NaMaster: Scientific Documentation

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Contents

1 Introduction

NaMaster is a C library, python module and standalone program to compute the pseudo-C_ℓ estimator of the angular power spectrum between two masked and contaminated fields (this is also the so-called "MASTER" algorithm). The contents of this scientific documentation describe the algorithm, drawing heavily from the methods presented by [1] and [2], extending their results to arbitrary cross-correlations between spin-0 and spin-2 fields (see also [3]).

2 Generalities and SHTs

Let $\mathbf{a}(\hat{\mathbf{n}})$ be a spin-s_a quantity defined on the sphere. Then we define its spherical harmonic coefficients as:

$$\mathbf{a}_{\ell m} \equiv \mathrm{SHT}(\mathbf{a}(\hat{\mathbf{n}}))^{s_a}_{\ell m} \equiv \int d\hat{\mathbf{n}}\, \hat{\mathsf{Y}}^{s_a\dagger}_{\ell m}(\hat{\mathbf{n}})\, \mathbf{a}(\hat{\mathbf{n}}), \quad \mathbf{a}(\hat{\mathbf{n}}) = \mathrm{SHT}^{-1}(\mathbf{a}_{\ell m})^{s_a}_{\hat{\mathbf{n}}} \equiv \sum_{\ell m} \hat{\mathsf{Y}}^{s_a}_{\ell m}(\hat{\mathbf{n}})\, \mathbf{a}_{\ell m}. \tag{1}$$

Note that here we will use a vector notation, such that for a complex spin-s_a field a we form the vector $\mathbf{a} \equiv (\mathrm{Re}(a), \mathrm{Im}(a))$. The harmonic coefficients above are decomposed in a similar manner into E and B modes: $\mathbf{a}_{\ell m} \equiv (a^E_{\ell m}, a^B_{\ell m})$. The spherical harmonic operators $\hat{\mathsf{Y}}^s$ are therefore matrix that we define in the following subsection.

2.1 Spin-weighed spherical harmonics

Let $\eth$ and $\bar{\eth}$ be the following complex differential operators defined on the sphere when acting on a spin-s quantity f_s:

$$\eth f_s \equiv -(\sin\theta)^s \left(\partial_\theta + i\frac{\partial_\phi}{\sin\theta}\right)(\sin\theta)^{-s} f_s(\theta,\phi), \quad \bar{\eth} f_s \equiv -(\sin\theta)^{-s}\left(\partial_\theta - i\frac{\partial_\phi}{\sin\theta}\right)(\sin\theta)^s f_s(\theta,\phi). \quad (2)$$

The following properties can be easily derived for the action of these operators:

- If f_s is a spin-s quantity, $(f_s)^*$ is a spin-$(-s)$ quantity.
- $\eth f_s$ is a spin-$(s+1)$ quantity, and $\bar{\eth} f_s$ is a spin-$(s-1)$ quantity.
- $(\eth^n f_s)^* = \bar{\eth}^n (f_s)^*$
- $\eth(f\,g) = f\eth g + g\eth f$
- $\eth^2(f\,g) = f\eth^2 g + g\eth^2 f + \eth f\eth g$

We start by defining the spin-weighed spherical harmonics with spin $s \geq 0$:

$$_sY_{\ell m} \equiv \alpha_{\ell,s}\eth^s Y_{\ell m}, \quad {}_{-s}Y_{\ell m} \equiv \alpha_{\ell,s}(-1)^s\bar{\eth}^s Y_{\ell m}, \quad \alpha_{l,s} \equiv \sqrt{\frac{(\ell-s)!}{(\ell+s)!}}, \tag{3}$$

which have the property: $({}_sY_{\ell m})^* = (-1)^{s+m}\,{}_{-s}Y_{\ell -m}$. We then define the E-mode and B-mode spherical harmonic vectors as:

$$_s\mathbf{Y}^E_{\ell m} \equiv \mathbf{D}^E_s Y_{\ell m} \equiv -\frac{\alpha_{\ell,s}}{2}\left(\begin{array}{c}\eth^s + \bar{\eth}^s \\ -i(\eth^s - \bar{\eth}^s)\end{array}\right) Y_{\ell m} \equiv -\frac{1}{2}\left(\begin{array}{c} {}_sY_{\ell m} + (-1)^s\,{}_{-s}Y_{\ell m} \\ -i({}_sY_{\ell m} - (-1)^s\,{}_{-s}Y_{\ell m})\end{array}\right) \tag{4}$$

$$_s\mathbf{Y}^B_{\ell m} \equiv \mathbf{D}^B_s Y_{\ell m} \equiv -\frac{\alpha_{\ell,s}}{2}\left(\begin{array}{c} i(\eth^s - \bar{\eth}^s) \\ \eth^s + \bar{\eth}^s\end{array}\right) Y_{\ell m} \equiv -\frac{1}{2}\left(\begin{array}{c} i({}_sY_{\ell m} - (-1)^s\,{}_{-s}Y_{\ell m}) \\ {}_sY_{\ell m} + (-1)^s\,{}_{-s}Y_{\ell m}\end{array}\right), \tag{5}$$

which also defines the differential operators $\mathbf{D}^{E,B}_s$. These functions, for $s = 0$ are simply $\mathbf{D}^E_0 = (Y_{\ell m}, 0)$ and $\mathbf{D}^B_0 = (0, Y_{\ell m})$.

The matrix operator $\hat{\mathsf{Y}}^s_{\ell m}$ is then defined as having $_s\mathbf{Y}^{E,B}_{\ell m}$ as columns:

$$\hat{\mathsf{Y}}^s_{\ell m} \equiv \left({}_s\mathbf{Y}^E_{\ell m}, {}_s\mathbf{Y}^B_{\ell m},\right) \equiv -\frac{1}{2}\left(\begin{array}{cc} {}_sY_{\ell m} + (-1)^s\,{}_{-s}Y_{\ell m} & i({}_sY_{\ell m} - (-1)^s\,{}_{-s}Y_{\ell m}) \\ -i({}_sY_{\ell m} - (-1)^s\,{}_{-s}Y_{\ell m}) & {}_sY_{\ell m} + (-1)^s\,{}_{-s}Y_{\ell m}\end{array}\right) \tag{6}$$

The matrices $\hat{\mathsf{Y}}^s_{\ell m}$ satisfy the following relations:

$$\hat{\mathsf{Y}}^{s\dagger}_{\ell m} = (-1)^{m+s}\hat{\mathsf{Y}}^{-s}_{\ell -m} \tag{7}$$

$$\int d\hat{\mathbf{n}}\hat{\mathsf{Y}}^s_{\ell m}\hat{\mathsf{Y}}^{s\dagger}_{\ell' m'} = \hat{1}\delta_{\ell\ell'}\delta_{mm'} \tag{8}$$

$$\hat{\mathsf{D}}^s_{\mathbf{l}\mathbf{l}_1\mathbf{l}_2} \equiv \int d\hat{\mathbf{n}}\left(\hat{\mathsf{Y}}^{s\dagger}_{\mathbf{l}}(\hat{\mathbf{n}})\,\hat{\mathsf{Y}}^s_{\mathbf{l}_1}(\hat{\mathbf{n}})\right)\hat{\mathsf{Y}}^0_{\mathbf{l}_2}(\hat{\mathbf{n}}), \tag{9}$$

$$= (-1)^{s+m}\sqrt{\frac{(2\ell+1)(2\ell_1+1)(2\ell_2+1)}{4\pi}}\left(\begin{array}{ccc}\ell & \ell_1 & \ell_2 \\ -m & m_1 & m_2\end{array}\right)\left(\begin{array}{ccc}\ell & \ell_1 & \ell_2 \\ s & -s & 0\end{array}\right)\hat{\mathsf{d}}_{\ell+\ell_1+\ell_2}, \tag{10}$$

$$\hat{\mathsf{d}}^2_n = \frac{1}{2}\left(\begin{array}{cc} 1+(-1)^n & -i[1-(-1)^n] \\ i[1-(-1)^n] & 1+(-1)^n\end{array}\right), \tag{11}$$

where we have abbreviated the pair (ℓ, m) as $\mathbf{l}$.

