

# Notes

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## 1 MASTER algorithm

Let  ${}_{s_a}\mathbf{I}^a(\hat{\mathbf{n}})$  be a spin- $s_a$  field, where  $s_a$  can be 0 (i.e. 1 single component - e.g.  $T$ ) or 2 (i.e. 2 components - e.g.  $(Q, U)$ ). The observed map is:

$${}_{s_a}\tilde{\mathbf{I}}^a(\hat{\mathbf{n}}) = w^a(\hat{\mathbf{n}}) [{}_{s_a}\mathbf{I}^a(\hat{\mathbf{n}}) + N^a(\hat{\mathbf{n}})], \quad (1)$$

where  $w(\hat{\mathbf{n}})$  is the weights map. The harmonic coefficients of the observed map can be written as:

$${}_{s_a}\tilde{\mathbf{I}}^a_{\ell_1 m_1} = \sum_{\ell_2, m_2} {}_{s_a}\mathbf{W}^a_{\ell_1 \ell_2, m_1 m_2} \cdot {}_{s_a}\mathbf{I}^a_{\ell_2 m_2}, \quad (2)$$

where the mixing matrix is

$${}_{s_a}\mathbf{W}^a_{\ell_1 \ell_2, m_1 m_2} \equiv (-1)^m \sum_{\ell_3, m_3} w^a_{\ell_3 m_3} \left[ \frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)}{4\pi} \right]^{1/2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -m_1 & m_2 & m_3 \end{pmatrix} {}_{s_a}\mathbf{J}_{\ell_1 \ell_2 \ell_3} \quad (3)$$

with

$${}_0\mathbf{J}_{\ell_1 \ell_2 \ell_3} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4)$$

$${}_2\mathbf{J}_{\ell_1 \ell_2 \ell_3} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 2 & -2 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 + (-1)^{\ell_1 + \ell_2 + \ell_3} & i[(-1)^{\ell_1 + \ell_2 + \ell_3} - 1] \\ -i[(-1)^{\ell_1 + \ell_2 + \ell_3} - 1] & 1 + (-1)^{\ell_1 + \ell_2 + \ell_3} \end{pmatrix} \quad (5)$$

Let us define the pseudo-power-spectrum  $\tilde{\mathbf{C}}_\ell^{ab}$

$$\tilde{\mathbf{C}}_\ell^{ab} \equiv \frac{1}{2\ell + 1} \sum_m {}_{s_a}\mathbf{I}^a_{\ell m} \cdot ({}_{s_b}\mathbf{I}^b_{\ell m})^\dagger \quad (6)$$

The relation between the pseudo-power-spectrum and the true power spectrum  $\mathbf{C}_\ell^{ab}$  can be derived to be of the form

$$\langle \tilde{\mathbf{C}}_\ell^{ab} \rangle = \sum_{\ell'} \mathbf{M}_{\ell \ell'}^{s_a s_b} \cdot \mathbf{C}_{\ell'}^{ab}, \quad (7)$$

where the mode-coupling matrix  $\mathbf{M}_{\ell \ell'}^{s_a s_b}$  takes the form:

- Case  $s_a = s_b = 0$ :

$$\langle \tilde{C}_\ell^{T_a T_b} \rangle = \sum_{\ell'} M_{\ell \ell'}^{00} C_{\ell'}^{T_a T_b} \quad (8)$$

with

$$M_{\ell \ell'}^{00} = \frac{2\ell' + 1}{4\pi} \sum_{\ell''} W_{\ell \ell'}^{ab} \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{pmatrix}^2 \quad (9)$$

