Machine Learning for Data Science (CS4786)
Lecture 24

Graphical Models: Approximate Inference

Course Webpage :
http://www.cs.cornell.edu/Courses/cs4786/2016sp/

Belief Propagation or Message Passing

- Each node passes messages to its parents and children
- Guaranteed to work on a tree
- To get marginal of node X_v (given evidence)

$$\sum_{\text{Parents}(X_v)} \left(\prod \text{messages to } X_v\right) \times P(X_v|\text{Parent}(X_v))$$

- For general graphs, belief propagation need not work
- Inference for general graphs can be computationally hard

Can we perform inference approximately?

What is Approximate Inference?

- Obtain $\hat{P}(X_v|\text{Observation})$ that is close to $P(X_v|\text{Observation})$
  - Additive approximation:

$$\left|\hat{P}(X_v|\text{Observation}) - P(X_v|\text{Observation})\right| \leq \epsilon$$

  - Multiplicative approximation:

$$(1-\epsilon) \leq \frac{\hat{P}(X_v|\text{Observation})}{P(X_v|\text{Observation})} \leq (1+\epsilon)$$

Approximate Inference

Two approaches:

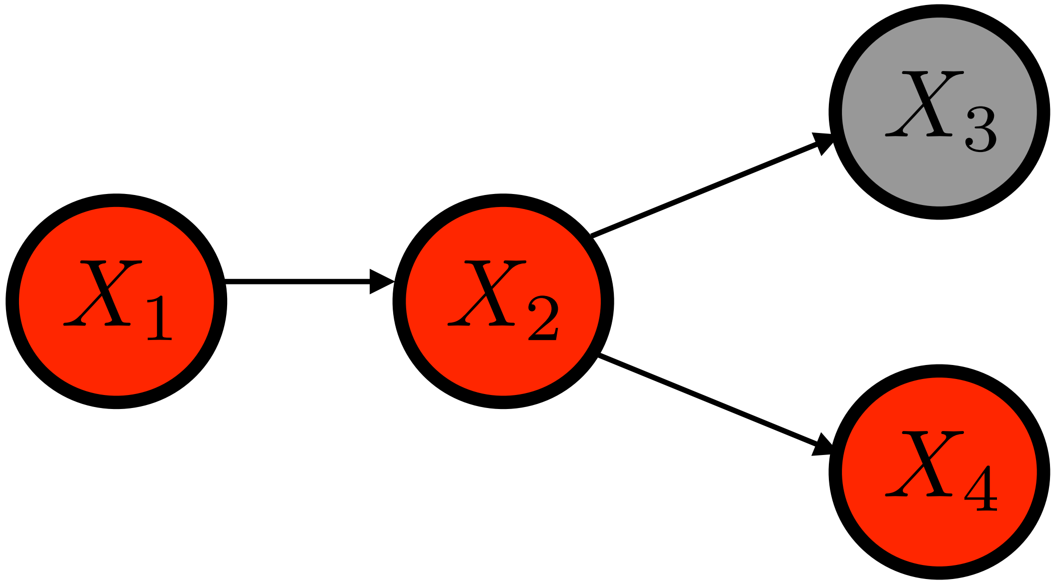
- Inference via sampling:
  generate instances from the model, compute marginals
- Use exact inference but move to a close enough simplified model

Inference Via Sampling

- Law of large numbers: empirical distribution using large samples approximates the true distribution
- Some approaches:
  - Rejection sampling: sample all the variables, retain only ones that match evidence
  - Importance sampling: Sample from a different distribution but then apply correction while computing empirical marginals
  - Gibbs sampling: iteratively sample from distributions closer and closer to the true one

Rejection Sampling

Example:



$$|\hat{P}(X_v = 1) - P(X_v = 1)| \approx \frac{1}{\sqrt{\text{\# of samples}}}$$

REJECTION SAMPLING

Algorithm:

Topologically sort variables (parents first children later)

