

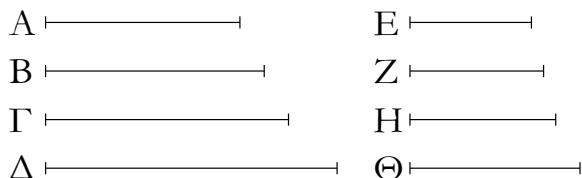
ELEMENTS BOOK 8

Continued Proportion[†]

[†]The propositions contained in Books 7–9 are generally attributed to the school of Pythagoras.

α'.

Ἐάν ὧσιν ὁσοιοποτοῦν ἀριθμοὶ ἐξῆς ἀνάλογον, οἱ δὲ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους ὧσιν, ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.



Ἐστωσαν ὁποιοιοῦν ἀριθμοὶ ἐξῆς ἀνάλογον οἱ A, B, Γ, Δ, οἱ δὲ ἄκροι αὐτῶν οἱ A, Δ, πρῶτοι πρὸς ἀλλήλους ἔστωσαν· λέγω, ὅτι οἱ A, B, Γ, Δ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Εἰ γὰρ μή, ἔστωσαν ἐλάττονες τῶν A, B, Γ, Δ οἱ E, Z, H, Θ ἐν τῷ αὐτῷ λόγῳ ὄντες αὐτοῖς. καὶ ἐπεὶ οἱ A, B, Γ, Δ ἐν τῷ αὐτῷ λόγῳ εἰσὶ τοῖς E, Z, H, Θ, καὶ ἔστιν ἴσον τὸ πλήθος [τῶν A, B, Γ, Δ] τῷ πλήθει [τῶν E, Z, H, Θ], δι' ἴσου ἄρα ἔστιν ὡς A πρὸς τὸν Δ, ὁ E πρὸς τὸν Θ. οἱ δὲ A, Δ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἐχοντας ἰσάκεις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον. μετρεῖ ἄρα ὁ A τὸν E ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἔστιν ἀδύνατον. οὐκ ἄρα οἱ E, Z, H, Θ ἐλάσσονες ὄντες τῶν A, B, Γ, Δ ἐν τῷ αὐτῷ λόγῳ εἰσὶν αὐτοῖς. οἱ A, B, Γ, Δ ἄρα ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς· ὅπερ ἔδει δεῖξαι.

β'.

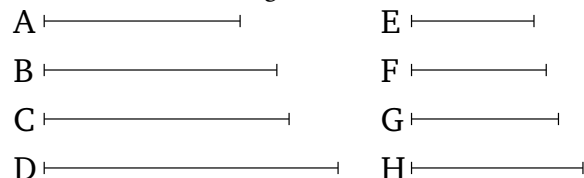
Αριθμοὺς εὑρεῖν ἐξῆς ἀνάλογον ἐλάχιστους, ὅσους ἂν ἐπιτάξῃ τις, ἐν τῷ δοθέντι λόγῳ.

Ἐστω ὁ δοθείς λόγος ἐν ἐλάχιστοις ἀριθμοῖς ὁ τοῦ A πρὸς τὸν B· δεῖ δὴ ἀριθμοὺς εὑρεῖν ἐξῆς ἀνάλογον ἐλάχιστους, ὅσους ἂν τις ἐπιτάξῃ, ἐν τῷ τοῦ A πρὸς τὸν B λόγῳ.

Ἐπιτετάχθωσαν δὴ τέσσαρες, καὶ ὁ A ἑαυτὸν πολλαπλασιάσας τὸν Γ ποιεῖτω, τὸν δὲ B πολλαπλασιάσας τὸν Δ ποιεῖτω, καὶ ἔτι ὁ B ἑαυτὸν πολλαπλασιάσας τὸν E ποιεῖτω, καὶ ἔτι ὁ A τοὺς Γ, Δ, E πολλαπλασιάσας τοὺς Z, H, Θ ποιεῖτω, ὁ δὲ B τὸν E πολλαπλασιάσας τὸν K ποιεῖτω.

Proposition 1

If there are any multitude whatsoever of continuously proportional numbers, and the outermost of them are prime to one another, then the (numbers) are the least of those (numbers) having the same ratio as them.



Let A, B, C, D be any multitude whatsoever of continuously proportional numbers. And let the outermost of them, A and D, be prime to one another. I say that A, B, C, D are the least of those (numbers) having the same ratio as them.

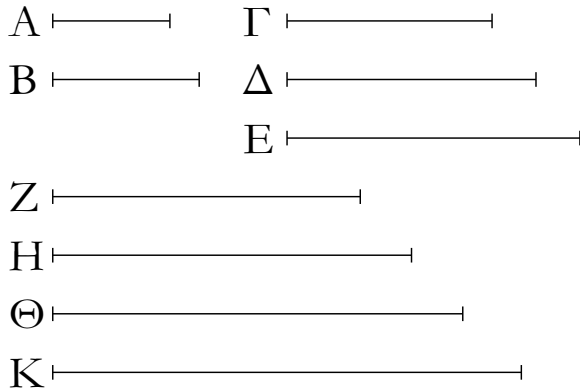
For if not, let E, F, G, H be less than A, B, C, D (respectively), being in the same ratio as them. And since A, B, C, D are in the same ratio as E, F, G, H, and the multitude [of A, B, C, D] is equal to the multitude [of E, F, G, H], thus, via equality, as A is to D, (so) E (is) to H [Prop. 7.14]. And A and D (are) prime (to one another). And prime (numbers are) also the least of those (numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, A measures E, the greater (measuring) the lesser. The very thing is impossible. Thus, E, F, G, H, being less than A, B, C, D, are not in the same ratio as them. Thus, A, B, C, D are the least of those (numbers) having the same ratio as them. (Which is) the very thing it was required to show.

Proposition 2

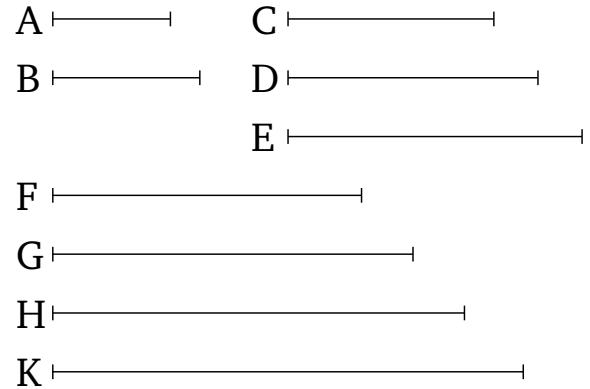
To find the least numbers, as many as may be prescribed, (which are) continuously proportional in a given ratio.

Let the given ratio, (expressed) in the least numbers, be that of A to B. So it is required to find the least numbers, as many as may be prescribed, (which are) in the ratio of A to B.

Let four (numbers) have been prescribed. And let A make C (by) multiplying itself, and let it make D (by) multiplying B. And, further, let B make E (by) multiplying itself. And, further, let A make F, G, H (by) multiplying C, D, E. And let B make K (by) multiplying E.



Καὶ ἐπεὶ ὁ A ἑαυτὸν μὲν πολλαπλασιάσας τὸν Γ πεποίηκεν, τὸν δὲ B πολλαπλασιάσας τὸν Δ πεποίηκεν, ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B , [οὕτως] ὁ Γ πρὸς τὸν Δ . πάλιν, ἐπεὶ ὁ μὲν A τὸν B πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ δὲ B ἑαυτὸν πολλαπλασιάσας τὸν E πεποίηκεν, ἐκάτερος ἄρα τῶν A, B τὸν B πολλαπλασιάσας ἐκάτερον τῶν Δ, E πεποίηκεν. ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Δ πρὸς τὸν E . ἀλλ' ὡς ὁ A πρὸς τὸν B , ὁ Γ πρὸς τὸν Δ · καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Δ , ὁ Δ πρὸς τὸν E . καὶ ἐπεὶ ὁ A τοὺς Γ, Δ πολλαπλασιάσας τοὺς Z, H πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ , [οὕτως] ὁ Z πρὸς τὸν H . ὡς δὲ ὁ Γ πρὸς τὸν Δ , οὕτως ἦν ὁ A πρὸς τὸν B · καὶ ὡς ἄρα ὁ A πρὸς τὸν B , ὁ Z πρὸς τὸν H . πάλιν, ἐπεὶ ὁ A τοὺς Δ, E πολλαπλασιάσας τοὺς H, Θ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν E , ὁ H πρὸς τὸν Θ . ἀλλ' ὡς ὁ Δ πρὸς τὸν E , ὁ A πρὸς τὸν B . καὶ ὡς ἄρα ὁ A πρὸς τὸν B , οὕτως ὁ H πρὸς τὸν Θ . καὶ ἐπεὶ οἱ A, B τὸν E πολλαπλασιάσαντες τοὺς Θ, K πεποιήκασιν, ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Θ πρὸς τὸν K . ἀλλ' ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Z πρὸς τὸν H καὶ ὁ H πρὸς τὸν Θ . καὶ ὡς ἄρα ὁ Z πρὸς τὸν H , οὕτως ὁ Θ πρὸς τὸν K · οἱ Γ, Δ, E ἄρα καὶ οἱ Z, H, Θ, K ἀνάλογόν εἰσιν ἐν τῷ τοῦ A πρὸς τὸν B λόγῳ. λέγω δὴ, ὅτι καὶ ἐλάχιστοι. ἐπεὶ γὰρ οἱ A, B ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, οἱ δὲ ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων πρῶτοι πρὸς ἀλλήλους εἰσίν, οἱ A, B ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐκάτερος μὲν τῶν A, B ἑαυτὸν πολλαπλασιάσας ἐκάτερον τῶν Γ, E πεποίηκεν, ἐκάτερον δὲ τῶν Γ, E πολλαπλασιάσας ἐκάτερον τῶν Z, K πεποίηκεν· οἱ Γ, E ἄρα καὶ οἱ Z, K πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐὰν δὲ ὦσιν ὅποιοι ἄριθμοι ἐξῆς ἀνάλογον, οἱ δὲ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους ὦσιν, ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς. οἱ Γ, Δ, E ἄρα καὶ οἱ Z, H, Θ, K ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς A, B · ὅπερ εἶδει δεῖξαι.



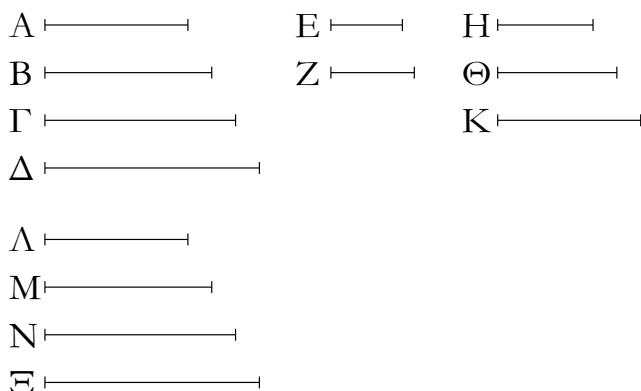
And since A has made C (by) multiplying itself, and has made D (by) multiplying B , thus as A is to B , [so] C (is) to D [Prop. 7.17]. Again, since A has made D (by) multiplying B , and B has made E (by) multiplying itself, A, B have thus made D, E , respectively, (by) multiplying B . Thus, as A is to B , so D (is) to E [Prop. 7.18]. But, as A (is) to B , (so) C (is) to D . And thus as C (is) to D , (so) D (is) to E . And since A has made F, G (by) multiplying C, D , thus as C is to D , [so] F (is) to G [Prop. 7.17]. And as C (is) to D , so A was to B . And thus as A (is) to B , (so) F (is) to G . Again, since A has made G, H (by) multiplying D, E , thus as D is to E , (so) G (is) to H [Prop. 7.17]. But, as D (is) to E , (so) A (is) to B . And thus as A (is) to B , so G (is) to H . And since A, B have made H, K (by) multiplying E , thus as A is to B , so H (is) to K . But, as A (is) to B , so F (is) to G , and G to H . And thus as F (is) to G , so G (is) to H , and H to K . Thus, C, D, E and F, G, H, K are (both continuously) proportional in the ratio of A to B . So I say that (they are) also the least (sets of numbers continuously proportional in that ratio). For since A and B are the least of those (numbers) having the same ratio as them, and the least of those (numbers) having the same ratio are prime to one another [Prop. 7.22], A and B are thus prime to one another. And A, B have made C, E , respectively, (by) multiplying themselves, and have made F, K by multiplying C, E , respectively. Thus, C, E and F, K are prime to one another [Prop. 7.27]. And if there are any multitude whatsoever of continuously proportional numbers, and the outermost of them are prime to one another, then the (numbers) are the least of those (numbers) having the same ratio as them [Prop. 8.1]. Thus, C, D, E and F, G, H, K are the least of those (continuously proportional sets of numbers) having the same ratio as A and B . (Which is) the very thing it was required to show.

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν τρεῖς ἀριθμοὶ ἐξῆς ἀνάλογον ἐλάχιστοι ἀνάλογον ἐλάχιστοι ὥσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, οἱ ἄκρον αὐτῶν τετράγωνοί εἰσιν, ἐὰν δὲ τέσσαρες, κύβοι.

Υ'.

Ἐὰν ὥσιν ὅποσοι οὖν ἀριθμοὶ ἐξῆς ἀνάλογον ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν.



Ἐστῶσαν ὅποσοι οὖν ἀριθμοὶ ἐξῆς ἀνάλογον ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς οἱ A, B, Γ, Δ· λέγω, ὅτι οἱ ἄκροι αὐτῶν οἱ A, Δ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰλήφθωσαν γὰρ δύο μὲν ἀριθμοὶ ἐλάχιστοι ἐν τῷ τῶν A, B, Γ, Δ λόγῳ οἱ E, Z, τρεῖς δὲ οἱ H, Θ, K, καὶ ἐξῆς ἐνὶ πλείους, ἕως τὸ λαμβανόμενον πλῆθος ἴσον γέννηται τῷ πλήθει τῶν A, B, Γ, Δ. εἰλήφθωσαν καὶ ἕστωσαν οἱ Λ, Μ, Ν, Ξ.

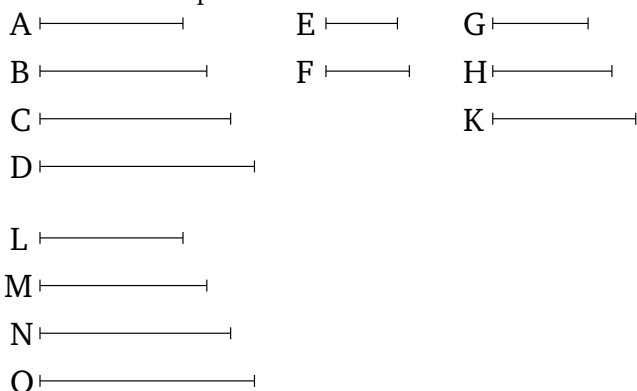
Καὶ ἐπεὶ οἱ E, Z ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ ἑκάτερος τῶν E, Z ἑαυτὸν μὲν πολλαπλασιάσας ἑκάτερον τῶν H, K πεποίηκεν, ἑκάτερον δὲ τῶν H, K πολλαπλασιάσας ἑκάτερον τῶν Λ, Ξ πεποίηκεν, καὶ οἱ H, K ἄρα καὶ οἱ Λ, Ξ πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ οἱ A, B, Γ, Δ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, εἰσὶ δὲ καὶ οἱ Λ, Μ, Ν, Ξ ἐλάχιστοι ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς A, B, Γ, Δ, καὶ ἐστὶν ἴσον τὸ πλῆθος τῶν A, B, Γ, Δ τῷ πλήθει τῶν Λ, Μ, Ν, Ξ, ἕκαστος ἄρα τῶν A, B, Γ, Δ ἑκάστῳ τῶν Λ, Μ, Ν, Ξ ἴσος ἐστίν· ἴσος ἄρα ἐστὶν ὁ μὲν A τῷ Λ, ὁ δὲ Δ τῷ Ξ. καὶ εἰσιν οἱ Λ, Ξ πρῶτοι πρὸς ἀλλήλους. καὶ οἱ A, Δ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

Corollary

So it is clear, from this, that if three continuously proportional numbers are the least of those (numbers) having the same ratio as them then the outermost of them are square, and, if four (numbers), cube.

Proposition 3

If there are any multitude whatsoever of continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them then the outermost of them are prime to one another.



Let A, B, C, D be any multitude whatsoever of continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them. I say that the outermost of them, A and D, are prime to one another.

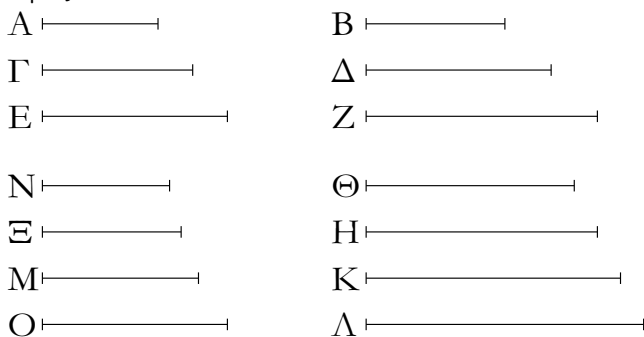
For let the two least (numbers) E, F (which are) in the same ratio as A, B, C, D have been taken [Prop. 7.33]. And the three (least numbers) G, H, K [Prop. 8.2]. And (so on), successively increasing by one, until the multitude of (numbers) taken is made equal to the multitude of A, B, C, D. Let them have been taken, and let them be L, M, N, O.