Finally, the following orthogonality relation for the Wigner $3j$ symbols is useful:

$$\sum_{mm_1}\left(\begin{array}{ccc}\ell & \ell_1 & \ell_2 \\ m & m_1 & m_2\end{array}\right)\left(\begin{array}{ccc}\ell & \ell_1 & \ell_3 \\ m & m_1 & m_3\end{array}\right) = \frac{\delta_{\ell_2\ell_3}\delta_{m_2m_3}}{2\ell_2+1} \tag{12}$$

2.2 E and B mode purification

We define a field $\mathbf{f}$ to be a B (E) mode if $(\mathbf{D}_s^{E(B)})^\dagger\mathbf{f}=0$. At the same time, and under the definition of the dot product:

$$(\mathbf{f},\mathbf{g})\equiv\int d\hat{\mathbf{n}}\,\mathbf{f}^\dagger\mathbf{g}, \tag{13}$$

we define a *pure* B (E) mode as a field that is orthogonal to all E (B) modes.

Since $\mathbf{D}_s^{E\dagger}\mathbf{D}_s^B=0$, one can always generate a B (E) mode by applying $\mathbf{D}_s^{B(E)}$ to a scalar field. It is then possible to show that E and B modes thus defined are orthogonal in the full sky:

$$(\mathbf{D}_s^E\phi,\mathbf{D}_s^B\psi)=\int d\hat{\mathbf{n}}\,(\mathbf{D}_s^E\phi)^\dagger\mathbf{D}_s^B\psi=0 \tag{14}$$

This can be done by integrating by parts and noting that the celestial sphere has no boundaries. On a cut sky, however, and for $s=2$, this is only true if the fields satisfy Neumann and Dirichlet boundary conditions simultaneously (i.e. vanishing value and first derivative on the boundary of the cut sky region).

Let $w(\hat{\mathbf{n}})$ be a sky window function defining the sky region to be analyzed (and the weight to be applied in each pixel). The standard pseudo B-mode of a field $\mathbf{P}$ is then given by

$$\tilde{B}_{\ell m}=\int d\hat{\mathbf{n}}\,w(\hat{\mathbf{n}})\left({}_s\mathbf{Y}^B_{\ell m}(\hat{\mathbf{n}})\right)^\dagger\mathbf{P}=\int d\hat{\mathbf{n}}\,w(\hat{\mathbf{n}})(\mathbf{D}_s^BY_{\ell m})^\dagger\mathbf{P}(\hat{\mathbf{n}}), \tag{15}$$

Now, since $\mathbf{D}_s^BY_{\ell m}$ is a B-mode, in the absence of w this expresion would correspond to a projection that filters out all the E-modes from $\mathbf{P}$. However, $w(\hat{\mathbf{n}})\mathbf{D}_s^BY_{\ell m}$ is not a B-mode, and therefore $\tilde{B}_{\ell m}$ receives contributions from ambiguous E modes (which then propagate into the variance of the pseudo-C_ℓ estimator of the power spectrum).

The idea behind B-mode purification is to move w to the right of $\mathbf{D}_s^B$, defining the field:

$$B^p_{\ell m}=\int d\hat{\mathbf{n}}\left(\mathbf{D}_2^B(w\,Y_{\ell m})\right)^\dagger\,\mathbf{P}(\hat{\mathbf{n}}). \tag{16}$$

Since $\mathbf{D}_2^B(wY_{\ell m})$ is a B-mode quantity, $B^p_{\ell m}$ should receive contributions only from B-modes.

Expanding $\mathbf{D}_2^B(wY_{\ell m})$, we can write $B^p_{\ell m}$ as:

$$B^p_{\ell m}=\left(\tilde{P}_2\right)^B_{\ell m}+2\frac{\alpha_{\ell,2}}{\alpha_{\ell,1}}\left(\tilde{P}_1\right)^B_{\ell m}+\alpha_{\ell,2}\left(\tilde{P}_2\right)^B_{\ell m}, \tag{17}$$

where $(f)^B_{\ell m}$ stands for the B-mode of field f, and we have defined the fields $\tilde{P}_n=(\eth^n w)^*(Q+iU)$, where Q and U are the real and imaginary parts of the field P.

Note that the derivatives of w can be computed as:

$$w\rightarrow w_{\ell m}=\mathrm{SHT}(w)\rightarrow\left\{{}_nw^E_{\ell m}=(-1)^H w_{\ell m}/\alpha_{\ell,n},\,{}_nw^B_{\ell m}=0\right\}\rightarrow\eth^n w=\mathrm{SHT}^{-1}(\{{}_nw^E_{\ell m},\,{}_nw^B_{\ell m}\}) \tag{18}$$

3 Contaminant cleaning

Let $\mathbf{a}$ be a random field defined on the sphere, let $v(\hat{\mathbf{n}})$ a mask for $\mathbf{a}$ and let $\mathbf{f}^i$ be a set of N_a contaminants of $\mathbf{a}$ such that the observed version of $\mathbf{a}$ be:

$$\mathbf{d}_a(\hat{\mathbf{n}})=\mathbf{a}^v(\hat{\mathbf{n}})+\sum_{i=1}^{N_a}\alpha_i\mathbf{f}^i(\hat{\mathbf{n}}), \tag{19}$$

where $\mathbf{a}^v(\hat{\mathbf{n}})=v(\hat{\mathbf{n}})\,\mathbf{a}(\hat{\mathbf{n}})$ (note that we have implicitly applied the same mask to $\mathbf{f}^i$). The best-fit value for the coefficients α_i assuming the same weights for all points in $\mathbf{d}_a$ can be found as

$$\tilde{\alpha}_i=\sum_j M_{ij}\int d\hat{\mathbf{n}}\,\mathbf{f}^{j\dagger}(\hat{\mathbf{n}})\,\mathbf{d}_a(\hat{\mathbf{n}}),\quad(\hat{\mathsf{M}}^{-1})_{ij}\equiv\int d\hat{\mathbf{n}}\,\mathbf{f}^{i\dagger}(\hat{\mathbf{n}})\,\mathbf{f}^j(\hat{\mathbf{n}}). \tag{20}$$

Thus we can find a cleaned version of $\mathbf{a}$ as:

$$\tilde{\mathbf{a}}(\hat{\mathbf{n}}) \equiv \mathbf{d}_a(\hat{\mathbf{n}}) - \mathbf{f}^i(\hat{\mathbf{n}})\, M_{ij} \int d\hat{\mathbf{n}}' \mathbf{f}^{j\dagger}(\hat{\mathbf{n}}')\mathbf{d}_a(\hat{\mathbf{n}}') \tag{21}$$

$$= \mathbf{a}^v(\hat{\mathbf{n}}) - \mathbf{f}^i(\hat{\mathbf{n}})\, M_{ij} \int d\hat{\mathbf{n}}' \mathbf{f}^{j\dagger}(\hat{\mathbf{n}}')\mathbf{a}^v(\hat{\mathbf{n}}'), \tag{22}$$

where there is an implicit summation sign over i and j (we will omit these from now on).

