Animation & Simulation

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Fluids: Liquids and Gases

- Computational Fluid Dynamics (CFD, no more cheating)
  - Molecules moving under factors such as density, temperature and momentum
  - Assumed to be continumm, properties are smooth throughout the field
  - Compressible (gases) vs Incompressible (fluids)
  - In *steady-state* flow, motion attributes (vel, acc) are constant in space
  - Vortices, circular swirls, are time invariant in steady-state flow, not in time-varying flows

Fluids: Liquids and Gases

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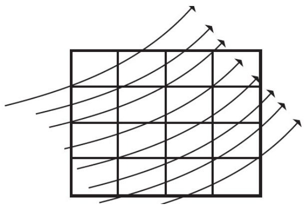
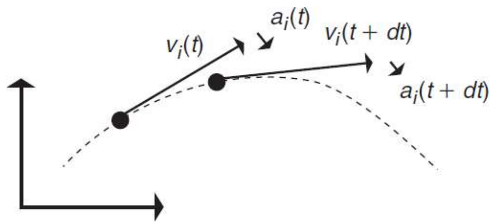
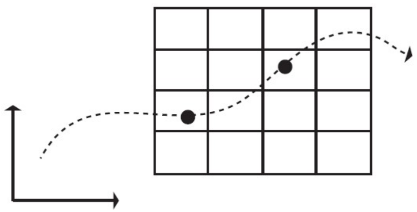
  - Euler method
    - Discretise space into grids, monitor grid cells in space and time



  - Lagrangian method
    - Discretise matter (into particles, e.g.), trace individual particles

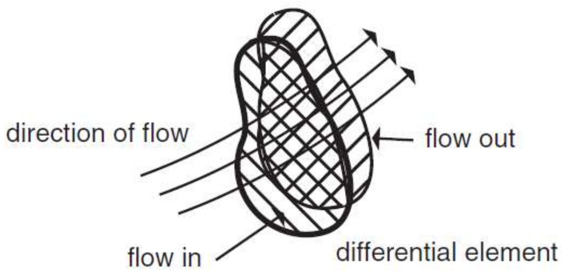


  - Hybrid



Fluids: Liquids and Gases

- Computational Fluid Dynamics (CFD, no more cheating)
  - Animating fluids mean recover the geometry and compute the rendering properties
    - Geometry->boundaries, rendering properties->density
  - CFD equations
    - Mass is conserved
    - Momentum is conserved
    - Energy is conserved



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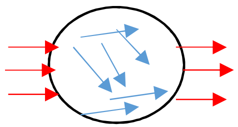
  - CFD equations
    - Mass is conserved
      - Density can change, the total mass cannot
      - Mass flow in/out of a cell
      - Divergence to describe

        the divergence for a field, $F = (Fx, Fy, Fz)$, is $div\, F = \nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

      - Divergence theorem: divergence over the volume is equal to the integral of the flow over the boundary



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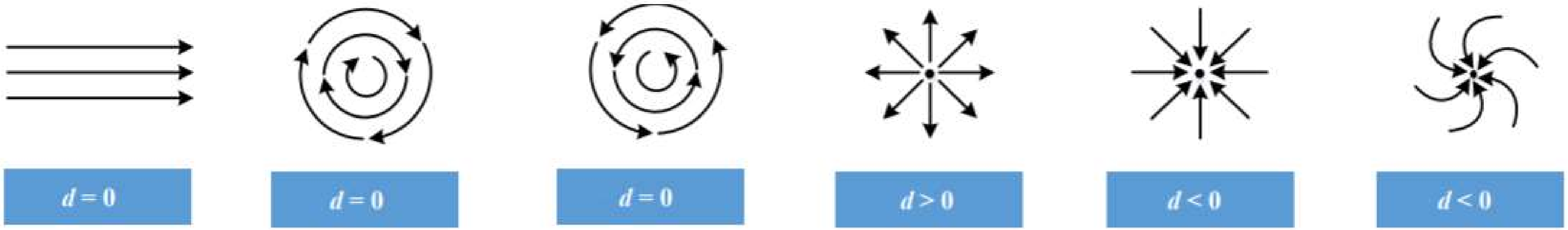