And since E and F are the least of those (numbers) having the same ratio as them they are prime to one another [Prop. 7.22]. And since E, F have made G, K, respectively, (by) multiplying themselves [Prop. 8.2 corr.], and have made L, O (by) multiplying G, K, respectively, G, K and L, O are thus also prime to one another [Prop. 7.27]. And since A, B, C, D are the least of those (numbers) having the same ratio as them, and L, M, N, O are also the least (of those numbers having the same ratio as them), being in the same ratio as A, B, C, D, and the multitude of A, B, C, D is equal to the multitude of L, M, N, O, thus A, B, C, D are equal to L, M, N, O, respectively. Thus, A is equal to L, and D to O. And L and O are prime to one another. Thus, A and D are also prime to one another. (Which is) the very thing it was

required to show.

δ'.

Λόγων δοθέντων ὁποσωνοῦν ἐν ἐλάχιστοις ἀριθμοῖς ἀριθμοὺς εὑρεῖν ἐξῆς ἀνάλογον ἐλάχιστους ἐν τοῖς δοθεῖσι λόγοις.

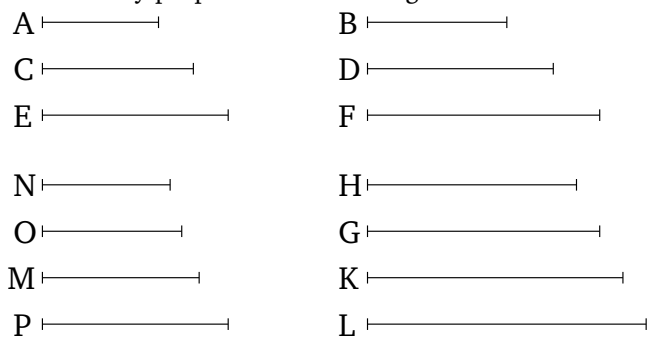


Ἐστωσαν οἱ δοθέντες λόγοι ἐν ἐλάχιστοις ἀριθμοῖς ὅ τε τοῦ A πρὸς τὸν B καὶ ὁ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι ὁ τοῦ E πρὸς τὸν Z· δεῖ δὴ ἀριθμοὺς εὑρεῖν ἐξῆς ἀνάλογον ἐλάχιστους ἐν τε τῷ τοῦ A πρὸς τὸν B λόγῳ καὶ ἐν τῷ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τῷ τοῦ E πρὸς τὸν Z.

Εἰλήφθω γὰρ ὁ ὑπὸ τῶν B, Γ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ H. καὶ ὁσάκις μὲν ὁ B τὸν H μετρεῖ, τοσαυτάκις καὶ ὁ A τὸν Θ μετρεῖται, ὁσάκις δὲ ὁ Γ τὸν H μετρεῖ, τοσαυτάκις καὶ ὁ Δ τὸν K μετρεῖται. ὁ δὲ E τὸν K ἢ τοι μετρεῖ ἢ οὐ μετρεῖ. μετρεῖται πρότερον. καὶ ὁσάκις ὁ E τὸν K μετρεῖ, τοσαυτάκις καὶ ὁ Z τὸν Λ μετρεῖται. καὶ ἐπεὶ ἰσάκις ὁ A τὸν Θ μετρεῖ καὶ ὁ B τὸν H, ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B, οὕτως ὁ Θ πρὸς τὸν H. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ H πρὸς τὸν K, καὶ ἔτι ὡς ὁ E πρὸς τὸν Z, οὕτως ὁ K πρὸς τὸν Λ· οἱ Θ, H, K, Λ ἄρα ἐξῆς ἀνάλογόν εἰσιν ἐν τε τῷ τοῦ A πρὸς τὸν B καὶ ἐν τῷ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι ἐν τῷ τοῦ E πρὸς τὸν Z λόγῳ. λέγω δὴ, ὅτι καὶ ἐλάχιστοι. εἰ γὰρ μὴ εἰσιν οἱ Θ, H, K, Λ ἐξῆς ἀνάλογον ἐλάχιστοι ἐν τε τοῖς τοῦ A πρὸς τὸν B καὶ τοῦ Γ πρὸς τὸν Δ καὶ ἐν τῷ τοῦ E πρὸς τὸν Z λόγοις, ἔστωσαν οἱ N, Ξ, M, O. καὶ ἐπεὶ ἔστιν ὡς ὁ A πρὸς τὸν B, οὕτως ὁ N πρὸς τὸν Ξ, οἱ δὲ A, B ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον, ὁ B ἄρα τὸν Ξ μετρεῖ. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν Ξ μετρεῖ· οἱ B, Γ ἄρα τὸν Ξ μετροῦσιν· καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν B, Γ μετρούμενος τὸν Ξ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν B, Γ μετρεῖται ὁ H· ὁ H ἄρα τὸν Ξ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἔσσονται τινες τῶν Θ, H, K, Λ ἐλάσσονες ἀριθμοὶ ἐξῆς ἐν τε τῷ τοῦ A πρὸς τὸν B καὶ τῷ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τῷ τοῦ E πρὸς τὸν Z λόγῳ.

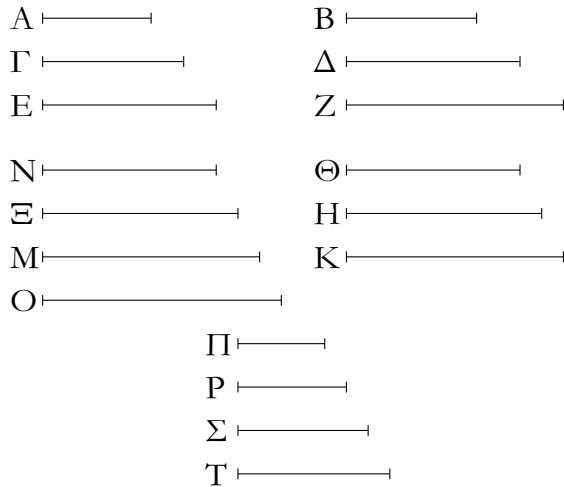
Proposition 4

For any multitude whatsoever of given ratios, (expressed) in the least numbers, to find the least numbers continuously proportional in these given ratios.



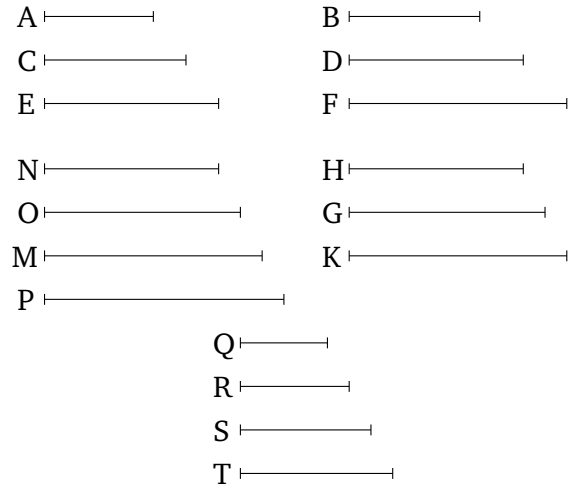
Let the given ratios, (expressed) in the least numbers, be the (ratios) of A to B , and of C to D , and, further, of E to F . So it is required to find the least numbers continuously proportional in the ratio of A to B , and of C to B , and, further, of E to F .

For let the least number, G , measured by (both) B and C have been taken [Prop. 7.34]. And as many times as B measures G , so many times let A also measure H . And as many times as C measures G , so many times let D also measure K . And E either measures, or does not measure, K . Let it, first of all, measure (K). And as many times as E measures K , so many times let F also measure L . And since A measures H the same number of times that B also (measures) G , thus as A is to B , so H (is) to G [Def. 7.20, Prop. 7.13]. And so, for the same (reasons), as C (is) to D , so G (is) to K , and, further, as E (is) to F , so K (is) to L . Thus, H, G, K, L are continuously proportional in the ratio of A to B , and of C to D , and, further, of E to F . So I say that (they are) also the least (numbers continuously proportional in these ratios). For if H, G, K, L are not the least numbers continuously proportional in the ratios of A to B , and of C to D , and of E to F , let N, O, M, P be (the least such numbers). And since as A is to B , so N (is) to O , and A and B are the least (numbers which have the same ratio as them), and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20], B thus measures O . So, for the same (reasons), C also measures O . Thus, B and C (both) measure O . Thus, the least number measured by (both) B and C will also measure O [Prop. 7.35]. And G (is) the least number measured by (both) B and C .



Μὴ μετρεῖται δὴ ὁ Ε τὸν Κ, καὶ εἰλήφθω ὑπὸ τῶν Ε, Κ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ Μ. καὶ ὅσας μὲν ὁ Κ τὸν Μ μετρεῖ, τοσαυτάκις καὶ ἑκάτερος τῶν Θ, Η ἑκάτερον τῶν Ν, Ξ μετρεῖται, ὅσας δὲ ὁ Ε τὸν Μ μετρεῖ, τοσαυτάκις καὶ ὁ Ζ τὸν Ο μετρεῖται. ἐπεὶ ἰσάκις ὁ Θ τὸν Ν μετρεῖ καὶ ὁ Η τὸν Ξ, ἔστιν ἄρα ὡς ὁ Θ πρὸς τὸν Η, οὕτως ὁ Ν πρὸς τὸν Ξ. ὡς δὲ ὁ Θ πρὸς τὸν Η, οὕτως ὁ Α πρὸς τὸν Β· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, οὕτως ὁ Ν πρὸς τὸν Ξ. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ξ πρὸς τὸν Μ. πάλιν, ἐπεὶ ἰσάκις ὁ Ε τὸν Μ μετρεῖ καὶ ὁ Ζ τὸν Ο, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Μ πρὸς τὸν Ο· οἱ Ν, Ξ, Μ, Ο ἄρα ἐξῆς ἀνάλογόν εἰσιν ἐν τοῖς τοῦ τε Α πρὸς τὸν Β καὶ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τοῦ Ε πρὸς τὸν Ζ λόγοις. λέγω δὴ, ὅτι καὶ ἐλάχιστοι ἐν τοῖς Α Β, Γ Δ, Ε Ζ λόγοις. εἰ γὰρ μὴ, ἔσονται τινες τῶν Ν, Ξ, Μ, Ο ἐλάσσονες ἀριθμοὶ ἐξῆς ἀνάλογον ἐν τοῖς Α Β, Γ Δ, Ε Ζ λόγοις. ἔστωσαν οἱ Π, Ρ, Σ, Τ. καὶ ἐπεὶ ἔστιν ὡς ὁ Π πρὸς τὸν Ρ, οὕτως ὁ Α πρὸς τὸν Β, οἱ δὲ Α, Β ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς ἰσάκις ὃ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον, ὁ Β ἄρα τὸν Ρ μετρεῖ. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν Ρ μετρεῖ· οἱ Β, Γ ἄρα τὸν Ρ μετροῦσιν. καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν Β, Γ μετρούμενος τὸν Ρ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Β, Γ μετρούμενος ἔστιν ὁ Η· ὁ Η ἄρα τὸν Ρ μετρεῖ. καὶ ἔστιν ὡς ὁ Η πρὸς τὸν Ρ, οὕτως ὁ Κ πρὸς τὸν Σ· καὶ ὁ Κ ἄρα τὸν Σ μετρεῖ. μετρεῖ δὲ καὶ ὁ Ε τὸν Σ· οἱ Ε, Κ ἄρα τὸν Σ μετροῦσιν. καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν Ε, Κ μετρούμενος τὸν Σ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Ε, Κ μετρούμενός ἐστιν ὁ Μ· ὁ Μ ἄρα τὸν Σ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἔστιν ἀδύνατον. οὐκ ἄρα ἔσονται τινες τῶν

Thus, G measures O , the greater (measuring) the lesser. The very thing is impossible. Thus, there cannot be any numbers less than H, G, K, L (which are) continuously (proportional) in the ratio of A to B , and of C to D , and, further, of E to F .



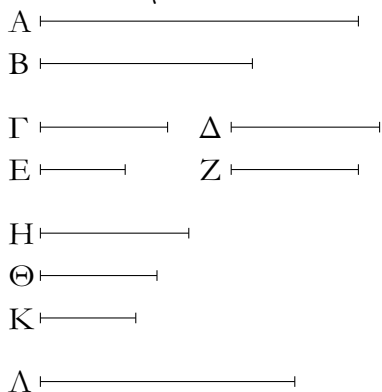
So let E not measure K . And let the least number, M , measured by (both) E and K have been taken [Prop. 7.34]. And as many times as K measures M , so many times let H, G also measure N, O , respectively. And as many times as E measures M , so many times let F also measure P . Since H measures N the same number of times as G (measures) O , thus as H is to G , so N (is) to O [Def. 7.20, Prop. 7.13]. And as H (is) to G , so A (is) to B . And thus as A (is) to B , so N (is) to O . And so, for the same (reasons), as C (is) to D , so O (is) to M . Again, since E measures M the same number of times as F (measures) P , thus as E is to F , so M (is) to P [Def. 7.20, Prop. 7.13]. Thus, N, O, M, P are continuously proportional in the ratios of A to B , and of C to D , and, further, of E to F . So I say that (they are) also the least (numbers) in the ratios of $A B, C D, E F$. For if not, then there will be some numbers less than N, O, M, P (which are) continuously proportional in the ratios of $A B, C D, E F$. Let them be Q, R, S, T . And since as Q is to R , so A (is) to B , and A and B (are) the least (numbers having the same ratio as them), and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20], B thus measures R . So, for the same (reasons), C also measures R . Thus, B and C (both) measure R . Thus, the least (number) measured by (both) B and C will also measure R [Prop. 7.35]. And G is the least number measured by (both) B and C . Thus, G measures R . And as G is to R , so K (is) to S . Thus,

N, Ξ, M, O ἐλάχιστονες ἀριθμοὶ ἐξῆς ἀνάλογον ἓν τε τοῖς τοῦ A πρὸς τὸν B καὶ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τοῦ E πρὸς τὸν Z λόγοις· οἱ N, Ξ, M, O ἄρα ἐξῆς ἀνάλογον ἐλάχιστοι εἰσιν ἐν τοῖς $A B, \Gamma \Delta, E Z$ λόγοις· ὅπερ ἔδει δεῖξαι.

K also measures S [Def. 7.20]. And E also measures S [Prop. 7.20]. Thus, E and K (both) measure S . Thus, the least (number) measured by (both) E and K will also measure S [Prop. 7.35]. And M is the least (number) measured by (both) E and K . Thus, M measures S , the greater (measuring) the lesser. The very thing is impossible. Thus there cannot be any numbers less than N, O, M, P (which are) continuously proportional in the ratios of A to B , and of C to D , and, further, of E to F . Thus, N, O, M, P are the least (numbers) continuously proportional in the ratios of $A B, C D, E F$. (Which is) the very thing it was required to show.

ε'.

Οἱ ἐπίπεδοι ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσι τὸν συγχεόμενον ἐκ τῶν πλευρῶν.



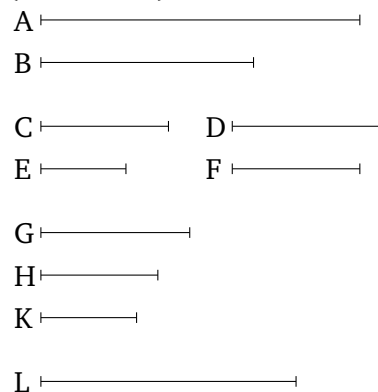
Ἐστωσαν ἐπίπεδοι ἀριθμοὶ οἱ A, B , καὶ τοῦ μὲν A πλευραὶ ἔστωσαν οἱ Γ, Δ ἀριθμοί, τοῦ δὲ B οἱ E, Z λέγω, ὅτι ὁ A πρὸς τὸν B λόγον ἔχει τὸν συγχεόμενον ἐκ τῶν πλευρῶν.

Λόγων γὰρ δοθέντων τοῦ τε ὃν ἔχει ὁ Γ πρὸς τὸν E καὶ ὁ Δ πρὸς τὸν Z εἰλήφθωσαν ἀριθμοὶ ἐξῆς ἐλάχιστοι ἐν τοῖς $\Gamma E, \Delta Z$ λόγοις, οἱ H, Θ, K , ὥστε εἶναι ὡς μὲν τὸν Γ πρὸς τὸν E , οὕτως τὸν H πρὸς τὸν Θ , ὡς δὲ τὸν Δ πρὸς τὸν Z , οὕτως τὸν Θ πρὸς τὸν K . καὶ ὁ Δ τὸν E πολλαπλασιάσας τὸν Λ ποιείτω.

