit Euler
      - Implicit Euler, midpoint Euler, Runge-Kutta

$$\mathbf{r}(t_2) = \mathbf{r}(t_1) + \mathbf{v}(t_1)\Delta t.$$

t_2

$$\mathbf{a}(t) = \frac{\mathbf{F}_{\text{net}}(t)}{m} = \dot{\mathbf{v}}(t)$$

$$\mathbf{v}(t_2) = \mathbf{v}(t_1) + \frac{\mathbf{F}_{\text{net}}(t_1)}{m}\Delta t$$

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Solving Motion Equations
    - Numerical Integration (compute the system step by step)
    - Alternative to Explicit Euler
    - Verlet (higher-order, cheap to evaluate)

$$\mathbf{r}(t_1+\Delta t)=\mathbf{r}(t_1)+\dot{\mathbf{r}}(t_1)\Delta t+\tfrac{1}{2}\ddot{\mathbf{r}}(t_1)\Delta t^2+\tfrac{1}{6}\dddot{\mathbf{r}}(t_1)\Delta t^3+O(\Delta t^4) \qquad \mathbf{r}(t_1-\Delta t)=\mathbf{r}(t_1)-\dot{\mathbf{r}}(t_1)\Delta t+\tfrac{1}{2}\ddot{\mathbf{r}}(t_1)\Delta t^2-\tfrac{1}{6}\dddot{\mathbf{r}}(t_1)\Delta t^3+O(\Delta t^4)$$

$$\mathbf{r}(t_1+\Delta t)=2\mathbf{r}(t_1)-\mathbf{r}(t_1-\Delta t)+\mathbf{a}(t_1)\Delta t^2+O(\Delta t^4)$$

$$\mathbf{r}(t_1+\Delta t)=2\mathbf{r}(t_1)-\mathbf{r}(t_1-\Delta t)+\frac{\mathbf{F}_{\text{net}}(t_1)}{m}\Delta t^2+O(\Delta t^4) \qquad \mathbf{v}(t_1+\Delta t)=\frac{\mathbf{r}(t_1+\Delta t)-\mathbf{r}(t_1)}{\Delta t}+O(\Delta t)$$

Collisions and Body Dynamics

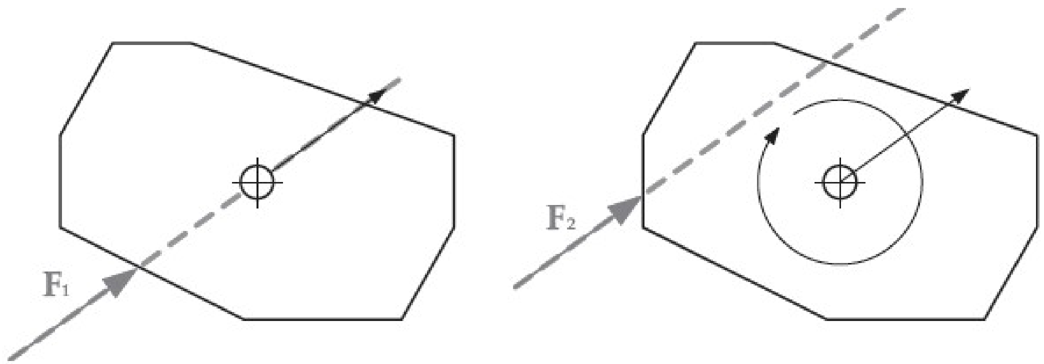
- Rigid Body Dynamics
  - Solving Motion Equations
    - Numerical Integration (compute the system step by step)
    - Alternative to Explicit Euler
    - Verlet (higher-order, cheap to evaluate)
    - Velocity Verlet

$$\mathbf{a}(t_1) = \frac{\mathbf{F}(t_1,\ \mathbf{r}(t_1),\ \mathbf{v}(t_1))}{m}$$

