

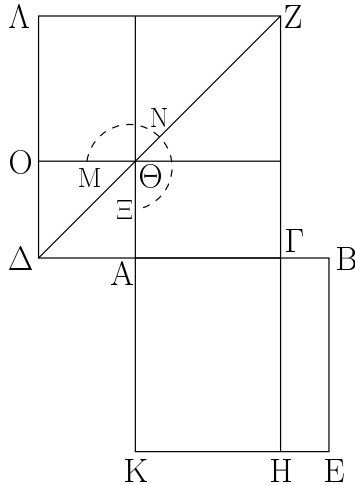
ELEMENTS BOOK 13

The Platonic Solids[†]

[†]The five regular solids—the cube, tetrahedron (*i.e.*, pyramid), octahedron, icosahedron, and dodecahedron—were probably discovered by the school of Pythagoras. They are generally termed “Platonic” solids because they feature prominently in Plato’s famous dialogue *Timaeus*. Many of the theorems contained in this book—particularly those which pertain to the last two solids—are ascribed to Theaetetus of Athens.

α'.

Ἐάν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ μείζον τμήμα προσλαβὼν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου.



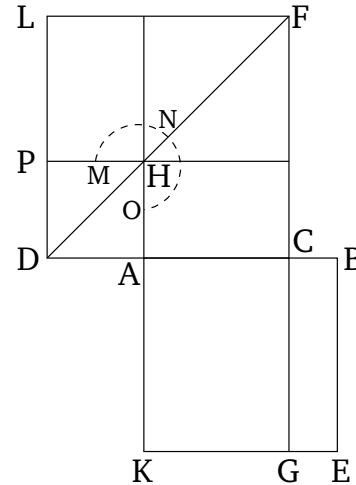
Εὐθεῖα γὰρ γραμμὴ ἡ AB ἄκρον καὶ μέσον λόγον τετμήσθω κατὰ τὸ Γ σημεῖον, καὶ ἔστω μείζον τμήμα τὸ AΓ, καὶ ἐκβεβλήσθω ἐπ' εὐθείας τῇ ΓΑ εὐθεῖα ἡ AΔ, καὶ κείσθω τῆς AB ἡμίσεια ἡ AΔ· λέγω, ὅτι πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΓΔ τοῦ ἀπὸ τῆς ΔΑ.

Ἀναγεγράφθωσαν γὰρ ἀπὸ τῶν AB, ΔΓ τετράγωνα τὰ AE, ΔZ, καὶ καταγεγράφθω ἐν τῷ ΔZ τὸ σχῆμα, καὶ διήχθω ἡ ZΓ ἐπὶ τὸ H. καὶ ἐπεὶ ἡ AB ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Γ, τὸ ἄρα ὑπὸ τῶν ABΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς AΓ. καὶ ἐστὶ τὸ μὲν ὑπὸ τῶν ABΓ τὸ ΓE, τὸ δὲ ἀπὸ τῆς AΓ τὸ ZΘ· ἴσον ἄρα τὸ ΓE τῷ ZΘ. καὶ ἐπεὶ διπλῇ ἐστὶν ἡ BA τῆς AΔ, ἴση δὲ ἡ μὲν BA τῇ KA, ἡ δὲ AΔ τῇ AΘ, διπλῇ ἄρα καὶ ἡ KA τῆς AΘ. ὥς δὲ ἡ KA πρὸς τὴν AΘ, οὕτως τὸ ΓK πρὸς τὸ ΓΘ· διπλάσιον ἄρα τὸ ΓK τοῦ ΓΘ. εἰσὶ δὲ καὶ τὰ ΛΘ, ΘΓ διπλάσια τοῦ ΓΘ. ἴσον ἄρα τὸ ΚΓ τοῖς ΛΘ, ΘΓ. ἐδείχθη δὲ καὶ τὸ ΓE τῷ ΘZ ἴσον· ὅλον ἄρα τὸ AE τετράγωνον ἴσον ἐστὶ τῷ MNΞ γνῶμονι. καὶ ἐπεὶ διπλῇ ἐστὶν ἡ BA τῆς AΔ, τετραπλάσιόν ἐστι τὸ ἀπὸ τῆς BA τοῦ ἀπὸ τῆς AΔ, τουτέστι τὸ AE τοῦ ΔΘ. ἴσον δὲ τὸ AE τῷ MNΞ γνῶμονι· καὶ ὁ MNΞ ἄρα γνῶμων τετραπλάσιός ἐστι τοῦ AO· ὅλον ἄρα τὸ ΔZ πενταπλάσιόν ἐστι τοῦ AO. καὶ ἐστὶ τὸ μὲν ΔZ τὸ ἀπὸ τῆς ΔΓ, τὸ δὲ AO τὸ ἀπὸ τῆς ΔΑ· τὸ ἄρα ἀπὸ τῆς ΓΔ πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΔΑ.

Ἐάν ἄρα εὐθεῖα ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ μείζον τμήμα προσλαβὼν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου· ὅπερ ἔδει δεῖξαι.

Proposition 1

If a straight-line is cut in extreme and mean ratio then the square on the greater piece, added to half of the whole, is five times the square on the half.



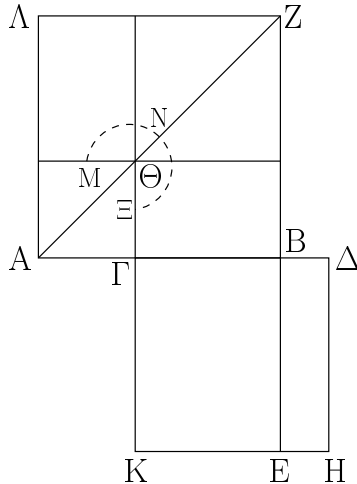
For let the straight-line AB have been cut in extreme and mean ratio at point C, and let AC be the greater piece. And let the straight-line AD have been produced in a straight-line with CA. And let AD be made (equal to) half of AB. I say that the (square) on CD is five times the (square) on DA.

For let the squares AE and DF have been described on AB and DC (respectively). And let the figure in DF have been drawn. And let FC have been drawn across to G. And since AB has been cut in extreme and mean ratio at C, the (rectangle contained) by ABC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. And CE is the (rectangle contained) by ABC, and FH the (square) on AC. Thus, CE (is) equal to FH. And since BA is double AD, and BA (is) equal to KA, and AD to AH, KA (is) thus also double AH. And as KA (is) to AH, so CK (is) to CH [Prop. 6.1]. Thus, CK (is) double CH. And LH plus HC is also double CH [Prop. 1.43]. Thus, KC (is) equal to LH plus HC. And CE was also shown (to be) equal to HF. Thus, the whole square AE is equal to the gnomon MNO. And since BA is double AD, the (square) on BA is four times the (square) on AD—that is to say, AE (is four times) DH. And AE (is) equal to gnomon MNO. And, thus, gnomon MNO is also four times AP. Thus, the whole of DF is five times AP. And DF is the (square) on DC, and AP the (square) on DA. Thus, the (square) on CD is five times the (square) on DA.

Thus, if a straight-line is cut in extreme and mean ratio then the square on the greater piece, added to half of

β'.

Ἐάν εὐθεῖα γραμμὴ τμήματος ἑαυτῆς πενταπλάσιον δύνηται, τῆς διπλασίας τοῦ εἰρημένου τμήματος ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζον τμήμα τὸ λοιπὸν μέρος ἐστὶ τῆς ἐξ ἄρχῃς εὐθείας.



Εὐθεῖα γὰρ γραμμὴ ἡ AB τμήματος ἑαυτῆς τοῦ AG πενταπλάσιον δυνάσθω, τῆς δὲ AG διπλῇ ἔστω ἡ $ΓΔ$. λέγω, ὅτι τῆς $ΓΔ$ ἄκρον καὶ μέσον λόγον τεμνομένου τὸ μείζον τμήμα ἐστὶν ἡ GB .

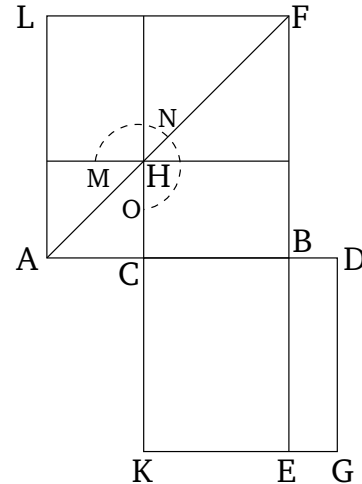
Ἀναγεγράφθω γὰρ ἀφ' ἐκατέρας τῶν AB , $ΓΔ$ τετράγωνα τὰ AZ , $ΓH$, καὶ καταγεγράφθω ἐν τῷ AZ τὸ σχῆμα, καὶ διήχθω ἡ BE . καὶ ἐπεὶ πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς BA τοῦ ἀπὸ τῆς AG , πενταπλάσιόν ἐστι τὸ AZ τοῦ $AΘ$. τετραπλάσιος ἄρα ὁ MNE γνῶμων τοῦ $AΘ$. καὶ ἐπεὶ διπλῇ ἐστὶν ἡ $ΔΓ$ τῆς $ΓΑ$, τετραπλάσιος ἄρα ἐστὶ τὸ ἀπὸ $ΔΓ$ τοῦ ἀπὸ $ΓΑ$, τουτέστι τὸ $ΓH$ τοῦ $AΘ$. ἐδείχθη δὲ καὶ ὁ MNE γνῶμων τετραπλάσιος τοῦ $AΘ$. ἴσος ἄρα ὁ MNE γνῶμων τῷ $ΓH$. καὶ ἐπεὶ διπλῇ ἐστὶν ἡ $ΔΓ$ τῆς $ΓΑ$, ἴση δὲ ἡ μὲν $ΔΓ$ τῇ $ΓK$, ἡ δὲ AG τῇ $ΓΘ$, [διπλῇ ἄρα καὶ ἡ $KΓ$ τῆς $ΓΘ$], διπλάσιος ἄρα καὶ τὸ KB τοῦ $BΘ$. εἰσὶ δὲ καὶ τὰ $ΛΘ$, $ΘB$ τοῦ $ΘB$ διπλάσια· ἴσον ἄρα τὸ KB τοῖς $ΛΘ$, $ΘB$. ἐδείχθη δὲ καὶ ὅλος ὁ MNE γνῶμων ὅλῳ τῷ $ΓH$ ἴσος· καὶ λοιπὸν ἄρα τὸ $ΘZ$ τῷ BH ἐστὶν ἴσον. καὶ ἐστὶ τὸ μὲν BH τὸ ὑπὸ τῶν $ΓΔB$. ἴση γὰρ ἡ $ΓΔ$ τῇ $ΔH$ · τὸ δὲ $ΘZ$ τὸ ἀπὸ τῆς $ΓB$ · τὸ ἄρα ὑπὸ τῶν $ΓΔB$ ἴσον ἐστὶ τῷ ἀπὸ τῆς $ΓB$. ἔστιν ἄρα ὡς ἡ $ΔΓ$ πρὸς τὴν $ΓB$, οὕτως ἡ $ΓB$ πρὸς τὴν BD . μείζων δὲ ἡ $ΔΓ$ τῆς $ΓB$ · μείζων ἄρα καὶ ἡ $ΓB$ τῆς BD . τῆς $ΓΔ$ ἄρα εὐθείας ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζον τμήμα ἐστὶν ἡ GB .

Ἐάν ἄρα εὐθεῖα γραμμὴ τμήματος ἑαυτῆς πενταπλάσιον δύνηται, τῆς διπλασίας τοῦ εἰρημένου τμήματος ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζον τμήμα τὸ λοιπὸν μέρος

the whole, is five times the square on the half. (Which is) the very thing it was required to show.

Proposition 2

If the square on a straight-line is five times the (square) on a piece of it, and double the aforementioned piece is cut in extreme and mean ratio, then the greater piece is the remaining part of the original straight-line.



For let the square on the straight-line AB be five times the (square) on the piece of it, AC . And let CD be double AC . I say that if CD is cut in extreme and mean ratio then the greater piece is CB .

For let the squares AF and CG have been described on each of AB and CD (respectively). And let the figure in AF have been drawn. And let BE have been drawn across. And since the (square) on BA is five times the (square) on AC , AF is five times AH . Thus, gnomon MNO (is) four times AH . And since DC is double CA , the (square) on DC is thus four times the (square) on CA —that is to say, CG (is four times) AH . And the gnomon MNO was also shown (to be) four times AH . Thus, gnomon MNO (is) equal to CG . And since DC is double CA , and DC (is) equal to CK , and AC to CH , [KC (is) thus also double CH], (and) KB (is) also double BH [Prop. 6.1]. And LH plus HB is also double HB [Prop. 1.43]. Thus, KB (is) equal to LH plus HB . And the whole gnomon MNO was also shown (to be) equal to the whole of CG . Thus, the remainder HF is also equal to (the remainder) BG . And BG is the (rectangle contained) by CDB . For CD (is) equal to DG . And HF (is) the square on CB . Thus, the (rectangle contained) by CDB is equal to the (square) on CB . Thus, as DC is to CB , so CB (is) to BD [Prop. 6.17]. And DC (is) greater than CB (see lemma). Thus, CB (is) also greater than BD [Prop. 5.14]. Thus, if the straight-line CD is cut

ἐστὶ τῆς ἐξ ἄρχῃς εὐθείας· ὅπερ ἔδει δεῖξαι.

Λήμμα.

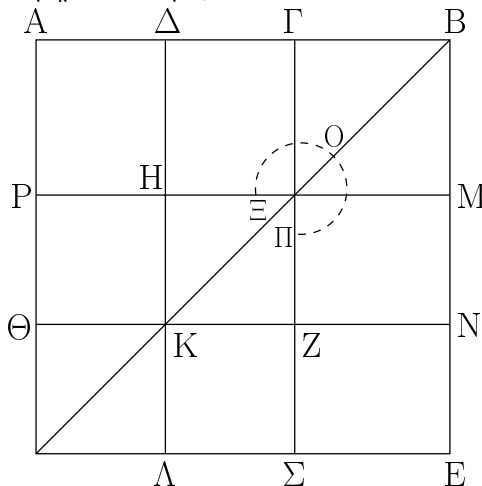
Ὅτι δὲ ἡ διπλῇ τῆς ΑΓ μείζων ἐστὶ τῆς ΒΓ, οὕτως δεικτέον.

Εἰ γὰρ μή, ἔστω, εἰ δυνατόν, ἡ ΒΓ διπλῇ τῆς ΓΑ. τετραπλάσιον ἄρα τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΑ· πενταπλάσια ἄρα τὰ ἀπὸ τῶν ΒΓ, ΓΑ τοῦ ἀπὸ τῆς ΓΑ. ὑπόκειται δὲ καὶ τὸ ἀπὸ τῆς ΒΑ πενταπλάσιον τοῦ ἀπὸ τῆς ΓΑ· τὸ ἄρα ἀπὸ τῆς ΒΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΓ, ΓΑ· ὅπερ ἀδύνατον. οὐκ ἄρα ἡ ΓΒ διπλασία ἐστὶ τῆς ΑΓ. ὁμοίως δὲ δεῖξομεν, ὅτι οὐδὲ ἡ ἐλάττω τῆς ΓΒ διπλασίον ἐστὶ τῆς ΓΑ· πολλῶ γὰρ [μείζον] τὸ ἄτοπον.

Ἡ ἄρα τῆς ΑΓ διπλῇ μείζων ἐστὶ τῆς ΓΒ· ὅπερ ἔδει δεῖξαι.

γ'.

Ἐάν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ ἔλασσον τμήμα προσλαβὼν τὴν ἡμίσειαν τοῦ μείζονος τμήματος πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμίσειας τοῦ μείζονος τμήματος τετραγώνου.



Εὐθεῖα γάρ τις ἡ ΑΒ ἄκρον καὶ μέσον λόγον τετμήσθω κατὰ τὸ Γ σημεῖον, καὶ ἔστω μείζον τμήμα τὸ ΑΓ, καὶ τετμήσθω ἡ ΑΓ δίχα κατὰ τὸ Δ· λέγω, ὅτι πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΔ τοῦ ἀπὸ τῆς ΔΓ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΕ, καὶ

in extreme and mean ratio then the greater piece is CB .

Thus, if the square on a straight-line is five times the (square) on a piece of itself, and double the aforementioned piece is cut in extreme and mean ratio, then the greater piece is the remaining part of the original straight-line. (Which is) the very thing it was required to show.

Lemma

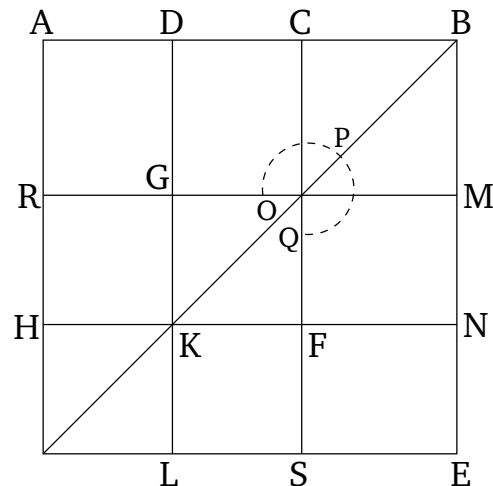
And it can be shown that double AC (i.e., DC) is greater than BC , as follows.

For if (double AC is) not (greater than BC), if possible, let BC be double CA . Thus, the (square) on BC (is) four times the (square) on CA . Thus, the (sum of) the (squares) on BC and CA (is) five times the (square) on CA . And the (square) on BA was assumed (to be) five times the (square) on CA . Thus, the (square) on BA is equal to the (sum of) the (squares) on BC and CA . The very thing (is) impossible [Prop. 2.4]. Thus, CB is not double AC . So, similarly, we can show that a (straight-line) less than CB is not double AC either. For (in this case) the absurdity is much [greater].

Thus, double AC is greater than CB . (Which is) the very thing it was required to show.

Proposition 3

If a straight-line is cut in extreme and mean ratio then the square on the lesser piece added to half of the greater piece is five times the square on half of the greater piece.

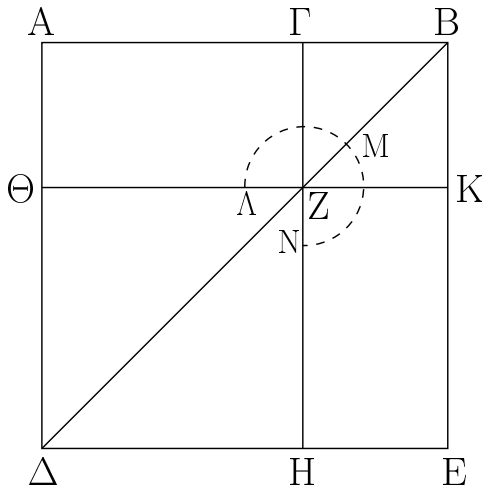


For let some straight-line AB have been cut in extreme and mean ratio at point C . And let AC be the greater piece. And let AC have been cut in half at D . I say that the (square) on BD is five times the (square) on DC .

καταγεγράφθω διπλοῦν τὸ σχῆμα. ἐπεὶ διπλῇ ἐστὶν ἡ ΑΓ τῆς ΔΓ, τετραπλάσιον ἄρα τὸ ἀπὸ τῆς ΑΓ τοῦ ἀπὸ τῆς ΔΓ, τουτέστι τὸ ΡΣ τοῦ ΖΗ. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ, καὶ ἐστὶ τὸ ὑπὸ τῶν ΑΒΓ τὸ ΓΕ, τὸ ἄρα ΓΕ ἴσον ἐστὶ τῷ ΡΣ. τετραπλάσιον δὲ τὸ ΡΣ τοῦ ΖΗ· τετραπλάσιον ἄρα καὶ τὸ ΓΕ τοῦ ΖΗ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΔΓ, ἴση ἐστὶ καὶ ἡ ΘΚ τῇ ΚΖ. ὥστε καὶ τὸ ΗΖ τετράγωνον ἴσον ἐστὶ τῷ ΘΑ τετραγώνῳ. ἴση ἄρα ἡ ΗΚ τῇ ΚΛ, τουτέστιν ἡ ΜΝ τῇ ΝΕ· ὥστε καὶ τὸ ΜΖ τῷ ΖΕ ἐστὶν ἴσον. ἀλλὰ τὸ ΜΖ τῷ ΓΗ ἐστὶν ἴσον· καὶ τὸ ΓΗ ἄρα τῷ ΖΕ ἐστὶν ἴσον. κοινὸν προσκείσθω τὸ ΓΝ· ὁ ἄρα ΞΟΠ γνῶμων ἴσος ἐστὶ τῷ ΓΕ. ἀλλὰ τὸ ΓΕ τετραπλάσιον ἐδείχθη τοῦ ΗΖ· καὶ ὁ ΞΟΠ ἄρα γνῶμων τετραπλάσιός ἐστι τοῦ ΖΗ τετραγώνου. ὁ ΞΟΠ ἄρα γνῶμων καὶ τὸ ΖΗ τετράγωνον πενταπλάσιός ἐστι τοῦ ΖΗ. ἀλλὰ ὁ ΞΟΠ γνῶμων καὶ τὸ ΖΗ τετράγωνόν ἐστι τὸ ΔΝ. καὶ ἐστὶ τὸ μὲν ΔΝ τὸ ἀπὸ τῆς ΔΒ, τὸ δὲ ΗΖ τὸ ἀπὸ τῆς ΔΓ. τὸ ἄρα ἀπὸ τῆς ΔΒ πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΔΓ· ὅπερ ἔδει δεῖξαι.

δ'.

Ἐὰν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ ἀπὸ τῆς ὅλης καὶ τοῦ ἐλάσσονος τμήματος, τὰ συναμφότερα τετράγωνα, τριπλάσιά ἐστι τοῦ ἀπὸ τοῦ μείζονος τμήματος τετραγώνου.



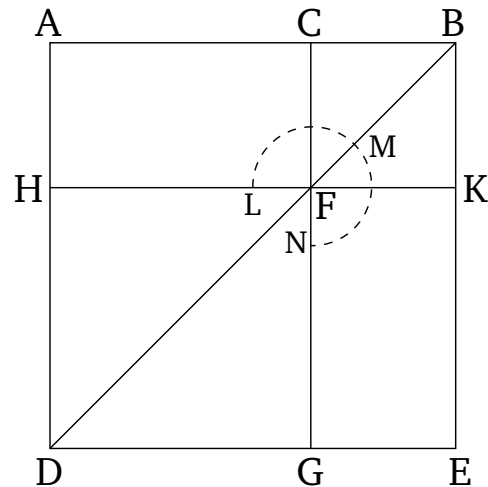
Ἐστω εὐθεῖα ἡ ΑΒ, καὶ τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ Γ, καὶ ἔστω μείζον τμήμα τὸ ΑΓ· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΒ, ΒΓ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΓΑ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔΕΒ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν ἡ ΑΒ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Γ, καὶ τὸ μείζον τμήμα ἐστὶν ἡ ΑΓ, τὸ ἄρα ὑπὸ τῶν ΑΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ. καὶ ἐστὶ τὸ μὲν ὑπὸ τῶν ΑΒΓ τὸ ΑΚ, τὸ δὲ ἀπὸ τῆς ΑΓ τὸ ΘΗ·

For let the square AE have been described on AB . And let the figure have been drawn double. Since AC is double DC , the (square) on AC (is) thus four times the (square) on DC —that is to say, RS (is four times) FG . And since the (rectangle contained) by ABC is equal to the (square) on AC [Def. 6.3, Prop. 6.17], and CE is the (rectangle contained) by ABC , CE is thus equal to RS . And RS (is) four times FG . Thus, CE (is) also four times FG . Again, since AD is equal to DC , HK is also equal to KF . Hence, square GF is also equal to square HL . Thus, GK (is) equal to KL —that is to say, MN to NE . Hence, MF is also equal to FE . But, MF is equal to CG . Thus, CG is also equal to FE . Let CN have been added to both. Thus, gnomon OPQ is equal to CE . But, CE was shown (to be) equal to four times GF . Thus, gnomon OPQ is also four times square FG . Thus, gnomon OPQ plus square FG is five times FG . But, gnomon OPQ plus square FG is (square) DN . And DN is the (square) on DB , and GF the (square) on DC . Thus, the (square) on DB is five times the (square) on DC . (Which is) the very thing it was required to show.

Proposition 4

If a straight-line is cut in extreme and mean ratio then the sum of the squares on the whole and the lesser piece is three times the square on the greater piece.



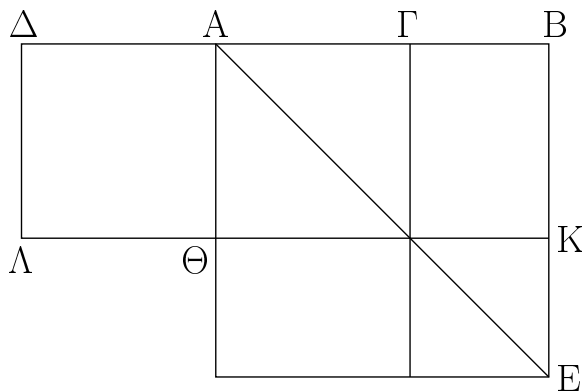
Let AB be a straight-line, and let it have been cut in extreme and mean ratio at C , and let AC be the greater piece. I say that the (sum of the squares) on AB and BC is three times the (square) on CA .