Καὶ ἐπεὶ ὁ Δ τὸν μὲν Γ πολλαπλασιάσας τὸν A πεποίηκεν, τὸν δὲ E πολλαπλασιάσας τὸν Λ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν E , οὕτως ὁ A πρὸς τὸν Λ . ὡς δὲ ὁ Γ πρὸς τὸν E , οὕτως ὁ H πρὸς τὸν Θ . καὶ ὡς ἄρα ὁ H πρὸς τὸν Θ , οὕτως ὁ A πρὸς τὸν Λ . πάλιν, ἐπεὶ ὁ E τὸν Δ πολλαπλασιάσας τὸν Λ πεποίηκεν, ἀλλὰ μὴν καὶ τὸν Z πολλαπλασιάσας τὸν B πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Z , οὕτως ὁ Λ πρὸς τὸν B . ἀλλ' ὡς ὁ Δ πρὸς τὸν Z , οὕτως ὁ Θ πρὸς τὸν K . καὶ ὡς ἄρα ὁ Θ πρὸς τὸν K , οὕτως ὁ Λ πρὸς τὸν B . ἐδείχθη δὲ καὶ ὡς ὁ H πρὸς τὸν Θ , οὕτως ὁ A πρὸς τὸν Λ . δι' ἴσου ἄρα ἔστιν ὡς ὁ H πρὸς τὸν K , [οὕτως] ὁ A πρὸς τὸν B . ὁ δὲ H πρὸς τὸν K λόγον ἔχει

Proposition 5

Plane numbers have to one another the ratio compounded† out of (the ratios of) their sides.



Let A and B be plane numbers, and let the numbers C, D be the sides of A , and (the numbers) E, F (the sides) of B . I say that A has to B the ratio compounded out of (the ratios of) their sides.

For given the ratios which C has to E , and D (has) to F , let the least numbers, G, H, K , continuously proportional in the ratios $C E, D F$ have been taken [Prop. 8.4], so that as C is to E , so G (is) to H , and as D (is) to F , so H (is) to K . And let D make L (by) multiplying E .

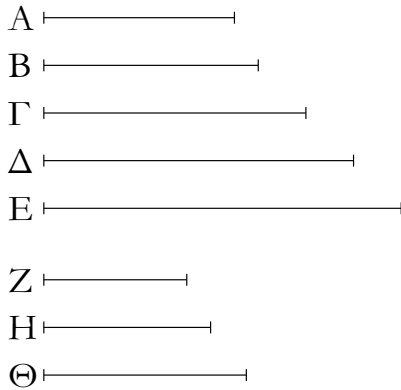
And since D has made A (by) multiplying C , and has made L (by) multiplying E , thus as C is to E , so A (is) to L [Prop. 7.17]. And as C (is) to E , so G (is) to H . And thus as G (is) to H , so A (is) to L . Again, since E has made L (by) multiplying D [Prop. 7.16], but, in fact, has also made B (by) multiplying F , thus as D is to F , so L (is) to B [Prop. 7.17]. But, as D (is) to F , so H (is) to K . And thus as H (is) to K , so L (is) to B . And it was also shown that as G (is) to H , so A (is) to L . Thus, via equality, as G is to K , [so] A (is) to B [Prop. 7.14]. And G has to K the ratio compounded out of (the ratios of) the sides (of A and B). Thus, A also has to B the ratio compounded out of (the ratios of) the sides (of A and B).

τὸν συγκείμενον ἐκ τῶν πλευρῶν· καὶ ὁ A ἄρα πρὸς τὸν B λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν· ὅπερ ἔδει δεῖξαι.

† i.e., multiplied.

ε'.

Ἐὰν ὦσιν ὁποσοιοῦν ἀριθμοὶ ἐξῆς ἀνάλογον, ὁ δὲ πρῶτος τὸν δεύτερον μὴ μετρήῃ, οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει.



Ἐστωσαν ὁποσοιοῦν ἀριθμοὶ ἐξῆς ἀνάλογον οἱ A, B, Γ, Δ, E , ὁ δὲ A τὸν B μὴ μετρεῖται· λέγω, ὅτι οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει.

Ὅτι μὲν οὖν οἱ A, B, Γ, Δ, E ἐξῆς ἀλλήλους οὐ μετροῦσιν, φανερόν· οὐδὲ γὰρ ὁ A τὸν B μετρεῖ. λέγω δὴ, ὅτι οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει. εἰ γὰρ δυνατόν, μετρεῖται ὁ A τὸν Γ . καὶ ὅσοι εἰσὶν οἱ A, B, Γ , τοσοῦτοι εἰληφθῶσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς A, B, Γ οἱ Z, H, Θ . καὶ ἐπεὶ οἱ Z, H, Θ ἐν τῷ αὐτῷ λόγῳ εἰσὶ τοῖς A, B, Γ , καὶ ἐστὶν ἴσον τὸ πλῆθος τῶν A, B, Γ τῷ πλῆθει τῶν Z, H, Θ , δι' ἴσου ἄρα ἐστὶν ὡς ὁ A πρὸς τὸν Γ , οὕτως ὁ Z πρὸς τὸν Θ . καὶ ἐπεὶ ἐστὶν ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Z πρὸς τὸν H , οὐ μετρεῖ δὲ ὁ A τὸν B , οὐ μετρεῖ ἄρα οὐδὲ ὁ Z τὸν H · οὐκ ἄρα μονὰς ἐστὶν ὁ Z · ἢ γὰρ μονὰς πάντα ἀριθμὸν μετρεῖ. καὶ εἰσὶν οἱ Z, Θ πρῶτοι πρὸς ἀλλήλους [οὐδὲ ὁ Z ἄρα τὸν Θ μετρεῖ]. καὶ ἐστὶν ὡς ὁ Z πρὸς τὸν Θ , οὕτως ὁ A πρὸς τὸν Γ · οὐδὲ ὁ A ἄρα τὸν Γ μετρεῖ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει· ὅπερ ἔδει δεῖξαι.

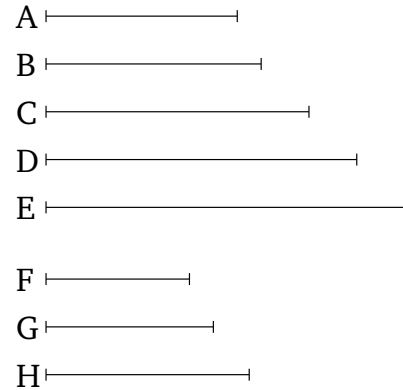
ζ'.

Ἐὰν ὦσιν ὁποσοιοῦν ἀριθμοὶ [ἐξῆς] ἀνάλογον, ὁ δὲ πρῶτος τὸν ἑσχάτον μετρήῃ, καὶ τὸν δεύτερον μετρήσει.

(Which is) the very thing it was required to show.

Proposition 6

If there are any multitude whatsoever of continuously proportional numbers, and the first does not measure the second, then no other (number) will measure any other (number) either.

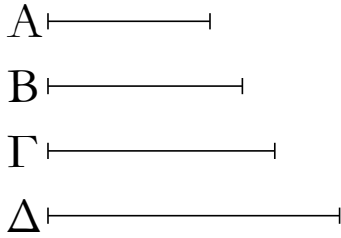


Let A, B, C, D, E be any multitude whatsoever of continuously proportional numbers, and let A not measure B . I say that no other (number) will measure any other (number) either.

Now, (it is) clear that A, B, C, D, E do not successively measure one another. For A does not even measure B . So I say that no other (number) will measure any other (number) either. For, if possible, let A measure C . And as many (numbers) as are A, B, C , let so many of the least numbers, F, G, H , have been taken of those (numbers) having the same ratio as A, B, C [Prop. 7.33]. And since F, G, H are in the same ratio as A, B, C , and the multitude of A, B, C is equal to the multitude of F, G, H , thus, via equality, as A is to C , so F (is) to H [Prop. 7.14]. And since as A is to B , so F (is) to G , and A does not measure B , F does not measure G either [Def. 7.20]. Thus, F is not a unit. For a unit measures all numbers. And F and H are prime to one another [Prop. 8.3] [and thus F does not measure H]. And as F is to H , so A (is) to C . And thus A does not measure C either [Def. 7.20]. So, similarly, we can show that no other (number) can measure any other (number) either. (Which is) the very thing it was required to show.

Proposition 7

If there are any multitude whatsoever of [continuously] proportional numbers, and the first measures the

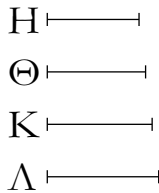
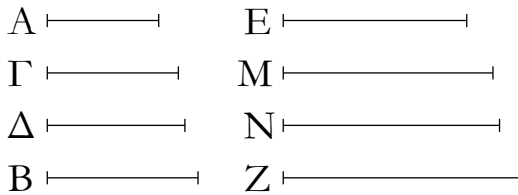


Ἐστωσαν ὁποσοιοῦν ἀριθμοὶ ἐξῆς ἀνάλογον οἱ A, B, Γ, Δ , ὁ δὲ A τὸν Δ μετρεῖτω· λέγω, ὅτι καὶ ὁ A τὸν B μετρεῖ.

Εἰ γὰρ οὐ μετρεῖ ὁ A τὸν B , οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει· μετρεῖ δὲ ὁ A τὸν Δ . μετρεῖ ἄρα καὶ ὁ A τὸν B · ὅπερ ἔδει δεῖξαι.

η'.

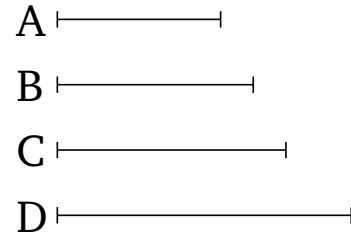
Ἐάν δύο ἀριθμῶν μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς τὸν αὐτὸν λόγον ἔχοντας [αὐτοῖς] μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται



Δύο γὰρ ἀριθμῶν τῶν A, B μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτέτωσαν ἀριθμοὶ οἱ Γ, Δ , καὶ πεποιήσθω ὡς ὁ A πρὸς τὸν B , οὕτως ὁ E πρὸς τὸν Z · λέγω, ὅτι ὅσοι εἰς τοὺς A, B μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς E, Z μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

Ὅσοι γὰρ εἰσι τῷ πλήθει οἱ A, B, Γ, Δ , τοσοῦτοι εἰληφθῶσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἔχόντων τοῖς A, Γ, Δ, B οἱ H, Θ, K, Λ · οἱ ἄρα ἄκροι αὐτῶν οἱ H, Λ πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ οἱ A, Γ, Δ, B τοῖς H, Θ, K, Λ ἐν τῷ αὐτῷ λόγῳ εἰσίν, καὶ ἐστὶν ἴσον τὸ πλήθος τῶν A, Γ, Δ, B τῷ πλήθει τῶν H, Θ, K, Λ , δι' ἴσου ἄρα ἐστὶν ὡς ὁ A πρὸς τὸν B , οὕτως ὁ H πρὸς τὸν Λ . ὡς δὲ ὁ A πρὸς τὸν B , οὕτως ὁ E πρὸς τὸν Z · καὶ

last, then (the first) will also measure the second.

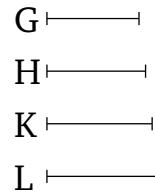
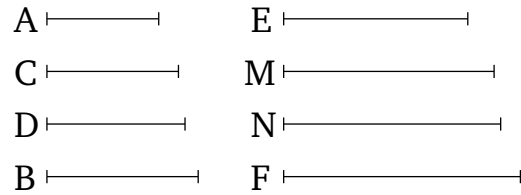


Let A, B, C, D be any number whatsoever of continuously proportional numbers. And let A measure D . I say that A also measures B .

For if A does not measure B then no other (number) will measure any other (number) either [Prop. 8.6]. But A measures D . Thus, A also measures B . (Which is) the very thing it was required to show.

Proposition 8

If between two numbers there fall (some) numbers in continued proportion then, as many numbers as fall in between them in continued proportion, so many (numbers) will also fall in between (any two numbers) having the same ratio [as them] in continued proportion.



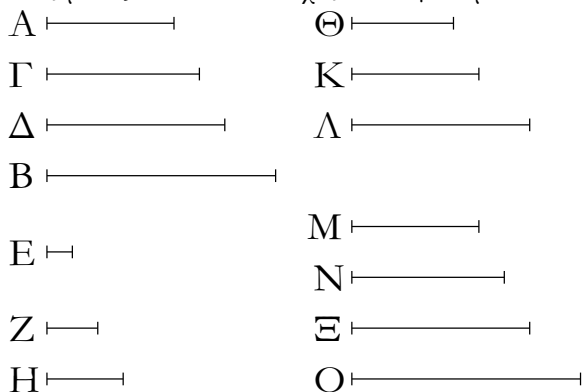
For let the numbers, C and D , fall between two numbers, A and B , in continued proportion, and let it have been contrived (that) as A (is) to B , so E (is) to F . I say that, as many numbers as have fallen in between A and B in continued proportion, so many (numbers) will also fall in between E and F in continued proportion.

For as many as A, B, C, D are in multitude, let so many of the least numbers, G, H, K, L , having the same ratio as A, B, C, D , have been taken [Prop. 7.33]. Thus, the outermost of them, G and L , are prime to one another [Prop. 8.3]. And since A, B, C, D are in the same ratio as G, H, K, L , and the multitude of A, B, C, D is equal to the multitude of G, H, K, L , thus, via equality, as A is to B , so G (is) to L [Prop. 7.14]. And as A (is) to B , so

ὥς ἄρα ὁ H πρὸς τὸν Λ , οὕτως ὁ E πρὸς τὸν Z . οἱ δὲ H , Λ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκεις ὃ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὃ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον. ἰσάκεις ἄρα ὁ H τὸν E μετρεῖ καὶ ὁ Λ τὸν Z . ὁσάκεις δὲ ὁ H τὸν E μετρεῖ, τοσαυτάκεις καὶ ἑκάτερος τῶν Θ , K ἑκάτερον τῶν M , N μετρεῖται· οἱ H , Θ , K , Λ ἄρα τοὺς E , M , N , Z ἰσάκεις μετροῦσιν. οἱ H , Θ , K , Λ ἄρα τοῖς E , M , N , Z ἐν τῷ αὐτῷ λόγῳ εἰσίν. ἀλλὰ οἱ H , Θ , K , Λ τοῖς A , Γ , Δ , B ἐν τῷ αὐτῷ λόγῳ εἰσίν· καὶ οἱ A , Γ , Δ , B ἄρα τοῖς E , M , N , Z ἐν τῷ αὐτῷ λόγῳ εἰσίν. οἱ δὲ A , Γ , Δ , B ἐξῆς ἀνάλογόν εἰσιν· καὶ οἱ E , M , N , Z ἄρα ἐξῆς ἀνάλογόν εἰσιν. ὅσοι ἄρα εἰς τοὺς A , B μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς E , Z μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί· ὅπερ ἔδει δεῖξαι.

θ'.

Ἐάν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, καὶ εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ ἑκατέρου αὐτῶν καὶ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.



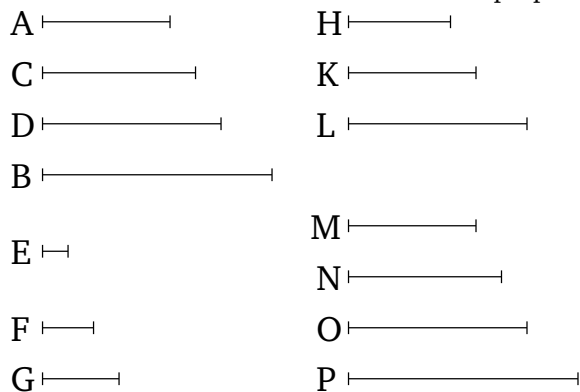
Ἐστῶσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ A , B , καὶ εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτέωσαν οἱ Γ , Δ , καὶ ἐκχείσθω ἡ E μονάς· λέγω, ὅτι ὅσοι εἰς τοὺς A , B μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ ἑκατέρου τῶν A , B καὶ τῆς μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

Εἰλήφθωσαν γὰρ δύο μὲν ἀριθμοὶ ἐλάχιστοι ἐν τῷ τῶν A , Γ , Δ , B λόγῳ ὄντες οἱ Z , H , τρεῖς δὲ οἱ Θ , K , Λ , καὶ αἰ

E (is) to F . And thus as G (is) to L , so E (is) to F . And G and L (are) prime (to one another). And (numbers) prime (to one another are) also the least (numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, G measures E the same number of times as L (measures) F . So as many times as G measures E , so many times let H , K also measure M , N , respectively. Thus, G , H , K , L measure E , M , N , F (respectively) an equal number of times. Thus, G , H , K , L are in the same ratio as E , M , N , F [Def. 7.20]. But, G , H , K , L are in the same ratio as A , C , D , B . Thus, A , C , D , B are also in the same ratio as E , M , N , F . And A , C , D , B are continuously proportional. Thus, E , M , N , F are also continuously proportional. Thus, as many numbers as have fallen in between A and B in continued proportion, so many numbers have also fallen in between E and F in continued proportion. (Which is) the very thing it was required to show.

Proposition 9

If two numbers are prime to one another and there fall in between them (some) numbers in continued proportion then, as many numbers as fall in between them in continued proportion, so many (numbers) will also fall between each of them and a unit in continued proportion.



Let A and B be two numbers (which are) prime to one another, and let the (numbers) C and D fall in between them in continued proportion. And let the unit E be set out. I say that, as many numbers as have fallen in between A and B in continued proportion, so many (numbers) will also fall between each of A and B and the unit in continued proportion.

For let the least two numbers, F and G , which are in the ratio of A , C , D , B , have been taken [Prop. 7.33].