- Case  $s_a = 0, s_b = 2$ :

$$\left\langle \begin{pmatrix} \tilde{C}_\ell^{T_a E_b} \\ \tilde{C}_\ell^{T_a B_b} \end{pmatrix} \right\rangle = \sum_{\ell'} \begin{pmatrix} M_{\ell \ell'}^{0+} & 0 \\ 0 & M_{\ell \ell'}^{0+} \end{pmatrix} \cdot \begin{pmatrix} C_{\ell'}^{T_a E_b} \\ C_{\ell'}^{T_a B_b} \end{pmatrix} \quad (10)$$

with

$$M_{\ell \ell'}^{0+} = \frac{2\ell' + 1}{4\pi} \sum_{\ell''} W_{\ell \ell'}^{ab} \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{pmatrix} \quad (11)$$

- Case  $s_a = 2, s_b = 2$ :

$$\left\langle \begin{pmatrix} \tilde{C}_\ell^{E_a E_b} \\ \tilde{C}_\ell^{E_a B_b} \\ \tilde{C}_\ell^{B_a E_b} \\ \tilde{C}_\ell^{B_a B_b} \end{pmatrix} \right\rangle = \sum_{\ell'} \begin{pmatrix} M_{\ell\ell'}^{++} & 0 & 0 & M_{\ell\ell'}^{--} \\ 0 & M_{\ell\ell'}^{++} & -M_{\ell\ell'}^{--} & 0 \\ 0 & -M_{\ell\ell'}^{--} & M_{\ell\ell'}^{++} & 0 \\ M_{\ell\ell'}^{--} & 0 & 0 & M_{\ell\ell'}^{++} \end{pmatrix} \cdot \begin{pmatrix} C_{\ell'}^{E_a E_b} \\ C_{\ell'}^{E_a B_b} \\ C_{\ell'}^{B_a E_b} \\ C_{\ell'}^{B_a B_b} \end{pmatrix} \quad (12)$$

with

$$M_{\ell\ell'}^{++} = \frac{2\ell' + 1}{4\pi} \sum_{\ell''} W_{\ell''}^{ab} \begin{pmatrix} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{pmatrix}^2 \frac{1 + (-1)^{\ell+\ell'+\ell''}}{2} \quad (13)$$

$$M_{\ell\ell'}^{--} = \frac{2\ell' + 1}{4\pi} \sum_{\ell''} W_{\ell''}^{ab} \begin{pmatrix} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{pmatrix}^2 \frac{1 - (-1)^{\ell+\ell'+\ell''}}{2}, \quad (14)$$

where in all these equations  $W_{\ell''}^{ab}$  is the cross-spectrum of the weights map (without the  $(2\ell + 1)$  normalization):

$$W_\ell^{ab} \equiv \sum_m w_{\ell m}^a (w_{\ell m}^b)^*. \quad (15)$$

Note that, in Eq. 7 one should add, on the right-hand side, the noise cross-power-spectrum:

$$\langle \tilde{\mathbf{N}}_\ell^{ab} \rangle \equiv \frac{1}{2\ell + 1} \sum_m \langle \mathbf{N}_{\ell m}^a \cdot (\mathbf{N}_{\ell m}^b)^\dagger \rangle \quad (16)$$

## 1.1 Beam

Adding the effect of a beam amounts to redefining:

$$\mathbf{M}_{\ell_1 \ell_2}^{s_a s_b} \rightarrow \mathbf{M}_{\ell_1 \ell_2}^{s_a s_b} b_{\ell_2}^{ab}, \quad (17)$$

where  $b_\ell^{ab}$  is the product of the harmonic transform of the beams for maps  $a$  and  $b$ .