For let the square $ADEB$ have been described on AB , and let the (remainder of the) figure have been drawn. Therefore, since AB has been cut in extreme and mean ratio at C , and AC is the greater piece, the (rectangle

ἴσον ἄρα ἐστὶ τὸ AK τῷ ΘH . καὶ ἐπεὶ ἴσον ἐστὶ τὸ AZ τῷ ZE , κοινὸν προσκείσθω τὸ $ΓΚ$. ὅλον ἄρα τὸ AK ὅλῳ τῷ $ΓΕ$ ἐστὶν ἴσον· τὰ ἄρα AK , $ΓΕ$ τοῦ AK ἐστὶ διπλάσια. ἀλλὰ τὰ AK , $ΓΕ$ ὁ AMN γνῶμων ἐστὶ καὶ τὸ $ΓΚ$ τετράγωνον· ὁ ἄρα AMN γνῶμων καὶ τὸ $ΓΚ$ τετράγωνον διπλάσιά ἐστι τοῦ AK . ἀλλὰ μὴν καὶ τὸ AK τῷ ΘH ἐδείχθη ἴσον· ὁ ἄρα AMN γνῶμων καὶ [τὸ $ΓΚ$ τετράγωνον διπλάσιά ἐστι τοῦ ΘH · ὥστε ὁ AMN γνῶμων καὶ] τὰ $ΓΚ$, ΘH τετράγωνα τριπλάσιά ἐστι τοῦ ΘH τετραγώνου. καὶ ἐστὶν ὁ [μὲν] AMN γνῶμων καὶ τὰ $ΓΚ$, ΘH τετράγωνα ὅλον τὸ AE καὶ τὸ $ΓΚ$, ἅπερ ἐστὶ τὰ ἀπὸ τῶν AB , $ΒΓ$ τετράγωνα, τὸ δὲ $H\Theta$ τὸ ἀπὸ τῆς $ΑΓ$ τετράγωνον. τὰ ἄρα ἀπὸ τῶν AB , $ΒΓ$ τετράγωνα τριπλάσιά ἐστι τοῦ ἀπὸ τῆς $ΑΓ$ τετραγώνου· ὅπερ ἔδει δεῖξαι.

ε'.

Ἐὰν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, καὶ προστεθῇ αὐτῇ ἴση τῷ μείζονι τμήματι, ἡ ὅλη εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μείζον τμήμα ἐστὶν ἡ ἐξ ἀρχῆς εὐθεῖα.



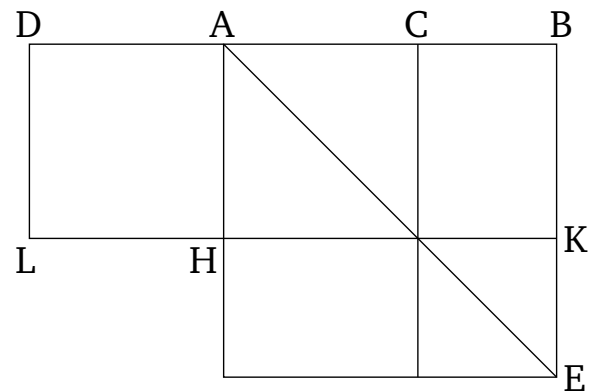
Εὐθεῖα γὰρ γραμμὴ ἡ AB ἄκρον καὶ μέσον λόγον τετμήσθω κατὰ τὸ Γ σημεῖον, καὶ ἔστω μείζον τμήμα ἡ $ΑΓ$, καὶ τῇ $ΑΓ$ ἴση [κείσθω] ἡ $ΑΔ$. λέγω, ὅτι ἡ $ΔΒ$ εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ A , καὶ τὸ μείζον τμήμα ἐστὶν ἡ ἐξ ἀρχῆς εὐθεῖα ἡ AB .

Ἀναγεγράφθω γὰρ ἀπὸ τῆς AB τετράγωνον τὸ AE , καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ ἡ AB ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Γ , τὸ ἄρα ὑπὸ $AB\Gamma$ ἴσον ἐστὶ τῷ ἀπὸ $ΑΓ$. καὶ ἐστὶ τὸ μὲν ὑπὸ $AB\Gamma$ τὸ $ΓΕ$, τὸ δὲ ἀπὸ τῆς $ΑΓ$ τὸ ΘH . ἴσον ἄρα τὸ $ΓΕ$ τῷ ΘH . ἀλλὰ τῷ μὲν $ΓΕ$ ἴσον ἐστὶ τὸ ΘE , τῷ δὲ ΘH ἴσον τὸ $\Delta\Theta$. καὶ τὸ $\Delta\Theta$ ἄρα ἴσον ἐστὶ τῷ ΘE [κοινὸν προσκείσθω τὸ ΘB]. ὅλον ἄρα τὸ ΔK ὅλῳ τῷ AE ἐστὶν ἴσον. καὶ ἐστὶ τὸ μὲν ΔK τὸ ὑπὸ τῶν $B\Delta$, ΔA ἴση

contained) by ABC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. And AK is the (rectangle contained) by ABC , and HG the (square) on AC . Thus, AK is equal to HG . And since AF is equal to FE [Prop. 1.43], let CK have been added to both. Thus, the whole of AK is equal to the whole of CE . Thus, AK plus CE is double AK . But, AK plus CE is the gnomon LMN plus the square CK . Thus, gnomon LMN plus square CK is double AK . But, indeed, AK was also shown (to be) equal to HG . Thus, gnomon LMN plus [square CK is double HG . Hence, gnomon LMN plus] the squares CK and HG is three times the square HG . And gnomon LMN plus the squares CK and HG is the whole of AE plus CK —which are the squares on AB and BC (respectively)—and GH (is) the square on AC . Thus, the (sum of the) squares on AB and BC is three times the square on AC . (Which is) the very thing it was required to show.

Proposition 5

If a straight-line is cut in extreme and mean ratio, and a (straight-line) equal to the greater piece is added to it, then the whole straight-line has been cut in extreme and mean ratio, and the original straight-line is the greater piece.



For let the straight-line AB have been cut in extreme and mean ratio at point C . And let AC be the greater piece. And let AD be [made] equal to AC . I say that the straight-line DB has been cut in extreme and mean ratio at A , and that the original straight-line AB is the greater piece.

For let the square AE have been described on AB , and let the (remainder of the) figure have been drawn. And since AB has been cut in extreme and mean ratio at C , the (rectangle contained) by ABC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. And CE is the (rectangle contained) by ABC , and CH the (square) on AC . But, HE is equal to CE [Prop. 1.43], and DH equal

γὰρ ἡ AD τῇ $ΔA$ · τὸ δὲ AE τὸ ἀπὸ τῆς AB · τὸ ἄρα ὑπὸ τῶν $BΔA$ ἴσον ἐστὶ τῷ ἀπὸ τῆς AB . ἔστιν ἄρα ὡς ἡ $ΔB$ πρὸς τὴν BA , οὕτως ἡ BA πρὸς τὴν AD . μείζων δὲ ἡ $ΔB$ τῆς BA · μείζων ἄρα καὶ ἡ BA τῆς AD .

Ἡ ἄρα $ΔB$ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ A , καὶ τὸ μείζον τμήμα ἐστὶν ἡ AB · ὅπερ ἔδει δεῖξαι.

to HC . Thus, DH is also equal to HE . [Let HB have been added to both.] Thus, the whole of DK is equal to the whole of AE . And DK is the (rectangle contained) by BD and DA . For AD (is) equal to DL . And AE (is) the (square) on AB . Thus, the (rectangle contained) by BDA is equal to the (square) on AB . Thus, as DB (is) to BA , so BA (is) to AD [Prop. 6.17]. And DB (is) greater than BA . Thus, BA (is) also greater than AD [Prop. 5.14].

Thus, DB has been cut in extreme and mean ratio at A , and the greater piece is AB . (Which is) the very thing it was required to show.

ε'.

Ἐὰν εὐθεῖα ῥητὴ ἄκρον καὶ μέσον λόγον τμηθῇ, ἐκάτερον τῶν τμημάτων ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.



Ἐστω εὐθεῖα ῥητὴ ἡ AB καὶ τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ $Γ$, καὶ ἔστω μείζον τμήμα ἡ $ΑΓ$ · λέγω, ὅτι ἐκάτερα τῶν $ΑΓ$, $ΓΒ$ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐκβεβλήσθω γὰρ ἡ BA , καὶ κείσθω τῆς BA ἡμίσεια ἡ AD . ἐπεὶ οὖν εὐθεῖα ἡ AB τέτμηται ἄκρον καὶ μέσον λόγον κατὰ τὸ $Γ$, καὶ τῷ μείζονι τμήματι τῷ $ΑΓ$ πρόσκειται ἡ AD ἡμίσεια οὕσα τῆς AB , τὸ ἄρα ἀπὸ $ΓΔ$ τοῦ ἀπὸ $ΔA$ πενταπλάσιόν ἐστιν. τὸ ἄρα ἀπὸ $ΓΔ$ πρὸς τὸ ἀπὸ $ΔA$ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμὸν· σύμμετρον ἄρα τὸ ἀπὸ $ΓΔ$ τῷ ἀπὸ $ΔA$. ῥητὸν δὲ τὸ ἀπὸ $ΔA$ · ῥητὴ γάρ [ἐστὶν] ἡ $ΔA$ ἡμίσεια οὕσα τῆς AB ῥητῆς οὕσης· ῥητὸν ἄρα καὶ τὸ ἀπὸ $ΓΔ$ · ῥητὴ ἄρα ἐστὶ καὶ ἡ $ΓΔ$. καὶ ἐπεὶ τὸ ἀπὸ $ΓΔ$ πρὸς τὸ ἀπὸ $ΔA$ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, ἀσύμμετρος ἄρα μήκει ἡ $ΓΔ$ τῇ $ΔA$ · αἱ $ΓΔ$, $ΔA$ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ $ΑΓ$. πάλιν, ἐπεὶ ἡ AB ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μείζον τμήμα ἐστὶν ἡ $ΑΓ$, τὸ ἄρα ὑπὸ AB , $ΒΓ$ τῷ ἀπὸ $ΑΓ$ ἴσον ἐστίν. τὸ ἄρα ἀπὸ τῆς $ΑΓ$ ἀποτομῆς παρὰ τὴν AB ῥητὴν παραβληθὲν πλάτος ποιεῖ τὴν $ΒΓ$. τὸ δὲ ἀπὸ ἀποτομῆς παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην· ἀποτομὴ ἄρα πρώτη ἐστὶν ἡ $ΓΒ$. ἐδείχθη δὲ καὶ ἡ $ΓA$ ἀποτομή.

Ἐὰν ἄρα εὐθεῖα ῥητὴ ἄκρον καὶ μέσον λόγον τμηθῇ, ἐκάτερον τῶν τμημάτων ἄλογός ἐστιν ἡ καλουμένη ἀποτομή· ὅπερ ἔδει δεῖξαι.

Proposition 6

If a rational straight-line is cut in extreme and mean ratio then each of the pieces is that irrational (straight-line) called an apotome.

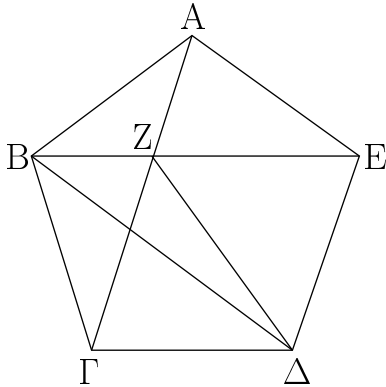


Let AB be a rational straight-line cut in extreme and mean ratio at C , and let AC be the greater piece. I say that AC and CB is each that irrational (straight-line) called an apotome.

For let BA have been produced, and let AD be made (equal) to half of BA . Therefore, since the straight-line AB has been cut in extreme and mean ratio at C , and AD , which is half of AB , has been added to the greater piece AC , the (square) on CD is thus five times the (square) on DA [Prop. 13.1]. Thus, the (square) on CD has to the (square) on DA the ratio which a number (has) to a number. The (square) on CD (is) thus commensurable with the (square) on DA [Prop. 10.6]. And the (square) on DA (is) rational. For DA [is] rational, being half of AB , which is rational. Thus, the (square) on CD (is) also rational [Def. 10.4]. Thus, CD is also rational. And since the (square) on CD does not have to the (square) on DA the ratio which a square number (has) to a square number, CD (is) thus incommensurable in length with DA [Prop. 10.9]. Thus, CD and DA are rational (straight-lines which are) commensurable in square only. Thus, AC is an apotome [Prop. 10.73]. Again, since AB has been cut in extreme and mean ratio, and AC is the greater piece, the (rectangle contained) by AB and BC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. Thus, the (square) on the apotome AC , applied to the rational (straight-line) AB , makes BC as width. And the (square) on an apotome, applied to a rational (straight-line), makes a first apotome as width [Prop. 10.97]. Thus, CB is a first apotome. And CA was also shown (to be) an apotome.

ζ'.

Ἐὰν πενταγώνου ἰσοπλεύρου αἱ τρεῖς γωνίαι ᾗτοι αἱ κατὰ τὸ ἐξῆς ἢ αἱ μὴ κατὰ τὸ ἐξῆς ἴσαι ᾧσιν, ἰσογώνιον ἔσται τὸ πεντάγωνον.



Πενταγώνου γὰρ ἰσοπλεύρου τοῦ ΑΒΓΔΕ αἱ τρεῖς γωνίαι πρότερον αἱ κατὰ τὸ ἐξῆς αἱ πρὸς τοῖς Α, Β, Γ ἴσαι ἀλλήλαις ἔστωσαν· λέγω, ὅτι ἰσογώνιον ἔστι τὸ ΑΒΓΔΕ πεντάγωνον.

Ἐπεζεύχθωσαν γὰρ αἱ ΑΓ, ΒΕ, ΖΔ. καὶ ἐπεὶ δύο αἱ ΓΒ, ΒΑ δυοὶ ταῖς ΒΑ, ΑΕ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ γωνία ἡ ὑπὸ ΓΒΑ γωνία τῇ ὑπὸ ΒΑΕ ἔστιν ἴση, βάσις ἄρα ἡ ΑΓ βάσει τῇ ΒΕ ἔστιν ἴση, καὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΒΕ τριγώνῳ ἴσον, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὅφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν, ἡ μὲν ὑπὸ ΒΓΑ τῇ ὑπὸ ΒΕΑ, ἡ δὲ ὑπὸ ΑΒΕ τῇ ὑπὸ ΓΑΒ· ὥστε καὶ πλευρὰ ἡ ΑΖ πλευρᾷ τῇ ΒΖ ἔστιν ἴση. ἐδείχθη δὲ καὶ ὅλη ἡ ΑΓ ὅλη τῇ ΒΕ ἴση· καὶ λοιπὴ ἄρα ἡ ΖΓ λοιπῇ τῇ ΖΕ ἔστιν ἴση. ἔστι δὲ καὶ ἡ ΓΔ τῇ ΔΕ ἴση. δύο δὴ αἱ ΖΓ, ΓΔ δυοὶ ταῖς ΖΕ, ΕΔ ἴσαι εἰσὶν· καὶ βάσις αὐτῶν κοινὴ ἡ ΖΔ· γωνία ἄρα ἡ ὑπὸ ΖΓΔ γωνία τῇ ὑπὸ ΖΕΔ ἔστιν ἴση. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΒΓΑ τῇ ὑπὸ ΑΕΒ ἴση· καὶ ὅλη ἄρα ἡ ὑπὸ ΒΓΔ ὅλη τῇ ὑπὸ ΑΕΔ ἴση. ἀλλ' ἡ ὑπὸ ΒΓΔ ἴση ὑπόκειται ταῖς πρὸς τοῖς Α, Β γωνίαις· καὶ ἡ ὑπὸ ΑΕΔ ἄρα ταῖς πρὸς τοῖς Α, Β γωνίαις ἴση ἔστί. ὁμοίως δὲ δείξομεν, ὅτι καὶ ἡ ὑπὸ ΓΔΕ γωνία ἴση ἔστί ταῖς πρὸς τοῖς Α, Β, Γ γωνίαις· ἰσογώνιον ἄρα ἔστί τὸ ΑΒΓΔΕ πεντάγωνον.

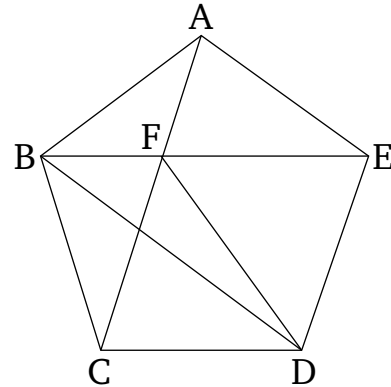
Ἀλλὰ δὴ μὴ ἔστωσαν ἴσαι αἱ κατὰ τὸ ἐξῆς γωνίαι, ἀλλ' ἔστωσαν ἴσαι αἱ πρὸς τοῖς Α, Γ, Δ σημείοις· λέγω, ὅτι καὶ οὕτως ἰσογώνιον ἔστι τὸ ΑΒΓΔΕ πεντάγωνον.

Ἐπεζεύχθω γὰρ ἡ ΒΔ. καὶ ἐπεὶ δύο αἱ ΒΑ, ΑΕ δυοὶ ταῖς ΒΓ, ΓΔ ἴσαι εἰσὶ καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΒΕ βάσει τῇ ΒΔ ἴση ἔστί, καὶ τὸ ΑΒΕ τρίγωνον τῷ ΒΓΔ τριγώνῳ ἴσον ἔστί, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὅφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν·

Thus, if a rational straight-line is cut in extreme and mean ratio then each of the pieces is that irrational (straight-line) called an apotome.

Proposition 7

If three angles, either consecutive or not consecutive, of an equilateral pentagon are equal then the pentagon will be equiangular.



For let three angles of the equilateral pentagon $ABCDE$ —first of all, the consecutive (angles) at A , B , and C —be equal to one another. I say that pentagon $ABCDE$ is equiangular.

For let AC , BE , and FD have been joined. And since the two (straight-lines) CB and BA are equal to the two (straight-lines) BA and AE , respectively, and angle CBA is equal to angle BAE , base AC is thus equal to base BE , and triangle ABC equal to triangle ABE , and the remaining angles will be equal to the remaining angles which the equal sides subtend [Prop. 1.4], (that is), BCA (equal) to BEA , and ABE to CAB . And hence side AF is also equal to side BF [Prop. 1.6]. And the whole of AC was also shown (to be) equal to the whole of BE . Thus, the remainder FC is also equal to the remainder FE . And CD is also equal to DE . So, the two (straight-lines) FC and CD are equal to the two (straight-lines) FE and ED (respectively). And FD is their common base. Thus, angle FCD is equal to angle FED [Prop. 1.8]. And BCA was also shown (to be) equal to AEB . And thus the whole of BCD (is) equal to the whole of AED . But, (angle) BCD was assumed (to be) equal to the angles at A and B . Thus, (angle) AED is also equal to the angles at A and B . So, similarly, we can show that angle CDE is also equal to the angles at A , B , C . Thus, pentagon $ABCDE$ is equiangular.

And so let consecutive angles not be equal, but let the (angles) at points A , C , and D be equal. I say that pentagon $ABCDE$ is also equiangular in this case.

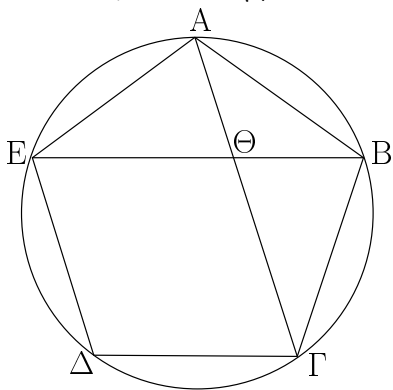
For let BD have been joined. And since the two

ἴση ἄρα ἐστὶν ἡ ὑπὸ AEB γωνία τῇ ὑπὸ $ΓΔΒ$. ἔστι δὲ καὶ ἡ ὑπὸ $ΒΕΔ$ γωνία τῇ ὑπὸ $ΒΔΕ$ ἴση, ἐπεὶ καὶ πλευρὰ ἡ $ΒΕ$ πλευρᾷ τῇ $ΒΔ$ ἐστὶν ἴση. καὶ ὅλη ἄρα ἡ ὑπὸ $ΑΕΔ$ γωνία ὅλη τῇ ὑπὸ $ΓΔΕ$ ἐστὶν ἴση. ἀλλὰ ἡ ὑπὸ $ΓΔΕ$ ταῖς πρὸς τοῖς $Α, Γ$ γωνίαις ὑπόκειται ἴση· καὶ ἡ ὑπὸ $ΑΕΔ$ ἄρα γωνία ταῖς πρὸς τοῖς $Α, Γ$ ἴση ἐστίν. διὰ τὰ αὐτὰ δὲ καὶ ἡ ὑπὸ $ΑΒΓ$ ἴση ἐστὶ ταῖς πρὸς τοῖς $Α, Γ, Δ$ γωνίαις. ἰσογώνιον ἄρα ἐστὶ τὸ $ΑΒΓΔΕ$ πεντάγωνον· ὅπερ ἔδει δεῖξαι.

(straight-lines) BA and AE are equal to the (straight-lines) BC and CD , and they contain equal angles, base BE is thus equal to base BD , and triangle ABE is equal to triangle BCD , and the remaining angles will be equal to the remaining angles which the equal sides subtend [Prop. 1.4]. Thus, angle AEB is equal to (angle) CDB . And angle BED is also equal to (angle) BDE , since side BE is also equal to side BD [Prop. 1.5]. Thus, the whole angle AED is also equal to the whole (angle) CDE . But, (angle) CDE was assumed (to be) equal to the angles at A and C . Thus, angle AED is also equal to the (angles) at A and C . So, for the same (reasons), (angle) ABC is also equal to the angles at A, C , and D . Thus, pentagon $ABCDE$ is equiangular. (Which is) the very thing it was required to show.

η'.

Ἐὰν πενταγώνου ἰσοπλευροῦ καὶ ἰσογωνίου τὰς κατὰ τὸ ἐξῆς δύο γωνίας ὑποτείνωσιν εὐθεῖαι, ἄκρον καὶ μέσον λόγον τέμνουσιν ἀλλήλας, καὶ τὰ μείζονα αὐτῶν τμήματα ἴσα ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ.

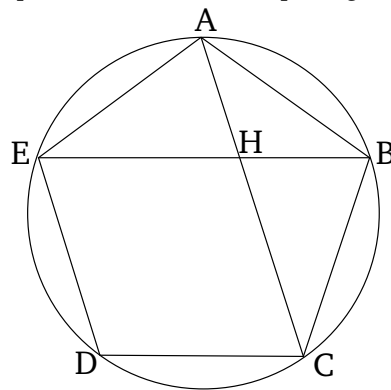


Πενταγώνου γὰρ ἰσοπλευρον καὶ ἰσογωνίου τοῦ $ΑΒΓΔΕ$ δύο γωνίας τὰς κατὰ τὸ ἐξῆς τὰς πρὸς τοῖς $Α, Β$ ὑποτείνεωσαν εὐθεῖαι αἱ $ΑΓ, ΒΕ$ τέμνουσαι ἀλλήλας κατὰ τὸ $Θ$ σημεῖον· λέγω, ὅτι ἑκάτερα αὐτῶν ἄκρον καὶ μέσον λόγον τέμνεται κατὰ τὸ $Θ$ σημεῖον, καὶ τὰ μείζονα αὐτῶν τμήματα ἴσα ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ.

Περιγεγράφθω γὰρ περὶ τὸ $ΑΒΓΔΕ$ πεντάγωνον κύκλος ὁ $ΑΒΓΔΕ$. καὶ ἐπεὶ δύο εὐθεῖαι αἱ $ΕΑ, ΑΒ$ δυοὶ ταῖς $ΑΒ, ΒΓ$ ἴσαι εἰσὶ καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ $ΒΕ$ βάσει τῇ $ΑΓ$ ἴση ἐστίν, καὶ τὸ $ΑΒΕ$ τρίγωνον τῷ $ΑΒΓ$ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκάτερα ἑκατέρῃ, ὅφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν. ἴση ἄρα ἐστὶν ἡ ὑπὸ $ΒΑΓ$ γωνία τῇ ὑπὸ $ΑΒΕ$ · διπλῇ ἄρα ἡ ὑπὸ $ΑΘΕ$ τῇς ὑπὸ $ΒΑΘ$. ἔστι δὲ καὶ ἡ ὑπὸ $ΕΑΓ$ τῇς ὑπὸ $ΒΑΓ$ διπλῇ, ἐπειδὴ περ καὶ περιφέρεια ἡ $ΕΔΓ$ περιφερείας τῇς $ΓΒ$ ἐστὶ διπλῇ· ἴση ἄρα ἡ ὑπὸ $ΘΑΕ$ γωνία τῇ ὑπὸ $ΑΘΕ$ · ὥστε καὶ ἡ $ΘΕ$ εὐθεῖα τῇ $ΕΑ$, τουτέστι τῇ $ΑΒ$

Proposition 8

If straight-lines subtend two consecutive angles of an equilateral and equiangular pentagon then they cut one another in extreme and mean ratio, and their greater pieces are equal to the sides of the pentagon.



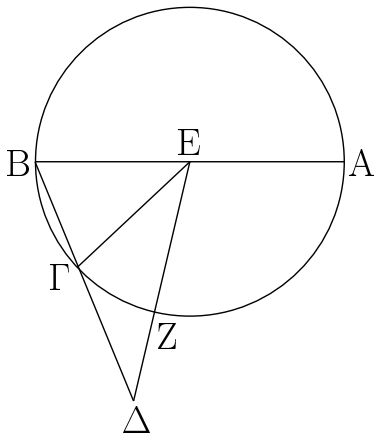
For let the two straight-lines, AC and BE , cutting one another at point H , have subtended two consecutive angles, at A and B (respectively), of the equilateral and equiangular pentagon $ABCDE$. I say that each of them has been cut in extreme and mean ratio at point H , and that their greater pieces are equal to the sides of the pentagon.

