NaMaster: Scientific Documentation

February 14, 2017

1 Generalities and SHTs

Let $\mathbf{a}(\hat{\mathbf{n}})$ be a spin-s_a quantity defined on the sphere, where s_a can be either 0 or 2. Then we define its spherical harmonic coefficients as:

$$\mathbf{a}_{\ell m} \equiv \mathrm{SHT}(\mathbf{a}(\hat{\mathbf{n}}))^{s_a}_{\ell m} \equiv \int d\hat{\mathbf{n}}\, \hat{\mathsf{Y}}^{s_a\dagger}_{\ell m}(\hat{\mathbf{n}})\,\mathbf{a}(\hat{\mathbf{n}}), \quad \mathbf{a}(\hat{\mathbf{n}}) = \mathrm{SHT}^{-1}(\mathbf{a}_{\ell m})^{s_a}_{\hat{\mathbf{n}}} \equiv \sum_{\ell m} \hat{\mathsf{Y}}^{s_a}_{\ell m}(\hat{\mathbf{n}})\,\mathbf{a}_{\ell m}. \tag{1}$$

Here, for a scalar quantity $\mathbf{a}(\hat{\mathbf{n}}) = a(\hat{\mathbf{n}})$, $\mathbf{a}_{\ell m} = a_{\ell m}$ are its spherical harmonic coefficients and $\hat{\mathsf{Y}}^0_{\ell m} = {}_0Y_{\ell m}$ are the scalar spherical harmonics. A spin-2 quantity can be decomposed in real space into so-called Q and U components, and in harmonic space into E and B components:

$$\mathbf{a}(\hat{\mathbf{n}}) = \left(\begin{array}{c} Q(\hat{\mathbf{n}}) \\ U(\hat{\mathbf{n}}) \end{array}\right), \quad \mathbf{a}_{\ell m} = \left(\begin{array}{c} E_{\ell m} \\ B_{\ell m} \end{array}\right), \tag{2}$$

and the transformation matrix is

$$\hat{\mathsf{Y}}^2_{\ell m} = \frac{1}{2}\left(\begin{array}{cc} -({}_2Y_{\ell m} +{}_{-2} Y_{\ell m}) & -i({}_2Y_{\ell m} -{}_{-2} Y_{\ell m}) \\ i({}_2Y_{\ell m} -{}_{-2} Y_{\ell m}) & -({}_2Y_{\ell m} +{}_{-2} Y_{\ell m}) \end{array}\right). \tag{3}$$

The matrices $\hat{\mathsf{Y}}^s_{\ell m}$ satisfy the following relations:

$$\hat{\mathsf{Y}}^{s\dagger}_{\ell m} = (-1)^m \hat{\mathsf{Y}}^{s\dagger}_{\ell -m} \tag{4}$$

$$\int d\hat{\mathbf{n}} \hat{\mathsf{Y}}^s_{\ell m} \hat{\mathsf{Y}}^{s\dagger}_{\ell' m'} = \hat{1}\delta_{\ell\ell'}\delta_{mm'} \tag{5}$$

$$\hat{\mathsf{D}}^s_{\mathbf{l}\mathbf{l}_1\mathbf{l}_2} \equiv \int d\hat{\mathbf{n}} \left(\hat{\mathsf{Y}}^{s\dagger}_{\mathbf{l}}(\hat{\mathbf{n}})\,\hat{\mathsf{Y}}^s_{\mathbf{l}_1}(\hat{\mathbf{n}})\right)\,\hat{\mathsf{Y}}^0_{\mathbf{l}_2}(\hat{\mathbf{n}}), \tag{6}$$

$$= (-1)^m \sqrt{\frac{(2\ell+1)(2\ell_1+1)(2\ell_2+1)}{4\pi}} \left(\begin{array}{ccc} \ell & \ell_1 & \ell_2 \\ -m & m_1 & m_2 \end{array}\right) \left(\begin{array}{ccc} \ell & \ell_1 & \ell_2 \\ s & -s & 0 \end{array}\right) \hat{\mathsf{d}}^s_{\ell+\ell_1+\ell_2}, \tag{7}$$

$$\hat{\mathsf{d}}^0_n = 1, \quad \hat{\mathsf{d}}^2_n = \frac{1}{2}\left(\begin{array}{cc} 1+(-1)^n & -i[1-(-1)^n] \\ i[1-(-1)^n] & 1+(-1)^n \end{array}\right), \tag{8}$$

where we have abbreviated the pair (ℓ, m) as $\mathbf{l}$.

