Modeling Shill Bidding in Online English Auctions

Nianzu Xiong

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Abstract

Shill bidding in auctions refers to the action of a seller placing bids on his own item. In this paper, we discuss two explanations for the purpose of shill bidding in online English auctions. The first is that shill bidding could serve as a substitute for setting a public reserve price, and the second is that shill bidding might help the seller extract additional revenue from the bidder with the highest valuation of the good. We develop two models to investigate these hypotheses. In the first model, we study a game of two sellers selling identical items simultaneously choosing between setting zero or his own valuation as the public reserve price. There are two types of sellers in the platform, an honest seller having these two choices as above while a dishonest seller having an extra choice of setting zero as the public reserve price and placing a shill bid equal to his own valuation during the auction. We show that for a dishonest seller, setting zero publicly and placing a shill bid equal to his own valuation would always give him an expected payoff at least as high as setting his own valuation as the public reserve price, and could give him a strictly higher expected payoff under some conditions that would make an honest seller also have incentives to set zero publicly. In the second model, we derive the condition under which it would be profitable to use shill bidding in order to extract additional revenue under a private value framework, and also give a formula for finding the profit-maximizing shill bid when profitable. Finally, we extend the second model to derive the profitability condition and profit-maximizing shill bid under an interdependent value framework.

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1 Introduction

Online auctions have become increasingly popular in recent decades owing to their accessibility to the general public. Most online auctions, including these on eBay and Alibaba, use English auctions as the format. An English auction starts from the seller's reserve price and accepts increasingly higher bids. The bidder with the highest bid at any time is considered the one with the outstanding bid at that time, and if no one outbids the outstanding bid before the auction ends, the item is sold to that bidder. Along with its popularity, fraud in online auction has also become prevalent due to the difficulty in detecting fraud. One of the major forms of online auction fraud is shill bidding, which refers to a seller placing bids on the item himself. Shill bidding is particularly popular because of the lack of authentication in online auctions (Wang et al., 2001). In this paper, we construct models to investigate two hypotheses related to shill bidding. First, it might be that shill bidding could serve as a substitute for a public reserve price. Second, shill bidding might help the seller extract additional revenue when there is only one real bidder left.

2 Literature Review

We first discuss literature related to the first hypothesis, and then review that related to the second hypothesis.

2.1 Using Shill Bidding as a Substitute for Reserve Price

The first hypothesis we investigate is that shill bidding could serve as a substitute for a public reserve price. Chen et al. (2018) consider the possibility of a seller starting with a low reserve price and participating in shill bidding. Their argument is built upon an empirical study that analyzes data from online auctions of 260,000 second-hand cars to show that a higher reserve price decreases the number of bidders (Choi et al., 2016). Thus, the seller may wish to set a low reserve price to attract more bidders, while placing a shill bid during the auction that is equal to the higher public reserve price that he would otherwise set.

Gerding et al. (2007) analyze a model in which two sellers simultaneously decide whether to set a public reserve price or set 0 as a reserve price and participate in shill bidding (not necessarily shill bidding the amount R). They study the existence of Nash equilibrium in three cases: both sellers set a public reserve price, one seller sets the public reserve price while the other sets 0 and shill bids, and both set 0 and shill bid. In their model, the reserve price is not necessarily the seller's valuation of his good, but rather a function that is his best response given the other seller's choices. They did not give explicit expressions of the sellers' utility in equilibrium, but rather used simulations.

Burguet and Sakovics (1999) also study a model where two sellers simultaneously sell an identical item and chooses the reserve price. They give a symmetric equilibrium of the buyers' game, and then derive the equilibrium behavior of the sellers given the buyers are following the symmetric equilibrium. They show that when the set of choices of reserve price by the sellers is continuous, there is no pure-strategy Nash equilibrium in the sellers' game.

One of the most defining characteristics of online auctions is the prevalence of setting zero as the reserve price. Therefore, building on these two models, we will consider the game with two sellers simultaneously each setting up an auction to sell identical items. Each could either choose to set his valuation of the good or zero as the reserve price before the auction. In the auction platform there are two types of sellers, one being honest and the other being dishonest, and dishonest sellers have the extra choice of setting 0 publicly and place a shill bid R. We give explicit expressions of honest and dishonest sellers' utility and analyze their equilibrium behavior. In addition, we incorporate buyers' suspicions into consideration, as each bidder would be suspicious about the type of seller when they see the 0 public reserve price. We derive the symmetric equilibrium of the buyers' game with suspicion, and then focus on the optimal choice of the public reserve price of a dishonest seller. We modify the model in the paper of Burguet and Sakovics (1999) by discretizing the choices of the sellers, and then show that for a dishonest seller, setting 0 publicly and placing shill bid R would always give the seller at least as high expected utility as setting R publicly, and could sometimes give higher expected utility if the valuation distribution and number of bidders satisfy some conditions. This provides support for the first hypothesis.

2.2 Using Shill Bidding to Extract Additional Revenue

The second hypothesis is that shill bidding could help the seller extract additional revenue from the bidder with the highest valuation in online English auctions. There are two common ways of modeling bidders' valuations in English auctions, which are called private value and common value frameworks. In a private value auction, each bidder knows his own valuation of the good, but does not know the valuations of other bidders. In contrast, bidders in a common value auction share the same valuation of the good. However, such valuation is unknown at the time of bidding, and each bidder estimates the true value of the good based on the private signal he receives before the auction starts.

Wang et al. (2001) analyze shill bidding in English auctions under the private value framework and give a profitability condition. Graham et al. (2002) analyze the profitability of shill bidding when the distributions of valuation across different bidders are heterogenous, and conclude that if the seller could gather information about the distribution of the highest valuation, then shill bidding would be profitable. In the second theorem, we use a similar approach by Wang et al. (2001) and show that the profit-maximizing shill bid is independent of the public reserve price. We also show that after placing the profit-maximizing shill bid, even if the bidder with the highest valuation stays in the auction, which gives new information about his valuation distribution, there could not exist another shill bid that the seller could use to further improve his expected utility.

In contrast to the private value setting, Chakraborty and Kosmopoulou (2004) analyze shill bidding in common value auctions. In their model, there is a common valuation and the reserve price is strategically set so that the bidder would place a bid if and only if he receives a "high" signal. A seller has a probability p of being able to participate in shill bidding without being detected. Each bidder has a belief that there is a probability p that the seller is participating in the auction. They derive theorems regarding the optimal bidding behaviors of the real bidders and the seller. They first show that the optimal bid for each bidder would decrease as p increases. They find that there are two types of equilibria for any p: in the first, the seller always participates in shill bidding if he is able; in the second, the seller never shill bids. They show that the sellers would have higher payoff when there is a lower participation rate of shill bidding. Therefore, the sellers face a dilemma: they wish to communicate the information that they are not involved in shill bidding to the bidders in order to get higher bids from them, but the impossibility of such communications makes them choose to place shill bids. They then conclude that the equilibrium where the sellers place shill bids whenever possible induces losses to both the sellers and the bidders.

The findings of Chakraborty and Kosmopoulou (2004) differ from those of Wang et al. (2001) and Graham et al. (2002). The main reason is the difference in the auction setting, as one analyzes common value auctions while others study private value auctions. In addition to private and common value auctions, there is another auction setting called interdependent value auctions. In an interdependent value auction, each bidder has a private signal before bidding, and the final valuation depends on signals of other bidders that are unknown at the time of placing bids. Therefore, the interdependent value framework could be viewed as a generalization of common value auctions. This framework is particularly suitable for modeling auctions of antiques and artworks, which are common in online auction platforms like Alibaba. Milgrom and Weber (1982) give the equilibrium strategy by bidders in an interdependent value auction. On the other hand, shill bidding in English auctions under interdependent value framework has not been studied. We build on our second model to study this, and add bidders' suspicion of shill bidding into the model by Milgrom and Weber (1982). We then analyze the profit-maximizing shill bid when real bidders are following the strategy.