1. Calculate $\mathbf{r}(t_1 + \Delta t) = \mathbf{r}(t_1) + \mathbf{v}(t_1)\Delta t + \frac{1}{2}\mathbf{a}(t_1)\Delta t^2$.
2. Calculate $\mathbf{v}(t_1 + \frac{1}{2}\Delta t) = \mathbf{v}(t_1) + \frac{1}{2}\mathbf{a}(t_1)\Delta t$.
3. Determine $\mathbf{a}(t_1 + \Delta t) = \mathbf{a}(t_2) = \dfrac{\mathbf{F}(t_2,\ \mathbf{r}(t_2),\ \mathbf{v}(t_2))}{m}$.
4. Calculate $\mathbf{v}(t_1 + \Delta t) = \mathbf{v}(t_1 + \frac{1}{2}\Delta t) + \frac{1}{2}\mathbf{a}(t_1 + \Delta t)\Delta t$

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Angular Dynamics
    - Moment of Inertia (rotational equivalent of mass)



Collisions and Body Dynamics

- Rigid Body Dynamics
  - Angular Dynamics
    - Angular Speed and Acceleration

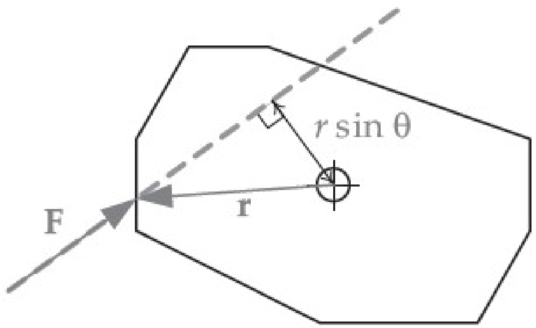
Angular: $\omega(t) = \dfrac{d\theta(t)}{dt} = \dot{\theta}(t);$ | Linear: $\mathbf{v}(t) = \dfrac{d\mathbf{r}(t)}{dt} = \dot{\mathbf{r}}(t).$

Angular: $\alpha(t) = \dfrac{d\omega(t)}{dt} = \dot{\omega}(t) = \ddot{\theta}(t);$ | Linear: $\mathbf{a}(t) = \dfrac{d\mathbf{v}(t)}{dt} = \dot{\mathbf{v}}(t) = \ddot{\mathbf{r}}(t).$

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Angular Dynamics
    - Moment of Inertia (rotational equivalent of mass)
    - Torque

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}.$$

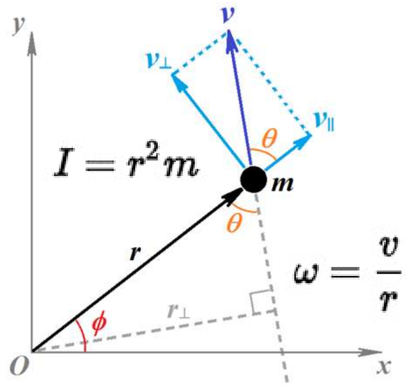


Angular: $$N_z = I\alpha(t) = I\dot{\omega}(t) = I\ddot{\theta}(t);$$

Linear: $$\mathbf{F} = m\mathbf{a}(t) = m\dot{\mathbf{v}}(t) = m\ddot{\mathbf{r}}(t)$$

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Angular Dynamics
    - Solving Angular Motion Equations



Angular: $$N_{\text{net}}(t) = I\dot{\omega}(t); \qquad \omega(t) = \dot{\theta}(t);$$

Linear: $$\mathbf{F}_{\text{net}}(t) = m\dot{\mathbf{v}}(t); \qquad \mathbf{v}(t) = \dot{\mathbf{r}}(t),$$

Angular: $$\omega(t_2) = \omega(t_1) + \frac{N_{\text{net}}(t_1)}{I}\Delta t; \qquad \theta(t_2) = \theta(t_1) + \omega(t_1)\Delta t;$$