Finally, the following orthogonality relation for the Wigner $3j$ symbols is useful:

$$\sum_{mm_1} \left(\begin{array}{ccc} \ell & \ell_1 & \ell_2 \\ m & m_1 & m_2 \end{array}\right) \left(\begin{array}{ccc} \ell & \ell_1 & \ell_3 \\ m & m_1 & m_3 \end{array}\right) = \frac{\delta_{\ell_2\ell_3}\delta_{m_2m_3}}{2\ell_2+1} \tag{9}$$

2 Contaminant cleaning

Let $\mathbf{a}$ be a random field defined on the sphere, let $v(\hat{\mathbf{n}})$ a mask for $\mathbf{a}$ and let $\mathbf{f}^i$ be a set of N_a contaminants of $\mathbf{a}$ such that the observed version of $\mathbf{a}$ be:

$$\mathbf{d}_a(\hat{\mathbf{n}}) = \mathbf{a}^v(\hat{\mathbf{n}}) + \sum_{i=1}^{N_a} \alpha_i \mathbf{f}^i(\hat{\mathbf{n}}), \tag{10}$$

where $\mathbf{a}^v(\hat{\mathbf{n}}) = v(\hat{\mathbf{n}})\,\mathbf{a}(\hat{\mathbf{n}})$ (note that we have implicitly applied the same mask to $\mathbf{f}^i$). The best-fit value for the coefficients α_i assuming the same weights for all points in $\mathbf{d}_a$ can be found as

$$\tilde{\alpha}_i = \sum_j M_{ij} \int d\hat{\mathbf{n}}\, \mathbf{f}^{j\dagger}(\hat{\mathbf{n}})\, \mathbf{d}_a(\hat{\mathbf{n}}), \quad (\hat{\mathsf{M}}^{-1})_{ij} \equiv \int d\hat{\mathbf{n}}\, \mathbf{f}^{i\dagger}(\hat{\mathbf{n}})\, \mathbf{f}^j(\hat{\mathbf{n}}). \tag{11}$$

Thus we can find a cleaned version of $\mathbf{a}$ as:

$$\tilde{\mathbf{a}}(\hat{\mathbf{n}}) \equiv \mathbf{d}_a(\hat{\mathbf{n}}) - \mathbf{f}^i(\hat{\mathbf{n}})\, M_{ij} \int d\hat{\mathbf{n}}' \mathbf{f}^{j\dagger}(\hat{\mathbf{n}}')\mathbf{d}_a(\hat{\mathbf{n}}') \tag{12}$$

$$= \mathbf{a}^v(\hat{\mathbf{n}}) - \mathbf{f}^i(\hat{\mathbf{n}})\, M_{ij} \int d\hat{\mathbf{n}}' \mathbf{f}^{j\dagger}(\hat{\mathbf{n}}')\mathbf{a}^v(\hat{\mathbf{n}}'), \tag{13}$$

where there is an implicit summation sign over i and j (we will omit these from now on).

The harmonic coefficients of the cleaned and masked field are:

$$\tilde{\mathbf{a}}_{\ell m} = \mathbf{a}^v_{\ell m} - \mathbf{f}^i_{\ell m} M_{ij} \sum_{\ell' m'} \mathbf{f}^{j\dagger}_{\ell' m'} \mathbf{a}^v_{\ell' m'}. \tag{14}$$

From now on we will simplify the notation by abbreviating the pair ℓm as $\mathbf{l}$, so that the previous equation reads:

$$\tilde{\mathbf{a}}_{\mathbf{l}} = \mathbf{a}^v_{\mathbf{l}} - \mathbf{f}^i_{\mathbf{l}} M_{ij} \sum_{\mathbf{l}'} \mathbf{f}^{j\dagger}_{\mathbf{l}'} \mathbf{a}^v_{\mathbf{l}'}. \tag{15}$$

The harmonic coefficients for the masked field can be related to those of the unmasked one and the mask v (understood as a spin-0 field) as:

$$\mathbf{a}^v_{\mathbf{l}} = \sum_{\mathbf{l}_1 \mathbf{l}_2} \hat{\mathsf{D}}^{s_a}_{\mathbf{l}\mathbf{l}_1\mathbf{l}_2} \mathbf{a}_{\mathbf{l}_1} v_{\mathbf{l}_2}. \tag{16}$$

3 Pseudo-C_ℓ estimators with mode deprojection

In what follows, for two fields $\mathbf{a}$ and $\mathbf{b}$ we will define their observed power spectrum as:

$$\tilde{\mathsf{C}}^{ab}_\ell \equiv \frac{1}{2\ell+1} \sum_m \mathbf{a}_{\ell m} \mathbf{b}^\dagger_{\ell m}. \tag{17}$$