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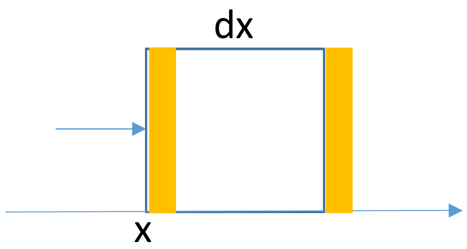
density volume Surface area

$$-\frac{\partial(\rho V)}{\partial t} = (\rho v_x A)|_{x+dx} - (\rho v_x A)|_x$$



V = dxdydz A = dydz

$$-\frac{d(\rho dxdydz)}{dt} = (\rho v_x dydz)|_{x+dx} - (\rho v_x dydz)|_x$$

$$\Downarrow$$

$$-\frac{\partial\rho}{\partial t} = \frac{\rho v_x}{dx}\Big|_{x+dx} - \frac{\rho v_x}{dx}\Big|_x \Rightarrow -\frac{\partial\rho}{\partial t} = \frac{\partial(\rho v_x)}{\partial x}$$

Extend to 3D $-\frac{\partial\rho}{\partial t} = \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} \xrightarrow{\nabla \cdot \mathrm{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}} -\frac{\partial\rho}{\partial t} = \nabla \cdot (\rho v) \xrightarrow[\text{Constant density thus divided on both sides}]{\text{If incompressible } \frac{\partial\rho}{\partial t} \text{ is zero}} 0 = \nabla \cdot \mathrm{v}$

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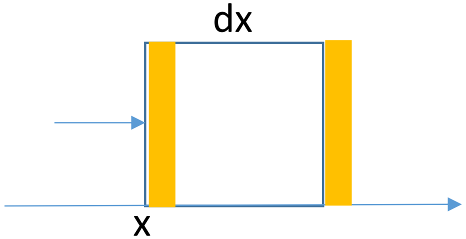
    - Momentum is conserved
      - P = mv, a change in time $\frac{d(mv)}{dt}$ caused by force $f = ma = m\frac{dv}{dt}$
      - Gravity, viscosity or change of pressure $\frac{dp}{dx}$
      - In V, change of momentum is $\frac{\partial(\rho V v)}{\partial t}$

Force — Momentum change inside + outside

$$\boxed{-\left(p|^{A}_{x+dx} - p|^{A}_{x}\right)} = \boxed{\frac{\partial(\rho V v)}{\partial t}} + \boxed{\left((\rho v_x A)v|_{x+dx} - (\rho v_x A)v|_{x}\right)}$$



V = dxdydz A = dydz $-\frac{\partial p}{\partial x} = \frac{\partial(\rho v_x v)}{\partial x} + \frac{\partial(\rho v)}{\partial t}$

Only consider v = v_x
The same for v_y and v_z

$$-\frac{\partial p}{\partial t} = \frac{\partial(\rho v_x^2)}{\partial x} + \frac{\partial(\rho v_x v_y)}{\partial y} + \frac{\partial(\rho v_x v_z)}{\partial z} + \frac{\partial(\rho v_x)}{\partial t}$$

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    - Geometry->boundaries, rendering properties->density
  - CFD equations
    - Navier-Stokes (NS) equations

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho v)$$

Mass conservation

$$-\frac{\partial p}{\partial t} = \frac{\partial(\rho v_x^2)}{\partial x} + \frac{\partial(\rho v_x v_y)}{\partial y} + \frac{\partial(\rho v_x v_z)}{\partial z} + \frac{\partial(\rho v_x)}{\partial t}$$

Momentum conservation

Velocity field

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \frac{\partial \mathbf{u}}{\partial t} &= -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \qquad \nabla^2 = \nabla \cdot \nabla \end{aligned}$$

viscosity

External force

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- Computational Fluid Dynamics (CFD, no more cheating)
  - Grid-based approach (Eulerian)
    - Discretise the space into grids, attributes are evaluated in grid cells
    - Discretise the NS equations accordingly
    - Numerically solve the system to simulate fluids by using Newton-like methods
    - Boundaries, solid/fluid, are formed as constraints
      - Dirichlet, Neumann, Cauchy

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  - Grid-based approach (Eulerian)
    - Stable fluids (Siggraph 98), inaccurate for engineering, good enough for graphics

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \frac{\partial \mathbf{u}}{\partial t} &= -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \end{aligned}$$

    - Helmholtz-Hodge Decomposition

Mass-conserving field Gradient field

Any vector field $\mathbf{w} = \mathbf{u} + \nabla q,$

where $\mathbf{u}$ has zero divergence: $\nabla \cdot \mathbf{u} = 0$ and q is a scalar field

