Notes

Dr. Frankenstein

January 25, 2017

1 Generalities and SHTs

Let $\mathbf{a}(\hat{\mathbf{n}})$ be a spin-s_a quantity defined on the sphere, where s_a can be either 0 or 2. Then we define its spherical harmonic coefficients as:

$$\mathbf{a}_{\ell m} \equiv \mathrm{SHT}(\mathbf{a}(\hat{\mathbf{n}}))^{s_a}_{\ell m} \equiv \int d\hat{\mathbf{n}}\, \hat{\mathsf{Y}}^{s_a\dagger}_{\ell m}(\hat{\mathbf{n}})\, \mathbf{a}(\hat{\mathbf{n}}), \quad \mathbf{a}(\hat{\mathbf{n}}) = \mathrm{SHT}^{-1}(\mathbf{a}_{\ell m})^{s_a}_{\hat{\mathbf{n}}} \equiv \sum_{\ell m} \hat{\mathsf{Y}}^{s_a}_{\ell m}(\hat{\mathbf{n}})\, \mathbf{a}_{\ell m}. \tag{1}$$

Here, for a scalar quantity $\mathbf{a}(\hat{\mathbf{n}}) = a(\hat{\mathbf{n}})$, $\mathbf{a}_{\ell m} = a_{\ell m}$ and $\hat{\mathsf{Y}}^0_{\ell m} = {}_0Y_{\ell m}$. A spin-2 quantity can be decomposed in real space into so-called Q and U components, and in harmonic space into E and B components:

$$\mathbf{a}(\hat{\mathbf{n}}) = \begin{pmatrix} Q(\hat{\mathbf{n}}) \\ U(\hat{\mathbf{n}}) \end{pmatrix}, \quad \mathbf{a}_{\ell m} = \begin{pmatrix} E_{\ell m} \\ B_{\ell m} \end{pmatrix}, \tag{2}$$

and the transform matrix is

$$\hat{\mathsf{Y}}^2_{\ell m} = \frac{1}{2} \begin{pmatrix} -({}_2Y_{\ell m} + {}_{-2}Y_{\ell m}) & -i({}_2Y_{\ell m} - {}_{-2}Y_{\ell m}) \\ i({}_2Y_{\ell m} - {}_{-2}Y_{\ell m}) & -({}_2Y_{\ell m} + {}_{-2}Y_{\ell m}) \end{pmatrix}. \tag{3}$$

The matrices $\hat{\mathsf{Y}}^s_{\ell m}$ satisfy the following relations:

$$\hat{\mathsf{Y}}^{s\dagger}_{\ell m} = (-1)^m \hat{\mathsf{Y}}^{s\dagger}_{\ell -m} \tag{4}$$

$$\int d\hat{\mathbf{n}} \hat{\mathsf{Y}}^s_{\ell m} \hat{\mathsf{Y}}^{s\dagger}_{\ell' m'} = \hat{1}\delta_{\ell\ell'}\delta_{mm'} \tag{5}$$

$$\hat{\mathsf{D}}^s_{\mathbf{l}\mathbf{l}_1\mathbf{l}_2} \equiv \int d\hat{\mathbf{n}} \left(\hat{\mathsf{Y}}^{s\dagger}_{\mathbf{l}}(\hat{\mathbf{n}})\, \hat{\mathsf{Y}}^s_{\mathbf{l}_1}(\hat{\mathbf{n}})\right) \hat{\mathsf{Y}}^0_{\mathbf{l}_2}(\hat{\mathbf{n}}), \tag{6}$$

$$= (-1)^m \sqrt{\frac{(2\ell+1)(2\ell_1+1)(2\ell_2+1)}{4\pi}} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ -m & m_1 & m_2 \end{pmatrix} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ s & -s & 0 \end{pmatrix} \hat{\mathsf{d}}^s_{\ell+\ell_1+\ell_2}, \tag{7}$$

$$\hat{\mathsf{d}}^0_n = 1, \quad \hat{\mathsf{d}}^2_n = \frac{1}{2} \begin{pmatrix} 1+(-1)^n & -i[1-(-1)^n] \\ i[1-(-1)^n] & 1+(-1)^n \end{pmatrix}, \tag{8}$$

where we have abbreviated the pair (ℓ, m) as $\mathbf{l}$.

Finally, the following orthogonality relation for the Wigner $3j$ symbols is useful:

$$\sum_{mm_1} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ m & m_1 & m_2 \end{pmatrix} \begin{pmatrix} \ell & \ell_1 & \ell_3 \\ m & m_1 & m_3 \end{pmatrix} = \frac{\delta_{\ell_2\ell_3}\delta_{m_2m_3}}{2\ell_2+1} \tag{9}$$

2 Contaminant cleaning

Let $\mathbf{a}$ be a random field defined on the sphere, let $v(\hat{\mathbf{n}})$ a mask for $\mathbf{a}$ and let $\mathbf{f}^i$ be a set of N_a contaminants of $\mathbf{a}$ such that the observed version of $\mathbf{a}$ be:

$$\mathbf{d}_a(\hat{\mathbf{n}}) = \mathbf{a}^v(\hat{\mathbf{n}}) + \sum_{i=1}^{N_a} \alpha_i \mathbf{f}^i(\hat{\mathbf{n}}), \tag{10}$$

where $\mathbf{a}^v(\hat{\mathbf{n}}) = v(\hat{\mathbf{n}})\,\mathbf{a}(\hat{\mathbf{n}})$ (note that we have implicitly applied the same mask to $\mathbf{f}^i$). The best-fit value for the coefficients α_i assuming the same weights for all points in $\mathbf{d}_a$ can be found as

$$\tilde{\alpha}_i = \sum_j M_{ij} \int d\hat{\mathbf{n}}\, \mathbf{f}^{j\dagger}(\hat{\mathbf{n}})\, \mathbf{d}_a(\hat{\mathbf{n}}), \quad (\hat{\mathsf{M}}^{-1})_{ij} \equiv \int d\hat{\mathbf{n}}\, \mathbf{f}^{i\dagger}(\hat{\mathbf{n}})\, \mathbf{f}^j(\hat{\mathbf{n}}). \tag{11}$$