This must not be confused with the true power spectrum defined as an ensemble average for isotropic fields:

$$\langle \mathbf{a}_{\mathbf{l}} \mathbf{b}^\dagger_{\mathbf{l}'} \rangle \equiv \hat{\mathsf{C}}^{ab}_\ell \delta_{\ell\ell'} \delta_{mm'}. \tag{18}$$

Now, let $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ be the contaminant-cleaned versions of two random fields $\mathbf{a}$ and $\mathbf{b}$ with contaminants $\mathbf{f}^i$ and $\mathbf{g}^j$ and masks v and w respectively, and let us define

$$(\hat{\mathsf{N}}^{-1})_{ij} \equiv \int d\hat{\mathbf{n}}\, \mathbf{g}^{i\dagger}(\hat{\mathbf{n}})\, \mathbf{g}^j(\hat{\mathbf{n}}). \tag{19}$$

The observed power spectrum of the contaminant-cleaned maps can be written as:

$$\begin{aligned}\tilde{\mathsf{C}}^{\tilde{a}\tilde{b}}_\ell =& \frac{1}{2\ell+1} \sum_m \mathbf{a}^v_{\ell m} \mathbf{b}^{w\dagger}_{\ell m} - \frac{N^*_{ij}}{2\ell+1} \sum_m \sum_{\mathbf{l}'} \mathbf{a}^v_{\mathbf{l}} \mathbf{b}^{w\dagger}_{\mathbf{l}'} \mathbf{g}^j_{\mathbf{l}'} \mathbf{g}^{i\dagger}_{\mathbf{l}} - \\ &- \frac{M_{ij}}{2\ell+1} \sum_m \sum_{\mathbf{l}'} \mathbf{f}^i_{\mathbf{l}} \mathbf{f}^{j\dagger}_{\mathbf{l}'} \mathbf{a}^v_{\mathbf{l}'} \mathbf{b}^{w\dagger}_{\mathbf{l}} + \frac{M_{ij} N^*_{pq}}{2\ell+1} \sum_m \sum_{\mathbf{l}'\mathbf{l}''} \mathbf{f}^i_{\mathbf{l}} \mathbf{f}^{j\dagger}_{\mathbf{l}'} \mathbf{a}^v_{\mathbf{l}'} \mathbf{b}^{w\dagger}_{\mathbf{l}''} \mathbf{g}^q_{\mathbf{l}''} \mathbf{g}^{p\dagger}_{\mathbf{l}}.\end{aligned} \tag{20}$$

In order to compute the bias of $\tilde{\mathsf{C}}^{\tilde{a}\tilde{b}}_\ell$ with respect to $\hat{\mathsf{C}}^{ab}_\ell$, we need to compute the ensemble average of the former, which we will write as:

$$\left\langle \tilde{\mathsf{C}}^{\tilde{a}\tilde{b}}_\ell \right\rangle = \hat{\mathsf{F}}^1_\ell - \hat{\mathsf{F}}^2_\ell - \hat{\mathsf{F}}^3_\ell + \hat{\mathsf{F}}^4_\ell, \tag{21}$$

where:

$$\hat{\mathsf{F}}^1_\ell \equiv \frac{1}{2\ell+1}\sum_m \langle \mathbf{a}^v_{\ell m}\mathbf{b}^{w\dagger}_{\ell m}\rangle, \quad \hat{\mathsf{F}}^2_\ell \equiv \frac{N^*_{ij}}{2\ell+1}\sum_m\sum_{\mathbf{l}'}\langle \mathbf{a}^v_{\mathbf{l}}\mathbf{b}^{w\dagger}_{\mathbf{l}'}\mathbf{g}^j_{\mathbf{l}'}\mathbf{g}^{i\dagger}_{\mathbf{l}}\rangle \tag{22}$$

$$\hat{\mathsf{F}}^3_\ell \equiv \frac{M_{ij}}{2\ell+1}\sum_m\sum_{\mathbf{l}'}\langle \mathbf{f}^i_{\mathbf{l}}\mathbf{f}^{j\dagger}_{\mathbf{l}'}\mathbf{a}^v_{\mathbf{l}'}\mathbf{b}^{w\dagger}_{\mathbf{l}}\rangle, \quad \hat{\mathsf{F}}^4_\ell \equiv \frac{M_{ij}N^*_{pq}}{2\ell+1}\sum_m\sum_{\mathbf{l}'\mathbf{l}''}\langle \mathbf{f}^i_{\mathbf{l}}\mathbf{f}^{j\dagger}_{\mathbf{l}'}\mathbf{a}^v_{\mathbf{l}'}\mathbf{b}^{w\dagger}_{\mathbf{l}''}\mathbf{g}^q_{\mathbf{l}''}\mathbf{g}^{p\dagger}_{\mathbf{l}}\rangle. \tag{23}$$

We will now compute the ensemble average of each of these terms.