3 Organization of the Paper

We discuss two main models in the rest of this paper. The first model investigates the hypothesis that shill bidding could serve as a substitute for public reserve prices, and we show that setting 0 publicly and placing a shill bid equal to R would always give dishonest sellers an expected payoff at least as high as setting R publicly, and could give them a strictly higher expected payoff under some conditions. In the second model, we derive the profitability condition for using shill bidding to extract additional revenues under private value framework. Finally, we build on the second model to derive the profitability condition and find the profit-maximizing shill bid under interdependent value framework.

4 Models

4.1 Shill Bidding as a Substitute for a Public Reserve Price

Let us consider the game between two sellers from an online auction platform. Suppose there are two types of sellers on the platform, one being honest sellers who choose between setting 0 and R as a public reserve price, where R equals his own valuation, and the other being dishonest sellers who have the extra choice of setting 0 publicly and place a shill bid R during the auction. Two sellers from the population are each organizing an auction that sells identical items, and they simultaneously choose between setting 0 and R as the public reserve price before the auction starts.

Suppose there are N people who view the auction page, each having valuation independent and identically distributed according to a differentiable probability distribution $F(\cdot)$. N and $F(\cdot)$ are known to sellers before deciding their public reserve prices. After observing the sellers' choices on the auction page, each person would decide to enter the auction and place bids or leave. Let $m < 1$ be the percentage of sellers who are dishonest in the auction platform. We assume N, m, and $F(\cdot)$ to be common knowledge among sellers and page viewers. We first consider the optimal strategy for each person. The utility of a person with valuation v_i in an auction with $r \in \{0, R\}$ as its reserve price and $V = \{v_1, v_2, ...v_n\}$ being the set of valuations of its participants (with the possibility of having a shill bidder among the participants with valuation R) is

$$U(v_i) = \begin{cases} 0 & \text{if } v_i < \max\limits_{j\in\{1,\dots,t\}, j\neq i} v_j \\ v_i - r & \text{if } V = \{v_i\} \\ v_i - \max\limits_{j\in\{1,\dots,t\}, j\neq i} v_j & \text{if } V \neq \{v_i\} \text{ and } v_i > \max\limits_{j\in\{1,\dots,t\}, j\neq i} v_j. \end{cases}$$

Moreover, upon setting an auction with 0 as its public reserve price, each page viewer have the belief that there is a probability p that the seller in this auction is a shill bidder. We have

$$p = \mathbb{P}(\text{dishonest seller} \mid 0 \text{ public reserve price}) = \frac{m \cdot q_s^*}{m \cdot q_s^* + (1-m) \cdot q_h^*}.$$

Here q_s^* is the equilibrium probability that 0 is the public reserve price given the seller is dishonest, and q_h^* is the equilibrium probability that 0 is the public reserve price given the seller is honest.

If one seller sets 0 as the public reserve price and the other sets R, let us call the auction with 0 as its reserve price auction 1, and the other auction 2. The following proposition analyzes the optimal choice of each page viewer in this setting when we fix a p that is less than 1.

Proposition 1: The following strategy profile is a Bayesian Nash equilibrium of the game: A person with valuation $v_i \leq R$ would attend auction 1. Furthermore, let v^* be the solution of the equation

$$\begin{aligned}
&(1-p) \cdot [v_i \cdot (\frac{1-F(v^*)}{2})^{N-1}] + p \cdot [(v_i - R) \cdot (\frac{1-F(v^*)}{2})^{N-1}] \\
&+(1-p) \cdot [\int_0^R (N-1) \cdot (v_i - u)(\frac{1-F(v^*)}{2} + F(u))^{N-2} \cdot f(u)\, du] \\
&+p \cdot [\int_0^R (N-1) \cdot (v_i - R)(\frac{1-F(v^*)}{2} + F(u))^{N-2} \cdot f(u)\, du] \\
&+\int_R^{v^*} (N-1) \cdot (v_i - u)(\frac{1-F(v^*)}{2} + F(u))^{N-2} \cdot f(u)\, du \\
&-(v_i - R) \cdot (\frac{1+F(v^*)}{2})^{N-1} = 0
\end{aligned}$$

if it has a unique solution such that $v^* \geq R$. Then a viewer with valuation v_i would attend auction 1 if $R \leq v_i \leq v^*$, and would mix between auction 1 and auction 2 each with probability $\frac{1}{2}$ if $v_i > v^*$. If there is no such v^*, then each person would attend auction 1.

Proof. Let us first consider the case when the equation above has a desired solution v^*. Let us consider the choices of a person with valuation $v_i < R$. There is a chance that he has the highest valuation among all the people who would go to auction 1 and the seller does not place a shill bid, in which case he will have positive utility by bidding his own valuation in auction 1. On the other hand, his expected utility of going to auction 2 is 0. Mathematically,

$$\mathbb{E}(U_1(v_i)) = \int_0^{v_i} (v_i - u)(F(u) + \frac{1-F(v^*)}{2})^{N-1} \cdot f(u)\, du > \mathbb{E}(U_2(v_i)) = 0.$$

This means he would go to auction 1. Suppose the equation in Theorem 2 has a solution v^*. Let $U_1(v_i)$ be the utility of a person with valuation v_i when attending auction 1, and let $U_1(v_i)$ be the utility of him attending auction 2. For a person with valuation v_i above v^*, we have

$$\begin{aligned}
\mathbb{E}(U_1(v_i)) &= (1-p)\cdot[v_i\cdot(\frac{1-F(v^*)}{2})^{N-1}] + p\cdot[(v_i-R)\cdot(\frac{1-F(v^*)}{2})^{N-1}] \\
&+(1-p)\cdot[\int_0^R (N-1)\cdot(v_i-u)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du] \\
&+p\cdot[\int_0^R (N-1)\cdot(v_i-R)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du] \\
&+\int_R^{v^*} (N-1)\cdot(v_i-u)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du \\
&+\int_{v^*}^{v_i} (N-1)\cdot(v_i-u)(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot\frac{f(u)}{2}\,du.
\end{aligned}$$

Here $(1-p)\cdot[v_i\cdot(\frac{1-F(v^*)}{2})^{N-1}]+p\cdot[(v_i-R)\cdot(\frac{1-F(v^*)}{2})^{N-1}]$ is the expected utility of the person when he is the only one attending auction 1 multiplied by the chance of this happening. $(1-p)\cdot[\int_0^R(N-1)\cdot(v_i-u)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du]+p\cdot[\int_0^R(N-1)\cdot(v_i-R)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du]$ is the expected utility when he wins in auction 1 and the second highest valuation is less than R. $\int_R^{v^*}(N-1)\cdot(v_i-u)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du$ is the expected utility when he wins in auction 1 and the second highest valuation is between R and v^*. $\int_{v^*}^{v_i}(N-1)\cdot(v_i-u)(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot\frac{f(u)}{2}\,du$ is the expected utility when he wins in auction 1 and the second highest valuation is higher than v^*.