Thus we can find a cleaned version of $\mathbf{a}$ as:

$$\tilde{\mathbf{a}}(\hat{\mathbf{n}}) \equiv \mathbf{d}_a(\hat{\mathbf{n}}) - \mathbf{f}^i(\hat{\mathbf{n}})\, M_{ij} \int d\hat{\mathbf{n}}' \mathbf{f}^{j\dagger}(\hat{\mathbf{n}}')\mathbf{d}_a(\hat{\mathbf{n}}') \tag{12}$$

$$= \mathbf{a}^v(\hat{\mathbf{n}}) - \mathbf{f}^i(\hat{\mathbf{n}})\, M_{ij} \int d\hat{\mathbf{n}}' \mathbf{f}^{j\dagger}(\hat{\mathbf{n}}')\mathbf{a}^v(\hat{\mathbf{n}}'), \tag{13}$$

where there is an implicit summation sign over i and j (we will omit these from now on).

The harmonic coefficients of the cleaned and masked field are:

$$\tilde{\mathbf{a}}_{\ell m} = \mathbf{a}^v_{\ell m} - \mathbf{f}^i_{\ell m} M_{ij} \sum_{\ell' m'} \mathbf{f}^{j\dagger}_{\ell' m'} \mathbf{a}^v_{\ell' m'}. \tag{14}$$

From now on we will simplify the notation by abbreviating the pair ℓm as $\mathbf{l}$, so that the previous equation reads:

$$\tilde{\mathbf{a}}_{\mathbf{l}} = \mathbf{a}^v_{\mathbf{l}} - \mathbf{f}^i_{\mathbf{l}} M_{ij} \sum_{\mathbf{l}'} \mathbf{f}^{j\dagger}_{\mathbf{l}'} \mathbf{a}^v_{\mathbf{l}'}. \tag{15}$$

The harmonic coefficients for the masked field can be related to those of the unmasked one and the mask v (understood as a spin-0 field) as:

$$\mathbf{a}^v_{\mathbf{l}} = \sum_{\mathbf{l}_1 \mathbf{l}_2} \hat{\mathsf{D}}^{s_a}_{\mathbf{l}\mathbf{l}_1\mathbf{l}_2} \mathbf{a}_{\mathbf{l}_1} v_{\mathbf{l}_2}. \tag{16}$$

3 Pseudo-C_ℓ estimators with mode deprojection

In what follows, for two fields $\mathbf{a}$ and $\mathbf{b}$ we will define their observed power spectrum as:

$$\tilde{\mathsf{C}}^{ab}_\ell \equiv \frac{1}{2\ell+1} \sum_m \mathbf{a}_{\ell m} \mathbf{b}^\dagger_{\ell m}. \tag{17}$$

This must not be confused with the true power spectrum defined as an ensemble average for isotropic fields:

$$\langle \mathbf{a}_{\mathbf{l}} \mathbf{b}^\dagger_{\mathbf{l}'} \rangle \equiv \hat{\mathsf{C}}^{ab}_\ell \delta_{\ell\ell'} \delta_{mm'}. \tag{18}$$

Now, let $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ be the contaminant-cleaned versions of two random fields $\mathbf{a}$ and $\mathbf{b}$ with contaminants $\mathbf{f}^i$ and $\mathbf{g}^j$ and masks v and w respectively, and let us define

$$(\hat{\mathsf{N}}^{-1})_{ij} \equiv \int d\hat{\mathbf{n}}\, \mathbf{g}^{i\dagger}(\hat{\mathbf{n}})\, \mathbf{g}^j(\hat{\mathbf{n}}). \tag{19}$$

The observed power spectrum of the contaminant-cleaned maps can be written as:

$$\begin{aligned} \tilde{\mathsf{C}}^{\tilde{a}\tilde{b}}_\ell = & \frac{1}{2\ell+1} \sum_m \mathbf{a}^v_{\ell m} \mathbf{b}^{w\dagger}_{\ell m} - \frac{N^*_{ij}}{2\ell+1} \sum_m \sum_{\mathbf{l}'} \mathbf{a}^v_{\mathbf{l}} \mathbf{b}^{w\dagger}_{\mathbf{l}'} \mathbf{g}^j_{\mathbf{l}'} \mathbf{g}^{i\dagger}_{\mathbf{l}} - \\ & - \frac{M_{ij}}{2\ell+1} \sum_m \sum_{\mathbf{l}'} \mathbf{f}^i_{\mathbf{l}} \mathbf{f}^{j\dagger}_{\mathbf{l}'} \mathbf{a}^v_{\mathbf{l}'} \mathbf{b}^{w\dagger}_{\mathbf{l}} + \frac{M_{ij} N^*_{pq}}{2\ell+1} \sum_m \sum_{\mathbf{l}'\mathbf{l}''} \mathbf{f}^i_{\mathbf{l}} \mathbf{f}^{j\dagger}_{\mathbf{l}'} \mathbf{a}^v_{\mathbf{l}'} \mathbf{b}^{w\dagger}_{\mathbf{l}''} \mathbf{g}^q_{\mathbf{l}''} \mathbf{g}^{p\dagger}_{\mathbf{l}}. \end{aligned} \tag{20}$$

