SGN – Assignment #1

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1 Impulsive guidance

Exercise 1

Let $\mathbf{x}(t) = \varphi(\mathbf{x}_0, t_0; t)$ be the flow of the geocentric two-body model. 1) Using one of Matlab's built-in integrators, implement and validate* a propagator that returns $\mathbf{x}(t)$ for given $\mathbf{x}_0$, t_0, t, and μ. 2) Given the pairs $\{\mathbf{r}_1, \mathbf{r}_2\}$ and $\{t_1, t_2\}$, develop a solver that finds $\mathbf{v}_1$ such that $\mathbf{r}(t_2) = \mathbf{r}_2$, where $(\mathbf{r}(t), \mathbf{v}(t))^\top = \varphi((\mathbf{r}_1, \mathbf{v}_1)^\top, t_1; t_2)$ (Lambert's problem). To compute the derivatives of the shooting function, use either a) finite differences or b) the state transition matrix $\Phi = \mathrm{d}\varphi/\mathrm{d}\mathbf{x}_0$. Validate the algorithms against the classic Lambert solver. 3) Using the propagator of point 1) in the heliocentric case, and reading the motion of the Earth and Mars from SPICE, solve the shooting problem

$$\min_{\mathbf{x}_1, t_1, t_2} \ \Delta v \quad \text{s.t.} \begin{cases} \mathbf{r}_1 = \mathbf{r}_E(t_1) \\ \mathbf{r}(t_2) = \mathbf{r}_M(t_2) \\ t_1^L \leq t_1 \leq t_1^U \\ t_2^L \leq t_2 \leq t_2^U \\ t_2 \geq t_1 \end{cases} \tag{1}$$

where $\Delta v = \Delta v_1 + \Delta v_2$, $\boldsymbol{\Delta}\mathbf{v}_1 = \mathbf{v}_1 - \mathbf{v}_E(t_1)$, $\boldsymbol{\Delta}\mathbf{v}_2 = \mathbf{v}(t_2) - \mathbf{v}_M(t_2)$. $\mathbf{x}_1 = (\mathbf{r}_1, \mathbf{v}_1)^\top$, and $(\mathbf{r}(t), \mathbf{v}(t))^\top = \varphi(\mathbf{x}_1, t_1; t_2)$. Define lower and upper bounds, and make sure to solve the problem stated in Eq. (1) for different initial guesses.

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1) In order to validate the propagator, it has been taken the position $\mathbf{r}_0 = [29597.43\ 0\ 0]^\mathrm{T}$ km and its corresponding circular velocity along the equatorial orbit computed as $\mathbf{v}_0 = \sqrt{\frac{\mu}{r}}$ with a time of propagation corresponding to the period $T = \sqrt{\frac{a^3}{\mu}}$. The error between initial and final instant is $\sim 10^{-9}$ km which is not even measurable, hence it validates the propagator.

2) Lambert's problem initial conditions: $\mathbf{r}_1 = \mathbf{r_0}$, $\mathbf{r}_2 = [-10000\ 8500\ 10]^\mathrm{T}$ km, $\mathbf{v}_{guess} = \mathbf{v_0}$ with $\{t_1 = 0, t_2 = \frac{T}{2}\}$. The problem has been solved with: a classic Lambert's solver, a shooting method with variational approach and with forward finite differences. The solution has been displayed in Figure 1.

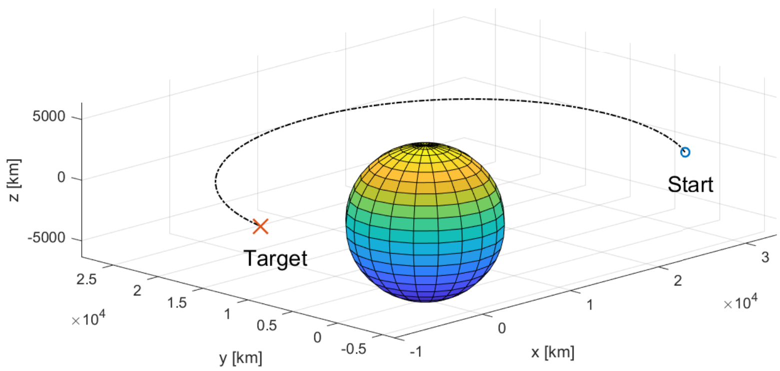

Figure 1: Lambert's problem solution.