The expected utility of attending auction 2 is

$$\mathbb{E}(U_2(v_i)) = (v_i-R)\cdot(\frac{1-F(v^*)}{2}+F(v^*))^{N-1} + \int_{v^*}^{v_i}(N-1)\cdot(v_i-u)(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot\frac{f(u)}{2}\,du.$$

Here $(v_i-R)\cdot(\frac{1-F(v^*)}{2}+F(v^*))^{N-1}$ is the expected utility when the person is the only one who is attending auction 2, and $\int_{v^*}^{v_i}(N-1)\cdot(v_i-u)(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot\frac{f(u)}{2}\,du$ is the expected utility when the second highest valuation is higher than v^*. Now, if v^* satisfies the equation

$$\begin{aligned}
&(1-p)\cdot[v_i\cdot(\frac{1-F(v^*)}{2})^{N-1}] + p\cdot[(v_i-R)\cdot(\frac{1-F(v^*)}{2})^{N-1}] \\
&+(1-p)\cdot[\int_0^R (N-1)\cdot(v_i-u)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du] \\
&+p\cdot[\int_0^R (N-1)\cdot(v_i-R)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du] \\
&+\int_R^{v^*} (N-1)\cdot(v_i-u)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du \\
&-(v_i-R)\cdot(\frac{1+F(v^*)}{2})^{N-1} = 0,
\end{aligned}$$

then we have

$$
\begin{aligned}
\mathbb{E}(U_2(v_i)) &= (v_i - R)\cdot(\frac{1-F(v^*)}{2} + F(v^*))^{N-1} + \int_{v^*}^{v_i}(N-1)\cdot(v_i-u)(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot\frac{f(u)}{2}\,du \\
&= (v_i - R)\cdot(\frac{1+F(v^*)}{2})^{N-1} + \int_{v^*}^{v_i}(N-1)\cdot(v_i-u)(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot\frac{f(u)}{2}\,du \\
&= (1-p)\cdot[v_i\cdot(\frac{1-F(v^*)}{2})^{N-1}] + p\cdot[(v_i-R)\cdot(\frac{1-F(v^*)}{2})^{N-1}] \\
&+(1-p)\cdot[\int_0^R(N-1)\cdot(v_i-u)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du] \\
&+p\cdot[\int_0^R(N-1)\cdot(v_i-R)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du] \\
&+\int_R^{v^*}(N-1)\cdot(v_i-u)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du \\
&+\int_{v^*}^{v_i}(N-1)\cdot(v_i-u)(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot\frac{f(u)}{2}\,du = \mathbb{E}(U_1(v_i)).
\end{aligned}
$$

Therefore, it would be optimal for each person with valuation above v^* to randomize between the two auctions. For a person with valuation between R and v^*, the expected utility of going to auction 1 is

$$
\begin{aligned}
\mathbb{E}(U_1(v_i)) &= p\cdot[(v_i-R)\cdot(\frac{1-F(v^*)}{2})^{N-1}] + (1-p)\cdot[v_i\cdot(\frac{1-F(v^*)}{2})^{N-1}] \\
&+p\cdot[\int_0^R(N-1)\cdot(v_i-R)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du] \\
&+(1-p)\cdot[\int_0^R(N-1)\cdot(v_i-u)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du] \\
&+\int_R^{v_i}(N-1)\cdot(v_i-u)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du,
\end{aligned}
$$

and the expected utility of going to auction 2 is

$$
\mathbb{E}(U_2(v_i)) = (v_i-R)\cdot(\frac{1-F(v^*)}{2}+F(v^*))^{N-1}.
$$

Notice

$$
\begin{aligned}
\frac{\partial}{\partial p}[\mathbb{E}(U_1(v_i))] &= (v_i-R)\cdot(\frac{1-F(v^*)}{2})^{N-1} - [v_i\cdot(\frac{1-F(v^*)}{2})^{N-1}] \\
&+\int_0^R(N-1)\cdot(v_i-R)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du \\
&-[\int_0^R(N-1)\cdot(v_i-u)(\frac{1-F(v^*)}{2}+F(u))^{N-2}\cdot f(u)\,du] < 0,
\end{aligned}
$$

and $\mathbb{E}(U_1(v_i)) = \mathbb{E}(U_2(v_i))$ when $p=1$. Therefore, we have that for $p\in(0,1)$, $\mathbb{E}(U_1(v_i)) > \mathbb{E}(U_2(v_i))$ if

$R < v_i < v^*$.

If no such v^* exists, then consider the expression

$$
\begin{gathered}
(1-p)\cdot[v_i\cdot(\frac{1-F(v)}{2})^{N-1}]+p\cdot[(v_i-R)\cdot(\frac{1-F(v)}{2})^{N-1}] \\
+(1-p)\cdot[\int_0^R(N-1)\cdot(v_i-u)(\frac{1-F(v)}{2}+F(u))^{N-2}\cdot f(u)\,du] \\
+p\cdot[\int_0^R(N-1)\cdot(v_i-R)(\frac{1-F(v)}{2}+F(u))^{N-2}\cdot f(u)\,du] \\
+\int_R^v(N-1)\cdot(v_i-u)(\frac{1-F(v)}{2}+F(u))^{N-2}\cdot f(u)\,du \\
-(v_i-R)\cdot(\frac{1+F(v)}{2})^{N-1} \text{ when } v=R.
\end{gathered}
$$

Since

$$
\begin{gathered}
(1-p)\cdot[v_i\cdot(\frac{1-F(R)}{2})^{N-1}]+p\cdot[(v_i-R)\cdot(\frac{1-F(R)}{2})^{N-1}] \\
+(1-p)\cdot[\int_0^R(N-1)\cdot(v_i-u)(\frac{1-F(R)}{2}+F(u))^{N-2}\cdot f(u)\,du] \\
+p\cdot[\int_0^R(N-1)\cdot(v_i-R)(\frac{1-F(R)}{2}+F(u))^{N-2}\cdot f(u)\,du] \\
-(v_i-R)\cdot(\frac{1+F(R)}{2})^{N-1} \\
>\int_0^R(N-1)\cdot(v_i-R)(\frac{1-F(R)}{2}+F(u))^{N-2}\cdot f(u)\,du \\
-(v_i-R)\cdot(\frac{1+F(R)}{2})^{N-1}=0,
\end{gathered}
$$

if the equation has no solution v^*, then it is optimal for a bidder with any valuation to always go to auction 1. we conclude that $\int_0^v u\cdot(N-1)\cdot(\frac{1-F(v)}{2}+F(u))^{N-2}\cdot f(u)\,du-R\cdot(\frac{1+F(v)}{2})^{N-1}<0$ when $v=R$. Therefore, it would always be optimal to attend auction 1. Finally, if v^* exists, then it is unique because

$$
\begin{gathered}
\frac{\partial}{\partial v}\{(1-p)\cdot[v_i\cdot(\frac{1-F(v)}{2})^{N-1}]+p\cdot[(v_i-R)\cdot(\frac{1-F(v)}{2})^{N-1}] \\
+(1-p)\cdot[\int_0^R(N-1)\cdot(v_i-u)(\frac{1-F(v)}{2}+F(u))^{N-2}\cdot f(u)\,du] \\
+p\cdot[\int_0^R(N-1)\cdot(v_i-R)(\frac{1-F(v)}{2}+F(u))^{N-2}\cdot f(u)\,du] \\
+\int_R^v(N-1)\cdot(v_i-u)(\frac{1-F(v)}{2}+F(u))^{N-2}\cdot f(u)\,du \\
-(v_i-R)\cdot(\frac{1+F(v)}{2})^{N-1}\}>0.
\end{gathered}
$$

■

The next proposition analyzes the scenario when both sellers set R publicly.

Proposition 2: If both sellers set R publicly, then the only symmetric Bayesian Nash equilibrium is the following: A person with valuation $v_i < R$ would not attend any auction. A person with valuation $v_i > R$ would randomize between the two auctions each with probability $\frac{1}{2}$.

