Notes

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1 MASTER algorithm

Let ${}_{s_a}\mathbf{I}^a(\hat{\mathbf{n}})$ be a spin-s_a field, where s_a can be 0 (i.e. 1 single component - e.g. T) or 2 (i.e. 2 components - e.g. (Q, U)). The observed map is:

$$ {}_{s_a}\tilde{\mathbf{I}}^a(\hat{\mathbf{n}}) = w^a(\hat{\mathbf{n}}) \left[{}_{s_a}\mathbf{I}^a(\hat{\mathbf{n}}) + N^a(\hat{\mathbf{n}}) \right], \tag{1} $$

where $w(\hat{\mathbf{n}})$ is the weights map. The harmonic coefficients of the observed map can be written as:

$$ {}_{s_a}\tilde{\mathbf{I}}^a_{\ell_1 m_1} = \sum_{\ell_2, m_2} {}_{s_a}\mathsf{W}^a_{\ell_1\ell_2, m_1 m_2} \cdot {}_{s_a}\mathbf{I}^a_{\ell_2 m_2}, \tag{2} $$

where the mixing matrix is

$$ {}_{s_a}\mathsf{W}^a_{\ell_1\ell_2, m_1 m_2} \equiv (-1)^m \sum_{\ell_3, m_3} w^a_{\ell_3 m_3} \left[\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi} \right]^{1/2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -m_1 & m_2 & m_3 \end{pmatrix} {}_{s_a}\mathsf{J}_{\ell_1\ell_2\ell_3} \tag{3} $$

with

$$ {}_0\mathsf{J}_{\ell_1\ell_2\ell_3} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4} $$

$$ {}_2\mathsf{J}_{\ell_1\ell_2\ell_3} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 2 & -2 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1+(-1)^{\ell_1+\ell_2+\ell_3} & i\left[(-1)^{\ell_1+\ell_2+\ell_3}-1\right] \\ -i\left[(-1)^{\ell_1+\ell_2+\ell_3}-1\right] & 1+(-1)^{\ell_1+\ell_2+\ell_3} \end{pmatrix} \tag{5} $$

Let us define the pseudo-power-spectrum $\tilde{\mathsf{C}}^{ab}_\ell$

$$ \tilde{\mathsf{C}}^{ab}_\ell \equiv \frac{1}{2\ell+1} \sum_m {}_{s_a}\mathbf{I}^a_{\ell m} \cdot \left({}_{s_b}\mathbf{I}^b_{\ell m} \right)^\dagger \tag{6} $$

The relation between the pseudo-power-spectrum and the true power spectrum C^{ab}_ℓ can be derived to be of the form

$$ \langle \tilde{\mathbf{C}}^{ab}_\ell \rangle = \sum_{\ell'} \mathsf{M}^{s_a s_b}_{\ell\ell'} \cdot \mathbf{C}^{ab}_{\ell'}, \tag{7} $$

where the mode-coupling matrix $\mathsf{M}^{s_a s_b}_{\ell\ell'}$ takes the form:

- Case $s_a = s_b = 0$:

$$ \langle \tilde{C}^{T_a T_b}_\ell \rangle = \sum_{\ell'} M^{00}_{\ell\ell'} C^{T_a T_b}_{\ell'} \tag{8} $$

  with

$$ M^{00}_{\ell\ell'} = \frac{2\ell'+1}{4\pi} \sum_{\ell''} W^{ab}_{\ell''} \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{pmatrix}^2 \tag{9} $$

- Case $s_a = 0$, $s_b = 2$:

$$ \left\langle \begin{pmatrix} \tilde{C}^{T_a E_b}_\ell \\ \tilde{C}^{T_a B_b}_\ell \end{pmatrix} \right\rangle = \sum_{\ell'} \begin{pmatrix} M^{0+}_{\ell\ell'} & 0 \\ 0 & M^{0+}_{\ell\ell'} \end{pmatrix} \cdot \begin{pmatrix} C^{T_a E_b}_{\ell'} \\ C^{T_a B_b}_{\ell'} \end{pmatrix} \tag{10} $$

  with

$$ M^{0+}_{\ell\ell'} = \frac{2\ell'+1}{4\pi} \sum_{\ell''} W^{ab}_{\ell''} \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{pmatrix} \tag{11} $$

