Notes

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1 MASTER algorithm

Let ${}_{s_a}\mathbf{I}^a(\hat{\mathbf{n}})$ be a spin-s_a field, where s_a can be 0 (i.e. 1 single component - e.g. T) or 2 (i.e. 2 components - e.g. (Q,U)). The observed map is:

$$ {}_{s_a}\tilde{\mathbf{I}}^a(\hat{\mathbf{n}}) = w^a(\hat{\mathbf{n}})\,\left[{}_{s_a}\mathbf{I}^a(\hat{\mathbf{n}}) + N^a(\hat{\mathbf{n}})\right], \tag{1} $$

where $w(\hat{\mathbf{n}})$ is the weights map. The harmonic coefficients of the observed map can be written as:

$$ {}_{s_a}\tilde{\mathbf{I}}^a_{\ell_1 m_1} = \sum_{\ell_2,m_2} {}_{s_a}\mathsf{W}^a_{\ell_1\ell_2,m_1m_2} \cdot {}_{s_a}\mathbf{I}^a_{\ell_2 m_2}, \tag{2} $$

where the mixing matrix is

$$ {}_{s_a}\mathsf{W}^a_{\ell_1\ell_2,m_1m_2} \equiv (-1)^m \sum_{\ell_3,m_3} w^a_{\ell_3 m_3}\left[\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}\right]^{1/2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -m_1 & m_2 & m_3 \end{pmatrix} {}_{s_a}\mathsf{J}_{\ell_1\ell_2\ell_3} \tag{3} $$

with

$$ {}_0\mathsf{J}_{\ell_1\ell_2\ell_3} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4} $$

$$ {}_2\mathsf{J}_{\ell_1\ell_2\ell_3} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 2 & -2 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1+(-1)^{\ell_1+\ell_2+\ell_3} & i\left[(-1)^{\ell_1+\ell_2+\ell_3}-1\right] \\ -i\left[(-1)^{\ell_1+\ell_2+\ell_3}-1\right] & 1+(-1)^{\ell_1+\ell_2+\ell_3} \end{pmatrix} \tag{5} $$

Let us define the pseudo-power-spectrum $\tilde{\mathsf{C}}^{ab}_\ell$

$$ \tilde{\mathsf{C}}^{ab}_\ell \equiv \frac{1}{2\ell+1}\sum_m {}_{s_a}\mathbf{I}^a_{\ell m} \cdot \left({}_{s_b}\mathbf{I}^b_{\ell m}\right)^\dagger \tag{6} $$

The relation between the pseudo-power-spectrum and the true power spectrum C^{ab}_ℓ can be derived to be of the form

$$ \langle \tilde{\mathbf{C}}^{ab}_\ell \rangle = \sum_{\ell'} \mathsf{M}^{s_a s_b}_{\ell\ell'} \cdot \mathbf{C}^{ab}_{\ell'}, \tag{7} $$

where the mode-coupling matrix $\mathsf{M}^{s_a s_b}_{\ell\ell'}$ takes the form:

- Case $s_a = s_b = 0$:

$$ \langle \tilde{C}^{T_a T_b}_\ell \rangle = \sum_{\ell'} M^{00}_{\ell\ell'} C^{T_a T_b}_{\ell'} \tag{8} $$

with

$$ M^{00}_{\ell\ell'} = \frac{2\ell'+1}{4\pi}\sum_{\ell''} W^{ab}_{\ell''} \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{pmatrix}^2 \tag{9} $$

- Case $s_a = 0$, $s_b = 2$:

$$\left\langle \begin{pmatrix} \tilde{C}_\ell^{T_a E_b} \\ \tilde{C}_\ell^{T_a B_b} \end{pmatrix} \right\rangle = \sum_{\ell'} \begin{pmatrix} M_{\ell\ell'}^{0+} & 0 \\ 0 & M_{\ell\ell'}^{0+} \end{pmatrix} \cdot \begin{pmatrix} C_{\ell'}^{T_a E_b} \\ C_{\ell'}^{T_a B_b} \end{pmatrix} \tag{10}$$

with

$$M_{\ell\ell'}^{0+} = \frac{2\ell'+1}{4\pi} \sum_{\ell''} W_{\ell''}^{ab} \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{pmatrix} \tag{11}$$

