207. Analysing physiological oscillations

What new skills will I possess after completing this laboratory?

- **Describing** the Rayleigh and van der Pol oscillatory systems
- **Applying** Kelso's hybrid oscillator for cognitive systems
- **Analysing and evaluating** general properties of limit cycles

Why do I need these skills?

We know about the cycles occurring in Lotka-Volterra systems, but those cycles are not robust: if we change the number of lynxes or hares, the system moves to a new trajectory and cannot by itself return to the old one. By contrast, we saw that a driven, damped pendulum exhibits *limit cycles*: using the Poincaré section we were able to see that trajectories lying outside the limit cycle will converge towards it over time. Can we find examples of such limit cycles in biology?

Limit cycles are actually an extremely important part of living systems – the cell cycle and the Krebs cycle are both examples of cycles whose importance lies in their ability to maintain their basic structure against perturbations from the outside world. At a more visible level, the human activity of walking uses robust limit cycles; if you stumble over a stone, you quickly adjust your stride back into the same regular stride which you had before the stumble.

Scott Kelso has studied rhythmic limb movements (clapping, walking, scratching, flapping wings, etc.) over the past 40 years, and concludes that we can best think of cognition (organisms' ability to choose) not as logic or neural networks, but as the dynamics of coupled oscillators. Kelso's experiments suggest that biological oscillators generally possess the following properties:

1. They are almost sinusoidal in form: they rise and fall at roughly equal speeds;
2. When human subjects increase the speed of a rhythmic limb movement, the amplitude of the movement decreases linearly with frequency.
3. A power spectrum analysis of the movement indicates that the movement contains only a single fundamental frequency.

We shall now seek a mathematical model of rhythmic limb movements. We start with the work of Rayleigh, who in 1877 derived the following DE describing oscillations of a violin string, where x is the displacement of the string from equilibrium:

(1) $\ddot{x} + \varepsilon(\dot{x}^2 - 1)\dot{x} + \omega^2 x = 0$

(i) Reduce equation (1) to the first-order form $\dot{x} = y$; $\dot{y} = -\omega^2 x - \varepsilon(y^2 - 1)y$, and build a dynamic model of the Rayleigh system. Phase-plot its general behaviour.

(ii) What happens if you change the value of ε? What stays the same? What is the physical meaning of ε and of the term $-\varepsilon(y^2 - 1)y$?

(iii) Pay special attention to the convex shape of the Rayleigh limit cycle, and describe how this shape influences the shape of a BOTG of displacement.

(iv) What is the meaning of the parameter ω? And of the term $-\omega^2 x$?

What is the structure of the skills?

In 1926, van der Pol derived the following DE to describe self-excitation oscillations in a non-linear LCR-circuit. Here x represents the voltage across the non-linear electronic component:

(2) $\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + \omega^2 x = 0$

(v) Phase-plot the behaviour of the van der Pol oscillator.

(vi) What now remains constant if you change the value of ε? What would you now say is the physical meaning of ε? And of the term $-\varepsilon(x^2 - 1)y$?

(vii) Notice the new, non-convex shape of the van der Pol limit cycle. Try to predict what effect this will have on a BOTG of x, before building a model to test your theory.

How can I extend my skills?

(viii) Neither the Rayleigh nor the van der Pol oscillator matches Kelso's empirical results 1 to 3 for rhythmic limb movements. As you have seen, both of these oscillators are asymmetric with respect to rising and falling. Now use your models to test item 2 above: how does the amplitude of the van der Pol and of the Rayleigh oscillators depend upon the oscillation frequency ω?

To describe rhythmic movements, Kelso constructed a hybrid oscillator which is a combination of the van der Pol and Rayleigh systems, and which fulfils all three experimental findings. Kelso's hybrid oscillator obeys the following equation:

(3) $$\ddot{x} + \varepsilon(x^2 + \dot{x}^2 - 1)\dot{x} + \omega^2 x = 0$$

(ix) Build the Kelso oscillator and check that it fulfils requirements 1 and 2 above. What aspect of the shape of the phase-plot indicates that this is the case?

How can I deepen my practice of the skills?

(x) In this exercise, you will use your models to test the empirical requirement 3 for the Rayleigh, van der Pol and Kelso oscillators. First, look up online how to use the Fast Fourier Transform to extract the component frequencies of oscillation data. Then use the julia FFTW library to extract the Fourier spectrum of the time series for each oscillator, and show that only the Kelso oscillator contains a single fundamental frequency.