*It can be done by taking $\mathbf{x}_0$ at the periapsis of elliptic orbits and t_f equal to their period; get μ from SPICE.

Since Newton's method is a local solver, a first guess solution sufficiently close to the exact one is needed. Luckily the zeros of this problem are just 2, those associated to the prograde and retrograde orbits, therefore the algorithm is quite robust.
As highlighted in Table 1, the variational approach takes much less steps and has also a higher accuracy because it uses the exact STM in the 2-body problem instead of an approximated one as finite differences do.

Table 1: Differences on $\mathbf{v}_1$ in shooting methods with respect to classic Lambert's solver.

Initial velocity guess	Lambert's solver	Error	N° of cycles
$\sqrt{\frac{\mu}{r}}$	Variational approach	$2.385 \cdot 10^{-7}$	11
	Finite differences	$1.278 \cdot 10^{-6}$	47

The shooting methods has stopping criteria based on the maximum number of cycles (100) and on the tolerance at the final position (set to 10^{-4} km).

3) The solution of Earth-Mars transfer has been recovered with the function `fmincon` of `MATLAB` with `'interior-point'` as algorithm (gradiental method) as minimization algorithm. It has been chosen the time span from 2033 to 2036 as shown in Table 2 with different departure and arrival time guesses.

Table 2: Different initial guesses solutions.

Dep. & Arr. time span	Dep. & Arr. guess	dv, km/s	Dep. & Arr. solution
01-04-2033/01-04-2036	01-10-2035	5.5719	26-06-2035
01-09-2033/01-09-2036	20-05-2036		13-01-2036
01-04-2033/01-04-2036	13-10-2033	6.9557	27-04-2033
01-09-2033/01-09-2036	20-06-2034		26-01-2034

The behaviour is typical of a local optimizer therefore the overall solution can change according to the dates provided as initial conditions. The iterations, displayed in Figure 2 and in Figure 3, highlight a fast and robust convergence thanks to the non-dimensionalization made on the minimizing variables which makes the solution spanning both in positions/velocities and times along the whole interval. Lower and upper boundaries have not been imposed (except for the ones stated in Equation 1), in order to use the algorithm in the most generic way.

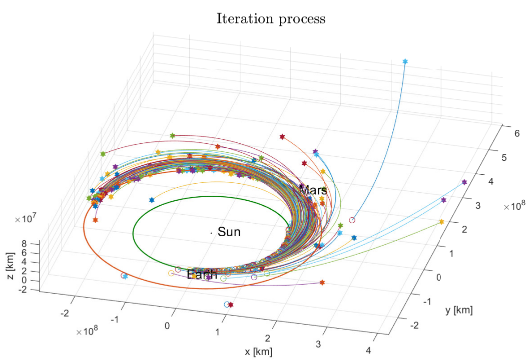

Figure 2: Iteration on the 1st guess.

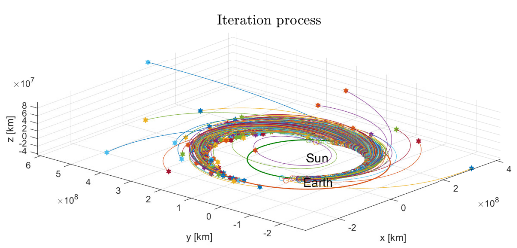

Figure 3: Iteration on the 2nd guess.

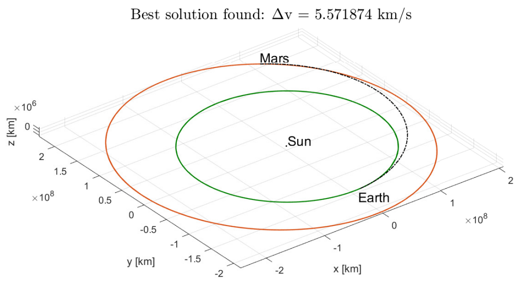

Figure 4: Solution of the 1st guess.