- Case $s_a = 2$, $s_b = 2$:

$$\left\langle \begin{pmatrix} \tilde{C}_\ell^{E_a E_b} \\ \tilde{C}_\ell^{E_a B_b} \\ \tilde{C}_\ell^{B_a E_b} \\ \tilde{C}_\ell^{B_a B_b} \end{pmatrix} \right\rangle = \sum_{\ell'} \begin{pmatrix} M_{\ell\ell'}^{++} & 0 & 0 & M_{\ell\ell'}^{--} \\ 0 & M_{\ell\ell'}^{++} & -M_{\ell\ell'}^{--} & 0 \\ 0 & -M_{\ell\ell'}^{--} & M_{\ell\ell'}^{++} & 0 \\ M_{\ell\ell'}^{--} & 0 & 0 & M_{\ell\ell'}^{++} \end{pmatrix} \cdot \begin{pmatrix} C_{\ell'}^{E_a E_b} \\ C_{\ell'}^{E_a B_b} \\ C_{\ell'}^{B_a E_b} \\ C_{\ell'}^{B_a B_b} \end{pmatrix} \tag{12}$$

with

$$M_{\ell\ell'}^{++} = \frac{2\ell'+1}{4\pi} \sum_{\ell''} W_{\ell''}^{ab} \begin{pmatrix} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{pmatrix}^2 \frac{1+(-1)^{\ell+\ell'+\ell''}}{2} \tag{13}$$

$$M_{\ell\ell'}^{--} = \frac{2\ell'+1}{4\pi} \sum_{\ell''} W_{\ell''}^{ab} \begin{pmatrix} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{pmatrix}^2 \frac{1-(-1)^{\ell+\ell'+\ell''}}{2}, \tag{14}$$

where in all these equations $W_{\ell''}^{ab}$ is the cross-spectrum of the weights map (without the $(2\ell+1)$ normalization):

$$W_\ell^{ab} \equiv \sum_m w_{\ell m}^a (w_{\ell m}^b)^*. \tag{15}$$

Note that, in Eq. 7 one should add, on the right-hand side, the noise cross-power-spectrum:

$$\langle \tilde{\mathsf{N}}_\ell^{ab} \rangle \equiv \frac{1}{2\ell+1} \sum_m \langle \mathbf{N}_{\ell m}^a \cdot (\mathbf{N}_{\ell m}^b)^\dagger \rangle \tag{16}$$

1.1 Beam

Adding the effect of a beam amounts to redefining:

$$\mathsf{M}_{\ell_1\ell_2}^{s_a s_b} \to \mathsf{M}_{\ell_1\ell_2}^{s_a s_b}\, b_{\ell_2}^{ab}, \tag{17}$$

where b_ℓ^{ab} is the product of the harmonic transform of the beams for maps a and b.

1.2 Bandpowers

Consider the case where you want to compute the power spectrum in band-powers given by

$$\mathbf{B}_k^{ab} \equiv \frac{1}{N_k} \sum_{\ell=\ell_k}^{\ell_k+N_k-1} f(\ell)\, \mathbf{C}_\ell^{ab}, \tag{18}$$

then Eq. 7 above becomes

$$\langle \tilde{\mathbf{B}}_k^{ab} \rangle = \sum_{k'} \mathsf{M}_{kk'}^{B,s_a s_b} \cdot \mathbf{B}_{k'}^{ab} + \langle \tilde{\mathsf{N}}_k^{B,ab} \rangle \tag{19}$$

where the binned coupling matrix $\mathsf{M}^{B,s_a s_b}$ is

$$\mathsf{M}_{k_1,k_2}^{B,s_a s_b} \equiv \frac{1}{N_{k_1}} \sum_{\ell_1=\ell_{k_1}}^{\ell_{k_1}+N_{k_1}-1} \sum_{\ell_2=\ell_{k_2}}^{\ell_{k_2}+N_{k_2}-1} \frac{f(\ell_1)}{f(\ell_2)} \mathsf{M}_{\ell_1\ell_2}^{s_a s_b} \tag{20}$$