- Case $s_a = 2$, $s_b = 2$:

$$\left\langle \begin{pmatrix} \tilde{C}_\ell^{E_a E_b} \\ \tilde{C}_\ell^{E_a B_b} \\ \tilde{C}_\ell^{B_a E_b} \\ \tilde{C}_\ell^{B_a B_b} \end{pmatrix} \right\rangle = \sum_{\ell'} \begin{pmatrix} M_{\ell\ell'}^{++} & 0 & 0 & M_{\ell\ell'}^{--} \\ 0 & M_{\ell\ell'}^{++} & -M_{\ell\ell'}^{--} & 0 \\ 0 & -M_{\ell\ell'}^{--} & M_{\ell\ell'}^{++} & 0 \\ M_{\ell\ell'}^{--} & 0 & 0 & M_{\ell\ell'}^{++} \end{pmatrix} \cdot \begin{pmatrix} C_{\ell'}^{E_a E_b} \\ C_{\ell'}^{E_a B_b} \\ C_{\ell'}^{B_a E_b} \\ C_{\ell'}^{B_a B_b} \end{pmatrix} \tag{12}$$

with

$$M_{\ell\ell'}^{++} = \frac{2\ell'+1}{4\pi} \sum_{\ell''} W_{\ell''}^{ab} \begin{pmatrix} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{pmatrix}^2 \frac{1+(-1)^{\ell+\ell'+\ell''}}{2} \tag{13}$$

$$M_{\ell\ell'}^{--} = \frac{2\ell'+1}{4\pi} \sum_{\ell''} W_{\ell''}^{ab} \begin{pmatrix} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{pmatrix}^2 \frac{1-(-1)^{\ell+\ell'+\ell''}}{2}, \tag{14}$$

where in all these equations $W_{\ell''}^{ab}$ is the cross-spectrum of the weights map (without the $(2\ell+1)$ normalization):

$$W_\ell^{ab} \equiv \sum_m w_{\ell m}^a (w_{\ell m}^b)^*. \tag{15}$$

Note that, in Eq. 7 one should add, on the right-hand side, the noise cross-power-spectrum:

$$\langle \tilde{\mathsf{N}}_\ell^{ab} \rangle \equiv \frac{1}{2\ell+1} \sum_m \langle \mathbf{N}_{\ell m}^a \cdot (\mathbf{N}_{\ell m}^b)^\dagger \rangle \tag{16}$$

1.1 Beam

Adding the effect of a beam amounts to redefining:

$$\mathsf{M}_{\ell_1 \ell_2}^{s_a s_b} \rightarrow \mathsf{M}_{\ell_1 \ell_2}^{s_a s_b} b_{\ell_2}^{ab}, \tag{17}$$

where b_ℓ^{ab} is the product of the harmonic transform of the beams for maps a and b.

1.2 Bandpowers

Consider the case where you want to compute the power spectrum in band-powers given by

$$\mathbf{B}_k^{ab} \equiv \frac{1}{N_k} \sum_{\ell=\ell_k}^{\ell_k+N_k-1} f(\ell)\, \mathbf{C}_\ell^{ab}, \tag{18}$$

then Eq. 7 above becomes

$$\langle \tilde{\mathbf{B}}_k^{ab} \rangle = \sum_{k'} \mathsf{M}_{kk'}^{B,s_a s_b} \cdot \mathbf{B}_{k'}^{ab} + \langle \tilde{\mathsf{N}}_k^{B,ab} \rangle \tag{19}$$

where the binned coupling matrix $\mathsf{M}^{B,s_a s_b}$ is

$$\mathsf{M}_{k_1,k_2}^{B,s_a s_b} \equiv \frac{1}{N_{k_1}} \sum_{\ell_1=\ell_{k_1}}^{\ell_{k_1}+N_{k_1}-1} \sum_{\ell_2=\ell_{k_2}}^{\ell_{k_2}+N_{k_2}-1} \frac{f(\ell_1)}{f(\ell_2)} \mathsf{M}_{\ell_1 \ell_2}^{s_a s_b} \tag{20}$$