3.1 $\hat{\mathsf{F}}^1_\ell$

$$\begin{aligned}
\hat{\mathsf{F}}^1_\ell &= \frac{1}{2\ell+1}\sum_{m\mathbf{l}_{1,2,3,4}} v_{\mathbf{l}_2}w^*_{\mathbf{l}_4}\hat{\mathsf{D}}^{s_a}_{\mathbf{l}\mathbf{l}_1\mathbf{l}_2}\langle \mathbf{a}_{\mathbf{l}_1}\mathbf{b}^\dagger_{\mathbf{l}_3}\rangle\hat{\mathsf{D}}^{s_b\dagger}_{\mathbf{l}\mathbf{l}_3\mathbf{l}_4}\\
&= \frac{1}{2\ell+1}\sum_{m\mathbf{l}_{1,2,3}} v_{\mathbf{l}_2}w^*_{\mathbf{l}_3}\hat{\mathsf{D}}^{s_a}_{\mathbf{l}\mathbf{l}_1\mathbf{l}_2}\hat{\mathsf{C}}^{ab}_{\mathbf{l}_1}\hat{\mathsf{D}}^{s_b\dagger}_{\mathbf{l}\mathbf{l}_1\mathbf{l}_3}\\
&= \frac{1}{2\ell+1}\sum_{\ell_1\mathbf{l}_{2,3}} v_{\mathbf{l}_2}w^*_{\mathbf{l}_3}\frac{(2\ell+1)(2\ell_1+1)}{4\pi}\sqrt{(2\ell_2+1)(2\ell_3+1)}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ s_a & -s_a & 0\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_3\\ s_b & -s_b & 0\end{pmatrix}\\
&\qquad\qquad \hat{\mathsf{d}}^{s_a}_{\ell+\ell_1+\ell_2}\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{d}}^{s_b\dagger}_{\ell+\ell_1+\ell_3}\sum_{mm_1}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ -m & m_1 & m_2\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_3\\ -m & m_1 & m_3\end{pmatrix}\\
&= \sum_{\ell_1\ell_2}\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi}\tilde{C}^{vw}_{\ell_2}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ s_a & -s_a & 0\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ s_b & -s_b & 0\end{pmatrix}\hat{\mathsf{d}}^{s_a}_{\ell+\ell_1+\ell_2}\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{d}}^{s_b\dagger}_{\ell+\ell_1+\ell_2}
\end{aligned} \tag{24}$$

For $v=w=1$ this reduces to $\tilde{C}^{vw}_\ell = 4\pi\delta_{\ell 0}$ and:

$$\begin{aligned}
\hat{\mathsf{F}}^1_\ell &= \sum_{\ell_1\ell_2}(2\ell_1+1)(2\ell_2+1)\delta_{\ell_2 0}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ s_a & -s_a & 0\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ s_b & -s_b & 0\end{pmatrix}\hat{\mathsf{d}}^{s_a}_{\ell+\ell_1+\ell_2}\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{d}}^{s_b\dagger}_{\ell+\ell_1+\ell_2}\\
&= \sum_{\ell_1}(2\ell_1+1)\begin{pmatrix}\ell & \ell_1 & 0\\ s_a & -s_a & 0\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & 0\\ s_b & -s_b & 0\end{pmatrix}\hat{\mathsf{d}}^{s_a}_{\ell+\ell_1}\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{d}}^{s_b\dagger}_{\ell+\ell_1}\\
&= \sum_{\ell_1}(2\ell_1+1)\delta_{\ell\ell_1}\frac{(-1)^{\ell-s_a}}{\sqrt{2\ell+1}}\delta_{\ell\ell_1}\frac{(-1)^{\ell-s_b}}{\sqrt{2\ell+1}}\hat{\mathsf{d}}^{s_a}_{\ell+\ell_1}\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{d}}^{s_b\dagger}_{\ell+\ell_1}\\
&= \hat{\mathsf{d}}^{s_a}_{2\ell}\hat{\mathsf{C}}^{ab}_\ell\hat{\mathsf{d}}^{s_b\dagger}_{2\ell}\\
&= \hat{\mathsf{C}}^{ab}_\ell
\end{aligned}$$