Proof. It is clear that here $\mathbb{E}(U_1(v_i)) = \mathbb{E}(U_2(v_i))$. To see that $\frac{1}{2}$ is the unique mixing probability, notice that for the indifference principle to hold, it must be that

$$\frac{\partial}{\partial v_i}[\mathbb{E}(U_1(v_i))] = \frac{\partial}{\partial v_i}[\mathbb{E}(U_2(v_i))].$$

This means the marginal increase in the expected utility with an increase in v_i must be the same between both auctions, so the probability of winning has to be the same across the two auctions. Therefore, the mixing probability must be $\frac{1}{2}$. ■

By the same logic as in Proposition 2, it is easy to see that $\frac{1}{2}$ is also the unique mixing probability when one seller sets 0 and the other sets R. In addition, if both sellers set 0, then the only symmetric Bayesian Nash equilibrium would be when each viewer participates and randomizes between the two auctions each with probability $\frac{1}{2}$.

When $p = 1$, page viewers with valuation above R would view auctions with a 0 public reserve price to be the same as the one with R as the reserve price, and people with valuation below R will be indifferent between whether or not to join the auction and bid their own valuation when they see an auction with 0 reserve price.

We now give the first main theorem of the paper.

Theorem 1: It is always profit-maximizing for a dishonest seller to choose 0 as the public reserve price and place shill bid equal to R, compared to setting R publicly.

Proof. We begin by studying the profit-maximizing choice of a dishonest seller when $p < 1$. A dishonest seller has three choices, and below $(0, s)$ represents the choice of setting 0 publicly and placing shill bid R, while 0 and R represent the choices of setting 0 and R as the public reserve price respectively. Say the dishonest seller is running auction 1, while the other seller is running auction 2. We consider two cases.

Case 1: the desired v^* exists for all $0 < p < 1$.

In this case, we use the desired v^* to express the expected utility of the choices of each type of sellers, and show that if $p < 1$, then it is strictly more profitable for a dishonest seller to set 0 and place shill bid R. Then we show that when $F(\cdot)$ and N satisfy some conditions, p would be equal to m, which is indeed less than 1.

In the game, given the other seller sets R publicly, if a dishonest seller set R publicly, then he has expected utility

$$\mathbb{E}[U_1(\{R, R\})] = \int_R^\infty (u - R) \cdot (\frac{1 - F(u)}{2}) \cdot (\frac{1 - F(u)}{2} + F(u))^{N-2} \cdot N \cdot (N-1) \cdot \frac{f(u)}{2}\, du.$$

If he sets 0 and does not place a shill bid, then his expected utility is

$$\begin{aligned}\mathbb{E}[U_1(\{0, R\})] = -R \cdot N \cdot (\frac{1 - F(v^*)}{2} + F(v^*)) \cdot (\frac{1 - F(v^*)}{2})^{N-1} \\ + \int_0^{v^*} (u - R) \cdot (\frac{1 - F(v^*)}{2} + F(v^*) - F(u)) \cdot N \cdot (N-1) \cdot (F(u) + \frac{1 - F(v^*)}{2})^{N-2} \cdot f(u)\, du \\ + \int_{v^*}^\infty (u - R) \cdot (\frac{1 - F(u)}{2}) \cdot (\frac{1 - F(u)}{2} + F(u))^{N-2} \cdot N \cdot (N-1) \cdot \frac{f(u)}{2}\, du.\end{aligned}$$

If he sets 0 and also places a shill bid R, then his expected utility would be

$$\begin{aligned}&\mathbb{E}[U_1(\{(0, s), R\})] \\ = &\int_R^{v^*} (u - R) \cdot (\frac{1 - F(v^*)}{2} + F(v^*) - F(u)) \cdot N \cdot (N-1) \cdot (F(u) + \frac{1 - F(v^*)}{2})^{N-2} \cdot f(u)\, du \\ &+ \int_{v^*}^\infty (u - R) \cdot (\frac{1 - F(u)}{2}) \cdot (\frac{1 - F(u)}{2} + F(u))^{N-2} \cdot N \cdot (N-1) \cdot \frac{f(u)}{2}\, du.\end{aligned}$$

Clearly

$$\mathbb{E}[U_1(\{(0, s), R\})] > \mathbb{E}[U_1(\{R, R\})]$$

because

$$\int_R^{v^*} (u - R) \cdot (\frac{1 - F(v^*)}{2} + F(v^*) - F(u)) \cdot N \cdot (N-1) \cdot (F(u) + \frac{1 - F(v^*)}{2})^{N-2} \cdot f(u)\, du > 0.$$

Furthermore, since

$$\begin{aligned}
&\mathbb{E}[U_1(\{(0,s),R\})]-\mathbb{E}[U_1(\{0,R\})]\\
&=R\cdot N\cdot(\frac{1-F(v^*)}{2}+F(v^*))\cdot(\frac{1-F(v^*)}{2})^{N-1}\\
&-\int_0^R(u-R)\cdot(\frac{1-F(v^*)}{2}+F(v^*)-F(u))\cdot N\cdot(N-1)\cdot(F(u)+\frac{1-F(v^*)}{2})^{N-2}\cdot f(u)\,du>0,
\end{aligned}$$

we have

$$\mathbb{E}[U_1(\{(0,s),R\})]>\mathbb{E}[U_1(\{0,R\})].$$

Therefore, when the other seller sets R, setting 0 publicly and placing shill bid equal to R would give the seller the highest expected utility.

We now consider the case when the seller of auction 2 sets 0. Notice the expected utility of the dishonest seller in auction 1 is the same whether the other seller plays $(0,s)$ or 0. If the dishonest seller sets R, then he could expect

$$\int_{v^*}^{\infty}(u-R)\cdot(\frac{1-F(u)}{2})\cdot(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot N\cdot(N-1)\cdot\frac{f(u)}{2}\,du.$$

If he sets 0 publicly and does not place a shill bid, then he could expect

$$\mathbb{E}[U_1(\{0,0\})]=\int_0^{\infty}(u-R)\cdot(\frac{1-F(u)}{2})\cdot(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot N\cdot(N-1)\cdot\frac{f(u)}{2}\,du.$$

If he also places a shill bid, then

$$\mathbb{E}[U_1(\{(0,s),0\})]=\int_R^{\infty}(u-R)\cdot(\frac{1-F(u)}{2})\cdot(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot N\cdot(N-1)\cdot\frac{f(u)}{2}\,du.$$

Since

$$\begin{aligned}
&\mathbb{E}[U_1(\{(0,s),0\})]-\mathbb{E}[U_1(\{R,0\})]\\
&=\int_R^{\infty}(u-R)\cdot(\frac{1-F(u)}{2})\cdot(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot N\cdot(N-1)\cdot\frac{f(u)}{2}\,du\\
&-\int_{v^*}^{\infty}(u-R)\cdot(\frac{1-F(u)}{2})\cdot(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot N\cdot(N-1)\cdot\frac{f(u)}{2}\,du\\
&=\int_R^{v^*}(u-R)\cdot(\frac{1-F(u)}{2})\cdot(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot N\cdot(N-1)\cdot\frac{f(u)}{2}\,du>0,
\end{aligned}$$

and

$$\mathbb{E}[U_1(\{(0,s),0\})] - \mathbb{E}[U_1(\{0,0\})]$$
$$= -\int_0^R (u-R)\cdot(\frac{1-F(u)}{2})\cdot(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot N\cdot(N-1)\cdot\frac{f(u)}{2}\,du > 0,$$

we conclude that given the other seller sets 0, the unique optimal choice is also to set 0 publicly and place a shill bid R.