For let the circle $ABCDE$ have been circumscribed about pentagon $ABCDE$ [Prop. 4.14]. And since the two straight-lines EA and AB are equal to the two (straight-lines) AB and BC (respectively), and they contain equal angles, the base BE is thus equal to the base AC , and triangle ABE is equal to triangle ABC , and the remaining angles will be equal to the remaining angles, respectively, which the equal sides subtend [Prop. 1.4]. Thus, angle BAC is equal to (angle) ABE . Thus, (angle) AHE (is) double (angle) BAH [Prop. 1.32]. And EAC is also dou-

ἐστὶν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ BA εὐθεΐα τῇ AE , ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ABE τῇ ὑπὸ AEB . ἀλλὰ ἡ ὑπὸ ABE τῇ ὑπὸ $BA\Theta$ ἐδείχθη ἴση· καὶ ἡ ὑπὸ BEA ἄρα τῇ ὑπὸ $BA\Theta$ ἐστὶν ἴση. καὶ κοινὴ τῶν δύο τριγώνων τοῦ τε ABE καὶ τοῦ $AB\Theta$ ἐστὶν ἡ ὑπὸ ABE · λοιπὴ ἄρα ἡ ὑπὸ BAE γωνία λοιπῇ τῇ ὑπὸ $A\Theta B$ ἐστὶν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ABE τρίγωνον τῷ $AB\Theta$ τριγώνῳ· ἀνάλογον ἄρα ἐστὶν ὡς ἡ EB πρὸς τὴν BA , οὕτως ἡ AB πρὸς τὴν $B\Theta$. ἴση δὲ ἡ BA τῇ $E\Theta$ · ὡς ἄρα ἡ BE πρὸς τὴν $E\Theta$, οὕτως ἡ $E\Theta$ πρὸς τὴν ΘB . μείζων δὲ ἡ BE τῆς $E\Theta$ · μείζων ἄρα καὶ ἡ $E\Theta$ τῆς ΘB . ἡ BE ἄρα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Θ , καὶ τὸ μείζον τμήμα τὸ ΘE ἴσον ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ. ὁμοίως δὲ δείξομεν, ὅτι καὶ ἡ AG ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Θ , καὶ τὸ μείζον αὐτῆς τμήμα ἡ $\Gamma\Theta$ ἴσον ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ· ὅπερ ἔδει δεῖξαι.

θ'.

Ἐὰν ἡ τοῦ ἑξαγώνου πλευρὰ καὶ ἡ τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων συντεθῶσιν, ἡ ὅλη εὐθεΐα ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μείζον αὐτῆς τμήμα ἐστὶν ἡ τοῦ ἑξαγώνου πλευρὰ.



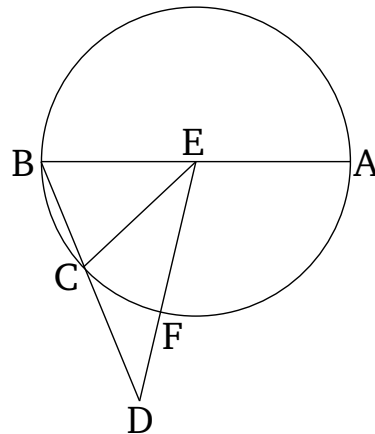
Ἐστω κύκλος ὁ $AB\Gamma$, καὶ τῶν εἰς τὸν $AB\Gamma$ κύκλον ἐγγραφομένων σχημάτων, δεκαγώνου μὲν ἔστω πλευρὰ ἡ $B\Gamma$, ἑξαγώνου δὲ ἡ $\Gamma\Delta$, καὶ ἔστωσαν ἐπ' εὐθείας· λέγω, ὅτι ἡ ὅλη εὐθεΐα ἡ BD ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μείζον αὐτῆς τμήμα ἐστὶν ἡ $\Gamma\Delta$.

Εἰλήφθω γάρ τὸ κέντρον τοῦ κύκλου τὸ E σημεῖον, καὶ ἐπεζεύχθωσαν αἱ EB , $E\Gamma$, $E\Delta$, καὶ διήχθω ἡ BE ἐπὶ τὸ

ble BAC , inasmuch as circumference EDC is also double circumference CB [Props. 3.28, 6.33]. Thus, angle HAE (is) equal to (angle) AHE . Hence, straight-line HE is also equal to (straight-line) EA —that is to say, to (straight-line) AB [Prop. 1.6]. And since straight-line BA is equal to AE , angle ABE is also equal to AEB [Prop. 1.5]. But, ABE was shown (to be) equal to BAH . Thus, BEA is also equal to BAH . And (angle) ABE is common to the two triangles ABE and ABH . Thus, the remaining angle BAE is equal to the remaining (angle) AHB [Prop. 1.32]. Thus, triangle ABE is equiangular to triangle ABH . Thus, proportionally, as EB is to BA , so AB (is) to BH [Prop. 6.4]. And BA (is) equal to EH . Thus, as BE (is) to EH , so EH (is) to HB . And BE (is) greater than EH . EH (is) thus also greater than HB [Prop. 5.14]. Thus, BE has been cut in extreme and mean ratio at H , and the greater piece HE is equal to the side of the pentagon. So, similarly, we can show that AC has also been cut in extreme and mean ratio at H , and that its greater piece CH is equal to the side of the pentagon. (Which is) the very thing it was required to show.

Proposition 9

If the side of a hexagon and of a decagon inscribed in the same circle are added together then the whole straight-line has been cut in extreme and mean ratio (at the junction point), and its greater piece is the side of the hexagon.[†]



Let ABC be a circle. And of the figures inscribed in circle ABC , let BC be the side of a decagon, and CD (the side) of a hexagon. And let them be (laid down) straight-on (to one another). I say that the whole straight-line BD has been cut in extreme and mean ratio (at C), and that CD is its greater piece.

For let the center of the circle, point E , have been

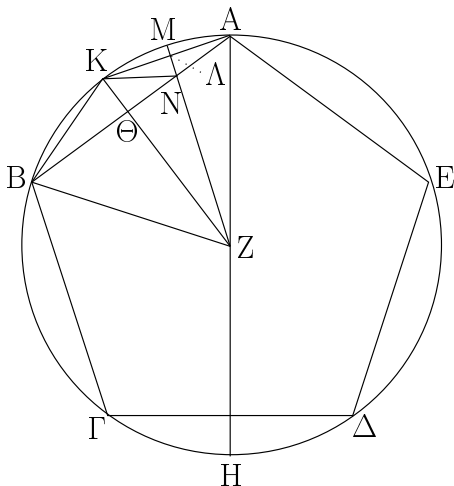
Α. ἐπεὶ δεκαγώνου ἰσοπλεύρου πλευρὰ ἐστὶν ἡ ΒΓ, πενταπλασίων ἄρα ἡ ΑΓΒ περιφέρεια τῆς ΒΓ περιφερείας· τετραπλασίων ἄρα ἡ ΑΓ περιφέρεια τῆς ΓΒ. ὥς δὲ ἡ ΑΓ περιφέρεια πρὸς τὴν ΓΒ, οὕτως ἡ ὑπὸ ΑΕΓ γωνία πρὸς τὴν ὑπὸ ΓΕΒ· τετραπλασίων ἄρα ἡ ὑπὸ ΑΕΓ τῆς ὑπὸ ΓΕΒ. καὶ ἐπεὶ ἴση ἡ ὑπὸ ΕΒΓ γωνία τῇ ὑπὸ ΕΓΒ, ἡ ἄρα ὑπὸ ΑΕΓ γωνία διπλασία ἐστὶ τῆς ὑπὸ ΕΓΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΕΓ εὐθεΐα τῇ ΓΔ· ἑκατέρα γὰρ αὐτῶν ἴση ἐστὶ τῇ τοῦ ἑξαγώνου πλευρᾷ τοῦ εἰς τὸν ΑΒΓ κύκλον [ἐγγραφομένου]· ἴση ἐστὶ καὶ ἡ ὑπὸ ΓΕΔ γωνία τῇ ὑπὸ ΓΔΕ γωνίᾳ· διπλασία ἄρα ἡ ὑπὸ ΕΓΒ γωνία τῆς ὑπὸ ΕΔΓ. ἀλλὰ τῆς ὑπὸ ΕΓΒ διπλασία ἐδείχθη ἡ ὑπὸ ΑΕΓ· τετραπλασία ἄρα ἡ ὑπὸ ΑΕΓ τῆς ὑπὸ ΕΔΓ. ἐδείχθη δὲ καὶ τῆς ὑπὸ ΒΕΓ τετραπλασία ἡ ὑπὸ ΑΕΓ· ἴση ἄρα ἡ ὑπὸ ΕΔΓ τῇ ὑπὸ ΒΕΓ. κοινὴ δὲ τῶν δύο τριγώνων, τοῦ τε ΒΕΓ καὶ τοῦ ΒΕΔ, ἡ ὑπὸ ΕΒΔ γωνία· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΒΕΔ τῇ ὑπὸ ΕΓΒ ἐστὶν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΕΒΔ τρίγωνον τῷ ΕΒΓ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΕΒ πρὸς τὴν ΒΓ. ἴση δὲ ἡ ΕΒ τῇ ΓΔ. ἐστὶν ἄρα ὡς ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΔΓ πρὸς τὴν ΓΒ. μείζων δὲ ἡ ΒΔ τῆς ΔΓ· μείζων ἄρα καὶ ἡ ΔΓ τῆς ΓΒ. ἡ ΒΔ ἄρα εὐθεΐα ἄκρον καὶ μέσον λόγον τέτμηται [κατὰ τὸ Γ], καὶ τὸ μείζον τμήμα αὐτῆς ἐστὶν ἡ ΔΓ· ὁπερ ἔδει δεῖξαι.

found [Prop. 3.1], and let EB , EC , and ED have been joined, and let BE have been drawn across to A . Since BC is a side on an equilateral decagon, circumference ACB (is) thus five times circumference BC . Thus, circumference AC (is) four times CB . And as circumference AC (is) to CB , so angle AEC (is) to CEB [Prop. 6.33]. Thus, (angle) AEC (is) four times CEB . And since angle EBC (is) equal to ECB [Prop. 1.5], angle AEC is thus double ECB [Prop. 1.32]. And since straight-line EC is equal to CD —for each of them is equal to the side of the hexagon [inscribed] in circle ABC [Prop. 4.15 corr.]—angle CED is also equal to angle CDE [Prop. 1.5]. Thus, angle ECB (is) double EDC [Prop. 1.32]. But, AEC was shown (to be) double ECB . Thus, AEC (is) four times EDC . And AEC was also shown (to be) four times BEC . Thus, EDC (is) equal to BEC . And angle EBD (is) common to the two triangles BEC and BED . Thus, the remaining (angle) BED is equal to the (remaining angle) ECB [Prop. 1.32]. Thus, triangle EBD is equiangular to triangle EBC . Thus, proportionally, as DB is to BE , so EB (is) to BC [Prop. 6.4]. And EB (is) equal to CD . Thus, as BD is to DC , so DC (is) to CB . And BD (is) greater than DC . Thus, DC (is) also greater than CB [Prop. 5.14]. Thus, the straight-line BD has been cut in extreme and mean ratio [at C], and DC is its greater piece. (Which is), the very thing it was required to show.

† If the circle is of unit radius then the side of the hexagon is 1, whereas the side of the decagon is $(1/2)(\sqrt{5} - 1)$.

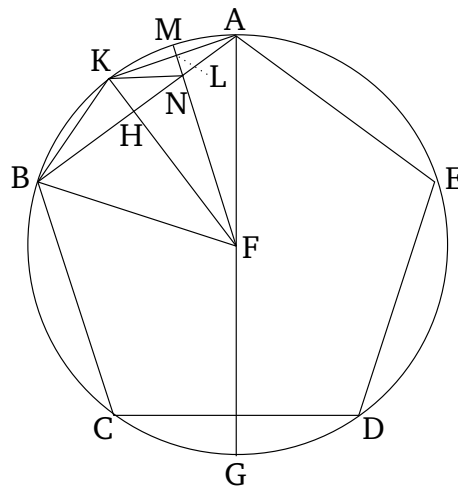
ι'.

Ἐὰν εἰς κύκλον πεντάγωνον ἰσόπλευρον ἐγγραφῇ, ἡ τοῦ πενταγώνου πλευρὰ δύναται τὴν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων.



Proposition 10

If an equilateral pentagon is inscribed in a circle then the square on the side of the pentagon is (equal to) the (sum of the squares) on the (sides) of the hexagon and of the decagon inscribed in the same circle.†



Ἐστω κύκλος ὁ $ABΓΔΕ$, καὶ εἰς τὸ $ABΓΔΕ$ κύκλον πεντάγωνον ἰσοπλευρον ἐγγεγράφω τὸ $ABΓΔΕ$. λέγω, ὅτι ἡ τοῦ $ABΓΔΕ$ πενταγώνου πλευρὰ δύναται τὴν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου πλευρὰν τῶν εἰς τὸν $ABΓΔΕ$ κύκλον ἐγγεγραμμένων.

Εἰλήφθω γάρ τὸ κέντρον τοῦ κύκλου τὸ Z σημεῖον, καὶ ἐπιζευχθεῖσα ἡ AZ διήχθω ἐπὶ τὸ H σημεῖον, καὶ ἐπεζεύχθω ἡ ZB , καὶ ἀπὸ τοῦ Z ἐπὶ τὴν AB κάθετος ἤχθω ἡ $ZΘ$, καὶ διήχθω ἐπὶ τὸ K , καὶ ἐπεζεύχθωσαν αἱ AK , KB , καὶ πάλιν ἀπὸ τοῦ Z ἐπὶ τὴν AK κάθετος ἤχθω ἡ $ZΛ$, καὶ διήχθω ἐπὶ τὸ M , καὶ ἐπεζεύχθω ἡ KN .

Ἐπεὶ ἴση ἐστὶν ἡ $ABΓH$ περιφέρεια τῇ $AEDH$ περιφέρειᾳ, ὧν ἡ $ABΓ$ τῇ AED ἐστὶν ἴση, λοιπὴ ἄρα ἡ $ΓH$ περιφέρεια λοιπῇ τῇ HD ἐστὶν ἴση. πενταγώνου δὲ ἡ $ΓΔ$ δεκαγώνου ἄρα ἡ $ΓH$. καὶ ἐπεὶ ἴση ἐστὶν ἡ ZA τῇ ZB , καὶ κάθετος ἡ $ZΘ$, ἴση ἄρα καὶ ἡ ὑπὸ AZK γωνία τῇ ὑπὸ KZB . ὥστε καὶ περιφέρεια ἡ AK τῇ KB ἐστὶν ἴση· διπλῇ ἄρα ἡ AB περιφέρεια τῆς BK περιφέρειας· δεκαγώνου ἄρα πλευρὰ ἐστὶν ἡ AK εὐθεῖα. διὰ τὰ αὐτὰ δὲ καὶ ἡ AK τῆς KM ἐστὶ διπλῇ. καὶ ἐπεὶ διπλῇ ἐστὶν ἡ AB περιφέρεια τῆς BK περιφέρειας, ἴση δὲ ἡ $ΓΔ$ περιφέρεια τῇ AB περιφέρειᾳ, διπλῇ ἄρα καὶ ἡ $ΓΔ$ περιφέρεια τῆς BK περιφέρειας. ἐστὶ δὲ ἡ $ΓΔ$ περιφέρεια καὶ τῆς $ΓH$ διπλῇ· ἴση ἄρα ἡ $ΓH$ περιφέρεια τῇ BK περιφέρειᾳ. ἀλλὰ ἡ BK τῆς KM ἐστὶ διπλῇ, ἐπεὶ καὶ ἡ KA · καὶ ἡ $ΓH$ ἄρα τῆς KM ἐστὶ διπλῇ. ἀλλὰ μὴν καὶ ἡ $ΓB$ περιφέρεια τῆς BK περιφέρειας ἐστὶ διπλῇ· ἴση γὰρ ἡ $ΓB$ περιφέρεια τῇ BA . καὶ ὅλη ἄρα ἡ HB περιφέρεια τῆς BM ἐστὶ διπλῇ· ὥστε καὶ γωνία ἡ ὑπὸ HZB γωνίας τῆς ὑπὸ BZM [ἐστὶ] διπλῇ. ἐστὶ δὲ ἡ ὑπὸ HZB καὶ τῆς ὑπὸ ZAB διπλῇ· ἴση γὰρ ἡ ὑπὸ ZAB τῇ ὑπὸ ABZ . καὶ ἡ ὑπὸ BZN ἄρα τῇ ὑπὸ ZAB ἐστὶν ἴση. κοινὴ δὲ τῶν δύο τριγώνων, τοῦ τε ABZ καὶ τοῦ BZN , ἡ ὑπὸ ABZ γωνία· λοιπὴ ἄρα ἡ ὑπὸ AZB λοιπῇ τῇ ὑπὸ BNZ ἐστὶν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ABZ τρίγωνον τῷ BZN τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ AB εὐθεῖα πρὸς τὴν BZ , οὕτως ἡ ZB πρὸς τὴν BN · τὸ ἄρα ὑπὸ τῶν ABN ἴσον ἐστὶ τῷ ἀπὸ BZ . πάλιν ἐπεὶ ἴση ἐστὶν ἡ AA τῇ AK , κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ AN , βάσις ἄρα ἡ KN βάσει τῇ AN ἐστὶν ἴση· καὶ γωνία ἄρα ἡ ὑπὸ AKN γωνία τῇ ὑπὸ LAN ἐστὶν ἴση. ἀλλὰ ἡ ὑπὸ LAN τῇ ὑπὸ KBN ἐστὶν ἴση· καὶ ἡ ὑπὸ AKN ἄρα τῇ ὑπὸ KBN ἐστὶν ἴση. καὶ κοινὴ τῶν δύο τριγώνων τοῦ τε AKB καὶ τοῦ AKN ἡ πρὸς τῷ A . λοιπὴ ἄρα ἡ ὑπὸ AKB λοιπῇ τῇ ὑπὸ KNA ἐστὶν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ KBA τρίγωνον τῷ KNA τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ BA εὐθεῖα πρὸς τὴν AK , οὕτως ἡ KA πρὸς τὴν AN · τὸ ἄρα ὑπὸ τῶν BAN ἴσον ἐστὶ τῷ ἀπὸ τῆς AK . ἐδείχθη δὲ καὶ τὸ ὑπὸ τῶν ABN ἴσον τῷ ἀπὸ τῆς BZ · τὸ ἄρα ὑπὸ τῶν ABN μετὰ τοῦ ὑπὸ BAN , ὅπερ ἐστὶ τὸ ἀπὸ τῆς BA , ἴσον ἐστὶ τῷ ἀπὸ τῆς BZ μετὰ τοῦ ἀπὸ τῆς AK . καὶ ἐστὶν ἡ μὲν BA πενταγώνου πλευρὰ, ἡ δὲ BZ ἑξαγώνου, ἡ δὲ AK δεκαγώνου.

Ἡ ἄρα τοῦ πενταγώνου πλευρὰ δύναται τὴν τε τοῦ

Let $ABCDE$ be a circle. And let the equilateral pentagon $ABCDE$ have been inscribed in circle $ABCDE$. I say that the square on the side of pentagon $ABCDE$ is the (sum of the squares) on the sides of the hexagon and of the decagon inscribed in circle $ABCDE$.

For let the center of the circle, point F , have been found [Prop. 3.1]. And, AF being joined, let it have been drawn across to point G . And let FB have been joined. And let FH have been drawn from F perpendicular to AB . And let it have been drawn across to K . And let AK and KB have been joined. And, again, let FL have been drawn from F perpendicular to AK . And let it have been drawn across to M . And let KN have been joined.

Since circumference $ABCG$ is equal to circumference $AEDG$, of which ABC is equal to AED , the remaining circumference CG is thus equal to the remaining (circumference) GD . And CD (is the side) of the pentagon. CG (is) thus (the side) of the decagon. And since FA is equal to FB , and FH is perpendicular (to AB), angle AFK (is) thus also equal to KFB [Props. 1.5, 1.26]. Hence, circumference AK is also equal to KB [Prop. 3.26]. Thus, circumference AB (is) double circumference BK . Thus, straight-line AK is the side of the decagon. So, for the same (reasons, circumference) AK is also double KM . And since circumference AB is double circumference BK , and circumference CD (is) equal to circumference AB , circumference CD (is) thus also double circumference BK . And circumference CD is also double CG . Thus, circumference CG (is) equal to circumference BK . But, BK is double KM , since KA (is) also (double KM). Thus, (circumference) CG is also double KM . But, indeed, circumference CB is also double circumference BK . For circumference CB (is) equal to BA . Thus, the whole circumference GB is also double BM . Hence, angle GFB [is] also double angle BFM [Prop. 6.33]. And GFB (is) also double FAB . For FAB (is) equal to ABF . Thus, BFN is also equal to FAB . And angle ABF (is) common to the two triangles ABF and BFN . Thus, the remaining (angle) AFB is equal to the remaining (angle) BNF [Prop. 1.32]. Thus, triangle ABF is equiangular to triangle BFN . Thus, proportionally, as straight-line AB (is) to BF , so FB (is) to BN [Prop. 6.4]. Thus, the (rectangle contained) by ABN is equal to the (square) on BF [Prop. 6.17]. Again, since AL is equal to LK , and LN is common and at right-angles (to KA), base KN is thus equal to base AN [Prop. 1.4]. And, thus, angle LKN is equal to angle LAN . But, LAN is equal to KBN [Props. 3.29, 1.5]. Thus, LKN is also equal to KBN . And the (angle) at A (is) common to the two triangles AKB and AKN . Thus, the remaining (angle) AKB is

ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγεγραμμένων· ὅπερ ἔδει δεῖξαι.

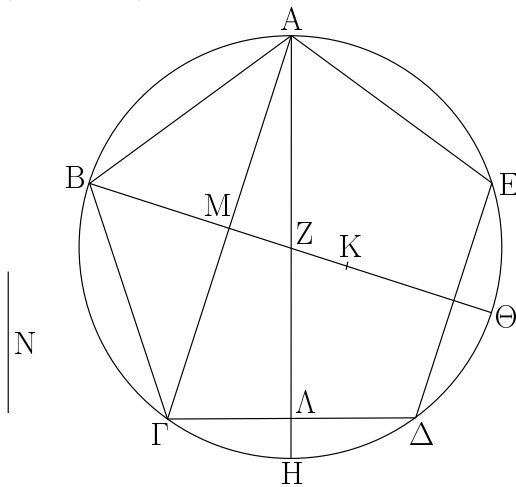
equal to the remaining (angle) KNA [Prop. 1.32]. Thus, triangle KBA is equiangular to triangle KNA . Thus, proportionally, as straight-line BA is to AK , so KA (is) to AN [Prop. 6.4]. Thus, the (rectangle contained) by BAN is equal to the (square) on AK [Prop. 6.17]. And the (rectangle contained) by ABN was also shown (to be) equal to the (square) on BF . Thus, the (rectangle contained) by ABN plus the (rectangle contained) by BAN , which is the (square) on BA [Prop. 2.2], is equal to the (square) on BF plus the (square) on AK . And BA is the side of the pentagon, and BF (the side) of the hexagon [Prop. 4.15 corr.], and AK (the side) of the decagon.

Thus, the square on the side of the pentagon (inscribed in a circle) is (equal to) the (sum of the squares) on the (sides) of the hexagon and of the decagon inscribed in the same circle.

† If the circle is of unit radius then the side of the pentagon is $(1/2) \sqrt{10 - 2\sqrt{5}}$.

ια'.

Ἐάν εἰς κύκλον ῥητὴν ἔχοντα τὴν διάμετρον πεντάγωνον ἰσόπλευρον ἐγγεγράφῃ, ἡ τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἢ καλουμένη ἐλάσσων.

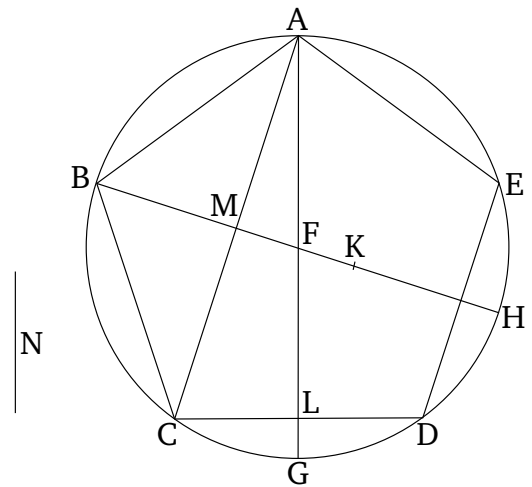


Εἰς γὰρ κύκλον τὸν $ABΓΔE$ ῥητὴν ἔχοντα τὴν διάμετρον πεντάγωνον ἰσόπλευρον ἐγγεγράφθω τὸ $ABΓΔE$. λέγω, ὅτι ἡ τοῦ $[ABΓΔE]$ πενταγώνου πλευρὰ ἄλογός ἐστιν ἢ καλουμένη ἐλάσσων.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ Z σημεῖον, καὶ ἐπεζεύχθωσαν αἱ AZ , ZB καὶ διήχθωσαν ἐπὶ τὰ H , Θ σημεῖα, καὶ ἐπεζεύχθω ἡ $ΑΓ$, καὶ κείσθω τῆς AZ τέταρτον μέρος ἡ ZK . ῥητὴ δὲ ἡ AZ . ῥητὴ ἄρα καὶ ἡ ZK . ἔστι δὲ καὶ ἡ BZ ῥητὴ· ὅλη ἄρα ἡ BK ῥητὴ ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ $ΑΓΗ$ περιφέρεια τῇ $ΑΔΗ$ περιφέρειᾳ, ὥν ἡ $ΑΒΓ$ τῇ $ΑΕΔ$ ἐστὶν ἴση, λοιπὴ ἄρα ἡ $ΓΗ$ λοιπὴ τῇ $ΗΔ$ ἐστὶν ἴση. καὶ ἐὰν ἐπιζεύξωμεν τὴν $ΑΔ$, συνάγονται ὀρθαὶ αἱ

Proposition 11

If an equilateral pentagon is inscribed in a circle which has a rational diameter then the side of the pentagon is that irrational (straight-line) called minor.



For let the equilateral pentagon $ABCDE$ have been inscribed in the circle $ABCDE$ which has a rational diameter. I say that the side of pentagon $[ABCDE]$ is that irrational (straight-line) called minor.