3.2 $\hat{\mathsf{F}}^2_\ell$

$$\begin{aligned}
\hat{\mathsf{F}}^2_\ell &= N^*_{ij}\int d\hat{\mathbf{n}}\,d\hat{\mathbf{n}}'\,\frac{\sum_{m\mathbf{l}'\mathbf{l}_{1,2,3,4}}}{2\ell+1}\hat{\mathsf{Y}}^{s_a\dagger}_{\mathbf{l}}(\hat{\mathbf{n}})\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}})\langle \mathbf{a}_{\mathbf{l}_1}\mathbf{b}^\dagger_{\mathbf{l}_3}\rangle\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_3}(\hat{\mathbf{n}}')\hat{\mathsf{Y}}^{s_b}_{\mathbf{l}'}(\hat{\mathbf{n}}')\mathbf{g}^j_{\mathbf{l}'}\mathbf{g}^{i\dagger}_{\mathbf{l}}v_{\mathbf{l}_2}w^*_{\mathbf{l}_4}Y_{\mathbf{l}_2}(\hat{\mathbf{n}})Y^*_{\mathbf{l}_4}(\hat{\mathbf{n}}')\\
&= N^*_{ij}\frac{\sum_m}{2\ell+1}\left\{\int d\hat{\mathbf{n}}\,v(\hat{\mathbf{n}})\,\hat{\mathsf{Y}}^{s_a\dagger}_{\mathbf{l}}(\hat{\mathbf{n}})\left[\sum_{\ell_1 m_1}\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}})\hat{\mathsf{C}}^{ab}_{\ell_1}\left(\int d\hat{\mathbf{n}}'\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_1}(\hat{\mathbf{n}}')\mathbf{g}^j(\hat{\mathbf{n}}')w(\hat{\mathbf{n}}')\right)\right]\mathbf{g}^{i\dagger}_{\mathbf{l}}\right\}\\
&= N^*_{ij}\frac{\sum_m}{2\ell+1}\mathrm{SHT}\left\{v(\hat{\mathbf{n}})\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(w\,\mathbf{g}^j\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}^{s_a}_{\mathbf{l}}\mathbf{g}^{i\dagger}_{\mathbf{l}}
\end{aligned} \tag{25}$$

For $v=w=1$ this reduces to:

$$\begin{aligned}
\hat{\mathsf{F}}^2_\ell &= N^*_{ij}\frac{\sum_m}{2\ell+1}\mathrm{SHT}\left\{\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(\mathbf{g}^j\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}^{s_a}_{\mathbf{l}}\mathbf{g}^{i\dagger}_{\mathbf{l}}\\
&= N^*_{ij}\,\hat{\mathsf{C}}^{ab}_\ell\frac{\sum_m \mathbf{g}^j_{\ell m}\mathbf{g}^{i\dagger}_{\ell m}}{2\ell+1}\\
&= N^*_{ij}\,\hat{\mathsf{C}}^{ab}_\ell\tilde{\mathsf{C}}^{g^j g^i}_\ell
\end{aligned}$$

3.3 $\hat{\mathsf{F}}^3_\ell$

$$\begin{aligned}
\hat{\mathsf{F}}^3_\ell &= M_{ij}\int d\hat{\mathbf{n}}\,d\hat{\mathbf{n}}'\,\frac{\sum_{m\mathbf{l}'\mathbf{l}_{1,2,3,4}}}{2\ell+1}\mathbf{f}^i_{\mathbf{l}}\mathbf{f}^{j\dagger}_{\mathbf{l}'}\hat{\mathsf{Y}}^{s_a\dagger}_{\mathbf{l}'}(\hat{\mathbf{n}}')\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_3}(\hat{\mathbf{n}}')\langle\mathbf{a}_{\mathbf{l}_3}\mathbf{b}^\dagger_{\mathbf{l}_1}\rangle\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_1}(\hat{\mathbf{n}})\hat{\mathsf{Y}}^{s_b}_{\mathbf{l}}(\hat{\mathbf{n}})v_{\mathbf{l}_4}w^*_{\mathbf{l}_2}Y_{\mathbf{l}_2}(\hat{\mathbf{n}})Y^*_{\mathbf{l}_4}(\hat{\mathbf{n}}')\\
&= M_{ij}\frac{\sum_m}{2\ell+1}\mathbf{f}^i_{\mathbf{l}}\left\{\int d\hat{\mathbf{n}}\,w(\hat{\mathbf{n}})\left[\sum_{\ell_1m_1}\left(\int d\hat{\mathbf{n}}'\,v(\hat{\mathbf{n}}')\mathbf{f}^{j\dagger}(\hat{\mathbf{n}}')\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}}')\right)\hat{\mathsf{C}}^{ab}_{\ell_1}\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_1}(\hat{\mathbf{n}})\right]\hat{\mathsf{Y}}^{s_b}_{\mathbf{l}}(\hat{\mathbf{n}})\right\}\\
&= M_{ij}\frac{\sum_m}{2\ell+1}\mathbf{f}^i_{\mathbf{l}}\,\mathrm{SHT}\left\{w(\hat{\mathbf{n}})\,\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab\dagger}_{\ell_1}\mathrm{SHT}\left(v\,\mathbf{f}^j\right)^{s_a}_{\mathbf{l}_1}\right]^{s_b}_{\hat{\mathbf{n}}}\right\}^{s_b\dagger}_{\mathbf{l}}
\end{aligned}\tag{26}$$