Linear: $$\mathbf{v}(t_2) = \mathbf{v}(t_1) + \frac{\mathbf{F}_{\text{net}}(t_1)}{m}\Delta t; \qquad \mathbf{r}(t_2) = \mathbf{r}(t_1) + \mathbf{v}(t_1)\Delta t.$$

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Angular Dynamics
    - Solving Angular Motion Equations

1. Calculate $\theta(t_1 + \Delta t) = \theta(t_1) + \omega(t_1)\Delta t + \frac{1}{2}\alpha(t_1)\Delta t^2$.
2. Calculate $\omega(t_1 + \frac{1}{2}\Delta t) = \omega(t_1) + \frac{1}{2}\alpha(t_1)\Delta t$.
3. Determine $\alpha(t_1 + \Delta t) = \alpha(t_2) = I^{-1}N_{\mathrm{net}}(t_2,\ \theta(t_2),\ \omega(t_2))$.
4. Calculate $\omega(t_1 + \Delta t) = \omega(t_1 + \frac{1}{2}\Delta t) + \frac{1}{2}\alpha(t_1 + \Delta t)\Delta t$.

1. Calculate $\mathbf{r}(t_1 + \Delta t) = \mathbf{r}(t_1) + \mathbf{v}(t_1)\Delta t + \frac{1}{2}\mathbf{a}(t_1)\Delta t^2$.
2. Calculate $\mathbf{v}(t_1 + \frac{1}{2}\Delta t) = \mathbf{v}(t_1) + \frac{1}{2}\mathbf{a}(t_1)\Delta t$.
3. Determine $\mathbf{a}(t_1 + \Delta t) = \mathbf{a}(t_2) = \dfrac{\mathbf{F}(t_2,\ \mathbf{r}(t_2),\ \mathbf{v}(t_2))}{m}$.
4. Calculate $\mathbf{v}(t_1 + \Delta t) = \mathbf{v}(t_1 + \frac{1}{2}\Delta t) + \frac{1}{2}\mathbf{a}(t_1 + \Delta t)\Delta t$

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Angular Dynamics in 3D
    - Inertia Tensor

$$\text{Angular:}\quad \begin{aligned} N_z &= I\alpha(t) \\ &= I\dot{\omega}(t) = I\ddot{\theta}(t); \end{aligned} \qquad \Bigg| \qquad \text{Linear:}\quad \begin{aligned} \mathbf{F} &= m\mathbf{a}(t) \\ &= m\dot{\mathbf{v}}(t) = m\ddot{\mathbf{r}}(t) \end{aligned}$$

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Angular Dynamics in 3D
    - Orientation

$$\mathrm{q} = [q_x \quad q_y \quad q_z \quad q_w] = [\mathbf{q} \quad q_w]$$
$$= \left[\mathbf{u}\sin\left(\tfrac{\theta}{2}\right) \quad \cos\left(\tfrac{\theta}{2}\right)\right].$$

Collisions and Body Dynamics

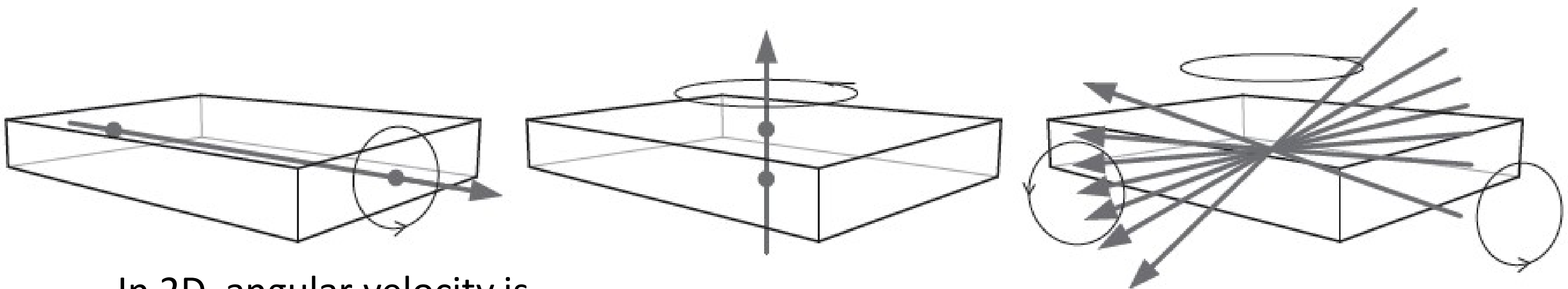
- Rigid Body Dynamics
  - Angular Dynamics in 3D
    - Angular Velocity and Momentum