For let the center of the circle, point F , have been found [Prop. 3.1]. And let AF and FB have been joined. And let them have been drawn across to points G and H (respectively). And let AC have been joined. And let FK made (equal) to the fourth part of AF . And AF (is) rational. FK (is) thus also rational. And BF is also rational. Thus, the whole of BK is rational. And since circumference ACG is equal to circumference ADG , of which

πρὸς τῷ Λ γωνίαι, καὶ διπλῇ ἢ $\Gamma\Delta$ τῆς $\Gamma\Lambda$. διὰ τὰ αὐτὰ δὴ καὶ αἱ πρὸς τῷ M ὀρθαί εἰσιν, καὶ διπλῇ ἢ $\Lambda\Gamma$ τῆς ΓM . ἐπεὶ οὖν ἴση ἐστὶν ἡ ὑπὸ $\Lambda\Lambda\Gamma$ γωνία τῇ ὑπὸ ΛMZ , κοινὴ δὲ τῶν δύο τριγώνων τοῦ τε $\Lambda\Gamma\Lambda$ καὶ τοῦ ΛMZ ἡ ὑπὸ $\Lambda\Lambda\Gamma$, λοιπὴ ἄρα ἡ ὑπὸ $\Lambda\Gamma\Lambda$ λοιπῇ τῇ ὑπὸ MZA ἐστὶν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ $\Lambda\Gamma\Lambda$ τρίγωνον τῷ ΛMZ τριγώνῳ· ἀνάλογον ἄρα ἐστὶν ὡς ἡ $\Lambda\Gamma$ πρὸς $\Gamma\Lambda$, οὕτως ἡ MZ πρὸς ZA · καὶ τῶν ἡγουμένων τὰ διπλάσια· ὡς ἄρα ἡ τῆς $\Lambda\Gamma$ διπλῇ πρὸς τὴν $\Gamma\Lambda$, οὕτως ἡ τῆς MZ διπλῇ πρὸς τὴν ZA . ὡς δὲ ἡ τῆς MZ διπλῇ πρὸς τὴν ZA , οὕτως ἡ MZ πρὸς τὴν ἡμίσειαν τῆς ZA · καὶ ὡς ἄρα ἡ τῆς $\Lambda\Gamma$ διπλῇ πρὸς τὴν $\Gamma\Lambda$, οὕτως ἡ MZ πρὸς τὴν ἡμίσειαν τῆς ZA · καὶ τῶν ἐπομένων τὰ ἡμίσεια· ὡς ἄρα ἡ τῆς $\Lambda\Gamma$ διπλῇ πρὸς τὴν ἡμίσειαν τῆς $\Gamma\Lambda$, οὕτως ἡ MZ πρὸς τὸ τέτατρον τῆς ZA . καὶ ἐστὶ τῆς μὲν $\Lambda\Gamma$ διπλῇ ἢ $\Delta\Gamma$, τῆς δὲ $\Gamma\Lambda$ ἡμίσεια ἢ ΓM , τῆς δὲ ZA τέτατρον μέρος ἢ ZK · ἐστὶν ἄρα ὡς ἡ $\Delta\Gamma$ πρὸς τὴν ΓM , οὕτως ἡ MZ πρὸς τὴν ZK . συνθέντι καὶ ὡς συναμφοτέρος ἢ $\Delta\Gamma M$ πρὸς τὴν ΓM , οὕτως ἢ MK πρὸς KZ · καὶ ὡς ἄρα τὸ ἀπὸ συναμφοτέρου τῆς $\Delta\Gamma M$ πρὸς τὸ ἀπὸ ΓM , οὕτως τὸ ἀπὸ MK πρὸς τὸ ἀπὸ KZ . καὶ ἐπεὶ τῆς ὑπὸ δύο πλευρᾶς τοῦ πενταγώνου ὑποτείνουσας, οἷον τῆς $\Lambda\Gamma$, ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζον τμήμα ἴσον ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ, τουτέστι τῇ $\Delta\Gamma$, τὸ δὲ μείζον τμήμα προσλαβὼν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τῆς ὅλης, καὶ ἐστὶν ὅλης τῆς $\Lambda\Gamma$ ἡμίσεια ἢ ΓM , τὸ ἄρα ἀπὸ τῆς $\Delta\Gamma M$ ὡς μιᾶς πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΓM . ὡς δὲ τὸ ἀπὸ τῆς $\Delta\Gamma M$ ὡς μιᾶς πρὸς τὸ ἀπὸ τῆς ΓM , οὕτως ἐδείχθη τὸ ἀπὸ τῆς MK πρὸς τὸ ἀπὸ τῆς KZ · πενταπλάσιον ἄρα τὸ ἀπὸ τῆς MK τοῦ ἀπὸ τῆς KZ . ῥητὸν δὲ τὸ ἀπὸ τῆς KZ · ῥητὴ γὰρ ἡ διάμετρος· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς MK · ῥητὴ ἄρα ἐστὶν ἡ MK [δυνάμει μόνον]. καὶ ἐπεὶ τετραπλάσια ἐστὶν ἡ BZ τῆς ZK , πενταπλάσια ἄρα ἐστὶν ἡ BK τῆς KZ · εἰκοσιπενταπλάσιον ἄρα τὸ ἀπὸ τῆς BK τοῦ ἀπὸ τῆς KZ . πενταπλάσιον δὲ τὸ ἀπὸ τῆς MK τοῦ ἀπὸ τῆς KZ · πενταπλάσιον ἄρα τὸ ἀπὸ τῆς BK τοῦ ἀπὸ τῆς KM · τὸ ἄρα ἀπὸ τῆς BK πρὸς τὸ ἀπὸ KM λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· ἀσύμμετρος ἄρα ἐστὶν ἡ BK τῇ KM μήκει. καὶ ἐστὶ ῥητὴ ἑκατέρα αὐτῶν. αἱ BK , KM ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ἐὰν δὲ ἀπὸ ῥητῆς ῥητὴ ἀφαιρεθῇ δυνάμει μόνον σύμμετρος οὕσα τῇ ὅλῃ, ἡ λοιπὴ ἄλογός ἐστιν ἀποτομή· ἀποτομή ἄρα ἐστὶν ἡ MB , προσαρμόζουσα δὲ αὐτῇ ἡ MK . λέγω δὴ, ὅτι καὶ τετάρτη. ὥ δὴ μείζον ἐστὶ τὸ ἀπὸ τῆς BK τοῦ ἀπὸ τῆς KM , ἐκείνῳ ἴσον ἔστω τὸ ἀπὸ τῆς N · ἢ BK ἄρα τῆς KM μείζον δύναται τῇ N . καὶ ἐπεὶ σύμμετρός ἐστὶν ἡ KZ τῇ ZB , καὶ συνθέντι σύμμετρός ἐστὶ ἡ KB τῇ ZB . ἀλλὰ ἡ BZ τῇ $B\Theta$ σύμμετρός ἐστιν· καὶ ἡ BK ἄρα τῇ $B\Theta$ σύμμετρός ἐστιν. καὶ ἐπεὶ πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς BK τοῦ ἀπὸ τῆς KM , τὸ ἄρα ἀπὸ τῆς BK πρὸς τὸ ἀπὸ τῆς KM λόγον ἔχει, ὃν ϵ πρὸς ϵ . ἀναστρέψαντι ἄρα τὸ ἀπὸ τῆς BK πρὸς τὸ ἀπὸ τῆς N λόγον ἔχει, ὃν ϵ πρὸς

ABC is equal to AED , the remainder CG is thus equal to the remainder GD . And if we join AD then the angles at L are inferred (to be) right-angles, and CD (is inferred to be) double CL [Prop. 1.4]. So, for the same (reasons), the (angles) at M are also right-angles, and AC (is) double CM . Therefore, since angle ALC (is) equal to AMF , and (angle) LAC (is) common to the two triangles ACL and AMF , the remaining (angle) ACL is thus equal to the remaining (angle) MFA [Prop. 1.32]. Thus, triangle ACL is equiangular to triangle AMF . Thus, proportionally, as LC (is) to CA , so MF (is) to FA [Prop. 6.4]. And (we can take) the doubles of the leading (magnitudes). Thus, as double LC (is) to CA , so double MF (is) to FA . And as double MF (is) to FA , so MF (is) to half of FA . And, thus, as double LC (is) to CA , so MF (is) to half of FA . And (we can take) the halves of the following (magnitudes). Thus, as double LC (is) to half of CA , so MF (is) to the fourth of FA . And DC is double LC , and CM half of CA , and FK the fourth part of FA . Thus, as DC is to CM , so MF (is) to FK . Via composition, as the sum of DCM (i.e., DC and CM) (is) to CM , so MK (is) to KF [Prop. 5.18]. And, thus, as the (square) on the sum of DCM (is) to the (square) on CM , so the (square) on MK (is) to the (square) on KF . And since the greater piece of a (straight-line) subtending two sides of a pentagon, such as AC , (which is) cut in extreme and mean ratio is equal to the side of the pentagon [Prop. 13.8]—that is to say, to DC —and the square on the greater piece added to half of the whole is five times the (square) on half of the whole [Prop. 13.1], and CM (is) half of the whole, AC , thus the (square) on DCM , (taken) as one, is five times the (square) on CM . And the (square) on DCM , (taken) as one, (is) to the (square) on CM , so the (square) on MK was shown (to be) to the (square) on KF . Thus, the (square) on MK (is) five times the (square) on KF . And the square on KF (is) rational. For the diameter (is) rational. Thus, the (square) on MK (is) also rational. Thus, MK is rational [in square only]. And since BF is four times FK , BK is thus five times KF . Thus, the (square) on BK (is) twenty-five times the (square) on KF . And the (square) on MK (is) five times the square on KF . Thus, the (square) on BK (is) five times the (square) on KM . Thus, the (square) on BK does not have to the (square) on KM the ratio which a square number (has) to a square number. Thus, BK is incommensurable in length with KM [Prop. 10.9]. And each of them is a rational (straight-line). Thus, BK and KM are rational (straight-lines which are) commensurable in square only. And if from a rational (straight-line) a rational (straight-line) is subtracted, which is commensurable in square only with the

$\bar{\delta}$, οὐχ ὃν τετράγωνος πρὸς τετράγωνον· ἀσύμμετρος ἄρα ἐστὶν ἡ BK τῇ N · ἡ BK ἄρα τῆς KM μείζον δύναται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ. ἐπεὶ οὖν ὅλη ἡ BK τῆς προσαρμοζούσης τῆς KM μείζον δύναται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ, καὶ ὅλη ἡ BK σύμμετρός ἐστι τῇ ἐκκειμένη ῥητῇ τῇ $B\Theta$, ἀποτομῇ ἄρα τετάρτη ἐστὶν ἡ MB . τὸ δὲ ὑπὸ ῥητῆς καὶ ἀποτομῆς τετάρτης περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν, καλεῖται δὲ ἐλάττων. δύναται δὲ τὸ ὑπὸ τῶν ΘBM ἢ AB διὰ τὸ ἐπιζευγνυμένης τῆς $A\Theta$ ἰσογώνιον γίνεσθαι τὸ $AB\Theta$ τρίγωνον τῷ ABM τριγώνῳ καὶ εἶναι ὡς τὴν ΘB πρὸς τὴν BA , οὕτως τὴν AB πρὸς τὴν BM .

Ἡ ἄρα AB τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων· ὁπερ ἔδει δεῖξαι.

whole, then the remainder is that irrational (straight-line called) an apotome [Prop. 10.73]. Thus, MB is an apotome, and MK its attachment. So, I say that (it is) also a fourth (apotome). So, let the (square) on N be (made) equal to that (magnitude) by which the (square) on BK is greater than the (square) on KM . Thus, the square on BK is greater than the (square) on KM by the (square) on N . And since KF is commensurable (in length) with FB then, via composition, KB is also commensurable (in length) with FB [Prop. 10.15]. But, BF is commensurable (in length) with BH . Thus, BK is also commensurable (in length) with BH [Prop. 10.12]. And since the (square) on BK is five times the (square) on KM , the (square) on BK thus has to the (square) on KM the ratio which 5 (has) to one. Thus, via conversion, the (square) on BK has to the (square) on N the ratio which 5 (has) to 4 [Prop. 5.19 corr.], which is not (that) of a square (number) to a square (number). BK is thus incommensurable (in length) with N [Prop. 10.9]. Thus, the square on BK is greater than the (square) on KM by the (square) on (some straight-line which is) incommensurable (in length) with (BK) . Therefore, since the square on the whole, BK , is greater than the (square) on the attachment, KM , by the (square) on (some straight-line which is) incommensurable (in length) with (BK) , and the whole, BK , is commensurable (in length) with the (previously) laid down rational (straight-line) BH , MB is thus a fourth apotome [Def. 10.14]. And the rectangle contained by a rational (straight-line) and a fourth apotome is irrational, and its square-root is that irrational (straight-line) called minor [Prop. 10.94]. And the square on AB is the rectangle contained by HBM , on account of joining AH , (so that) triangle ABH becomes equiangular with triangle ABM [Prop. 6.8], and (proportionally) as HB is to BA , so AB (is) to BM .

Thus, the side AB of the pentagon is that irrational (straight-line) called minor.[†] (Which is) the very thing it was required to show.

[†] If the circle has unit radius then the side of the pentagon is $(1/2)\sqrt{10 - 2\sqrt{5}}$. However, this length can be written in the “minor” form (see Prop. 10.94) $(\rho/\sqrt{2})\sqrt{1 + k/\sqrt{1 + k^2}} - (\rho/\sqrt{2})\sqrt{1 - k/\sqrt{1 + k^2}}$, with $\rho = \sqrt{5}/2$ and $k = 2$.

ιβ'.

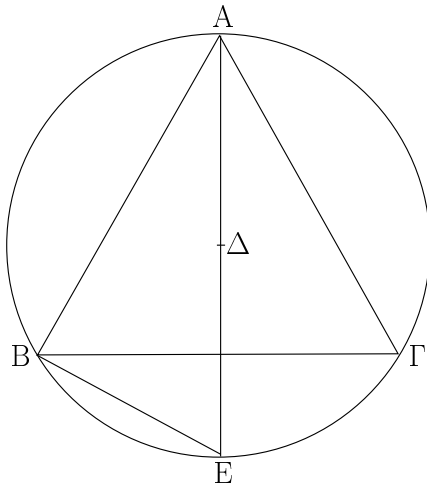
Ἐὰν εἰς κύκλον τρίγωνον ἰσόπλευρον ἐγγραφῇ, ἡ τοῦ τριγώνου πλευρὰ δυνάμει τριπλασίῳ ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ κύκλου.

Ἐστω κύκλος ὁ $AB\Gamma$, καὶ εἰς αὐτὸν τρίγωνον ἰσόπλευρον ἐγγεγράφῃ τὸ $AB\Gamma$. λέγω, ὅτι τοῦ $AB\Gamma$ τριγώνου μία πλευρὰ δυνάμει τριπλασίῳ ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ $AB\Gamma$ κύκλου.

Proposition 12

If an equilateral triangle is inscribed in a circle then the square on the side of the triangle is three times the (square) on the radius of the circle.

Let there be a circle ABC , and let the equilateral triangle ABC have been inscribed in it [Prop. 4.2]. I say that the square on one side of triangle ABC is three times the (square) on the radius of circle ABC .



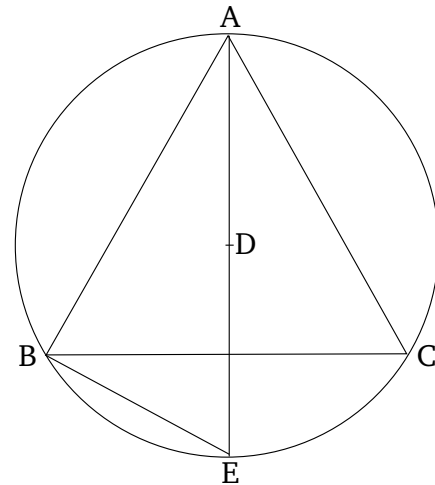
Εἰλήφθω γὰρ τὸ κέντρον τοῦ $AB\Gamma$ κύκλου τὸ Δ , καὶ ἐπιζευχθεῖσα ἡ $A\Delta$ διήχθω ἐπὶ τὸ E , καὶ ἐπεζεύχθω ἡ BE .

Καὶ ἐπεὶ ἰσόπλευρόν ἐστι τὸ $AB\Gamma$ τρίγωνον, ἡ BEG ἄρα περιφέρεια τρίτον μέρος ἐστὶ τῆς τοῦ $AB\Gamma$ κύκλου περιφέρειας. ἡ ἄρα BE περιφέρεια ἕκτον ἐστὶ μέρος τῆς τοῦ κύκλου περιφέρειας· ἐξαγώνου ἄρα ἐστὶν ἡ BE εὐθεῖα· ἴση ἄρα ἐστὶ τῇ ἐκ τοῦ κέντρου τῇ ΔE . καὶ ἐπεὶ διπλῇ ἐστὶν ἡ AE τῆς ΔE , τετραπλάσιον ἐστὶ τὸ ἀπὸ τῆς AE τοῦ ἀπὸ τῆς $E\Delta$, τούτέστι τοῦ ἀπὸ τῆς BE . ἴσον δὲ τὸ ἀπὸ τῆς AE τοῖς ἀπὸ τῶν AB , BE · τὰ ἄρα ἀπὸ τῶν AB , BE τετραπλάσιά ἐστι τοῦ ἀπὸ τῆς BE . διελόντι ἄρα τὸ ἀπὸ τῆς AB τριπλάσιόν ἐστι τοῦ ἀπὸ BE . ἴση δὲ ἡ BE τῇ ΔE · τὸ ἄρα ἀπὸ τῆς AB τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΔE .

Ἡ ἄρα τοῦ τριγώνου πλευρὰ δυνάμει τριπλασία ἐστὶ τῆς ἐκ τοῦ κέντρου [τοῦ κύκλου]· ὅπερ ἔδει δεῖξαι.

ιγ'.

Πυραμίδα συστήσασθαι καὶ σφαῖρα περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει ἡμιολία ἐστὶ τῆς πλευρᾶς τῆς πυραμίδος.



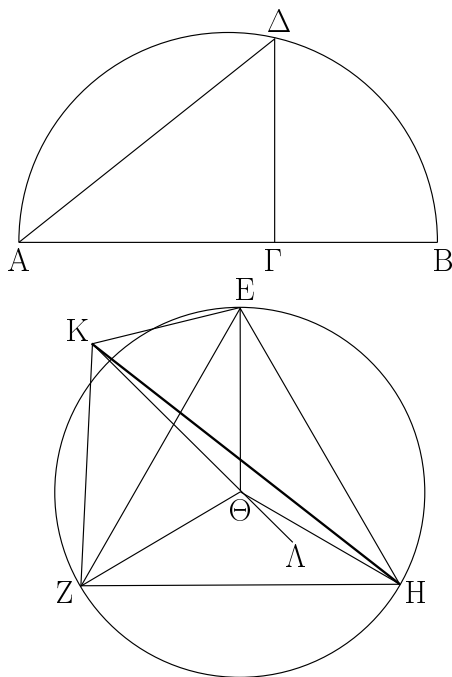
For let the center, D , of circle ABC have been found [Prop. 3.1]. And AD (being) joined, let it have been drawn across to E . And let BE have been joined.

And since triangle ABC is equilateral, circumference BEC is thus the third part of the circumference of circle ABC . Thus, circumference BE is the sixth part of the circumference of the circle. Thus, straight-line BE is (the side) of a hexagon. Thus, it is equal to the radius DE [Prop. 4.15 corr.]. And since AE is double DE , the (square) on AE is four times the (square) on ED —that is to say, of the (square) on BE . And the (square) on AE (is) equal to the (sum of the squares) on AB and BE [Props. 3.31, 1.47]. Thus, the (sum of the squares) on AB and BE is four times the (square) on BE . Thus, via separation, the (square) on AB is three times the (square) on BE . And BE (is) equal to DE . Thus, the (square) on AB is three times the (square) on DE .

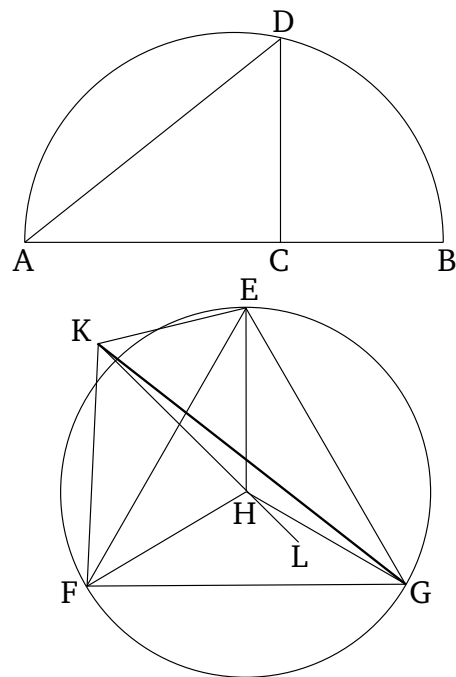
Thus, the square on the side of the triangle is three times the (square) on the radius [of the circle]. (Which is) the very thing it was required to show.

Proposition 13

To construct a (regular) pyramid (i.e., a tetrahedron), and to enclose (it) in a given sphere, and to show that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.



Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ AB , καὶ τετμήσθω κατὰ τὸ Γ σημεῖον, ὥστε διπλασίαν εἶναι τὴν AG τῆς GB , καὶ γεγράφθω ἐπὶ τῆς AB ἡμικύκλιον τὸ ADB , καὶ ἤχθω ἀπὸ τοῦ Γ σημείου τῇ AB πρὸς ὀρθὰς ἡ ΓD , καὶ ἐπεζεύχθω ἡ ΔA , καὶ ἐκκείσθω κύκλος ὁ EZH ἴσην ἔχων τὴν ἐκ τοῦ κέντρου τῇ $\Delta\Gamma$, καὶ ἐγγεγράφθω εἰς τὸν EZH κύκλον τρίγωνον ἰσοπλευρον τὸ EZH , καὶ εἰλήφθω τὸ κέντρον τοῦ κύκλου τὸ Θ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ $E\Theta$, ΘZ , ΘH , καὶ ἀνεστάτω ἀπὸ τοῦ Θ σημείου τῷ τοῦ EZH κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΘK , καὶ ἀφηρήσθω ἀπὸ τῆς ΘK τῇ AG εὐθείᾳ ἴση ἡ ΘK , καὶ ἐπεζεύχθωσαν αἱ KE , KZ , KH . καὶ ἐπεὶ ἡ $K\Theta$ ὀρθὴ ἐστὶ πρὸς τὸ τοῦ EZH κύκλου ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὐσας ἐν τῷ τοῦ EZH κύκλου ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἄπτεται δὲ αὐτῆς ἐκάστη τῶν ΘE , ΘZ , ΘH , ἡ ΘK ἄρα πρὸς ἐκάστη τῶν ΘE , ΘZ , ΘH ὀρθὴ ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν AG τῇ ΘK , ἡ δὲ ΓD τῇ ΘE , καὶ ὀρθὰς γωνίας περιέχουσιν, βάσις ἄρα ἡ ΔA βάσει τῇ KE ἐστὶν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἐκατέρω τῶν KZ , KH τῇ ΔA ἐστὶν ἴση· αἱ τρεῖς ἄρα αἱ KE , KZ , KH ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ διπλὴ ἐστὶν ἡ AG τῆς GB , τριπλὴ ἄρα ἡ AB τῆς BF . ὥς δὲ ἡ AB πρὸς τὴν BF , οὕτως τὸ ἀπὸ τῆς AD πρὸς τὸ ἀπὸ τῆς $\Delta\Gamma$, ὥς ἐξῆς δειχθήσεται. τριπλάσιον ἄρα τὸ ἀπὸ τῆς AD τοῦ ἀπὸ τῆς $\Delta\Gamma$. ἐστὶ δὲ καὶ τὸ ἀπὸ τῆς ZE τοῦ ἀπὸ τῆς $E\Theta$ τριπλάσιον, καὶ ἐστὶν ἴση ἡ $\Delta\Gamma$ τῇ $E\Theta$, ἴση ἄρα καὶ ἡ ΔA τῇ EZ . ἀλλὰ ἡ ΔA ἐκάστη τῶν KE , KZ , KH ἐδείχθη ἴση· καὶ ἐκάστη ἄρα τῶν EZ , ZH , HE ἐκάστη τῶν KE , KZ , KH ἐστὶν ἴση· ἰσοπλευρὰ ἄρα ἐστὶ τὰ τέσσαρα τρίγωνα τὰ EZH , KEZ , KZH , KEH . πυραμὶς ἄρα συνέσταται ἐκ τεσσάρων τριγώνων ἰσοπλεύρων, ἧς βάσις μὲν ἐστὶ τὸ EZH τρίγωνον,



Let the diameter AB of the given sphere be laid out, and let it have been cut at point C such that AC is double CB [Prop. 6.10]. And let the semi-circle ADB have been drawn on AB . And let CD have been drawn from point C at right-angles to AB . And let DA have been joined. And let the circle EFG be laid down having a radius equal to DC , and let the equilateral triangle EFG have been inscribed in circle EFG [Prop. 4.2]. And let the center of the circle, point H , have been found [Prop. 3.1]. And let EH , HF , and HG have been joined. And let HK have been set up, at point H , at right-angles to the plane of circle EFG [Prop. 11.12]. And let HK , equal to the straight-line AC , have been cut off from HK . And let KE , KF , and KG have been joined. And since KH is at right-angles to the plane of circle EFG , it will thus also make right-angles with all of the straight-lines joining it (which are) also in the plane of circle EFG [Def. 11.3]. And HE , HF , and HG each join it. Thus, HK is at right-angles to each of HE , HF , and HG . And since AC is equal to HK , and CD to HE , and they contain right-angles, the base DA is thus equal to the base KE [Prop. 1.4]. So, for the same (reasons), KF and KG is each equal to DA . Thus, the three (straight-lines) KE , KF , and KG are equal to one another. And since AC is double CB , AB (is) thus triple BC . And as AB (is) to BC , so the (square) on AD (is) to the (square) on DC , as will be shown later [see lemma]. Thus, the (square) on AD (is) three times the (square) on DC . And the (square) on FE is also three times the (square) on EH [Prop. 13.12], and DC is equal to EH . Thus, DA (is)

κορυφή δὲ τὸ Κ σημεῖον.

Δεῖ δὴ αὐτὴν καὶ σφαῖρα περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ δυνάμει τῆς πλευρᾶς τῆς πυραμίδος.