ἐξῆς ἐνὶ πλείους, ἕως ἂν ἴσον γένηται τὸ πλῆθος αὐτῶν τῷ πλήθει τῶν A, Γ, Δ, B . εἰλήφθωσαν, καὶ ἕστωσαν οἱ M, N, Ξ, O . φανερόν δὲ, ὅτι ὁ μὲν Z ἑαυτὸν πολλαπλασιάσας τὸν Θ πεποίηκεν, τὸν δὲ Θ πολλαπλασιάσας τὸν M πεποίηκεν, καὶ ὁ H ἑαυτὸν μὲν πολλαπλασιάσας τὸν Λ πεποίηκεν, τὸν δὲ Λ πολλαπλασιάσας τὸν O πεποίηκεν. καὶ ἐπεὶ οἱ M, N, Ξ, O ἐλάχιστοι εἰσι τῶν τὸν αὐτὸν λόγον ἔχόντων τοῖς Z, H , εἰσὶ δὲ καὶ οἱ A, Γ, Δ, B ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἔχόντων τοῖς Z, H , καὶ ἔστιν ἴσον τὸ πλῆθος τῶν M, N, Ξ, O τῷ πλήθει τῶν A, Γ, Δ, B , ἕκαστος ἄρα τῶν M, N, Ξ, O ἐκάστῳ τῶν A, Γ, Δ, B ἴσος ἐστίν· ἴσος ἄρα ἐστὶν ὁ μὲν M τῷ A , ὁ δὲ O τῷ B . καὶ ἐπεὶ ὁ Z ἑαυτὸν πολλαπλασιάσας τὸν Θ πεποίηκεν, ὁ Z ἄρα τὸν Θ μετρεῖ κατὰ τὰς ἐν τῷ Z μονάδας. μετρεῖ δὲ καὶ ἡ E μονὰς τὸν Z κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκεις ἄρα ἡ E μονὰς τὸν Z ἀριθμὸν μετρεῖ καὶ ὁ Z τὸν Θ . ἔστιν ἄρα ὡς ἡ E μονὰς πρὸς τὸν Z ἀριθμὸν, οὕτως ὁ Z πρὸς τὸν Θ . πάλιν, ἐπεὶ ὁ Z τὸν Θ πολλαπλασιάσας τὸν M πεποίηκεν, ὁ Θ ἄρα τὸν M μετρεῖ κατὰ τὰς ἐν τῷ Z μονάδας. μετρεῖ δὲ καὶ ἡ E μονὰς τὸν Z ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκεις ἄρα ἡ E μονὰς τὸν Z ἀριθμὸν μετρεῖ καὶ ὁ Θ τὸν M . ἔστιν ἄρα ὡς ἡ E μονὰς πρὸς τὸν Z ἀριθμὸν, οὕτως ὁ Θ πρὸς τὸν M . ἐδείχθη δὲ καὶ ὡς ἡ E μονὰς πρὸς τὸν Z ἀριθμὸν, οὕτως ὁ Z πρὸς τὸν Θ . καὶ ὡς ἄρα ἡ E μονὰς πρὸς τὸν Z ἀριθμὸν, οὕτως ὁ Z πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν M . ἴσος δὲ ὁ M τῷ A · ἔστιν ἄρα ὡς ἡ E μονὰς πρὸς τὸν Z ἀριθμὸν, οὕτως ὁ Z πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν A . διὰ τὰ αὐτὰ δὴ καὶ ὡς ἡ E μονὰς πρὸς τὸν H ἀριθμὸν, οὕτως ὁ H πρὸς τὸν Λ καὶ ὁ Λ πρὸς τὸν B . ὅσοι ἄρα εἰς τοὺς A, B μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ ἐκατέρου τῶν A, B καὶ μονάδος τῆς E μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί· ὅπερ ἔδει δεῖξαι.

ι'.

Ἐάν δύο ἀριθμῶν ἑκατέρου καὶ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι ἑκατέρου αὐτῶν καὶ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

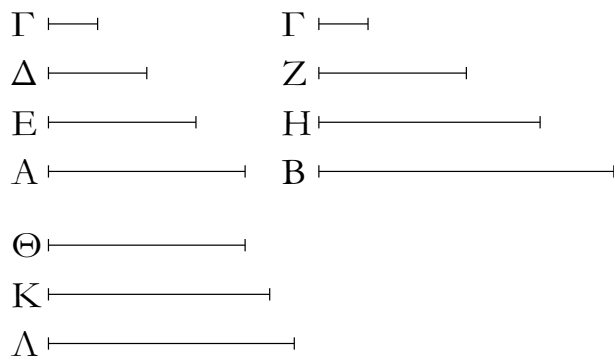
Δύο γὰρ ἀριθμῶν τῶν A, B καὶ μονάδος τῆς Γ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτέωσαν ἀριθμοί οἱ Δ, E καὶ οἱ Z, H . λέγω, ὅτι ὅσοι ἑκατέρου τῶν A, B καὶ μονάδος τῆς Γ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς A, B μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

And the (least) three (numbers), H, K, L . And so on, successively increasing by one, until the multitude of the (least numbers taken) is made equal to the multitude of A, C, D, B [Prop. 8.2]. Let them have been taken, and let them be M, N, O, P . So (it is) clear that F has made H (by) multiplying itself, and has made M (by) multiplying H . And G has made L (by) multiplying itself, and has made P (by) multiplying L [Prop. 8.2 corr.]. And since M, N, O, P are the least of those (numbers) having the same ratio as F, G , and A, C, D, B are also the least of those (numbers) having the same ratio as F, G [Prop. 8.2], and the multitude of M, N, O, P is equal to the multitude of A, C, D, B , thus M, N, O, P are equal to A, C, D, B , respectively. Thus, M is equal to A , and P to B . And since F has made H (by) multiplying itself, F thus measures H according to the units in F [Def. 7.15]. And the unit E also measures F according to the units in it. Thus, the unit E measures the number F as many times as F (measures) H . Thus, as the unit E is to the number F , so F (is) to H [Def. 7.20]. Again, since F has made M (by) multiplying H , H thus measures M according to the units in F [Def. 7.15]. And the unit E also measures the number F according to the units in it. Thus, the unit E measures the number F as many times as H (measures) M . Thus, as the unit E is to the number F , so H (is) to M [Prop. 7.20]. And it was shown that as the unit E (is) to the number F , so F (is) to H . And thus as the unit E (is) to the number F , so F (is) to H , and H (is) to M . And M (is) equal to A . Thus, as the unit E is to the number F , so F (is) to H , and H to A . And so, for the same (reasons), as the unit E (is) to the number G , so G (is) to L , and L to B . Thus, as many (numbers) as have fallen in between A and B in continued proportion, so many numbers have also fallen between each of A and B and the unit E in continued proportion. (Which is) the very thing it was required to show.

Proposition 10

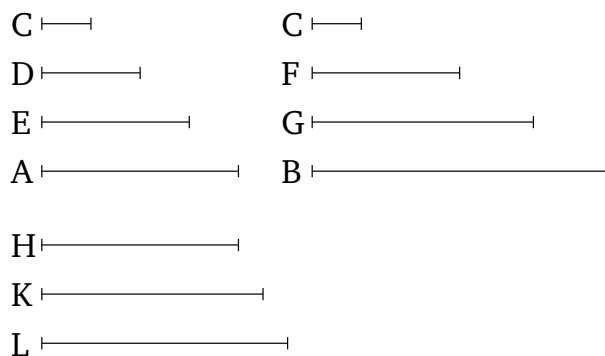
If (some) numbers fall between each of two numbers and a unit in continued proportion then, as many (numbers) as fall between each of the (two numbers) and the unit in continued proportion, so many (numbers) will also fall in between the (two numbers) themselves in continued proportion.

For let the numbers D, E and F, G fall between the numbers A and B (respectively) and the unit C in continued proportion. I say that, as many numbers as have fallen between each of A and B and the unit C in continued proportion, so many will also fall in between A and B in continued proportion.



Ὁ Δ γὰρ τὸν Ζ πολλαπλασιάσας τὸν Θ ποιείτω, ἑκάτερος δὲ τῶν Δ, Ζ τὸν Θ πολλαπλασιάσας ἑκάτερον τῶν Κ, Λ ποιείτω.

Καὶ ἐπεὶ ἔστιν ὡς ἡ Γ μονὰς πρὸς τὸν Δ ἀριθμὸν, οὕτως ὁ Δ πρὸς τὸν Ε, ἰσάκεις ἄρα ἡ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ ὁ Δ τὸν Ε. ἡ δὲ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· καὶ ὁ Δ ἄρα ἀριθμὸς τὸν Ε μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· ὁ Δ ἄρα ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν. πάλιν, ἐπεὶ ἔστιν ὡς ἡ Γ [μονὰς] πρὸς τὸν Δ ἀριθμὸν, οὕτως ὁ Ε πρὸς τὸν Α, ἰσάκεις ἄρα ἡ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ ὁ Ε τὸν Α. ἡ δὲ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· καὶ ὁ Ε ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· ὁ Δ ἄρα τὸν Ε πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ μὲν Ζ ἑαυτὸν πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Η πολλαπλασιάσας τὸν Β πεποίηκεν. καὶ ἐπεὶ ὁ Δ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε πεποίηκεν, τὸν δὲ Ζ πολλαπλασιάσας τὸν Θ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Ε πρὸς τὸν Θ. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Θ πρὸς τὸν Η. καὶ ὡς ἄρα ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Θ πρὸς τὸν Η. πάλιν, ἐπεὶ ὁ Δ ἑκάτερον τῶν Ε, Θ πολλαπλασιάσας ἑκάτερον τῶν Α, Κ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Κ. ἀλλ' ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Δ πρὸς τὸν Ζ· καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Κ. πάλιν, ἐπεὶ ἑκάτερος τῶν Δ, Ζ τὸν Θ πολλαπλασιάσας ἑκάτερον τῶν Κ, Λ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Κ πρὸς τὸν Λ. ἀλλ' ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Κ· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Κ, οὕτως ὁ Κ πρὸς τὸν Λ. ἔτι ἐπεὶ ὁ Ζ ἑκάτερον τῶν Θ, Η πολλαπλασιάσας ἑκάτερον τῶν Α, Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Θ πρὸς τὸν Η, οὕτως ὁ Α πρὸς τὸν Β. ὡς δὲ ὁ Θ πρὸς τὸν Η, οὕτως ὁ Δ πρὸς τὸν Ζ· καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Κ καὶ ὁ Κ πρὸς τὸν Λ· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Κ, οὕτως ὁ Κ πρὸς τὸν Λ καὶ ὁ Λ πρὸς τὸν Β. οἱ Α, Κ, Λ, Β ἄρα κατὰ τὸ συνεχὲς ἐξῆς εἰσιν ἀνάλογον. ὅσοι ἄρα ἑκατέρου τῶν Α, Β καὶ τῆς Γ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἐμπεσοῦνται· ὅπερ ἔδει



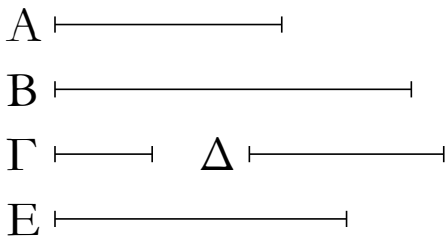
For let D make H (by) multiplying F . And let D, F make K, L , respectively, by multiplying H .

As since as the unit C is to the number D , so D (is) to E , the unit C thus measures the number D as many times as D (measures) E [Def. 7.20]. And the unit C measures the number D according to the units in D . Thus, the number D also measures E according to the units in D . Thus, D has made E (by) multiplying itself. Again, since as the [unit] C is to the number D , so E (is) to A , the unit C thus measures the number D as many times as E (measures) A [Def. 7.20]. And the unit C measures the number D according to the units in D . Thus, E also measures A according to the units in D . Thus, D has made A (by) multiplying E . And so, for the same (reasons), F has made G (by) multiplying itself, and has made B (by) multiplying G . And since D has made E (by) multiplying itself, and has made H (by) multiplying F , thus as D is to F , so E (is) to H [Prop 7.17]. And so, for the same reasons, as D (is) to F , so H (is) to G [Prop. 7.18]. And thus as E (is) to H , so H (is) to G . Again, since D has made A, K (by) multiplying E, H , respectively, thus as E is to H , so A (is) to K [Prop 7.17]. But, as E (is) to H , so D (is) to F . And thus as D (is) to F , so A (is) to K . Again, since D, F have made K, L , respectively, (by) multiplying H , thus as D is to F , so K (is) to L [Prop. 7.18]. But, as D (is) to F , so A (is) to K . And thus as A (is) to K , so K (is) to L . Further, since F has made L, B (by) multiplying H, G , respectively, thus as H is to G , so L (is) to B [Prop 7.17]. And as H (is) to G , so D (is) to F . And thus as D (is) to F , so L (is) to B . And it was also shown that as D (is) to F , so A (is) to K , and K to L . And thus as A (is) to K , so K (is) to L , and L to B . Thus, A, K, L, B are successively in continued proportion. Thus, as many numbers as fall between each of A and B and the unit C in continued proportion, so many will also fall in between A and B in continued proportion. (Which is) the very thing it was required to show.

δείξαι.

ια'.

Δύο τετραγώνων ἀριθμῶν εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ τετράγωνος πρὸς τὸν τετράγωνον διπλασίονα λόγον ἔχει ἥπερ ἡ πλευρὰ πρὸς τὴν πλευράν.



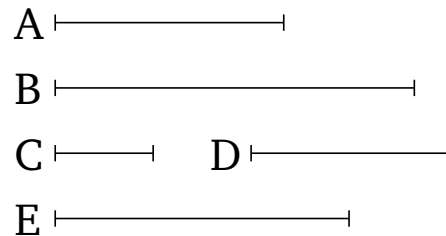
Ἐστωσαν τετράγωνοι ἀριθμοὶ οἱ A, B, καὶ τοῦ μὲν A πλευρὰ ἔστω ὁ Γ, τοῦ δὲ B ὁ Δ· λέγω, ὅτι τῶν A, B εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ A πρὸς τὸν B διπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Δ.

Ὁ Γ γὰρ τὸν Δ πολλαπλασιάσας τὸν E ποιεῖται. καὶ ἐπεὶ τετράγωνός ἐστιν ὁ A, πλευρὰ δὲ αὐτοῦ ἐστιν ὁ Γ, ὁ Γ ἄρα ἑαυτὸν πολλαπλασιάσας τὸν A πεποιήκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Δ ἑαυτὸν πολλαπλασιάσας τὸν B πεποιήκεν. ἐπεὶ οὖν ὁ Γ ἑκάτερον τῶν Γ, Δ πολλαπλασιάσας ἑκάτερον τῶν A, E πεποιήκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ A πρὸς τὸν E. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ E πρὸς τὸν B. καὶ ὡς ἄρα ὁ A πρὸς τὸν E, οὕτως ὁ E πρὸς τὸν B. τῶν A, B ἄρα εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός.

Λέγω δὴ, ὅτι καὶ ὁ A πρὸς τὸν B διπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Δ. ἐπεὶ γὰρ τρεῖς ἀριθμοὶ ἀνάλογόν εἰσιν οἱ A, E, B, ὁ A ἄρα πρὸς τὸν B διπλασίονα λόγον ἔχει ἥπερ ὁ A πρὸς τὸν E. ὡς δὲ ὁ A πρὸς τὸν E, οὕτως ὁ Γ πρὸς τὸν Δ. ὁ A ἄρα πρὸς τὸν B διπλασίονα λόγον ἔχει ἥπερ ἡ Γ πλευρὰ πρὸς τὴν Δ· ὅπερ ἔδει δείξαι.

Proposition 11

There exists one number in mean proportion to two (given) square numbers.[†] And (one) square (number) has to the (other) square (number) a squared[‡] ratio with respect to (that) the side (of the former has) to the side (of the latter).



Let A and B be square numbers, and let C be the side of A, and D (the side) of B. I say that there exists one number in mean proportion to A and B, and that A has to B a squared ratio with respect to (that) C (has) to D.

For let C make E (by) multiplying D. And since A is square, and C is its side, C has thus made A (by) multiplying itself. And so, for the same (reasons), D has made B (by) multiplying itself. Therefore, since C has made A, E (by) multiplying C, D, respectively, thus as C is to D, so A (is) to E [Prop. 7.17]. And so, for the same (reasons), as C (is) to D, so E (is) to B [Prop. 7.18]. And thus as A (is) to E, so E (is) to B. Thus, one number (namely, E) is in mean proportion to A and B.

So I say that A also has to B a squared ratio with respect to (that) C (has) to D. For since A, E, B are three (continuously) proportional numbers, A thus has to B a squared ratio with respect to (that) A (has) to E [Def. 5.9]. And as A (is) to E, so C (is) to D. Thus, A has to B a squared ratio with respect to (that) side C (has) to (side) D. (Which is) the very thing it was required to show.

[†] In other words, between two given square numbers there exists a number in continued proportion.

[‡] Literally, “double”.

ιβ'.

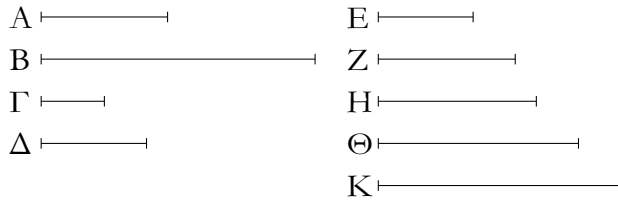
Δύο κύβων ἀριθμῶν δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί, καὶ ὁ κύβος πρὸς τὸν κύβον τριπλασίονα λόγον ἔχει ἥπερ ἡ πλευρὰ πρὸς τὴν πλευράν.