We can define an operator P that projects w onto u, u = Pw = w - ∇q

It is implicitly defined by multiply both sides with ∇

Field boundary

$\nabla \cdot \mathbf{w} = \nabla^2 q$ a Poisson equation for *q* with Neumann boundary $\frac{\partial q}{\partial n} = 0$ on ∂D

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Mass-conserving field

Gradient field

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$\nabla \cdot \mathbf{w} = \nabla^2 q$ a Poisson equation for *q* with Neumann boundary $\frac{\partial q}{\partial n} = 0$ on ∂D

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \Rightarrow \frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(-(\mathbf{u} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \mathbf{u} + \mathbf{f}\right)$$

$$\mathbf{P}\mathbf{u} = \mathbf{u} \text{ and } \mathbf{P}\nabla p = 0$$

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$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \nu\nabla^2\mathbf{u} + \mathbf{f}\right) \quad \mathbf{P}\mathbf{u} = \mathbf{u} \text{ and } \mathbf{P}\nabla p = 0$$

    - How to solve?
      - Given the initial state $\mathbf{u}_0 = \mathbf{u}(\mathbf{x}, 0)$, for any time t $\mathbf{w}_0(\mathbf{x}) = \mathbf{u}(\mathbf{x}, t)$

$$\mathbf{w}_0(\mathbf{x}) \overbrace{\longrightarrow}^{\text{add force}} \mathbf{w}_1(\mathbf{x}) \overbrace{\longrightarrow}^{\text{advect}} \mathbf{w}_2(\mathbf{x}) \overbrace{\longrightarrow}^{\text{diffuse}} \mathbf{w}_3(\mathbf{x}) \overbrace{\longrightarrow}^{\text{project}} \mathbf{w}_4(\mathbf{x})$$

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    - First step, assume force stays constant within the time step

$$\mathbf{w}_1(\mathbf{x}) = \mathbf{w}_0(\mathbf{x}) + \Delta t\ \mathbf{f}(\mathbf{x}, t)$$

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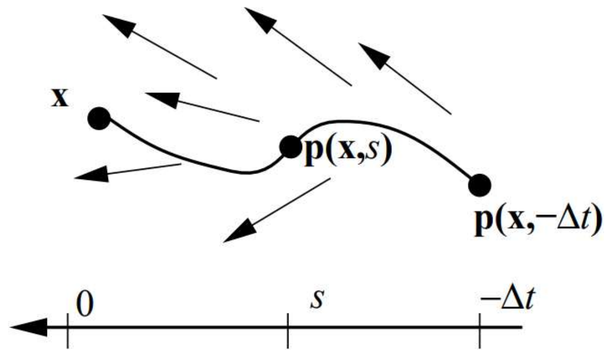
$$\mathbf{w}_0(\mathbf{x}) \overset{\text{add force}}{\longrightarrow} \mathbf{w}_1(\mathbf{x}) \overset{\text{advect}}{\longrightarrow} \mathbf{w}_2(\mathbf{x}) \overset{\text{diffuse}}{\longrightarrow} \mathbf{w}_3(\mathbf{x}) \overset{\text{project}}{\longrightarrow} \mathbf{w}_4(\mathbf{x})$$

- Second step, advection or convection, $-(\mathbf{u}\cdot\nabla)\mathbf{u}$--how disturbances propagate in fluids

$$\mathbf{w}_1(\mathbf{x}) = \mathbf{w}_0(\mathbf{x}) + \Delta t\ \mathbf{f}(\mathbf{x},t)$$

$$\mathbf{w}_2(\mathbf{x}) = \mathbf{w}_1(\mathbf{p}(\mathbf{x}, -\Delta t))$$



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- Third step, solve for viscosity, a diffuse equation

$$\mathbf{w}_1(\mathbf{x}) = \mathbf{w}_0(\mathbf{x}) + \Delta t\ \mathbf{f}(\mathbf{x}, t)$$
$$\mathbf{w}_2(\mathbf{x}) = \mathbf{w}_1(\mathbf{p}(\mathbf{x}, -\Delta t))$$

$$\frac{\partial \mathbf{w}_2}{\partial t} = \nu\nabla^2\mathbf{w}_2.$$

$\mathbf{I}$ is the identity operator.
diffusion operator $\nabla^2 = \nabla\cdot\nabla$

to use an implicit method: $\left(\mathbf{I} - \nu\Delta t\nabla^2\right)\mathbf{w}_3(\mathbf{x}) = \mathbf{w}_2(\mathbf{x})$