Ἐκβεβλήσθω γὰρ ἐπ' εὐθείας τῇ ΚΘ εὐθεία ἡ ΘΑ, καὶ κείσθω τῇ ΓΒ ἴση ἡ ΘΑ. καὶ ἐπεὶ ἐστὶν ὡς ἡ ΑΓ πρὸς τὴν ΓΔ, οὕτως ἡ ΓΔ πρὸς τὴν ΓΒ, ἴση δὲ ἡ μὲν ΑΓ τῇ ΚΘ, ἡ δὲ ΓΔ τῇ ΘΕ, ἡ δὲ ΓΒ τῇ ΘΑ, ἔστιν ἄρα ὡς ἡ ΚΘ πρὸς τὴν ΘΕ, οὕτως ἡ ΕΘ πρὸς τὴν ΘΑ· τὸ ἄρα ὑπὸ τῶν ΚΘ, ΘΑ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ. καὶ ἐστὶν ὀρθὴ ἑκατέρα τῶν ὑπὸ ΚΘΕ, ΕΘΑ γωνιῶν· τὸ ἄρα ἐπὶ τῆς ΚΑ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Ε [ἐπειδὴ περ ἐὰν ἐπιζευζόμεν τὴν ΕΑ, ὀρθὴ γίνεται ἡ ὑπὸ ΑΕΚ γωνία διὰ τὸ ἰσογώνιον γίνεσθαι τὸ ΕΑΚ τρίγωνον ἑκατέρω τῶν ΕΑΘ, ΕΘΚ τριγώνων]. ἐὰν δὴ μενούσης τῆς ΚΑ περιεγεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἥξει καὶ διὰ τῶν Ζ, Η σημείων ἐπιζευγνυμένων τῶν ΖΑ, ΔΗ καὶ ὀρθῶν ὁμοίως γινομένων τῶν πρὸς τοῖς Ζ, Η γωνιῶν· καὶ ἔσται ἡ πυραμὶς σφαῖρα περιειλημμένη τῇ δοθείσῃ. ἡ γὰρ ΚΑ τῆς σφαίρας διάμετρος ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμετρῷ τῇ ΑΒ, ἐπειδὴ περ τῇ μὲν ΑΓ ἴση κεῖται ἡ ΚΘ, τῇ δὲ ΓΒ ἡ ΘΑ.

Λέγω δὴ, ὅτι ἡ τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ δυνάμει τῆς πλευρᾶς τῆς πυραμίδος.

Ἐπεὶ γὰρ διπλὴ ἐστὶν ἡ ΑΓ τῆς ΓΒ, τριπλὴ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΓ· ἀναστρέψαντι ἡμιολία ἄρα ἐστὶν ἡ ΒΑ τῆς ΑΓ. ὡς δὲ ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΔ [ἐπειδὴ περ ἐπιζευγνύμενης τῆς ΔΒ ἐστὶν ὡς ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΔΑ πρὸς τὴν ΑΓ διὰ τὴν ὁμοιότητα τῶν ΔΑΒ, ΔΑΓ τριγώνων, καὶ εἶναι ὡς τὴν πρώτην πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας]. ἡμιόλιον ἄρα καὶ τὸ ἀπὸ τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΔ. καὶ ἐστὶν ἡ μὲν ΒΑ ἡ τῆς δοθείσης σφαίρας διάμετρος, ἡ δὲ ΑΔ ἴση τῇ πλευρᾷ τῆς πυραμίδος.

Ἡ ἄρα τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ τῆς πλευρᾶς τῆς πυραμίδος· ὅπερ ἔδει δεῖξαι.

also equal to EF . But, DA was shown (to be) equal to each of KE , KF , and KG . Thus, EF , FG , and GE are equal to KE , KF , and KG , respectively. Thus, the four triangles EFG , KEF , KFG , and KEG are equilateral. Thus, a pyramid, whose base is triangle EFG , and apex the point K , has been constructed from four equilateral triangles.

So, it is also necessary to enclose it in the given sphere, and to show that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

For let the straight-line HL have been produced in a straight-line with KH , and let HL be made equal to CB . And since as AC (is) to CD , so CD (is) to CB [Prop. 6.8 corr.], and AC (is) equal to KH , and CD to HE , and CB to HL , thus as KH is to HE , so EH (is) to HL . Thus, the (rectangle contained) by KH and HL is equal to the (square) on EH [Prop. 6.17]. And each of the angles KHE and EHL is a right-angle. Thus, the semi-circle drawn on KL will also pass through E [inasmuch as if we join EL then the angle LEK becomes a right-angle, on account of triangle ELK becoming equiangular to each of the triangles ELH and EHK [Props. 6.8, 3.31]]. So, if KL remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, it will also pass through points F and G , (because) if FL and LG are joined, the angles at F and G will similarly become right-angles. And the pyramid will have been enclosed by the given sphere. For the diameter, KL , of the sphere is equal to the diameter, AB , of the given sphere—inasmuch as KH was made equal to AC , and HL to CB .

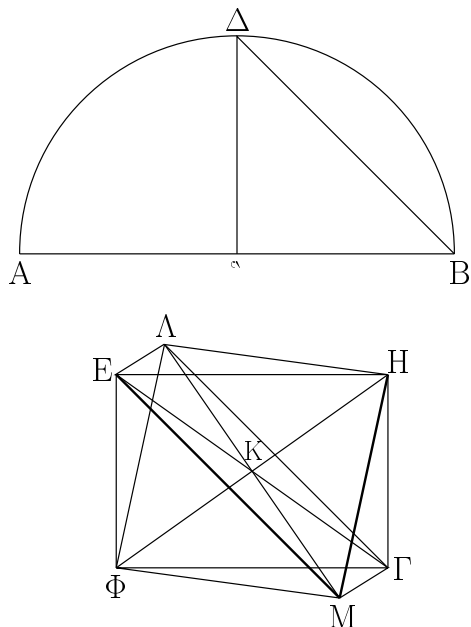
So, I say that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

For since AC is double CB , AB is thus triple BC . Thus, via conversion, BA is one and a half times AC . And as BA (is) to AC , so the (square) on BA (is) to the (square) on AD [inasmuch as if DB is joined then as BA is to AD , so DA (is) to AC , on account of the similarity of triangles DAB and DAC . And as the first is to the third (of four proportional magnitudes), so the (square) on the first (is) to the (square) on the second.] Thus, the (square) on BA (is) also one and a half times the (square) on AD . And BA is the diameter of the given sphere, and AD (is) equal to the side of the pyramid.

Thus, the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.[†] (Which is) the very thing it was required to show.

[†] If the radius of the sphere is unity then the side of the pyramid (i.e., tetrahedron) is $\sqrt{8/3}$.

ἐπεξεύχθωσαν αἱ ΘΖ, ΕΗ, καὶ ἀνεστάτω ἀπὸ τοῦ Κ σημείου τῷ τοῦ ΕΖΗΘ τετραγώνου ἐπιπέδῳ πρὸς ὀρθὰς εὐθεΐα ἡ ΚΛ καὶ διήχθω ἐπὶ τὰ ἕτερα μέρη τοῦ ἐπιπέδου ὡς ἡ ΚΜ, καὶ ἀφηρήσθω ἀφ' ἑκατέρας τῶν ΚΛ, ΚΜ μιᾶ τῶν ΕΚ, ΖΚ, ΗΚ, ΘΚ ἴση ἑκατέρα τῶν ΚΛ, ΚΜ, καὶ ἐπεξεύχθωσαν αἱ ΑΕ, ΑΖ, ΑΗ, ΑΘ, ΜΕ, ΜΖ, ΜΗ, ΜΘ.

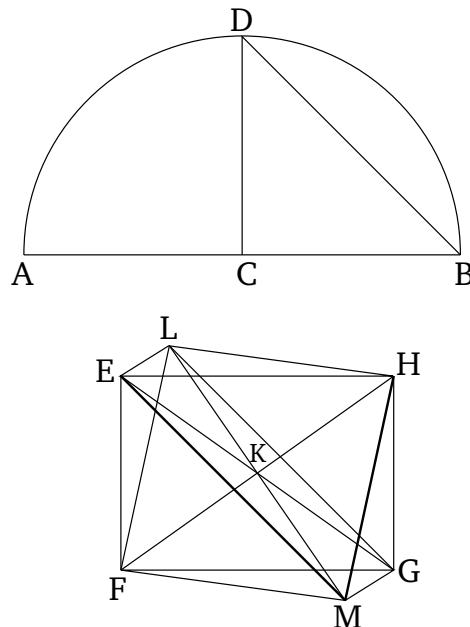


Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΚΕ τῇ ΚΘ, καὶ ἐστὶν ὀρθὴ ἡ ὑπὸ ΕΚΘ γωνία, τὸ ἄρα ἀπὸ τῆς ΘΕ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΚ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΑΚ τῇ ΚΕ, καὶ ἐστὶν ὀρθὴ ἡ ὑπὸ ΑΚΕ γωνία, τὸ ἄρα ἀπὸ τῆς ΕΑ διπλάσιόν ἐστι τοῦ ἀπὸ ΕΚ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΘΕ διπλάσιον τοῦ ἀπὸ τῆς ΕΚ· τὸ ἄρα ἀπὸ τῆς ΑΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ· ἴση ἄρα ἐστὶν ἡ ΑΕ τῇ ΕΘ. διὰ τὰ αὐτὰ δὲ καὶ ἡ ΑΘ τῇ ΘΕ ἐστὶν ἴση· ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΕΘ τρίγωνον. ὁμοίως δὲ δείξομεν, ὅτι καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ὧν βάσεις μὲν εἰσιν αἱ τοῦ ΕΖΗΘ τετραγώνου πλευраί, κορυφαὶ δὲ τὰ Α, Μ σημεία, ἰσόπλευρόν ἐστιν· ὀκτάεδρον ἄρα συνέσταται ὑπὸ ὀκτὼ τριγώνων ἰσοπλευρῶν περιεχόμενον.

Δεῖ δὴ αὐτὸ καὶ σφαῖρα περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίῳ ἐστὶ τῆς τοῦ ὀκταέδρου πλευρᾶς.

Ἐπεὶ γὰρ αἱ τρεῖς αἱ ΑΚ, ΚΜ, ΚΕ ἴσαι ἀλλήλαις εἰσίν, τὸ ἄρα ἐπὶ τῆς ΑΜ γραφόμενον ἡμικύκλιον ἤξει καὶ διὰ τοῦ Ε. καὶ διὰ τὰ αὐτά, ἐὰν μενούσης τῆς ΑΜ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἤξει καὶ διὰ τῶν Ζ, Η, Θ σημείων, καὶ ἔσται σφαῖρα περιελημμένη τὸ ὀκτάεδρον. λέγω δὲ, ὅτι καὶ τῇ δοθείσῃ. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΑΚ τῇ ΚΜ, κοινὴ δὲ ἡ ΚΕ,

been joined. And let the square $EFGH$, having each of its sides equal to DB , be laid out. And let HF and EG have been joined. And let the straight-line KL have been set up, at point K , at right-angles to the plane of square $EFGH$ [Prop. 11.12]. And let it have been drawn across on the other side of the plane, like KM . And let KL and KM , equal to one of EK , FK , GK , and HK , have been cut off from KL and KM , respectively. And let LE , LF , LG , LH , ME , MF , MG , and MH have been joined.



And since KE is equal to KH , and angle EKH is a right-angle, the (square) on the HE is thus double the (square) on EK [Prop. 1.47]. Again, since LK is equal to KE , and angle LKE is a right-angle, the (square) on EL is thus double the (square) on EK [Prop. 1.47]. And the (square) on HE was also shown (to be) double the (square) on EK . Thus, the (square) on LE is equal to the (square) on EH . Thus, LE is equal to EH . So, for the same (reasons), LH is also equal to HE . Triangle LEH is thus equilateral. So, similarly, we can show that each of the remaining triangles, whose bases are the sides of the square $EFGH$, and apexes the points L and M , are equilateral. Thus, an octahedron contained by eight equilateral triangles has been constructed.

So, it is also necessary to enclose it by the given sphere, and to show that the square on the diameter of the sphere is double the (square) on the side of the octahedron.

For since the three (straight-lines) LK , KM , and KE are equal to one another, the semi-circle drawn on LM will thus also pass through E . And, for the same (reasons), if LM remains (fixed), and the semi-circle is car-

καὶ γωνίας ὀρθὰς περιέχουσιν, βάσις ἄρα ἡ $\Lambda\epsilon$ βάσει τῇ $\epsilon\mu$ ἐστὶν ἴση. καὶ ἐπεὶ ὀρθὴ ἐστὶν ἡ ὑπὸ $\Lambda\epsilon\mu$ γωνία· ἐν ἡμικυκλίῳ γάρ· τὸ ἄρα ἀπὸ τῆς $\Lambda\mu$ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς $\Lambda\epsilon$. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ $\alpha\Gamma$ τῇ $\Gamma\beta$, διπλασία ἐστὶν ἡ $\alpha\beta$ τῆς $\beta\Gamma$. ὥς δὲ ἡ $\alpha\beta$ πρὸς τὴν $\beta\Gamma$, οὕτως τὸ ἀπὸ τῆς $\alpha\beta$ πρὸς τὸ ἀπὸ τῆς $\beta\Delta$ · διπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς $\alpha\beta$ τοῦ ἀπὸ τῆς $\beta\Delta$. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς $\Lambda\mu$ διπλάσιον τοῦ ἀπὸ τῆς $\Lambda\epsilon$. καὶ ἐστὶν ἴσον τὸ ἀπὸ τῆς $\Delta\beta$ τῷ ἀπὸ τῆς $\Lambda\epsilon$ · ἴση γὰρ κεῖται ἡ $\epsilon\Theta$ τῇ $\Delta\beta$. ἴσον ἄρα καὶ τὸ ἀπὸ τῆς $\alpha\beta$ τῷ ἀπὸ τῆς $\Lambda\mu$ · ἴση ἄρα ἡ $\alpha\beta$ τῇ $\Lambda\mu$. καὶ ἐστὶν ἡ $\alpha\beta$ ἡ τῆς δοθείσης σφαίρας διάμετρος· ἡ $\Lambda\mu$ ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ.

Περιεῖληπται ἄρα τὸ ὀκτάεδρον τῇ δοθείσῃ σφαίρᾳ. καὶ συναποδέδεικται, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίῳ ἐστὶ τῆς τοῦ ὀκταέδρου πλευρᾶς· ὅπερ ἔδει δεῖξαι.

ried around, and again established at the same (position) from which it began to be moved, then it will also pass through points F , G , and H , and the octahedron will have been enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For since LK is equal to KM , and KE (is) common, and they contain right-angles, the base LE is thus equal to the base EM [Prop. 1.4]. And since angle LEM is a right-angle—for (it is) in a semi-circle [Prop. 3.31]—the (square) on LM is thus double the (square) on LE [Prop. 1.47]. Again, since AC is equal to CB , AB is double BC . And as AB (is) to BC , so the (square) on AB (is) to the (square) on BC [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is double the (square) on BC . And the (square) on LM was also shown (to be) double the (square) on LE . And the (square) on DB is equal to the (square) on LE . For EH was made equal to DB . Thus, the (square) on AB (is) also equal to the (square) on LM . Thus, AB (is) equal to LM . And AB is the diameter of the given sphere. Thus, LM is equal to the diameter of the given sphere.

Thus, the octahedron has been enclosed by the given sphere, and it has been simultaneously proved that the square on the diameter of the sphere is double the (square) on the side of the octahedron.[†] (Which is) the very thing it was required to show.

[†] If the radius of the sphere is unity then the side of octahedron is $\sqrt{2}$.

ιε'.

Proposition 15

Κύβον συστήσασθαι καὶ σφαίρα περιλαβεῖν, ἥ καὶ τὴν πυραμίδα, καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίῳ ἐστὶ τῆς τοῦ κύβου πλευρᾶς.

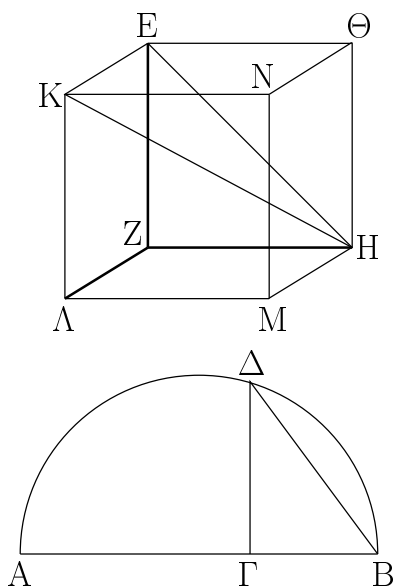
Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ $\alpha\beta$ καὶ τετμήσθω κατὰ τὸ Γ ὥστε διπλῆν εἶναι τὴν $\alpha\Gamma$ τῆς $\Gamma\beta$, καὶ γεγράψθω ἐπὶ τῆς $\alpha\beta$ ἡμικύκλιον τὸ $\alpha\Delta\beta$, καὶ ἀπὸ τοῦ Γ τῇ $\alpha\beta$ πρὸς ὀρθὰς ἤχθω ἡ $\Gamma\Delta$, καὶ ἐπεζεύχθω ἡ $\Delta\beta$, καὶ ἐκκείσθω τετράγωνον τὸ $\epsilon\zeta\eta\theta$ ἴσην ἔχον τὴν πλευρὰν τῇ $\Delta\beta$, καὶ ἀπὸ τῶν ϵ , ζ , η , θ τῷ τοῦ $\epsilon\zeta\eta\theta$ τετραγώνου ἐπιπέδῳ πρὸς ὀρθὰς ἤχθωσαν αἱ $\epsilon\kappa$, $\zeta\lambda$, $\eta\mu$, $\theta\eta$, καὶ ἀφῆρῆσθω ἀπὸ ἐκάστης τῶν $\epsilon\kappa$, $\zeta\lambda$, $\eta\mu$, $\theta\eta$ μιᾶ τῶν $\epsilon\zeta$, $\zeta\eta$, $\eta\theta$, $\theta\epsilon$ ἴση ἐκάστη τῶν $\epsilon\kappa$, $\zeta\lambda$, $\eta\mu$, $\theta\eta$, καὶ ἐπεζεύχθωσαν αἱ $\kappa\lambda$, $\lambda\mu$, $\mu\eta$, $\eta\kappa$ · κύβος ἄρα συνέσταται ὁ $\zeta\eta$ ὑπὸ $\epsilon\zeta$ τετραγώνων ἴσων περιεχόμενος.

Δεῖ δὴ αὐτὸν καὶ σφαίρα περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασία ἐστὶ τῆς πλευρᾶς τοῦ κύβου.

To construct a cube, and to enclose (it) in a sphere, like in the (case of the) pyramid, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube.

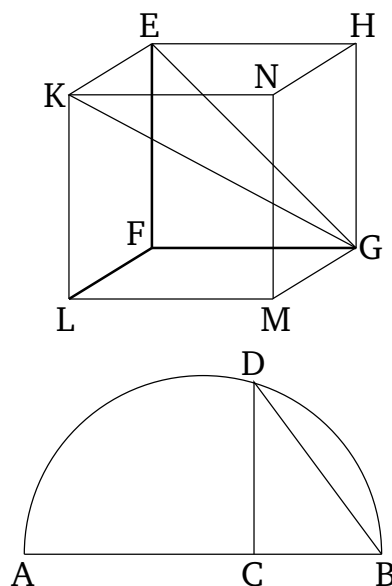
Let the diameter AB of the given sphere be laid out, and let it have been cut at C such that AC is double CB . And let the semi-circle ADB have been drawn on AB . And let CD have been drawn from C at right-angles to AB . And let DB have been joined. And let the square $EFGH$, having (its) side equal to DB , be laid out. And let $E\kappa$, FL , GM , and HN have been drawn from (points) E , F , G , and H , (respectively), at right-angles to the plane of square $EFGH$. And let $E\kappa$, FL , GM , and HN , equal to one of EF , FG , GH , and HE , have been cut off from $E\kappa$, FL , GM , and HN , respectively. And let KL , LM , MN , and NK have been joined. Thus, a cube contained by six equal squares has been constructed.

So, it is also necessary to enclose it by the given sphere, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube.



Ἐπεζεύχθωσαν γὰρ αἱ KH , EH . καὶ ἐπεὶ ὀρθὴ ἐστὶν ἡ ὑπὸ KEH γωνία διὰ τὸ καὶ τὴν KE ὀρθὴν εἶναι πρὸς τὸ EH ἐπίπεδον δηλαδὴ καὶ πρὸς τὴν EH εὐθεῖαν, τὸ ἄρα ἐπὶ τῆς KH γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ E σημείου. πάλιν, ἐπεὶ ἡ HZ ὀρθὴ ἐστὶ πρὸς ἑκατέραν τῶν $ZΛ$, ZE , καὶ πρὸς τὸ ZK ἄρα ἐπίπεδον ὀρθὴ ἐστὶν ἡ HZ . ὥστε καὶ ἐὰν ἐπιζεύξωμεν τὴν ZK , ἡ HZ ὀρθὴ ἔσται καὶ πρὸς τὴν ZK . καὶ διὰ τοῦτο πάλιν τὸ ἐπὶ τῆς HK γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Z . ὁμοίως καὶ διὰ τῶν λοιπῶν τοῦ κύβου σημείων ἥξει. ἐὰν δὲ μενούσης τῆς KH περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἔσται σφαῖρα περιειλημμένος ὁ κύβος. λέγω δὴ, ὅτι καὶ τῇ δοθείῃ. ἐπεὶ γὰρ ἴση ἐστὶν ἡ HZ τῇ ZE , καὶ ἐστὶν ὀρθὴ ἡ πρὸς τῷ Z γωνία, τὸ ἄρα ἀπὸ τῆς EH διπλάσιόν ἐστι τοῦ ἀπὸ τῆς EZ . ἴση δὲ ἡ EZ τῇ EK . τὸ ἄρα ἀπὸ τῆς EH διπλάσιόν ἐστι τοῦ ἀπὸ τῆς EK . ὥστε τὰ ἀπὸ τῶν HE , EK , τουτέστι τὸ ἀπὸ τῆς HK , τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς EK . καὶ ἐπεὶ τριπλασίον ἐστὶν ἡ AB τῆς $BΓ$, ὡς δὲ ἡ AB πρὸς τὴν $BΓ$, οὕτως τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς $BΔ$, τριπλάσιον ἄρα τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς $BΔ$. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς HK τοῦ ἀπὸ τῆς KE τριπλάσιον. καὶ κεῖται ἴση ἡ KE τῇ $ΔB$. ἴση ἄρα καὶ ἡ KH τῇ AB . καὶ ἐστὶν ἡ AB τῆς δοθείσης σφαίρας διάμετρος. καὶ ἡ KH ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ.

Τῇ δοθείῃ ἄρα σφαῖρα περιεῖληπται ὁ κύβος. καὶ συναποδεδεικται, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίον ἐστὶ τῆς τοῦ κύβου πλευρᾶς. ὅπερ ἔδει δεῖξαι.



For let KG and EG have been joined. And since angle KEG is a right-angle—on account of KE also being at right-angles to the plane EG , and manifestly also to the straight-line EG [Def. 11.3]—the semi-circle drawn on KG will thus also pass through point E . Again, since GF is at right-angles to each of FL and FE , GF is thus also at right-angles to the plane FK [Prop. 11.4]. Hence, if we also join FK then GF will also be at right-angles to FK . And, again, on account of this, the semi-circle drawn on GK will also pass through point F . Similarly, it will also pass through the remaining (angular) points of the cube. So, if KG remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, then the cube will have been enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For since GF is equal to FE , and the angle at F is a right-angle, the (square) on EG is thus double the (square) on EF [Prop. 1.47]. And EF (is) equal to EK . Thus, the (square) on EG is double the (square) on EK . Hence, the (sum of the squares) on GE and EK —that is to say, the (square) on GK [Prop. 1.47]—is three times the (square) on EK . And since AB is three times BC , and as AB (is) to BC , so the (square) on AB (is) to the (square) on BD [Prop. 6.8, Def. 5.9], the (square) on AB (is) thus three times the (square) on BD . And the (square) on GK was also shown (to be) three times the (square) on KE . And KE was made equal to DB . Thus, KG (is) also equal to AB . And AB is the radius of the given sphere. Thus, KG is also equal to the diameter of the given sphere.

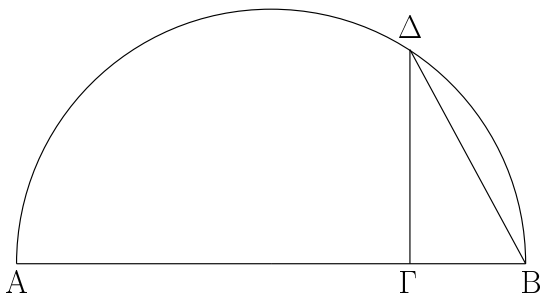
Thus, the cube has been enclosed by the given sphere. And it has simultaneously been shown that the square on the diameter of the sphere is three times the (square) on

the side of the cube.[†] (Which is) the very thing it was required to show.

[†] If the radius of the sphere is unity then the side of the cube is $\sqrt{4/3}$.

ιγ'.

Εἰκοσάεδρον συστήσασθαι καὶ σφαῖρα περιλαβεῖν, ἥ καὶ τὰ προειρημένα σχήματα, καὶ δεῖξαι, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων.

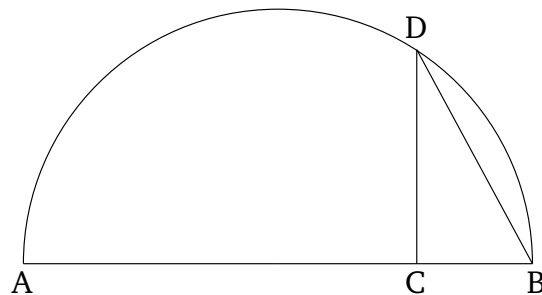


Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ AB καὶ τετμήσθω κατὰ τὸ Γ ὥστε τετραπλῆν εἶναι τὴν AG τῆς GB, καὶ γεγράψθω ἐπὶ τῆς AB ἡμικύκλιον τὸ ADB, καὶ ἦχθω ἀπὸ τοῦ Γ τῇ AB πρὸς ὀρθὰς γωνίας εὐθεῖα γραμμὴ ἡ ΓΔ, καὶ ἐπεζεύχθω ἡ ΔB, καὶ ἐκκείσθω κύκλος ὁ EZHΘK, οὗ ἡ ἐν τοῦ κέντρου ἴση ἔστω τῇ ΔB, καὶ ἐγγεγράφθω εἰς τὸν EZHΘK κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον τὸ EZHΘK, καὶ τετμήσθωσαν αἱ EZ, ZH, HΘ, ΘK, KE περιφέρειαι δίχα κατὰ τὰ Λ, Μ, Ν, Ξ, Ο σημεία, καὶ ἐπεζεύχθωσαν αἱ ΑΜ, ΜΝ, ΝΞ, ΞΟ, ΟΑ, ΕΟ. ἰσόπλευρον ἄρα ἐστὶ καὶ τὸ ΑΜΝΞΟ πεντάγωνον, καὶ δεκαγώνου ἡ ΕΟ εὐθεῖα. καὶ ἀνεστάτωσαν ἀπὸ τῶν Ε, Ζ, Η, Θ, Κ σημείων τῶ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς γωνίας εὐθεῖαι αἱ ΕΠ, ΖΡ, ΗΣ, ΘΤ, ΚΥ ἴσαι οὖσαι τῇ ἐκ τοῦ κέντρου τοῦ EZHΘK κύκλου, καὶ ἐπεζεύχθωσαν αἱ ΠΡ, ΡΣ, ΣΤ, ΤΥ, ΥΠ, ΠΑ, ΑΡ, ΡΜ, ΜΣ, ΣΝ, ΝΤ, ΤΞ, ΞΥ, ΥΟ, ΟΠ.