In order to compute the bias of $\tilde{\mathsf{C}}^{\tilde{a}\tilde{b}}_\ell$ with respect to $\hat{\mathsf{C}}^{ab}_\ell$, we need to compute the ensemble average of the former, which we will write as:

$$\langle \tilde{\mathsf{C}}^{ab}_\ell \rangle = \hat{\mathsf{F}}^1_\ell - \hat{\mathsf{F}}^2_\ell - \hat{\mathsf{F}}^3_\ell + \hat{\mathsf{F}}^4_\ell, \tag{21}$$

where:

$$\hat{\mathsf{F}}^1_\ell \equiv \frac{1}{2\ell+1}\sum_m \langle \mathbf{a}^v_{\ell m}\mathbf{b}^{w\dagger}_{\ell m}\rangle, \quad \hat{\mathsf{F}}^2_\ell \equiv \frac{N^*_{ij}}{2\ell+1}\sum_m\sum_{\mathbf{l}'}\langle \mathbf{a}^v_{\mathbf{l}}\mathbf{b}^{w\dagger}_{\mathbf{l}'}\mathbf{g}^j_{\mathbf{l}'}\mathbf{g}^{i\dagger}_{\mathbf{l}}\rangle \tag{22}$$

$$\hat{\mathsf{F}}^3_\ell \equiv \frac{M_{ij}}{2\ell+1}\sum_m\sum_{\mathbf{l}'}\langle \mathbf{f}^i_{\mathbf{l}}\mathbf{f}^{j\dagger}_{\mathbf{l}'}\mathbf{a}^v_{\mathbf{l}'}\mathbf{b}^{w\dagger}_{\mathbf{l}}\rangle, \quad \hat{\mathsf{F}}^4_\ell \equiv \frac{M_{ij}N^*_{pq}}{2\ell+1}\sum_m\sum_{\mathbf{l}'\mathbf{l}''}\langle \mathbf{f}^i_{\mathbf{l}}\mathbf{f}^{j\dagger}_{\mathbf{l}'}\mathbf{a}^v_{\mathbf{l}'}\mathbf{b}^{w\dagger}_{\mathbf{l}''}\mathbf{g}^q_{\mathbf{l}''}\mathbf{g}^{p\dagger}_{\mathbf{l}}\rangle. \tag{23}$$

We will now compute the ensemble average of each of these terms.

3.1 $\hat{\mathsf{F}}^1_\ell$

$$\begin{aligned}
\hat{\mathsf{F}}^1_\ell &= \frac{1}{2\ell+1}\sum_{m\mathbf{l}_{1,2,3,4}} v_{\mathbf{l}_2} w^*_{\mathbf{l}_4}\hat{\mathsf{D}}^{s_a}_{\mathbf{l}\mathbf{l}_1\mathbf{l}_2}\langle \mathbf{a}_{\mathbf{l}_1}\mathbf{b}^\dagger_{\mathbf{l}_3}\rangle\hat{\mathsf{D}}^{s_b\dagger}_{\mathbf{l}\mathbf{l}_3\mathbf{l}_4}\\
&= \frac{1}{2\ell+1}\sum_{m\mathbf{l}_{1,2,3}} v_{\mathbf{l}_2} w^*_{\mathbf{l}_3}\hat{\mathsf{D}}^{s_a}_{\mathbf{l}\mathbf{l}_1\mathbf{l}_2}\hat{\mathsf{C}}^{ab}_{\mathbf{l}_1}\hat{\mathsf{D}}^{s_b\dagger}_{\mathbf{l}\mathbf{l}_1\mathbf{l}_3}\\
&= \frac{1}{2\ell+1}\sum_{\ell_1\mathbf{l}_{2,3}} v_{\mathbf{l}_2} w^*_{\mathbf{l}_3}\frac{(2\ell+1)(2\ell_1+1)}{4\pi}\sqrt{(2\ell_2+1)(2\ell_3+1)}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ s_a & -s_a & 0\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_3\\ s_b & -s_b & 0\end{pmatrix}\\
&\qquad\qquad \hat{\mathsf{d}}^{s_a}_{\ell+\ell_1+\ell_2}\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{d}}^{s_b\dagger}_{\ell+\ell_1+\ell_3}\sum_{mm_1}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ -m & m_1 & m_2\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_3\\ -m & m_1 & m_3\end{pmatrix}\\
&= \sum_{\ell_1\ell_2}\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi}\tilde{C}^{vw}_{\ell_2}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ s_a & -s_a & 0\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ s_b & -s_b & 0\end{pmatrix}\hat{\mathsf{d}}^{s_a}_{\ell+\ell_1+\ell_2}\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{d}}^{s_b\dagger}_{\ell+\ell_1+\ell_2}
\end{aligned}$$

For $v = w = 1$ this reduces to $\tilde{C}^{vw}_\ell = 4\pi\delta_{\ell 0}$ and:

$$\begin{aligned}
\hat{\mathsf{F}}^1_\ell &= \sum_{\ell_1\ell_2}(2\ell_1+1)(2\ell_2+1)\delta_{\ell_2 0}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ s_a & -s_a & 0\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ s_b & -s_b & 0\end{pmatrix}\hat{\mathsf{d}}^{s_a}_{\ell+\ell_1+\ell_2}\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{d}}^{s_b\dagger}_{\ell+\ell_1+\ell_2}\\
&= \sum_{\ell_1}(2\ell_1+1)\begin{pmatrix}\ell & \ell_1 & 0\\ s_a & -s_a & 0\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & 0\\ s_b & -s_b & 0\end{pmatrix}\hat{\mathsf{d}}^{s_a}_{\ell+\ell_1}\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{d}}^{s_b\dagger}_{\ell+\ell_1}\\
&= \sum_{\ell_1}(2\ell_1+1)\delta_{\ell\ell_1}\frac{(-1)^{\ell-s_a}}{\sqrt{2\ell+1}}\delta_{\ell\ell_1}\frac{(-1)^{\ell-s_b}}{\sqrt{2\ell+1}}\hat{\mathsf{d}}^{s_a}_{\ell+\ell_1}\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{d}}^{s_b\dagger}_{\ell+\ell_1}\\
&= \hat{\mathsf{d}}^{s_a}_{2\ell}\hat{\mathsf{C}}^{ab}_\ell\hat{\mathsf{d}}^{s_b\dagger}_{2\ell}\\
&= \hat{\mathsf{C}}^{ab}_\ell
\end{aligned} \tag{24}$$