Ἐστωσαν κύβοι ἀριθμοὶ οἱ A, B καὶ τοῦ μὲν A πλευρὰ ἔστω ὁ Γ, τοῦ δὲ B ὁ Δ· λέγω, ὅτι τῶν A, B δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί, καὶ ὁ A πρὸς τὸν B τριπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Δ.

Proposition 12

There exist two numbers in mean proportion to two (given) cube numbers.[†] And (one) cube (number) has to the (other) cube (number) a cubed[‡] ratio with respect to (that) the side (of the former has) to the side (of the latter).

Let A and B be cube numbers, and let C be the side of A, and D (the side) of B. I say that there exist two numbers in mean proportion to A and B, and that A has



Ὁ γὰρ Γ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε ποιεῖτω, τὸν δὲ Δ πολλαπλασιάσας τὸν Ζ ποιεῖτω, ὁ δὲ Δ ἑαυτὸν πολλαπλασιάσας τὸν Η ποιεῖτω, ἑκάτερος δὲ τῶν Γ, Δ τὸν Ζ πολλαπλασιάσας ἑκάτερον τῶν Θ, Κ ποιεῖτω.

Καὶ ἐπεὶ κύβος ἐστὶν ὁ Α, πλευρὰ δὲ αὐτοῦ ὁ Γ, καὶ ὁ Γ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε πεποίηκεν, ὁ Γ ἄρα ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε πεποίηκεν, τὸν δὲ Ε πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Δ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Η πολλαπλασιάσας τὸν Β πεποίηκεν. καὶ ἐπεὶ ὁ Γ ἑκάτερον τῶν Γ, Δ πολλαπλασιάσας ἑκάτερον τῶν Ε, Ζ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Ζ. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ζ πρὸς τὸν Η. πάλιν, ἐπεὶ ὁ Γ ἑκάτερον τῶν Ε, Ζ πολλαπλασιάσας ἑκάτερον τῶν Α, Θ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Θ. ὡς δὲ ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Γ πρὸς τὸν Δ· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Θ. πάλιν, ἐπεὶ ἑκάτερος τῶν Γ, Δ τὸν Ζ πολλαπλασιάσας ἑκάτερον τῶν Θ, Κ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Θ πρὸς τὸν Κ. πάλιν, ἐπεὶ ὁ Δ ἑκάτερον τῶν Ζ, Η πολλαπλασιάσας ἑκάτερον τῶν Κ, Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ζ πρὸς τὸν Η, οὕτως ὁ Κ πρὸς τὸν Β. ὡς δὲ ὁ Ζ πρὸς τὸν Η, οὕτως ὁ Γ πρὸς τὸν Δ· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν Κ καὶ ὁ Κ πρὸς τὸν Β. τῶν Α, Β ἄρα δύο μέσοι ἀνάλογόν εἰσιν οἱ Θ, Κ.

Λέγω δὴ, ὅτι καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Δ. ἐπεὶ γὰρ τέσσαρες ἀριθμοὶ ἀνάλογόν εἰσιν οἱ Α, Θ, Κ, Β, ὁ Α ἄρα πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Α πρὸς τὸν Θ. ὡς δὲ ὁ Α πρὸς τὸν Θ, οὕτως ὁ Γ πρὸς τὸν Δ· καὶ ὁ Α [ἄρα] πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Δ· ὅπερ ἔδει δεῖξαι.

† In other words, between two given cube numbers there exist two numbers in continued proportion.

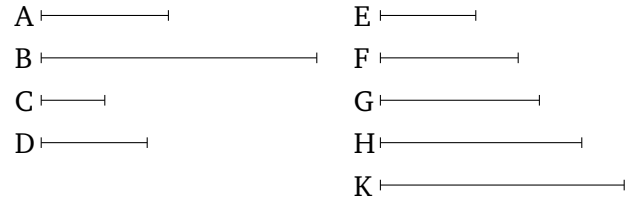
‡ Literally, "triple".

ιγ'.

Ἐὰν ὧσιν ὁποιοῦν ἀριθμοὶ ἐξῆς ἀνάλογον, καὶ πολλαπλασιάσας ἕκαστος ἑαυτὸν ποιῇ τινα, οἱ γεγόμενοι ἐξ αὐτῶν ἀνάλογον ἔσονται· καὶ ἐὰν οἱ ἐξ ἀρχῆς τοὺς γενομένους πολλαπλασιάσαντες ποιῶσιν τινας, καὶ αὐτοὶ ἀνάλογον ἔσονται [καὶ αἰεὶ περὶ τοὺς ἄκρους τοῦτο συμβαίνει].

Ἔστωσαν ὁποιοῦν ἀριθμοὶ ἐξῆς ἀνάλογον, οἱ Α, Β,

to B a cubed ratio with respect to (that) C (has) to D.



For let C make E (by) multiplying itself, and let it make F (by) multiplying D. And let D make G (by) multiplying itself, and let C, D make H, K, respectively, (by) multiplying F.

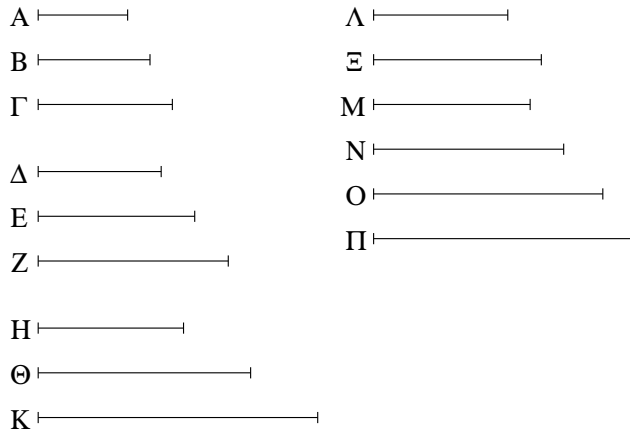
And since A is cube, and C (is) its side, and C has made E (by) multiplying itself, C has thus made E (by) multiplying itself, and has made A (by) multiplying E. And so, for the same (reasons), D has made G (by) multiplying itself, and has made B (by) multiplying G. And since C has made E, F (by) multiplying C, D, respectively, thus as C is to D, so E (is) to F [Prop. 7.17]. And so, for the same (reasons), as C (is) to D, so F (is) to G [Prop. 7.18]. Again, since C has made A, H (by) multiplying E, F, respectively, thus as E is to F, so A (is) to H [Prop. 7.17]. And as E (is) to F, so C (is) to D. And thus as C (is) to D, so A (is) to H. Again, since C, D have made H, K, respectively, (by) multiplying F, thus as C is to D, so H (is) to K [Prop. 7.18]. Again, since D has made K, B (by) multiplying F, G, respectively, thus as F is to G, so K (is) to B [Prop. 7.17]. And as F (is) to G, so C (is) to D. And thus as C (is) to D, so A (is) to H, and H to K, and K to B. Thus, H and K are two (numbers) in mean proportion to A and B.

So I say that A also has to B a cubed ratio with respect to (that) C (has) to D. For since A, H, K, B are four (continuously) proportional numbers, A thus has to B a cubed ratio with respect to (that) A (has) to H [Def. 5.10]. And as A (is) to H, so C (is) to D. And [thus] A has to B a cubed ratio with respect to (that) C (has) to D. (Which is) the very thing it was required to show.

Proposition 13

If there are any multitude whatsoever of continuously proportional numbers, and each makes some (number by) multiplying itself, then the (numbers) created from them will (also) be (continuously) proportional. And if the original (numbers) make some (more numbers by) multiplying the created (numbers) then these will also

Γ, ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς τὸν Γ, καὶ οἱ Α, Β, Γ ἑαυτοὺς μὲν πολλαπλασιάσαντες τοὺς Δ, Ε, Ζ ποιείωσαν, τοὺς δὲ Δ, Ε, Ζ πολλαπλασιάσαντες τοὺς Η, Θ, Κ ποιείωσαν· λέγω, ὅτι οἱ τε Δ, Ε, Ζ καὶ οἱ Η, Θ, Κ ἐξῆς ἀνάλογον εἰσιν.



Ὁ μὲν γὰρ Α τὸν Β πολλαπλασιάσας τὸν Λ ποιείτω, ἑκάτερος δὲ τῶν Α, Β τὸν Λ πολλαπλασιάσας ἑκάτερον τῶν Μ, Ν ποιείτω. καὶ πάλιν ὁ μὲν Β τὸν Γ πολλαπλασιάσας τὸν Ξ ποιείτω, ἑκάτερος δὲ τῶν Β, Γ τὸν Ξ πολλαπλασιάσας ἑκάτερον τῶν Ο, Π ποιείτω.

Ὁμοίως δὴ τοῖς ἐπάνω δεῖξομεν, ὅτι οἱ Δ, Α, Ε καὶ οἱ Η, Μ, Ν, Θ ἐξῆς εἰσιν ἀνάλογον ἐν τῷ τοῦ Α πρὸς τὸν Β λόγῳ, καὶ ἔτι οἱ Ε, Ξ, Ζ καὶ οἱ Θ, Ο, Π, Κ ἐξῆς εἰσιν ἀνάλογον ἐν τῷ τοῦ Β πρὸς τὸν Γ λόγῳ. καὶ ἐστὶν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς τὸν Γ· καὶ οἱ Δ, Α, Ε ἄρα τοῖς Ε, Ξ, Ζ ἐν τῷ αὐτῷ λόγῳ εἰσὶ καὶ ἔτι οἱ Η, Μ, Ν, Θ τοῖς Θ, Ο, Π, Κ. καὶ ἐστὶν ἴσον τὸ μὲν τῶν Δ, Α, Ε πλήθος τῶν τῶν Ε, Ξ, Ζ πλήθει, τὸ δὲ τῶν Η, Μ, Ν, Θ τῶν Θ, Ο, Π, Κ δι' ἴσου ἄρα ἐστὶν ὡς μὲν ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Ε πρὸς τὸν Ζ, ὡς δὲ ὁ Η πρὸς τὸν Θ, οὕτως ὁ Θ πρὸς τὸν Κ· ὅπερ εἶδει δεῖξαι.

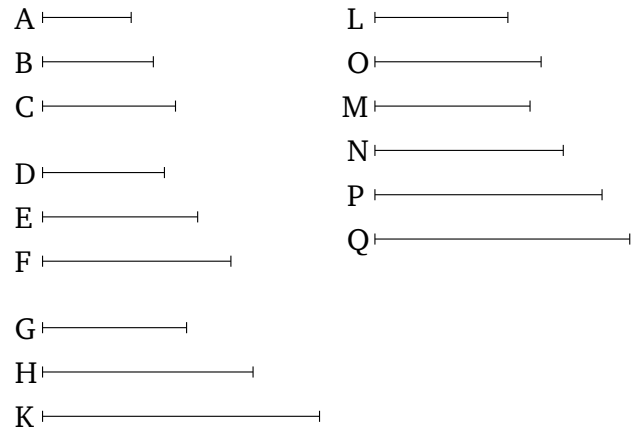
ιδ'.

Ἐὰν τετράγωνος τετράγωνον μετρήῃ, καὶ ἡ πλευρὰ τὴν πλευρὰν μετρήσῃ· καὶ ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρήῃ, καὶ ὁ τετράγωνος τὸν τετράγωνον μετρήσῃ.

Ἐστωσαν τετράγωνοι ἀριθμοὶ οἱ Α, Β, πλευραὶ δὲ αὐτῶν ἔστωσαν οἱ Γ, Δ, ὁ δὲ Α τὸν Β μετρεῖτω· λέγω, ὅτι καὶ ὁ Γ τὸν Δ μετρεῖ.

be (continuously) proportional [and this always happens with the extremes].

Let A, B, C be any multitude whatsoever of continuously proportional numbers, (such that) as A (is) to B , so B (is) to C . And let A, B, C make D, E, F (by) multiplying themselves, and let them make G, H, K (by) multiplying D, E, F . I say that D, E, F and G, H, K are continuously proportional.



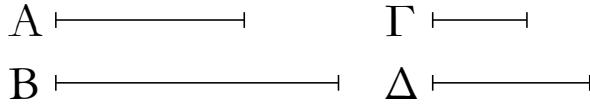
For let A make L (by) multiplying B . And let A, B make M, N , respectively, (by) multiplying L . And, again, let B make O (by) multiplying C . And let B, C make P, Q , respectively, (by) multiplying O .

So, similarly to the above, we can show that D, L, E and G, M, N, H are continuously proportional in the ratio of A to B , and, further, (that) E, O, F and H, P, Q, K are continuously proportional in the ratio of B to C . And as A is to B , so B (is) to C . And thus D, L, E are in the same ratio as E, O, F , and, further, G, M, N, H (are in the same ratio) as H, P, Q, K . And the multitude of D, L, E is equal to the multitude of E, O, F , and that of G, M, N, H to that of H, P, Q, K . Thus, via equality, as D is to E , so E (is) to F , and as G (is) to H , so H (is) to K [Prop. 7.14]. (Which is) the very thing it was required to show.

Proposition 14

If a square (number) measures a(nother) square (number) then the side (of the former) will also measure the side (of the latter). And if the side (of a square number) measures the side (of another square number) then the (former) square (number) will also measure the (latter) square (number).

Let A and B be square numbers, and let C and D be their sides (respectively). And let A measure B . I say that C also measures D .



Ὁ Γ γὰρ τὸν Δ πολλαπλασιάσας τὸν Ε ποιεῖται· οἱ Α, Ε, Β ἄρα ἐξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ οἱ Α, Ε, Β ἐξῆς ἀνάλογόν εἰσιν, καὶ μετρεῖ ὁ Α τὸν Β, μετρεῖ ἄρα καὶ ὁ Α τὸν Ε. καὶ ἐστὶν ὡς ὁ Α πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Δ· μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ.

Πάλιν δὴ ὁ Γ τὸν Δ μετρεῖται· λέγω, ὅτι καὶ ὁ Α τὸν Β μετρεῖ.

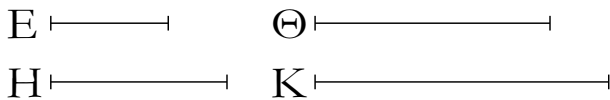
Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι οἱ Α, Ε, Β ἐξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ ἐστὶν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Ε, μετρεῖ δὲ ὁ Γ τὸν Δ, μετρεῖ ἄρα καὶ ὁ Α τὸν Ε. καὶ εἰσιν οἱ Α, Ε, Β ἐξῆς ἀνάλογον· μετρεῖ ἄρα καὶ ὁ Α τὸν Β.

Ἐὰν ἄρα τετράγωνος τετράγωνον μετρήῃ, καὶ ἡ πλευρὰ τὴν πλευρὰν μετρήσῃ· καὶ ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρήῃ, καὶ ὁ τετράγωνος τὸν τετράγωνον μετρήσῃ· ὅπερ ἔδει δεῖξαι.

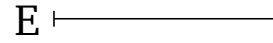
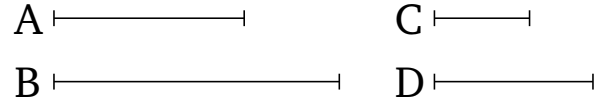
ιε'.

Ἐὰν κύβος ἀριθμὸς κύβον ἀριθμὸν μετρήῃ, καὶ ἡ πλευρὰ τὴν πλευρὰν μετρήσῃ· καὶ ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρήῃ, καὶ ὁ κύβος τὸν κύβον μετρήσῃ.

Κύβος γὰρ ἀριθμὸς ὁ Α κύβον τὸν Β μετρεῖται, καὶ τοῦ μὲν Α πλευρὰ ἔστω ὁ Γ, τοῦ δὲ Β ὁ Δ· λέγω, ὅτι ὁ Γ τὸν Δ μετρεῖ.



Ὁ Γ γὰρ ἑαυτὸν πολλαπλασιάσας τὸν Ε ποιεῖται, ὁ δὲ Δ



For let C make E (by) multiplying D . Thus, A , E , B are continuously proportional in the ratio of C to D [Prop. 8.11]. And since A , E , B are continuously proportional, and A measures B , A thus also measures E [Prop. 8.7]. And as A is to E , so C (is) to D . Thus, C also measures D [Def. 7.20].

So, again, let C measure D . I say that A also measures B .

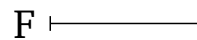
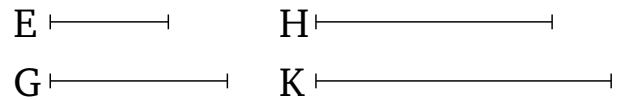
For similarly, with the same construction, we can show that A , E , B are continuously proportional in the ratio of C to D . And since as C is to D , so A (is) to E , and C measures D , A thus also measures E [Def. 7.20]. And A , E , B are continuously proportional. Thus, A also measures B .

Thus, if a square (number) measures a(nother) square (number) then the side (of the former) will also measure the side (of the latter). And if the side (of a square number) measures the side (of another square number) then the (former) square (number) will also measure the (latter) square (number). (Which is) the very thing it was required to show.

Proposition 15

If a cube number measures a(nother) cube number then the side (of the former) will also measure the side (of the latter). And if the side (of a cube number) measures the side (of another cube number) then the (former) cube (number) will also measure the (latter) cube (number).

