Q1: Linear Algebra

From the hypothesis, we know that vector $\hat{x}$ and $\hat{y}$ are non-zero vectors. Thus, we get:

$$\|x\| \neq 0 \tag{1}$$

and

$$\|y\| \neq 0 \tag{2}$$

From the fact that $\hat{x}$ and $\hat{y}$ are orthogonal we have:

$$\overrightarrow{x} \cdot \overrightarrow{y} = 0 \implies \mathbf{x}^T\mathbf{y} = 0 \tag{3}$$

Let us assume we have the following class of matrices:

$$A = \mathbf{x}\mathbf{y}^T + kI \text{ for } k \in \mathbb{Z} \tag{4}$$

Then we get:

$$\begin{aligned}
(A\overrightarrow{x}) \cdot (A\overrightarrow{y}) =& (A\mathbf{x})^T A\mathbf{y} \\
=& \mathbf{x}^T A^T A\mathbf{y} \\
\overset{(4)}{=}& \mathbf{x}^T(\mathbf{x}\mathbf{y}^T + kI)^T(\mathbf{x}\mathbf{y}^T + kI)\mathbf{y} \\
=& \mathbf{x}^T(\mathbf{y}\mathbf{x}^T + kI)(\mathbf{x}\mathbf{y}^T + kI)\mathbf{y} \\
=& (\mathbf{x}^T\mathbf{y}\mathbf{x}^T + k\mathbf{x}^T)(\mathbf{x}\mathbf{y}^T + kI)\mathbf{y} \\
\overset{(3)}{=}& k\mathbf{x}^T(\mathbf{x}\mathbf{y}^T + kI)\mathbf{y} \\
=& (k\mathbf{x}^T\mathbf{x}\mathbf{y}^T + k^2\mathbf{x}^T)\mathbf{y} \\
=& k\|x\|^2\mathbf{y}^T\mathbf{y} + k^2\mathbf{x}^T\mathbf{y} \\
\overset{(3)}{=}& k\|x\|^2\|y\|^2
\end{aligned}$$

Finally for $k \neq 0$ we have:

$$k\|x\|^2\|y\|^2 \overset{(1,2)}{>} 0$$

Thus, for this class (4) of matrices A, $A\overrightarrow{x}$ and $A\overrightarrow{y}$ are not orthogonal.

NOTE: Many of you gave a specific example in this question by specifying x and y vectors. This is fine but we actually wanted you to find such a matrix A for all x and y vectors.

Q2: Partial Derivatives

We first calculate the first partial derivative:

$$\begin{aligned}
\frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_i} =& \frac{\partial}{\partial x_i}\left[-\sum_{j=1}^{n} x_j \log(x_j)\right] \\
=& \frac{\partial}{\partial x_i}\left(-x_i \log(x_i)\right) \\
=& -\log(x_i) - x_i\frac{1}{x_i} \\
=& -\log(x_i) - 1
\end{aligned}$$

The partial derivative is defined for $x_i > 0$ (the same goes for the original function f).

For the second partial derivative there are two cases. If $j \neq i$ then it is 0. Otherwise, we have:

$$\begin{aligned}\frac{\partial^2 f(x_1, x_2, \ldots, x_n)}{\partial x_i^2} =& \frac{\partial}{\partial x_i} \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_i} \\ =& \frac{\partial}{\partial x_i} \left(-\log\left(x_i\right) - 1\right) \\ =& -\frac{1}{x_i}\end{aligned}$$

The second order partial derivative is defined for $x_i > 0$ (the same goes for the original function f).

Thus, the result is the following:

$$\frac{\partial^2 f(x_1, x_2, \ldots, x_n)}{\partial x_i \partial x_j} = \begin{cases} -\frac{1}{x_i} & \text{, if } i = j. \\ 0 & \text{, otherwise.} \end{cases}$$

Q3: Conditional Probability

Let us define the following events:

$$\begin{aligned} R \triangleq& \text{ It rains in Ms. Y's neighborhood at any given day.} \\ S \triangleq& \text{ The sprinkler is turned on next door.} \\ W \triangleq& \text{ Ms. Y wears a poncho.} \end{aligned}$$

Then we need to calculate the probabilities of R given W and S given W. For the first one we have:

$$\begin{aligned} Pr\left[R/W\right] =& \frac{Pr\left[W/R\right] Pr\left[R\right]}{Pr\left[W\right]} \\ =& \frac{0.6 \times 0.1}{Pr\left[W\right]} \\ =& \frac{0.06}{Pr\left[W\right]} \end{aligned}$$

The first equality holds from ”Probability Theory”.

For the second one we have:

$$\begin{aligned} Pr\left[S/W\right] =& \frac{Pr\left[W/S\right] Pr\left[S\right]}{Pr\left[W\right]} \\ =& \frac{0.2 \times 0.3}{Pr\left[W\right]} \\ =& \frac{0.06}{Pr\left[W\right]} \end{aligned}$$

The first equality holds from ”Probability Theory”.
Thus, the first event (it rained at that day) and the second event (the sprinkler was on) are equally likely.

COMMON MISTAKE: Many of you tried to calculate P[W] writing Pr[W]=Pr[W/S]*Pr[S]+Pr[W/R]*Pr[R]
You cannot calculate Pr[W] this way because to be able to write this equation, you need to know that R and S do not intersect; however,this information is not given in the question.

Q4: Linearity of Expectation

Let us define the following events:

$$F_i \triangleq \text{Fan } i \text{ is a baseball fan}$$
$$J_i \triangleq \text{Fan } i \text{ gets a baseball jersey}$$

Let us suppose that we have a variable X_i defined as follows:

$$X_i = \begin{cases} 1 & \text{, if } F_i \text{ and } J_i. \\ 0 & \text{, otherwise.} \end{cases}$$

Then the expected number of baseball fans that will get a baseball shirt is the following:

$$\begin{aligned} E\left[\sum_{i=1}^{100} X_i\right] &= \sum_{i=1}^{100} E[X_i] \\ &= \sum_{i=1}^{100} Pr\left[F_i \cap J_i\right] \\ &= \sum_{i=1}^{100} Pr\left[F_i\right] Pr\left[J_i\right] \\ &= \sum_{i=1}^{100} 0.1 \times 0.1 \\ &= \sum_{i=1}^{100} 0.01 = 1 \end{aligned}$$

The first equality holds from the ”Linearity of Expectation” and the third one from the fact that F_i and J_i are completely independent of each other (random order from hypothesis).