3.2 $\hat{\mathsf{F}}^2_\ell$

$$\hat{\mathsf{F}}^2_\ell = N^*_{ij}\int d\hat{\mathbf{n}}\,d\hat{\mathbf{n}}'\,\frac{\sum_{m\mathbf{l}'\mathbf{l}_{1,2,3,4}}}{2\ell+1}\hat{\mathsf{Y}}^{s_a\dagger}_{\mathbf{l}}(\hat{\mathbf{n}})\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}})\langle\mathbf{a}_{\mathbf{l}_1}\mathbf{b}^\dagger_{\mathbf{l}_3}\rangle\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_3}(\hat{\mathbf{n}}')\hat{\mathsf{Y}}^{s_b}_{\mathbf{l}'}(\hat{\mathbf{n}}')\mathbf{g}^j_{\mathbf{l}'}\mathbf{g}^{i\dagger}_{\mathbf{l}}v_{\mathbf{l}_2}w^*_{\mathbf{l}_4}Y_{\mathbf{l}_2}(\hat{\mathbf{n}})Y^*_{\mathbf{l}_4}(\hat{\mathbf{n}}')$$

$$= N^*_{ij}\frac{\sum_m}{2\ell+1}\left\{\int d\hat{\mathbf{n}}\,v(\hat{\mathbf{n}})\,\hat{\mathsf{Y}}^{s_a\dagger}_{\mathbf{l}}(\hat{\mathbf{n}})\left[\sum_{\ell_1 m_1}\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}})\hat{\mathsf{C}}^{ab}_{\ell_1}\left(\int d\hat{\mathbf{n}}'\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_1}(\hat{\mathbf{n}}')\mathbf{g}^j(\hat{\mathbf{n}}')w(\hat{\mathbf{n}}')\right)\right]\mathbf{g}^{i\dagger}_{\mathbf{l}}\right\} \tag{25}$$

$$= N^*_{ij}\frac{\sum_m}{2\ell+1}\mathrm{SHT}\left\{v(\hat{\mathbf{n}})\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(w\,\mathbf{g}^j\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}^{s_a}_{\mathbf{l}}\mathbf{g}^{i\dagger}_{\mathbf{l}} \tag{26}$$

For $v = w = 1$ this reduces to:

$$\begin{aligned}
\hat{\mathsf{F}}^2_\ell &= N^*_{ij}\frac{\sum_m}{2\ell+1}\mathrm{SHT}\left\{\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(\mathbf{g}^j\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}^{s_a}_{\mathbf{l}}\mathbf{g}^{i\dagger}_{\mathbf{l}}\\
&= N^*_{ij}\,\hat{\mathsf{C}}^{ab}_\ell\frac{\sum_m \mathbf{g}^j_{\ell m}\mathbf{g}^{i\dagger}_{\ell m}}{2\ell+1}\\
&= N^*_{ij}\,\hat{\mathsf{C}}^{ab}_\ell\tilde{\mathsf{C}}^{g^j g^i}_\ell
\end{aligned}$$

3.3 $\hat{\mathsf{F}}^3_\ell$

$$\begin{aligned}
\hat{\mathsf{F}}^3_\ell &= M_{ij}\int d\hat{\mathbf{n}}\,d\hat{\mathbf{n}}'\,\frac{\sum_{m\mathbf{l}'\mathbf{l}_{1,2,3,4}}}{2\ell+1}\mathbf{f}^i_{\mathbf{l}}\mathbf{f}^{j\dagger}_{\mathbf{l}'}\hat{\mathsf{Y}}^{s_a\dagger}_{\mathbf{l}'}(\hat{\mathbf{n}}')\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_3}(\hat{\mathbf{n}}')\langle\mathbf{a}_{\mathbf{l}_3}\mathbf{b}^\dagger_{\mathbf{l}_1}\rangle\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_1}(\hat{\mathbf{n}})\hat{\mathsf{Y}}^{s_b}_{\mathbf{l}}(\hat{\mathbf{n}})v_{\mathbf{l}_4}w^*_{\mathbf{l}_2}Y_{\mathbf{l}_2}(\hat{\mathbf{n}})Y^*_{\mathbf{l}_4}(\hat{\mathbf{n}}')\\
&= M_{ij}\frac{\sum_m}{2\ell+1}\mathbf{f}^i_{\mathbf{l}}\left\{\int d\hat{\mathbf{n}}\,w(\hat{\mathbf{n}})\left[\sum_{\ell_1 m_1}\left(\int d\hat{\mathbf{n}}'\,v(\hat{\mathbf{n}}')\mathbf{f}^{j\dagger}(\hat{\mathbf{n}}')\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}}')\right)\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_1}(\hat{\mathbf{n}})\right]\hat{\mathsf{Y}}^{s_b}_{\mathbf{l}}(\hat{\mathbf{n}})\right\} \qquad (27)\\
&= M_{ij}\frac{\sum_m}{2\ell+1}\mathbf{f}^i_{\mathbf{l}}\,\mathrm{SHT}\left\{w(\hat{\mathbf{n}})\,\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab\dagger}_{\ell_1}\mathrm{SHT}\left(v\,\mathbf{f}^j\right)^{s_a}_{\mathbf{l}_1}\right]^{s_b}_{\hat{\mathbf{n}}}\right\}^{s_b\dagger}_{\mathbf{l}} \qquad (28)
\end{aligned}$$