## 1.2 Bandpowers

Consider the case where you want to compute the power spectrum in band-powers given by

$$\mathbf{B}_k^{ab} \equiv \frac{1}{N_k} \sum_{\ell=\ell_k}^{\ell_k + N_k - 1} f(\ell) \mathbf{C}_\ell^{ab}, \quad (18)$$

then Eq. 7 above becomes

$$\langle \tilde{\mathbf{B}}_k^{ab} \rangle = \sum_{k'} \mathbf{M}_{kk'}^{B, s_a s_b} \cdot \mathbf{B}_{k'}^{ab} + \langle \tilde{\mathbf{N}}_k^{B, ab} \rangle \quad (19)$$

where the binned coupling matrix  $\mathbf{M}^{B, s_a s_b}$  is

$$\mathbf{M}_{k_1, k_2}^{B, s_a s_b} \equiv \frac{1}{N_{k_1}} \sum_{\ell_1=\ell_{k_1}}^{\ell_{k_1} + N_{k_1} - 1} \sum_{\ell_2=\ell_{k_2}}^{\ell_{k_2} + N_{k_2} - 1} \frac{f(\ell_1)}{f(\ell_2)} \mathbf{M}_{\ell_1 \ell_2}^{s_a s_b} \quad (20)$$

## 2 Minimum variance quadratic estimator

Let  $\mathbf{d}$  be the data, given as a pixelized full-sky map, with covariance matrix

$$\langle \mathbf{d} \mathbf{d}^T \rangle \equiv C(\hat{\mu}) \equiv S(\hat{\mu}) + N(\hat{\mu}) = \sum_\ell C_\ell P_\ell(\hat{\mu}) + N(\hat{\mu}), \quad (21)$$

where  $\hat{\mu} = \mathbf{n} \mathbf{n}^T$ ,  $\mathbf{n}$  is the vector of unit vectors containing the angular coordinates to each pixel and  $P_\ell(x) \equiv (2\ell + 1)L_\ell(x)/(4\pi)$ , where  $L_\ell$  are the Legendre polynomials. Note that we can expand  $P_\ell$  as

$$P_\ell(\hat{\mu}) \equiv \sum_{m=-\ell}^{\ell} Y_{\ell m}(\mathbf{n}) \cdot Y_{\ell m}^\dagger(\mathbf{n}). \quad (22)$$

We will parametrize the power spectrum with a set of step functions, such that

$$C_\ell = \sum_b c_b \Theta_b(\ell), \quad \leftarrow \quad \Theta_b(\ell) = 1 \text{ if } \ell \in [\ell_{\min}^b, \ell_{\max}^b), 0 \text{ otherwise}, \quad (23)$$

thus we can write

$$C(\hat{\mu}) = \sum_b c_b Q_b(\hat{\mu}) + N(\hat{\mu}), \quad \text{with } Q_b(\mu) = \sum_\ell \Theta_b(\ell) P_\ell(\mu). \quad (24)$$

Let us write the most general quadratic estimator for the coefficients  $c_b$ :

$$\tilde{c}_b \equiv \mathbf{d}^T \hat{E}_b \mathbf{d} - B_b. \quad (25)$$

The mean value of  $\tilde{c}_b$  is:

$$\langle \tilde{c}_b \rangle = \sum_a c_a \text{Tr}(\hat{E}_b \hat{Q}_a) + \text{Tr}(\hat{E}_b \hat{N}) - B_b, \quad (26)$$

and therefore we can remove the noise bias by defining

$$B_b \equiv \text{Tr}(\hat{E}_b \hat{N}). \quad (27)$$

The covariance matrix of this estimator is:

$$\langle (\tilde{c}_a - \langle \tilde{c}_a \rangle) (\tilde{c}_b - \langle \tilde{c}_b \rangle) \rangle = 2 \text{Tr}(\hat{C} \hat{E}_a \hat{C} \hat{E}_b). \quad (28)$$

Minimizing the variance of  $\tilde{c}_a$  under the constrain that the coupling matrix  $W_{ab} \equiv \text{Tr}(\hat{E}_b \hat{Q}_a)$  have unit diagonal, we find the following minimization problem:

$$L = \text{Tr} \left[ \hat{E}_b \hat{C} \hat{E}_b \hat{C} - 2(\lambda \hat{E}_b \hat{Q}_b - 1) \right], \quad (29)$$

where  $\lambda$  is the lagrange multiplier related to the constraint. The solution to this problem is:

$$\hat{E}_b = \frac{\hat{C}^{-1} \hat{Q}_b \hat{C}^{-1}}{\text{Tr}(\hat{C}^{-1} \hat{Q}_b \hat{C}^{-1} \hat{Q}_b)}. \quad (30)$$