The harmonic coefficients of the cleaned and masked field are:

$$\tilde{\mathbf{a}}_{\ell m} = \mathbf{a}^v_{\ell m} - \mathbf{f}^i_{\ell m} M_{ij} \sum_{\ell' m'} \mathbf{f}^{j\dagger}_{\ell' m'} \mathbf{a}^v_{\ell' m'}. \tag{23}$$

From now on we will simplify the notation by abbreviating the pair ℓm as $\mathbf{l}$, so that the previous equation reads:

$$\tilde{\mathbf{a}}_{\mathbf{l}} = \mathbf{a}^v_{\mathbf{l}} - \mathbf{f}^i_{\mathbf{l}} M_{ij} \sum_{\mathbf{l}'} \mathbf{f}^{j\dagger}_{\mathbf{l}'} \mathbf{a}^v_{\mathbf{l}'}. \tag{24}$$

The harmonic coefficients for the masked field can be related to those of the unmasked one and the mask v (understood as a spin-0 field) as:

$$\mathbf{a}^v_{\mathbf{l}} = \sum_{\mathbf{l}_1\mathbf{l}_2} \hat{\mathsf{D}}^{s_a}_{\mathbf{l}\mathbf{l}_1\mathbf{l}_2} \mathbf{a}_{\mathbf{l}_1} v_{\mathbf{l}_2}. \tag{25}$$

4 Pseudo-C_ℓ estimators with mode deprojection

In what follows, for two fields $\mathbf{a}$ and $\mathbf{b}$ we will define their observed power spectrum as:

$$\tilde{\mathsf{C}}^{ab}_\ell \equiv \frac{1}{2\ell+1} \sum_m \mathbf{a}_{\ell m} \mathbf{b}^\dagger_{\ell m}. \tag{26}$$

This must not be confused with the true power spectrum defined as an ensemble average for isotropic fields:

$$\langle \mathbf{a}_{\mathbf{l}} \mathbf{b}^\dagger_{\mathbf{l}'} \rangle \equiv \hat{\mathsf{C}}^{ab}_\ell \delta_{\ell\ell'} \delta_{mm'}. \tag{27}$$

Now, let $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ be the contaminant-cleaned versions of two random fields $\mathbf{a}$ and $\mathbf{b}$ with contaminants $\mathbf{f}^i$ and $\mathbf{g}^j$ and masks v and w respectively, and let us define

$$(\hat{\mathsf{N}}^{-1})_{ij} \equiv \int d\hat{\mathbf{n}}\, \mathbf{g}^{i\dagger}(\hat{\mathbf{n}})\, \mathbf{g}^j(\hat{\mathbf{n}}). \tag{28}$$

The observed power spectrum of the contaminant-cleaned maps can be written as:

$$\begin{aligned}\tilde{\mathsf{C}}^{\tilde{a}\tilde{b}}_\ell =& \frac{1}{2\ell+1} \sum_m \mathbf{a}^v_{\ell m} \mathbf{b}^{w\dagger}_{\ell m} - \frac{N^*_{ij}}{2\ell+1} \sum_m \sum_{\mathbf{l}'} \mathbf{a}^v_{\mathbf{l}} \mathbf{b}^{w\dagger}_{\mathbf{l}'} \mathbf{g}^j_{\mathbf{l}'} \mathbf{g}^{i\dagger}_{\mathbf{l}} - \\ &- \frac{M_{ij}}{2\ell+1} \sum_m \sum_{\mathbf{l}'} \mathbf{f}^i_{\mathbf{l}} \mathbf{f}^{j\dagger}_{\mathbf{l}'} \mathbf{a}^v_{\mathbf{l}'} \mathbf{b}^{w\dagger}_{\mathbf{l}} + \frac{M_{ij} N^*_{pq}}{2\ell+1} \sum_m \sum_{\mathbf{l}'\mathbf{l}''} \mathbf{f}^i_{\mathbf{l}} \mathbf{f}^{j\dagger}_{\mathbf{l}'} \mathbf{a}^v_{\mathbf{l}'} \mathbf{b}^{w\dagger}_{\mathbf{l}''} \mathbf{g}^q_{\mathbf{l}''} \mathbf{g}^{p\dagger}_{\mathbf{l}}.\end{aligned} \tag{29}$$

In order to compute the bias of $\tilde{\mathsf{C}}^{\tilde{a}\tilde{b}}_\ell$ with respect to $\hat{\mathsf{C}}^{ab}_\ell$, we need to compute the ensemble average of the former, which we will write as:

$$\left\langle \tilde{\mathsf{C}}^{\tilde{a}\tilde{b}}_\ell \right\rangle = \hat{\mathsf{F}}^1_\ell - \hat{\mathsf{F}}^2_\ell - \hat{\mathsf{F}}^3_\ell + \hat{\mathsf{F}}^4_\ell, \tag{30}$$

where:

$$\hat{\mathsf{F}}^1_\ell \equiv \frac{1}{2\ell+1} \sum_m \langle \mathbf{a}^v_{\ell m} \mathbf{b}^{w\dagger}_{\ell m} \rangle, \quad \hat{\mathsf{F}}^2_\ell \equiv \frac{N^*_{ij}}{2\ell+1} \sum_m \sum_{\mathbf{l}'} \langle \mathbf{a}^v_{\mathbf{l}} \mathbf{b}^{w\dagger}_{\mathbf{l}'} \mathbf{g}^j_{\mathbf{l}'} \mathbf{g}^{i\dagger}_{\mathbf{l}} \rangle \tag{31}$$

$$\hat{\mathsf{F}}^3_\ell \equiv \frac{M_{ij}}{2\ell+1} \sum_m \sum_{\mathbf{l}'} \langle \mathbf{f}^i_{\mathbf{l}} \mathbf{f}^{j\dagger}_{\mathbf{l}'} \mathbf{a}^v_{\mathbf{l}'} \mathbf{b}^{w\dagger}_{\mathbf{l}} \rangle, \quad \hat{\mathsf{F}}^4_\ell \equiv \frac{M_{ij} N^*_{pq}}{2\ell+1} \sum_m \sum_{\mathbf{l}'\mathbf{l}''} \langle \mathbf{f}^i_{\mathbf{l}} \mathbf{f}^{j\dagger}_{\mathbf{l}'} \mathbf{a}^v_{\mathbf{l}'} \mathbf{b}^{w\dagger}_{\mathbf{l}''} \mathbf{g}^q_{\mathbf{l}''} \mathbf{g}^{p\dagger}_{\mathbf{l}} \rangle. \tag{32}$$

We will now compute the ensemble average of each of these terms.