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    - Fourth step, projection to make the field divergence free

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$$\mathbf{w}_2(\mathbf{x}) = \mathbf{w}_1(\mathbf{p}(\mathbf{x}, -\Delta t))$$
$$\left(\mathbf{I} - \nu\Delta t\nabla^2\right)\mathbf{w}_3(\mathbf{x}) = \mathbf{w}_2(\mathbf{x})$$

$$\nabla^2 q = \nabla\cdot\mathbf{w}_3 \qquad \mathbf{w}_4 = \mathbf{w}_3 - \nabla q$$

Sparse Linear system

Helmholtz-Hodge Decomposition

$$\mathbf{w} = \mathbf{u} + \nabla q,$$

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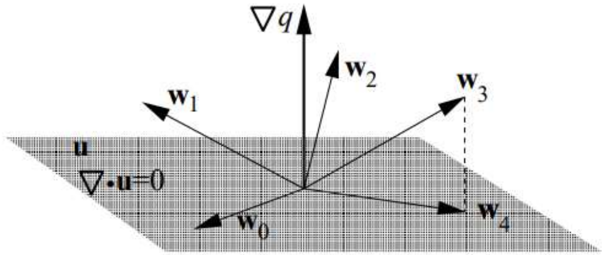

    - Other topics: periodic boundaries, moving substances in fluids

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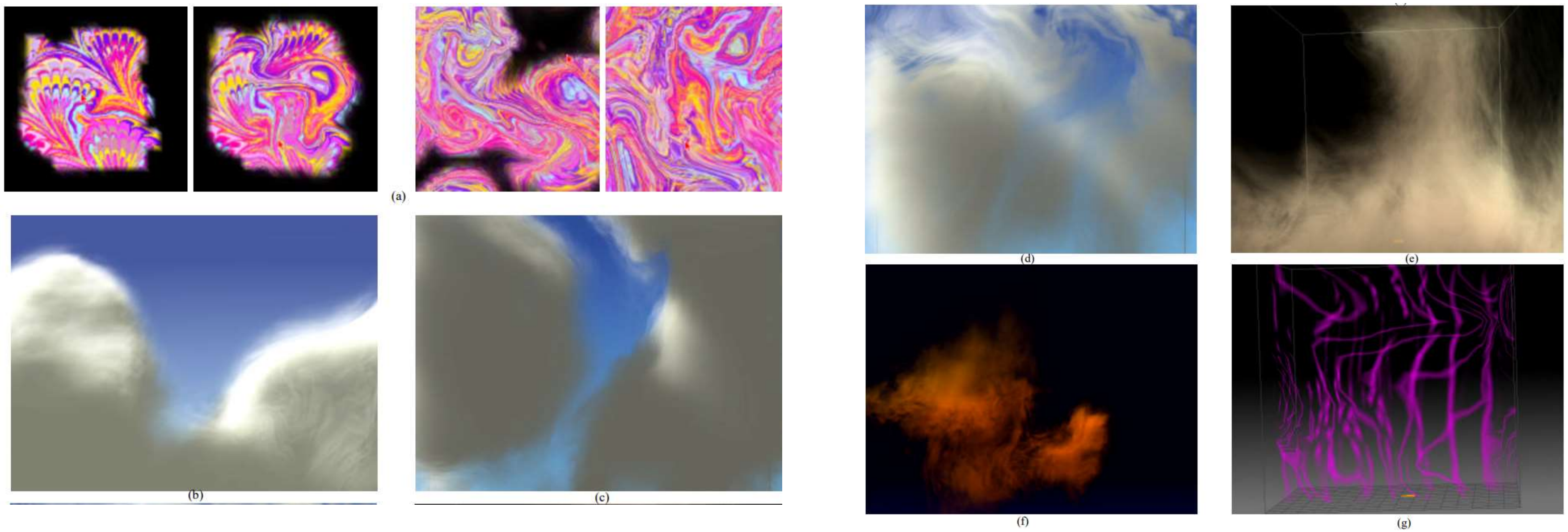

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- Computational Fluid Dynamics (CFD, no more cheating)
  - Grid-based approach (Eulerian)
    - More accurate, easy to render the surface
    - Slow