2 Minimum variance quadratic estimator

Let $\mathbf{d}$ be the data, given as a pixelized full-sky map, with covariance matrix

$$\langle \mathbf{d}\, \mathbf{d}^T \rangle \equiv C(\hat{\mu}) \equiv S(\hat{\mu}) + N(\hat{\mu}) = \sum_\ell C_\ell\, P_\ell(\hat{\mu}) + N(\hat{\mu}), \tag{21}$$

where $\hat{\mu} = \mathbf{n}\,\mathbf{n}^T$, $\mathbf{n}$ is the vector of unit vectors containing the angular coordinates to each pixel and $P_\ell(x) \equiv (2\ell+1)L_\ell(x)/(4\pi)$, where L_ℓ are the Legendre polynomials. Note that we can expand P_ℓ as

$$P_\ell(\hat{\mu}) \equiv \sum_{m=-\ell}^{\ell} Y_{\ell m}(\mathbf{n}) \cdot Y^\dagger_{\ell m}(\mathbf{n}). \tag{22}$$

We will parametrize the power spectrum with a set of step functions, such that

$$C_\ell = \sum_b c_b\,\Theta_b(\ell), \qquad \leftarrow \qquad \Theta_b(\ell) = 1 \,\text{if}\, \ell \in [\ell^b_{\rm min}, \ell^b_{\rm max}),\; 0\,\text{otherwise}, \tag{23}$$

thus we can write

$$C(\hat{\mu}) = \sum_b c_b\,Q_b(\hat{\mu}) + N(\hat{\mu}), \;\; \text{with} \;\; Q_b(\mu) = \sum_\ell \Theta_b(\ell) P_\ell(\mu). \tag{24}$$

Let us write the most general quadratic estimator for the coefficients c_b:

$$\tilde{c}_b \equiv \mathbf{d}^T \hat{E}_b \mathbf{d} - B_b. \tag{25}$$

The mean value of $\tilde{c}_b$ is:

$$\langle \tilde{c}_b \rangle = \sum_a c_a \mathrm{Tr}(\hat{E}_b\,\hat{Q}_a) + \mathrm{Tr}(\hat{E}_b \hat{N}) - B_b, \tag{26}$$

and therefore we can remove the noise bias by defining

$$B_b \equiv \mathrm{Tr}(\hat{E}_b \hat{N}). \tag{27}$$

The covariance matrix of this estimator is:

$$\langle (\tilde{c}_a - \langle \tilde{c}_a \rangle)\,(\tilde{c}_b - \langle \tilde{c}_b \rangle) \rangle = 2\mathrm{Tr}(\hat{C}\hat{E}_a\hat{C}\hat{E}_b). \tag{28}$$

Minimizing the variance of $\tilde{c}_a$ under the constrain that the coupling matrix $W_{ab} \equiv \mathrm{Tr}(\hat{E}_b\,\hat{Q}_a)$ have unit diagonal, we find the following minimization problem:

$$L = \mathrm{Tr}\left[\hat{E}_b\hat{C}\hat{E}_b\hat{C} - 2(\lambda\hat{E}_b\hat{Q}_b - 1)\right], \tag{29}$$

where λ is the lagrange multiplier related to the constraint. The solution to this problem is:

$$\hat{E}_b = \frac{\hat{C}^{-1}\hat{Q}_b\hat{C}^{-1}}{\mathrm{Tr}(\hat{C}^{-1}\hat{Q}_b\hat{C}^{-1}\hat{Q}_b)}. \tag{30}$$

Thus, a decoupled, unbiased and minimum-variance estimator, $\hat{c}_b$, for c_b can be found as:

$$\frac{\mathbf{d}^T\hat{C}^{-1}\hat{Q}_b\hat{C}^{-1}\mathbf{d} - \mathrm{Tr}(\hat{C}^{-1}\hat{Q}_b^{-1}\hat{C}^{-1}\hat{N})}{\mathrm{Tr}(\hat{C}^{-1}\hat{Q}_b\hat{C}^{-1}\hat{Q}_b)} = \sum_a \hat{c}_a \frac{\mathrm{Tr}(\hat{C}^{-1}\hat{Q}_b\hat{C}^{-1}\hat{Q}_a)}{\mathrm{Tr}(\hat{C}^{-1}\hat{Q}_b\hat{C}^{-1}\hat{Q}_b)}. \tag{31}$$

Note that this estimator requires a guess for the covariance matrix of the data $C(\hat{\mu})$, and exact minimum variance will only be attained for the true covariance (which itself depends on the power spectrum coefficients c_b). The choice of prior covariance will define the type of estimator. Note that this formalism immediately encompasses cut-skies by setting to zero all elements of the full covariance matrix involving unobserved pixels (this would correspond to the limit of infinite uncorrelated noise for those pixels).