Rotational axis

$$\boldsymbol{\omega}(t) = \omega_u(t)\,\mathbf{u}(t) = \dot{\theta}_u(t)\,\mathbf{u}(t).$$

Angular: $\mathbf{L}(t) = \mathbf{I}\,\boldsymbol{\omega}(t)$; | Linear: $\mathbf{p}(t) = m\,\mathbf{v}(t)$.

$$\mathrm{q} = [q_x \quad q_y \quad q_z \quad q_w] = [\mathbf{q} \quad q_w]$$
$$= \left[\mathbf{u}\sin\left(\tfrac{\theta}{2}\right) \quad \cos\left(\tfrac{\theta}{2}\right)\right].$$



In 2D, angular velocity is constant under 0 forces

In 3D, angular velocity might NOT be constant under 0 forces. Angular Velocity is NOT conserved

Collisions and Body Dynamics

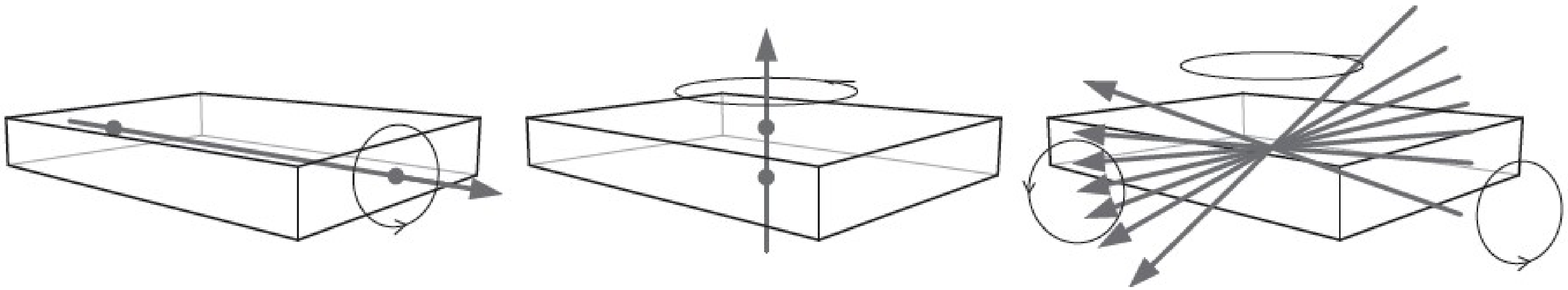
- Rigid Body Dynamics
  - Angular Dynamics in 3D
    - Angular Velocity and Momentum

Angular Momentum is conserved

$$\boldsymbol{\omega}(t) = \boldsymbol{\omega}_u(t)\,\mathbf{u}(t) = \dot{\theta}_u(t)\,\mathbf{u}(t).$$

Angular: $\mathbf{L}(t) = \mathbf{I}\,\boldsymbol{\omega}(t)$; | Linear: $\mathbf{p}(t) = m\,\mathbf{v}(t)$.

$$\begin{bmatrix} L_x(t) \\ L_y(t) \\ L_z(t) \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x(t) \\ \omega_y(t) \\ \omega_z(t) \end{bmatrix}$$