4.1 $\hat{\mathsf{F}}^1_\ell$

$$
\begin{aligned}
\hat{\mathsf{F}}^1_\ell &= \frac{1}{2\ell+1}\sum_{m\mathbf{l}_{1,2,3,4}} v_{\mathbf{l}_2} w^*_{\mathbf{l}_4}\hat{\mathsf{D}}^{s_a}_{\mathbf{l}\mathbf{l}_1\mathbf{l}_2}\langle \mathbf{a}_{\mathbf{l}_1}\mathbf{b}^\dagger_{\mathbf{l}_3}\rangle \hat{\mathsf{D}}^{s_b\dagger}_{\mathbf{l}\mathbf{l}_3\mathbf{l}_4} \\
&= \frac{1}{2\ell+1}\sum_{m\mathbf{l}_{1,2,3}} v_{\mathbf{l}_2} w^*_{\mathbf{l}_3}\hat{\mathsf{D}}^{s_a}_{\mathbf{l}\mathbf{l}_1\mathbf{l}_2}\hat{\mathsf{C}}^{ab}_{\mathbf{l}_1}\hat{\mathsf{D}}^{s_b\dagger}_{\mathbf{l}\mathbf{l}_1\mathbf{l}_3} \\
&= \frac{1}{2\ell+1}\sum_{\ell_1\mathbf{l}_{2,3}} v_{\mathbf{l}_2} w^*_{\mathbf{l}_3}\frac{(2\ell+1)(2\ell_1+1)}{4\pi}\sqrt{(2\ell_2+1)(2\ell_3+1)}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ s_a & -s_a & 0\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_3\\ s_b & -s_b & 0\end{pmatrix} \\
&\qquad \hat{\mathsf{d}}^{s_a}_{\ell+\ell_1+\ell_2}\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{d}}^{s_b\dagger}_{\ell+\ell_1+\ell_3}\sum_{mm_1}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ -m & m_1 & m_2\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_3\\ -m & m_1 & m_3\end{pmatrix} \\
&= \sum_{\ell_1\ell_2}\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi}\tilde{C}^{vw}_{\ell_2}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ s_a & -s_a & 0\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ s_b & -s_b & 0\end{pmatrix}\hat{\mathsf{d}}^{s_a}_{\ell+\ell_1+\ell_2}\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{d}}^{s_b\dagger}_{\ell+\ell_1+\ell_2}
\end{aligned}
\tag{33}
$$

For $v=w=1$ this reduces to $\tilde{C}^{vw}_\ell = 4\pi\delta_{\ell 0}$ and:

$$
\begin{aligned}
\hat{\mathsf{F}}^1_\ell &= \sum_{\ell_1\ell_2}(2\ell_1+1)(2\ell_2+1)\delta_{\ell_2 0}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ s_a & -s_a & 0\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ s_b & -s_b & 0\end{pmatrix}\hat{\mathsf{d}}^{s_a}_{\ell+\ell_1+\ell_2}\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{d}}^{s_b\dagger}_{\ell+\ell_1+\ell_2} \\
&= \sum_{\ell_1}(2\ell_1+1)\begin{pmatrix}\ell & \ell_1 & 0\\ s_a & -s_a & 0\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & 0\\ s_b & -s_b & 0\end{pmatrix}\hat{\mathsf{d}}^{s_a}_{\ell+\ell_1}\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{d}}^{s_b\dagger}_{\ell+\ell_1} \\
&= \sum_{\ell_1}(2\ell_1+1)\delta_{\ell\ell_1}\frac{(-1)^{\ell-s_a}}{\sqrt{2\ell+1}}\delta_{\ell\ell_1}\frac{(-1)^{\ell-s_b}}{\sqrt{2\ell+1}}\hat{\mathsf{d}}^{s_a}_{\ell+\ell_1}\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{d}}^{s_b\dagger}_{\ell+\ell_1} \\
&= \hat{\mathsf{d}}^{s_a}_{2\ell}\hat{\mathsf{C}}^{ab}_{\ell}\hat{\mathsf{d}}^{s_b\dagger}_{2\ell} \\
&= \hat{\mathsf{C}}^{ab}_\ell
\end{aligned}
$$

4.2 $\hat{\mathsf{F}}^2_\ell$

$$
\begin{aligned}
\hat{\mathsf{F}}^2_\ell &= N^*_{ij}\int d\hat{\mathbf{n}}\, d\hat{\mathbf{n}}'\,\frac{\sum_{m\mathbf{l}'\mathbf{l}_{1,2,3,4}}}{2\ell+1}\hat{\mathsf{Y}}^{s_a\dagger}_{\mathbf{l}}(\hat{\mathbf{n}})\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}})\langle \mathbf{a}_{\mathbf{l}_1}\mathbf{b}^\dagger_{\mathbf{l}_3}\rangle\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_3}(\hat{\mathbf{n}}')\hat{\mathsf{Y}}^{s_b}_{\mathbf{l}'}(\hat{\mathbf{n}}')\mathbf{g}^j_{\mathbf{l}'}\mathbf{g}^{i\dagger}_{\mathbf{l}} v_{\mathbf{l}_2} w^*_{\mathbf{l}_4} Y_{\mathbf{l}_2}(\hat{\mathbf{n}}) Y^*_{\mathbf{l}_4}(\hat{\mathbf{n}}') \\
&= N^*_{ij}\frac{\sum_m}{2\ell+1}\left\{\int d\hat{\mathbf{n}}\, v(\hat{\mathbf{n}})\,\hat{\mathsf{Y}}^{s_a\dagger}_{\mathbf{l}}(\hat{\mathbf{n}})\left[\sum_{\ell_1 m_1}\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}})\hat{\mathsf{C}}^{ab}_{\ell_1}\left(\int d\hat{\mathbf{n}}'\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_1}(\hat{\mathbf{n}}')\mathbf{g}^j(\hat{\mathbf{n}}')w(\hat{\mathbf{n}}')\right)\right]\mathbf{g}^{i\dagger}_{\mathbf{l}}\right\} \\
&= N^*_{ij}\frac{\sum_m}{2\ell+1}\mathrm{SHT}\left\{v(\hat{\mathbf{n}})\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(w\,\mathbf{g}^j\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}^{s_a}_{\mathbf{l}}\mathbf{g}^{i\dagger}_{\mathbf{l}}
\end{aligned}
\tag{34}
$$