When $p = 1$, then setting 0 publicly and placing shill bid equal to R would clearly give the same expected utility as setting R publicly, because each page viewer with valuation above R would view these auctions as identical. Setting 0 without shill bid is clearly suboptimal because it gives the extra risk of having only people with valuation below R (who are indifferent between joining or not) placing bids.

Thus, depending on whether p is less than 1 or equal to 1, setting 0 publicly with shill bid R could lead to higher expected utility or equal utility compared to setting R publicly. We now discuss the conditions on N and $F(\cdot)$ that would make $p < 1$ and therefore make shill bidding R strictly more profitable than setting R publicly.

Since

$$p = \mathbb{P}(\text{dishonest seller} \mid 0 \text{ public reserve price}) = \frac{m\cdot q_s^*}{m\cdot q_s^* + (1-m)\cdot q_h^*},$$

we know that we want to find the condition that would make $q_h^* > 0$, that is, we need to find the condition under which an honest seller has incentive to set 0 publicly. Here we assume that R is common knowledge among bidders even if they see two auctions with 0 reserve price.

There are two scenarios to consider. The first is that page viewers see one seller setting R publicly while the other setting 0. To make it an equilibrium, the honest seller who sets R should not have incentive to deviate to 0. Furthermore, to make the equilibrium suspicion level less than 1 so that a shill bidder would have strict gain from shill bidding, it must be that both honest and dishonest sellers have incentive to set 0 given the other seller sets R. So if the seller who sets 0 publicly is honest, he also should not have incentive to deviate to R. Let us call the auction with R as reserve price auction 1 and the one with 0 auction 2. The conditions are

$$\begin{aligned}\mathbb{E}[U_1(\{R,0\})] &= \int_{v^*}^{\infty}(u-R)\cdot(\frac{1-F(u)}{2})\cdot(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot N\cdot(N-1)\cdot\frac{f(u)}{2}\,du\\ &> \mathbb{E}[U_1(\{0,0\})] = -R\cdot N\cdot(\frac{1-F(v^*)}{2}+F(v^*))\cdot(\frac{1-F(v^*)}{2})^{N-1}\\ &+\int_{0}^{\infty}(u-R)\cdot(\frac{1-F(u)}{2})\cdot(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot N\cdot(N-1)\cdot\frac{f(u)}{2}\,du,\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}[U_2(\{R,0\})] = -R\cdot N\cdot(\frac{1-F(v^*)}{2}+F(v^*))\cdot(\frac{1-F(v^*)}{2})^{N-1}\\ &+\int_{0}^{v^*}(u-R)\cdot(\frac{1-F(v^*)}{2}+F(v^*)-F(u))\cdot N\cdot(N-1)\cdot(F(u)+\frac{1-F(v^*)}{2})^{N-2}\cdot f(u)\,du\\ &+\int_{v^*}^{\infty}(u-R)\cdot(\frac{1-F(u)}{2})\cdot(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot N\cdot(N-1)\cdot\frac{f(u)}{2}\,du\\ &> \mathbb{E}[U_2(\{R,R\})] = \int_{R}^{\infty}(u-R)\cdot(\frac{1-F(u)}{2})\cdot(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot N\cdot(N-1)\cdot\frac{f(u)}{2}\,du.\end{aligned}$$

In this case, we have

$$p^* = \frac{m\cdot q_s^*}{m\cdot q_s^*+(1-m)\cdot q_h^*} = \frac{m}{m+1-m} = m < 1.$$

If the above conditions are satisfied, then upon seeing the two auctions, bidders will have the belief that there is a chance m that the seller with 0 reserve price is dishonest, which is consistent with setting 0 publicly being strictly profitable for the dishonest seller.

The other scenario is that both sellers set 0. If an honest seller has incentive to set 0 publicly, then clearly p would also be equal to m, which means the 0 public reserve price does not give bidders information about the honesty of sellers. For this scenario to happen, by symmetry, we need

$$\begin{aligned}\mathbb{E}[U_1(\{0,0\})] &= \int_{0}^{\infty}(u-R)\cdot(\frac{1-F(u)}{2})\cdot(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot N\cdot(N-1)\cdot\frac{f(u)}{2}\,du\\ &> \mathbb{E}[U_1(\{R,0\})] = \int_{v^*}^{\infty}(u-R)\cdot(\frac{1-F(u)}{2})\cdot(\frac{1-F(u)}{2}+F(u))^{N-2}\cdot N\cdot(N-1)\cdot\frac{f(u)}{2}\,du.\end{aligned}$$

Intuitively, if the valuation distribution is such that bids are very unlikely to be below R and quite likely to be between R and v^*, then honest sellers have incentive to set 0 publicly to attract these bidders with valuation between R and v^*, without having to worry much about the possibility of selling the item with a price below R.

The scenarios could still be in equilibrium if some conditions are held with equality. For example, if $\mathbb{E}[U_1(0,0)] = \mathbb{E}[U_1(\{R,0\})]$, then honest sellers would be indifferent between setting 0 and R. As long as their mixing probability of setting 0, which is q_h^*, is less than 1, we still have

$$p^* = \frac{m}{m + (1-m) \cdot q_h^*} < 1,$$

and here shill bidding is still strictly profitable, which is consistent to $p_s^* = 1$ in the calculation of p^*.

Case 2: no such v^* exists for some $p_0 < 1$.

We now consider the case where the equation in Proposition 1 has no desired solution when $p = p_0$, for some $p_0 < 1$.

Here when bidders hold the belief p_0, if one seller sets 0 and the other R, then it is always optimal for anyone with valuation above R to attend the auction with 0 reserve price since the suspicion level is less than 1. Following the same logic as the derivation above, if a seller sets 0 publicly, it would always be optimal to place a shill bid R to eliminate the chance of having negative profit. Thus it is still always strictly profit-maximizing to set 0 publicly and shill bid R.

Given $q_s^* = 1$ when $p_0 < 1$, we first analyze the scenario when $p_0 = m$. Following the same logic as in case 1, sustaining the equilibrium $p^* = m$ when bidders see one seller setting 0 and the other setting R requires N and $F(\cdot)$ to satisfy

$$\mathbb{E}[U_1(\{R,0\})] = 0 > \mathbb{E}[U_1(\{0,0\})] = \int_0^\infty (u-R) \cdot (\frac{1-F(u)}{2}) \cdot (\frac{1-F(u)}{2} + F(u))^{N-2} \cdot N \cdot (N-1) \cdot \frac{f(u)}{2} \, du,$$

and

$$\begin{aligned} \mathbb{E}[U_2(\{R,0\})] &= \int_0^\infty (u-R)N \cdot (F(u)^{N-1} \cdot (1-F(u)) \, du \\ > \mathbb{E}[U_2(\{R,R\})] &= \int_R^\infty (u-R) \cdot (\frac{1-F(u)}{2}) \cdot (\frac{1-F(u)}{2} + F(u))^{N-2} \cdot N \cdot (N-1) \cdot \frac{f(u)}{2} \, du. \end{aligned}$$

Sustaining the equilibrium with $p^* = m$ when bidders see both sellers setting 0 requires

$$\begin{aligned} \mathbb{E}[U_1(\{0,0\})] = \int_0^\infty (u-R) \cdot (\frac{1-F(u)}{2}) \cdot (\frac{1-F(u)}{2} + F(u))^{N-2} \cdot N \cdot (N-1) \cdot \frac{f(u)}{2} \, du \\ > \mathbb{E}[U_1(\{R,0\})] = 0. \end{aligned}$$

