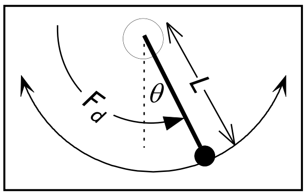
204. Archie and the chaotic swing



What new skills will I possess after completing this laboratory?

- **Describing** the meaning of a phase diagram.
- **Applying** Poincaré section to a dynamical system.
- **Generalising** the idea of periodicity to deterministic chaos in dynamical systems.

Why do I need these skills?

I mentioned in the previous lab that for chaotic dynamics to occur in continuous systems, the system must be non-linear and possess at least three degrees of freedom – this is the content of the ***Poincaré-Bendixson*** theorem. In this lab, we will use the Julia module **Pendulum** to analyse the onset of dynamical chaos in a damped, driven pendulum.

(i) The diagram above shows a child (Archie) of mass m on a swing supported by rigid rods with negligible mass and length L, swinging round a vertical circle of radius L. Use Newton's Second Law to derive the following equation of motion for Archie:

$$m\ddot{\theta} + \beta\dot{\theta} + {\omega_n}^2 \sin\theta = \boldsymbol{F}_d = \alpha\cos(\omega t)\,,$$

where $\omega_n \equiv \sqrt{g/L}$ is the ***natural frequency*** of the pendulum, θ is the anticlockwise angle measured from the vertical downward position, β is the ***damping coefficient***, and ω is the angular frequency of the force $\boldsymbol{F}_d$ that I apply rhythmically to make Archie swing higher.

(ii) Set $m = L = g = 1$ and draw an SPD of the resulting simpler equation:

(1) $\ddot{\theta} + \beta\dot{\theta} + \sin\theta = \alpha\cos(\omega t)$

(iii) We will work with this simplified equation (1) in all exercises of this lab. Set $\alpha = \beta = 0$ to obtain the ***undamped pendulum*** equation: $\ddot{\theta} + \sin\theta = 0$, and implement this equation as a dynamical model. Note that in order to do this, you must first reduce this *single second*-order equation into *two first*-order equations by introducing the additional phase variable $\dot{\theta}$.

(iv) Test your implementation: use initial conditions $\theta(0) = 0$ and $\dot{\theta}(0) = 1$ to generate both a BOTG and a phase-plot of $\dot{\theta}$ against θ over 10 seconds. Describe Archie's motion over time in words. *Make sure you understand how points in the BOTG relate to the phase plot points!*

(v) Raise the initial value $\dot{\theta}(0)$ to just under 2.0 (say, 1.99) and repeat exercise (iv) over 20 seconds. What is happening at points where θ approaches its maximum absolute value?

(vi) Set $\dot{\theta}(0) = 2.001$ and redo exercise (v) over 50 seconds. What is happening to poor Archie in this motion? Again, be sure you understand the relationship between corresponding points in your BOTG and you phase plot.

(vii) Explain the critical value of $\dot{\theta}(0)$ by considering the total energy of the system ($\frac{1}{2}mv^2 + mgh$) at the top and bottom of the swing.

(viii) The ***damped pendulum*** equation is $\ddot{\theta} + \beta\dot{\theta} + \sin\theta = 0$. Set the damping constant to $\beta = 0.2$, run your model with $\theta(0) = 0$ and $\dot{\theta}(0) = 1$ for 20 seconds, and interpret your plots.

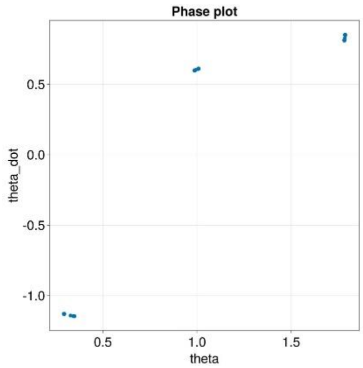
What is the structure of the skills?

OK, so now we know how to interpret Archie's motion, let's start driving the swing a little …

(ix) Equation (1) is the full equation for the ***damped, driven pendulum***. Adapt your model to solve (1) numerically over 500 seconds with $\beta = 0.2$, $\alpha = 0.52$, $\omega = 0.694$ and initial conditions $(0) = 0.8$ and $\dot{\theta}(0) = 0.8$, then display and interpret your plots of this system.

(x) Although the rotation angle θ can become very large, the swing's orientation at each instant always lies within the range $]-\pi, \pi]$. Use the provided function **orient()** to convert the rotation angle θ to a new value in this range. Display a phase-plot of $\dot{\theta}$ against this orientation value.

(xi) Your plot should exhibit a ***period-3 limit cycle***. What *exactly* does this phrase mean? You will find this easier to see if you first allow Archie's motion to settle down by setting the transient time to **Ttr=100** in the call to **trajectory()** (Look this up in Help).



(xii) ***Poincaré sections*** help us analyse periodic motion by sampling just *one* data point in each driving period $T = 2\pi/\omega$. Calculate the value of T for our current driving frequency ω.

(xiii) Generate a Poincaré section by setting the parameter $\mathbf{\Delta t}$ in the call to **trajectory()** equal to the value of T that you have just calculated. Create a **scatter()** plot of the values of $\dot{\theta}$ against the orientation angle to obtain the diagram on the right.

(xiv) The three groups of dots in the plot represent a period-3 **attractor**: Archie's oscillation contains three periods – one for each group. But if each oscillation repeats *exactly*, surely each group should contain only one point! Why do our groups each contain several dots?

How can I extend my skills?

Now we will drive the swing's motion into chaos using the full dynamical equation (1).

(xv) Use your model to generate a scatter plot of $\dot{\theta}$ against orientation with $\beta = 0.1, \alpha = 0.6, \omega = 0.694$ and initial conditions $\theta(0) = 0.8$ and $\dot{\theta}(0) = 0.8$ over 500 seconds. What is the main difference between this plot and that of exercise (ix)?

(xvi) Archie's motion is no longer periodic, but wanders unpredictably round the phase plane. Explain why we cannot call this motion *random*.

Our specific choices of parameter values in exercise (xv) produce a wild and unpredictable form of motion in the swing, which is in fact *chaotic*. To demonstrate this, we need to show that Archie's motion is *sensitive to initial conditions*:

(xvii) Using the parameter settings from exercise (xv), create a phase plot of the driven pendulum for the following three initial values of $\theta(0)$: $\theta(0) = 0.799999$, $\theta(0) = 0.8$ and $\theta(0) = 0.800001$. What do you notice about the difference between these three plots?

We see here the importance of sensitivity to initial conditions in chaotic systems:

Definition: We call the motion of a dynamical system ***chaotic*** if altering its initial conditions $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ causes the resulting trajectory $\boldsymbol{x}(t)$ to change *exponentially* in the following sense: a small perturbation $\Delta\boldsymbol{x}_0$ in $\boldsymbol{x}_0$ leads to a change $\Delta\boldsymbol{x}(t) \approx e^t \Delta\boldsymbol{x}_0$ at later times t.

Sensitivity to initial conditions is extremely important in evolution, since it enables developmental processes to react rapidly to very small changes in genes or environment.

How can I deepen my practice of the skills?

Chaotic motion is periodic motion with an infinitely long period; that is, the motion repeats itself only after an infinitely long time. In this case, we should be able to use a Poincaré map to generate a picture of the periodic components of the motion. That is the goal of our last exercise:

(xviii) Generate a Poincaré section of Archie's chaotic motion over 400 seconds (use **markersize=5**). Notice that the points seem to have only random structure. Now repeat the simulation over 100_000 seconds, and discover the ***fractal*** structure characteristic of all chaotic attractors.