For $v=w=1$ this reduces to:

$$\begin{aligned}
\hat{\mathsf{F}}^3_\ell &= M_{ij}\frac{\sum_m}{2\ell+1}\mathbf{f}^i_{\mathbf{l}}\,\mathrm{SHT}\left\{\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab\dagger}_{\ell_1}\mathrm{SHT}\left(\mathbf{f}^j\right)^{s_a}_{\mathbf{l}_1}\right]^{s_b}_{\hat{\mathbf{n}}}\right\}^{s_b\dagger}_{\mathbf{l}}\\
&= M_{ij}\frac{\sum_m\mathbf{f}^i_{\ell m}\mathbf{f}^{j\dagger}_{\ell m}}{2\ell+1}\hat{\mathsf{C}}^{ab}_\ell\\
&= M_{ij}\tilde{\mathsf{C}}^{f^if^j}_\ell\hat{\mathsf{C}}^{ab}_\ell
\end{aligned}$$

3.4 $\hat{\mathsf{F}}^4_\ell$

$$\begin{aligned}
\hat{\mathsf{F}}^4_\ell &= \frac{M_{ij}N^*_{pq}}{2\ell+1}\int d\hat{\mathbf{n}}\,d\hat{\mathbf{n}}'\sum_{m\mathbf{l}'\mathbf{l}''\mathbf{l}_{1,2,3,4}}\mathbf{f}^i_{\mathbf{l}}\mathbf{f}^{j\dagger}_{\mathbf{l}'}\hat{\mathsf{Y}}^{s_a\dagger}_{\mathbf{l}'}(\hat{\mathbf{n}})\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}})\langle\mathbf{a}_{\mathbf{l}_1}\mathbf{b}^\dagger_{\mathbf{l}_3}\rangle\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_3}(\hat{\mathbf{n}}')\hat{\mathsf{Y}}^{s_b}_{\mathbf{l}''}(\hat{\mathbf{n}}')\mathbf{g}^q_{\mathbf{l}''}\mathbf{g}^{p\dagger}_{\mathbf{l}}v_{\mathbf{l}_2}w^*_{\mathbf{l}_4}Y_{\mathbf{l}_2}(\hat{\mathbf{n}})Y^*_{\mathbf{l}_4}(\hat{\mathbf{n}}')\\
&= \frac{M_{ij}N^*_{pq}}{2\ell+1}\sum_m\mathbf{f}^i_{\mathbf{l}}\left\{\int d\hat{\mathbf{n}}\,v(\hat{\mathbf{n}})\,\mathbf{f}^{j\dagger}(\hat{\mathbf{n}})\left[\sum_{\ell_1m_1}\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}})\hat{\mathsf{C}}^{ab}_{\ell_1}\left(\int d\hat{\mathbf{n}}'\,\hat{\mathsf{Y}}^{s_b\dagger}_{\mathbf{l}_1}(\hat{\mathbf{n}}')\mathbf{g}^q(\hat{\mathbf{n}}')w(\hat{\mathbf{n}}')\right)\right]\right\}\mathbf{g}^{p\dagger}_{\mathbf{l}}\\
&= M_{ij}N^*_{pq}\left\{\int d\hat{\mathbf{n}}\,v(\hat{\mathbf{n}})\,\mathbf{f}^{j\dagger}(\hat{\mathbf{n}})\,\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(w\,\mathbf{g}^q\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}\tilde{\mathsf{C}}^{f^ig^p}_\ell
\end{aligned}\tag{27}$$