Thus, a decoupled, unbiased and minimum-variance estimator,  $\hat{c}_b$ , for  $c_b$  can be found as:

$$\frac{\mathbf{d}^T \hat{C}^{-1} \hat{Q}_b \hat{C}^{-1} \mathbf{d} - \text{Tr}(\hat{C}^{-1} \hat{Q}_b \hat{C}^{-1} \hat{N})}{\text{Tr}(\hat{C}^{-1} \hat{Q}_b \hat{C}^{-1} \hat{Q}_b)} = \sum_a \hat{c}_a \frac{\text{Tr}(\hat{C}^{-1} \hat{Q}_b \hat{C}^{-1} \hat{Q}_a)}{\text{Tr}(\hat{C}^{-1} \hat{Q}_b \hat{C}^{-1} \hat{Q}_b)}. \quad (31)$$

Note that this estimator requires a guess for the covariance matrix of the data  $C(\hat{\mu})$ , and exact minimum variance will only be attained for the true covariance (which itself depends on the power spectrum coefficients  $c_b$ ). The choice of prior covariance will define the type of estimator. Note that this formalism immediately encompasses cut-skies by setting to zero all elements of the full covariance matrix involving unobserved pixels (this would correspond to the limit of infinite uncorrelated noise for those pixels).

## 2.1 Strategies for the different terms

- $\hat{C}^{-1} \mathbf{d}$ . This term can be computed by solving the linear system:

$$\hat{C} \mathbf{z} = \mathbf{d} \quad (32)$$

via conjugate gradients. The action of  $\hat{C} = \hat{S} + \hat{N}$  can be computed as follows:

$$[\hat{S}\mathbf{z}](\hat{\mathbf{n}}_1) \equiv \sum_{\hat{\mathbf{n}}_2} S(\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2) z(\hat{\mathbf{n}}_2) \quad (33)$$

$$= \frac{1}{\Delta\Omega} \int d\hat{\mathbf{n}}_2 \sum_{\ell m} C_\ell Y_{\ell m}^*(\hat{\mathbf{n}}_1) Y_{\ell m}^*(\hat{\mathbf{n}}_2) \tilde{z}(\hat{\mathbf{n}}_2) \quad (34)$$

$$= \frac{1}{\Delta\Omega} \sum_{\ell m} Y_{\ell m}^*(\hat{\mathbf{n}}_1) C_\ell \tilde{z}_{\ell m} \quad (35)$$

$$= \frac{1}{\Delta\Omega} \text{SHT}^{-1} [C_\ell \text{SHT}[\tilde{z}(\hat{\mathbf{n}})]_{\ell m}], \quad (36)$$

where  $\tilde{z}$  is the extension of  $z$  to the whole sphere, with  $\tilde{z} = 0$  in all unobserved pixels.

In most cases the noise power spectrum will be white, in which case  $\hat{N}\mathbf{z}$  is just  $\sum_{\hat{\mathbf{n}}} \sigma_N^2(\hat{\mathbf{n}}) z(\hat{\mathbf{n}})$ , with  $\sigma^2(\hat{\mathbf{n}})$  the per-pixel variance.

- $\mathbf{d}^T \hat{C}^{-1} \hat{Q}_b \hat{C}^{-1} \mathbf{d}$ . With  $\mathbf{z}$  defined as above, this is just:

$$\mathbf{z}^T \hat{Q}_b \mathbf{z} = \frac{1}{(\Delta\Omega)^2} \sum_{\ell} \Theta_b(\ell) \sum_m |\tilde{z}_{\ell m}|^2 \quad (37)$$