Καὶ ἐπεὶ ἑκάτερα τῶν ΕΠ, ΚΥ τῶ αὐτῶ ἐπιπέδῳ πρὸς ὀρθὰς ἐστίν, παράλληλος ἄρα ἐστὶν ἡ ΕΠ τῇ ΚΥ. ἐστὶ δὲ αὐτῇ καὶ ἴση· αἱ δὲ τὰς ἴσας τε καὶ παραλλήλους ἐπιζευγνύουσαι ἐπὶ τὰ αὐτὰ μέρη εὐθεῖαι ἴσαι τε καὶ παράλληλοί εἰσιν. ἡ ΠΥ ἄρα τῇ ΕΚ ἴση τε καὶ παράλληλός ἐστιν. πενταγώνου δὲ ἰσοπλεύρου ἡ ΕΚ· πενταγώνου ἄρα ἰσοπλεύρου καὶ ἡ ΠΥ τοῦ εἰς τὸν EZHΘK κύκλον ἐγγεγραμμένου. διὰ τὰ αὐτὰ δὴ καὶ ἑκάστη τῶν ΠΡ, ΡΣ, ΣΤ, ΤΥ πενταγώνου ἐστὶν ἰσοπλεύρου τοῦ εἰς τὸν EZHΘK κύκλον ἐγγεγραμμένου· ἰσόπλευρον ἄρα τὸ ΠΡΣΤΥ πεντάγωνον. καὶ ἐπεὶ ἐξαγώνου μὲν ἐστὶν ἡ ΠΕ, δεκαγώνου δὲ ἡ ΕΟ, καὶ ἐστὶν ὀρθὴ ἡ ὑπὸ ΠΕΟ, πενταγώνου ἄρα ἐστὶν ἡ ΠΟ· ἡ γὰρ τοῦ πενταγώνου πλευρὰ δύναται τὴν τε τοῦ ἐξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγεγραμμένων. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΟΥ πενταγώνου ἐστὶ

Proposition 16

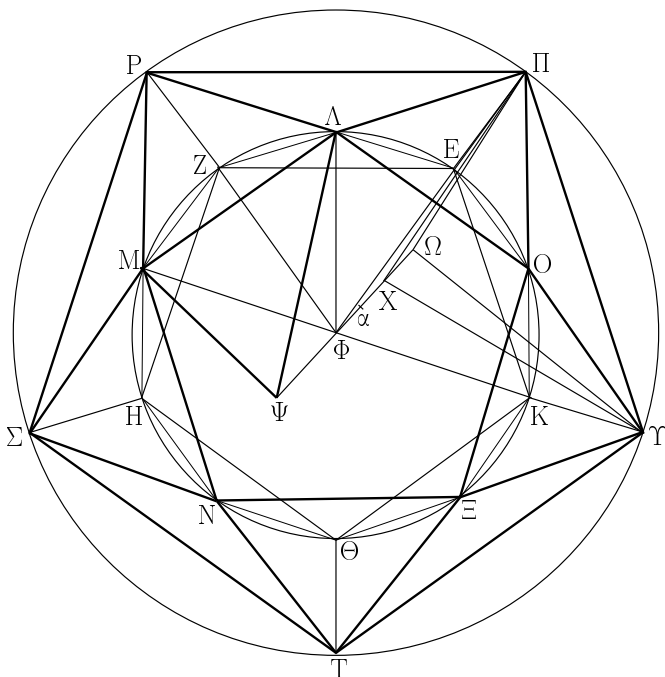
To construct an icosahedron, and to enclose (it) in a sphere, like the aforementioned figures, and to show that the side of the icosahedron is that irrational (straight-line) called minor.



Let the diameter AB of the given sphere be laid out, and let it have been cut at C such that AC is four times CB [Prop. 6.10]. And let the semi-circle ADB have been drawn on AB . And let the straight-line CD have been drawn from C at right-angles to AB . And let DB have been joined. And let the circle $EFGHK$ be set down, and let its radius be equal to DB . And let the equilateral and equiangular pentagon $EFGHK$ have been inscribed in circle $EFGHK$ [Prop. 4.11]. And let the circumferences EF , FG , GH , HK , and KE have been cut in half at points L , M , N , O , and P (respectively). And let LM , MN , NO , OP , PL , and EP have been joined. Thus, pentagon $LMNOP$ is also equilateral, and EP (is) the side of the decagon (inscribed in the circle). And let the straight-lines EQ , FR , GS , HT , and KU , which are equal to the radius of circle $EFGHK$, have been set up at right-angles to the plane of the circle, at points E , F , G , H , and K (respectively). And let QR , RS , ST , TU , UQ , QL , LR , RM , MS , SN , NT , TO , OU , UP , and PQ have been joined.

And since EQ and KU are each at right-angles to the same plane, EQ is thus parallel to KU [Prop. 11.6]. And it is also equal to it. And straight-lines joining equal and parallel (straight-lines) on the same side are (themselves) equal and parallel [Prop. 1.33]. Thus, QU is equal and parallel to EK . And EK (is the side) of an equilateral pentagon (inscribed in circle $EFGHK$). Thus, QU (is) also the side of an equilateral pentagon inscribed in circle $EFGHK$. So, for the same (reasons), QR , RS , ST , and TU are also the sides of an equilateral pentagon inscribed in circle $EFGHK$. Pentagon $QRSTU$ (is) thus equilat-

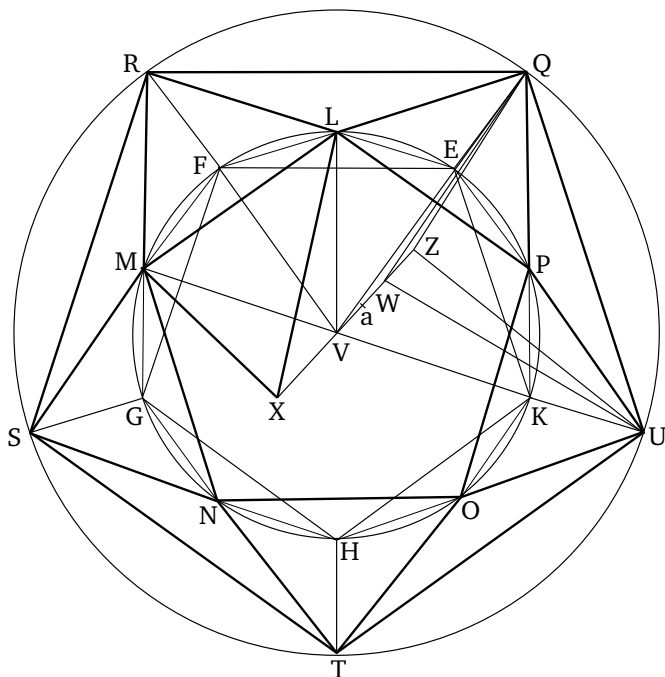
πλευρά. ἔστι δὲ καὶ ἡ ΠΥ πενταγώνου· ἰσόπλευρον ἄρα ἔστι τὸ ΠΟΥ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν ΠΑΡ, ΡΜΣ, ΣΝΤ, ΤΞΥ ἰσόπλευρόν ἐστιν. καὶ ἐπεὶ πενταγώνου ἐδείχθη ἑκατέρα τῶν ΠΛ, ΠΟ, ἔστι δὲ καὶ ἡ ΛΟ πενταγώνου, ἰσόπλευρον ἄρα ἔστι τὸ ΠΛΟ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν ΛΡΜ, ΜΣΝ, ΝΤΞ, ΞΥΟ τριγώνων ἰσόπλευρόν ἐστιν.



Εἰλήφθω τὸ κέντρον τοῦ ΕΖΗΘΚ κύκλου τὸ Φ σημεῖον· καὶ ἀπὸ τοῦ Φ τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἀνεστάτω ἡ ΦΩ, καὶ ἐκβεβλήσθω ἐπὶ τὰ ἔτερα μέρη ὡς ἡ ΦΨ, καὶ ἀφῆρησθω ἑξαγώνου μὲν ἡ ΦΧ, δεκαγώνου δὲ ἑκατέρα τῶν ΦΨ, ΧΩ, καὶ ἐπεξεύχθωσαν αἱ ΠΩ, ΠΧ, ΥΩ, ΕΦ, ΛΦ, ΛΨ, ΨΜ.

Καὶ ἐπεὶ ἑκατέρα τῶν ΦΧ, ΠΕ τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἐστίν, παράλληλος ἄρα ἐστὶν ἡ ΦΧ τῇ ΠΕ. εἰσὶ δὲ καὶ ἴσαι· καὶ αἱ ΕΦ, ΠΧ ἄρα ἴσαι τε καὶ παράλληλοί εἰσιν. ἑξαγώνου δὲ ἡ ΕΦ· ἑξαγώνου ἄρα καὶ ἡ ΠΧ. καὶ ἐπεὶ ἑξαγώνου μὲν ἐστὶν ἡ ΠΧ, δεκαγώνου δὲ ἡ ΧΩ, καὶ ὀρθὴ ἐστὶν ἡ ὑπὸ ΠΧΩ γωνία, πενταγώνου ἄρα ἐστὶν ἡ ΠΩ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΥΩ πενταγώνου ἐστίν, ἐπειδὴ περ,

eral. And side QE is (the side) of a hexagon (inscribed in circle $EFGHK$), and EP (the side) of a decagon, and (angle) QEP is a right-angle, thus QP is (the side) of a pentagon (inscribed in the same circle). For the square on the side of a pentagon is (equal to the sum of) the (squares) on (the sides of) a hexagon and a decagon inscribed in the same circle [Prop. 13.10]. So, for the same (reasons), PU is also the side of a pentagon. And QU is also (the side) of a pentagon. Thus, triangle QPU is equilateral. So, for the same (reasons), (triangles) QLR , RMS , SNT , and TOU are each also equilateral. And since QL and QP were each shown (to be the sides) of a pentagon, and LP is also (the side) of a pentagon, triangle QLP is thus equilateral. So, for the same (reasons), triangles LRM , MSN , NTU , and OUV are each also equilateral.



Let the center, point V , of circle $EFGHK$ have been found [Prop. 3.1]. And let VZ have been set up, at (point) V , at right-angles to the plane of the circle. And let it have been produced on the other side (of the circle), like VX . And let VW have been cut off (from XZ so as to be equal to the side) of a hexagon, and each of VX and WZ (so as to be equal to the side) of a decagon. And let QZ , QW , UZ , EV , LV , LX , and XM have been joined.

And since VW and QE are each at right-angles to the plane of the circle, VW is thus parallel to QE [Prop. 11.6]. And they are also equal. EV and QW are thus equal and parallel (to one another) [Prop. 1.33].

ἐὰν ἐπιζεύξωμεν τὰς ΦΚ, ΧΥ, ἴσαι καὶ ἀπεναντίον ἔσονται, καὶ ἔστιν ἡ ΦΚ ἐκ τοῦ κέντρου οὕσα ἐξαγώνου. ἐξαγώνου ἄρα καὶ ἡ ΧΥ. δεκαγώνου δὲ ἡ ΧΩ, καὶ ὀρθή ἡ ὑπὸ ΥΧΩ· πενταγώνου ἄρα ἡ ΥΩ. ἔστι δὲ καὶ ἡ ΠΥ πενταγώνου· ἰσόπλευρον ἄρα ἔστι τὸ ΠΥΩ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ὧν βάσεις μὲν εἰσιν αἱ ΠΡ, ΡΣ, ΣΤ, ΤΥ εὐθεῖαι, κορυφή δὲ τὸ Ω σημεῖον, ἰσόπλευρόν ἐστιν. πάλιν, ἐπεὶ ἐξαγώνου μὲν ἡ ΦΛ, δεκαγώνου δὲ ἡ ΦΨ, καὶ ὀρθή ἔστιν ἡ ὑπὸ ΛΦΨ γωνία, πενταγώνου ἄρα ἔστιν ἡ ΛΨ. διὰ τὰ αὐτὰ δὴ ἐὰν ἐπιζεύξωμεν τὴν ΜΦ οὕσαν ἐξαγώνου, συνάγεται καὶ ἡ ΜΨ πενταγώνου. ἔστι δὲ καὶ ἡ ΑΜ πενταγώνου· ἰσόπλευρον ἄρα ἔστι τὸ ΑΜΨ τρίγωνον. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ὧν βάσεις μὲν εἰσιν αἱ ΜΝ, ΝΞ, ΞΟ, ΟΛ, κορυφή δὲ τὸ Ψ σημεῖον, ἰσόπλευρόν ἐστιν. συνέσταται ἄρα εἰκοσάεδρον ὑπὸ εἴκοσι τριγώνων ἰσοπλευρῶν περιεχόμενον.

Δεῖ δὴ αὐτὸ καὶ σφαῖρα περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Ἐπεὶ γὰρ ἐξαγώνου ἔστιν ἡ ΦΧ, δεκαγώνου δὲ ἡ ΧΩ, ἡ ΦΩ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Χ, καὶ τὸ μείζον αὐτῆς τμήμα ἐστιν ἡ ΦΧ· ἔστιν ἄρα ὡς ἡ ΩΦ πρὸς τὴν ΦΧ, οὕτως ἡ ΦΧ πρὸς τὴν ΧΩ. ἴση δὲ ἡ μὲν ΦΧ τῇ ΦΕ, ἡ δὲ ΧΩ τῇ ΦΨ· ἔστιν ἄρα ὡς ἡ ΩΦ πρὸς τὴν ΦΕ, οὕτως ἡ ΕΦ πρὸς τὴν ΦΨ. καὶ εἰσιν ὀρθαὶ αἱ ὑπὸ ΩΦΕ, ΕΦΨ γωνίαι· ἐὰν ἄρα ἐπιζεύξωμεν τὴν ΕΩ εὐθεῖαν, ὀρθή ἔσται ἡ ὑπὸ ΨΕΩ γωνία διὰ τὴν ὁμοιότητα τῶν ΨΕΩ, ΦΕΩ τριγώνων. διὰ τὰ αὐτὰ δὴ ἐπεὶ ἔστιν ὡς ἡ ΩΦ πρὸς τὴν ΦΧ, οὕτως ἡ ΦΧ πρὸς τὴν ΧΩ, ἴση δὲ ἡ μὲν ΩΦ τῇ ΨΧ, ἡ δὲ ΦΧ τῇ ΧΠ, ἔστιν ἄρα ὡς ἡ ΨΧ πρὸς τὴν ΧΠ, οὕτως ἡ ΠΧ πρὸς τὴν ΧΩ. καὶ διὰ τοῦτο πάλιν ἐὰν ἐπιζεύξωμεν τὴν ΠΨ, ὀρθή ἔσται ἡ πρὸς τῷ Π γωνία· τὸ ἄρα ἐπὶ τῆς ΨΩ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Π. καὶ ἐὰν μενούσης τῆς ΨΩ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἥξει καὶ διὰ τοῦ Π καὶ τῶν λοιπῶν σημείων τοῦ εἰκοσαέδρου, καὶ ἔσται σφαῖρα περιειλημμένη τὸ εἰκοσάεδρον. λέγω δὴ, ὅτι καὶ τῇ δοθείσῃ. τετμησθῶ γὰρ ἡ ΦΧ δίχα κατὰ τὸ α. καὶ ἐπεὶ εὐθεῖα γραμμὴ ἡ ΦΩ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Χ, καὶ τὸ ἐλάσσον αὐτῆς τμήμα ἐστιν ἡ ΩΧ, ἡ ἄρα ΩΧ προσλαβοῦσα τὴν ἡμίσειαν τοῦ μείζονος τμήματος τὴν Χα πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τοῦ μείζονος τμήματος· πενταπλάσιον ἄρα ἔστι τὸ ἀπὸ τῆς Ωα τοῦ ἀπὸ τῆς αΧ. καὶ ἔστι τῆς μὲν Ωα διπλῆ ἡ ΩΨ, τῆς δὲ αΧ διπλῆ ἡ ΦΧ· πενταπλάσιον ἄρα ἔστι τὸ ἀπὸ τῆς ΩΨ τοῦ ἀπὸ τῆς ΧΦ. καὶ ἐπεὶ τετραπλῆ ἔστιν ἡ ΑΓ τῆς ΓΒ, πενταπλῆ ἄρα ἔστιν ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΔ· πενταπλάσιον ἄρα ἔστι τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΩΨ πενταπλάσιον τοῦ ἀπὸ τῆς ΦΧ. καὶ ἔστιν ἴση ἡ ΔΒ τῇ

And EV (is the side) of a hexagon. Thus, QW (is) also (the side) of a hexagon. And since QW is (the side) of a hexagon, and WZ (the side) of a decagon, and angle QWZ is a right-angle [Def. 11.3, Prop. 1.29], QZ is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), UZ is also (the side) of a pentagon—inasmuch as, if we join VK and WU then they will be equal and opposite. And VK , being (equal) to the radius (of the circle), is (the side) of a hexagon [Prop. 4.15 corr.]. Thus, WU (is) also the side of a hexagon. And WZ (is the side) of a decagon, and (angle) UWZ (is) a right-angle. Thus, UZ (is the side) of a pentagon [Prop. 13.10]. And QU is also (the side) of a pentagon. Triangle QUZ is thus equilateral. So, for the same (reasons), each of the remaining triangles, whose bases are the straight-lines QR , RS , ST , and TU , and apexes the point Z , are also equilateral. Again, since VL (is the side) of a hexagon, and VX (the side) of a decagon, and angle LVX is a right-angle, LX is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), if we join MV , which is (the side) of a hexagon, MX is also inferred (to be the side) of a pentagon. And LM is also (the side) of a pentagon. Thus, triangle LMX is equilateral. So, similarly, it can be shown that each of the remaining triangles, whose bases are the (straight-lines) MN , NO , OP , and PL , and apexes the point X , are also equilateral. Thus, an icosahedron contained by twenty equilateral triangles has been constructed.

So, it is also necessary to enclose it in the given sphere, and to show that the side of the icosahedron is that irrational (straight-line) called minor.

For, since VW is (the side) of a hexagon, and WZ (the side) of a decagon, VZ has thus been cut in extreme and mean ratio at W , and VW is its greater piece [Prop. 13.9]. Thus, as ZV is to VW , so VW (is) to WZ . And VW (is) equal to VE , and WZ to VX . Thus, as ZV is to VE , so EV (is) to VX . And angles ZVE and EVX are right-angles. Thus, if we join straight-line EZ then angle XEZ will be a right-angle, on account of the similarity of triangles XEZ and VEZ . [Prop. 6.8]. So, for the same (reasons), since as ZV is to VW , so VW (is) to WZ , and ZV (is) equal to XW , and VW to WQ , thus as XW is to WQ , so QW (is) to WZ . And, again, on account of this, if we join QX then the angle at Q will be a right-angle [Prop. 6.8]. Thus, the semi-circle drawn on XZ will also pass through Q [Prop. 3.31]. And if XZ remains fixed, and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, then it will also pass through (point) Q , and (through) the remaining (angular) points of the icosahedron. And the icosahedron will have been en-

ΦΧ· ἑκατέρα γὰρ αὐτῶν ἴση ἐστὶ τῇ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου· ἴση ἄρα καὶ ἡ ΑΒ τῇ ΨΩ. καὶ ἐστὶν ἡ ΑΒ ἡ τῆς δοθείσης σφαίρας διάμετρος· καὶ ἡ ΨΩ ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ· τῇ ἄρα δοθείσῃ σφαίρᾳ περιείληπται τὸ εἰκοσάεδρον.

Λέγω δὴ, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων. ἐπεὶ γὰρ ῥητὴ ἐστὶν ἡ τῆς σφαίρας διάμετρος, καὶ ἐστὶ δυνάμει πενταπλασίῳ τῆς ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου, ῥητὴ ἄρα ἐστὶ καὶ ἡ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου· ὥστε καὶ ἡ διάμετρος αὐτοῦ ῥητὴ ἐστὶν. ἐὰν δὲ εἰς κύκλον ῥητὴν ἔχοντα τὴν διάμετρον πεντάγωνον ἰσόπλευρον ἐγγραφῇ, ἡ τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων. ἡ δὲ τοῦ ΕΖΗΘΚ πενταγώνου πλευρὰ ἡ τοῦ εἰκοσαέδρου ἐστίν. ἡ ἄρα τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων.

closed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For let VW have been cut in half at a . And since the straight-line VZ has been cut in extreme and mean ratio at W , and ZW is its lesser piece, then the square on ZW added to half of the greater piece, Wa , is five times the (square) on half of the greater piece [Prop. 13.3]. Thus, the (square) on Za is five times the (square) on aW . And ZX is double Za , and VW double aW . Thus, the (square) on ZX is five times the (square) on WV . And since AC is four times CB , AB is thus five times BC . And as AB (is) to BC , so the (square) on AB (is) to the (square) on BD [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is five times the (square) on BD . And the (square) on ZX was also shown (to be) five times the (square) on VW . And DB is equal to VW . For each of them is equal to the radius of circle $EFGHK$. Thus, AB (is) also equal to XZ . And AB is the diameter of the given sphere. Thus, XZ is equal to the diameter of the given sphere. Thus, the icosahedron has been enclosed by the given sphere.

So, I say that the side of the icosahedron is that irrational (straight-line) called minor. For since the diameter of the sphere is rational, and the square on it is five times the (square) on the radius of circle $EFGHK$, the radius of circle $EFGHK$ is thus also rational. Hence, its diameter is also rational. And if an equilateral pentagon is inscribed in a circle having a rational diameter then the side of the pentagon is that irrational (straight-line) called minor [Prop. 13.11]. And the side of pentagon $EFGHK$ is (the side) of the icosahedron. Thus, the side of the icosahedron is that irrational (straight-line) called minor.

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει πενταπλασίῳ ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ἀφ' οὗ τὸ εἰκοσάεδρον ἀναγέγραπται, καὶ ὅτι ἡ τῆς σφαίρας διάμετρος σύγκειται ἐκ τε τῆς τοῦ ἑξαγώνου καὶ δύο τῶν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων. ὅπερ ἔδει δεῖξαι.

Corollary

So, (it is) clear, from this, that the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron has been described, and that the diameter of the sphere is the sum of (the side) of the hexagon, and two of (the sides) of the decagon, inscribed in the same circle.[†]

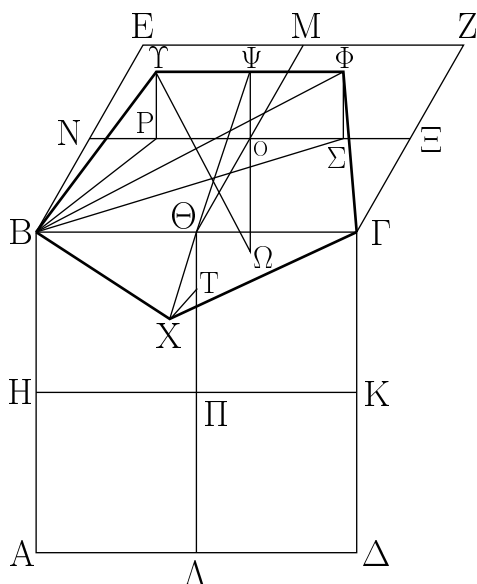
[†] If the radius of the sphere is unity then the radius of the circle is $2/\sqrt{5}$, and the sides of the hexagon, decagon, and pentagon/icosahedron are $2/\sqrt{5}$, $1 - 1/\sqrt{5}$, and $(1/\sqrt{5})\sqrt{10 - 2\sqrt{5}}$, respectively.

ιζ'.

Δωδεκάεδρον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν, ἥ καὶ τὰ προειρημένα σχήματα, καὶ δεῖξαι, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

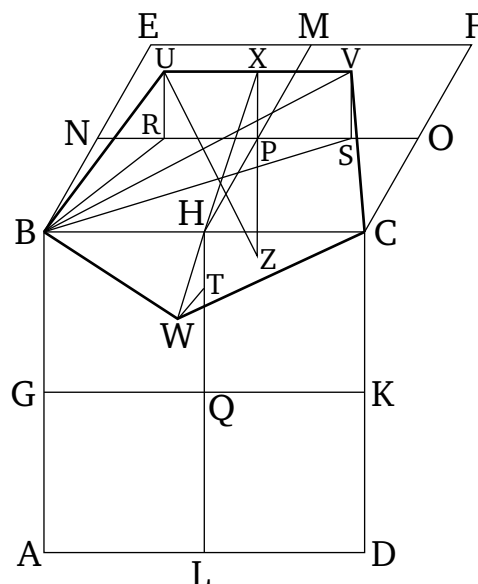
Proposition 17

To construct a dodecahedron, and to enclose (it) in a sphere, like the aforementioned figures, and to show that the side of the dodecahedron is that irrational (straight-line) called an apotome.