For $v=w=1$ this reduces to:

$$
\begin{aligned}
\hat{\mathsf{F}}^2_\ell &= N^*_{ij}\frac{\sum_m}{2\ell+1}\mathrm{SHT}\left\{\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(\mathbf{g}^j\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}^{s_a}_{\mathbf{l}}\mathbf{g}^{i\dagger}_{\mathbf{l}} \\
&= N^*_{ij}\,\hat{\mathsf{C}}^{ab}_\ell\frac{\sum_m \mathbf{g}^j_{\ell m}\mathbf{g}^{i\dagger}_{\ell m}}{2\ell+1} \\
&= N^*_{ij}\,\hat{\mathsf{C}}^{ab}_\ell\tilde{\mathsf{C}}^{g^j g^i}_\ell
\end{aligned}
$$

4.3 $\hat{\mathsf{F}}^3_\ell$

$$\begin{aligned}
\hat{\mathsf{F}}^3_\ell &= M_{ij}\int d\hat{\mathbf{n}}\,d\hat{\mathbf{n}}'\,\frac{\sum_{m\mathbf{l}'\mathbf{l}_{1,2,3,4}}}{2\ell+1}\mathbf{f}^i_{\mathbf{l}}\mathbf{f}^{j\dagger}_{\mathbf{l}'}\hat{\mathsf{Y}}^{s_a\dagger}_{\mathbf{l}'}(\hat{\mathbf{n}}')\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_3}(\hat{\mathbf{n}}')\langle\mathbf{a}_{\mathbf{l}_3}\mathbf{b}^\dagger_{\mathbf{l}_1}\rangle\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_1}(\hat{\mathbf{n}})\hat{\mathsf{Y}}^{s_b}_{\mathbf{l}}(\hat{\mathbf{n}})v_{\mathbf{l}_4}w^*_{\mathbf{l}_2}Y_{\mathbf{l}_2}(\hat{\mathbf{n}})Y^*_{\mathbf{l}_4}(\hat{\mathbf{n}}') \\
&= M_{ij}\frac{\sum_m}{2\ell+1}\mathbf{f}^i_{\mathbf{l}}\left\{\int d\hat{\mathbf{n}}\,w(\hat{\mathbf{n}})\left[\sum_{\ell_1m_1}\left(\int d\hat{\mathbf{n}}'\,v(\hat{\mathbf{n}}')\mathbf{f}^{j\dagger}(\hat{\mathbf{n}}')\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}}')\right)\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_1}(\hat{\mathbf{n}})\right]\hat{\mathsf{Y}}^{s_b}_{\mathbf{l}}(\hat{\mathbf{n}})\right\} \\
&= M_{ij}\frac{\sum_m}{2\ell+1}\mathbf{f}^i_{\mathbf{l}}\,\mathrm{SHT}\left\{w(\hat{\mathbf{n}})\,\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab\dagger}_{\ell_1}\mathrm{SHT}\left(v\,\mathbf{f}^j\right)^{s_a}_{\mathbf{l}_1}\right]^{s_b}_{\hat{\mathbf{n}}}\right\}^{s_b\dagger}_{\mathbf{l}}
\end{aligned}\tag{35}$$

For $v = w = 1$ this reduces to:

$$\begin{aligned}
\hat{\mathsf{F}}^3_\ell &= M_{ij}\frac{\sum_m}{2\ell+1}\mathbf{f}^i_{\mathbf{l}}\,\mathrm{SHT}\left\{\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab\dagger}_{\ell_1}\mathrm{SHT}\left(\mathbf{f}^j\right)^{s_a}_{\mathbf{l}_1}\right]^{s_b}_{\hat{\mathbf{n}}}\right\}^{s_b\dagger}_{\mathbf{l}} \\
&= M_{ij}\frac{\sum_m \mathbf{f}^i_{\ell m}\mathbf{f}^{j\dagger}_{\ell m}}{2\ell+1}\hat{\mathsf{C}}^{ab}_\ell \\
&= M_{ij}\tilde{\mathsf{C}}^{f^if^j}_\ell\hat{\mathsf{C}}^{ab}_\ell
\end{aligned}$$

4.4 $\hat{\mathsf{F}}^4_\ell$

$$\begin{aligned}
\hat{\mathsf{F}}^4_\ell &= \frac{M_{ij}N^*_{pq}}{2\ell+1}\int d\hat{\mathbf{n}}\,d\hat{\mathbf{n}}'\sum_{m\mathbf{l}'\mathbf{l}''\mathbf{l}_{1,2,3,4}}\mathbf{f}^i_{\mathbf{l}}\mathbf{f}^{j\dagger}_{\mathbf{l}'}\hat{\mathsf{Y}}^{s_a\dagger}_{\mathbf{l}'}(\hat{\mathbf{n}})\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}})\langle\mathbf{a}_{\mathbf{l}_1}\mathbf{b}^\dagger_{\mathbf{l}_3}\rangle\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_3}(\hat{\mathbf{n}}')\hat{\mathsf{Y}}^{s_b}_{\mathbf{l}''}(\hat{\mathbf{n}}')\mathbf{g}^q_{\mathbf{l}''}\mathbf{g}^{p\dagger}_{\mathbf{l}}v_{\mathbf{l}_2}w^*_{\mathbf{l}_4}Y_{\mathbf{l}_2}(\hat{\mathbf{n}})Y^*_{\mathbf{l}_4}(\hat{\mathbf{n}}') \\
&= \frac{M_{ij}N^*_{pq}}{2\ell+1}\sum_m\mathbf{f}^i_{\mathbf{l}}\left\{\int d\hat{\mathbf{n}}\,v(\hat{\mathbf{n}})\,\mathbf{f}^{j\dagger}(\hat{\mathbf{n}})\left[\sum_{\ell_1m_1}\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}})\hat{\mathsf{C}}^{ab}_{\ell_1}\left(\int d\hat{\mathbf{n}}'\,\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_1}(\hat{\mathbf{n}}')\mathbf{g}^q(\hat{\mathbf{n}}')w(\hat{\mathbf{n}}')\right)\right]\right\}\mathbf{g}^{p\dagger}_{\mathbf{l}} \\
&= M_{ij}N^*_{pq}\left\{\int d\hat{\mathbf{n}}\,v(\hat{\mathbf{n}})\,\mathbf{f}^{j\dagger}(\hat{\mathbf{n}})\,\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(w\,\mathbf{g}^q\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}\tilde{\mathsf{C}}^{f^ig^p}_\ell
\end{aligned}\tag{36}$$

For $v = w = 1$ this reduces to:

$$\begin{aligned}
\hat{\mathsf{F}}^4_\ell &= M_{ij}N^*_{pq}\left\{\int d\hat{\mathbf{n}}\,\mathbf{f}^{j\dagger}(\hat{\mathbf{n}})\,\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(\mathbf{g}^q\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}\tilde{\mathsf{C}}^{f^ig^p}_\ell \\
&= M_{ij}N^*_{pq}\left\{\int d\hat{\mathbf{n}}\sum_{\mathbf{l}_2}\mathbf{f}^{j\dagger}_{\mathbf{l}_2}\hat{\mathsf{Y}}^{s_a\dagger}_{\mathbf{l}_2}(\hat{\mathbf{n}})\sum_{\mathbf{l}_1}\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}})\hat{\mathsf{C}}^{ab}_{\ell_1}\mathbf{g}^q_{\mathbf{l}_1}\right\}\tilde{\mathsf{C}}^{f^ig^p}_\ell \\
&= M_{ij}N^*_{pq}\mathbf{f}^{j\dagger}_{\mathbf{l}_1}\hat{\mathsf{C}}^{ab}_{\ell_1}\mathbf{g}^q_{\mathbf{l}_1}\tilde{\mathsf{C}}^{f^ig^p}_\ell \\
&= M_{ij}N^*_{pq}\left[\sum_{\ell_1}(2\ell_1+1)\mathrm{Tr}\left(\hat{\mathsf{C}}^{ab}_{\ell_1}\tilde{\mathsf{C}}^{g^qf^j}_{\ell_1}\right)\right]\tilde{\mathsf{C}}^{f^ig^p}_\ell
\end{aligned}$$