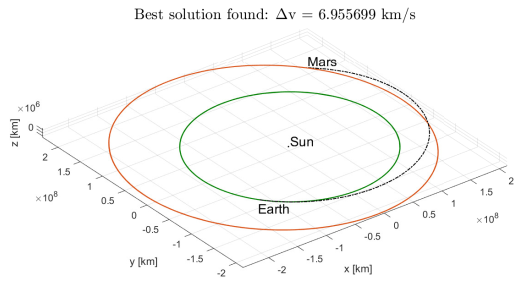

Figure 5: Solution of the 2nd guess.

Exercise 2

A spacecraft is initially parked on a 200 km circular orbit around the Earth. The final destination is the Earth–Moon Lagrange point L_2 (EML2)†. The spacecraft is transferred by applying a sequence of three impulsive maneuvers that mimic a bi-elliptic transfer, namely: i) a first impulse injects the spacecraft into an "elliptic orbit 1" with apogee higher than 500,000 km; ii) a second impulse is carried out at the apogee of elliptic orbit 1 to achieve "elliptic orbit 2", having the perigee at EML2; iii) a third impulse is carried out at the perigee of elliptic orbit 2 to achieve the EML2 state. 1) Using multiple shooting, formalize an unambiguous statement of the problem (akin to the one of Eq. (1)) that minimizes the cost of the transfer by considering that the spacecraft is subject to the gravitational attractions of the Earth, Moon, and Sun all acting simultaneously. 2) Develop the code that solves the problem formulated in point 1) and solve it for arbitrary values of the departure epoch, transfer time, and number of segments. 3) Repeat point 2) for departure epochs spanning an entire year (two points per month of suggested discretization). Plot the optimal transfer cost as a function of departure epoch.

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1)

$$\min_{\mathbf{x}_1,\mathbf{x}_m,t_1,t_2} \Delta v \quad \text{s.t.} \begin{cases} \mathbf{r}_1(t_1) = \mathbf{r}_{park} \\ \mathbf{r}_1(t_2) = \mathbf{r}_2(t_2) \\ \mathbf{r}_2(t_3) = \mathbf{r}_{L_2}(t_3) \\ \mathbf{r}_2(t_2) \geq 500000\,\text{km} \\ t_1^L \leq t_1 \leq t_1^U \\ t_3^L \leq t_3 \leq t_3^U \\ t_1 \leq t_2 \leq t_3 \end{cases} \tag{2}$$

2) The NLP solution of the problem stated in Equation 2 has been solved with 2 segments using `MATLAB fmincon` with `'interior-point'` algorithm. Since the optimizer needs a good initial guess, it has been chosen as initial condition a trajectory whose maneuvering point is very far from Earth both because an external capture from the Moon requires less Δv [1] and in order to use the L_2 of Earth-Sun system (ESL2) ($\sim 1.5 \cdot 10^6$ km) [2] to perform the change of plane with the lowest fuel consumption possible. The best trajectory, whose features have been resumed in table below, has been shown in Figure 6 and in Figure 7 with all the wanted properties previously written down. For the integration it has been used `ode113` with the system in the Earth reference frame to whom it has been added the non-inertial term of Equation 3.

$$\ddot{\mathbf{r}} = -G\sum_{j=1}^{3} m_j \left(\frac{\mathbf{d}_j}{d_j^3} + \frac{\boldsymbol{\rho}_j}{\rho_j^3}\right) \tag{3}$$

†EML2 can be modeled as a point aligned with the Earth–Moon line, 60,000 km beyond the Moon, and having the velocity of the Moon.