For $v=w=1$ this reduces to:

$$\begin{aligned}
\hat{\mathsf{F}}^4_\ell &= M_{ij}N^*_{pq}\left\{\int d\hat{\mathbf{n}}\,\mathbf{f}^{j\dagger}(\hat{\mathbf{n}})\,\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(\mathbf{g}^q\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}\tilde{\mathsf{C}}^{f^ig^p}_\ell\\
&= M_{ij}N^*_{pq}\left\{\int d\hat{\mathbf{n}}\sum_{\mathbf{l}_2}\mathbf{f}^{j\dagger}_{\mathbf{l}_2}\hat{\mathsf{Y}}^{s_a\dagger}_{\mathbf{l}_2}(\hat{\mathbf{n}})\sum_{\mathbf{l}_1}\hat{\mathsf{Y}}^{s_a}_{\mathbf{l}_1}(\hat{\mathbf{n}})\hat{\mathsf{C}}^{ab}_{\ell_1}\mathbf{g}^q_{\mathbf{l}_1}\right\}\tilde{\mathsf{C}}^{f^ig^p}_\ell\\
&= M_{ij}N^*_{pq}\mathbf{f}^{j\dagger}_{\mathbf{l}_1}\hat{\mathsf{C}}^{ab}_{\ell_1}\mathbf{g}^q_{\mathbf{l}_1}\tilde{\mathsf{C}}^{f^ig^p}_\ell\\
&= M_{ij}N^*_{pq}\left[\sum_{\ell_1}(2\ell_1+1)\mathrm{Tr}\left(\hat{\mathsf{C}}^{ab}_{\ell_1}\tilde{\mathsf{C}}^{g^qf^j}_{\ell_1}\right)\right]\tilde{\mathsf{C}}^{f^ig^p}_\ell
\end{aligned}$$

3.5 Final form of the estimator

Putting together the results from Equations 24, 25, 26 and 27, we can write down an unbiased estimator for the pseudo-C_ℓ of the cut-sky maps free from contamination from $\mathbf{f}$ and $\mathbf{g}$:

$$\begin{aligned}
\tilde{\mathsf{C}}^{ab}_\ell =&\tilde{\mathsf{C}}^{\tilde{a}\tilde{b}}_\ell + N^*_{ij}\frac{\sum_m}{2\ell+1}\mathrm{SHT}\left\{v(\hat{\mathbf{n}})\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(w\,\mathbf{g}^j\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}^{s_a}_{\mathbf{l}}\mathbf{g}^{i\dagger}_{\mathbf{l}}+\\
&+M_{ij}\frac{\sum_m}{2\ell+1}\mathbf{f}^i_{\mathbf{l}}\,\mathrm{SHT}\left\{w(\hat{\mathbf{n}})\,\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab\dagger}_{\ell_1}\mathrm{SHT}\left(v\,\mathbf{f}^j\right)^{s_a}_{\mathbf{l}_1}\right]^{s_b}_{\hat{\mathbf{n}}}\right\}^{s_b\dagger}_{\mathbf{l}}-\\
&-M_{ij}N^*_{pq}\left\{\int d\hat{\mathbf{n}}\,v(\hat{\mathbf{n}})\,\mathbf{f}^{j\dagger}(\hat{\mathbf{n}})\,\mathrm{SHT}^{-1}\left[\hat{\mathsf{C}}^{ab}_{\ell_1}\mathrm{SHT}\left(w\,\mathbf{g}^q\right)^{s_b}_{\mathbf{l}_1}\right]^{s_a}_{\hat{\mathbf{n}}}\right\}\tilde{\mathsf{C}}^{f^ig^p}_\ell
\end{aligned}\tag{28}$$

Once $\tilde{\mathsf{C}}^{ab}$ is calculated, it can be corrected for the effects of masking by inverting the linear transformation in Eq 24. This transformation can be written explicitly by first transforming the power spectrum matrices into vectors ${}_v\mathsf{C}$. E.g. for $s_a = s_b = 2$ we transform:

$$\hat{\mathsf{C}}^{ab}_\ell \equiv \begin{pmatrix} C^{E_aE_b}_\ell & C^{E_aB_b}_\ell \\ C^{B_aE_b}_\ell & C^{B_aB_b}_\ell \end{pmatrix} \quad \text{into} \quad {}_v\hat{\mathsf{C}}^{ab}_\ell \equiv \begin{pmatrix} C^{E_aE_b}_\ell \\ C^{E_aB_b}_\ell \\ C^{B_aE_b}_\ell \\ C^{B_aB_b}_\ell \end{pmatrix}. \tag{29}$$