4.5 Final form of the estimator

Putting together the results from Equations 33, 34, 35 and 36, we can write down an unbiased estimator for the pseudo-C_ℓ of the cut-sky maps free from contamination from $\mathbf{f}$ and $\mathbf{g}$:

$$\begin{aligned}
\tilde{\mathsf{C}}^{ab}_\ell =&\tilde{\mathsf{C}}^{\tilde{a}\tilde{b}}_\ell + N^*_{ij}\frac{\sum_m}{2\ell+1}\mathrm{SHT}\left\{v(\hat{\mathbf{n}})\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(w\,\mathbf{g}^j\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}^{s_a}_{\mathbf{l}}\mathbf{g}^{i\dagger}_{\mathbf{l}}+ \\
&+ M_{ij}\frac{\sum_m}{2\ell+1}\mathbf{f}^i_{\mathbf{l}}\,\mathrm{SHT}\left\{w(\hat{\mathbf{n}})\,\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab\dagger}_{\ell_1}\mathrm{SHT}\left(v\,\mathbf{f}^j\right)^{s_a}_{\mathbf{l}_1}\right]^{s_b}_{\hat{\mathbf{n}}}\right\}^{s_b\dagger}_{\mathbf{l}} - \\
&- M_{ij}N^*_{pq}\left\{\int d\hat{\mathbf{n}}\,v(\hat{\mathbf{n}})\,\mathbf{f}^{j\dagger}(\hat{\mathbf{n}})\,\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(w\,\mathbf{g}^q\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}\tilde{\mathsf{C}}^{f^ig^p}_\ell
\end{aligned}\tag{37}$$

Once $\tilde{\mathsf{C}}^{ab}$ is calculated, it can be corrected for the effects of masking by inverting the linear transformation in Eq 33. This transformation can be written explicitly by first transforming the power spectrum matrices into vectors ${}_v\mathsf{C}$. E.g. for $s_a = s_b = 2$ we transform:

$$\hat{\mathsf{C}}^{ab}_\ell \equiv \left(\begin{array}{cc} C^{E_aE_b}_\ell & C^{E_aB_b}_\ell \\ C^{B_aE_b}_\ell & C^{B_aB_b}_\ell \end{array}\right) \quad \text{into} \quad {}_v\hat{\mathsf{C}}^{ab}_\ell \equiv \left(\begin{array}{c} C^{E_aE_b}_\ell \\ C^{E_aB_b}_\ell \\ C^{B_aE_b}_\ell \\ C^{B_aB_b}_\ell \end{array}\right). \tag{38}$$

We can then write, in general:

$${}_v\tilde{\mathsf{C}}^{ab}_\ell = \sum_{\ell'} \mathsf{M}^{s_as_b}_{\ell\ell'} \cdot {}_v\hat{\mathsf{C}}^{ab}_{\ell'}, \tag{39}$$

where:

$$\mathsf{M}^{00}_{\ell\ell'} = \frac{2\ell'+1}{4\pi}\sum_{\ell''}(2\ell''+1)C^{vw}_{\ell''}\left(\begin{array}{ccc} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{array}\right)^2 \tag{40}$$

$$\mathsf{M}^{02}_{\ell\ell'} = M^{0+}_{\ell\ell'}\,\hat{1}, \quad M^{0+}_{\ell\ell'} = \frac{2\ell'+1}{4\pi}\sum_{\ell''}(2\ell''+1)C^{vw}_{\ell''}\left(\begin{array}{ccc} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{array}\right)\left(\begin{array}{ccc} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{array}\right) \tag{41}$$

$$\mathsf{M}^{22}_{\ell\ell'} = \left(\begin{array}{cccc} M^{++}_{\ell\ell'} & 0 & 0 & M^{--}_{\ell\ell'} \\ 0 & M^{++}_{\ell\ell'} & -M^{--}_{\ell\ell'} & 0 \\ 0 & -M^{--}_{\ell\ell'} & M^{++}_{\ell\ell'} & 0 \\ M^{--}_{\ell\ell'} & 0 & 0 & M^{++}_{\ell\ell'} \end{array}\right) \tag{42}$$

$$M^{\pm\pm}_{\ell\ell'} = \frac{2\ell'+1}{4\pi}\sum_{\ell''}(2\ell''+1)C^{vw}_{\ell''}\left(\begin{array}{ccc} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{array}\right)^2 \frac{1\pm(-1)^{\ell+\ell'+\ell''}}{2} \tag{43}$$

If either the B or E modes of a spin-2 field has been purified, the equations above must be modified by carrying out the following modification in the equations above:

$$\left(\begin{array}{ccc} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{array}\right) \longrightarrow \tag{44}$$

$$\left(\begin{array}{ccc} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{array}\right) + 2\sqrt{\frac{(\ell+1)!(\ell-2)!(\ell''+1)!}{(\ell-1)!(\ell+2)!(\ell''-1)!}}\left(\begin{array}{ccc} \ell & \ell' & \ell'' \\ 1 & -2 & 1 \end{array}\right) + \sqrt{\frac{(\ell-2)!(\ell''+2)!}{(\ell+2)!(\ell''-2)!}}\left(\begin{array}{ccc} \ell & \ell' & \ell'' \\ 0 & -2 & 2 \end{array}\right).$$

This change must be applied to the corresponding factors of $\left(\begin{array}{ccc} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{array}\right)$.

4.6 Beam

Adding the effect of a beam amounts to redefining:

$$\mathsf{M}^{s_as_b}_{\ell_1\ell_2} \rightarrow \mathsf{M}^{s_as_b}_{\ell_1\ell_2}\, b^{ab}_{\ell_2}, \tag{45}$$

where b^{ab}_ℓ is the product of the harmonic transform of the beams for maps a and b.