If $p_0 \neq m$, then it is still possible to sustain equilibria in the combined buyers' and sellers' game, but this would require honest bidders to be indifferent between R and 0. For example, if N and $F(\cdot)$ are such that $\mathbb{E}[U_1(\{0,0\})] = \mathbb{E}[U_1(\{R,0\})]$, then the equilibrium where dishonest sellers chooses to shill bid, honest bidders choose to mix, and when seeing two auctions both having 0 reserve price, bidders have suspicion p_0 and mix evenly across the two auctions could be sustained if each honest bidder assigns probability

$$p_h^* = \frac{(1-p_0)m}{p_0(1-m)}$$

to setting 0, which would make

$$p^* = \frac{m}{m + (1-m) \cdot \frac{(1-p_0)m}{p_0(1-m)}} = p_0.$$

Therefore, we have shown that setting 0 publicly and placing a shill bid equal to R would always give dishonest sellers an expected payoff at least as high as setting R publicly, and could give them a strictly higher expected payoff under some conditions when honest sellers also have incentive to set 0 publicly. ■

The theorem above provides direct support for the first hypothesis. The lack of endogenous incentive for a dishonest seller to set the reserve price at his own valuation shows the necessity of having monetary punishment when shill bidding is detected.

4.2 Extracting Additional Revenue with Shill Bidding

We now discuss the second possible motivation for shill bidding: sellers might want to extract additional revenue from the bidder with the highest valuation. We start by constructing the model under a private value framework, and then extend the model to study the profitability condition and profit-maximizing shill bid under an interdependent value framework.

4.2.1 Private Value Auctions

Suppose a seller with valuation v_0 sets a public reserve price R, which does not necessarily equal v_0, and attracts N people to view the auction page. Each page viewer would place bids if and only if his valuation is above R. With the automatic extension rule, online auctions prevents bid sniping (which refers to placing a bid slightly higher than the outstanding bid a few seconds before the auction ends) from happening and allows bidders to follow their weakly dominant strategy to bid until the amount reaches their true valuations. Under the private value framework, the suspicion level is irrelevant to the bid that buyers would place because there is no profitable unilateral deviation of bid in the auction for any suspicion level. Then at some time during the auction the bid will stop increasing and the outstanding bid would be the second highest valuation. Looking at this outstanding bid, the seller would decide whether to place a shill bid, and if so, how high the shill bid should be.

The following theorem gives the condition under which a seller would find it optimal to place a shill bid to extract additional revenue, and gives an expression for the profit-maximizing shill bid.

Theorem 2: When there are at least two people who place bids, a seller would find it profitable to place a shill bid s if and only if there exists some s such that $\max_{s>v_2}\{(s-v_0)\cdot(\frac{1-F(s)}{1-F(v_2)})\} \geq v_2 - v_0$, where v_2 is the second highest valuation observed (i.e., the highest bid when there is no shill bidding). When there is only one bidder, the profitability condition is $\max_{s>R}(s-v_0)\cdot(\frac{1-F(s)}{1-F(R)}) > R - v_0$. If shill bidding is optimal, the profit-maximizing shill bid solves $\arg\max_{s>v_2}\{(s-v_0)\cdot(1-F(s))\}$ when there are at least two bidders, and $\arg\max_{s>R}\{(s-v_0)\cdot(1-F(s))\}$ when there is one bidder.

Proof. We denote the cumulative density distribution of valuation V conditional on $V > R$ by $F_{V|V>R}(v)$, and

$$F_{V|V>R}(v) = P(V < v|V > R) = \frac{F(v) - F(R)}{1 - F(R)}.$$

Then the probability density function is

$$f_{V|V>R}(v) = \frac{\partial}{\partial v}[\frac{F(v) - F(R)}{1 - F(R)}] = \frac{f(v)}{1 - F(R)}.$$

Suppose that out of the N people who would view the auction, m of them would place a bid. Let us first consider the case when $m \geq 2$. Let $f_{(2,m)V|V>R}(v_2)$ be the p.d.f. of the second highest valuation among m bidders. We have

$$f_{(2,m)V|V>R}(v_2) = m\cdot(m-1)\cdot f_{V|V>R}(v_2)\cdot(F_{V|V>R}(v_2))^{m-2}\cdot(1 - F_{V|V>R}(v_2)).$$

Let $f_{\{(2,m),(1,m)\}V|V>R}(v_2, v_1)$ be the joint p.d.f. of the second highest valuation and the highest valuation conditional on all valuations being above R. $f_{\{(2,m),(1,m)\}V|V>R}(v_2, v_1)dv_2dv_1$ is the probability that $m-2$ valuations are below v_2, one valuation $\in dv_2$, and one valuation $\in dv_1$. Therefore, we have

$$f_{\{(2,m),(1,m)\}V|V>R}(v_2, v_1) = m\cdot(m-1)\cdot(F_{V|V>R}(v_2))^{m-2}\cdot f_{V|V>R}(v_2)\cdot f_{V|V>R}(v_1).$$

Let $f_{\{(1,m)|(2,m)\}V|V>R}(v_1|V_{(2,m)} = v_2)$ be the p.d.f. of the highest valuation (conditional on all valuations above R) given the value of the second highest valuations. We have

$$\begin{aligned} & f_{\{(1,m)|(2,m)\}V|V>R}(v_1|V_{(2,m)} = v_2) \\ = & \frac{m \cdot (m-1) \cdot (F_{V|V>R}(v_2))^{m-2} \cdot f_{V|V>R}(v_2) \cdot f_{V|V>R}(v_1)}{m \cdot (m-1) \cdot f_{V|V>R}(v_2) \cdot (F_{V|V>R}(v_2))^{m-2} \cdot (1 - F_{V|V>R}(v_2))} \\ = & \frac{f_{V|V>R}(v_1)}{1 - F_{V|V>R}(v_2)}. \end{aligned}$$

Let$F_{\{(1,m)|(2,m)\}V|V>R}(v_1|V_{(2,m)} = v_2)$ be the c.d.f. of the highest valuation conditional on the value of the second highest valuation. We have

$$\begin{aligned} & F_{\{(1,m)|(2,m)\}V|V>R}(v_1|V_{(2,m)} = v_2) \\ = & \int_{v_2}^{v_1} \frac{f_{V|V>R}(u)}{1 - F_{V|V>R}(v_2)}\, du = \frac{F_{V|V>R}(v_1) - F_{V|V>R}(v_2)}{1 - F_{V|V>R}(v_2)}. \end{aligned}$$

Since $F_{V|V>R}(v_2) = \frac{F(v_2)-F(R)}{1-F(R)}$ and $F_{V|V>R}(v_1) = \frac{F(v_1)-F(R)}{1-F(R)}$, we have

$$F_{\{(1,m)|(2,m)\}V|V>R}(v_1|V_{(2,m)} = v_2) = \frac{\frac{F(v_1)-F(R)}{1-F(R)} - \frac{F(v_2)-F(R)}{1-F(R)}}{1 - \frac{F(v_2)-F(R)}{1-F(R)}} = \frac{F(v_1) - F(v_2)}{1 - F(v_2)}.$$

Let U_h denote the utility of the honest seller who does not place any shill bids, and U_s denote the utility of the seller who places shill bids. The utility of the honest seller is the difference between the final price and his own valuation. Therefore, $U_h = v_2 - v_0$.

