Point Vortices and Vortex Shedding: A Computational Approach

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Introduction

We experience the effects of vortices in the world around us every day of our lives, from the scale of a gentle beat of a butterfly's wing to the strength of a tornado. Their effects can be profound, playing an influential role in structure design.

Our focus will be centred around the motion of irrotational vortices, building up a simulation, starting with fairly simple motion with the aim of modelling the effects of vortex shedding on a cylindrical structure.

The code for this project, as well as the animations from which all of these plots are taken, can be found on the GitHub page here: https://github.com/NiallOswald/Python-Vortex-Sim

Point Vortex

Before being able to model point vortices, we need to find some equation of motion. We start by considering the following complex potential [1],

$$h(z) = \frac{\Gamma}{2\pi i}\log(z - z_0),$$

where $z = x + iy = re^{i\theta}$.

We can take the real and imaginary parts of $h(z)$ to give the velocity potential and stream functions,

$$\phi = \frac{\Gamma}{2\pi}\arg(z - z_0), \qquad \psi = -\frac{\Gamma}{2\pi}\log|z - z_0|.$$

Vorticity, given by ω, is a measure of a fluid's rotation. As we want to model an irrotational point vortex, to verify that $h(z)$ is correct, we must check that the only point of non-zero vorticity is at $z = z_0$, using [1],

$$\nabla^2\psi = -\omega.$$

We can then calculate the velocity vector of the fluid at any point written in polar coordinates [1],

$$\vec{u} = \begin{pmatrix} \frac{1}{r}\frac{\partial\psi}{\partial\theta} \\ -\frac{\partial\psi}{\partial r} \end{pmatrix}.$$

We then apply a change of coordinats argument, using the Jacobian, to write $\vec{u}$ in Cartesian coordinates,

$$\vec{u} = \frac{\Gamma}{2\pi((x - x_0)^2 + (y - y_0)^2)}\begin{pmatrix} -(y - y_0) \\ x - x_0 \end{pmatrix}. \qquad (1)$$

Systems of Multiple Point Vortices

Now that we have a formula for the local flow velocity for a single vortex, we can consider what the velocities may look like for a system containing multiple point vortices.

We can first displace a point vortex away from the origin by setting (x_0, y_0) to the centre of the point vortex in (1). Then simply summing the vector fields of the velocity of the point vortices will give us the resulting vector field of any number of point vortices placed in the plane.

Using these ideas, we will now model a few specific examples below.

Pairs of Vortices

We will now look at a simple example of what the vector field of a pair of vortices may look like, and also consider the dynamics of how the system may evolve over time.

Using the result (1), we displace two point vortices with circulation $\Gamma = 2\pi$ to $(-1, 0)$, $(1, 0)$ respectively,

$$\vec{u}_1 = \frac{1}{(x+1)^2 + y^2}\begin{pmatrix} -y \\ x+1 \end{pmatrix}, \qquad \vec{u}_2 = \frac{1}{(x-1)^2 + y^2}\begin{pmatrix} -y \\ x-1 \end{pmatrix}. \qquad (2)$$

Then summing the local flow velocities $\vec{u}_1$, $\vec{u}_2$ in (2) we get the new velocity $\vec{u} = \vec{u}_1 + \vec{u}_2$. It is then possible to plot the vector field of $\vec{u}$ as a quiver plot, giving a clear visualisation of the overall flow.

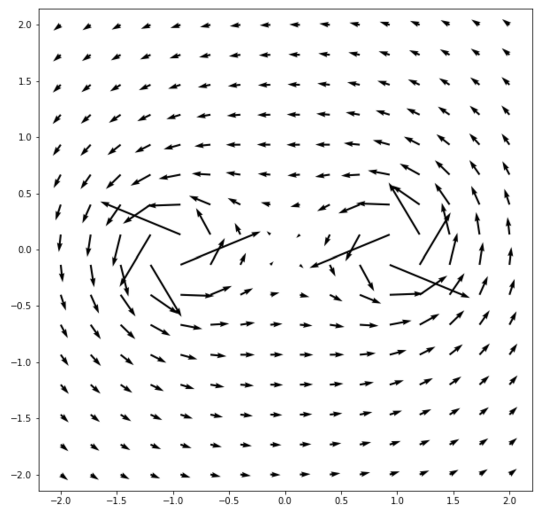

Figure 1: Quiver plot of $\vec{u}$

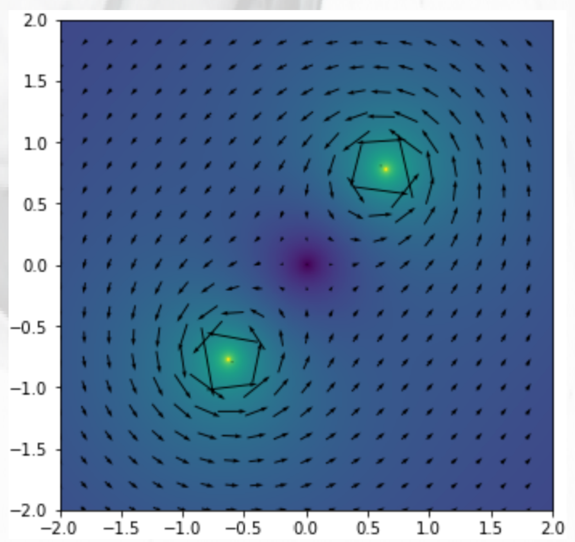

Figure 2: Python simulation of $\vec{u}$

Considering the vector field of each vortex individually, we can evaluate the local flow velocity at the centre of the other vortex, thus,

$$\vec{u}_1(1, 0) = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, \qquad \vec{u}_2(-1, 0) = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}.$$

So we would expect the vortex at $(-1, 0)$ to move downwards and the vortex at $(1, 0)$ to move upwards. We can model this using Python by repeating the same procedure as above and then displacing each vortex a small distance in the direction of the local velocity. From the model we can then produce visualisations of the interacting vortices, producing animations of plots like in figure 2.