Table 3: Characteristics of the best bielliptic trajectory achieved

Departure	11/11/2021-17:32	Δv_1	3185 m/s
Manoeuvre	04/12/2021-09:20	Δv_2	466 m/s
Arrival	29/12/2021-19:07	Δv_3	20 m/s
Duration	48.3 days	Δv_{tot}	3671 m/s

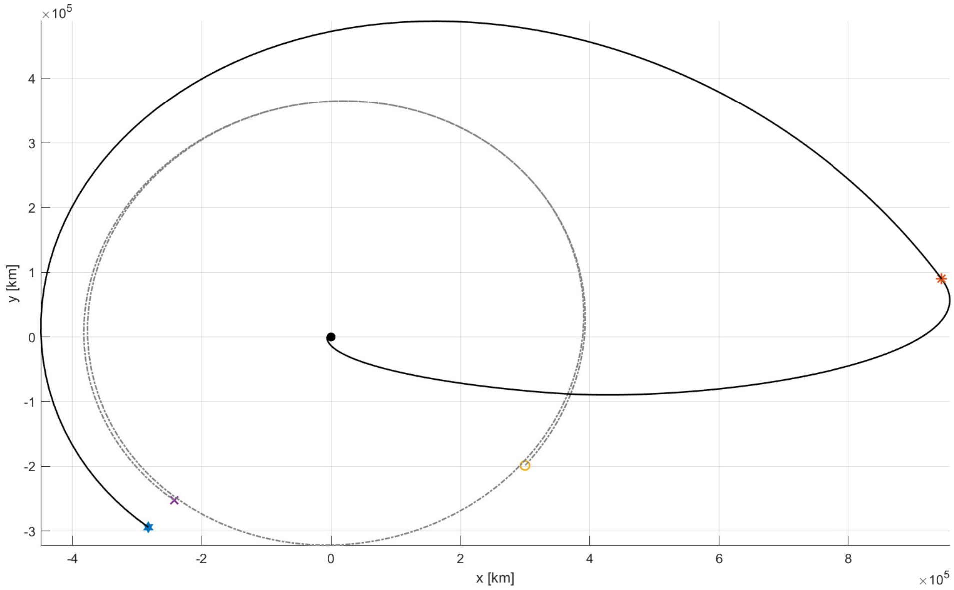

Figure 6: Front view of the best bielliptic trajectory for Earth-L$_2$ (EML2).

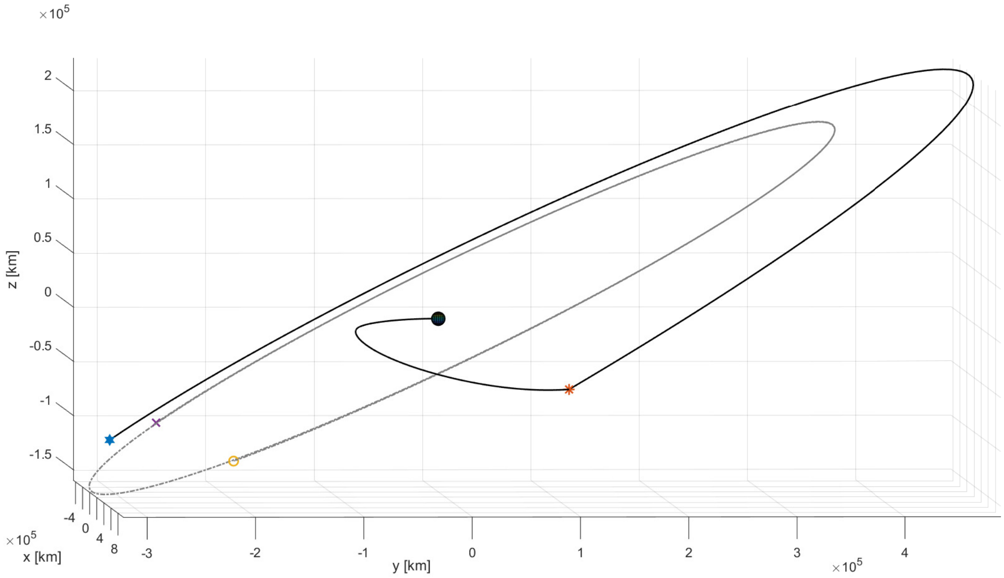

Figure 7: Lateral view of the best bielliptic trajectory for Earth-L$_2$ (EML2), highlighting the change of plane.

It has been noticed that the solution can be achieved only if the `'reltol'` and `'abstol'` have been set at least 10^{-10}, otherwise `fmincon` will converge to a higher Δv. On the other hand, decreasing the tolerance involves much more `'MaxFunctionEvaluations'`, hence 10^{-10} is the best tradeoff in accuracies.