If he places a shill bid s, then he would get $s - v_0$ if the highest valuation is above s, and 0 if the highest valuation is below or equal to s. Therefore, his expected utility when placing a shill bid s is given by

$$\mathbb{E}(U_s(s)) = (s - v_0) \cdot (\frac{1 - F(s)}{1 - F(v_2)}) + 0 \cdot (1 - \frac{1 - F(s)}{1 - F(v_2)}) = (s - v_0) \cdot (\frac{1 - F(s)}{1 - F(v_2)}).$$

Therefore, he would place a shill bid if and only if $\max_{s>v_2} \mathbb{E}(U_s(s)) \geq U_h$. The optimality condition is, therefore,

$$\max_{s>v_2}\{(s - v_0) \cdot (\frac{1 - F(s)}{1 - F(v_2)})\} \geq v_2 - v_0.$$

And the optimal shill bid solves $\arg\max_{s>v_2}\{(s - v_0) \cdot (1 - F(s))\}$.

If $m = 1$, then the person who has valuation above R would place a bid equal to R. Without shill bidding, the seller gets $R - v_0$ as his payoff. His expected utility when place shill bid s equals $(s - v_0) \cdot (1 - \frac{F(s)-F(R)}{1-F(R)})$.

Then he would find shill bidding profitable if and only if

$$\max_{s>R}(s-v_0)\cdot(\frac{1-F(s)}{1-F(R)}) > R-v_0,$$

and when profitable, the profit-maximizing shill bid still solves $\arg\max_{s>v_2}\{(s-v_0)\cdot(1-F(s))\}$. ■

The profit-maximizing shill bid is independent of the public reserve price, and is also independent of the number of page viewers.

Corollary 1: After placing the profit-maximizing shill bid, even if the bidder with the highest valuation still has not left the auction, there would not exist any new shill bid that could provide the seller with higher expected utility despite this new information.

Proof. Let $s_1 \in \arg\max_{s>v_2}\{(s-v_0)\cdot(1-F(s))\}$ be the first shill bid. If the bidder with the highest valuation still stays, then

$$\mathbb{P}(v_1 < v | v_1 > s_1) = \frac{F(v_1)-F(s_1)}{1-F(s_1)}.$$

We want to investigate the existence of some $s_2 > s_1$ such that

$$(s_2-v_0)\cdot(\frac{1-F(s_2)}{1-F(s_1)}) > s_1 - v_0.$$

Multiplying each side by $\frac{1-F(s_1)}{1-F(v_2)}$ gives

$$(s_2-v_0)\cdot(\frac{1-F(s_2)}{1-F(v_2)}) > (s_1-v_0)\cdot(\frac{1-F(s_1)}{1-F(v_2)}),$$

which clearly contradicts the fact that s_1 is a global maximizer of $(s-v_0)\cdot(1-F(s))$ in the region $s > v_0$. ■

4.2.2 Interdependent Value Auctions

We now consider a setting of an interdependent value auction instead of private value. Now, suppose that the auction has a public reserve price normalized to 0, and there are N bidders. The seller could choose to either not join the auction or join the auction and pretend to be another bidder. Each bidder receives a private signal, and we label them in increasing order, so that $x_i < x_j$ if $i < j$. Suppose that the population percentage of shill bidders is p, which is common knowledge. It is also a common belief among bidders that a shill bidder would join the auction and leave no earlier than the time that the real bidder with the second highest valuation leaves. We study the game with N real bidders and a shill bidder. Say the real bidders have signals $x_1, ..., x_N$, and then the valuation of bidder i would be $u_N(x_i, x_1, ..., x_N)$. If there are $N+1$ real bidders with signals $x_1, ..., x_{N+1}$, then the valuation of bidder i would be $u_{N+1}(x_i, x_1, ..., x_{N+1})$. Each bidder believes that there is a p chance that there are N real bidders, in which case his ex-post valuation would be $u_N(x_i, x_1, ..., x_{i-1}, x_{i+1}, ..., x_N)$, and $1-p$ chance there are $N+1$ real bidders, where his valuation would be $u_{N+1}(x_i, x_1, ..., x_{i-1}, x_{i+1}, ..., x_{N+1})$. We assume u to be strictly increasing in the first component (i.e., one's own signal) and symmetric in the other components.

Proposition 3: The following bidding strategy $\beta = (\beta^1, ..., \beta^N)$ is a symmetric equilibrium of the buyers' game:

1. When no bidder has left the auction, let each bidder with valuation x follow the strategy of staying until the price reaches $\beta^N(x) = p \cdot u_N(x, x) + (1-p) \cdot u_{N+1}(x, x)$ or someone leaves the auction. Here $u_N(x, x)$ is the valuation when there are N real bidders and each bidder has signal equal to x and $u_{N+1}(x, x)$ is the valuation when there are $N+1$ real bidders. Since u_N and u_{N+1} are both strictly increasing in the first component, all the sellers would know the lowest signal x_1 as x_1 uniquely solves $\beta_N = p \cdot u_N(x_1, x_1) + (1-p) \cdot u_{N+1}(x_1, x_1)$, where β_N is the price at which the first leaver appears. Therefore, the first leaver will be the one with the lowest signal.

2. When one bidder has left, each of the remaining bidders stays until the price reaches $\beta^{N-1}(x) = p \cdot u_N(x, (x, x_1)) + (1-p) \cdot u_{N+1}(x, (x, x_1))$ or someone leaves the auction. Here $u_N(x, (x, x_1))$ is the valuation when there are N real bidders and all bidders except the first leaver have signal x, and $u_{N+1}(x, (x, x_1))$ is defined similarly. As soon as the second person leaves the auction, his valuation also becomes common knowledge: it would be the unique solution to $\beta_{N-1} = p \cdot u_N(x_2, (x_2, x_1)) + (1-p) \cdot u_{N+1}(x_2, (x_2, x_1))$, where β_{N-1} is the price at which the second leaver appears.

3. When $k \leq N-2$ bidders have left, then the leavers have valuation $x_1 < x_2 ... < x_k$ respectively. Now, each of the remaining bidders stay until the price reaches $\beta^{N-k}(x) = p \cdot u_N(x, (x, x_k, ..., x_1)) + (1-p) \cdot u_{N+1}(x, (x, x_k, ..., x_1))$ or someone leaves the auction. Here $u_N(x, (x, x_k, ..., x_1))$ is the valuation when there are N real bidders and all remaining bidders, including himself, have signal x. As soon as he leaves, his signal would also become common knowledge.

4. When there is only one real bidder left, he would place bids until the price reaches $\beta^1(x) = p \cdot u_N(x, x_{N-1}, ..., x_1) + (1-p) \cdot u_{N+1}(x, x, x_{N-1}, ..., x_1)$.

Proof. Suppose all except one bidder is following the strategy above. If he follows this strategy, then he will win if and only if he has the highest signal, and his expected payoff would be

$$\mathbb{E}[U(x)] = p \cdot u_N(x, x_{N-1}, ..., x_1) + (1-p) \cdot u_{N+1}(x, x, x_{N-1}, ..., x_1)$$
$$-\{p \cdot \min\{s, p \cdot u_N(x, x_{N-1}, ..., x_1) + (1-p) \cdot u_{N+1}(x, x, x_{N-1}, ..., x_1)\}$$
$$+(1-p) \cdot [p \cdot u_N(x_{N-1}, x_{N-1}, ..., x_1) + (1-p) \cdot u_{N+1}(x_{N-1}, x_{N-1}, x_{N-1}, ..., x_1)]\},$$

where s is the profit-maximizing shill bid placed by the seller as derived in Theorem 3 below when shill bidding is profitable (the profitability condition and profit-maximizing shill bid are known to the real bidder with the highest valuation as $x_1, ..., x_{N-1}$ became known), and when not profitable, $s = p \cdot u_N(x, x_{N-1}, ..., x_1) + (1-p) \cdot u_{N+1}(x, x, x_{N-1}, ..., x_1)$.