Ἐκκείσθωσαν τοῦ προειρημένου κύβου δύο ἐπίπεδα πρὸς ὀρθὰς ἀλλήλοις τὰ $AB\Gamma\Delta$, ΓBEZ , καὶ τετμήσθω ἑκάστη τῶν AB , $B\Gamma$, $\Gamma\Delta$, ΔA , EZ , EB , $Z\Gamma$ πλευρῶν δίχα κατὰ τὰ H , Θ , K , Λ , M , N , Ξ , καὶ ἐπεζεύχθωσαν αἱ HK , $\Theta\Lambda$, $M\Theta$, $N\Xi$, καὶ τετμήσθω ἑκάστη τῶν NO , $O\Xi$, $\Theta\Pi$ ἄκρον καὶ μέσον λόγον κατὰ τὰ P , Σ , T σημεῖα, καὶ ἔστω αὐτῶν μείζονα τμήματα τὰ PO , $O\Sigma$, $T\Pi$, καὶ ἀνεστάτωσαν ἀπὸ τῶν P , Σ , T σημείων τοῖς τοῦ κύβου ἐπιπέδοις πρὸς ὀρθὰς ἐπὶ τὰ ἐκτὸς μέρη τοῦ κύβου αἱ PY , $\Sigma\Phi$, TX , καὶ κείσθωσαν ἴσαι ταῖς PO , $O\Sigma$, $T\Pi$, καὶ ἐπεζεύχθωσαν αἱ YB , BX , $X\Gamma$, $\Gamma\Phi$, ΦY .

Λέγω, ὅτι τὸ $YBX\Gamma\Phi$ πεντάγωνον ἰσόπλευρόν τε καὶ ἐν ἐνὶ ἐπιπέδῳ καὶ ἔτι ἰσογώνιον ἐστίν. ἐπεζεύχθωσαν γὰρ αἱ PB , ΣB , ΦB . καὶ ἐπεὶ εὐθεῖα ἡ NO ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ P , καὶ τὸ μείζον τμήμα ἐστίν ἡ PO , τὰ ἄρα ἀπὸ τῶν ON , NP τριπλάσια ἐστὶ τοῦ ἀπὸ τῆς PO . ἴση δὲ ἡ μὲν ON τῇ NB , ἡ δὲ OP τῇ PY . τὰ ἄρα ἀπὸ τῶν BN , NP τριπλάσια ἐστὶ τοῦ ἀπὸ τῆς PY . τοῖς δὲ ἀπὸ τῶν BN , NP τὸ ἀπὸ τῆς BP ἐστὶν ἴσον· τὸ ἄρα ἀπὸ τῆς BP τριπλάσιόν ἐστὶ τοῦ ἀπὸ τῆς PY . ὥστε τὰ ἀπὸ τῶν BP , PY τετραπλάσια ἐστὶ τοῦ ἀπὸ τῆς PY . τοῖς δὲ ἀπὸ τῶν BP , PY ἴσον ἐστὶ τὸ ἀπὸ τῆς BY . τὸ ἄρα ἀπὸ τῆς BY τετραπλάσιόν ἐστὶ τοῦ ἀπὸ τῆς YP . διπλὴ ἄρα ἐστὶν ἡ BY τῆς PY . ἐστὶ δὲ καὶ ἡ ΦY τῆς YP διπλὴ, ἐπειδὴ περ καὶ ἡ ΣP τῆς OP , τουτέστι τῆς PY , ἐστὶ διπλὴ· ἴση ἄρα ἡ BY τῇ $Y\Phi$. ὁμοίως δὲ δεῖχθήσεται, ὅτι καὶ ἑκάστη τῶν BX , $X\Gamma$, $\Gamma\Phi$ ἑκατέρω τῶν BY , $Y\Phi$ ἐστὶν ἴση. ἰσόπλευρον ἄρα ἐστὶ τὸ $YBX\Gamma\Phi$ πεντάγωνον. λέγω δὴ, ὅτι καὶ ἐν ἐνὶ ἐστὶν ἐπιπέδῳ. ἤχθω γὰρ ἀπὸ τοῦ O ἑκατέρω τῶν PY , $\Sigma\Phi$ παράλληλος ἐπὶ τὰ ἐκτὸς τοῦ κύβου μέρη ἡ $O\Psi$, καὶ ἐπεζεύχθωσαν αἱ $\Psi\Theta$, ΘX . λέγω, ὅτι ἡ $\Psi\Theta X$ εὐθεῖα ἐστίν. ἐπεὶ γὰρ ἡ $\Theta\Pi$ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ T , καὶ τὸ μείζον αὐτῆς τμήμα ἐστὶν ἡ $T\Pi$, ἔστιν ἄρα ὡς ἡ $\Theta\Pi$ πρὸς τὴν $T\Pi$, οὕτως ἡ $T\Pi$ πρὸς τὴν



Let two planes of the aforementioned cube [Prop. 13.15], $ABCD$ and $CBEF$, (which are) at right-angles to one another, be laid out. And let the sides AB , BC , CD , DA , EF , EB , and FC have each been cut in half at points G , H , K , L , M , N , and O (respectively). And let GK , HL , MH , and NO have been joined. And let NP , PO , and HQ have each been cut in extreme and mean ratio at points R , S , and T (respectively). And let their greater pieces be RP , PS , and TQ (respectively). And let RU , SV , and TW have been set up on the exterior side of the cube, at points R , S , and T (respectively), at right-angles to the planes of the cube. And let them be made equal to RP , PS , and TQ . And let UB , BW , WC , CV , and VU have been joined.

I say that the pentagon $UBWCV$ is equilateral, and in one plane, and, further, equiangular. For let RB , SB , and VB have been joined. And since the straight-line NP has been cut in extreme and mean ratio at R , and RP is the greater piece, the (sum of the squares) on PN and NR is thus three times the (square) on RP [Prop. 13.4]. And PN (is) equal to NB , and PR to RU . Thus, the (sum of the squares) on BN and NR is three times the (square) on RU . And the (square) on BR is equal to the (sum of the squares) on BN and NR [Prop. 1.47]. Thus, the (square) on BR is three times the (square) on RU . Hence, the (sum of the squares) on BR and RU is four times the (square) on RU . And the (square) on BU is equal to the (sum of the squares) on BR and RU [Prop. 1.47]. Thus, the (square) on BU is four times the (square) on RU . Thus, BU is double RU . And VU is also double UR , inasmuch as SR is also double PR —that is to say, RU . Thus, BU (is) equal to UV . So, similarly, it can be shown that each of BW , WC , CV is equal to each

ΤΘ. ἴση δὲ ἡ μὲν ΘΠ τῇ ΘΟ, ἡ δὲ ΠΤ ἑκατέρᾳ τῶν ΤΧ, ΟΨ· ἔστιν ἄρα ὡς ἡ ΘΟ πρὸς τὴν ΟΨ, οὕτως ἡ ΧΤ πρὸς τὴν ΤΘ. καὶ ἔστι παράλληλος ἡ μὲν ΘΟ τῇ ΤΧ· ἑκατέρα γὰρ αὐτῶν τῷ ΒΔ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν· ἡ δὲ ΤΘ τῇ ΟΨ· ἑκατέρα γὰρ αὐτῶν τῷ ΒΖ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. ἐὰν δὲ δύο τρίγωνα συντεθῇ κατὰ μίαν γωνίαν, ὡς τὰ ΨΟΘ, ΘΤΧ, τὰς δύο πλευρὰς ταῖς δυὸν ἀνάλογον ἔχοντα, ὥστε τὰς ὁμολόγους αὐτῶν πλευρὰς καὶ παραλλήλους εἶναι, αἱ λοιπαὶ εὐθεῖαι ἐπ' εὐθείας ἔσσονται· ἐπ' εὐθείας ἄρα ἐστὶν ἡ ΨΘ τῇ ΘΧ. πᾶσα δὲ εὐθεῖα ἐν ἐνὶ ἐστὶν ἐπιπέδῳ· ἐν ἐνὶ ἄρα ἐπιπέδῳ ἐστὶ τὸ ΥΒΧΓΦ πεντάγωνον.

Λέγω δὴ, ὅτι καὶ ἰσογώνιον ἐστίν.

Ἐπεὶ γὰρ εὐθεῖα γραμμὴ ἡ ΝΟ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ρ, καὶ τὸ μείζον τμήμα ἐστὶν ἡ ΟΡ [ἔστιν ἄρα ὡς συναμφότερος ἡ ΝΟ, ΟΡ πρὸς τὴν ΟΝ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΡ], ἴση δὲ ἡ ΟΡ τῇ ΟΣ [ἔστιν ἄρα ὡς ἡ ΣΝ πρὸς τὴν ΝΟ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΣ], ἡ ΝΣ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ο, καὶ τὸ μείζον τμήμα ἐστὶν ἡ ΝΟ· τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΟ τριπλάσια ἐστὶ τοῦ ἀπὸ τῆς ΝΟ. ἴση δὲ ἡ μὲν ΝΟ τῇ ΝΒ, ἡ δὲ ΟΣ τῇ ΣΦ· τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΦ τετράγωνα τριπλάσια ἐστὶ τοῦ ἀπὸ τῆς ΝΒ· ὥστε τὰ ἀπὸ τῶν ΦΣ, ΣΝ, ΝΒ τετραπλάσια ἐστὶ τοῦ ἀπὸ τῆς ΝΒ. τοῖς δὲ ἀπὸ τῶν ΣΝ, ΝΒ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΣΒ· τὰ ἄρα ἀπὸ τῶν ΒΣ, ΣΦ, τουτέστι τὸ ἀπὸ τῆς ΒΦ [ὀρθὴ γὰρ ἡ ὑπὸ ΦΣΒ γωνία], τετραπλάσιον ἐστὶ τοῦ ἀπὸ τῆς ΝΒ· διπλῆ ἄρα ἐστὶν ἡ ΦΒ τῆς ΒΝ. ἔστι δὲ καὶ ἡ ΒΓ τῆς ΒΝ διπλῆ· ἴση ἄρα ἐστὶν ἡ ΒΦ τῇ ΒΓ. καὶ ἐπεὶ δύο αἱ ΒΥ, ΥΦ δυοὶ ταῖς ΒΧ, ΧΓ ἴσαι εἰσίν, καὶ βάσις ἡ ΒΦ βάσει τῇ ΒΓ ἴση, γωνία ἄρα ἡ ὑπὸ ΒΥΦ γωνία τῇ ὑπὸ ΒΧΓ ἐστὶν ἴση. ὁμοίως δὲ δεῖξομεν, ὅτι καὶ ἡ ὑπὸ ΥΦΓ γωνία ἴση ἐστὶ τῇ ὑπὸ ΒΧΓ· αἱ ἄρα ὑπὸ ΒΧΓ, ΒΥΦ, ΥΦΓ τρεῖς γωνίαι ἴσαι ἀλλήλαις εἰσίν. ἐὰν δὲ πενταγώνου ἰσοπλευροῦ αἱ τρεῖς γωνίαι ἴσαι ἀλλήλαις ὦσιν, ἰσογώνιον ἔσται τὸ πεντάγωνον· ἰσογώνιον ἄρα ἐστὶ τὸ ΒΥΦΓΧ πεντάγωνον. ἐδείχθη δὲ καὶ ἰσόπλευρον· τὸ ἄρα ΒΥΦΓΧ πεντάγωνον ἰσόπλευρόν ἐστι καὶ ἰσογώνιον, καὶ ἐστὶν ἐπὶ μιᾷ τοῦ κύβου πλευρᾶς τῆς ΒΓ. ἐὰν ἄρα ἐφ' ἑκάστης τῶν τοῦ κύβου δώδεκα πλευρῶν τὰ αὐτὰ κατασκευάσωμεν, συσταθήσεται τι σχῆμα στερεὸν ὑπὸ δώδεκα πενταγόνων ἰσοπλευρῶν τε καὶ ἰσογώνιων περιεχόμενον, ὃ καλεῖται δωδεκάεδρον.

Δεῖ δὴ αὐτὸ καὶ σφαῖρα περιλαβεῖν τῇ δοθείῃ καὶ δεῖξαι, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐκβεβλήσθω γὰρ ἡ ΨΟ, καὶ ἔστω ἡ ΨΩ· συμβάλλει ἄρα ἡ ΟΩ τῇ τοῦ κύβου διαμέτρῳ, καὶ δίχα τέμνουσιν ἀλλήλας· τοῦτο γὰρ δέδεικται ἐν τῷ παρατελείτῳ θεωρηματι τοῦ ἐνδεκάτου βιβλίου. τεμνέτωσαν κατὰ τὸ Ω· τὸ Ω ἄρα κέντρον ἐστὶ τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον, καὶ ἡ ΩΟ ἡμίσεια τῆς πλευρᾶς τοῦ κύβου. ἐπεζεύχθω δὲ ἡ ΥΩ. καὶ ἐπεὶ εὐθεῖα γραμμὴ ἡ ΝΣ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ο, καὶ τὸ μείζον αὐτῆς τμήμα ἐστὶν ἡ ΝΟ,

of BU and UV . Thus, pentagon $BUVCW$ is equilateral. So, I say that it is also in one plane. For let PX have been drawn from P , parallel to each of RU and SV , on the exterior side of the cube. And let XH and HW have been joined. I say that XHW is a straight-line. For since HQ has been cut in extreme and mean ratio at T , and QT is its greater piece, thus as HQ is to QT , so QT (is) to TH . And HQ (is) equal to HP , and QT to each of TW and PX . Thus, as HP is to PX , so WT (is) to TH . And HP is parallel to TW . For of each of them is at right-angles to the plane BD [Prop. 11.6]. And TH (is parallel) to PX . For each of them is at right-angles to the plane BF [Prop. 11.6]. And if two triangles, like XPH and HTW , having two sides proportional to two sides, are placed together at a single angle such that their corresponding sides are also parallel then the remaining sides will be straight-on (to one another) [Prop. 6.32]. Thus, XH is straight-on to HW . And every straight-line is in one plane [Prop. 11.1]. Thus, pentagon $UBWCV$ is in one plane.

So, I say that it is also equiangular.

For since the straight-line NP has been cut in extreme and mean ratio at R , and PR is the greater piece [thus as the sum of NP and PR is to PN , so NP (is) to PR], and PR (is) equal to PS [thus as SN is to NP , so NP (is) to PS], NS has thus also been cut in extreme and mean ratio at P , and NP is the greater piece [Prop. 13.5]. Thus, the (sum of the squares) on NS and SP is three times the (square) on NP [Prop. 13.4]. And NP (is) equal to NB , and PS to SV . Thus, the (sum of the) squares on NS and SV is three times the (square) on NB . Hence, the (sum of the squares) on VS , SN , and NB is four times the (square) on NB . And the (square) on SB is equal to the (sum of the squares) on SN and NB [Prop. 1.47]. Thus, the (sum of the squares) on BS and SV —that is to say, the (square) on BV [for angle $VSΒ$ (is) a right-angle]—is four times the (square) on NB [Def. 11.3, Prop. 1.47]. Thus, VB is double BN . And BC (is) also double BN . Thus, BV is equal to BC . And since the two (straight-lines) BU and UV are equal to the two (straight-lines) BW and WC (respectively), and the base BV (is) equal to the base BC , angle BUV is thus equal to angle BWC [Prop. 1.8]. So, similarly, we can show that angle UVC is equal to angle BWC . Thus, the three angles BWC , BUV , and UVC are equal to one another. And if three angles of an equilateral pentagon are equal to one another then the pentagon is equiangular [Prop. 13.7]. Thus, pentagon $BUVCW$ is equiangular. And it was also shown (to be) equilateral. Thus, pentagon $BUVCW$ is equilateral and equiangular, and it is on one of the sides, BC , of the cube. Thus, if we make the

τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΟ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΟ. ἴση δὲ ἡ μὲν ΝΣ τῇ ΨΩ, ἐπειδὴ περ καὶ ἡ μὲν ΝΟ τῇ ΟΩ ἐστὶν ἴση, ἡ δὲ ΨΟ τῇ ΟΣ. ἀλλὰ μὴν καὶ ἡ ΟΣ τῇ ΨΥ, ἐπεὶ καὶ τῇ ΡΟ· τὰ ἄρα ἀπὸ τῶν ΩΨ, ΨΥ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΟ. τοῖς δὲ ἀπὸ τῶν ΩΨ, ΨΥ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΥΩ· τὸ ἄρα ἀπὸ τῆς ΥΩ τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΝΟ. ἐστὶ δὲ καὶ ἡ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περιλαμβανοῦσης τὸν κύβον δυνάμει τριπλασίῳ τῆς ἡμισείας τῆς τοῦ κύβου πλευρᾶς· προδεδεικται γὰρ κύβον συστήσασθαι καὶ σφαῖρα περιλαβεῖν καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίῳ ἐστὶ τῆς πλευρᾶς τοῦ κύβου. εἰ δὲ ὅλη τῆς ὅλης, καὶ [ἡ] ἡμίσεια τῆς ἡμισείας· καὶ ἐστὶν ἡ ΝΟ ἡμίσεια τῆς τοῦ κύβου πλευρᾶς· ἡ ἄρα ΥΩ ἴση ἐστὶ τῇ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περιλαμβανοῦσης τὸν κύβον. καὶ ἐστὶ τὸ Ω κέντρον τῆς σφαίρας τῆς περιλαμβανοῦσης τὸν κύβον· τὸ Υ ἄρα σημεῖον πρὸς τῇ ἐπιφανείᾳ ἐστὶ τῆς σφαίρας. ὁμοίως δὲ δεῖξομεν, ὅτι καὶ ἐκάστη τῶν λοιπῶν γωνιῶν τοῦ δωδεκαέδρου πρὸς τῇ ἐπιφανείᾳ ἐστὶ τῆς σφαίρας· περιεῖληπται ἄρα τὸ δωδεκαέδρον τῇ δοθείσῃ σφαίρᾳ.

Λέγω δὴ, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐπεὶ γὰρ τῆς ΝΟ ἄκρον καὶ μέσον λόγον τετμημένης τὸ μείζον τμημὰ ἐστὶν ὁ ΡΟ, τῆς δὲ ΟΞ ἄκρον καὶ μέσον λόγον τετμημένης τὸ μείζον τμημὰ ἐστὶν ἡ ΟΣ, ὅλης ἄρα τῆς ΝΞ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζον τμημὰ ἐστὶν ἡ ΡΣ. [οἷον ἐπεὶ ἐστὶν ὡς ἡ ΝΟ πρὸς τὴν ΟΡ, ἡ ΟΡ πρὸς τὴν ΡΝ, καὶ τὰ διπλάσια· τὰ γὰρ μέρη τοῖς ἰσάκεις πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον· ὡς ἄρα ἡ ΝΞ πρὸς τὴν ΡΣ, οὕτως ἡ ΡΣ πρὸς συναμφοτέρον τὴν ΝΡ, ΣΞ. μείζων δὲ ἡ ΝΞ τῆς ΡΣ· μείζων ἄρα καὶ ἡ ΡΣ συναμφοτέρου τῆς ΝΡ, ΣΞ· ἡ ΝΞ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μείζον αὐτῆς τμημὰ ἐστὶν ἡ ΡΣ.] ἴση δὲ ἡ ΡΣ τῇ ΥΦ· τῆς ἄρα ΝΞ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζον τμημὰ ἐστὶν ἡ ΥΦ. καὶ ἐπεὶ ῥητὴ ἐστὶν τῆς σφαίρας διάμετρος καὶ ἐστὶ δυνάμει τριπλασίῳ τῆς τοῦ κύβου πλευρᾶς, ῥητὴ ἄρα ἐστὶν ἡ ΝΞ πλευρὰ οὔσα τοῦ κύβου. ἐὰν δὲ ῥητὴ γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, ἐκάτερον τῶν τμημάτων ἄλογός ἐστιν ἀποτομή.

Ἡ ΥΦ ἄρα πλευρὰ οὔσα τοῦ δωδεκαέδρου ἄλογός ἐστιν ἀποτομή.

same construction on each of the twelve sides of the cube then some solid figure contained by twelve equilateral and equiangular pentagons will have been constructed, which is called a dodecahedron.

So, it is necessary to enclose it in the given sphere, and to show that the side of the dodecahedron is that irrational (straight-line) called an apotome.

For let XP have been produced, and let (the produced straight-line) be XZ . Thus, PZ meets the diameter of the cube, and they cut one another in half. For, this has been proved in the penultimate theorem of the eleventh book [Prop. 11.38]. Let them cut (one another) at Z . Thus, Z is the center of the sphere enclosing the cube, and ZP (is) half the side of the cube. So, let UZ have been joined. And since the straight-line NS has been cut in extreme and mean ratio at P , and its greater piece is NP , the (sum of the squares) on NS and SP is thus three times the (square) on NP [Prop. 13.4]. And NS (is) equal to XZ , inasmuch as NP is also equal to PZ , and XP to PS . But, indeed, PS (is) also (equal) to XU , since (it is) also (equal) to RP . Thus, the (sum of the squares) on ZX and XU is three times the (square) on NP . And the (square) on UZ is equal to the (sum of the squares) on ZX and XU [Prop. 1.47]. Thus, the (square) on UZ is three times the (square) on NP . And the square on the radius of the sphere enclosing the cube is also three times the (square) on half the side of the cube. For it has previously been demonstrated (how to) construct the cube, and to enclose (it) in a sphere, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube [Prop. 13.15]. And if the (square on the) whole (is three times) the (square on the) whole, then the (square on the) half (is) also (three times) the (square on the) half. And NP is half of the side of the cube. Thus, UZ is equal to the radius of the sphere enclosing the cube. And Z is the center of the sphere enclosing the cube. Thus, point U is on the surface of the sphere. So, similarly, we can show that each of the remaining angles of the dodecahedron is also on the surface of the sphere. Thus, the dodecahedron has been enclosed by the given sphere.

So, I say that the side of the dodecahedron is that irrational straight-line called an apotome.

For since RP is the greater piece of NP , which has been cut in extreme and mean ratio, and PS is the greater piece of PO , which has been cut in extreme and mean ratio, RS is thus the greater piece of the whole of NO , which has been cut in extreme and mean ratio. [Thus, since as NP is to PR , (so) PR (is) to RN , and (the same is also true) of the doubles. For parts have the same ratio as similar multiples (taken in corresponding

order) [Prop. 5.15]. Thus, as NO (is) to RS , so RS (is) to the sum of NR and SO . And NO (is) greater than RS . Thus, RS (is) also greater than the sum of NR and SO [Prop. 5.14]. Thus, NO has been cut in extreme and mean ratio, and RS is its greater piece.] And RS (is) equal to UV . Thus, UV is the greater piece of NO , which has been cut in extreme and mean ratio. And since the diameter of the sphere is rational, and the square on it is three times the (square) on the side of the cube, NO , which is the side of the cube, is thus rational. And if a rational (straight)-line is cut in extreme and mean ratio then each of the pieces is the irrational (straight-line called) an apotome.

Thus, UV , which is the side of the dodecahedron, is the irrational (straight-line called) an apotome [Prop. 13.6].

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι τῆς τοῦ κύβου πλευρᾶς ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζον τμήμα ἐστὶν ἡ τοῦ δωδεκαέδρου πλευρά. ὅπερ ἔδει δείξαι.

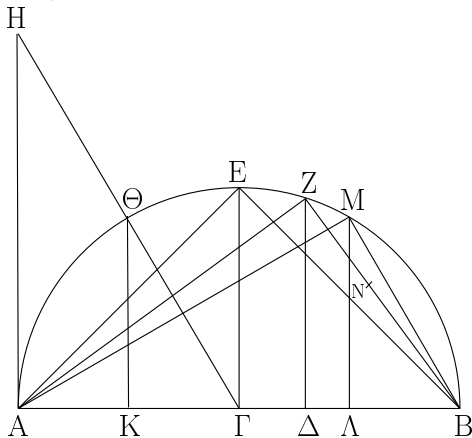
Corollary

So, (it is) clear, from this, that the side of the dodecahedron is the greater piece of the side of the cube, when it is cut in extreme and mean ratio.[†] (Which is) the very thing it was required to show.

[†] If the radius of the circumscribed sphere is unity then the side of the cube is $\sqrt{4/3}$, and the side of the dodecahedron is $(1/3)(\sqrt{15} - \sqrt{3})$.

ιη'.

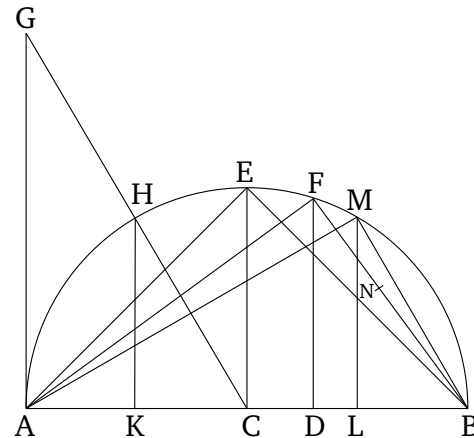
Τὰς πλευρὰς τῶν πέντε σχημάτων ἐκθέσθαι καὶ συγκρίναι πρὸς ἀλλήλας.



Ἐκκεῖσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ AB , καὶ τετμήσθω κατὰ τὸ Γ ὥστε ἴσην εἶναι τὴν $A\Gamma$ τῇ ΓB , κατὰ δὲ τὸ Δ ὥστε διπλασίονα εἶναι τὴν $A\Delta$ τῆς ΔB , καὶ γεγράφθω ἐπὶ τῆς AB ἡμικύκλιον τὸ AEB , καὶ ἀπὸ τῶν Γ, Δ τῇ AB πρὸς ὀρθὰς ἤχθωσαν αἱ $\Gamma E, \Delta Z$, καὶ ἐπεζεύχθωσαν αἱ AZ, ZB, EB . καὶ ἐπεὶ διπλὴ ἐστὶν ἡ $A\Delta$ τῆς ΔB , τριπλὴ ἄρα ἐστὶν ἡ AB τῆς $B\Delta$. ἀναστρέψαντι ἡμιολία ἄρα ἐστὶν ἡ BA τῆς $A\Delta$. ὥς δὲ ἡ BA πρὸς τὴν $A\Delta$, οὕτως τὸ ἀπὸ τῆς BA

Proposition 18

To set out the sides of the five (aforementioned) figures, and to compare (them) with one another.[†]



Let the diameter, AB , of the given sphere be laid out. And let it have been cut at C , such that AC is equal to CB , and at D , such that AD is double DB . And let the semi-circle AEB have been drawn on AB . And let CE and DF have been drawn from C and D (respectively), at right-angles to AB . And let AF, FB , and EB have been joined. And since AD is double DB , AB is thus triple BD . Thus, via conversion, BA is one and a half

πρὸς τὸ ἀπὸ τῆς AZ · ἰσογώνιον γάρ ἐστι τὸ AZB τρίγωνον τῷ $AZ\Delta$ τριγώνῳ· ἡμιόλιον ἄρα ἐστὶ τὸ ἀπὸ τῆς BA τοῦ ἀπὸ τῆς AZ . ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει ἡμιολία τῆς πλευρᾶς τῆς πυραμίδος. καὶ ἐστὶν ἡ AB ἡ τῆς σφαίρας διάμετρος· ἡ AZ ἄρα ἴση ἐστὶ τῇ πλευρᾷ τῆς πυραμίδος.