For $v=w=1$ this reduces to:

$$\begin{aligned}
\hat{\mathsf{F}}^3_\ell &= M_{ij}\frac{\sum_m}{2\ell+1}\mathbf{f}^i_{\mathbf{l}}\,\mathrm{SHT}\left\{\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab\dagger}_{\ell_1}\mathrm{SHT}\left(\mathbf{f}^j\right)^{s_a}_{\mathbf{l}_1}\right]^{s_b}_{\hat{\mathbf{n}}}\right\}^{s_b\dagger}_{\mathbf{l}}\\
&= M_{ij}\frac{\sum_m \mathbf{f}^i_{\ell m}\mathbf{f}^{j\dagger}_{\ell m}}{2\ell+1}\hat{\mathsf{C}}^{ab}_\ell\\
&= M_{ij}\tilde{\mathsf{C}}^{f^i f^j}_\ell\hat{\mathsf{C}}^{ab}_\ell
\end{aligned}$$

3.4 $\hat{\mathsf{F}}^4_\ell$

$$\begin{aligned}
\hat{\mathsf{F}}^4_\ell &= \frac{M_{ij}N^*_{pq}}{2\ell+1}\int d\hat{\mathbf{n}}\,d\hat{\mathbf{n}}'\sum_{m\mathbf{l}'\mathbf{l}''\mathbf{l}_{1,2,3,4}}\mathbf{f}^i_{\mathbf{l}}\mathbf{f}^{j\dagger}_{\mathbf{l}'}\hat{\mathsf{Y}}^{s_a\dagger}_{\mathbf{l}'}(\hat{\mathbf{n}})\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}})\langle\mathbf{a}_{\mathbf{l}_1}\mathbf{b}^\dagger_{\mathbf{l}_3}\rangle\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_3}(\hat{\mathbf{n}}')\hat{\mathsf{Y}}^{s_b}_{\mathbf{l}''}(\hat{\mathbf{n}}')\mathbf{g}^q_{\mathbf{l}''}\mathbf{g}^{p\dagger}_{\mathbf{l}}v_{\mathbf{l}_2}w^*_{\mathbf{l}_4}Y_{\mathbf{l}_2}(\hat{\mathbf{n}})Y^*_{\mathbf{l}_4}(\hat{\mathbf{n}}')\\
&= \frac{M_{ij}N^*_{pq}}{2\ell+1}\sum_m\mathbf{f}^i_{\mathbf{l}}\left\{\int d\hat{\mathbf{n}}\,v(\hat{\mathbf{n}})\,\mathbf{f}^{j\dagger}(\hat{\mathbf{n}})\left[\sum_{\ell_1 m_1}\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}})\hat{\mathsf{C}}^{ab}_{\ell_1}\left(\int d\hat{\mathbf{n}}'\,\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_1}(\hat{\mathbf{n}}')\mathbf{g}^q(\hat{\mathbf{n}}')w(\hat{\mathbf{n}}')\right)\right]\right\}\mathbf{g}^{p\dagger}_{\mathbf{l}} \qquad (29)\\
&= \frac{M_{ij}N^*_{pq}}{2\ell+1}\sum_m\mathbf{f}^i_{\mathbf{l}}\left\{\int d\hat{\mathbf{n}}\,v(\hat{\mathbf{n}})\,\mathbf{f}^{j\dagger}(\hat{\mathbf{n}})\,\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(w\,\mathbf{g}^q\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}\mathbf{g}^{p\dagger}_{\mathbf{l}} \qquad (30)
\end{aligned}$$

For $v=w=1$ this reduces to:

$$\begin{aligned}
\hat{\mathsf{F}}^4_\ell &= \frac{M_{ij}N^*_{pq}}{2\ell+1}\sum_m\mathbf{f}^i_{\mathbf{l}}\left\{\int d\hat{\mathbf{n}}\,\mathbf{f}^{j\dagger}(\hat{\mathbf{n}})\,\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(\mathbf{g}^q\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}\mathbf{g}^{p\dagger}_{\mathbf{l}}\\
&= \frac{M_{ij}N^*_{pq}}{2\ell+1}\sum_m\mathbf{f}^i_{\mathbf{l}}\left\{\int d\hat{\mathbf{n}}\sum_{\mathbf{l}_2}\mathbf{f}^{j\dagger}_{\mathbf{l}_2}\hat{\mathsf{Y}}^{s_a\dagger}_{\mathbf{l}_2}(\hat{\mathbf{n}})\sum_{\mathbf{l}_1}\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}})\hat{\mathsf{C}}^{ab}_{\ell_1}\mathbf{g}^q_{\mathbf{l}_1}\right\}\mathbf{g}^{p\dagger}_{\mathbf{l}}\\
&= \frac{M_{ij}N^*_{pq}}{2\ell+1}\sum_{m\ell_1 m_1}\mathbf{f}^i_{\mathbf{l}}\mathbf{f}^{j\dagger}_{\mathbf{l}_1}\hat{\mathsf{C}}^{ab}_{\ell_1}\mathbf{g}^q_{\mathbf{l}_1}\mathbf{g}^{p\dagger}_{\mathbf{l}}\\
&= M_{ij}N^*_{pq}\tilde{\mathsf{C}}^{f^i g^p}_\ell\sum_{\ell_1}(2\ell_1+1)\mathrm{Tr}\left(\hat{\mathsf{C}}^{ab}_{\ell_1}\tilde{\mathsf{C}}^{g^q f^j}_{\ell_1}\right)
\end{aligned}$$