Clearly $\mathbb{E}[U(x)] > 0$ because

$$p \cdot u_N(x, x_{N-1}, ..., x_1) + (1-p) \cdot u_{N+1}(x, x, x_{N-1}, ..., x_1)$$
$$\geq \min\{s, p \cdot u_N(x, x_{N-1}, ..., x_1) + (1-p) \cdot u_{N+1}(x, x, x_{N-1}, ..., x_1)\},$$

and

$$p \cdot u_N(x, x_{N-1}, ..., x_1) + (1-p) \cdot u_{N+1}(x, x, x_{N-1}, ..., x_1)$$
$$> p \cdot u_N(x_{N-1}, x_{N-1}, x_{N-2}, ..., x_1) + (1-p) \cdot u_{N+1}(x_{N-1}, x_{N-1}, x_{N-1}, x_{N-2}, ..., x_1).$$

If he deviates and loses the auction, then compared to following the equilibrium strategy, he gets less payoff if he has the highest signal, and the payoff is the same if he does not have the highest signal. If he deviates and wins the auction, then he would still pay the same price and therefore have the same utility if he has the highest signal, and his utility would be negative if he is not the one with the highest signal, because his valuation would not exceed the price at which the bidder with the second highest signal leaves in the equilibrium strategy. ■

Let us now examine the profit-maximizing choice of the seller.

Theorem 3: In this interdependent value framework, the condition under which shill bidding is profitable is the existence of an x_s such that

$$\max_{x_s > x_{N-1}} [p \cdot u_N(x_s, x_{N-1}, x_{N-2}, ..., x_1) + (1-p) \cdot u_{N+1}(x_s, x_s, x_{N-1}, x_{N-2}, ..., x_1) - v_0] \cdot \frac{1 - F(x_s)}{1 - F(x_{N-1})}$$
$$> p \cdot u_N(x_{N-1}, x_{N-1}, x_{N-2}, ..., x_1) + (1-p) \cdot u_{N+1}(x_{N-1}, x_{N-1}, x_{N-1}, x_{N-2}, ..., x_1) - v_0,$$

and when profitable, the profit-maximizing shill bid equals

$$p \cdot u_N(x_s, x_{N-1}, x_{N-2}, ..., x_1) + (1-p) \cdot u_{N+1}(x_s, x_s, x_{N-1}, x_{N-2}, ..., x_1),$$

where x_s solves

$$\max_{x_s > x_{N-1}} [p \cdot u_N(x_s, x_{N-1}, x_{N-2}, ..., x_1) + (1-p) \cdot u_{N+1}(x_s, x_s, x_{N-1}, x_{N-2}, ..., x_1) - v_0] \cdot \frac{1 - F(x_s)}{1 - F(x_{N-1})}.$$

Proof. Let us study the choices of the seller. When participating in the auction, it is clearly not optimal to leave when there are at least two real bidders, because this would give them information of low signal. When the real bidder with the second highest signal has left, suppose the shill bidder strategically chooses a leaving price that would send the fake signal x_s. Upon observing the second highest signal, he knows that the highest signal has the c.d.f.

$$F(x_N | X_{N-1} = x_{N-1}) = \frac{F(x_N) - F(x_{N-1})}{1 - F(x_{N-1})}.$$

Upon participating in the auction, the seller also has a risk-free choice, which is to leave simultaneously as the bidder with the second highest signal leaves. Then the condition for shill bidding being profitable would be that shill bidding could give him higher expected payoff than if he joins but leaves at the same time when the bidder with the second highest valuation leaves. When profitable, choosing the profit-maximizing shill bid involves sending a fake signal x_s. If the shill bidder's leaving price is lower than the leaving price of the bidder with the highest signal, then his utility would be

$$p \cdot u_N(x_s, x_{N-1}, x_{N-2}, ..., x_1) + (1-p) \cdot u_{N+1}(x_s, x_s, x_{N-1}, x_{N-2}, ..., x_1) - v_0.$$

We also have

$$\mathbb{P}(\text{the highest signal is higher than } x_s | \text{the second highest signal is } x_{N-1}) = \frac{1 - F(x_s)}{1 - F(x_{N-1})}.$$

If x_s is higher than the highest real signal, then the shill bidder would get the item himself, resulting in zero utility when we assume participating in the auction does not induce any cost for him. Therefore, the profitability condition is the existence of x_s such that

$$\max_{x_s > x_{N-1}} [p \cdot u_N(x_s, x_{N-1}, x_{N-2}, ..., x_1) + (1-p) \cdot u_{N+1}(x_s, x_s, x_{N-1}, x_{N-2}, ..., x_1) - v_0] \cdot \frac{1 - F(x_s)}{1 - F(x_{N-1})}$$
$$> p \cdot u_N(x_{N-1}, x_{N-1}, x_{N-2}, ..., x_1) + (1-p) \cdot u_{N+1}(x_{N-1}, x_{N-1}, x_{N-1}, x_{N-2}, ..., x_1) - v_0.$$

When profitable, the profit-maximizing shill bid equals

$$p \cdot u_N(x_s, x_{N-1}, x_{N-2}, ..., x_1) + (1-p) \cdot u_{N+1}(x_s, x_s, x_{N-1}, x_{N-2}, ..., x_1),$$

where x_s solves

$$\max_{x_s > x_{N-1}} [p \cdot u_N(x_s, x_{N-1}, x_{N-2}, ..., x_1) + (1-p) \cdot u_{N+1}(x_s, x_s, x_{N-1}, x_{N-2}, ..., x_1) - v_0] \cdot \frac{1 - F(x_s)}{1 - F(x_{N-1})}.$$

■

5 Conclusion

In this paper, we analyzed two hypotheses for the motivation of shill bidding. The first was that shill bidding could serve as a substitute for a public reserve price, and we showed that if a dishonest seller sets 0 publicly and places shill bid equal to R, then he could expect higher utility than setting R publicly in some cases and expect the same utility in other cases. Intuitively, if bidders have a suspicion level below 1, which means they do not have the belief that any seller setting 0 public reserve price is dishonest, then those bidders with valuations moderately above R would prefer attending the auction with 0 reserve price to the auction with R as the reserve price, while those with very high valuations would be indifferent. This would give a dishonest seller incentives to set 0 publicly to attract bidders with moderately high valuations while placing a shill bid R to eliminate the possibility of monetary loss in case there are only bidders with low valuations in his auction. If the number of participants and the distribution of valuation give an honest seller incentives to set 0 publicly, then bidders could not be certain about the honesty of a seller when they see an auction with 0 as the reserve price, which would make the suspicion level below 1, therefore sustaining an equilibrium in the combined buyers' and sellers' game. On the other hand, if the number of participants and the distribution of valuation make an honest seller prefer setting R, then bidders will know that the seller is dishonest when they see an auction with 0 reserve price, and they would anticipate the shill bid R. With suspicion level 1, bidders with valuation above R would view auctions with 0 and R reserve prices as identical, and then setting 0 with shill biding would give the dishonest seller an expected payoff that is the same as setting R publicly.

The second hypothesis was that shill bidding could help sellers extract additional revenue. We gave profitability conditions and profit-maximizing shill bid under private and interdependent value frameworks in Theorems 2 and 3, respectively.

Future work can extend the first model to compare the sellers' expected utility of choosing the optimal reserve price as derived in Riley and Samuelson (1981), versus setting 0 as the public reserve price and placing shill bid equal to the optimal reserve price. It would also be interesting to incorporate the possibility of bidders leaving the auction before the price reaches his own valuation. Additionally, based on the second model, future work could also study the sufficient amount of monetary punishments when successfully detecting shill bids in order to deter shill bidding.

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