3) Figure 8 has been recovered with one iteration every 2.5 days and clearly shows how frequent are peaks in the Δv required for the transfer. This is due to a bad alignment of the three bodies, in particular at those dates the Earth-Maneuvering point direction is opposite to the Earth-Sun direction because the Sun decelerates the spacecraft in its first branch of the bielliptic and accelerates it in the second one making the Moon capture much more difficult.

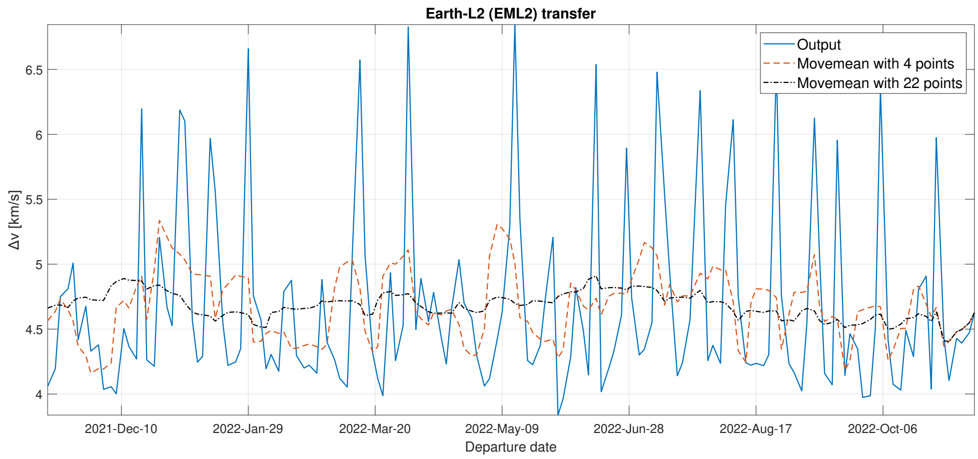

Figure 8: Transfer cost change in a year, results filtered also with a moving average filter.

Best maneuvering points are indeed the ones which are helped by the Sun in closing the last branch, hence the Moon capture requires much less Δv. Those points have usually the Earth-Moon-Maneuvering point aligned and orthogonal to the Sun. The higher frequency oscillations happen every $\sim$15 days and they are far more important than the lower frequency ones whose period is every $\sim$6 months with peaks in June and December, which correspond to the furthest distance and the minimum distance respectively with respect to the Earth.

2 Continuous guidance

Exercise 3

A spacecraft equipped with low-thrust propulsion moves in the geocentric two-body problem. The spacecraft has to accomplish a *time-optimal* transfer from the initial point $(\mathbf{r}_0, \mathbf{v}_0, m_0)^\top$ to the final point $(\mathbf{r}_f, \mathbf{v}_f)^\top$. 1) Write down the spacecraft equations of motion, the costate dynamics, and the zero-finding problem for the unknowns $\{\boldsymbol{\lambda}_0, t_f\}$. 2) Solve the problem with the following data: $\mathbf{r}_0 = (0, -29597.43, 0)^\top$ km, $\mathbf{v}_0 = (1.8349, 0.0002, 3.1783)^\top$ km/s, $m_0 = 735$ kg; $\mathbf{r}_f = (0, -29617.43, 0)^\top$ km, $\mathbf{v}_f = (1.8371, 0.0002, 3.1755)^\top$ km/s; $T_{\max} = 100$ mN, $I_{sp} = 3000$ s. 3) Optional: solve the problem in 2) for several values of $T_{\max}$ and plot $t_f(T_{\max})$.