4 MASTER algorithm

Let ${}_{s_a}\mathbf{I}^a(\hat{\mathbf{n}})$ be a spin-s_a field, where s_a can be 0 (i.e. 1 single component - e.g. T) or 2 (i.e. 2 components - e.g. (Q,U)). The observed map is:

$$ {}_{s_a}\tilde{\mathbf{I}}^a(\hat{\mathbf{n}}) = w^a(\hat{\mathbf{n}})\left[{}_{s_a}\mathbf{I}^a(\hat{\mathbf{n}})+N^a(\hat{\mathbf{n}})\right], \qquad (31)$$

where $w(\hat{\mathbf{n}})$ is the weights map. The harmonic coefficients of the observed map can be written as:

$$ {}_{s_a}\tilde{\mathbf{I}}^a_{\ell_1 m_1} = \sum_{\ell_2,m_2} {}_{s_a}\mathsf{W}^a_{\ell_1\ell_2,m_1m_2}\cdot{}_{s_a}\mathbf{I}^a_{\ell_2 m_2}, \qquad (32)$$

where the mixing matrix is

$$ {}_{s_a}\mathsf{W}^a_{\ell_1\ell_2,m_1m_2} \equiv (-1)^m \sum_{\ell_3,m_3} w^a_{\ell_3 m_3} \left[\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}\right]^{1/2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -m_1 & m_2 & m_3 \end{pmatrix} {}_{s_a}\mathsf{J}_{\ell_1\ell_2\ell_3} \quad (33) $$

with

$$ {}_0\mathsf{J}_{\ell_1\ell_2\ell_3} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}, \quad (34) $$

$$ {}_2\mathsf{J}_{\ell_1\ell_2\ell_3} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 2 & -2 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1+(-1)^{\ell_1+\ell_2+\ell_3} & i\left[(-1)^{\ell_1+\ell_2+\ell_3}-1\right] \\ -i\left[(-1)^{\ell_1+\ell_2+\ell_3}-1\right] & 1+(-1)^{\ell_1+\ell_2+\ell_3} \end{pmatrix} \quad (35) $$

Let us define the pseudo-power-spectrum $\tilde{\mathsf{C}}^{ab}_\ell$

$$ \tilde{\mathsf{C}}^{ab}_\ell \equiv \frac{1}{2\ell+1} \sum_m {}_{s_a}\mathbf{I}^a_{\ell m} \cdot \left({}_{s_b}\mathbf{I}^b_{\ell m}\right)^\dagger \quad (36) $$

The relation between the pseudo-power-spectrum and the true power spectrum C^{ab}_ℓ can be derived to be of the form

$$ \langle \tilde{\mathbf{C}}^{ab}_\ell \rangle = \sum_{\ell'} \mathsf{M}^{s_a s_b}_{\ell\ell'} \cdot \mathbf{C}^{ab}_{\ell'}, \quad (37) $$

where the mode-coupling matrix $\mathsf{M}^{s_a s_b}_{\ell\ell'}$ takes the form:

- Case $s_a = s_b = 0$:

$$ \langle \tilde{C}^{T_aT_b}_\ell \rangle = \sum_{\ell'} M^{00}_{\ell\ell'} C^{T_aT_b}_{\ell'} \quad (38) $$

  with

$$ M^{00}_{\ell\ell'} = \frac{2\ell'+1}{4\pi} \sum_{\ell''} W^{ab}_{\ell''} \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{pmatrix}^2 \quad (39) $$

- Case $s_a = 0$, $s_b = 2$:

$$ \left\langle \begin{pmatrix} \tilde{C}^{T_aE_b}_\ell \\ \tilde{C}^{T_aB_b}_\ell \end{pmatrix} \right\rangle = \sum_{\ell'} \begin{pmatrix} M^{0+}_{\ell\ell'} & 0 \\ 0 & M^{0+}_{\ell\ell'} \end{pmatrix} \cdot \begin{pmatrix} C^{T_aE_b}_{\ell'} \\ C^{T_aB_b}_{\ell'} \end{pmatrix} \quad (40) $$

  with

$$ M^{0+}_{\ell\ell'} = \frac{2\ell'+1}{4\pi} \sum_{\ell''} W^{ab}_{\ell''} \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{pmatrix} \quad (41) $$

- Case $s_a = 2$, $s_b = 2$:

$$ \left\langle \begin{pmatrix} \tilde{C}^{E_aE_b}_\ell \\ \tilde{C}^{E_aB_b}_\ell \\ \tilde{C}^{B_aE_b}_\ell \\ \tilde{C}^{B_aB_b}_\ell \end{pmatrix} \right\rangle = \sum_{\ell'} \begin{pmatrix} M^{++}_{\ell\ell'} & 0 & 0 & M^{--}_{\ell\ell'} \\ 0 & M^{++}_{\ell\ell'} & -M^{--}_{\ell\ell'} & 0 \\ 0 & -M^{--}_{\ell\ell'} & M^{++}_{\ell\ell'} & 0 \\ M^{--}_{\ell\ell'} & 0 & 0 & M^{++}_{\ell\ell'} \end{pmatrix} \cdot \begin{pmatrix} C^{E_aE_b}_{\ell'} \\ C^{E_aB_b}_{\ell'} \\ C^{B_aE_b}_{\ell'} \\ C^{B_aB_b}_{\ell'} \end{pmatrix} \quad (42) $$

  with

$$ M^{++}_{\ell\ell'} = \frac{2\ell'+1}{4\pi} \sum_{\ell''} W^{ab}_{\ell''} \begin{pmatrix} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{pmatrix}^2 \frac{1+(-1)^{\ell+\ell'+\ell''}}{2} \quad (43) $$

$$ M^{--}_{\ell\ell'} = \frac{2\ell'+1}{4\pi} \sum_{\ell''} W^{ab}_{\ell''} \begin{pmatrix} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{pmatrix}^2 \frac{1-(-1)^{\ell+\ell'+\ell''}}{2}, \quad (44) $$

where in all these equations $W^{ab}_{\ell''}$ is the cross-spectrum of the weights map (without the $(2\ell+1)$ normalization):

$$ W^{ab}_\ell \equiv \sum_m w^a_{\ell m} (w^b_{\ell m})^*. \quad (45) $$

Note that, in Eq. 37 one should add, on the right-hand side, the noise cross-power-spectrum:

$$ \langle \tilde{\mathsf{N}}^{ab}_\ell \rangle \equiv \frac{1}{2\ell+1} \sum_m \langle \mathbf{N}^a_{\ell m} \cdot (\mathbf{N}^b_{\ell m})^\dagger \rangle \quad (46) $$

4.1 Beam

Adding the effect of a beam amounts to redefining:

$$\mathsf{M}^{s_a s_b}_{\ell_1 \ell_2} \rightarrow \mathsf{M}^{s_a s_b}_{\ell_1 \ell_2}\, b^{ab}_{\ell_2}, \tag{47}$$

where b^{ab}_ℓ is the product of the harmonic transform of the beams for maps a and b.