For $t = 1$ to n

For $i = 1$ to N

Sample $x_i^t \sim P(X_i | X_1 = x_1^t, \ldots, X_{i-1} = x_{i-1}^t)$

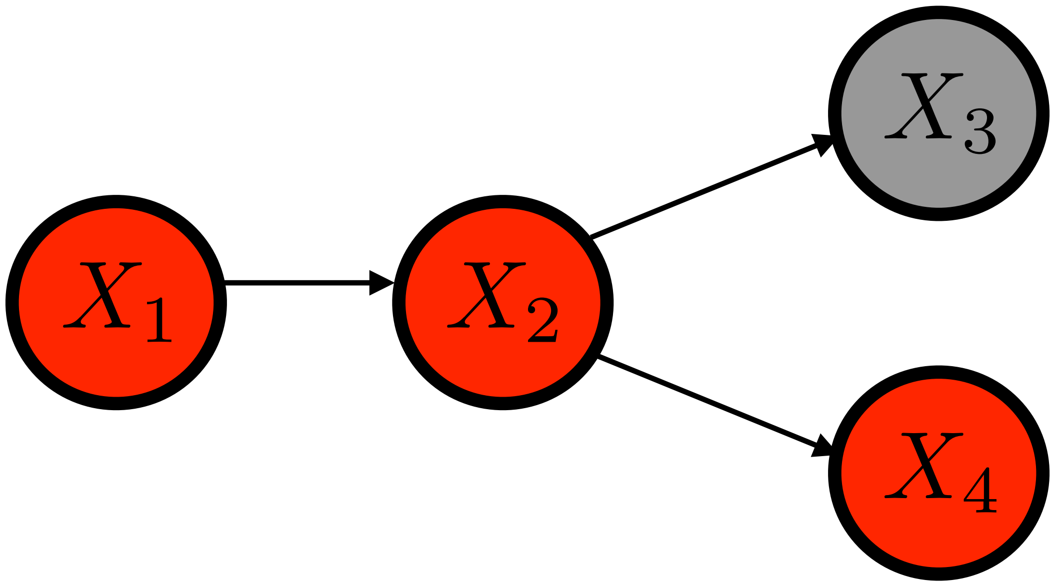
End For

End For

Throw away x^t's that do not match observations

Rejection Sampling

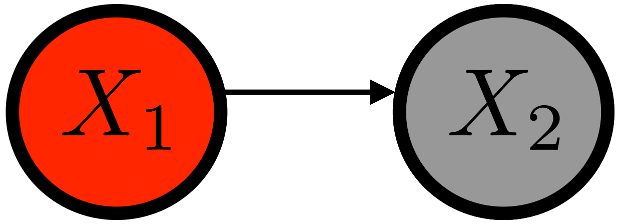
Example:



What about $\left|\hat{P}(X_v = 1|\text{observation}) - P(X_v = 1|\text{observation})\right|$?

Importance Sampling

Example: Likelihood weighting



Importance Sampling

Likelihood weighting:

Topologically sort variables (parents first children later)

For $t = 1$ to n

Set $w_t = 1$
For $i = 1$ to N

If X_i is observed, set $w_t \leftarrow w_t \cdot P(X_i = x_i | X_1 = x_1^t, \ldots, X_{i-1} = x_{i-1}^t)$
Else, sample $x_i^t \sim P(X_i | X_1 = x_1^t, \ldots, X_{i-1} = x_{i-1}^t)$

End For

End For

To compute $P(\text{Variable}|\text{Observation})$ set,

$$P(\text{Variable} = \text{value}|\text{Observation}) = \frac{\sum_{t=1}^n w_t \mathbf{1}\{\text{Variable} = \text{value}\}}{\sum_{t=1}^n w_t}$$

Importance Sampling

- More generally importance sampling is given by:
- Draw $x_1, \ldots, x_n \sim Q$
- Notice that

$$\mathbb{E}_{X \sim P}[f(X)] = \mathbb{E}_{X \sim Q}\left[\frac{P(X)}{Q(X)} f(X)\right]$$

- Hence,

$$\hat{P}(X = x) \approx \frac{1}{n} \sum_{t=1}^{n} \mathbf{1}\{x_t = x\} \frac{P(X = x)}{Q(X = x)}$$

- Idea draw samples from Q but re-weight them

Gibbs Sampling

- Fix values of observed variables v to the observations ($x_v^1 = x_v$)
- Randomly initialize all other variables u by randomly sampling x_u^1
- For $t = 2$ to n
- For $i = 1$ to N

  If X_i is observed set

$$x_i^t = x_i^{t-1}$$

  Else sample x_i^{t+1} from

$$x_i^{t+1} \sim P(X_i | X_1 = x_1^{t+1}, \ldots, X_{i-1} = x_{i-1}^{t+1}, X_{i+1} = x_{i+1}^t, \ldots, X_N = x_N^t)$$

  - End For
- End For
- Take $(x_1^n, \ldots, x_N^n)$ as one sample and repeat

MCMC Sampling in General

- Gibbs sampling belongs o a class of methods called Markov Chain Monte Carlo methods
- We start by sampling from some simple distribution
- Set up a markov chain whose stationary distribution is the target distribution
- That is, based on previous sample (state) we transit to the next state, and then to the next state and so on
- If the transition probabilities are set up right, after multiple transitions, our sample looks like one from target distribution