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1) A *time-optimal* problem involves the control to be always at its maximum, hence it has been imposed $u = 1$. The problem has been stated as a shooting problem solved with `fsolve` of `MATLAB` with `'Levenberg-Marquardt'` algorithm and formulated as:

$$\begin{cases} \dot{\mathbf{r}} = \mathbf{v} \\ \dot{\mathbf{v}} = -\frac{\mu}{r^3}\mathbf{r} - u\frac{T_{max}}{m}\frac{\boldsymbol{\lambda_v}}{||\boldsymbol{\lambda_v}||} \\ \dot{m} = -u\frac{T_{max}}{I_{sp}g_0} \\ \dot{\boldsymbol{\lambda}}_{\boldsymbol{r}} = -\frac{3\mu}{r^5}(\mathbf{r}\cdot\boldsymbol{\lambda_v})\mathbf{r} + \frac{\mu}{r^3}\boldsymbol{\lambda_v} \\ \dot{\boldsymbol{\lambda}}_{\boldsymbol{v}} = -\boldsymbol{\lambda_r} \\ \dot{\lambda}_m = -u||\boldsymbol{\lambda_v}||\frac{T_{max}}{m^2} \end{cases} \quad \text{with I.Cs.:} \quad \begin{cases} \mathbf{r}(t_0) = \mathbf{r}_0 \\ \mathbf{v}(t_0) = \mathbf{v}_0 \\ m(t_0) = m_0 \\ \boldsymbol{\lambda_r}(t_0) = \boldsymbol{\lambda_{r_0}} \\ \boldsymbol{\lambda_v}(t_0) = \boldsymbol{\lambda_{v_0}} \\ \lambda_m(t_0) = \lambda_{r_0} \end{cases}$$

$$\text{and s.t.:} \quad \begin{cases} \mathbf{r}(t_f) = \mathbf{r}_f \\ \mathbf{v}(t_f) = \mathbf{v}_f \\ \lambda_m(t_f) = 0 \end{cases}$$

2) The guesses of $\boldsymbol{\lambda}_\mathbf{0}^{(\mathbf{0})}$ have been taken as random number in the range:

$$-10 \leq \boldsymbol{\lambda^{(0)}}_{\boldsymbol{r_0}} \leq 10 \qquad -10^4 \leq \boldsymbol{\lambda^{(0)}}_{\boldsymbol{v_0}} \leq 10^4 \qquad 50 \leq \lambda^{(0)}{}_{m_0} \leq 100$$

Indeed final solution is:

$$\boldsymbol{\lambda}_0 = \begin{bmatrix} 18.3278 \\ 97.2887 \\ 31.5962 \\ 5186.0074 \\ 9729.7723 \\ 6490.0222 \\ 23.2475 \end{bmatrix} \quad \text{and} \quad t_f = 50790\,\text{s}$$

which is close to the initial guess $\boldsymbol{\lambda}_0^{(\mathbf{0})}$ and has a time higher than the one of an impulsive manoeuvre with the same semi-major axis. These I.Cs. lead to the low thrust manoeuvre whose initial and final point have been shown in Figure 9.

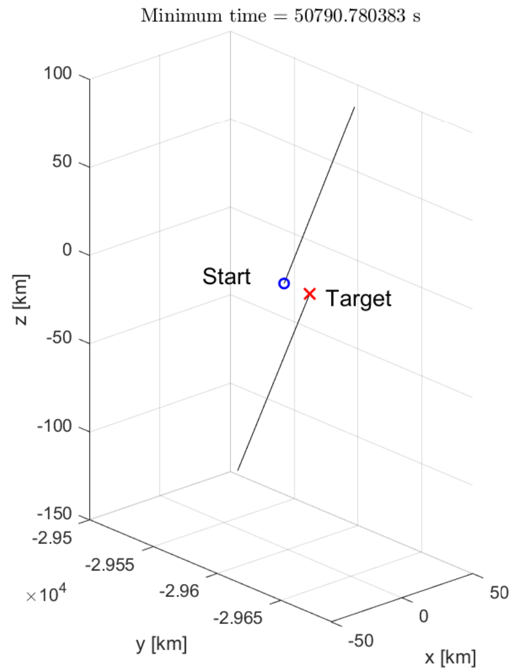

Figure 9: Low thrust manoeuvre starting and arrival point.

References

[1] N. Assadian and S. H. Pourtakdoust. "Multiobjective genetic optimization of Earth–Moon trajectories in the restricted four-body problem". In: *Advances in Space Research* 45.3 (2010), pp. 398–409.

[2] F. Topputo. "On optimal two-impulse Earth–Moon transfers in a four-body model". In: *Celestial Mechanics and Dynamical Astronomy* 117.3 (2013), pp. 279–313.