For let the cube number A measure the cube (number) B , and let C be the side of A , and D (the side) of B . I say that C measures D .



For let C make E (by) multiplying itself. And let

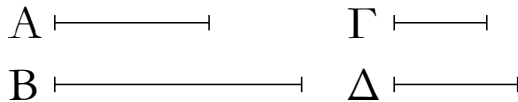
ἑαυτὸν πολλαπλασιάσας τὸν Η ποιεῖτω, καὶ ἔτι ὁ Γ τὸν Δ πολλαπλασιάσας τὸν Ζ [ποιεῖτω], ἑκάτερος δὲ τῶν Γ, Δ τὸν Ζ πολλαπλασιάσας ἑκάτερον τῶν Θ, Κ ποιεῖτω. φανερόν δὴ, ὅτι οἱ Ε, Ζ, Η καὶ οἱ Α, Θ, Κ, Β ἐξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ οἱ Α, Θ, Κ, Β ἐξῆς ἀνάλογόν εἰσιν, καὶ μετρεῖ ὁ Α τὸν Β, μετρεῖ ἄρα καὶ τὸν Θ. καὶ ἔστιν ὡς ὁ Α πρὸς τὸν Θ, οὕτως ὁ Γ πρὸς τὸν Δ· μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ.

Ἀλλὰ δὴ μετρεῖτω ὁ Γ τὸν Δ· λέγω, ὅτι καὶ ὁ Α τὸν Β μετρήσει.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δὴ δεῖξομεν, ὅτι οἱ Α, Θ, Κ, Β ἐξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ ὁ Γ τὸν Δ μετρεῖ, καὶ ἔστιν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Θ, καὶ ὁ Α ἄρα τὸν Θ μετρεῖ· ὥστε καὶ τὸν Β μετρεῖ ὁ Α· ὅπερ ἔδει δεῖξαι.

ιϛ'.

Ἐὰν τετράγωνος ἀριθμὸς τετράγωνον ἀριθμὸν μὴ μετρήῃ, οὐδὲ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· καὶ ἡ πλευρὰ τὴν πλευρὰν μὴ μετρήῃ, οὐδὲ ὁ τετράγωνος τὸν τετράγωνον μετρήσει.



Ἐστωσαν τετράγωνοι ἀριθμοὶ οἱ Α, Β, πλευραὶ δὲ αὐτῶν ἔστωσαν οἱ Γ, Δ, καὶ μὴ μετρεῖτω ὁ Α τὸν Β· λέγω, ὅτι οὐδὲ ὁ Γ τὸν Δ μετρεῖ.

Εἰ γὰρ μετρεῖ ὁ Γ τὸν Δ, μετρήσει καὶ ὁ Α τὸν Β. οὐ μετρεῖ δὲ ὁ Α τὸν Β· οὐδὲ ἄρα ὁ Γ τὸν Δ μετρήσει.

Μὴ μετρεῖτω [δὴ] πάλιν ὁ Γ τὸν Δ· λέγω, ὅτι οὐδὲ ὁ Α τὸν Β μετρήσει.

Εἰ γὰρ μετρεῖ ὁ Α τὸν Β, μετρήσει καὶ ὁ Γ τὸν Δ. οὐ μετρεῖ δὲ ὁ Γ τὸν Δ· οὐδ' ἄρα ὁ Α τὸν Β μετρήσει· ὅπερ ἔδει δεῖξαι.

ιζ'.

Ἐὰν κύβος ἀριθμὸς κύβον ἀριθμὸν μὴ μετρήῃ, οὐδὲ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· καὶ ἡ πλευρὰ τὴν πλευρὰν μὴ μετρήῃ, οὐδὲ ὁ κύβος τὸν κύβον μετρήσει.

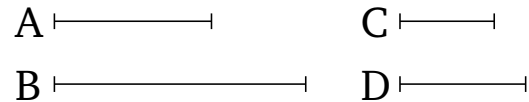
D make *G* (by) multiplying itself. And, further, [let] *C* [make] *F* (by) multiplying *D*, and let *C*, *D* make *H*, *K*, respectively, (by) multiplying *F*. So it is clear that *E*, *F*, *G* and *A*, *H*, *K*, *B* are continuously proportional in the ratio of *C* to *D* [Prop. 8.12]. And since *A*, *H*, *K*, *B* are continuously proportional, and *A* measures *B*, (*A*) thus also measures *H* [Prop. 8.7]. And as *A* is to *H*, so *C* (is) to *D*. Thus, *C* also measures *D* [Def. 7.20].

And so let *C* measure *D*. I say that *A* will also measure *B*.

For similarly, with the same construction, we can show that *A*, *H*, *K*, *B* are continuously proportional in the ratio of *C* to *D*. And since *C* measures *D*, and as *C* is to *D*, so *A* (is) to *H*, *A* thus also measures *H* [Def. 7.20]. Hence, *A* also measures *B*. (Which is) the very thing it was required to show.

Proposition 16

If a square number does not measure a(nother) square number then the side (of the former) will not measure the side (of the latter) either. And if the side (of a square number) does not measure the side (of another square number) then the (former) square (number) will not measure the (latter) square (number) either.



Let *A* and *B* be square numbers, and let *C* and *D* be their sides (respectively). And let *A* not measure *B*. I say that *C* does not measure *D* either.

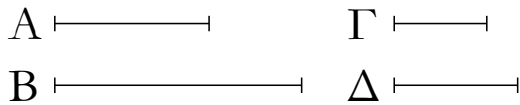
For if *C* measures *D* then *A* will also measure *B* [Prop. 8.14]. And *A* does not measure *B*. Thus, *C* will not measure *D* either.

[So], again, let *C* not measure *D*. I say that *A* will not measure *B* either.

For if *A* measures *B* then *C* will also measure *D* [Prop. 8.14]. And *C* does not measure *D*. Thus, *A* will not measure *B* either. (Which is) the very thing it was required to show.

Proposition 17

If a cube number does not measure a(nother) cube number then the side (of the former) will not measure the side (of the latter) either. And if the side (of a cube number) does not measure the side (of another cube number) then the (former) cube (number) will not measure the (latter) cube (number) either.



Κύβος γὰρ ἀριθμὸς ὁ A κύβον ἀριθμὸν τὸν B μὴ μετρεῖτω, καὶ τοῦ μὲν A πλευρὰ ἔστω ὁ Γ , τοῦ δὲ B ὁ Δ . λέγω, ὅτι ὁ Γ τὸν Δ οὐ μετρήσει.

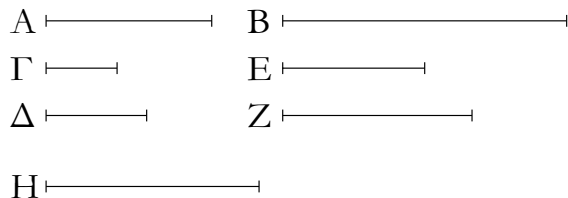
Εἰ γὰρ μετρεῖ ὁ Γ τὸν Δ , καὶ ὁ A τὸν B μετρήσει. οὐ μετρεῖ δὲ ὁ A τὸν B : οὐδ' ἄρα ὁ Γ τὸν Δ μετρεῖ.

Ἀλλὰ δὴ μὴ μετρεῖτω ὁ Γ τὸν Δ . λέγω, ὅτι οὐδὲ ὁ A τὸν B μετρήσει.

Εἰ γὰρ ὁ A τὸν B μετρεῖ, καὶ ὁ Γ τὸν Δ μετρήσει. οὐ μετρεῖ δὲ ὁ Γ τὸν Δ : οὐδ' ἄρα ὁ A τὸν B μετρήσει· ὅπερ ἔδει δεῖξαι.

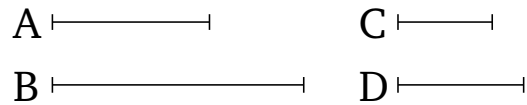
ιη'.

Δύο ὁμοίων ἐπιπέδων ἀριθμῶν εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός· καὶ ὁ ἐπίπεδος πρὸς τὸν ἐπίπεδον διπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν.



Ἐστώσαν δύο ὅμοιοι ἐπίπεδοι ἀριθμοὶ οἱ A, B , καὶ τοῦ μὲν A πλευραὶ ἔστωσαν οἱ Γ, Δ ἀριθμοί, τοῦ δὲ B οἱ E, Z . καὶ ἐπεὶ ὅμοιοι ἐπίπεδοι εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ , οὕτως ὁ E πρὸς τὸν Z . λέγω οὖν, ὅτι τῶν A, B εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ A πρὸς τὸν B διπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν E ἢ ὁ Δ πρὸς τὸν Z , τουτέστιν ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον [πλευράν].

Καὶ ἐπεὶ ἔστιν ὡς ὁ Γ πρὸς τὸν Δ , οὕτως ὁ E πρὸς τὸν Z , ἐναλλάξ ἄρα ἐστὶν ὡς ὁ Γ πρὸς τὸν E , ὁ Δ πρὸς τὸν Z . καὶ ἐπεὶ ἐπίπεδός ἐστιν ὁ A , πλευραὶ δὲ αὐτοῦ οἱ Γ, Δ , ὁ Δ ἄρα τὸν Γ πολλαπλασιάσας τὸν A πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ E τὸν Z πολλαπλασιάσας τὸν B πεποίηκεν. ὁ Δ δὴ τὸν E πολλαπλασιάσας τὸν H ποιεῖτω. καὶ ἐπεὶ ὁ Δ τὸν μὲν Γ πολλαπλασιάσας τὸν A πεποίηκεν, τὸν δὲ E πολλαπλασιάσας τὸν H πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν E , οὕτως ὁ A πρὸς τὸν H . ἀλλ' ὡς ὁ Γ πρὸς τὸν E , [οὕτως] ὁ Δ πρὸς τὸν Z : καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Z , οὕτως ὁ A πρὸς τὸν H . πάλιν, ἐπεὶ ὁ E τὸν μὲν Δ πολλαπλασιάσας τὸν H πεποίηκεν, τὸν δὲ Z πολλαπλασιάσας τὸν B πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Z , οὕτως ὁ H πρὸς τὸν B . ἐδείχθη δὲ καὶ ὡς ὁ Δ πρὸς τὸν Z , οὕτως ὁ A πρὸς τὸν



For let the cube number A not measure the cube number B . And let C be the side of A , and D (the side) of B . I say that C will not measure D .

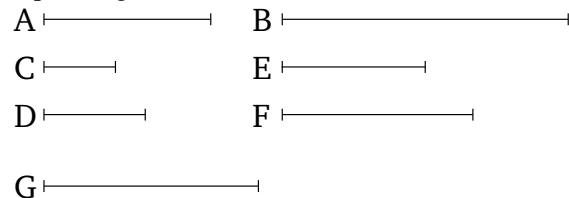
For if C measures D then A will also measure B [Prop. 8.15]. And A does not measure B . Thus, C does not measure D either.

And so let C not measure D . I say that A will not measure B either.

For if A measures B then C will also measure D [Prop. 8.15]. And C does not measure D . Thus, A will not measure B either. (Which is) the very thing it was required to show.

Proposition 18

There exists one number in mean proportion to two similar plane numbers. And (one) plane (number) has to the (other) plane (number) a squared[†] ratio with respect to (that) a corresponding side (of the former has) to a corresponding side (of the latter).



Let A and B be two similar plane numbers. And let the numbers C, D be the sides of A , and E, F (the sides) of B . And since similar numbers are those having proportional sides [Def. 7.21], thus as C is to D , so E (is) to F . Therefore, I say that there exists one number in mean proportion to A and B , and that A has to B a squared ratio with respect to that C (has) to E , or D to F —that is to say, with respect to (that) a corresponding side (has) to a corresponding [side].

For since as C is to D , so E (is) to F , thus, alternately, as C is to E , so D (is) to F [Prop. 7.13]. And since A is plane, and C, D its sides, D has thus made A (by) multiplying C . And so, for the same (reasons), E has made B (by) multiplying F . So let D make G (by) multiplying E . And since D has made A (by) multiplying C , and has made G (by) multiplying E , thus as C is to E , so A (is) to G [Prop. 7.17]. But as C (is) to E , [so] D (is) to F . And thus as D (is) to F , so A (is) to G . Again, since E has made G (by) multiplying D , and has made B (by) multiplying F , thus as D is to F , so G (is) to B [Prop. 7.17]. And it was also shown that as D (is) to F , so A (is) to G . And thus as A (is) to G , so G (is) to B . Thus, A, G, B are

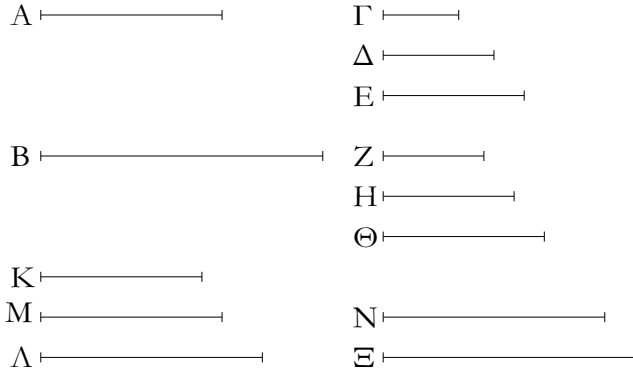
Η· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Η, οὕτως ὁ Η πρὸς τὸν Β. οἱ Α, Η, Β ἄρα ἐξῆς ἀνάλογόν εἰσιν. τῶν Α, Β ἄρα εἷς μέσος ἀνάλογόν ἐστὶν ἀριθμός.

Λέγω δὴ, ὅτι καὶ ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἥπερ ὁ Γ πρὸς τὸν Ε ἢ ὁ Δ πρὸς τὸν Ζ. ἐπεὶ γὰρ οἱ Α, Η, Β ἐξῆς ἀνάλογόν εἰσιν, ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἥπερ πρὸς τὸν Η. καὶ ἐστὶν ὡς ὁ Α πρὸς τὸν Η, οὕτως ὁ τε Γ πρὸς τὸν Ε καὶ ὁ Δ πρὸς τὸν Ζ. καὶ ὁ Α ἄρα πρὸς τὸν Β διπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Ε ἢ ὁ Δ πρὸς τὸν Ζ· ὅπερ ἔδει δεῖξαι.

† Literally, “double”.

ιθ'.

Δύο ὁμοίων στερεῶν ἀριθμῶν δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί· καὶ ὁ στερεὸς πρὸς τὸν ὅμοιον στερεὸν τριπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν.



Ἐστωσαν δύο ὅμοιοι στερεοὶ οἱ Α, Β, καὶ τοῦ μὲν Α πλευραὶ ἔστωσαν οἱ Γ, Δ, Ε, τοῦ δὲ Β οἱ Ζ, Η, Θ. καὶ ἐπεὶ ὅμοιοι στερεοὶ εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς, ἔστιν ἄρα ὡς μὲν ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ζ πρὸς τὸν Η, ὡς δὲ ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Η πρὸς τὸν Θ. λέγω, ὅτι τῶν Α, Β δύο μέσοι ἀνάλογόν ἐμπίπτουσιν ἀριθμοί, καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ.

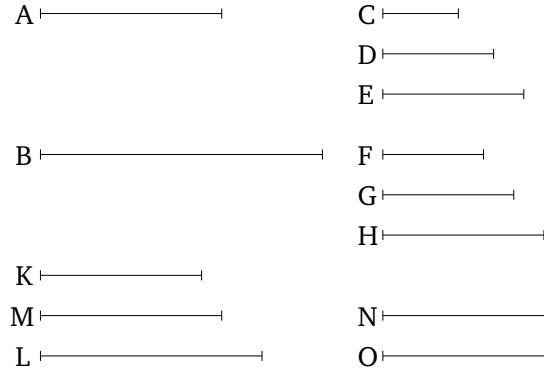
Ὁ Γ γὰρ τὸν Δ πολλαπλασιάσας τὸν Κ ποιεῖτω, ὁ δὲ Ζ τὸν Η πολλαπλασιάσας τὸν Λ ποιεῖτω. καὶ ἐπεὶ οἱ Γ, Δ τοῖς Ζ, Η ἐν τῷ αὐτῷ λόγῳ εἰσιν, καὶ ἐκ μὲν τῶν Γ, Δ ἐστὶν ὁ Κ, ἐκ δὲ τῶν Ζ, Η ὁ Λ, οἱ Κ, Λ [ἄρα] ὅμοιοι ἐπίπεδοι εἰσιν ἀριθμοί· τῶν Κ, Λ ἄρα εἷς μέσος ἀνάλογόν ἐστὶν ἀριθμός. ἔστω ὁ Μ. ὁ Μ ἄρα ἐστὶν ὁ ἐκ τῶν Δ, Ζ, ὡς ἐν τῷ πρὸ τούτου θεωρήματι ἐδείχθη. καὶ ἐπεὶ ὁ Δ τὸν μὲν Γ πολλαπλασιάσας τὸν Κ πεποίηκεν, τὸν δὲ Ζ πολλαπλασιάσας τὸν Μ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Ζ, οὕτως ὁ Κ πρὸς τὸν Μ. ἀλλ' ὡς ὁ Κ πρὸς τὸν Μ, ὁ Μ πρὸς τὸν Λ. οἱ Κ, Μ, Λ ἄρα ἐξῆς εἰσιν ἀνάλογον ἐν

continuously proportional. Thus, there exists one number (namely, G) in mean proportion to A and B .