4.2 Bandpowers

Consider the case where you want to compute the power spectrum in band-powers given by

$$\mathbf{B}^{ab}_k \equiv \frac{1}{N_k} \sum_{\ell=\ell_k}^{\ell_k+N_k-1} f(\ell)\, \mathbf{C}^{ab}_\ell, \tag{48}$$

then Eq. 37 above becomes

$$\langle \tilde{\mathbf{B}}^{ab}_k \rangle = \sum_{k'} \mathsf{M}^{B,s_a s_b}_{kk'} \cdot \mathbf{B}^{ab}_{k'} + \langle \tilde{\mathsf{N}}^{B,ab}_k \rangle \tag{49}$$

where the binned coupling matrix $\mathsf{M}^{B,s_a s_b}$ is

$$\mathsf{M}^{B,s_a s_b}_{k_1,k_2} \equiv \frac{1}{N_{k_1}} \sum_{\ell_1=\ell_{k_1}}^{\ell_{k_1}+N_{k_1}-1} \sum_{\ell_2=\ell_{k_2}}^{\ell_{k_2}+N_{k_2}-1} \frac{f(\ell_1)}{f(\ell_2)} \mathsf{M}^{s_a s_b}_{\ell_1 \ell_2} \tag{50}$$

5 Mode deprojection

Let $\mathbf{t}$ be a set of contaminant templates such that we can write:

$$\mathbf{d} = \mathbf{s} + \mathbf{n} + \hat{\mathbf{t}} \cdot \mathbf{a}. \tag{51}$$

A maximum-likelihood estimate for $\mathbf{a}$ can be found as

$$\tilde{\mathbf{a}} = (\hat{\mathbf{t}}^T \hat{\mathbf{C}}^{-1} \hat{\mathbf{t}})^{-1} \hat{\mathbf{t}} \hat{\mathbf{C}}^{-1} \mathbf{d} \tag{52}$$

6 Minimum variance quadratic estimator

Let $\mathbf{d}_1$ and $\mathbf{d}_2$ be the data, given as a pixelized full-sky maps, with covariance matrix

$$\langle \mathbf{d}_1\, \mathbf{d}_1^T \rangle \equiv C^{12}(\hat{\mu}) \equiv S^{12}(\hat{\mu}) + N^{12}(\hat{\mu}) = \sum_\ell C^{12}_\ell\, P_\ell(\hat{\mu}) + N^{12}(\hat{\mu}), \tag{53}$$

where $\hat{\mu} = \mathbf{n}\,\mathbf{n}^T$, $\mathbf{n}$ is the vector of unit vectors containing the angular coordinates to each pixel and $P_\ell(x) \equiv (2\ell+1)L_\ell(x)/(4\pi)$, where L_ℓ are the Legendre polynomials. Note that we can expand P_ℓ as

$$P_\ell(\hat{\mu}) \equiv \sum_{m=-\ell}^{\ell} Y_{\ell m}(\mathbf{n}) \cdot Y^\dagger_{\ell m}(\mathbf{n}). \tag{54}$$

We will parametrize the power spectrum with a set of step functions, such that

$$C^{12}_\ell = \sum_b c_b\, \Theta_b(\ell), \quad \leftarrow \quad \Theta_b(\ell) = 1\,\mathrm{if}\, \ell \in [\ell^b_{\rm min}, \ell^b_{\rm max}),\ 0\,\mathrm{otherwise}, \tag{55}$$

thus we can write

$$C^{12}(\hat{\mu}) = \sum_b c_b\, Q_b(\hat{\mu}) + N^{12}(\hat{\mu}), \quad \text{with} \quad Q_b(\mu) = \sum_\ell \Theta_b(\ell) P_\ell(\mu). \tag{56}$$

Let us write the most general quadratic estimator for the coefficients c_b:

$$\tilde{c}_b \equiv \mathbf{d}_1^T \hat{E}_b \mathbf{d}_2 - B_b. \tag{57}$$

The mean value of $\tilde{c}_b$ is:

$$\langle \tilde{c}_b \rangle = \sum_a c_a \mathrm{Tr}(\hat{E}_b\, \hat{Q}_a) + \mathrm{Tr}(\hat{E}_b \hat{N}^{12}) - B_b, \tag{58}$$

and therefore we can remove the noise bias by defining

$$B_b \equiv \mathrm{Tr}(\hat{E}_b \hat{N}^{12}). \tag{59}$$

The covariance matrix of this estimator is:

$$\langle (\tilde{c}_a - \langle \tilde{c}_a \rangle)\, (\tilde{c}_b - \langle \tilde{c}_b \rangle) \rangle = \mathrm{Tr}(\hat{C}^{11} \hat{E}_a \hat{C}^{22} \hat{E}_b), \tag{60}$$

where we have approximated $C_\ell^{12} \ll [C_\ell^{11}, C_\ell^{22}]$. Minimizing the variance of $\tilde{c}_a$ under the constrain that the coupling matrix $W_{ab} \equiv \mathrm{Tr}(\hat{E}_b\, \hat{Q}_a)$ have unit diagonal, we find the following minimization problem:

$$L = \mathrm{Tr}\left[\hat{E}_b \hat{C}^{11} \hat{E}_b \hat{C}^{22} - 2(\lambda \hat{E}_b \hat{Q}_b - 1)\right], \tag{61}$$

where λ is the lagrange multiplier related to the constraint. The solution to this problem is:

$$\hat{E}_b = \frac{\hat{C}_{11}^{-1} \hat{Q}_b \hat{C}_{22}^{-1}}{\mathrm{Tr}(\hat{C}_{11}^{-1} \hat{Q}_b \hat{C}_{22}^{-1} \hat{Q}_b)}. \tag{62}$$