4.7 Binning into bandpowers

Given the loss of information implicit in masking the originally full-sky field, it is in general not possible to invert Eq. 39 directly. De usual approach to doing so is by binning the pseudo-C_ℓ into bandpowers. A bandpower k is defined by a set of N_k multipoles $\vec{\ell}_k \equiv (\ell^1_k, ..., \ell^{N_k}_k)$ and a set of weights $\vec{w}_k \equiv (w^1_k, ..., w^{N_k}_k)$ normalized such that $\sum_{i=1}^{N_k} w^i_k = 1$. The $k-$th bandpower for the coupled pseudo-C_ℓ is then defined as:

$${}_v\tilde{\mathsf{B}}^{ab}_k \equiv \sum_{i=1}^{N_k} w^i_k\, {}_v\tilde{\mathsf{C}}^{ab}_{\ell^i_k} = \sum_{i=1}^{N_k} w^i_k \sum_{\ell'} \mathsf{M}^{s_as_b}_{\ell^i_k\ell'}\, {}_v\hat{\mathsf{C}}^{ab}_{\ell}. \tag{46}$$

One then proceeds by assuming that the true power spectrum is a step-wise function, taking constant values over the multipoles corresponding to each bandpower: ${}_v\hat{\mathsf{C}}^{ab}_\ell = \sum_k {}_v\hat{\mathsf{B}}^{ab}_k \Theta(\ell \in \vec{\ell}_k)$ (where Θ is a binary step function). The previous equation then reads:

$$ {}_v\tilde{\mathsf{B}}^{ab}_k = \sum_{k'} \mathcal{M}^{s_a s_b}_{kk'}\, {}_v\hat{\mathsf{B}}^{ab}_{kk'} \equiv \sum_{k'} \left(\sum_{\ell\in\vec{\ell}_k} \sum_{\ell'\in\vec{\ell}_{k'}} w^\ell_k \mathsf{M}^{s_a s_b}_{\ell\ell'} \right) {}_v\hat{\mathsf{B}}^{ab}_{k'}, \tag{47} $$

which defines the binned coupling matrix $\mathcal{M}^{ab}_{kk'}$. The decoupled bandpowers are then estimated by inverting $\mathcal{M}^{ab}$:

$$ {}_v\hat{\mathsf{B}}^{ab}_k = \sum_{k'} \left(\mathcal{M}^{ab}\right)^{-1}_{kk'}\, {}_v\tilde{\mathsf{B}}^{ab}_{k'}. \tag{48} $$

Note that, even though this procedure is based on the assumption that the true power spectrum is step-wise constant, the bandpowers computed this way should be compared with the theoretical prediction subjected to the same type of transformation. I.e. the theoretical prediction for the bandpowers is:

$$ {}_v\bar{\mathsf{B}}^{ab}_k = \sum_{k'} \left(\mathcal{M}^{ab}\right)^{-1}_{kk'} \sum_{\ell'\in\vec{\ell}_{k'}} w^{\ell'}_{k'} \sum_{\ell''} \mathsf{M}^{s_a s_b}_{\ell'\ell''}\, {}_v\bar{\mathsf{C}}^{ab}_{\ell''}, \tag{49} $$

where we the overline $^-$ denotes theoretical predictions.

5 Flat-sky

5.1 Fourier transforms

In the flat sky we will write the directional vector $\hat{\mathbf{n}}$ as $\mathbf{x}$. Let $\mathbf{a}(\mathbf{x})$ be a spin-s_a quantity. Under the approximation $\sin\theta \sim 1$, $\Delta\theta \to \Delta x$, $\Delta\phi \to \Delta y$, the differential operator $\eth$ now takes the form:

$$ \eth = -(\partial_x + i\partial_y), \quad \bar{\eth} = -(\partial_x - i\partial_y), \tag{50} $$

and acts on a plane wave $e^{i\mathbf{kx}}$ as:

$$ \eth^s e^{i\mathbf{kx}} = (-ik)^s e^{i\,s\phi_k} e^{i\mathbf{kx}}, \quad \bar{\eth} e^{i\mathbf{kx}} = (-ik)^s e^{-i\,s\phi_k} e^{i\mathbf{kx}} \tag{51} $$

Let us define the basis functions:

$$ {}_s\mathcal{Y}_{\mathbf{k}}(\mathbf{x}) \equiv k^{-s}\eth^s e^{i\mathbf{kx}} = (-i)^s e^{is\phi_k} e^{i\mathbf{kx}}, \quad {}_{-s}\mathcal{Y}_{\mathbf{k}}(\mathbf{x}) \equiv (-k)^{-s}\bar{\eth}^s e^{i\mathbf{kx}} = i^s e^{-is\phi_k} e^{i\mathbf{kx}}, \tag{52} $$

and the spin-s Fourier coefficients:

$$ a(\mathbf{x}) = \int \frac{d\mathbf{l}^2}{2\pi}\, {}_s\mathcal{Y}_{\mathbf{l}}(\mathbf{x})\, {}_s a_{\mathbf{l}}, \quad a^*(\mathbf{x}) = \int \frac{d\mathbf{l}^2}{2\pi}\, {}_{-s}\mathcal{Y}_{\mathbf{l}}(\mathbf{x})\, {}_{-s} a_{\mathbf{l}}, \tag{53} $$

$$ {}_s a_{\mathbf{l}} \equiv \int \frac{d\mathbf{x}^2}{2\pi}\, {}_s\mathcal{Y}^*_{\mathbf{l}}(\mathbf{x}) a(\mathbf{x}), \quad {}_{-s} a_{\mathbf{l}} \equiv \int \frac{d\mathbf{x}^2}{2\pi}\, {}_{-s}\mathcal{Y}^*_{\mathbf{l}}(\mathbf{x}) a^*(\mathbf{x}). \tag{54} $$

The E and B-mode coefficients are then defined as:

$$ {}_s E_{\mathbf{l}} \equiv -\frac{1}{2}\left[{}_s a_{\mathbf{l}} + (-1)^s\, {}_{-s} a_{\mathbf{l}}\right], \quad i\, {}_s B_{\mathbf{l}} \equiv -\frac{1}{2}\left[{}_s a_{\mathbf{l}} - (-1)^s\, {}_{-s} a_{\mathbf{l}}\right] \tag{55} $$

Let us now write a as a vector such that in real space $\mathbf{a}(\mathbf{x}) \equiv (\mathrm{Re}(a), \mathrm{Im}(a))$, and in Fourier space $\mathbf{a}_{\mathbf{l}} \equiv ({}_s E_{\mathbf{l}}, {}_s B_{\mathbf{l}})$. We can rewrite the equations above in vectorial form:

$$ \mathbf{a}(\mathbf{x}) \equiv \int \frac{d\mathbf{l}^2}{2\pi}\, {}_s\mathsf{E}_{\mathbf{l}}(\mathbf{x}) \mathbf{a}_{\mathbf{l}}, \quad \mathbf{a}_{\mathbf{l}} \equiv \int \frac{d\mathbf{x}^2}{2\pi}\, {}_s\mathsf{E}^\dagger_{\mathbf{l}}(\mathbf{x}) \mathbf{a}(\mathbf{x}), \tag{56} $$

where we have defined the matrix basis functions:

$$ {}_s\mathsf{E}_{\mathbf{l}}(\mathbf{x}) \equiv -\frac{1}{2}\begin{pmatrix} {}_s\mathcal{Y}_{\mathbf{l}} + (-1)^s\, {}_{-s}\mathcal{Y}_{\mathbf{l}} & i({}_s\mathcal{Y}_{\mathbf{l}} - (-1)^s\, {}_{-s}\mathcal{Y}_{\mathbf{l}}) \\ -i({}_s\mathcal{Y}_{\mathbf{l}} - (-1)^s\, {}_{-s}\mathcal{Y}_{\mathbf{l}}) & {}_s\mathcal{Y}_{\mathbf{l}} + (-1)^s\, {}_{-s}\mathcal{Y}_{\mathbf{l}} \end{pmatrix} \tag{57} $$

$$ = -\frac{1}{2l^s}\begin{pmatrix} \eth^s + \bar{\eth}^s & i(\eth^s - \bar{\eth}^s) \\ -i(\eth^s - \bar{\eth}^s) & \eth^s + \bar{\eth}^s \end{pmatrix} e^{i\mathbf{lx}} \tag{58} $$

$$ = -(-i)^s \begin{pmatrix} \cos(s\phi_l) & \sin(s\phi_l) \\ -\sin(s\phi_l) & \cos(s\phi_l) \end{pmatrix} e^{i\mathbf{lx}} \tag{59} $$

$$ = -(-i)^s \mathsf{R}(\phi_l) e^{i\mathbf{lx}}, \tag{60} $$

where $\mathsf{R}(\phi)$ is a rotation matrix.