Leap-frogging Vortices

We begin as we did with the vortex pair, displacing four point vortices to $(-1, -1)$, $(-1, 1)$, $(1, -1)$, and $(1, 1)$. However this time we set the circulation of the bottom two vortices to $\Gamma = -2\pi$.

Again we can consider groups of three vortices to find the local flow velocity induced on each vortex individually. With these we can model the movement of the point vortices as before. Again we will use our Python simulation to plot the vector fields and produce an animation of the plots as they evolve over time.

We observe that the pairs of vortices interchange places periodically as they move to the right, in effect 'leap-frogging' one another.

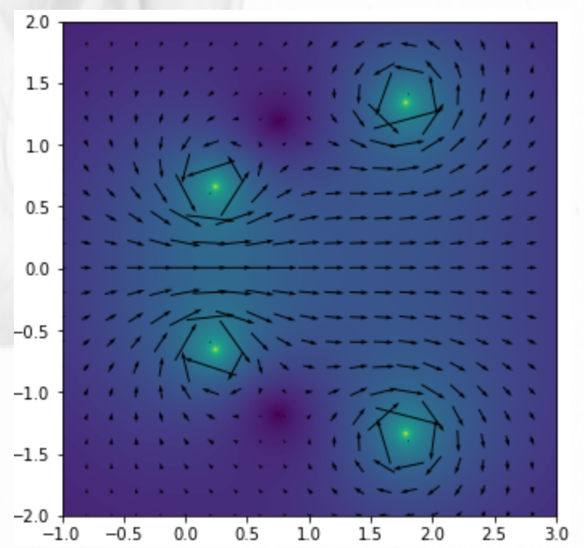

Figure 3: Image from the Python animation

Now that our Python simulation can model the basic movement of any number of vortices, we will turn to attempting to model vortex shedding on a cylinder.

Vortex Shedding on a Cylinder

Vortex shedding can occur when a fluid flows across a surface resulting in vortices forming alternately on opposite sides of the body [2].

Using Bernoulli's equation we can get the formula for the pressures, p_1 and p_2, at a pair of points,

$$\frac{1}{2}\rho v_1^2 + \rho g h_1 + p_1 = \frac{1}{2}\rho v_2^2 + \rho g h_2 + p_2.$$

So for a constant height, with some simple algebra we can attain an equation for the pressure difference between the two points,

$$\Delta p = p_1 - p_2 = \frac{1}{2}\rho\left(v_2^2 - v_1^2\right).$$

To model the pressure difference caused by vortex shedding, we place vortices alternately on opposite sides of the cylinder, measuring the velocity at two points on its edge with our simulation.

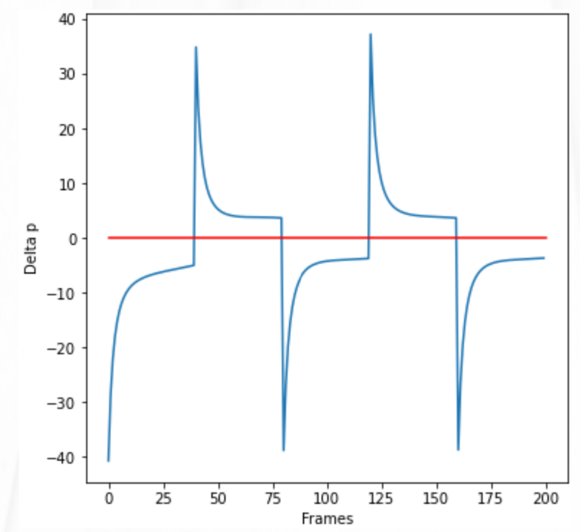

Figure 4: Plot of Δp over time

As vortices form periodically they induce a pressure difference, which in turn results in an alternating force being exerted on the body, causing it to oscillate orthogonal to the direction of the wind [2].

These effects can be mitigated by inducing turbulence around the structure which impedes the formations of vortices, so the vortex shedding effect is less pronounced or eliminated completely [3].

References

[1] Newton PK. Introduction. In: *The N-Vortex Problem: Analytical Techniques*. New York, NY, United States: Springer; 2001. p.1-64.

[2] Fu F. Fundamentals of Tall Building Design In: *Design and Analysis of Tall and Complex Structures*. Oxford, UK: Elsevier; 2018. p.5-80.

[3] Irwin P. Vortices and tall buildings: A recipe for resonance. *Physics Today*. 2010; 63(9): 68. Available from: https://doi.org/10.1063/1.3490510.

[4] Background image: Wagner J (Self-photographed). [Image on the Internet]. 2014. Available from: https://commons.wikimedia.org/wiki/File:Karmansche_Wirbelstr_kleine_Re.JPG