Thus, a decoupled, unbiased and minimum-variance estimator, $\hat{c}_b$, for c_b can be found as:

$$\mathbf{d}_1^T \hat{C}_{11}^{-1} \hat{Q}_b \hat{C}_{22}^{-1} \mathbf{d}_2 - \mathrm{Tr}(\hat{C}_{11}^{-1} \hat{Q}_b \hat{C}_{22}^{-1} \hat{N}) = \sum_a \hat{c}_a \mathrm{Tr}(\hat{C}_{11}^{-1} \hat{Q}_b \hat{C}_{22}^{-1} \hat{Q}_a). \tag{63}$$

Note that this estimator requires a guess for the covariance matrix of the data $C^{ij}(\hat{\mu})$, and exact minimum variance will only be attained for the true covariance (which itself depends on the power spectrum coefficients c_b). The choice of prior covariance will define the type of estimator. Note that this formalism immediately encompasses cut-skies by setting to zero all elements of the full covariance matrix involving unobserved pixels (this would correspond to the limit of infinite uncorrelated noise for those pixels).

6.1 Strategies for the different terms

- $\hat{C}_{ii}^{-1}\, \mathbf{d}_i$. This term can be computed by solving the linear system:

  $$\hat{C}_{ii} \mathbf{z}_i = \mathbf{d}_i \tag{64}$$

  via conjugate gradients. The action of $\hat{C} = \hat{S} + \hat{N}$ can be computed as follows:

  $$[\hat{S}\mathbf{z}](\hat{\mathbf{n}}_1) \equiv \sum_{\hat{\mathbf{n}}_2} S(\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2) z(\hat{\mathbf{n}}_2) \tag{65}$$

  $$= \frac{1}{\Delta\Omega} \int d\hat{\mathbf{n}}_2 \sum_{\ell m} C_\ell Y_{\ell m}^*(\hat{\mathbf{n}}_1) Y_{\ell m}^*(\hat{\mathbf{n}}_1)\, \tilde{z}(\hat{\mathbf{n}}_2) \tag{66}$$

  $$= \frac{1}{\Delta\Omega} \sum_{\ell m} Y_{\ell m}^*(\hat{\mathbf{n}}_1) C_\ell \tilde{z}_{\ell m} \tag{67}$$

  $$= \frac{1}{\Delta\Omega} \mathrm{SHT}^{-1}\left[C_\ell\, \mathrm{SHT}[\tilde{z}(\hat{\mathbf{n}})]_{\ell m}\right], \tag{68}$$

  where $\tilde{z}$ is the extension of z to the whole sphere, with $\tilde{z} = 0$ in all unobserved pixels.

  In most cases the noise power spectrum will be white, in which case $\hat{N}\mathbf{z}$ is just $\sum_{\hat{\mathbf{n}}} \sigma_N^2(\hat{\mathbf{n}}) z(\hat{\mathbf{n}})$, with $\sigma^2(\hat{\mathbf{n}})$ the per-pixel variance.

  Note that this procedure works for any $\hat{C}^{-1}\mathbf{v}$-type operation (see below).

- $\mathbf{d}_1^T \hat{C}_{11}^{-1} \hat{Q}_b \hat{C}_{22}^{-1} \mathbf{d}_2$. With $\mathbf{z}_i$ defined as above, this is just:

$$\mathbf{z}_i^T \hat{Q}_b \mathbf{z}_j = \frac{1}{(\Delta\Omega)^2} \sum_\ell \Theta_b(\ell) \sum_m (\tilde{z}^i_{\ell m})^* \tilde{z}^i_{\ell m} \tag{69}$$

- $\mathrm{Tr}(\hat{C}_{11}^{-1} \hat{Q}_b \hat{C}_{22}^{-1} \hat{Q}_a)$. Let $\mathbf{v}$ be a random vector with covariance $\langle \mathbf{v}\mathbf{v}^T \rangle = \hat{1}$, then one can calculate traces by averaging over realizations of such vectors:

$$\mathrm{Tr}\hat{A} = \left\langle \mathbf{v}^T \hat{A} \mathbf{v} \right\rangle . \tag{70}$$

  Then, the only thing to bear in mind in order to compute the trace above is that the action of the $\hat{Q}_a$ operator is:

$$\left(\hat{Q}_a \mathbf{v} \right)_{\hat{\mathbf{n}}} = \frac{1}{\Delta\Omega} \mathrm{SHT}^{-1} \left[\Theta_b(\ell) \mathrm{SHT} \left[\tilde{v} \right] \right]_{\hat{\mathbf{n}}} , \tag{71}$$

  where, as before $\tilde{v}$ is the extension of v with zeros in all unobserved pixels.

  In order to minimize the number of operations needed to compute this trace we can compute it as:

$$\mathrm{Tr}(\hat{C}_{11}^{-1} \hat{Q}_b \hat{C}_{22}^{-1} \hat{Q}_a) = \left\langle (\hat{C}_{11}^{-1}\, \mathbf{v})^T \hat{Q}_b (\hat{C}_{22}^{-1} \hat{Q}_a\, \mathbf{v}) \right\rangle \tag{72}$$