Thus:

$$\begin{pmatrix} Q(\mathbf{x}) \\ U(\mathbf{x}) \end{pmatrix} = -(-i)^s \int \frac{d\mathbf{l}^2}{2\pi} \begin{pmatrix} \cos(s\phi_l) & \sin(s\phi_l) \\ -\sin(s\phi_l) & \cos(s\phi_l) \end{pmatrix} \begin{pmatrix} {}_sE_{\mathbf{l}} \\ {}_sB_{\mathbf{l}} \end{pmatrix} e^{i\mathbf{l}\mathbf{x}} \tag{61}$$

$$\begin{pmatrix} {}_sE_{\mathbf{l}} \\ {}_sB_{\mathbf{l}} \end{pmatrix} = -i^s \begin{pmatrix} \cos(s\phi_l) & -\sin(s\phi_l) \\ \sin(s\phi_l) & \cos(s\phi_l) \end{pmatrix} \begin{pmatrix} Q_{\mathbf{l}} \\ U_{\mathbf{l}} \end{pmatrix}, \tag{62}$$

where $Q_{\mathbf{l}}$ and $U_{\mathbf{l}}$ are the standard Fourier transforms of Q and U.

The functions ${}_s\mathsf{E}_{\mathbf{l}}(\mathbf{x})$ satisfy the following orthogonality and completeness relations:

$$\int \frac{d\mathbf{x}^2}{(2\pi)^2}\, {}_s\mathsf{E}_{\mathbf{l}}(\mathbf{x})_s\mathsf{E}^\dagger_{\mathbf{l}'}(\mathbf{x}) = 1\,\delta(\mathbf{l}-\mathbf{l}'), \qquad \int \frac{d\mathbf{l}^2}{(2\pi)^2}\, {}_s\mathsf{E}_{\mathbf{l}}(\mathbf{x})_s\mathsf{E}^\dagger_{\mathbf{l}}(\mathbf{x}') = 1\,\delta(\mathbf{x}-\mathbf{x}'). \tag{63}$$

5.2 Contaminant cleaning

Using the same notation as in Section 3, the contaminant-cleaned version of $\mathbf{a}$ is:

$$\tilde{\mathbf{a}}(\mathbf{x}) = \mathbf{a}^v(\mathbf{x}) - \mathbf{f}^i(\mathbf{x})M_{ij}\int d\mathbf{x}^2\, \mathbf{f}^{j\dagger}(\mathbf{x}')\mathbf{a}^v(\mathbf{x}'), \quad \tilde{\mathbf{a}}_{\mathbf{l}} = \mathbf{a}^v_{\mathbf{l}} - \mathbf{f}^i_{\mathbf{l}} M_{ij}\int d\mathbf{k}^2\, \mathbf{f}^{j\dagger}_{\mathbf{k}}\mathbf{a}^v_{\mathbf{k}} \tag{64}$$

5.3 Pseudo-C_ℓ estimator

The Fourier coefficients of the masked field are:

$$\begin{aligned} \mathbf{a}^v_{\mathbf{l}} &= \int\int \frac{d\mathbf{k}^2 d\mathbf{q}^2}{2\pi}\left[\int \frac{d\mathbf{x}^2}{(2\pi)^2}\, {}_{s_a}\mathsf{E}^\dagger_{\mathbf{l}}(\mathbf{x})\, {}_{s_a}\mathsf{E}_{\mathbf{k}}(\mathbf{x})\, {}_0E_{\mathbf{q}}(\mathbf{x})\right]\mathbf{a}_{\mathbf{k}}\, v_{\mathbf{q}} && (65) \\ &= \int \frac{d\mathbf{k}^2}{2\pi}\mathsf{R}^\dagger\left(s(\phi_{\mathbf{l}}-\phi_{\mathbf{k}})\right)\mathbf{a}_{\mathbf{k}} v_{\mathbf{l}-\mathbf{k}} && (66) \end{aligned}$$

Then:

$$\begin{aligned} \left\langle \tilde{\mathbf{a}}_{\mathbf{l}}\, \tilde{\mathbf{b}}^\dagger_{\mathbf{l}} \right\rangle &= \int\int \frac{dk^D dq^D}{(2\pi)^D}\mathsf{R}^\dagger(s_a(\phi_{\mathbf{l}}-\phi_{\mathbf{k}}))\left\langle \mathbf{a}_{\mathbf{k}}\mathbf{b}^\dagger_{\mathbf{q}}\right\rangle \mathsf{R}(s_b(\phi_{\mathbf{l}}-\phi_{\mathbf{q}}))v_{\mathbf{l}-\mathbf{k}}w^*_{\mathbf{l}-\mathbf{q}} && (67) \\ &= \int \frac{dk^D}{(2\pi)^D}\mathsf{R}^\dagger(s_a(\phi_{\mathbf{l}}-\phi_{\mathbf{k}}))\mathsf{C}^{ab}_k\mathsf{R}(s_b(\phi_{\mathbf{l}}-\phi_{\mathbf{k}}))v_{\mathbf{l}-\mathbf{k}}w^*_{\mathbf{l}-\mathbf{k}} && (68) \end{aligned}$$

$$\begin{aligned} \left(\frac{L}{2\pi}\right)^D \tilde{\mathsf{C}}^{\tilde{a}\tilde{b}}_\ell &\equiv \int \frac{d\phi_{\mathbf{l}}}{2\pi}\left\langle \tilde{\mathbf{a}}_{\mathbf{l}}\, \tilde{\mathbf{b}}^\dagger_{\mathbf{l}}\right\rangle && (69) \\ &= \int \frac{kdk\, qdq}{(2\pi)^D}\left[\int d\phi_{\mathbf{l}} d\phi_{\mathbf{k}} d\phi_{\mathbf{q}}\mathsf{R}^\dagger(s_a(\phi_{\mathbf{l}}-\phi_{\mathbf{k}}))\mathsf{C}^{ab}_k\mathsf{R}(s_b(\phi_{\mathbf{l}}-\phi_{\mathbf{k}}))v_{\mathbf{q}}w^*_{\mathbf{q}}\delta(\mathbf{l}-\mathbf{k}-\mathbf{q})\right] && (70) \end{aligned}$$

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