We can then write, in general:

$${}_v\tilde{\mathsf{C}}^{ab}_\ell = \sum_{\ell'} \mathsf{M}^{s_as_b}_{\ell\ell'} \cdot {}_v\hat{\mathsf{C}}^{ab}_{\ell'}, \tag{30}$$

where:

$$\mathsf{M}^{00}_{\ell\ell'} = \frac{2\ell'+1}{4\pi} \sum_{\ell''} (2\ell''+1) C^{vw}_{\ell''} \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{pmatrix}^2 \tag{31}$$

$$\mathsf{M}^{02}_{\ell\ell'} = M^{0+}_{\ell\ell'}\,\hat{1}, \quad M^{0+}_{\ell\ell'} = \frac{2\ell'+1}{4\pi} \sum_{\ell''} (2\ell''+1) C^{vw}_{\ell''} \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{pmatrix} \tag{32}$$

$$\mathsf{M}^{22}_{\ell\ell'} = \begin{pmatrix} M^{++}_{\ell\ell'} & 0 & 0 & M^{--}_{\ell\ell'} \\ 0 & M^{++}_{\ell\ell'} & -M^{--}_{\ell\ell'} & 0 \\ 0 & -M^{--}_{\ell\ell'} & M^{++}_{\ell\ell'} & 0 \\ M^{--}_{\ell\ell'} & 0 & 0 & M^{++}_{\ell\ell'} \end{pmatrix} \tag{33}$$

$$M^{\pm\pm}_{\ell\ell'} = \frac{2\ell'+1}{4\pi} \sum_{\ell''} (2\ell''+1) C^{vw}_{\ell''} \begin{pmatrix} \ell & \ell' & \ell'' \\ 2 & -2 & 0 \end{pmatrix}^2 \frac{1 \pm (-1)^{\ell+\ell'+\ell''}}{2} \tag{34}$$

3.6 Binning into bandpowers

Given the loss of information implicit in masking the originally full-sky field, it is in general not possible to invert Eq. 30 directly. De usual approach to doing so is by binning the pseudo-C_ℓ into bandpowers. A bandpower k is defined by a set of N_k multipoles $\vec{\ell}_k \equiv (\ell^1_k, ..., \ell^{N_k}_k)$ and a set of weights $\vec{w}_k \equiv (w^1_k, ..., w^{N_k}_k)$ normalized such that $\sum_{i=1}^{N_k} w^i_k = 1$. The $k-$th bandpower for the coupled pseudo-C_ℓ is then defined as:

$${}_v\tilde{\mathsf{B}}^{ab}_k \equiv \sum_{i=1}^{N_k} w^i_k\, {}_v\tilde{\mathsf{C}}^{ab}_{\ell^i_k} = \sum_{i=1}^{N_k} w^i_k \sum_{\ell'} \mathsf{M}^{s_as_b}_{\ell^i_k\ell'}\, {}_v\hat{\mathsf{C}}^{ab}_\ell. \tag{35}$$

One then proceeds by assuming that the true power spectrum is a step-wise function, taking constant values over the multipoles corresponding to each bandpower: ${}_v\hat{\mathsf{C}}^{ab}_\ell = \sum_k {}_v\hat{\mathsf{B}}^{ab}_k \Theta(\ell \in \vec{\ell}_k)$ (where Θ is a binary step function). The previous equation then reads:

$${}_v\tilde{\mathsf{B}}^{ab}_k = \sum_{k'} \mathcal{M}^{s_as_b}_{kk'}\, {}_v\hat{\mathsf{B}}^{ab}_{kk'} \equiv \sum_{k'} \left(\sum_{\ell\in\vec{\ell}_k} \sum_{\ell'\in\vec{\ell}_{k'}} w^\ell_k \mathsf{M}^{s_as_b}_{\ell\ell'} \right) {}_v\hat{\mathsf{B}}^{ab}_{k'}, \tag{36}$$

which defines the binned coupling matrix $\mathcal{M}^{ab}_{kk'}$. The decoupled bandpowers are then estimated by inverting $\mathcal{M}^{ab}$:

$${}_v\hat{\mathsf{B}}^{ab}_k = \sum_{k'} \left(\mathcal{M}^{ab}\right)^{-1}_{kk'}\, {}_v\tilde{\mathsf{B}}^{ab}_{k'}. \tag{37}$$

Note that, even though this procedure is based on the assumption that the true power spectrum is step-wise constant, the bandpowers computed this way should be compared with the theoretical prediction subjected to the same type of transformation. I.e. the theoretical prediction for the bandpowers is:

$${}_v\bar{\mathsf{B}}^{ab}_k = \sum_{k'} \left(\mathcal{M}^{ab}\right)^{-1}_{kk'} \sum_{\ell'\in\vec{\ell}_{k'}} w^{\ell'}_k \sum_{\ell''} \mathsf{M}^{s_as_b}_{\ell'\ell''}\, {}_v\bar{\mathsf{C}}^{ab}_{\ell''}, \tag{38}$$

where we the overline $^-$ denotes theoretical predictions.