So I say that A also has to B a squared ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) C (has) to E , or D to F . For since A, G, B are continuously proportional, A has to B a squared ratio with respect to (that A has) to G [Prop. 5.9]. And as A is to G , so C (is) to E , and D to F . And thus A has to B a squared ratio with respect to (that) C (has) to E , or D to F . (Which is) the very thing it was required to show.

Proposition 19

Two numbers fall (between) two similar solid numbers in mean proportion. And a solid (number) has to a similar solid (number) a cubed† ratio with respect to (that) a corresponding side (has) to a corresponding side.



Let A and B be two similar solid numbers, and let C, D, E be the sides of A , and F, G, H (the sides) of B . And since similar solid (numbers) are those having proportional sides [Def. 7.21], thus as C is to D , so F (is) to G , and as D (is) to E , so G (is) to H . I say that two numbers fall (between) A and B in mean proportion, and (that) A has to B a cubed ratio with respect to (that) C (has) to F , and D to G , and, further, E to H .

For let C make K (by) multiplying D , and let F make L (by) multiplying G . And since C, D are in the same ratio as F, G , and K is the (number created) from (multiplying) C, D , and L the (number created) from (multiplying) F, G , [thus] K and L are similar plane numbers [Def. 7.21]. Thus, there exists one number in mean proportion to K and L [Prop. 8.18]. Let it be M . Thus, M is the (number created) from (multiplying) D, F , as shown in the theorem before this (one). And since D has made K (by) multiplying C , and has made M (by) multiplying F , thus as C is to F , so K (is) to M [Prop. 7.17]. But, as

τῷ τοῦ Γ πρὸς τὸν Ζ λόγῳ. καὶ ἐπεὶ ἐστὶν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ζ πρὸς τὸν Η, ἐναλλάξ ἄρα ἐστὶν ὡς ὁ Γ πρὸς τὸν Ζ, οὕτως ὁ Δ πρὸς τὸν Η. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Δ πρὸς τὸν Η, οὕτως ὁ Ε πρὸς τὸν Θ. οἱ Κ, Μ, Λ ἄρα ἐξῆς εἰσὶν ἀνάλογον ἐν τε τῷ τοῦ Γ πρὸς τὸν Ζ λόγῳ καὶ τῷ τοῦ Δ πρὸς τὸν Η καὶ ἔτι τῷ τοῦ Ε πρὸς τὸν Θ. ἑκάτερος δὴ τῶν Ε, Θ τὸν Μ πολλαπλασιάσας ἑκάτερον τῶν Ν, Ξ ποιεῖτω. καὶ ἐπεὶ στερεὸς ἐστὶν ὁ Α, πλευραὶ δὲ αὐτοῦ εἰσὶν οἱ Γ, Δ, Ε, ὁ Ε ἄρα τὸν ἐκ τῶν Γ, Δ πολλαπλασιάσας τὸν Α πεποίηκεν. ὁ δὲ ἐκ τῶν Γ, Δ ἐστὶν ὁ Κ· ὁ Ε ἄρα τὸν Κ πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Θ τὸν Λ πολλαπλασιάσας τὸν Β πεποίηκεν. καὶ ἐπεὶ ὁ Ε τὸν Κ πολλαπλασιάσας τὸν Α πεποίηκεν, ἀλλὰ μὴν καὶ τὸν Μ πολλαπλασιάσας τὸν Ν πεποίηκεν, ἔστιν ἄρα ὡς ὁ Κ πρὸς τὸν Μ, οὕτως ὁ Α πρὸς τὸν Ν. ὡς δὲ ὁ Κ πρὸς τὸν Μ, οὕτως ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Ν. πάλιν, ἐπεὶ ἑκάτερος τῶν Ε, Θ τὸν Μ πολλαπλασιάσας ἑκάτερον τῶν Ν, Ξ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Ν πρὸς τὸν Ξ. ἀλλ' ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Ν καὶ ὁ Ν πρὸς τὸν Ξ. πάλιν, ἐπεὶ ὁ Θ τὸν Μ πολλαπλασιάσας τὸν Ξ πεποίηκεν, ἀλλὰ μὴν καὶ τὸν Λ πολλαπλασιάσας τὸν Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Μ πρὸς τὸν Λ, οὕτως ὁ Ξ πρὸς τὸν Β. ἀλλ' ὡς ὁ Μ πρὸς τὸν Λ, οὕτως ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ. καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ, οὕτως οὐ μόνον ὁ Ξ πρὸς τὸν Β, ἀλλὰ καὶ ὁ Α πρὸς τὸν Ν καὶ ὁ Ν πρὸς τὸν Ξ. οἱ Α, Ν, Ξ, Β ἄρα ἐξῆς εἰσὶν ἀνάλογον ἐν τοῖς εἰρημένους τῶν πλευρῶν λόγοις.

Λέγω, ὅτι καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἥπερ ὁ Γ ἀριθμὸς πρὸς τὸν Ζ ἢ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ. ἐπεὶ γὰρ τέσσαρες ἀριθμοὶ ἐξῆς ἀνάλογον εἰσὶν οἱ Α, Ν, Ξ, Β, ὁ Α ἄρα πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Α πρὸς τὸν Ν. ἀλλ' ὡς ὁ Α πρὸς τὸν Ν, οὕτως ἐδείχθη ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ. καὶ ὁ Α ἄρα πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἥπερ ὁ Γ ἀριθμὸς πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ· ὅπερ εἶδει δεῖξαι.

† Literally, "triple".

κ'.

Ἐὰν δύο ἀριθμῶν εἷς μέσος ἀνάλογον ἐμπίπτῃ ἀριθμός, ὁμοιοὶ ἐπίπεδοι ἔσονται οἱ ἀριθμοί.

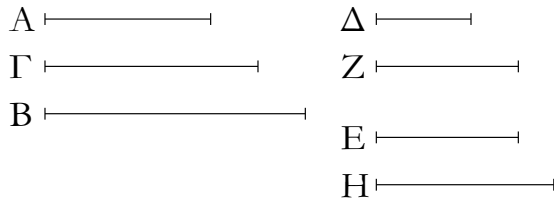
K (is) to *M*, (so) *M* (is) to *L*. Thus, *K*, *M*, *L* are continuously proportional in the ratio of *C* to *F*. And since as *C* is to *D*, so *F* (is) to *G*, thus, alternately, as *C* is to *F*, so *D* (is) to *G* [Prop. 7.13]. And so, for the same (reasons), as *D* (is) to *G*, so *E* (is) to *H*. Thus, *K*, *M*, *L* are continuously proportional in the ratio of *C* to *F*, and of *D* to *G*, and, further, of *E* to *H*. So let *E*, *H* make *N*, *O*, respectively, (by) multiplying *M*. And since *A* is solid, and *C*, *D*, *E* are its sides, *E* has thus made *A* (by) multiplying the (number created) from (multiplying) *C*, *D*. And *K* is the (number created) from (multiplying) *C*, *D*. Thus, *E* has made *A* (by) multiplying *K*. And so, for the same (reasons), *H* has made *B* (by) multiplying *L*. And since *E* has made *A* (by) multiplying *K*, but has, in fact, also made *N* (by) multiplying *M*, thus as *K* is to *M*, so *A* (is) to *N* [Prop. 7.17]. And as *K* (is) to *M*, so *C* (is) to *F*, and *D* to *G*, and, further, *E* to *H*. And thus as *C* (is) to *F*, and *D* to *G*, and *E* to *H*, so *A* (is) to *N*. Again, since *E*, *H* have made *N*, *O*, respectively, (by) multiplying *M*, thus as *E* is to *H*, so *N* (is) to *O* [Prop. 7.18]. But, as *E* (is) to *H*, so *C* (is) to *F*, and *D* to *G*. And thus as *C* (is) to *F*, and *D* to *G*, and *E* to *H*, so (is) *A* to *N*, and *N* to *O*. Again, since *H* has made *O* (by) multiplying *M*, but has, in fact, also made *B* (by) multiplying *L*, thus as *M* (is) to *L*, so *O* (is) to *B* [Prop. 7.17]. But, as *M* (is) to *L*, so *C* (is) to *F*, and *D* to *G*, and *E* to *H*. And thus as *C* (is) to *F*, and *D* to *G*, and *E* to *H*, so not only (is) *O* to *B*, but also *A* to *N*, and *N* to *O*. Thus, *A*, *N*, *O*, *B* are continuously proportional in the aforementioned ratios of the sides.

So I say that *A* also has to *B* a cubed ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) the number *C* (has) to *F*, or *D* to *G*, and, further, *E* to *H*. For since *A*, *N*, *O*, *B* are four continuously proportional numbers, *A* thus has to *B* a cubed ratio with respect to (that) *A* (has) to *N* [Def. 5.10]. But, as *A* (is) to *N*, so it was shown (is) *C* to *F*, and *D* to *G*, and, further, *E* to *H*. And thus *A* has to *B* a cubed ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) the number *C* (has) to *F*, and *D* to *G*, and, further, *E* to *H*. (Which is) the very thing it was required to show.

Proposition 20

If one number falls between two numbers in mean proportion then the numbers will be similar plane (num-

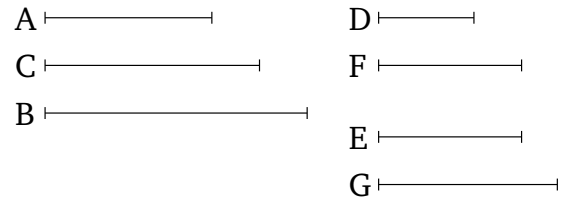
Δύο γὰρ ἀριθμῶν τῶν A, B εἷς μέσος ἀνάλογον ἐμπίπττω ἀριθμὸς ὁ Γ . λέγω, ὅτι οἱ A, B ὅμοιοι ἐπίπεδοι εἰσιν ἀριθμοί.



Εἰλήφθωσαν [γὰρ] ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς A, Γ οἱ Δ, E . ἰσάκεις ἄρα ὁ Δ τὸν A μετρεῖ καὶ ὁ E τὸν Γ . ὁσάκεις δὴ ὁ Δ τὸν A μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Z . ὁ Z ἄρα τὸν Δ πολλαπλασιάσας τὸν A πεποίηκεν. ὥστε ὁ A ἐπίπεδός ἐστιν, πλευραὶ δὲ αὐτοῦ οἱ Δ, Z . πάλιν, ἐπεὶ οἱ Δ, E ἐλάχιστοι εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Γ, B , ἰσάκεις ἄρα ὁ Δ τὸν Γ μετρεῖ καὶ ὁ E τὸν B . ὁσάκεις δὴ ὁ E τὸν B μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ H . ὁ H ἄρα τὸν E πολλαπλασιάσας τὸν B πεποίηκεν. ὁ B ἄρα ἐπίπεδος ἐστι, πλευραὶ δὲ αὐτοῦ εἰσιν οἱ E, H . οἱ A, B ἄρα ἐπίπεδοι εἰσιν ἀριθμοί. λέγω δὴ, ὅτι καὶ ὅμοιοι. ἐπεὶ γὰρ ὁ Z τὸν μὲν Δ πολλαπλασιάσας τὸν A πεποίηκεν, τὸν δὲ E πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν E , οὕτως ὁ A πρὸς τὸν Γ , τουτέστιν ὁ Γ πρὸς τὸν B . πάλιν, ἐπεὶ ὁ E ἐκάτερον τῶν Z, H πολλαπλασιάσας τοὺς Γ, B πεποίηκεν, ἔστιν ἄρα ὡς ὁ Z πρὸς τὸν H , οὕτως ὁ Γ πρὸς τὸν B . ὡς δὲ ὁ Γ πρὸς τὸν B , οὕτως ὁ Δ πρὸς τὸν E . καὶ ὡς ἄρα ὁ Δ πρὸς τὸν E , οὕτως ὁ Z πρὸς τὸν H . καὶ ἐναλλάξ ὡς ὁ Δ πρὸς τὸν Z , οὕτως ὁ E πρὸς τὸν H . οἱ A, B ἄρα ὅμοιοι ἐπίπεδοι ἀριθμοὶ εἰσιν· αἱ γὰρ πλευραὶ αὐτῶν ἀνάλογόν εἰσιν· ὅπερ ἔδει δεῖξαι.

bers).

For let one number C fall between the two numbers A and B in mean proportion. I say that A and B are similar plane numbers.



[For] let the least numbers, D and E , having the same ratio as A and C have been taken [Prop. 7.33]. Thus, D measures A as many times as E (measures) C [Prop. 7.20]. So as many times as D measures A , so many units let there be in F . Thus, F has made A (by) multiplying D [Def. 7.15]. Hence, A is plane, and D, F (are) its sides. Again, since D and E are the least of those (numbers) having the same ratio as C and B , D thus measures C as many times as E (measures) B [Prop. 7.20]. So as many times as E measures B , so many units let there be in G . Thus, G has made B (by) multiplying E [Def. 7.15]. Thus, B is plane, and E, G are its sides. Thus, A and B are (both) plane numbers. So I say that (they are) also similar. For since F has made A (by) multiplying D , and has made C (by) multiplying E , thus as D is to E , so A (is) to C —that is to say, C to B [Prop. 7.17].[†] Again, since E has made C, B (by) multiplying F, G , respectively, thus as F is to G , so C (is) to B [Prop. 7.17]. And as C (is) to B , so D (is) to E . And thus as D (is) to E , so F (is) to G . And, alternately, as D (is) to F , so E (is) to G [Prop. 7.13]. Thus, A and B are similar plane numbers. For their sides are proportional [Def. 7.21]. (Which is) the very thing it was required to show.

[†] This part of the proof is defective, since it is not demonstrated that $F \times E = C$. Furthermore, it is not necessary to show that $D : E :: A : C$, because this is true by hypothesis.

κα'.

Proposition 21

Ἐὰν δύο ἀριθμῶν δύο μέσοι ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅμοιοι στερεοὶ εἰσιν οἱ ἀριθμοί.

Δύο γὰρ ἀριθμῶν τῶν A, B δύο μέσοι ἀνάλογον ἐμπίπτέωσαν ἀριθμοὶ οἱ Γ, Δ . λέγω, ὅτι οἱ A, B ὅμοιοι στερεοὶ εἰσιν.

Εἰλήφθωσαν γὰρ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς A, Γ , Δ τρεῖς οἱ E, Z, H . οἱ ἄρα ἄκροι αὐτῶν οἱ E, H πρῶτοι πρὸς ἀλλήλους εἰσιν. καὶ ἐπεὶ τῶν E, H εἷς μέσος ἀνάλογον ἐμπίπτωκεν ἀριθμὸς ὁ Z , οἱ E, H ἄρα ἀριθμοὶ ὅμοιοι ἐπίπεδοι εἰσιν. ἔστωσαν οὖν τοῦ μὲν

If two numbers fall between two numbers in mean proportion then the (latter) are similar solid (numbers).

For let the two numbers C and D fall between the two numbers A and B in mean proportion. I say that A and B are similar solid (numbers).

For let the three least numbers E, F, G having the same ratio as A, C, D have been taken [Prop. 8.2]. Thus, the outermost of them, E and G , are prime to one another [Prop. 8.3]. And since one number, F , has fallen (between) E and G in mean proportion, E and G are

Ε πλευραὶ οἱ Θ, Κ, τοῦ δὲ Η οἱ Λ, Μ. φανερόν ἄρα ἐστὶν ἐκ τοῦ πρὸ τούτου, ὅτι οἱ Ε, Ζ, Η ἐξῆς εἰσὶν ἀνάλογον ἔν τε τῷ τοῦ Θ πρὸς τὸν Α λόγῳ καὶ τῷ τοῦ Κ πρὸς τὸν Μ. καὶ ἐπεὶ οἱ Ε, Ζ, Η ἐλάχιστοι εἰσὶ τῶν τὸν αὐτὸν λόγον ἔχόντων τοῖς Α, Γ, Δ, καὶ ἐστὶν ἴσον τὸ πλῆθος τῶν Ε, Ζ, Η τῷ πλῆθει τῶν Α, Γ, Δ, δι' ἴσου ἄρα ἐστὶν ὡς ὁ Ε πρὸς τὸν Η, οὕτως ὁ Α πρὸς τὸν Δ. οἱ δὲ Ε, Η πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς ἰσάκεις ὃ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὃ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον· ἰσάκεις ἄρα ὁ Ε τὸν Α μετρεῖ καὶ ὁ Η τὸν Δ. ὁσάκεις δὴ ὁ Ε τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ν. ὁ Ν ἄρα τὸν Ε πολλαπλασιάσας τὸν Α πεποίηκεν. ὁ δὲ Ε ἐστὶν ὁ ἐκ τῶν Θ, Κ· ὁ Ν ἄρα τὸν ἐκ τῶν Θ, Κ πολλαπλασιάσας τὸν Α πεποίηκεν. στερεὸς ἄρα ἐστὶν ὁ Α, πλευραὶ δὲ αὐτοῦ εἰσὶν οἱ Θ, Κ, Ν. πάλιν, ἐπεὶ οἱ Ε, Ζ, Η ἐλάχιστοι εἰσὶ τῶν τὸν αὐτὸν λόγον ἔχόντων τοῖς Γ, Δ, Β, ἰσάκεις ἄρα ὁ Ε τὸν Γ μετρεῖ καὶ ὁ Η τὸν Β. ὁσάκεις δὴ ὁ Ε τὸν Γ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ξ. ὁ Ξ ἄρα τὸν Η πολλαπλασιάσας τὸν Β πεποίηκεν. ὁ δὲ Η ἐστὶν ὁ ἐκ τῶν Λ, Μ· ὁ Ξ ἄρα τὸν ἐκ τῶν Λ, Μ πολλαπλασιάσας τὸν Β πεποίηκεν. στερεὸς ἄρα ἐστὶν ὁ Β, πλευραὶ δὲ αὐτοῦ εἰσὶν οἱ Λ, Μ, Ξ· οἱ Α, Β ἄρα στερεοὶ εἰσιν.