Πάλιν, ἐπεὶ διπλασίον ἐστὶν ἡ AD τῆς ΔB , τριπλῇ ἄρα ἐστὶν ἡ AB τῆς $B\Delta$. ὥς δὲ ἡ AB πρὸς τὴν $B\Delta$, οὕτως τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς BZ · τριπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς BZ . ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίον τῆς τοῦ κύβου πλευρᾶς. καὶ ἐστὶν ἡ AB ἡ τῆς σφαίρας διάμετρος· ἡ BZ ἄρα τοῦ κύβου ἐστὶ πλευρά.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ AG τῇ GB , διπλῇ ἄρα ἐστὶν ἡ AB τῆς BG . ὥς δὲ ἡ AB πρὸς τὴν BG , οὕτως τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς BE · διπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς BE . ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίον τῆς τοῦ ὀκταέδρου πλευρᾶς. καὶ ἐστὶν ἡ AB ἡ τῆς δοθείσης σφαίρας διάμετρος· ἡ BE ἄρα τοῦ ὀκταέδρου ἐστὶ πλευρά.

Ἦχθω δὲ ἀπὸ τοῦ A σημείου τῇ AB εὐθείᾳ πρὸς ὀρθὰς ἡ AH , καὶ κείσθω ἡ AH ἴση τῇ AB , καὶ ἐπεξεύχθω ἡ $H\Gamma$, καὶ ἀπὸ τοῦ Θ ἐπὶ τὴν AB κάθετος ἡχθω ἡ ΘK . καὶ ἐπεὶ διπλῇ ἐστὶν ἡ HA τῆς AG · ἴση γὰρ ἡ HA τῇ AB · ὥς δὲ ἡ HA πρὸς τὴν AG , οὕτως ἡ ΘK πρὸς τὴν $K\Gamma$, διπλῇ ἄρα καὶ ἡ ΘK τῆς $K\Gamma$. τετραπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΘK τοῦ ἀπὸ τῆς $K\Gamma$ · τὰ ἄρα ἀπὸ τῶν ΘK , $K\Gamma$, ὅπερ ἐστὶ τὸ ἀπὸ τῆς $\Theta\Gamma$, πενταπλάσιον ἐστὶ τοῦ ἀπὸ τῆς $K\Gamma$. ἴση δὲ ἡ $\Theta\Gamma$ τῇ GB · πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς $B\Gamma$ τοῦ ἀπὸ τῆς $ΓK$. καὶ ἐπεὶ διπλῇ ἐστὶν ἡ AB τῆς GB , ὣν ἡ AD τῆς ΔB ἐστὶ διπλῇ, λοιπὴ ἄρα ἡ $B\Delta$ λοιπῆς τῆς $\Delta\Gamma$ ἐστὶ διπλῇ. τριπλῇ ἄρα ἡ $B\Gamma$ τῆς $\Gamma\Delta$ · ἐνναπλάσιον ἄρα τὸ ἀπὸ τῆς $B\Gamma$ τοῦ ἀπὸ τῆς $\Gamma\Delta$. πενταπλάσιον δὲ τὸ ἀπὸ τῆς $B\Gamma$ τοῦ ἀπὸ τῆς ΓK · μείζον ἄρα τὸ ἀπὸ τῆς ΓK τοῦ ἀπὸ τῆς $\Gamma\Delta$. μείζων ἄρα ἐστὶν ἡ ΓK τῆς $\Gamma\Delta$. κείσθω τῇ ΓK ἴση ἡ $\Gamma\Lambda$, καὶ ἀπὸ τοῦ Λ τῇ AB πρὸς ὀρθὰς ἡχθω ἡ ΛM , καὶ ἐπεξεύχθω ἡ MB . καὶ ἐπεὶ πενταπλάσιον ἐστὶ τὸ ἀπὸ τῆς $B\Gamma$ τοῦ ἀπὸ τῆς ΓK , καὶ ἐστὶ τῆς μὲν $B\Gamma$ διπλῇ ἡ AB , τῆς δὲ ΓK διπλῇ ἡ $K\Lambda$, πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς $K\Lambda$. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει πενταπλάσιον τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ἀφ' οὗ τὸ εἰκοσάεδρον ἀναγέγραπται. καὶ ἐστὶν ἡ AB ἡ τῆς σφαίρας διάμετρος· ἡ $K\Lambda$ ἄρα ἐκ τοῦ κέντρου ἐστὶ τοῦ κύκλου, ἀφ' οὗ τὸ εἰκοσάεδρον ἀναγέγραπται· ἡ $K\Lambda$ ἄρα ἐξαγώνου ἐστὶ πλευρὰ τοῦ εἰρημένου κύκλου. καὶ ἐπεὶ ἡ τῆς σφαίρας διάμετρος σύγκειται ἐκ τε τῆς τοῦ ἐξαγώνου καὶ δύο τῶν τοῦ δεκαγώνου τῶν εἰς τὸν εἰρημένον κύκλον ἐγγραφομένων, καὶ ἐστὶν ἡ μὲν AB ἡ τῆς σφαίρας διάμετρος, ἡ δὲ $K\Lambda$ ἐξαγώνου πλευρά, καὶ ἴση ἡ AK τῇ AB , ἑκατέρα ἄρα τῶν AK , AB δεκαγώνου ἐστὶ πλευρὰ τοῦ ἐγγραφομένου εἰς τὸν κύκλον, ἀφ' οὗ τὸ εἰκοσάεδρον ἀναγέγραπται. καὶ ἐπεὶ δεκαγώνου μὲν ἡ AB , ἐξαγώνου

times AD . And as BA (is) to AD , so the (square) on BA (is) to the (square) on AF [Def. 5.9]. For triangle AFB is equiangular to triangle AFD [Prop. 6.8]. Thus, the (square) on BA is one and a half times the (square) on AF . And the square on the diameter of the sphere is also one and a half times the (square) on the side of the pyramid [Prop. 13.13]. And AB is the diameter of the sphere. Thus, AF is equal to the side of the pyramid.

Again, since AD is double DB , AB is thus triple BD . And as AB (is) to BD , so the (square) on AB (is) to the (square) on BF [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is three times the (square) on BF . And the square on the diameter of the sphere is also three times the (square) on the side of the cube [Prop. 13.15]. And AB is the diameter of the sphere. Thus, BF is the side of the cube.

And since AC is equal to CB , AB is thus double BC . And as AB (is) to BC , so the (square) on AB (is) to the (square) on BE [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is double the (square) on BE . And the square on the diameter of the sphere is also double the (square) on the side of the octagon [Prop. 13.14]. And AB is the diameter of the given sphere. Thus, BE is the side of the octagon.

So let AG have been drawn from point A at right-angles to the straight-line AB . And let AG be made equal to AB . And let GC have been joined. And let HK have been drawn from H , perpendicular to AB . And since GA is double AC . For GA (is) equal to AB . And as GA (is) to AC , so HK (is) to KC [Prop. 6.4]. HK (is) thus also double KC . Thus, the (square) on HK is four times the (square) on KC . Thus, the (sum of the squares) on HK and KC , which is the (square) on HC [Prop. 1.47], is five times the (square) on KC . And HC (is) equal to CB . Thus, the (square) on BC (is) five times the (square) on CK . And since AB is double CB , of which AD is double DB , the remainder BD is thus double the remainder DC . BC (is) thus triple CD . The (square) on BC (is) thus nine times the (square) on CD . And the (square) on BC (is) five times the (square) on CK . Thus, the (square) on CK (is) greater than the (square) on CD . CK is thus greater than CD . Let CL be made equal to CK . And let LM have been drawn from L at right-angles to AB . And let MB have been joined. And since the (square) on BC is five times the (square) on CK , and AB is double BC , and KL double CK , the (square) on AB is thus five times the (square) on KL . And the square on the diameter of the sphere is also five times the (square) on the radius of the circle from which the icosahedron has been described [Prop. 13.16 corr.]. And AB is the diameter of the sphere. Thus, KL is the radius of the circle from

δὲ ἡ $ΜΑ$ ἴση γάρ ἐστι τῇ $ΚΛ$, ἐπεὶ καὶ τῇ $ΘΚ$ ἴσον γὰρ ἀπέχουσιν ἀπὸ τοῦ κέντρου· καὶ ἐστὶν ἑκατέρω τῶν $ΘΚ$, $ΚΛ$ διπλασίων τῆς $ΚΓ$ · πενταγώνου ἄρα ἐστὶν ἡ $ΜΒ$. ἡ δὲ τοῦ πενταγώνου ἐστὶν ἡ τοῦ εἰκοσαέδρου· εἰκοσαέδρου ἄρα ἐστὶν ἡ $ΜΒ$.

Καὶ ἐπεὶ ἡ ZB κύβου ἐστὶ πλευρά, τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ N , καὶ ἔστω μείζων τμήμα τὸ NB · ἡ NB ἄρα δωδεκαέδρου ἐστὶ πλευρά.

Καὶ ἐπεὶ ἡ τῆς σφαίρας διάμετρος ἐδείχθη τῆς μὲν AZ πλευρᾶς τῆς πυραμίδος δυνάμει ἡμιολία, τῆς δὲ τοῦ ὀκταέδρου τῆς BE δυνάμει διπλασίων, τῆς δὲ τοῦ κύβου τῆς ZB δυνάμει τριπλασίων, οἷων ἄρα ἡ τῆς σφαίρας διάμετρος δυνάμει ἕξ, τοιούτων ἡ μὲν τῆς πυραμίδος τεσσάρων, ἡ δὲ τοῦ ὀκταέδρου τριῶν, ἡ δὲ τοῦ κύβου δύο. ἡ μὲν ἄρα τῆς πυραμίδος πλευρὰ τῆς μὲν τοῦ ὀκταέδρου πλευρᾶς δυνάμει ἐστὶν ἐπίτριστος, τῆς δὲ τοῦ κύβου δυνάμει διπλῇ, ἡ δὲ τοῦ ὀκταέδρου τῆς τοῦ κύβου δυνάμει ἡμιολία. αἱ μὲν οὖν εἰρημέναι τῶν τριῶν σχημάτων πλευραί, λέγω δὴ πυραμίδος καὶ ὀκταέδρου καὶ κύβου, πρὸς ἀλλήλας εἰσὶν ἐν λόγοις ῥητοῖς. αἱ δὲ λοιπαὶ δύο, λέγω δὴ ἡ τε τοῦ εἰκοσαέδρου καὶ ἡ τοῦ δωδεκαέδρου, οὔτε πρὸς ἀλλήλας οὔτε πρὸς τὰς προειρημένας εἰσὶν ἐν λόγοις ῥητοῖς· ἄλογοι γάρ εἰσιν, ἡ μὲν ἐλάττω, ἡ δὲ ἀποτομή.

Ὅτι μείζων ἐστὶν ἡ τοῦ εἰκοσαέδρου πλευρὰ ἡ $ΜΒ$ τῆς τοῦ δωδεκαέδρου τῆς NB , δείξομεν οὕτως.

Ἐπεὶ γὰρ ἰσογώνιον ἐστὶ τὸ $ZΔB$ τρίγωνον τῷ ZAB τριγώνῳ, ἀνάλογόν ἐστιν ὡς ἡ $ΔB$ πρὸς τὴν BZ , οὕτως ἡ BZ πρὸς τὴν BA . καὶ ἐπεὶ τρεῖς εὐθεῖαι ἀνάλογόν εἰσιν, ἔστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας· ἔστιν ἄρα ὡς ἡ $ΔB$ πρὸς τὴν BA , οὕτως τὸ ἀπὸ τῆς $ΔB$ πρὸς τὸ ἀπὸ τῆς BZ · ἀνάπαλιν ἄρα ὡς ἡ AB πρὸς τὴν $BΔ$, οὕτως τὸ ἀπὸ τῆς ZB πρὸς τὸ ἀπὸ τῆς $BΔ$. τριπλῇ δὲ ἡ AB τῆς $BΔ$ · τριπλάσιον ἄρα τὸ ἀπὸ τῆς ZB τοῦ ἀπὸ τῆς $BΔ$. ἔστι δὲ καὶ τὸ ἀπὸ τῆς $ΑΔ$ τοῦ ἀπὸ τῆς $ΔB$ τετραπλάσιον· διπλῇ γὰρ ἡ $ΑΔ$ τῆς $ΔB$ · μείζων ἄρα τὸ ἀπὸ τῆς $ΑΔ$ τοῦ ἀπὸ τῆς ZB · μείζων ἄρα ἡ $ΑΔ$ τῆς ZB · πολλῶ ἄρα ἡ $ΑΔ$ τῆς ZB μείζων ἐστὶν. καὶ τῆς μὲν $ΑΔ$ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζων τμήμα ἐστὶν ἡ $ΚΛ$, ἐπειδὴ περ ἡ μὲν $ΑΚ$ ἐξαγώνου ἐστὶν, ἡ δὲ $ΚΑ$ δεκαγώνου· τῆς δὲ ZB ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζων τμήμα ἐστὶν ἡ NB · μείζων ἄρα ἡ $ΚΛ$ τῆς NB . ἴση δὲ ἡ $ΚΛ$ τῇ $ΑΜ$ · μείζων ἄρα ἡ $ΑΜ$ τῆς NB [τῆς δὲ $ΑΜ$ μείζων ἐστὶν ἡ $ΜΒ$]. πολλῶ ἄρα ἡ $ΜΒ$ πλευρὰ οὔσα τοῦ εἰκοσαέδρου μείζων ἐστὶ τῆς NB πλευρᾶς οὔσης τοῦ δωδεκαέδρου· ὅπερ ἔδει δεῖξαι.

which the icosahedron has been described. Thus, KL is (the side) of the hexagon (inscribed) in the aforementioned circle [Prop. 4.15 corr.]. And since the diameter of the sphere is composed of (the side) of the hexagon, and two of (the sides) of the decagon, inscribed in the aforementioned circle, and AB is the diameter of the sphere, and KL the side of the hexagon, and AK (is) equal to LB , thus AK and LB are each sides of the decagon inscribed in the circle from which the icosahedron has been described. And since LB is (the side) of the decagon. And ML (is the side) of the hexagon—for (it is) equal to KL , since (it is) also (equal) to HK , for they are equally far from the center. And HK and KL are each double KC . MB is thus (the side) of the pentagon (inscribed in the circle) [Props. 13.10, 1.47]. And (the side) of the pentagon is (the side) of the icosahedron [Prop. 13.16]. Thus, MB is (the side) of the icosahedron.

And since FB is the side of the cube, let it have been cut in extreme and mean ratio at N , and let NB be the greater piece. Thus, NB is the side of the dodecahedron [Prop. 13.17 corr.].

And since the (square) on the diameter of the sphere was shown (to be) one and a half times the square on the side, AF , of the pyramid, and twice the square on (the side), BE , of the octagon, and three times the square on (the side), FB , of the cube, thus, of whatever (parts) the (square) on the diameter of the sphere (makes) six, of such (parts) the (square) on (the side) of the pyramid (makes) four, and (the square) on (the side) of the octagon three, and (the square) on (the side) of the cube two. Thus, the (square) on the side of the pyramid is one and a third times the square on the side of the octagon, and double the square on (the side) of the cube. And the (square) on (the side) of the octahedron is one and a half times the square on (the side) of the cube. Therefore, the aforementioned sides of the three figures—I mean, of the pyramid, and of the octahedron, and of the cube—are in rational ratios to one another. And (the sides of) the remaining two (figures)—I mean, of the icosahedron, and of the dodecahedron—are neither in rational ratios to one another, nor to the (sides) of the aforementioned (three figures). For they are irrational (straight-lines): (namely), a minor [Prop. 13.16], and an apotome [Prop. 13.17].

(And), we can show that the side, MB , of the icosahedron is greater than the (side), NB , or the dodecahedron, as follows.

For, since triangle FDB is equiangular to triangle FAB [Prop. 6.8], proportionally, as DB is to BF , so BF (is) to BA [Prop. 6.4]. And since three straight-lines are (continually) proportional, as the first (is) to the third,

so the (square) on the first (is) to the (square) on the second [Def. 5.9, Prop. 6.20 corr.]. Thus, as DB is to BA , so the (square) on DB (is) to the (square) on BF . Thus, inversely, as AB (is) to BD , so the (square) on FB (is) to the (square) on BD . And AB (is) triple BD . Thus, the (square) on FB (is) three times the (square) on BD . And the (square) on AD is also four times the (square) on DB . For AD (is) double DB . Thus, the (square) on AD (is) greater than the (square) on FB . Thus, AD (is) greater than FB . Thus, AL is much greater than FB . And KL is the greater piece of AL , which is cut in extreme and mean ratio—inasmuch as LK is (the side) of the hexagon, and KA (the side) of the decagon [Prop. 13.9]. And NB is the greater piece of FB , which is cut in extreme and mean ratio. Thus, KL (is) greater than NB . And KL (is) equal to LM . Thus, LM (is) greater than NB [and MB is greater than LM]. Thus, MB , which is (the side) of the icosahedron, is much greater than NB , which is (the side) of the dodecahedron. (Which is) the very thing it was required to show.

† If the radius of the given sphere is unity then the sides of the pyramid (i.e., tetrahedron), octahedron, cube, icosahedron, and dodecahedron, respectively, satisfy the following inequality: $\sqrt{8/3} > \sqrt{2} > \sqrt{4/3} > (1/\sqrt{5}) \sqrt{10 - 2\sqrt{5}} > (1/3) (\sqrt{15} - \sqrt{3})$.

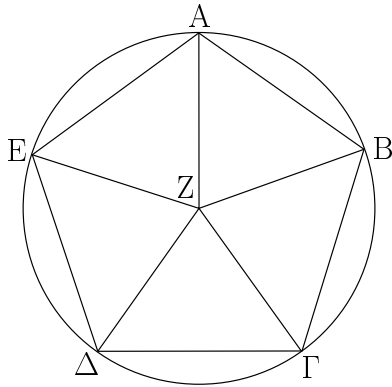
Λέγω δὴ, ὅτι παρὰ τὰ εἰρημένα πέντε σχήματα οὐ συσταθήσεται ἕτερον σχῆμα περιεχόμενον ὑπὸ ἰσοπλευρῶν τε καὶ ἰσογωνίων ἴσων ἀλλήλοις.

Ὑπὸ μὲν γὰρ δύο τριγώνων ἢ ὅλως ἐπιπέδων στερεὰ γωνία οὐ συνίσταται. ὑπὸ δὲ τριῶν τριγώνων ἢ τῆς πυραμίδος, ὑπὸ δὲ τεσσάρων ἢ τοῦ ὀκταέδρου, ὑπὸ δὲ πέντε ἢ τοῦ εἰκοσαέδρου· ὑπὸ δὲ ἑξ τριγώνων ἰσοπλευρῶν τε καὶ ἰσογωνίων πρὸς ἐνὶ σημείῳ συνισταμένων οὐκ ἔσται στερεὰ γωνία· οὕσης γὰρ τῆς τοῦ ἰσοπλευροῦ τριγώνου γωνίας διμοίρου ὀρθῆς ἔσονται αἱ ἑξ τέσσαρσιν ὀρθαῖς ἴσαι· ὅπερ ἀδύνατον· ἅπαντα γὰρ στερεὰ γωνία ὑπὸ ἐλασσόνων ἢ τεσσάρων ὀρθῶν περέχεται. διὰ τὰ αὐτὰ δὲ οὐδὲ ὑπὸ πλειόνων ἢ ἑξ γωνιῶν ἐπιπέδων στερεὰ γωνία συνίσταται. ὑπὸ δὲ τετραγώνων τριῶν ἢ τοῦ κύβου γωνία περιέχεται· ὑπὸ δὲ τεσσάρων ἀδύνατον· ἔσονται γὰρ πάλιν τέσσαρες ὀρθαί. ὑπὸ δὲ πενταγώνων ἰσοπλευρῶν καὶ ἰσογωνίων, ὑπὸ μὲν τριῶν ἢ τοῦ δωδεκαέδρου· ὑπὸ δὲ τεσσάρων ἀδύνατον· οὕσης γὰρ τῆς τοῦ πενταγώνου ἰσοπλευροῦ γωνίας ὀρθῆς καὶ πέμπτου, ἔσονται αἱ τέσσαρες γωνίαι τεσσάρων ὀρθῶν μείζους· ὅπερ ἀδύνατον. οὐδὲ μὴν ὑπὸ πολυγώνων ἐτέρων σχημάτων περισχεθήσεται στερεὰ γωνία διὰ τὸ αὐτὸ ἄτοπον.

Οὐκ ἄρα παρὰ τὰ εἰρημένα πέντε σχήματα ἕτερον σχῆμα στερεὸν συσταθήσεται ὑπὸ ἰσοπλευρῶν τε καὶ ἰσογωνίων περιεχόμενον· ὅπερ ἔδει δεῖξαι.

So, I say that, beside the five aforementioned figures, no other (solid) figure can be constructed (which is) contained by equilateral and equiangular (planes), equal to one another.

For a solid angle cannot be constructed from two triangles, or indeed (two) planes (of any sort) [Def. 11.11]. And (the solid angle) of the pyramid (is constructed) from three (equiangular) triangles, and (that) of the octahedron from four (triangles), and (that) of the icosahedron from (five) triangles. And a solid angle cannot be (made) from six equilateral and equiangular triangles set up together at one point. For, since the angles of a equilateral triangle are (each) two-thirds of a right-angle, the (sum of the) six (plane) angles (containing the solid angle) will be four right-angles. The very thing (is) impossible. For every solid angle is contained by (plane angles whose sum is) less than four right-angles [Prop. 11.21]. So, for the same (reasons), a solid angle cannot be constructed from more than six plane angles (equal to two-thirds of a right-angle) either. And the (solid) angle of a cube is contained by three squares. And (a solid angle contained) by four (squares is) impossible. For, again, the (sum of the plane angles containing the solid angle) will be four right-angles. And (the solid angle) of a dodecahedron (is contained) by three equilateral and equiangular pentagons. And (a solid angle contained) by four



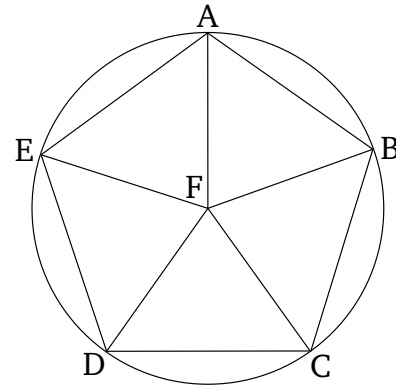
Λήμμα.

Ὅτι δὲ ἡ τοῦ ἰσοπλεύρου καὶ ἰσογώνιου πενταγώνου γωνία ὀρθή ἐστι καὶ πέμπτου, οὕτω δεικτέον.

Ἐστω γὰρ πεντάγωνον ἰσόπλευρον καὶ ἰσογώνιον τὸ ΑΒΓΔΕ, καὶ περιγεγράφθω περὶ αὐτὸ κύκλος ὁ ΑΒΓΔΕ, καὶ εἰληφθῇ αὐτοῦ τὸ κέντρον τὸ Ζ, καὶ ἐπεζεύχθωσαν αἱ ΖΑ, ΖΒ, ΖΓ, ΖΔ, ΖΕ. δίχα ἄρα τέμνουσι τὰς πρὸς τοῖς Α, Β, Γ, Δ, Ε τοῦ πενταγώνου γωνίας. καὶ ἐπεὶ αἱ πρὸς τῷ Ζ πέντε γωνίαι τέσσαρσιν ὀρθαῖς ἴσαι εἰσὶ καὶ εἰσιν ἴσαι, μία ἄρα αὐτῶν, ὡς ἡ ὑπὸ ΑΖΒ, μιᾶς ὀρθῆς ἐστι παρὰ πέμπτου· λοιπαὶ ἄρα αἱ ὑπὸ ΖΑΒ, ΑΒΖ μιᾶς εἰσιν ὀρθῆς καὶ πέμπτου. ἴση δὲ ἡ ὑπὸ ΖΑΒ τῇ ὑπὸ ΖΒΓ· καὶ ὅλη ἄρα ἡ ὑπὸ ΑΒΓ τοῦ πενταγώνου γωνία μιᾶς ἐστὶν ὀρθῆς καὶ πέμπτου· ὅπερ ἔδει δεῖξαι.

(equiangular pentagons is) impossible. For, the angle of an equilateral pentagon being one and one-fifth of right-angle, four (such) angles will be greater (in sum) than four right-angles. The very thing (is) impossible. And, on account of the same absurdity, a solid angle cannot be constructed from any other (equiangular) polygonal figures either.

Thus, beside the five aforementioned figures, no other solid figure can be constructed (which is) contained by equilateral and equiangular (planes). (Which is) the very thing it was required to show.



Lemma

It can be shown that the angle of an equilateral and equiangular pentagon is one and one-fifth of a right-angle, as follows.

For let $ABCDE$ be an equilateral and equiangular pentagon, and let the circle $ABCDE$ have been circumscribed about it [Prop. 4.14]. And let its center, F , have been found [Prop. 3.1]. And let FA , FB , FC , FD , and FE have been joined. Thus, they cut the angles of the pentagon in half at (points) A , B , C , D , and E [Prop. 1.4]. And since the five angles at F are equal (in sum) to four right-angles, and are also equal (to one another), (any) one of them, like AFB , is thus one less a fifth of a right-angle. Thus, the (sum of the) remaining (angles in triangle ABF), FAB and ABF , is one plus a fifth of a right-angle [Prop. 1.32]. And FAB (is) equal to FBC . Thus, the whole angle, ABC , of the pentagon is also one and one-fifth of a right-angle. (Which is) the very thing it was required to show.