Collisions and Body Dynamics

- Rigid Body Dynamics
  - Angular Dynamics in 3D
    - Torque

$$\mathbf{N} = \mathbf{I}\boldsymbol{\alpha}(t) = \mathbf{I}\frac{d\boldsymbol{\omega}(t)}{dt} = \frac{d}{dt}\left(\mathbf{I}\boldsymbol{\omega}(t)\right) \quad = \frac{d\mathbf{L}(t)}{dt}$$

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Angular Dynamics in 3D
    - Solving Angular Motion Equations

$$\text{A3D(?):}\ \begin{aligned}&\boldsymbol{\omega}(t_2)=\boldsymbol{\omega}(t_1)+\mathbf{I}^{-1}\,\mathbf{N}_{\text{net}}(t_1)\,\Delta\ ;\\&\boldsymbol{\theta}(t_2)=\boldsymbol{\theta}(t_1)+\boldsymbol{\omega}(t_1)\,\Delta t;\end{aligned}\quad\Bigg|\quad \text{L:}\ \begin{aligned}&\mathbf{v}(t_2)=\mathbf{v}(t_1)+\frac{\mathbf{F}_{\text{net}}(t_1)}{m}\Delta t;\\&\mathbf{r}(t_2)=\mathbf{r}(t_1)+\mathbf{v}(t_1)\,\Delta t.\end{aligned}$$

No! 1: Angular moment (not angular velocity) needs to be conserved 2: Needs to map quaternions to 3D vector

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Angular Dynamics in 3D
    - Solving Angular Motion Equations

$$\mathbf{N}_{\text{net}}(t) = \dot{\mathbf{L}}(t);$$

$$\boldsymbol{\omega}(t) = \mathbf{I}^{-1}\mathbf{L}(t);$$

$$\omega(t) = [\boldsymbol{\omega}(t) \quad 0];$$

$$\tfrac{1}{2}\omega(t)\,\mathrm{q}(t) = \dot{\mathrm{q}}(t);$$

$$\mathbf{F}_{\text{net}}(t) = \dot{\mathbf{p}}(t);$$

$$\text{L:} \quad \mathbf{v}(t) = \frac{\mathbf{p}(t)}{m};$$

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t).$$

$$\omega = \begin{bmatrix} \omega_x & \omega_y & \omega_z & 0 \end{bmatrix}$$

Quaternion Product

$$\frac{d\,\mathrm{q}(t)}{dt} = \dot{\mathrm{q}}(t) = \underbrace{\boxed{\tfrac{1}{2}\omega(t)\,\mathrm{q}(t)}}_{\text{Angular Velocity Quaternion}}$$

Collisions and Body Dynamics

- Rigid Body Dynamics
  - Angular Dynamics in 3D
    - Solving Angular Motion Equations

$$\mathbf{N}_{\text{net}}(t) = \dot{\mathbf{L}}(t); \qquad \mathbf{L}(t_2) = \mathbf{L}(t_1) + \mathbf{N}_{\text{net}}(t_1)\Delta t$$

$$\boldsymbol{\omega}(t) = \mathbf{I}^{-1}\mathbf{L}(t); \qquad = \mathbf{L}(t_1) + \Delta t \sum \left(\mathbf{r}_i \times \mathbf{F}_i(t_1)\right); \qquad \text{(vectors)}$$

$$\omega(t) = [\boldsymbol{\omega}(t) \quad 0]; \qquad \omega(t_2) = [\mathbf{I}^{-1}\,\mathbf{L}(t_2) \quad 0]; \qquad \text{(quaternion)}$$

$$\tfrac{1}{2}\omega(t)\,\mathrm{q}(t) = \dot{\mathrm{q}}(t); \qquad \mathrm{q}(t_2) = \mathrm{q}(t_1) + \tfrac{1}{2}\omega(t_1)\,\mathrm{q}(t_1)\,\Delta t. \qquad \text{(quaternions)}$$