Α	—————	Θ	—————
Γ	—————	Κ	—————
Δ	—————	Ν	—————
Β	—————		

Ε	—————	Λ	—————
Ζ	—————	Μ	—————
Η	—————	Ξ	—————

Λέγω [δὴ], ὅτι καὶ ὁμοιοί. ἐπεὶ γὰρ οἱ Ν, Ξ τὸν Ε πολλαπλασιάσαντες τοὺς Α, Γ πεποίηκασιν, ἔστιν ἄρα ὡς ὁ Ν πρὸς τὸν Ξ, ὁ Α πρὸς τὸν Γ, τουτέστιν ὁ Ε πρὸς τὸν Ζ. ἀλλ' ὡς ὁ Ε πρὸς τὸν Ζ, ὁ Θ πρὸς τὸν Α καὶ ὁ Κ πρὸς τὸν Μ· καὶ ὡς ἄρα ὁ Θ πρὸς τὸν Α, οὕτως ὁ Κ πρὸς τὸν Μ καὶ ὁ Ν πρὸς τὸν Ξ. καὶ εἰσὶν οἱ μὲν Θ, Κ, Ν πλευραὶ τοῦ Α,

thus similar plane numbers [Prop. 8.20]. Therefore, let H, K be the sides of E , and L, M (the sides) of G . Thus, it is clear from the (proposition) before this (one) that E, F, G are continuously proportional in the ratio of H to L , and of K to M . And since E, F, G are the least (numbers) having the same ratio as A, C, D , and the multitude of E, F, G is equal to the multitude of A, C, D , thus, via equality, as E is to G , so A (is) to D [Prop. 7.14]. And E and G (are) prime (to one another), and prime (numbers) are also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, E measures A the same number of times as G (measures) D . So as many times as E measures A , so many units let there be in N . Thus, N has made A (by) multiplying E [Def. 7.15]. And E is the (number created) from (multiplying) H and K . Thus, N has made A (by) multiplying the (number created) from (multiplying) H and K . Thus, A is solid, and its sides are H, K, N . Again, since E, F, G are the least (numbers) having the same ratio as C, D, B , thus E measures C the same number of times as G (measures) B [Prop. 7.20]. So as many times as E measures C , so many units let there be in O . Thus, G measures B according to the units in O . Thus, O has made B (by) multiplying G . And G is the (number created) from (multiplying) L and M . Thus, O has made B (by) multiplying the (number created) from (multiplying) L and M . Thus, B is solid, and its sides are L, M, O . Thus, A and B are (both) solid.

Α	—————	Η	—————
Κ	—————	Κ	—————
Δ	—————	Ν	—————
Β	—————		

Ε	—————	Λ	—————
Ζ	—————	Μ	—————
Η	—————	Ο	—————

[So] I say that (they are) also similar. For since N, O have made A, C (by) multiplying E , thus as N is to O , so A (is) to C —that is to say, E to F [Prop. 7.18]. But, as E (is) to F , so H (is) to L , and K to M . And thus as H (is) to L , so K (is) to M , and N to O . And H, K, N are the sides of A , and L, M, O the sides of B . Thus, A and

οἱ δὲ Ξ , Λ , M πλευραὶ τοῦ B . οἱ A , B ἄρα ἀριθμοὶ ὅμοιοι στερεοὶ εἰσιν· ὅπερ ἔδει δεῖξαι.

B are similar solid numbers [Def. 7.21]. (Which is) the very thing it was required to show.

† The Greek text has “ O , L , M ”, which is obviously a mistake.

κβ'.

Ἐὰν τρεῖς ἀριθμοὶ ἐξῆς ἀνάλογον ᾧσιν, ὁ δὲ πρῶτος τετράγωνος ᾗ, καὶ ὁ τρίτος τετράγωνος ἔσται.

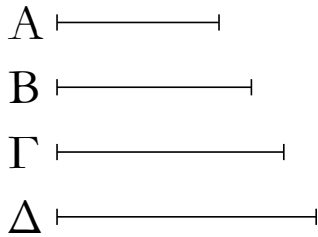


Ἐστῶσαν τρεῖς ἀριθμοὶ ἐξῆς ἀνάλογον οἱ A , B , Γ , ὁ δὲ πρῶτος ὁ A τετράγωνος ἔστω· λέγω, ὅτι καὶ ὁ τρίτος ὁ Γ τετράγωνός ἐστιν.

Ἐπεὶ γὰρ τῶν A , Γ εἷς μέσος ἀναλογόν ἐστιν ἀριθμὸς ὁ B , οἱ A , Γ ἄρα ὅμοιοι ἐπίπεδοι εἰσιν. τετράγωνος δὲ ὁ A · τετράγωνος ἄρα καὶ ὁ Γ · ὅπερ ἔδει δεῖξαι.

κγ'.

Ἐὰν τέσσαρες ἀριθμοὶ ἐξῆς ἀνάλογον ᾧσιν, ὁ δὲ πρῶτος κύβος ᾗ, καὶ ὁ τέταρτος κύβος ἔσται.

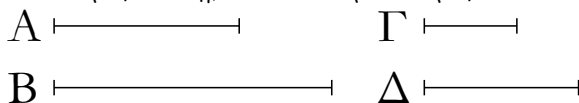


Ἐστῶσαν τέσσαρες ἀριθμοὶ ἐξῆς ἀνάλογον οἱ A , B , Γ , Δ , ὁ δὲ A κύβος ἔστω· λέγω, ὅτι καὶ ὁ Δ κύβος ἐστίν.

Ἐπεὶ γὰρ τῶν A , Δ δύο μέσοι ἀναλογόν εἰσιν ἀριθμοὶ οἱ B , Γ , οἱ A , Δ ἄρα ὅμοιοι εἰσι στερεοὶ ἀριθμοί. κύβος δὲ ὁ A · κύβος ἄρα καὶ ὁ Δ · ὅπερ ἔδει δεῖξαι.

κδ'.

Ἐὰν δύο ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχωσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, ὁ δὲ πρῶτος τετράγωνος ᾗ, καὶ ὁ δεύτερος τετράγωνος ἔσται.



Δύο γὰρ ἀριθμοὶ οἱ A , B πρὸς ἀλλήλους λόγον

Proposition 22

If three numbers are continuously proportional, and the first is square, then the third will also be square.

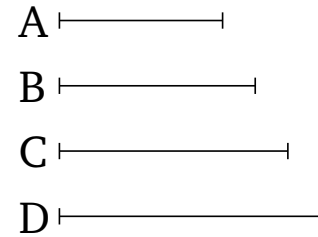


Let A , B , C be three continuously proportional numbers, and let the first A be square. I say that the third C is also square.

For since one number, B , is in mean proportion to A and C , A and C are thus similar plane (numbers) [Prop. 8.20]. And A is square. Thus, C is also square [Def. 7.21]. (Which is) the very thing it was required to show.

Proposition 23

If four numbers are continuously proportional, and the first is cube, then the fourth will also be cube.

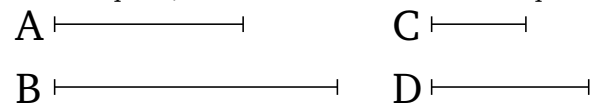


Let A , B , C , D be four continuously proportional numbers, and let A be cube. I say that D is also cube.

For since two numbers, B and C , are in mean proportion to A and D , A and D are thus similar solid numbers [Prop. 8.21]. And A (is) cube. Thus, D (is) also cube [Def. 7.21]. (Which is) the very thing it was required to show.

Proposition 24

If two numbers have to one another the ratio which a square number (has) to a(nother) square number, and the first is square, then the second will also be square.



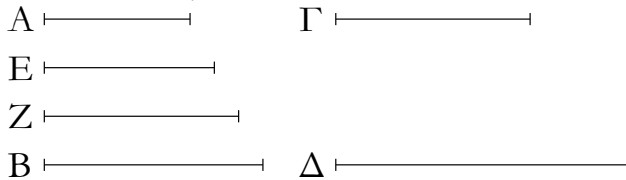
For let two numbers, A and B , have to one another

ἐχέτωσαν, ὃν τετράγωνος ἀριθμὸς ὁ Γ πρὸς τετράγωνον ἀριθμὸν τὸν Δ, ὁ δὲ Α τετράγωνος ἔστω· λέγω, ὅτι καὶ ὁ Β τετράγωνός ἐστιν.

Ἐπεὶ γὰρ οἱ Γ, Δ τετράγωνοι εἰσιν, οἱ Γ, Δ ἄρα ὅμοιοι ἐπίπεδοι εἰσιν. τῶν Γ, Δ ἄρα εἰς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. καὶ ἐστὶν ὡς ὁ Γ πρὸς τὸν Δ, ὁ Α πρὸς τὸν Β· καὶ τῶν Α, Β ἄρα εἰς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. καὶ ἐστὶν ὁ Α τετράγωνος· καὶ ὁ Β ἄρα τετράγωνός ἐστιν· ὅπερ ἔδει δεῖξαι.

κε'.

Ἐὰν δύο ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχωσιν, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμὸν, ὁ δὲ πρῶτος κύβος ᾗ, καὶ ὁ δεύτερος κύβος ἔσται.

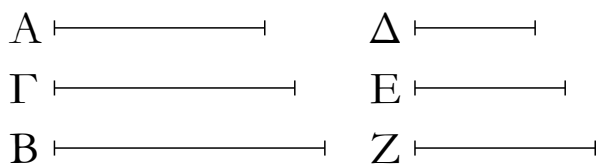


Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρὸς ἀλλήλους λόγον ἔχέτωσαν, ὃν κύβος ἀριθμὸς ὁ Γ πρὸς κύβον ἀριθμὸν τὸν Δ, κύβος δὲ ἔστω ὁ Α· λέγω [δὴ], ὅτι καὶ ὁ Β κύβος ἐστίν.

Ἐπεὶ γὰρ οἱ Γ, Δ κύβοι εἰσίν, οἱ Γ, Δ ὅμοιοι στερεοὶ εἰσιν· τῶν Γ, Δ ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. ὅσοι δὲ εἰς τοὺς Γ, Δ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν, τοσοῦτοι καὶ εἰς τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς· ὥστε καὶ τῶν Α, Β δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. ἐμπίπτέτωσαν οἱ Ε, Ζ. ἐπεὶ οὖν τέσσαρες ἀριθμοὶ οἱ Α, Ε, Ζ, Β ἐξῆς ἀνάλογόν εἰσιν, καὶ ἐστὶ κύβος ὁ Α, κύβος ἄρα καὶ ὁ Β· ὅπερ ἔδει δεῖξαι.

κε'.

Οἱ ὅμοιοι ἐπίπεδοι ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν.



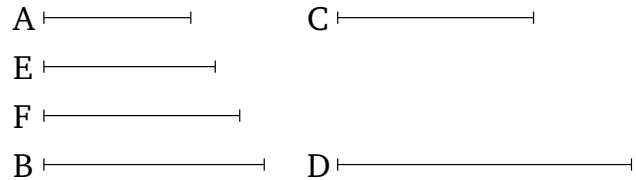
Ἐστώσαν ὅμοιοι ἐπίπεδοι ἀριθμοὶ οἱ Α, Β· λέγω, ὅτι ὁ Α πρὸς τὸν Β λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς

the ratio which the square number C (has) to the square number D . And let A be square. I say that B is also square.

For since C and D are square, C and D are thus similar plane (numbers). Thus, one number falls (between) C and D in mean proportion [Prop. 8.18]. And as C is to D , (so) A (is) to B . Thus, one number also falls (between) A and B in mean proportion [Prop. 8.8]. And A is square. Thus, B is also square [Prop. 8.22]. (Which is) the very thing it was required to show.

Proposition 25

If two numbers have to one another the ratio which a cube number (has) to a(nother) cube number, and the first is cube, then the second will also be cube.

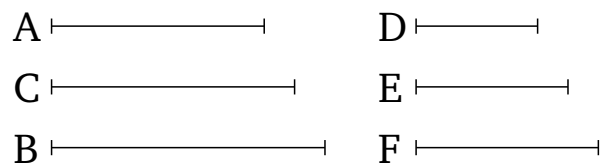


For let two numbers, A and B , have to one another the ratio which the cube number C (has) to the cube number D . And let A be cube. [So] I say that B is also cube.

For since C and D are cube (numbers), C and D are (thus) similar solid (numbers). Thus, two numbers fall (between) C and D in mean proportion [Prop. 8.19]. And as many (numbers) as fall in between C and D in continued proportion, so many also (fall) in (between) those (numbers) having the same ratio as them (in continued proportion) [Prop. 8.8]. And hence two numbers fall (between) A and B in mean proportion. Let E and F (so) fall. Therefore, since the four numbers A, E, F, B are continuously proportional, and A is cube, B (is) thus also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

Proposition 26

Similar plane numbers have to one another the ratio which (some) square number (has) to a(nother) square number.



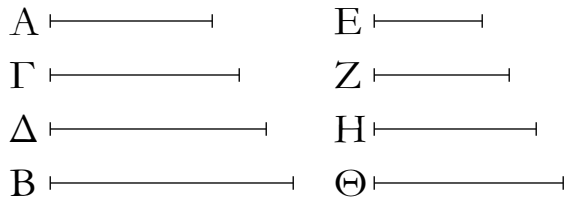
Let A and B be similar plane numbers. I say that A has to B the ratio which (some) square number (has) to

τετράγωνον ἀριθμόν.

Ἐπεὶ γὰρ οἱ A, B ὅμοιοι ἐπίπεδοι εἰσιν, τῶν A, B ἄρα εἷς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. ἐμπιπτέτω καὶ ἔστω ὁ Γ , καὶ εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἔχόντων τοῖς A, Γ, B οἱ Δ, E, Z · οἱ ἄρα ἄκροι αὐτῶν οἱ Δ, Z τετράγωνοι εἰσιν. καὶ ἐπεὶ ἔστιν ὡς ὁ Δ πρὸς τὸν Z , οὕτως ὁ A πρὸς τὸν B , καὶ εἰσιν οἱ Δ, Z τετράγωνοι, ὁ A ἄρα πρὸς τὸν B λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ὅπερ ἔδει δεῖξαι.

κζ'.

Οἱ ὅμοιοι στερεοὶ ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν.



Ἐστώσαν ὅμοιοι στερεοὶ ἀριθμοὶ οἱ A, B · λέγω, ὅτι ὁ A πρὸς τὸν B λόγον ἔχει, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν.

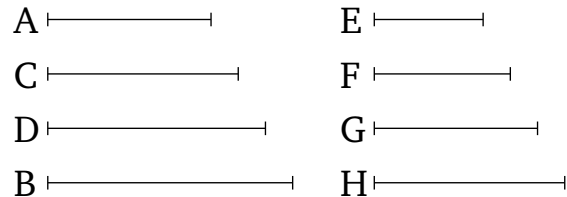
Ἐπεὶ γὰρ οἱ A, B ὅμοιοι στερεοὶ εἰσιν, τῶν A, B ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. ἐμπιπτέτωσαν οἱ Γ, Δ , καὶ εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἔχόντων τοῖς A, Γ, Δ, B ἴσοι αὐτοῖς τὸ πλῆθος οἱ E, Z, H, Θ · οἱ ἄρα ἄκροι αὐτῶν οἱ E, Θ κύβοι εἰσίν. καὶ ἔστιν ὡς ὁ E πρὸς τὸν Θ , οὕτως ὁ A πρὸς τὸν B · καὶ ὁ A ἄρα πρὸς τὸν B λόγον ἔχει, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν· ὅπερ ἔδει δεῖξαι.

a(nother) square number.

For since A and B are similar plane numbers, one number thus falls (between) A and B in mean proportion [Prop. 8.18]. Let it (so) fall, and let it be C . And let the least numbers, D, E, F , having the same ratio as A, C, B have been taken [Prop. 8.2]. The outermost of them, D and F , are thus square [Prop. 8.2 corr.]. And since as D is to F , so A (is) to B , and D and F are square, A thus has to B the ratio which (some) square number (has) to a(nother) square number. (Which is) the very thing it was required to show.

Proposition 27

Similar solid numbers have to one another the ratio which (some) cube number (has) to a(nother) cube number.



Let A and B be similar solid numbers. I say that A has to B the ratio which (some) cube number (has) to a(nother) cube number.

For since A and B are similar solid (numbers), two numbers thus fall (between) A and B in mean proportion [Prop. 8.19]. Let C and D have (so) fallen. And let the least numbers, E, F, G, H , having the same ratio as A, C, D, B , (and) equal in multitude to them, have been taken [Prop. 8.2]. Thus, the outermost of them, E and H , are cube [Prop. 8.2 corr.]. And as E is to H , so A (is) to B . And thus A has to B the ratio which (some) cube number (has) to a(nother) cube number. (Which is) the very thing it was required to show.

